Impact-adjusted valuation and the criticality of leverage

Marking whole positions to the current clearing price as in mark-to-market accounting ignores the effect that liquidating a position can have. Such valuations overstate the cash that will be received and underestimate a position's leverage. Simple parametric models for price impact can capture the effect – and serve as an early warning system against over-leveraging a position, according to ***Jean-Philippe Bouchaud**, **Fabio Caccioli** and **Doyne Farmer***

Mark-to-market or 'fair-value' accounting is standard industry practice. It consists of assigning a value to a position held in a financial instrument based on the current market clearing price for the relevant instrument or similar instruments. This is commonly justified by the theory of efficient markets, which posits that at any given time market prices faithfully reflect all known information about the value of an asset. However, mark-to-market prices are only marginal prices, reflecting the value of selling an infinitesimal number of shares.

Obviously, traders are typically concerned with selling more than an infinitesimal number of shares, and are intuitively aware that this practice is flawed. Selling has an impact on the market, depressing the price by an amount that increases with the quantity sold. The first part of a sale will be sold near the current price, but as more is liquidated the clearing price may drop substantially. This counter-intuitively implies the value of 10% of a company is less than 10 times the value of 1% of that company. We take advantage of what has been learned recently about market impact to propose an impact-adjusted valuation method that results in better risk control than mark-to-market valuation. This is in line with other recent proposals that valuation should be based on liquidation prices (Acerbi & Scandolo, 2008, and Caccioli *et al*, 2011).

The need for a better alternative to marking-to-market is most evident for positions with leverage, that is, when assets are purchased with borrowed money. As a leveraged position is sold, the price tends to drop due to market impact. As it is gradually unwound, the depression in prices due to impact overwhelms the decrease in position size, and leverage can initially rise rather than fall. As more of the position is sold, provided the initial leverage and initial position are not too large, it will eventually come back down and the position retains some value. However, if the initial leverage and position are too large, the leverage diverges during unwinding, and the resulting liquidation value is less than zero, that is, the debt to the creditors exceeds the resale value of the asset. The upshot is that under mark-to-market accounting a leveraged position that appears to be worth billions of dollars may predictably be worth less than nothing by the time it is liquidated. Under fire-sale conditions or in very illiquid markets, things are even worse.

From the point of view of a risk manager or regulator, this makes it clear that an alternative to mark-to-market accounting is badly needed. Neglecting impact allows huge positions in illiquid instruments to appear profitable when this is not the case. We propose such an alternative based on the known functional form of market impact, and that valuation should be based on expected liquidation value. While mark-to-market valuation only indicates problems with excessive leverage after they have occurred, this makes them clear before positions are entered into. At the macro level, this could be extremely useful for damping the leverage cycle and coping with pro-cyclical behaviour (Thurner, Geanakoplos & Farmer, 2012, and Geanakoplos, 2010). An extended discussion of our proposal that treats extensions to the problem of risky execution can be found in Caccioli, Bouchaud & Farmer (2012).

Market impact and liquidation accounting

Accounting based on liquidation prices requires a quantitative model of market impact. Because market impact is very noisy, and because it usually requires proprietary data to be studied empirically, a good picture of market impact has emerged only gradually in the literature (for recent reviews, see Bouchaud, Farmer & Lillo, 2009, Moro *et al*, 2009, and Toth *et al*, 2011). Here, we are particularly concerned with the liquidation of large positions, which must either be sold in a block market or broken into pieces and executed incrementally. Our interest is therefore in the impact of a so-called meta-order, that is, a single large trade that must be executed in pieces. This is in contrast to the impact of a single small trade in the order book, or the impact of the average order flow, both of which have different functional forms, and different time dependencies (see Toth *et al*, 2011). Empirical studies on meta-orders now make it clear that the market impact $I = E[\varepsilon \cdot (p_f - p_0)/p_0]$, defined as the expected shift in price from the price p_0 observed before a buy trade ($\varepsilon = +1$) or a sell trade ($\varepsilon = -1$) to the price p_f at which the last share is executed, is a concave function of position size Q normalised by the trading volume V. When liquidation occurs in normal conditions, that is, at a reasonable pace that does not attempt to remove liquidity too quickly from the order book, the expected impact I due to liquidating Q shares is to a large extent universal, independent of the asset, time period, tick size, execution style, etc. It is given by:

$$I(Q) = Y\sigma\sqrt{\frac{Q}{V}} \qquad (1)$$

where σ is the daily volatility, V is daily share transaction volume and Y is a numerical constant of order unity (see Toth *et al*, 2011, for a detailed discussion). A crucial observation for the validity of our further analysis is that the above formula holds approximately true within each meta-order as well, that is, the impact of the first q shares is simply given by $I(q)$. After completion of the meta-order the behaviour of impact is less clear (Farmer *et al*, 2011, and Toth *et al*, 2011).

The earliest theory of market impact (Kyle, 1985) predicted that expected impact should be linear and permanent. This was further supported by the work of Huberman & Stanzl (2004), who argued that providing certain assumptions are met, such as lack of correlation in order flow, impact has to be linear in order to avoid arbitrage. However, more recent empirical studies have made it clear that these assumptions are not met (see, for example, Toth *et al*, 2011), and the overwhelming empirical evidence that impact is concave has driven the development of alternative theories. For example, Farmer *et al* (2011) have proposed a theory based on a strategic equilibrium between liquidity demanders and liquidity providers, in which uncertainty about Q on the part of liquidity providers dictates the functional form of the impact. Toth *et al* (2011), in contrast, derive a square-root impact function within a stochastic order flow model. Assuming prices are diffusions, they show that this implies a locally linear latent order book, and provide a proof-of-principle using a simple agent-based model. Both of these theories roughly predict square-root impact, though with some differences.

We should stress the formulas above for market impact hold only in relatively calm market conditions, when execution is slow enough for the order book to replenish between successive trades (Weber & Rosenow, 2005, and Bouchaud, Farmer & Lillo, 2009). If the execution schedule is so aggressive that Q becomes comparable to V, liquidity may dry up, in which case the parameters σ and V can no longer be considered fixed, but themselves react to the trade – σ increases and V drops. Impact in such extreme conditions, such as the so-called flash crashes, is expected to be much larger than the square-root formula above. In these cases, the expected impact becomes less concave and it can become linear or even super-linear (Gatheral, 2010). For the above impact formula to be valid, the execution time T needs to be large enough that Q remains much smaller than V (20% is a typical upper limit). The execution time should not be too long either, otherwise impact is necessarily linear in Q: beyond the time scale for the market to remember linkages between individual trades, trades must necessarily become independent and impact must be additive (see Toth *et al*, 2011).

The establishment of a quantitative theory for expected impact makes it possible to do impact-adjusted accounting. Rather than using the mark-to-market price, which is the marginal price of an infinitesimal liquidation, we propose using the expected price under complete liquidation, depressed by the impact. Using the approximation that shares are executed continuously and integrating the impact, it is easy to see that this is given by:

$$\tilde{p} = p_0\left(1 - \frac{2}{3}I(Q)\right) \qquad (2)$$

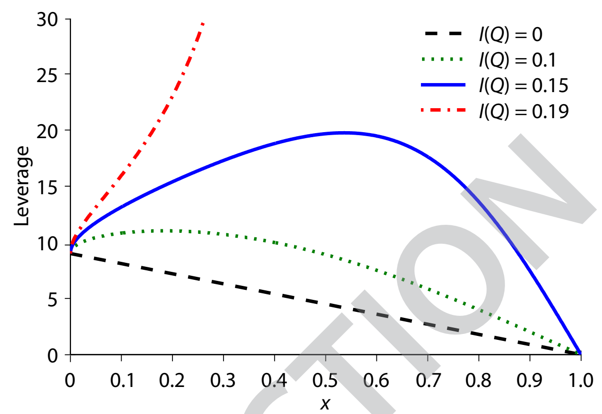
where p_0 is the initial mark-to-market price.

The critical nature of leverage

When leverage is used, it becomes particularly important to take impact into account and value assets based on their expected liquidation prices. Consider an asset manager taking on liabilities L to hold Q shares of an asset with price p. For simplicity, we consider the case of a single asset. We define the leverage λ as the ratio of the value of the asset to the total equity:

$$\lambda = \frac{Qp}{Qp - L} \qquad (3)$$

1 Possible deleveraging trajectories



Note: possible deleveraging trajectories, showing the leverage $\lambda(x)$ based on mark-to-market accounting as a function of the fraction x of the position that has been liquidated. We hold the initial leverage $\lambda_0 = 9$ constant and show four trajectories for different values of the market impact parameter $\mathcal{I} = I(Q) = Y\sigma\sqrt{Q/V}$, that is, $\mathcal{I} = 0$ (black dashed line, corresponding to the no-impact case), $\mathcal{I} = 0.1$ (green dotted line), 0.15 (blue solid line) and 0.19 (red dotted-dashed line). If the market impact is too high, the leverage diverges before the position can be liquidated, implying that the position is bankrupt

In the absence of market impact, selling q shares always decreases leverage linearly, because the denominator remains constant – the cash generated by selling the asset reduces the liability by the same amount, that is, $Q_p - L \to (Q - q)p - (L - qp)$. So $\lambda \to \lambda(1 - x)$ where $x = q/Q$ is the fraction of assets sold.

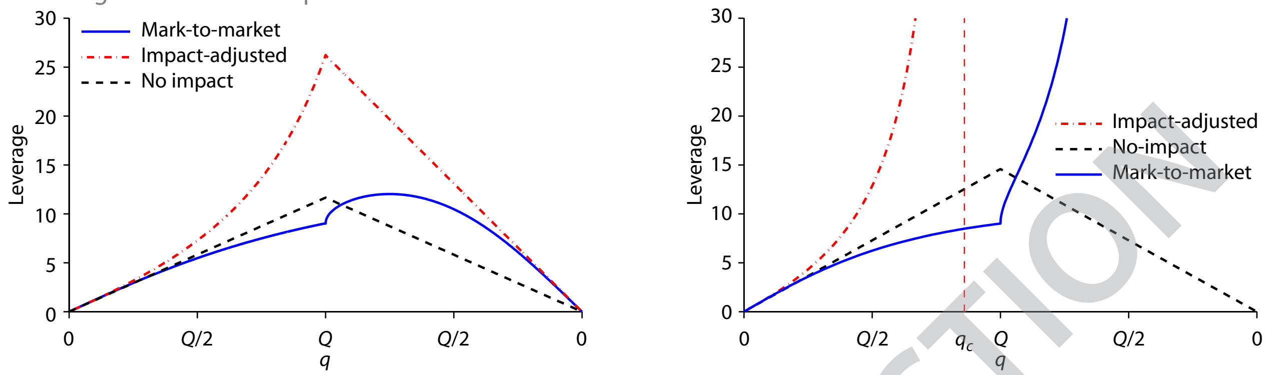
This changes when impact is considered in deleveraging. Selling q shares pushes current trading prices down, which under mark-to-market accounting decreases the value of the remaining $Q - q$ unsold shares. As we will show, this generally overwhelms the effect of selling the shares, increasing the leverage even as the overall position is reduced. Letting λ_0 be the initial leverage before selling begins, the leverage as a function of the fraction x sold can be shown to be:

$$\lambda(x) = \lambda_0\left(\frac{(1-x)\left(1-\mathcal{I}\sqrt{x}\right)}{1-\lambda_0\mathcal{I}\sqrt{x}(1-x/3)}\right) \qquad (4)$$

where $\mathcal{I} \equiv I(Q) = Y\sigma\sqrt{Q/V}$ is the impact of selling the entire position. From this expression, one deduces that for small x and any $\mathcal{I} > 0$, $\lambda(x)$ is larger than λ_0 for $\lambda_0 > 1$, that is, whenever any leverage is used. This means, seemingly paradoxically, that when selling a leveraged position, the expected leverage under mark-to-market accounting always initially increases. When $\lambda_0\mathcal{I} > 3/2$, the leverage $\lambda(x)$ in fact diverges during liquidation.

Three representative deleveraging trajectories $\lambda(x)$ are illustrated in figure 1, together with the trajectory obtained in the absence of market impact. We assume a fixed starting mark-to-market leverage $\lambda_0 = 9$ and show three cases corresponding to different values of the overall market impact parameter $\mathcal{I}$. For the two cases where the leverage is subcritical, that is, with $\lambda_0\mathcal{I} < 3/2$, the manager unwinds the position without bankruptcy. However, due to the rise in leverage during the course of liquidation, they may have trouble with their prime broker. For example, in the case where $\mathcal{I} = 0.15$ at its peak, $\lambda(x)$ is more than twice its starting value (see figure 1). The case where the leverage is allowed to become supercritical is a disaster. If $\lambda_0\mathcal{I} > 3/2$, which for $\lambda_0 = 9$ implies $\mathcal{I} > 0.16$, the manager is trapped, and the likely outcome of attempting to deleverage is bankruptcy.

2 Leverage as a function of position size



Note: The x axes denote number of shares held. The position $q(t)$ varies from zero to Q in the left half of each panel and from Q to zero in the right. The black line shows the leverage without price; the blue line shows the leverage including impact under mark-to-market accounting; and the red line shows the leverage using impact-adjusted valuation. The left panel is a case in which Q is small enough that the leverage never becomes critical; the right panel is a case where the leverage becomes supercritical. In this case, the impact-adjusted leverage diverges as the position is entered, warning the manager of the impending disaster. The dashed red vertical line shows the critical position q_c

Risk management is improved by impact-adjusted accounting, simply by using the average impact-adjusted valuation price $\tilde{p}$ in the formula for leverage. In figure 2, we show how the leverage behaves when a manager first steadily assumes a position $0 \leq q(t) \leq Q$ and then steadily liquidates it. We compare three different notions of leverage:

- No-impact leverage is represented by the dashed black line. This is the leverage that would exist if the price remained constant on average. It rises and falls linearly proportional to the position $q(t)$.
- Mark-to-market leverage is represented by the solid blue line. While the position is building, it rises more slowly than linearly, because impact causes the price to increase, by partially offsetting the increasing position size. When the position is exited, the expected leverage initially shoots up. In the subcritical case (left graph) it eventually returns to zero, but in the supercritical case (right graph) it diverges, making the position bankrupt.
- Impact-adjusted leverage is represented by the dashed red line, and is always greater than the other two measures. It is particularly useful in the supercritical case – its rapid increase is a clear warning that a problem is developing, in contrast to the decreasing mark-to-market leverage. In particular, this shows how dangerous the mark-to-market case is – it overestimates profits and depresses the time leverage value. Over-leveraging is only revealed when it is too late. A prudent risk manager would use impact-adjusted leverage to avoid bankruptcies.

So far it is not clear whether the effects we have illustrated in the preceding sections happen under realistic conditions, or whether they require such extreme conditions as to be practically unimportant. In this section, we plug in some numbers for different assets and show that indeed these effects can happen under realistic conditions.

Let us first give some orders of magnitude for stock markets. The daily volume of a typical stock is roughly 5×10^{-3} of its market cap, while its volatility is of the order of 2% a day. Suppose the portfolio to be liquidated owns $Q = 5\%$ of the market cap of a given stock. Taking $Y = 0.5$, the impact discount is:

$$I(Q) \approx 2\% \times \sqrt{\frac{0.05}{0.005}} \approx 6\% \qquad (5)$$

A 6% haircut on the value of a portfolio of very liquid stocks is already quite large, and it is obviously much larger for less liquid/more volatile markets.

Let us now turn to the question of the critical leverage λ_c under mark-to-market accounting. From the discussion above, the condition reads:

$$\lambda_c \mathcal{I} = \frac{3}{2} \to \lambda_c = \frac{3}{2Y\sigma}\sqrt{V/Q} \qquad (6)$$

To get a feeling for whether or not these conditions can be met, we present representative values for several different assets. For futures we assume $Q = V$, implying that it would take five days to trade out of the position with a participation rate of 20%. For stocks we assume $Q = 10V$, which assuming the same participation rate implies a position that would take 50 trading days to unwind. Such positions might seem large, but they do occur for large funds; for instance, Warren Buffet was recently reported to have taken more than eight months to buy a 5.5% share of IBM. The results are given in table A.

We see that for liquid futures, such as the Bund or S&P 500, the critical leverage is large enough that the phenomenon we discuss here is unlikely to ever occur. As soon as we enter the world of equities, however, the situation looks quite different. For over-the-counter markets, the effect is certainly very real. Using reasonable estimates, we find that the impact of deleveraging a position can easily reach 20% on these markets, corresponding to a critical leverage $\lambda_c \approx 7.5$.

Conclusion

Positions need to be based on liquidation prices rather than mark-to-market prices. For small, unleveraged positions in liquid markets there is no problem, but as soon as any of these conditions are violated, the problem can become severe. As we have shown, standard valuations, which do nothing to take impact into account, can be wildly overoptimistic.

Impact-adjusted accounting gives a more realistic value by estimating liquidation prices based on recent advances in understanding market impact. If one believes – as we do – that equation (1) is a reasonable representation of the impact that an asset manager will unavoidably incur when liquidating his/her position, our procedure has the key virtue of being extremely easy to implement. It is based on quantities such as volatility, trading volume or the spread, which are all relatively easy to measure. Risk estimates can be calculated for the typical expected behaviour or for the probability of a loss of a given magnitude (see Caccioli, Bouchaud & Farmer, 2012).

The worst negative side-effects of mark-to-market valuations occur when leverage is used. As we have shown here, when liquidity is low, leverage can become critical. By this we mean that as a position is being entered there is a critical value of the leverage λ_c above which it becomes very likely that liquidation will result in bankruptcy, that is, liquidation value less than money owed to creditors. This does not require bad luck or unusual price fluctuations – it is a nearly mechanical consequence of using too much leverage.

Standard mark-to-market accounting gives no warning of this problem, in fact quite the opposite: mark-to-market prices rise as a position is purchased, causing leverage to be underestimated. However, as a position is unwound the situation is reversed. The impact of unwinding causes leverage to rise, and if the initial leverage is at or above a critical value, the leverage becomes infinite and the position is bankrupt. Under mark-to-market accounting this comes as a complete surprise. Under impact-adjusted accounting, in contrast, the warning is clear. As the critical point is approached, the impact-adjusted leverage diverges, telling any sensible portfolio manager that it is time to stop buying.

The method of valuation that we propose here could potentially be used both by individual risk managers as well as by regulators. Had such procedures been in place in the past, we believe many previous disasters could have been avoided. As demonstrated in the previous section, the values where leverage becomes critical are not unreasonable compared with those used before, such as the leverages of 50–100 used by Long-Term Capital Management in 1998, or 30–40 used by Lehman Brothers and other investment banks in 2008.

However, one should worry about other potentially destabilising feedback loops that our impact-adjusted valuation could trigger. For example, in a crisis situation, spreads and volatilities increase while the liquidity of the market decreases. Updating the parameters entering the impact formula (volatility, spread and available volumes) too quickly would predict a deeper discount on the asset valuation, potentially leading to further fire sales, fuelling more panic, etc. It is therefore important to estimate parameters using a slow-moving average to avoid any overreaction to temporary liquidity droughts. This observation is in fact quite general: recalibrating models after every market hiccup often leads to instabilities.

The failure of marginal prices as a useful means of valuation is part of an emerging view of markets as dynamic, endogenously driven and self-referential (Bouchaud, 2010), as suggested long ago by Keynes and more recently by Soros. For example, recent studies suggest that exogenous news plays a minor role in explaining major price jumps (Joulin *et al*, 2008), while self-referential feedback effects are strong (Filimonov & Sornette, 2012). Market prices are moulded and shaped by trading, just as trading is moulded and shaped by prices, with intricate and sometimes destabilising feedback. Because the liquidity of markets is so low, the impact of trades is essential to understand why prices move (Bouchaud, Farmer & Lillo, 2009). ■

A. Rough orders of magnitude for numerical parameters entering the impact formulas given in equation (1), with the corresponding estimates of impact and critical leverage

Asset	σ (daily)	V (\$bn)	$\mathcal{I}$ *	λ_c
Bund**	0.4%	140	0.4%	~ 300
S&P 500**	1.6%	150	1.6%	~ 100
MSFT***	2%	1.25	6.3%	~ 25
AAPL***	2.8%	0.5	8.9%	~ 17
KKR****	2.5%	2****	7.9%	~ 16
ClubMed*****	4.3%	1*****	13.5%	~ 11

Note: except as otherwise noted, numbers are based on data for first-quarter 2008
* Impact $\mathcal{I}_1 = I(Q)$ based on volatility and volume, calculated with equation (1), with $Y = 1$ and $Q = V$ for futures and $Q = 10V$ for stocks, roughly 5% of the market capitalisation
** For futures, we refer to the nearest maturity; the numbers for the 10-year US note are very similar to those for the Bund
*** Large cap US stocks, $Q = 10V$
**** Krispy Kreme Doughnuts, a small cap stock, March 2012, with $Q = 10V$, V in \$m and v in \$'000
***** ClubMed, a small cap French stock, $Q = 10V$ with V in €m and v in €'000

Fabio Caccioli is a post-doctoral fellow at the Santa Fe Institute in Santa Fe, New Mexico. Jean-Philippe Bouchaud is chairman of Capital Fund Management Paris and a professor at Ecole Polytechnique Paris. Doyne Farmer is a professor at the Institute for New Economic Thinking and Mathematical Institute at the University of Oxford. This work was supported by the National Science Foundation under grant 0965673, the European Union Seventh Framework Programme FP7/2007-2013 under grant agreement CRISIS-ICT-2011-288501 and the Sloan Foundation. Jean-Philippe Bouchaud acknowledges important discussions with Xavier Brockman, Julien Kockelkoren, Yves Lempérière and Bence Toth. Email: caccioli@santafe.edu, Jean-Philippe.Bouchaud@cfm.fr, jdf@santafe.edu

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