Journal of Banking & Finance 46 (2014) 233–245

Contents lists available at ScienceDirect

Journal of Banking & Finance

journal homepage: www.elsevier.com/locate/jbf

Stability analysis of financial contagion due to overlapping portfolios

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ARTICLE INFO

Article history:
Received 12 September 2013
Accepted 22 May 2014
Available online 6 June 2014

JEL classification:
G01
G11

Keywords:
Systemic risk
Network models
Contagion

ABSTRACT

Common asset holdings are widely believed to have been the primary vector of contagion in the recent financial crisis. We develop a network approach to the amplification of financial contagion due to the combination of overlapping portfolios and leverage, and we show how it can be understood in terms of a generalized branching process. This can be used to compute the stability for any particular configuration of portfolios. By studying a stylized model we estimate the circumstances under which systemic instabilities are likely to occur as a function of parameters such as leverage, market crowding, diversification, and market impact. Although diversification may be good for individual institutions, it can create dangerous systemic effects, and as a result financial contagion gets worse with too much diversification. There is a critical threshold for leverage; below it financial networks are always stable, and above it the unstable region grows as leverage increases. Note that our model assumes passive portfolio management during a crisis; however, we show that dynamic deleveraging during a crisis can amplify instabilities. The financial system exhibits "robust yet fragile" behavior, with regions of the parameter space where contagion is rare but catastrophic whenever it occurs. Our model and methods of analysis can be calibrated to real data and provide simple yet powerful tools for macroprudential stress testing.

1. Introduction

The 2007–2009 financial crisis highlighted the complex interconnections between financial institutions and made it clear that we need a better understanding of how financial contagion propagates and the circumstances under which it is amplified (Gai and Kapadia, 2010; Haldane and May, 2011; Allen et al., 2012; Ibragimov et al., 2011; Gai et al., 2011; May and Arinaminpathy, 2010; Amini et al., 2013; Georg, 2013; Caccioli et al., 2012a). Financial contagion comes through different channels, including (i) counterparty risk, (ii) roll-over risk, and (iii) common asset holdings, i.e. *overlapping portfolios*. Of these the first two have so far received the most attention, even though the primary problem is believed by many to have been due to the third. Our goal in this paper is to remedy this by gaining a better understanding of the problem of overlapping portfolios. To do this we develop a method of computing the stability of financial networks under contagion due to overlapping portfolios. We first show how to compute the stability of any particular set of overlapping portfolios. Then, to understand the factors that determine network stability, we develop and study a stylized model, and suggest how it can be extended to be more realistic. This model can be regarded as a multiple asset extension of the single asset model developed in reference Caccioli et al. (2012b).[1]

Inter-institutional lending drives the problem of counterparty and roll-over risk. Counterparty risk occurs when a bankrupt institution is unable to pay its debts and consequently causes other institutions to fail (Staum, 2013). Roll-over risk occurs when financial institutions depend on short term lending for liquidity and their creditors stop lending because they fail or are under stress, so that they are no longer able to borrow and consequently fail or become under stress (Gai et al., 2011). These have now been extensively studied and we are rapidly developing better insight into the circumstances where interbank lending causes problems (see for instance Gai and Kapadia, 2010, May and Arinaminpathy, 2010, Staum, 2013).

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[1] Reference Caccioli et al. (2012b) considered the properties of leveraged single asset portfolios. It was shown that under deleveraging market impact can cause bankruptcy if leverage is too large.

http://dx.doi.org/10.1016/j.jbankfin.2014.05.021
0378-4266/

Financial contagion due to overlapping portfolios is driven by common asset holdings (May and Arinaminpathy, 2010; Beale et al., 2011). In the event that an asset price fluctuation causes an institution to fail, the resulting "fire sale" of assets by that institution further depresses prices, which in turn may cause other institutions to fail, causing a spiral of selling and further asset price decreases. This also induces correlations between different assets that further exacerbate the problem (Cont and Wagalath, 2012).

The problem of overlapping portfolios is very general. It occurs even without inter-institutional lending, and applies to any institutions that manage money. Although this can occur even without leverage,[2] the use of leverage makes it particularly acute. We are particularly interested in the banking system, where it is not uncommon for investments to be leveraged by a factor of 30 or more, but our analysis applies equally well to hedge funds or any other financial institutions that make leveraged investments. For convenience we will use the word *bank* to refer to institutions in general, but the reader should bear in mind that our model applies equally well to any leveraged financial institution.

The problem of overlapping portfolios has previously been considered in the literature. Cifuentes et al. (2005), as well as Nier et al. (2007) and Gai and Kapadia (2010), consider for instance the situation where one asset is held by banks engaged in interbank lending. In these models contagion occurs because of fire-sales and counterparty loss, while May and Arinaminpathy (2010) and Arinaminpathy et al. (2012) introduce further refinements to account also for loss of confidence effects. In these papers, however, liquidation effects are considered on top of counterparty or roll-over risk, and all banks are usually assumed to invest in the same asset class. Here we are interested in the situation in which shocks can propagate between different financial institutions through a pattern of local portfolio overlaps (e.g. a bank has assets in common with some banks, that have in turn other assets in common with other banks, etc.). In this respect, our model can be seen as a generalization of the fire-sale dynamics considered in Cifuentes et al. (2005) to the case of many assets, and may represent a theoretical complement to recent empirical investigations carried on by Cai et al. (2011), who study the relation between overlapping portfolios and systemic risk in the context of syndicated lending, and Huang et al. (2013), who use balance sheet data to study the effect of fire-sales for US commercial banks.

Our work is also related to the model introduced by Wagner (2011). In his paper Wagner considers a three-period model in which investors are allowed to invest in two assets, and shows that in equilibrium investors choose to invest in heterogeneous portfolios because of the risk of joint portfolio liquidation. Our model complements and builds on this earlier work. Unlike Wagner, we allow for an arbitrary portfolio structure, and thus from a structural point of view our model can be considered a generalization. This general setting, however, prevents us from being able to compute the equilibrium. From this point of view our model has a different intent: We address the question of how to compute the stability of the banking network with respect to overlapping portfolios, and solve some idealized cases with varying network structure, without asking what portfolios banks should hold in an equilibrium setting. The microfoundation provided by Wagner's model justifies the assumption that investors have different portfolios. The last point was also recently made by Gordy and Lutkebohmert (2013), who suggest a procedure to adjust capital requirements to account for banks not being perfectly diversified. Another possible justification would be that of the existence of a diversification cost, as assumed for instance by Corsi et al. (2013), who study the effect of financial innovation on systemic stability.

Our model is simple: we assume that banks own a portfolio of assets, that when a bank goes bankrupt due to a loss in the value of its portfolio it sells its assets, and that this in turn causes these assets to be devalued according to a simple market impact function relating the size of the sale to the change in price.

The model we consider is purely mechanistic, i.e. we do not attempt to describe decision-making processes by banks. Our goal is instead to understand the stability of a given configuration of bank portfolios, without trying to understand why the banks might have formulated their portfolios in any particular configuration. As an approximation we assume portfolios are fixed unless there is a default, in which case all the assets in the portfolio are fully liquidated. This causes a depression in the price of the assets, which can trigger further defaults. Under this assumption we perform a macroprudential stress test by applying localized shocks affecting either a single bank or a single asset. After the initial shock is applied we test to see whether it causes any defaults; if so we iterate the process as needed until either there are no more failures or all banks have failed. The only trades during the course of the dynamics are fire sales triggered by insolvency. This basic method has a long history in the study of counterparty risk (Nier et al., 2007; Gai et al., 2011; Upper, 2011; Staum, 2013).

We justify the assumption of fixed portfolios on that basis that during times of crisis banks have insufficient time to deleverage or rebalance their portfolios. This is justified by the fact that most markets are illiquid in comparison to the size of positions that are typically held by large institutions. Even for relatively liquid assets such as large cap stocks, for a typical case the daily volume is a small fraction of market cap. Bouchaud (2011) reports, for example, that for a large cap stock the daily volume is the order of 10^{-3} of market cap. Studies of large institutional trades have shown that during normal times investors can trade up to 20% of daily volume without suffering extreme market impact; the fact that there are very few examples of investors trading more than this suggests that it is widely viewed as a limit on what is possible. Combining these numbers indicates that an investor holding 1% of market cap requires at least 20 days to unwind a position. (This issue is discussed extensively in Caccioli et al. (2012b)).

In times of crisis the situation is worse. Beltran et al. (2013) make the point that during the subprime crisis large banks kept CDOs and other mortgage backed securities in their books because current prices were well below the fundamental value of the underlying assets and "selling in this distressed market would have crystallized losses and eroded their capital position even further". Furthermore, during a crisis it may become difficult for potential buyers to raise the funds needed to purchase the assets, even at depressed prices (Krishnamurthy, 2010). The result can cause a self-reinforcing, sharp drop in liquidity, in which no one is able to trade. The fact that prices are below their fundamental value may trigger a de-leveraging trap, in which leveraged investors cannot liquidate their positions without incurring large losses or even insolvency (see Caccioli et al. (2012b)).

Of course, in some circumstances banks may be able to see crises coming and with sufficient warning, may be able to take action to prevent it. The resulting deleveraging can induce correlations in asset prices that are interesting in and of themselves; for an example see Corsi et al. (2013). A detailed analysis is beyond the scope of the current model, which provides a good approximation only in situations where the market moves sufficiently fast that deleveraging is impossible. Nonetheless, at the end of the paper we present a simple extension in which we relax this assumption by allowing banks to sell a fraction of their portfolios in response to a devaluation of assets. In this context, we show that the attempt by banks to

[2] In this paper we assume that institutions sell assets only when they become bankrupt, in which case the problem of financial contagion occurs only when leverage is used. In general asset sales may be triggered by losses that are less severe, for example if investment funds are forced to liquidate even when they are solvent, as occurred during the stat-arb meltdown in 2007.

reduce their risk actually increases the instability of the system with respect to the benchmark case of banks with a passive strategy, due to the fact that the decrease in the risk of individual banks is overwhelmed by the systemic effect of their selling in tandem. Moreover, we also discuss how the results of this paper can lay the framework for an early warning capability, in which a trusted authority such as a central bank might monitor portfolios to detect when the financial system is on the verge of becoming unstable.

Our focus in this paper is in understanding the specific role of market impact and portfolio overlaps as a contagion mechanism between leveraged financial institutions. To this end, we consider a network of banks and assets, and we test how the average level of diversification in bank portfolios, the ratio of the number of banks to the number of assets (crowding), and the leverage attained by banks impact the stability of the system with respect to an initial shock affecting a single asset or bank.

The stability of the system will be measured in terms of the probability of observing a global cascade of failures, with a smaller probability being associated with a higher stability. A global cascade of failures, in this context, refers to the failure of a significant fraction of the banks: that is, a non-zero fraction in the limit of infinite network size. By mapping our model onto a generalized branching process, we show analytically that there is a region in parameter space where global cascades of failures occur. One advantage of this mechanistic approach is that it can in principle be calibrated against real data and used to perform stress tests on real financial systems.

We find that, as the diversification of the banks' portfolios increases, the system undergoes two phase transitions, with a region in between where global cascades occur. Below the first transition, banks are not interconnected enough for shocks to propagate in the network. Above the second transition, banks are robust to devaluations in a few of their assets. In between these two transitions, banks are both vulnerable to shocks in their asset prices, and interconnected enough for these shocks to spread. We also find that more leverage increases the overall instability of the network and that the system exhibits a "robust yet fragile" behavior, with regions of parameter space where contagion is rare but the whole system is brought down whenever it occurs.

The paper is organized as follows. In the next section we introduce the model. In Section 3 we map the model into a generalized branching process and present the analytical approach that allows us to identify the region of phase space where global cascades occur. In Section 4 we report results from numerical simulations exploring how stability of banking systems depends on parameters and network properties. In Section 5 we compare the results of numerical simulations to those of stability analysis. In Section 6 we extend the model allowing banks to de-leverage in response to on-going fire sales, and we present our conclusions in the last section.

2. The model

2.1. Banks, assets, and cascades of bankruptcies

We consider a representation of a financial system given in terms of a network of N banks and M assets. Whenever a bank invests in an asset, we draw a link in the network connecting that bank to that asset. The resulting network is bipartite (see Fig. 1 for a cartoon representation of a bipartite network), meaning that there are two groups of nodes (banks and assets) and that there are links only between these two groups.[3]

[3] The model we consider is in principle not restricted to a specific network, but rather can be studied on bipartite networks with different features (for instance different degree distributions). We therefore introduce the model in its full generality in this section, and specialize to the case of Erdős–Renyi random networks in Section 3.3.

The number of assets in the portfolio of bank i, i.e. the number of links of the corresponding node, is its degree k_i. The *average diversification*, i.e. the average degree of banks in the network, is then

$$\mu_b = \frac{1}{N}\sum_{i=1}^{N} k_i, \tag{1}$$

where the sum runs over all N banks. Conversely, the number of banks that hold asset j in their portfolio is its degree ℓ_j, and the average degree of the assets is

$$\mu_a = \frac{1}{M}\sum_{j=1}^{M} \ell_j. \tag{2}$$

Since each link connects a bank to an asset, the total degree of the banks must equal the total degree of the assets, so

$$\mu_b N = \mu_a M. \tag{3}$$

Although a complete description of the network's topology would require more information, a rough characterization can be given in terms of two parameters, μ_b and $n = N/M$. The *crowding parameter* n is a measure of the density of institutions choosing their investments from the same pool of assets. If $n \ll 1$ the market is not crowded because the number of banks is much smaller than the number of assets available for investment. In this case there is a small chance for portfolios to overlap. If $n \gg 1$, in contrast, the market is crowded because the number of banks is much larger than the number of assets, and their portfolios are strongly overlapping.

Each solvent bank i holds a portfolio $\{Q_{i,1}, \ldots, Q_{i,M}\}$. Its value at time t is

$$A_i^t = \sum_{j=1}^{M} Q_{ij} p_j^t,$$

where Q_{ij} is the number of shares of asset j held by bank i and p_j^t the price of asset j at time t. In our dynamics a bank holds onto its portfolio as long as it is solvent, so Q_{ij} is independent of time. Notice that, given that bank i invests in k_i assets, only k_i of the M portfolio weights Q_{ij} will be non-zero for bank i.

Each solvent bank also holds cash C_i, and we denote by L_i its total liabilities; neither of these quantities depend on time. If A_i^0 is the initial value of bank i's portfolio, its initial equity (or capital) is therefore $E_i^0 = A_i^0 + C_i - L_i$. The leverage of a bank is the ratio between the amount of risky assets on its balance sheet and its equity. Assuming no risk associated with cash holdings, the initial leverage of bank i is $\lambda_i = A_i^0/E_i^0$.

The condition for bank i to be solvent at time t is

$$\sum_{j=1}^{M} Q_{ij} p_j^t + C_i \geqslant L_i. \tag{4}$$

Given that $E_i^0 = A_i^0 + C_i - L_i$, the above condition can be expressed as

$$A_i^0 - \sum_{j=1}^{M} Q_{ij} p_j^t \leqslant E_i^0. \tag{5}$$

The left hand side represents the loss with respect to the initial investment. If such a loss happens to be greater than the initial capital of the bank, the bank is out of business.

Note that leverage is a necessary condition for banks to fail. A bank investing only its own capital always satisfies condition (5), since its maximal loss is equal to its equity. We can write (5) as a condition on the leverage,

$$\lambda_i \leqslant \frac{\sum_{j=1}^{M} Q_{ij} p_j^t}{E_i^0} + 1. \tag{6}$$

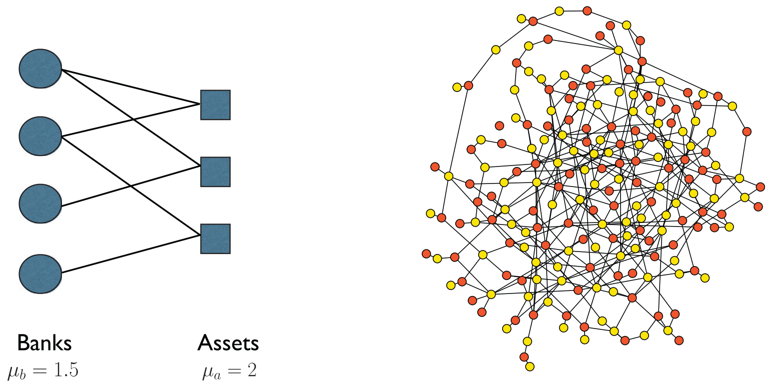

Fig. 1. *Left panel*: Graphical representation of a bipartite network of banks and assets. Banks are denoted by circles, assets by squares. Links connect banks to the assets they have in their portfolios. In this toy example $N = 4, M = 3$, the average banks' degree is $\mu_b = 1.5$ and the average assets' degree is $\mu_a = 2$. This figure is simply meant to give an intuition of the bipartite structure of our model, but is not representative of the actual networks used in our numerical simulations and analytic calculations. *Right panel:* One of the actual networks used in the simulations reported in Fig. 6. In this case $N = 100, M = 100$ and $\mu_b = 3$. Red nodes represent banks, yellow ones represent assets. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Even in the worst case scenario where $p_j^t = 0$ for all assets, this condition can be violated only if $\lambda_i > 1$, i.e. if the bank is leveraged. The use of leverage is common practice in the financial industry. Typical values of leverage may be around 10 for commercial banks, while large investment banks usually have leverage in the range $10 - 30$.

Whenever a bank does not satisfy the solvency condition (5), we assume its portfolio undergoes a fire sale, i.e. all its assets are immediately liquidated. The fire sale causes the price of the assets in the bank's portfolio to drop. This causes a loss for other banks that invest in the same assets because we consider a mark-to-market accounting framework in which all banks update the value of their portfolios according to the most recent observed prices.

If x_j is the fraction of asset j that has been liquidated, the price is updated as[4]

$$p_j \rightarrow p_j f_j(x_j). \tag{7}$$

We are interested in the response of the system to an initial shock. We consider two kinds of initial shocks:

- *Presence of a toxic asset.* We select a random asset j and devalue it at time 0.
- *Initial failure of a bank.* We select a random bank i and cause it to go bankrupt.

In each case we follow the chain of events caused by the initial shock. The dynamics we consider is very simple: after shocking the system at time $t = 0$, at each time step $t = 1, 2, \ldots$ the solvency condition (5) is checked for each bank, the portfolios of newly insolvent (bankrupted) banks are liquidated, and new prices are computed for each asset. The dynamics stops when no new bankruptcies occur between two consecutive time steps. This can be expressed with the following algorithm:

1. Introduce the initial shock in the system;
2. Liquidate the portfolio of insolvent banks;
3. Recompute prices of assets;
4. If new banks are insolvent go to step 2, otherwise end.

[4] An implicit assumption of our model is that external investors will be able to absorb the increased supply of assets due to banks liquidating their shares. In particular we consider here the situation of assets with independent market-impacts, which is equivalent to assuming independent demands for the different assets. A generalization of the model could include a common factor to account for the fact that if external investors buy shares in one of the assets there may be less liquidity available to buy the other assets. A similar propagation of illiquidity between assets occurred for instance during the subprime crisis within the market for asset backed securities, as described in Gorton (2009).

Note that we don't allow for new banks to enter the system, so that once a bank has gone bankrupt it remains in this state for the rest of the process.

In the limit of large systems, when $N, M \rightarrow \infty$ while the parameters μ_b and $n = N/M$ remain finite, the initial shock we consider only affects an infinitesimal (of order $\mathcal{O}(1/N)$) fraction of the banking system. We are interested in understanding if and when such infinitesimal shocks can trigger global cascades of failures. A *global cascade* of failures is defined as a cascade affecting a finite fraction of banks in the infinite system. In the following we will measure the probability and the average extent of contagion. We define the *probability of contagion* as the probability that a global cascade of failures occurs, and the *average extent of contagion* as the average size of a global cascade.

3. Stability analysis

In this section we develop a theoretical approach that allows us to compute a bound on stability, which as we will show is a good estimate of when cascading bank failures are likely to occur. We show how this can be applied to understand the stability of specific banking networks (i.e. a given set of banks and their balance sheets), and we also show how it can be used to understand how stability depends on the parameters of the network, such as diversification, crowding, and leverage.

Let us start by discussing what happens if there is an external shock that causes a particular bank to go bankrupt. Through the combination of leverage and impact, this failure can trigger the failure of other banks investing in the same assets. If the parameters of the system amplify shocks, this can generate a cascading failure that propagates through the system. One of our main points is that, while the likelihood of the first bank failure depends on the nature of the shocks, whether or not this propagates depends on whether the financial system is stable, which in turn depends on parameters such as the leverage, market impact and network structure. We begin with a general discussion of branching processes. We then discuss how it can be applied to understand a given banking network, and make some specific assumptions that allow us to demonstrate how the stability of the banking system depends on parameters.

3.1. The Galton–Watson process

The expression "financial contagion" is adopted from epidemiology, and comparisons between systemic risk and epidemics are natural (see for instance Gai and Kapadia (2010), Gai et al. (2011)

and Arinaminpathy et al. (2012)). The intuition behind the analogy is clear: Just as infected individuals can pass a disease to the people they interact with, stressed banks can pass stress to the banks they interact with. In this section we show that the analogy is more than a metaphor, and that mathematical techniques used to study the spread of epidemics can be adapted to study financial contagion.

The mechanism at the basis of epidemiological models is that infected individuals can infect healthy ones with whom they enter into contact. Depending on the specific properties of the disease under study, the *basic reproduction number* (often denoted as R_0) is computed. The basic reproduction number is simply the average number of new cases that are generated by an infected individual over the course of an infection period, and is a measure of the likelihood that the disease will spread among the population. In the context of financial contagion this means that, if we want to know whether a cascade of bankruptcies will occur, we need to compute the average number of new bankruptcies triggered by a failed bank.

A tool that is commonly used to model epidemic outbreaks is the branching process that Galton and Watson originally introduced to study the survival probability of family names over generations (Watson and Galton, 1875). This process is formulated in terms of a progenitor (or first infected individual for epidemic spreading) that gives rise to x children (new cases of infection), where x is a non-negative integer drawn from a probability distribution $g(x)$. Each of the children, in turn, independently generates a number of offspring distributed according to $g(x)$, and the same process is repeated at each generation. The question is whether such a process is doomed to extinction or not, i.e. if the population drops to zero after a finite number of generations, so that the total number of descendants is finite. A fundamental result in the theory of branching processes states that such a process goes extinct with probability one if $\mathbb{E}[x] < 1$, where $\mathbb{E}[x]$ is the expected number of offspring per individual, i.e. the basic reproduction number.

For our purposes it is essential to consider a generalized Galton–Watson process with individuals of different types $i \in 1, 2, \ldots, q$. The key quantities are then, for each pair of types i, j, the expected number of offspring of type i produced by an individual of type j. We denote these as a $q \times q$ matrix $\mathcal{N}_{ij}$. The condition for extinction is then that the largest eigenvalue ξ_1 of $\mathcal{N}$ is smaller than one (Mode, 1971). Conversely, if this eigenvalue is greater than one, then with positive probability this process lasts forever, producing an infinite number of offspring. We say that the branching process is *subcritical* or *supercritical* if this eigenvalue is less than or greater than one, respectively.

The scenario of cascading failures for banks closely resembles the branching process studied by Galton and Watson. In the context of our model, we are interested in computing the expected number of banks that go bankrupt because of the previous failure of another bank. Consider for example the case in which a random bank i receives a shock at time $t = 0$ that causes it to become bankrupt. This bank is equivalent to the progenitor of the Galton–Watson process, and banks whose bankruptcy is triggered by that of i are equivalent to its offspring (or new cases of infection epidemic analogy). In the language of branching processes, banks failing at time t correspond to individuals in the t-th generation. We are interested in understanding when there is a non-zero chance that financial contagion keeps on spreading over time, which is equivalent to asking whether shocks will be amplified rather than dying out. If the branching process is supercritical, then this initial shock results in a global cascade with non-zero probability, affecting a non-zero fraction of all the banks in the limit of infinite system size.

Note that in our model there are banks with different properties (degree, leverage, size…) that can be considered as individuals of different types in the generalized Galton–Watson process. Thus $\mathcal{N}_{hk}$ is the expected number of banks of type h that fail because of the failure of a bank of type k. There is obviously considerable flexibility in how we classify banks into types, which at its most fine-grained extreme allows the "types" to correspond to individual banks. From an economic point of view it is possible to interpret the sum of the elements belonging to the h-th row/column of N as a measure of the contagiousness/vulnerability of banks of type h.

It is important to stress at this point that the process here considered is more complex than the usual generalized Galton–Watson process. In particular, in our case, the ultimate fate of bank i depends not only on its properties, but also on those of all the other banks whose portfolio overlaps with i. This happens because the price drop that follows the fire sale liquidation of an asset depends on the fraction of total shares of that asset being liquidated, which changes from bank to bank.

A second important difference is that the Galton–Watson process occurs on a tree, so that individuals of a given generation are independent. For banks the failure process is not necessarily a tree, but is rather a more general graph which may have loops. To see this, consider a simple example of three banks i, j, and k with one asset in common. Let us suppose that i is robust with respect to the failure of j by itself, but not with respect to the failure of both j and k together. Now, if the failure of j is enough to trigger that of k, then i is effectively vulnerable to the failure of j. Such situations, which occur whenever assets have degree higher than 2, are neglected under the analytical calculation that we perform here. Therefore, our analytical treatment gives a sufficient but not a necessary condition for global cascades to occur, and gives only an upper bound on the stability of the banking system. We will see, however, that it is nonetheless a good approximation, in rough agreement with the results of numerical simulations.[5]

It is in principle possible to improve this approximation to account for the non-linearities induced by loops in the branching process by considering multiple time-step dynamics. This method is commonly used in dynamical system theory: the t-th iteration of the dynamics converts cycles of length t into fixed points. For instance, to properly treat triplets one can compute a two-step matrix that counts the average number of banks of type i whose failure is triggered by the bankruptcy of a bank of type j within two time steps of the dynamics. Comparing to the example given above, if an initial shock causes j to fail, k will fail after one iteration, and since both j and k have now failed, i will fail in the second time step. While our one-step approximation is already quite accurate, this approach provides a path for systematically improving the degree of approximation, which deserves further investigation.

3.2. Stability of a given system

If we have complete information about the banking system, i.e. if we know the portfolios Q_{ba} of all the banks and, in addition, the market impact function for their assets, then we can describe the stability of the system through a matrix $\mathcal{B}$, where $\mathcal{B}_{ij}$ is the probability that bank i will fail under the failure of bank j. Bank i becomes insolvent when bank j fails if and only if the market impact due to the sale of their overlapping assets causes a loss to bank i that exceeds its equity E_i. As described above, we focus for now on the direct effect on bank i of the failure of bank j. Since we assume that bank j's entire portfolio $\{Q_{ja}\}$ is liquidated, the new price for asset a is $p_a(1 - f_a(Q_{ja}))$. Using the shorthand

[5] Exact results based on branching processes have been recently demonstrated by Campbell (2013) in the context of word-of-mouth information spreading and consumer decision making. In that context loops of length three are shown not to change the qualitative features of the system with respect to the benchmark case of locally tree-like networks.

Prob(x) to indicate the probability that condition x is satisfied, the stability matrix $\mathcal{B}_{ij}$ is defined as

$$\mathcal{B}_{ij} = \text{Prob}\left[\sum_{a=1}^{M} Q_{ia} p_a \left(1 - f_a(Q_{ja})\right) - E_i > 0\right]. \tag{8}$$

In order to understand whether a cascade of failures will spread, we compute $\mathcal{B}_{ij}$ in the case where the assets shared by banks i and j have not yet been devalued, and still have their initial prices. That is, we focus on the "boundary" of the cascade, with failures and devaluations spreading outward through the network through banks and assets that have not yet been touched by the crisis. In that case, since the dynamics themselves are deterministic, $\mathcal{B}_{ij}$ depends only on the initial structure of the banks' portfolios, and in particular on the network structure. The stability of the banking system can then be estimated by simply computing the largest eigenvalue ξ_1 of $\mathcal{B}$ and determining whether ξ_1 is greater than or less than one.

Note that, rather than using the simplifying approximation that the market impact function is deterministic, one could more realistically use a stochastic market impact function as in Caccioli et al. (2012b). Similarly, imperfect knowledge about bank portfolios and equity can be coped with using probabilities to represent uncertainties in their values. In either case, we can still bound the stability of the network by computing $\mathcal{B}$'s largest eigenvalue.

3.3. Simplifying assumptions

The approach described above makes it possible to estimate the stability of the banking system when it is in a particular state, corresponding to a particular configuration of the balance sheets of each bank. One of our main goals here, however, is to understand more generically how the stability of the banking system depends on its network properties. To make a high-level characterization it is necessary to think in terms of ensembles of networks, and to understand how stability varies as properties of the ensemble are varied. As a first step in this direction we will make some specific assumptions in order to simplify the problem and gain intuition. While these assumptions are rather arbitrary, the basic method used here is easily generalized, as discussed later.

- *Network topology:* We will consider random networks with Poisson degree distributions for both banks and assets. Specifically, for each possible bank-asset pair a link is drawn with probability μ_b/M. The resulting network is drawn from the bipartite Erdős–Renyi ensemble of random networks with average degrees μ_b and $\mu_a = \mu_b N/M$ for the banks and assets respectively.
- *Structure of balance sheets:* We will assume all banks have the same amount of money $A_i^0 = A^0$ available for investment, and that each bank uniformly splits its investment in the assets that are in its portfolio. The asset side of bank's balance sheets will be composed of 80% assets and 20% cash. For bank i each link thus corresponds to an investment of $0.8A^0/k_i$, where k_i is the number of assets in i's portfolio. Unless otherwise stated, we assume for each bank an initial equity $E_i^0 = E^0$ corresponding to 4% of its total assets. This corresponds to all banks having initial leverage $\lambda = A^0/E^0 = 20$.
- *Market impact function:* We will assume that the market impact function has the form $f_j(x_j^t) = e^{-\alpha x_j^t}$, where x_j^t is the fraction of asset j liquidated up to time t. The parameter α is chosen such that the price drops by 10% when 10% of the asset is liquidated, i.e. $\alpha = 1.0536$. All prices are set to $p_j^0 = 1$ at time 0. This choice corresponds to linear market impact for log-prices, as originally used to describe price dynamics in Bouchaud and Cont (1998), Farmer (2002). It should be noted that recent empirical and theoretical evidence indicates that market impact for large trades is a concave function of the number of traded shares, which under normal conditions impact is well approximated by a square-root function (Bouchaud et al., 2009). By normal conditions we mean that execution is slow enough for the order book to replenish between successive trades. Under extreme conditions, like those of a fire sale, market impact is expected to become less concave and even linear or super-linear (Gatheral, 2010), which motivates our choice of functional form here.

Altering these assumptions does not change the qualitative behavior of the system. In particular, our methods generalize easily to degree distributions other than Poisson, e.g. power laws, and also to multiple types of banks with different sizes, portfolio structures, and amounts of leverage, or multiple types of assets with different initial prices and market impact functions.[6]

Using these assumptions we can compute the stability matrix through the following steps: First we divide nodes in classes of degree, so that the matrix element $\mathcal{N}_{ij}$ represents the expected number of banks of degree i that fail because of the failure of a bank of degree j. We then compute the expected number of banks of degree i that are connected to a bank of degree j through an asset of degree a, we multiply this quantity by the probability that the devaluation of the asset causes a bank of degree i to fail and we integrate over the degree distribution for assets. Notice that the devaluation of an asset depends on the degree of all banks connected to it. Therefore the probability that a bank triggers the failure of another bank through an asset does not depends only on the degrees of the two banks, but on the properties of the other banks investing in the asset. This makes our calculation more complex with respect to the case of the standard branching process.

The explicit calculation of the stability matrix under the simplifying assumptions outlined above is carried on in Appendix A. We will compare results obtained from the analytic approach to numerical simulations in Section 5.

4. Dependence on leverage and network properties

We now explore how the stability of the banking network depends on parameters. We first show results based on numerical simulations and then compare them to results based on the stability matrix $\mathcal{N}$.

In numerical simulations N and M are both finite, and global cascades can be defined as cascades for which the fraction of bankrupted banks exceeds a fixed threshold. For consistency with previous work on counterparty loss (Gai and Kapadia, 2010; Gleeson et al., 2011; Hurd and Gleeson, 2011), we set this threshold to 5%. The contagion probability is then measured as the fraction of runs in which a global cascade results from the initial shock. The conditional average extent of contagion is the fraction of failed banks, averaged *only* over those runs where a global cascade occurs.

4.1. Effect of diversification and crowding

We begin with the case where the initial shock consists of devaluing a random asset, and examine the dependence on

[6] We tested the robustness of our results with respect to changes in the market-impact function. We considered in particular functions of the form $f_j(x_j^t) = 1 - \alpha x_j^t$ and $f_j(x_j^t) = 1 - \alpha\sqrt{x_j^t}$. We found that the qualitative features discussed in the next sections (non-monotonicity of contagion probabilities, robust yet-fragile behaviors, increased instability due to higher levels of leverage) are preserved. More specific properties, like the location of the phase transitions and the width of the contagion window, depend in general on the details of the model.

diversification and crowding. In the left panel of Fig. 2, we plot the probability and conditional extent of contagion measured for a system of $N = 10^4$ banks and $M = 10^4$ assets as a function of the average banks' degree μ_b. Results refer to 1000 runs in which a random asset is initially devalued by 35%. We observe phase transitions at two critical values μ_1, μ_2 of μ_b, with a *contagion window* in between where global cascades of failures occur with non-zero probability. Above and below this window, where $\mu_b > \mu_2$ or $\mu_b < \mu_1$, global cascades do not occur.

The existence of a contagion window, and the nonmonotonicity of the contagion probability as a function of μ_b, can be understood with the following arguments. On one hand, for sufficiently low values of μ_b, stress cannot propagate through the system because the network is poorly connected; there is not enough overlap between the banks' portfolios to spread the cascade. In particular, for small enough μ_b the network of banks and assets consists of small components disconnected from one another, so even if every bank is extremely vulnerable to collapse, an initial shock will only affect one of these components. Thus there is a critical μ_1 below which the cascade cannot propagate; the initial shock might affect a few nearby banks, but the cascade quickly dies out.

On the other hand, if the banks' portfolios are sufficiently diverse, they are robust with respect to devaluing any single asset in their portfolio. Moreover, a larger average bank degree μ_b also implies a larger average asset degree μ_a, so each institution typically holds a smaller fraction of the shares of any given asset. As a consequence, each bank failure has a relatively small effect on asset prices, and most banks remain solvent even if some of their assets are devalued. Thus there is a critical μ_2 above which cascades quickly die out even though the network is highly connected.

The left panel of Fig. 2 also shows that the system displays a "robust yet fragile" behavior (Gai and Kapadia, 2010) for some values of the parameters. Specifically, if μ_b is slightly less than μ_2, just inside the upper end of the contagion window, the probability of a global cascade is very small, tending continuously to zero as μ_b approaches μ_2 from below. But when a global cascade does occur, it affects almost all the banks: the conditional extent of the contagion is almost 1.

In the right panel of Fig. 2 we plot the contagion probability for different values of the crowding parameter $n = N/M$. As n increases, the contagion window shifts to the left, decreasing both μ_1 and μ_2.

The shift in μ_1 can be understood in terms of the appearance of a giant connected component in the network. In the ensemble of random networks considered here, the emergence of the giant component corresponds to the situation where the average number of banks to which a given bank b is exposed, i.e. the average number of banks whose portfolios share at least one asset with b, is one. Equivalently, this is the average degree of the *projected network* where two banks are connected if they share an asset. For this ensemble (essentially the bipartite version of the Erdős–Rényi model) this degree is $\mu_a\mu_b = \mu_b^2 n$, giving $\mu_1 = 1/\sqrt{n}$.

To explain the shift in the second transition point μ_2, we note that the drop in price of an asset caused by the liquidation of a portfolio is a decreasing function of n. This is because the average number of banks investing in a given asset is $\mu_a = \mu_b n$. If each bank owns a smaller fraction of an asset, the market impact of a fire sale on that asset is smaller. When n is larger, this effect takes over at a smaller value of μ_b.

Note that, as a result, changing the crowding parameter n has different effects on the network's stability depending on the value of μ_b. If μ_b is close to μ_1, increasing n while keeping μ_b fixed increases the instability of the system, moving it into the contagion window by increasing the connectivity of the network. The opposite is true if μ_b is close to μ_2, where increasing n moves us outside the contagion window by making assets and banks more robust. Thus the contagion probability is not a monotonic function of n.

4.2. Dependence on shocks

The above simulations started with an initial shock consisting of devaluing a random asset. We now consider the case where we begin with the failure of a random bank. Fig. 3 shows a comparison between simulations with shocks of these two types. We observe that the probability of contagion depends on the type of shock, but the contagion window and the conditional extent of contagion are the same for both types of shock. The reason is simple: while the initial conditions of these two processes are different, their dynamics are the same. Once a cascade has begun, it doesn't matter what kind of shock began it. Thus the region where the dynamics

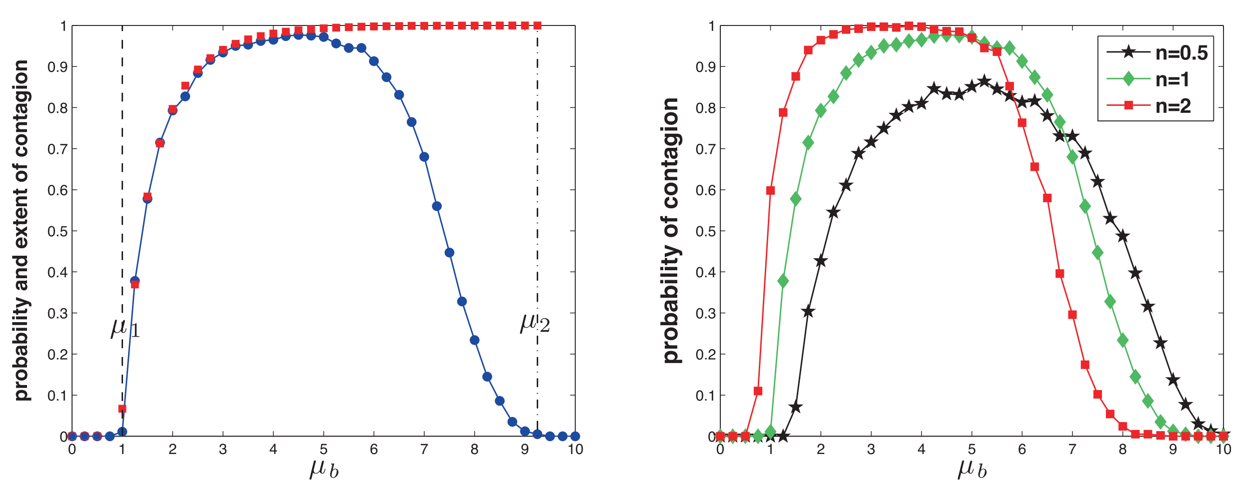

Fig. 2. *Left panel:* contagion probability (blue dots, the solid line is a guide for the eye) and conditional extent of contagion (red squares) measured from 1000 simulations of a system with $N = M = 10^4$. In each run, the initial shock consists of dropping the price of a random asset by 35% at the beginning of the simulation. We vary the average degree of diversification $\mu_b = \mu_a$. The two vertical dashed lines mark our numerical estimates for the critical values μ_1 and μ_2 where phase transitions occur, and show the existence of a contagion window between these transitions where global cascades occur with non-zero probability. The system also displays a "robust yet fragile" behavior for μ_b slightly below μ_2: the probability of a global cascade is small, but when one occurs it affects almost all the banks. *Right panel*: contagion probability for systems with $N = 10^4$ and $M = 5 \times 10^3$ (red squares), $M = 10^4$ (green diamonds) and $M = 2 \times 10^4$ (black stars) as a function of the average banks' degree. Solid lines are a guide for the eye. The boundaries μ_1, μ_2 of the contagion window depend on the value of the crowding parameter $n = N/M$: for larger n both phase transitions are shifted to the left. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

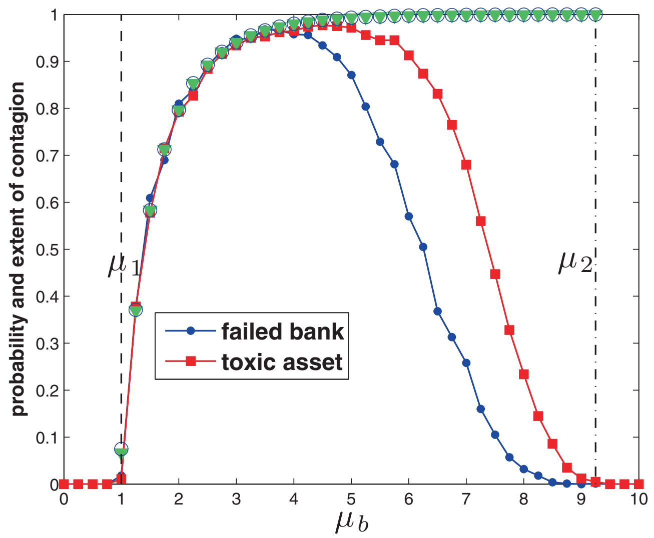

Fig. 3. The probability of contagion, and the average conditional extent of contagion, as a function of μ_b for the two types of initial shock (failed asset vs. failed bank). Red squares: contagion probability where a random asset is devalued by 35%. Blue dots: the contagion probability when a random bank fails. Blue circles and green triangles: conditional extent of contagion for asset shocks and bank shocks respectively. We see that while the probability of contagion differs between the two types of shocks, the window $\mu_1 < \mu_b < \mu_2$ in which they occur with non-zero probability is the same. Moreover, when a global cascade does occur, its average size is the same for both types of shocks. Results refer to 1000 simulations of systems with $N = M = 10^4$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

cause a cascade to spread rather than die out is the same in both cases, as is the eventual size of a global cascade if one occurs.

4.3. Leverage

We now show what happens for different values of initial leverage. In the left panel of Fig. 4 we plot the contagion probability for different values of μ_b as a function of λ. We observe that, for each μ_b, there is a critical value of λ above which global cascades occur with non-zero probability, and below which they do not. This is of interest for regulatory purposes, since it implies the existence of a critical level of leverage below which systemic stability is guaranteed. In addition, the critical value of λ increases as μ_b increases: in other words, increasing diversification allows for a greater degree of leverage without creating systemic events.

In the right panel of Fig. 4, we show that a similar behavior occurs as we change the parameter α that appears in the market impact function while keeping the leverage fixed. That is, for a given value of μ_b and λ, there is a critical value of α above which contagion occurs. This is not unexpected, since under the assumptions specified in Section 3.3 the solvency condition for bank i can be written as

$$\lambda_i \leqslant \frac{\sum_{j=1}^{M} Q_{ij} e^{-\alpha x_j^t}}{E_i^0} + 1, \qquad (9)$$

where x_j^t is the fraction of shares of asset j liquidated up to time t. When α is larger, the market impact of a fire sale is greater, causing a sharper drop in asset prices. On the other hand, increasing diversification μ_b increases this critical value of α, showing that diversification allows banks to survive a larger price impact.

Summarizing, we presented in this section results of numerical simulations for bipartite networks with Poisson degree distributions for both banks and assets. The probability and the average extent of contagion have been measured for two different types of shocks, namely the initial depreciation of a random asset or the initial failure of a random bank. Our simulations suggest that:

- As a function of the average diversification of banks' portfolios, represented by their average degree μ_b, the system is characterized by two phase transitions that define a contagion window where global cascades occurs with non-zero probability.
- Changing the crowding parameter n, i.e. the ratio of the number of banks to the number of assets available for investment, can increase or decrease the contagion probability depending on which of these transitions we are close to.
- Although the contagion probability is different for the two types of initial shocks, the contagion window within which global cascades occur, and the average extent of these cascades when they occur, are the same.
- The system displays a "robust yet fragile" behavior, with regions in parameter space where global cascades are very unlikely, but where almost the entire system is affected if one occurs.

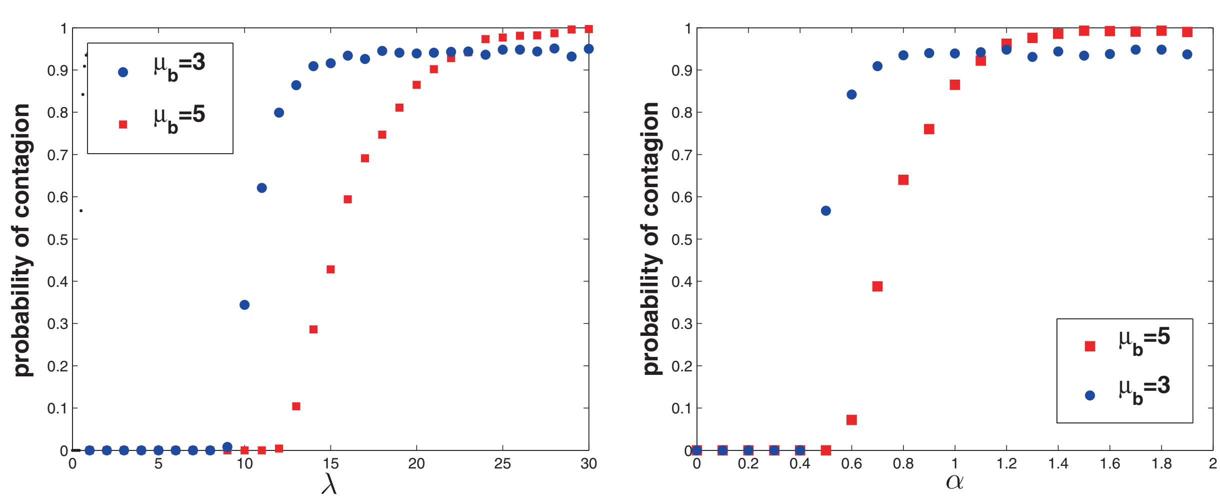

Fig. 4. *Left panel:* Contagion probability as a function of leverage measured from 1000 simulations of a system with $N = 10,000$ for different values of μ_b. The initial shock is the failure of a random bank. Contagion probability is a monotonic function of leverage, and a phase transition separates a regime where no global cascades are observed from one where they occur with non-zero probability. *Right panel:* Contagion probability as a function of the market impact parameter α. Increasing market impact has a similar effect to increasing leverage.

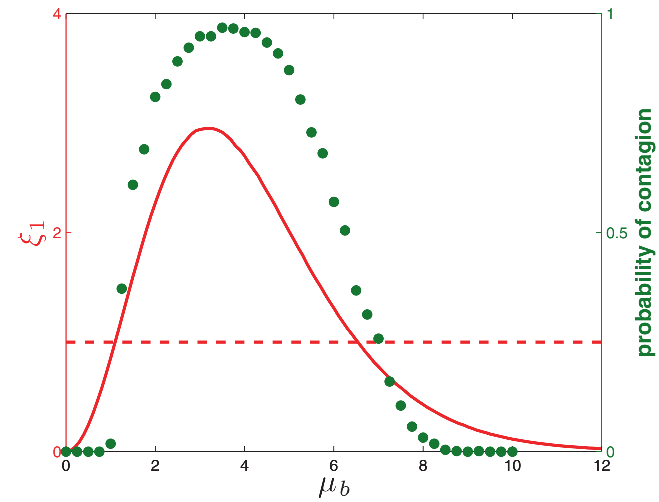

Fig. 5. Contagion probability (green dots, right axis) as computed from numerical simulations of a system of size $N = M = 10^4$. The red solid line (left axis) represent the largest eigenvalue ξ_1 of the matrix $\mathcal{N}$. The dashed horizontal line is in correspondence to $\xi_1 = 1$. If $\xi_1 > 1$ global cascades are observed in numerical simulations. The theory underestimates the width of the contagion window, as it only gives a sufficient condition for global cascades to occur. However, it is a good approximation, and the discrepancy between theory and numerical results is partly due to finite size effects (see Fig. 6). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

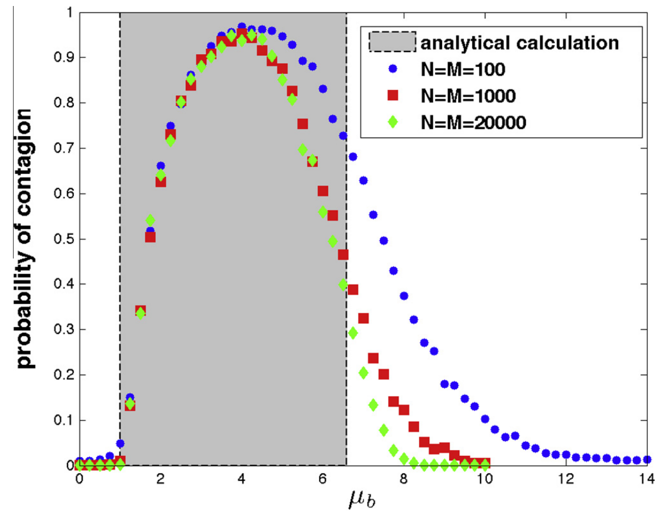

Fig. 6. Simulation results for $N = 100$ (blue circles), $N = 1000$ (red squares), $N = 20000$ (green triangles), $n = 1$. The vertical dashed lines are drawn in correspondence to the phase transitions predicted by the analytic calculation, and the shaded region is the unstable region predicted by the analytic calculation. As the size of the system increases the agreement between theory and simulations improves. Finite size effects are expected given that the theory is valid in the limit $\{N, M\} \to \infty$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

- For each fixed μ_b and n, there is a critical value of the leverage λ above which the system becomes unstable. This critical value of λ increases with μ_b.

5. Comparison to predictions from stability analysis

We now compare the numerical results presented in the previous section to those based on stability analysis. The stability analysis depends on two assumptions that are not necessarily well-satisfied in the simulation. The first is that M and N are both infinite (even though their ratio $n = N/M$ is finite), and the second is that the failure process can be described through a branching process on a tree.

We estimated $F(h, k, \ell)$ through monte-carlo methods and assumed that the contribution coming from banks with degree higher than 200 is negligible. We then numerically diagonalized the 200 by 200 matrix $\mathcal{N}$. We discuss in the following the results obtained in the case where $P_a(\ell)$ and $P_b(h)$ are Poisson distributions.

In Fig. 5 we plot for $n = 1$ the largest eigenvalue of $\mathcal{N}$ and we compare it with the contagion probability as computed from numerical simulations. As expected, when the largest eigenvalue of $\mathcal{N}$ is greater than 1 global cascades are observed. We see from the figure that the analytic calculation underestimates the size of the contagion window. This is partly due to finite size effects, as observed for instance in Watts (2002). We plot in Fig. 6 the contagion probability as measured from numerical simulations for $n = 1$ and different values of N. From the figure we clearly see that by increasing the size of the system the discrepancy between analytic and numerical calculations gets smaller,[7] and that the analytic solution, although giving only a sufficient condition for global cascades to occur, produces a reasonable estimate of the contagion window when N and M are large.

As a summary in Figs. 7 and 8 we show the region where the system is unstable based on the analytic approach. There are three

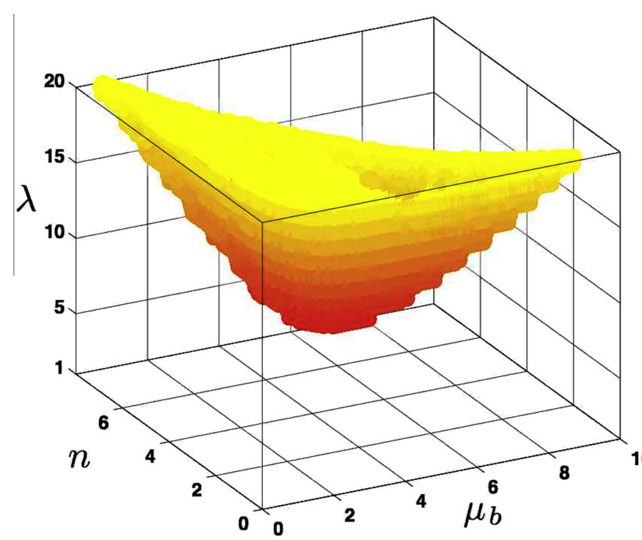

Fig. 7. 3D visualization of the unstable region of parameter space as predicted with the analytical approach, shown as a function of the diversification parameter μ_b, the crowding parameter $n = N/M$ and the leverage λ. The unstable region is shown in yellow and red. We note that there is critical value of leverage below which the system remains stable for any values of diversification and crowding parameter.

relevant parameters, μ_b, n and λ; we begin with a three dimensional visualization in Fig. 7. The red and yellow region denotes instability, i.e. within this region the initial shock is amplified and propagates through the system, while outside it the system is subcritical and is able to absorb the shock.

Two dimensional slices of the phase diagram showing the unstable region are given in Fig. 8. The panel on the left holds the leverage fixed at $\lambda = 20$ and varies μ_b and n, corresponding to a horizontal slice through Fig. 7. We see some of the features already observed in numerical simulations. In particular, for fixed n, we see the existence of two phase transitions that define a window of connectivities where global cascades occur with non-zero probability. As n is decreased the analytic calculation also predicts

[7] From the plot we see that the analytic phase transition is around $\mu_b \simeq 6.5$, while the numerical one is around $\mu_b \simeq 13$ for $N = 100$, $\mu_b \simeq 10$ for $N = 1000$ and $\mu_b \simeq 8.5$ for $N = 20000$.

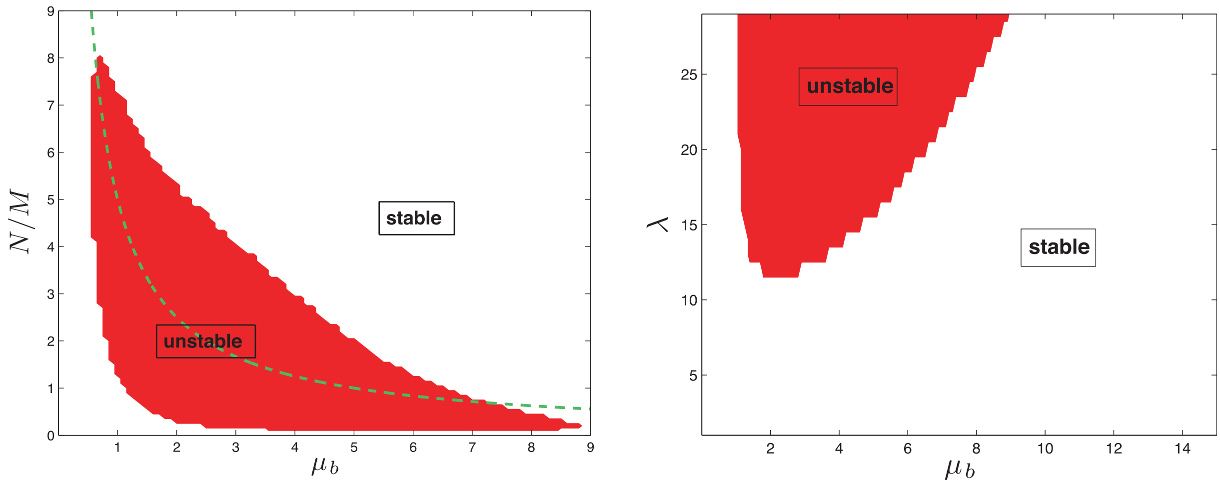

Fig. 8. *Left panel*: The red region is the region of phase space where the financial system is unstable for a system with $\lambda = 20$ as a function of μ_b and n. The dashed green line is drawn in correspondence to the curve $n = 5/\mu_b$, and represents the case of constant $\mu_a = 5$. *Right panel*: The red region corresponds to the unstable parameter values for $n = 1$, shown as a function of μ_b and λ. There is a critical value of λ below which the system is stable for any value of μ_b. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the shift in the transition points toward higher values of μ_b. From the left panel of Fig. 7 we can also deduce the behavior of the system when μ_a and μ_b are allowed to vary independently. The parameter μ_a is constant along the equilateral hyperboles defined by $n = \mu_a/\mu_b$, e.g. as illustrated by the dashed green curve. The non-monotonicity of the contagion probability is also preserved in this case.

In the right panel of Fig. 8 we slice through Fig. 7 vertically, holding the crowding parameter constant at $n = 1$ while varying leverage λ and diversification μ_b. As expected, we see that the contagion window widens as λ increases.

Interestingly, we observe the existence of a leverage threshold at roughly $\lambda \simeq 12$: if the leverage is below this critical level the system is always stable and shocks are not amplified. It is important to stress here that this has been determined under the simplifying assumptions described in Section 3.3. The critical leverage depends, in general, on the specific properties of the system under study; obtaining a realistic value would require calibrating with real data. Just as one example of factors that can affect leverage, in his model of global cascades in complex networks Watts (2002) showed that for scale-free networks the critical value of leverage is higher than for Erdős–Renyi networks. We could therefore expect that in presence of a more heterogeneous degree distribution the critical leverage below which the system is overall stable will be higher than the one shown in Fig. 8.

The existence of a critical leverage threshold is of potential interest for regulators in relation to the debate concerning capital requirements, such as those discussed in the Basel agreements. Loosely speaking, capital requirements introduce a constraint on leverage. The details of the implementation are complex, but the idea is the following: Assets are divided into classes according to their risk in order to define the risk weighted leverage ratio. Banks are then required to keep their risk-weighted leverage below a regulatory threshold. The leverage λ we consider in our model represents a very simple implementation of the concept of risk weighted leverage, where the M illiquid assets are assigned the same weight, while cash (or, in general, any asset other than the M explicitly modeled) is assigned weight zero.

A critical decision faced by regulators is that of selecting the "right" leverage threshold. Our analysis may be of assistance in this respect, as it shows the existence of a critical leverage below which the system is overall stable, regardless of other conditions. Of course the model will have to be calibrated with real data to provide a good estimation of this critical leverage, but once this is done the methodology developed here can be used to compute the critical leverage.

Another point of potential interest for regulators is the eigenvalue of the matrix $\mathcal{N}$. In a dynamic setting one expects ξ_1 to change over time as banks trade to rebalance their portfolios. By monitoring the time behavior of ξ_1, a regulator will know that the system is approaching a dangerous regime because ξ_1 gets too close to 1, and could then act to increase the stability of the system.

A source of instability during the subprime crisis was the procyclicality of leverage identified by Adrian and Shin (2010). Before the crisis, when market conditions were good, capital requirements implemented through a VaR constraint allowed banks to expand their balance sheet (basically because risk was perceived to be low). This induced a build-up of systemic risk that became apparent after the burst of the housing bubble, when VaR constraints forced banks to liquidate their positions, making the situation even worse.

Monitoring the largest eigenvalue of the stability matrix would have made it possible to see the accumulation of systemic risk before the crisis actually occurred. A method of controlling systemic risk could employ a dynamical capital requirement framework based on a fixed value of ξ_1. This could take into account network effects, and thus go beyond the countercyclical capital requirements for individual banks discussed in Basel III. These requirements increase banks' capital requirements in good times (therefore reducing the potential for the occurrence of asset bubbles) and relax capital requirements in bad times (therefore reducing fire-sales triggered by banks trying to de-leverage). Our analysis potentially provides a more accurate monitoring tool; as we have shown here, the critical leverage is a function of systemic factors, such as the degree to which portfolios are overlapping, and thus a fixed requirement may be unduly strict in some circumstances and dangerous in others.

6. Portfolio rebalancing can increase systemic instability

The model considered so far assumes a passive strategy for banks, in which they hold their portfolios constant until they are forced to liquidate due to bankruptcy. We have argued in the introduction that this may well be a realistic assumption when events unfold rapidly and banks are not able to reduce exposure to toxic assets. Alternatively, this might be regarded as a worst case scenario; were banks to instead perform active risk management, one might think that the system would necessarily become more stable. In this section we show that this is not necessarily the case, and indeed, for the example we construct here, the opposite happens: Preventative portfolio balancing is destabilizing.

We consider an extension of our model in which banks are allowed to reduce the size of their portfolios in response to stress. When assets are devalued we assume that banks react by liquidating a fraction of their portfolio on a pro rata basis. The size of the liquidation is chosen based on a target leverage, which is the leverage λ_i before the initial shock hits the system. More specifically, at any time t a solvent bank liquidates a fraction ΔA_i^t of its portfolio corresponding to

$$\Delta A_i^t = \gamma A_i^t \left(1 - \frac{\lambda_i E_i^t}{A_i^t}\right), \tag{10}$$

where $\gamma \in [0,1]$ is a parameter representing how aggressive the bank is in trying to reach its target leverage. As before, when a bank becomes insolvent its entire portfolio is liquidated. If $\gamma = 0$ the bank's strategy reduces to the passive strategy considered in the previous sections of the paper. If $\gamma = 1$ the bank tries to reach its target in one time-step, while for $0 < \gamma < 1$ the timescale for reaching the target is of order $1/\gamma$ (in absence of market impact). As in Section 3.3, the price of asset j at time t is $p_j^t = p_j^0 e^{-\alpha x_j^t}$, where x_j^t is the fraction of asset j liquidated up to time t.

In Fig. 9 we plot the contagion probability obtained from numerical simulations for the cases $\gamma = 0, \gamma = 10^{-3}$ and $\gamma = 10^{-2}$. The initial shock applied is the bankruptcy of a random bank. Network and balance sheet structures are those specified in Section 3.3.

It is clear that in this case preventative liquidation is actually destabilizing. Even for a very small value of the parameter γ, such as 10^{-3}, we see that the contagion window is larger than the benchmark case $\gamma = 0$. The increased instability is related to the fact that banks that de-leverage in a falling market put additional pressure on already stressed assets (Geanakoplos, 2003; Brunnermeier and Pedersen, 2009; Thurner et al., 2012). In fact, if a bank is sufficiently leveraged it can drive itself bankrupt by de-leveraging due to its own market impact, as described in Caccioli et al. (2012b).

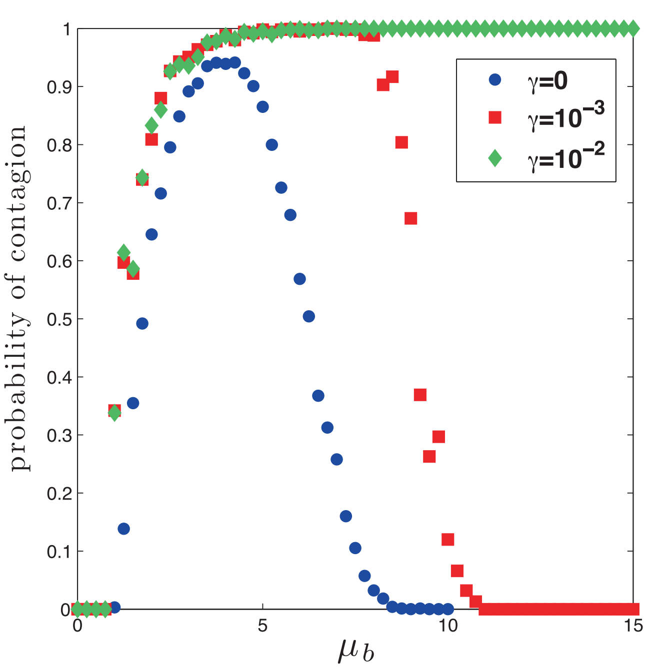

Fig. 9. Comparison of passive banks to active portfolio management. As before, we plot the contagion probability as a function of average bank diversification. Blue dots: benchmark case of passive banks ($\gamma = 0.0$). Red squares: $\gamma = 10^{-3}$, Green diamonds: $\gamma = 10^{-2}$. This shows that, at least for the model considered here, the attempt by banks to reduce their risk by de-leveraging is destabilizing. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

A complete characterization of the effect of preventative deleveraging in a dynamical setting is beyond the scope of this paper (and indeed is the subject of future work). The preliminary result discussed in this section makes it clear that stability analysis based on passive banks should not be regarded as a worst case scenario, but rather as a useful benchmark.

7. Conclusion

We have developed a framework for thinking about the stability properties of banking networks due to overlapping portfolios. This framework emphasizes that the key property is stability: If the system is stable, shocks will not propagate; if it is unstable, a shock can be amplified and trigger cascading bankruptcies. This can be discussed in terms of a branching process that gives insight into the dynamics of failure. While we have called these "banking networks" for simplicity, the basic ideas are relevant for any leveraged financial institutions.

To understand how the stability of banking networks might depend on parameters such as diversification, leverage and crowding, we formulated a stylized model of a financial system in which N banks with average diversification μ_b invest in a common pool of M assets. The system can be conveniently described in terms of a bipartite network, with banks being connected through links to the assets in their portfolios. Links have a twofold role in such a network. On one hand, they allow individual banks to diversify their investment and reduce their exposure to a specific asset. On the other hand, they are channels for the propagation of financial contagion. We characterized the response of such systems to initial shocks affecting a single asset or bank.

The relevant parameters for the model are the average diversification μ_b, the crowding parameter $n = N/M$ (that measures the proportion of banks to assets), and the initial leverage λ. By means of numerical simulations we showed the existence of phase transitions separating a region in parameter space where global cascades occur from a region where global cascades never occur. In particular, the double role played by links in the bipartite network representing the system is reflected in a nonmonotonic behavior of the contagion probability as a function of μ_b. We observe the existence of two phase transitions at $\mu_b = \mu_1$ and $\mu_b = \mu_2$, defining a window of connectivities such that global cascades occur if $\mu_1 \leqslant \mu_b \leqslant \mu_2$. Changing the crowding parameter n has the effect of shifting the location of the phase transitions. Finally, our model shows that while increasing leverage implies greater instability, there is a critical level of leverage below which global cascades do not occur for any value of diversification or crowding.

Using an analytical approach based on generalized branching processes on networks, we are able to analytically estimate the region of parameter space where global cascades occur. This branching process is different from standard ones in the fact that the fate of a node depends on its degree and on the degree of all its neighbors. This greatly increases the difficulty of the problem. We are nonetheless able to solve it by generalizing existing methods. Thus, apart from their specific application to financial contagion, our methods can be applied to a wide variety of contagion models, where susceptibility and transmission probabilities depend on node degrees.

The mechanistic model considered in this paper can be extended in several directions. First of all, it would be interesting to relax some of the specific assumptions considered in this paper (homogeneity of banks' balance sheets, Poisson degree distributions, market impact function) in order to understand how different choices for the network topology or the statistical properties of balance sheets impact the stability of the system. Although we do not expect different results from a qualitative point of view, it should nonetheless be possible to assess the relative stability of systems with different properties, similarly to what has been done for counterparty loss in Caccioli et al. (2012a). In particular, it would be very useful to empirically characterize real systems and calibrate the model with real data. This could potentially make it possible to test the effectiveness of new policies aimed at reducing systemic risk.

A further direction we plan to pursue in the future is to go beyond the mechanistic model by considering more realistic price dynamics and allowing banks to react to price fluctuations by rebalancing their portfolios. This should allow the system to develop endogenous crisis similar to the ones observed in Thurner et al. (2012), and to generate the systemic instabilities induced by leverage and mark-to-market accounting practices discussed in Caccioli et al. (2012b). A first step in this direction was presented in Section 6, where we allowed solvent banks to reduce their exposure to falling assets. In the setting we considered we showed how banks' attempts to reduce their individual risk is actually destabilizing.

Acknowledgments

This work was supported by the National Science Foundation under grant 0965673, by the European Union Seventh Framework Programme FP7/2007–2013 under grant agreement CRISIS-ICT-2011–288501 and by the Sloan Foundation. C.M. is supported by the AFOSR and DARPA under grant #FA9550-12-1-0432. The authors would like to thank Adam Ashcraf, Morten Bech, Simon Potter and Martin Summer for useful discussions.The authors would also like to thank Goetz Von Peter and Alfred Lehar for discussing our paper and providing useful feedback during the conferences "Interlinkages and systemic risk", held in Ancona, and "Financial regulation and systemic risk", held in Paris.

Appendix A. Explicit calculation of the stability matrix

In order to understand how stability depends on network properties, we lump banks into equivalence classes according to their degree, equating their degree with their type in the generalized Galton–Watson process. We define the following notation:

- N_h is the number of banks of degree h.
- $\mathcal{P}(h,k|a)$ is the probability that a given bank of degree h and a given bank of degree k share a given asset a, i.e., are both connected to a in the network.
- $F(h|k,a)$ is the probability that a bank of degree h fails given that it is connected to a failed bank of degree k through asset a.

Under the assumption that we are in the limit where $M\to\infty, N\to\infty$ while μ_b and $n=N/M$ are finite, the network is sparse, and we can easily compute the probability $\mathcal{B}_{ij}$ by summing over each asset one at a time. If i has degree h and j has degree k, the probability that the failure of bank j causes bank i to fail can be written

$$\mathcal{B}_{ij} = \sum_a \mathcal{P}(h,k|a)\,F(h|k,a). \tag{A.1}$$

Summing over all banks of degree h, the expected number of failures of banks of degree h caused by the failure of a bank of degree k, is

$$\mathcal{N}_{hk} = N_h \sum_{a=1}^{M} \mathcal{P}(h,k|a)F(h|k,a). \tag{A.2}$$

This is the matrix defining the branching process, i.e. the expected number of offspring of type h from an individual of type k.

We can now compute each of the entries of $\mathcal{N}_{hk}$ in turn. Since the degree distribution of our network ensemble is Poisson, the number of banks of degree h is simply $N_h = NP_b(h)$ where

$$P_b(h) = \frac{e^{-\mu_b}\mu_b^h}{h!} \tag{A.3}$$

is the probability that a bank has degree h. A given bank of degree h is connected to a given asset a with degree ℓ_a with probability $h\ell_a/(\mu_b N)$, where $\mu_b N$ is the total number of edges in the network. The probability that a failed bank of degree k is also connected to the same asset a is $h(k-1)\ell_a(\ell_a-1)/(\mu_b N)^2$, where the factor of $k-1$ comes from the fact that one of the k edges of the failed bank is already connected to the asset that caused its failure. This gives

$$\mathcal{P}(h,k|a) = \frac{h\ell_a(k-1)(\ell_a-1)}{\mu_b^2 N}. \tag{A.4}$$

We now compute the probability $F(h|k,a)$ that a bank i of degree h fails due to failure of a bank j of degree k given that they share an asset a. The shift in price when a fraction x_a of an asset is sold is $(1-f_a(x_a))$. (Recall that the initial price is set to one for convenience). Thus the condition for a bank of degree k to fail because bank j sells a fraction x_a of asset a is

$$\frac{A^0}{k}(1-f_a(x_a)) > E^0. \tag{A.5}$$

If $v(a)$ denotes the set of banks investing in asset a, the fraction of a that is liquidated when j fails is

$$x_a = \frac{A^0/k}{\sum_{m\in v(a)} A^0/k_m} = \frac{1/k}{\sum_{m\in v(a)} 1/k_m} = \frac{1/h + 1/k}{1/h+1/k+\sum_{m\in v'(a)} 1/k_m}, \tag{A.6}$$

where $v'(a)$ denotes the set of banks, *other than i and j*, that invest in a and k_m the degree of bank m.

To compute $F(h|k,a)$ we must add up the probability of failure for each possible configuration of banks that are compatible with the condition of choosing a specific pair of banks of degrees h and k that are connected through asset a. If a has degree ℓ_a, there are $\ell_a - 2$ remaining banks. Letting i index these banks, we must average over the possible configurations $\{m_1,\dots,m_{\ell_a-2}\}$. Fortunately the degrees of the banks are independent. The probability that bank i has degree m_i is the ratio of the number of edges for banks of degree m to the total number of edges. Since $N_m = NP_b(m)$, the number of edges for banks of degree m is mN_m, and the total number of edges in the network is $\mu_b N$. Thus each bank has degree m with probability $mN_m/(\mu_b N) = mP_b(m)/\mu_b$ and the probability of any given configuration of bank degrees is

$$\prod_{i=1}^{\ell_a-2} \frac{m_i P_b(m_i)}{\mu_b}. \tag{A.7}$$

Combining equations (A.5–A.7) and summing over all the possible configurations $\{m_1,\dots,m_{\ell_a-2}\}$ gives

$$F(h|k,a) = \sum_{m_1=1}^{\infty}\cdots\sum_{m_{(\ell_a-2)}=1}^{\infty}\prod_{i=1}^{\ell_a-2}\frac{m_i P_b(m_i)}{\mu_b}\;\Theta\left[\frac{A^0}{h}\left(1-f_a\left(\frac{1/k}{1/h+1/k+\sum_{i=1}^{\ell_a-2}1/m_i}\right)\right)-E^0\right], \tag{A.8}$$

where Θ is the Heaviside step function, $\Theta(x) = 1$ if $x > 0$ and zero otherwise.

After summing over assets Eq. (A.2) becomes

$$\mathcal{N}_{hk} = \frac{e^{-\mu_b}\mu_b^h}{h!}\frac{h(k-1)}{\mu_b^2 n}\sum_{\ell}\frac{e^{-\mu_a}\mu_a^{\ell}}{\ell!}\ell(\ell-1)F(h,k,\ell), \quad (A.9)$$

where we have used the fact that the number of assets with given degree ℓ is $MP_a(\ell)$ and explicitly introduced the Poisson degree distributions of banks and assets.

The form of the matrix $\mathcal{N}$ confirms that the independent parameters of the model are μ_b, n, λ and α. We can see in particular that, although leverage has a similar effect on stability to the market impact constant α, the two are not related through a simple relation that allows us to eliminate one of the two dependencies. However, if we had used a market impact that was linear in the price, instead of the log-price, i.e. of the form $f_a(x) = \alpha' x$, then the stability would depend only on the product $\alpha'\lambda$ and not on the two parameters separately.

For networks in which all the banks have the same degree k we can compute the largest eigenvalue of $\mathcal{N}$ in closed form. In this case the matrix $\mathcal{N}$ reduces to the scalar quantity

$$\mathcal{N} = \xi_1 = (k-1)kn\frac{\Gamma(l^*-1, kn)}{\Gamma(l^*-1)}, \quad (A.10)$$

where

$$l^* = \frac{1}{\log\left(\frac{\lambda}{\lambda-k}\right)}, \quad (A.11)$$

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt \quad (A.12)$$

is the gamma function, and

$$\Gamma(x,z) = \int_z^{\infty} t^{x-1}e^{-t}dt \quad (A.13)$$

is the incomplete Gamma function.

If we make the approximation $\frac{1}{m_i} \to \mathbb{E}\left[\frac{1}{k}\right]$ in the denominator of the argument of the market-impact function f_a we can obtain a closed expression for $\mathcal{N}_{h,k}$. This approximation (sometimes called an annealed approximation) is equivalent to replacing the ensemble of binary networks with a fully connected weighted network, where the weight associated to a link connecting two nodes i and j in the new network is equal to the probability that i and j are connected in the original ensemble, i.e. $k_i k_j/\mu_b N$ (Dorogovtsev et al., 2008).

However, given that this approximation is uncontrolled, we do not give an explicit form for the matrix elements, but rather compute them exactly via Montecarlo methods.

References

Adrian, T., Shin, H.S., 2010. Liquidity and leverage. Journal of Financial Intermediation 19, 418–437.

Allen, F., Babus, A., Carletti, E., 2012. Asset commonality, debt maturity and systemic risk. Journal of Financial Economics 104, 519–534.

Amini, H., Cont, R., Minca, A., 2013. Resilience to contagion in financial networks. Mathematical Finance.

Arinaminpathy, N., Kapadia, S., May, R.M., 2012. Size and complexity in model financial systems. Proceedings of the National Academy of Sciences 109, 18338–18343.

Beale, N., Rand, D.G., Battey, H., Croxson, K., May, R.M., Nowak, M.A., 2011. Individual versus systemic risk and the regulator's dilemma. Proceedings of the National Academy of Sciences 108, 12647–12652.

Beltran, D.O., Cordell, L., Thomas, C.P., 2013. Asymmetric Information and the Death of ABS CDOs. International Finance Discussion Papers, Washington, DC : Board of Governors of the Federal Reserve System.

Bouchaud, J.P., 2011. The endogenous dynamics of markets: price impact, feedback loops and instabilities. In: Berd, A. (Ed.), Lessons from the 2008 crisis. Risk Publications, pp. 57–156.

Bouchaud, J.P., Cont, R., 1998. A langevin approach to stock market fluctuations and crashes. European Physics Journal B 6, 543–550.

Bouchaud, J.P., Farmer, J.D., Lillo, F., 2009. How markets slowly digest changes in supply and demand. In: Hens, T., Schenk-Hoppe, K. (Eds.), Handbook of Financial Markets: Dynamics and Evolution. Elsevier, pp. 57–156.

Brunnermeier, M.K., Pedersen, L.H., 2009. Market liquidity and funding liquidity. Review of Financial Studies 22, 2201–2238, <http://rfs.oxfordjournals.org/content/22/6/2201.full.pdf+html>.

Caccioli, F., Catanach, T.A., Farmer, J.D., 2012a. Heterogeneity, correlations and financial contagion. Advances in Complex Systems 15, 1250058.

Caccioli, F., Bouchaud, J.P., Doyne Farmer, J., 2012b. A proposal for impact-adjusted valuation: critical leverage and execution risk. ArXiv e-prints, 1204.0922.

Cai, J., Saunders, A., Steffen, S., 2011. Syndication, Interconnectedness, and Systemic Risk. Social Science Research Network Working Paper Series.

Campbell, A., 2013. Word-of-mouth communication and percolation in social networks. American Economic Review 103, 2466–2498.

Cifuentes, R., Ferrucci, G., England, B.O., Shin, H.S., 2005. Liquidity risk and contagion. Journal of the European Economic Association, 556–566.

Cont, R., Wagalath, L., 2012. Running for the exit: Distressed selling and endogenous correlation in financial markets. Mathematical Finance, no–no.

Corsi, F., Marmi, S., Lillo, F., 2013. When micro prudence increases macro risk: the destabilizing effects of financial innovation, leverage, and diversification. Available at SSRN: <http://ssrn.com/abstract=2278298>.

Dorogovtsev, S.N., Goltsev, A.V., Mendes, J.F.F., 2008. Critical phenomena in complex networks. Reviews of Modern Physics 80, 1275–1335.

Farmer, J.D., 2002. Market force, ecology and evolution. Industrial and Corporate Change 11, 895–953.

Gai, P., Haldane, A., Kapadia, S., 2011. Complexity, concentration and contagion. Journal of Monetary Economics 58, 453–470.

Gai, P., Kapadia, S., 2010. Contagion in financial networks. Proceedings of the Royal Society A 466, 2401–2423.

Gatheral, J., 2010. No-dynamic-arbitrage and market impact. Quantitative Finance 10, 749–759.

Geanakoplos, J., 2003. Liquidity, default, and crashes. In: Dewatripont, M., Hansen, L., Turnovsky, S. (Eds.), Advances in economics and econometrics: theory and applications, Eighth World Congress. Cambridge University Press, Cambridge, pp. 170–205.

Georg, C.P., 2013. The effect of the interbank network structure on contagion and common shocks. Journal of Banking and Finance 37, 2216–2228.

Gleeson, J., Hurd, T.R., Melnik, S., Hackett, A., 2011. Systemic risk in banking networks without monte carlo simulation. In: Kranakis, E. (Ed.), Advances in Network Analysis and its Applications. Springer Verlag, pp. 57–156.

Gordy, M.B., Lutkebohmert, E., 2013. Granularity adjustment for regulatory capital assessment. International Journal of Central Banking 9, 38–77.

Gorton, G., 2009. The subprime panic*. European Financial Management 15, 10–46.

Haldane, A.G., May, R.M., 2011. Systemic risk in banking ecosystems. Nature, 351.

Huang, X., Vodenska, I., Havlin, S., Stanley, H.E., 2013. Cascading Failures in Bipartite Graphs: Model for Systemic Risk Propagation. Scientific Reports, 1219.

Hurd, T.R., Gleeson, J.P., 2011. A framework for analyzing contagion in banking networks. ArXiv e-prints 1110.4312.

Ibragimov, R., Jaffee, D., Walden, J., 2011. Diversification disasters. Journal of Financial Economics 99, 333–348.

Krishnamurthy, A., 2010. How debt markets have malfunctioned in the crisis. Journal of Economic Perspectives 24, 3–28.

May, R.M., Arinaminpathy, N., 2010. Systemic risk: the dynamics of model banking systems. Journal of the Royal Society Interface 7, 823–838.

Mode, C.J., 1971. Multitype Branching Processes: Theory and Applications. American Elsevier Pub. Co., New York.

Nier, E., Yang, J., Yorulmazer, T., Alentorn, A., 2007. Network models and financial stability. Journal of Economic Dynamics and Control 31, 2033–2060.

Staum, J.C., 2013. Counterparty contagion in context: contributions to systemic risk. In: Fouque, J.P., Langsam, J.A. (Eds.), Handbook on Systemic Risk. Cambridge University Press, pp. 512–544.

Thurner, S., Geanakoplos, G., Farmer, J.D., 2012. Leverage causes fat tails and clustered volatility. Quantitative Finance 12, 695–707.

Upper, C., 2011. Simulations methods to assess the danger of contagion in the interbank markets. Journal of Financial Stability 7, 111–125.

Wagner, W., 2011. Systemic liquidation risk and the diversity diversification trade-off. The Journal of Finance 66, 1141–1175.

Watson, H.W., Galton, F., 1875. On the probability of the extinction of families. The Journal of the Anthropological Institute of Great Britain and Ireland 4, 138–144.

Watts, D., 2002. A simple model of global cascades on random networks. Proceedings of the National Academy of Science, 5766–5771.