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The impact of systemic and illiquidity risk on financing with risky collateral

Fabrizio Lillo [a,b,c,*], Davide Pirino [a]

[a] Scuola Normale Superiore, Pisa, Italy
[b] Dipartimento di Fisica e Chimica, University of Palermo, Italy
[c] Santa Fe Institute, United States

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ABSTRACT

Repurchase agreements (repos) are one of the most important sources of funding liquidity for many financial investors and intermediaries. In a repo, some assets are given by a borrower as collateral in exchange of funding. The capital given to the borrower is the market value of the collateral, reduced by an amount termed as haircut (or margin). The haircut protects the capital lender from loss of value of the collateral contingent on the borrower's default. For this reason, the haircut is typically calculated with a simple Value at Risk estimation of the collateral for the purpose of preventing the risk associated to volatility. However, other risk factors should be included in the haircut and a severe undervaluation of them could result in a significant loss of value of the collateral if the borrower defaults. In this paper we present a stylized model of the financial system, which allows us to compute the haircut incorporating the liquidity risk of the collateral and, most important, possible systemic effects. These are mainly due to the similarity of bank portfolios, excessive leverage of financial institutions, and illiquidity of assets. The model is analytically solvable under some simplifying assumptions and robust to the relaxation of these assumptions, as shown through Monte Carlo simulations. We also show which are the most critical model parameters for the determination of haircuts.

1. Introduction

Funding is one of the most critical activities for the functioning of financial institutions. When an institution needs funding, it might use a large variety of contracts with other counterparties, including other financial institutions or central banks. Such contracts are mainly defined by their duration and by the presence of a collateral. In the last case, which is the object of this paper, the contract is termed as repo. Specifically, repurchase agreements (repo, plural repos) are deals stipulated by two parties: a borrower (the seller of the repo) and a lender or financier (the buyer of the repo). At the beginning of the contract the borrower lends a security (usually a bond or an equity) to the financier against cash collateral. On the other side, the financier lends cash against the security as collateral. At the maturity of the repo the borrower returns the capital lent (plus some interests) to the financier and receives back the security. Nevertheless, as a consequence of the uncertainty of the collateral, the capital lent by the financier is always strictly smaller than the market value of the security

* Corresponding author.
E-mail addresses: fabrizio.lillo@sns.it (F. Lillo), davide.pirino@sns.it (D. Pirino).

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at the beginning of the contract. This extra quantity required from the financier is usually termed as margin or haircut and, as defined in Gorton and Metrick (2012), it *[…] reflects the perceived underlying risk of the collateral and protects the lender against a change in its value*.

Repo agreements are the main source of funds for securitized-banking activities, both for investment and commercial banks. Further, they are a very important segment of security finance because, quoting Fabozzi and Mann (2005), they *[…] enable market makers to take long and short positions in a flexible manner, buying and selling according to customer demand on a relatively small capital base. In addition, repos are used extensively to facilitate hedging and speculation. Repo is also a flexible and relatively safe investment opportunity for short-term investors. The ability to execute repo is particularly important to firms in less-developed countries that might not have access to a deposit base*.

There are three possible ways in which a financial institution can get funding through repos. In the first one, the repo is stipulated on a bilateral basis between the lender and the borrower over the counter. In the second way, which is becoming increasingly popular in recent years, the two counter-parties are met by a clearing house, i.e. a financial institution that is in charge of setting the proper margins and bears the burden of counter-party risk. Consider, for example, the case of Eurex Clearing: the informative note *Risk Based Margining*[1] reports that the margin is set in order to cover 99% of all possible price variations, given the current forecast of volatility. Translated in formula this means that, if $\hat{\sigma}$ is a forecast of the daily volatility, the margin m_{clearing} (in unit of the current value of the collateral) set by the clearing house is[2]

$$m_{\text{clearing}} = \sqrt{2T}\hat{\sigma}\,\text{erfc}^{-1}(2p), \quad p = 0.01, \tag{1}$$

where T is the maturity in days. Eq. (1) implicitly assumes that the value of the collateral follows a standard Brownian motion. The third way of collateralized financing, when available, is to open a repo with a central bank. For example, those banks with at least an agency in the Euro Zone can currently sell a repo to the European Central Bank (ECB) that provides low haircuts (see, for example, the table in the publicly available document ECB, 2013). Nevertheless, with the gradual reduction of the recent economic crisis, this preferential channel is likely to be closed, and the volume of the repos traded with ECB reduced accordingly.

It is important to understand the effect of the expected decline of the availability of ECB repo funding and the increasing activity in competitive markets. Let us focus on the case of government bonds. For repo bonds the European Central Bank (see ECB, 2013) applies haircuts based on the bond rating, the residual maturity and the type of the coupon (zero/fixed). Most important, the margin required is applied to the market value of the bond, hence a drastic reduction of the bond's value is translated in a reduction of the margin. On the contrary, clearing houses, such as Eurex Clearing, set margins according to a volatility forecast and apply them to the nominal value of the bond. In periods of distress, highly illiquid bonds are likely to rapidly lose value and thus the margining procedure applied by the clearing house can severely reduce their potential as collaterals.

The relevant role of margins in affecting financial equilibria is analyzed, from a theoretical point of view, by Brunnermeier and Pedersen (2009). This paper presents a three periods equilibrium model where margins can potentially increase with illiquidity and create liquidity spirals. The impact on haircuts of balance sheet constraints and, more generally, on the capacity of bearing risk by lenders is analyzed in a continuous-time model in Oehmke (2013). The key-point of the paper by Oehmke (2013) is that, if lenders face a maximum level of risk (volatility) on their position, their optimal strategies are deeply affected by this threshold and, moreover, *rather than relying on purely statistical models, repo lenders should take into account creditor structure, strategic interaction, and their own balance sheet constraints when setting margins [haircuts] to manage counter-party risk […]*.

The future transformation of the practice of repo funding calls for a better understanding of the risks associated with this type of contracts. As we have discussed above, the current practice of setting margins considers only the volatility risk, and in this case the volatility measure does not take into account the fact that it should be computed *conditionally* on the default of the borrower. Moreover other sources of risk should be considered. The first is the liquidity risk, i.e. the fact that the price obtained by the financier when liquidating the collateral in the case of borrower's default might be lower than the mark to market price at the time of liquidation. This is due to market impact and to the finite liquidity of markets. More important, this paper focuses on the impact of systemic risk on collateral financing. In fact, in case of default, the borrower will liquidate its position, depressing the price of other assets and thus triggering a subset of the rest of the financial system to rebalance their positions. This in turn will modify the price of a certain number of assets: the larger the degree of portfolio overlap between the borrower and the rest of the financial world, the larger the number of assets affected. In particular, the asset used as collateral in the repo may be hit by this sort of avalanche effect. The resulting depreciation is clearly not captured by the standard practice adopted in setting haircuts (which is summarized by Eq. (1)). Here, we develop a model and provide a closed form expression for the haircut, which takes explicitly into account all the aforementioned sources of risk. Specifically, we take the point of view of a financier that has to decide the proper margin that must be applied at the beginning of the contract. Our model shows that the value of the haircut, taking into account liquidity and portfolio overlap, depends critically on the linear combination of an illiquidity weighted scalar product between the portfolio of the borrower and those of other financial institutions. The coefficients of the linear combination are the leverage of each financial

[1] Downloadable at: http://www.cftc.gov/ucm/groups/public/@otherif/documents/ifdocs/eurexmcoriskbasedmargin.pdf

[2] The derivation of formula (1) is reported in Appendix A.

institution whose portfolio partially overlaps with that of the borrower. Therefore asset commonality, illiquidity, and leverage of financial institutions contribute significantly to the value of the haircut, which takes into account volatility, liquidity, and systemic risk.

Having a good comprehension of the main sources of systemic risk is thus of crucial relevance for our research topic. Although the literature on systemic risk and on liquidity is huge and having a satisfactory survey on it is far away from the purposes of this paper, we are particularly interested in the impact of distressed selling on the depreciation of assets' value and on the determination of haircuts. Our paper is connected with two different strands of recent literature. On one hand, we are considering distressed selling (fire sales) and its impact on the market price dynamics (Kyle and Xiong, 2001; Wagner, 2011; Cont and Wagalath, 2013; Thurner et al., 2012). In particular, the recent paper by Cont and Wagalath (2013) witnesses the growing interest of the academic community in understanding the role of liquidation of large funds on the stability of financial markets. The model proposed in Cont and Wagalath (2013) allows a better comprehension of excess covariance and excess volatility induced by distressed selling. Further, the impact of fire sales as a source of systemic risk is analyzed in Duarte and Eisenbach (2013). The framework developed by Duarte and Eisenbach (2013) is designed to give a systemic risk measure of bank vulnerability using data on regulatory balance sheets. Moreover, it provides useful insights on which kind of data can be used to assess systemic risk and a simple framework to analyze them. Target-leveraging is the key-mechanism that creates systemic risk in Duarte and Eisenbach (2013) and it is deeply discussed in the empirical analysis by Adrian and Shin (2010). Here it is shown that firms such as commercial banks, security brokers and dealers operate in the market rebalancing their balance sheets in order to reach a firm-specific target leverage. Adrian and Shin (2010) stress the fact that target-leveraging lays the basis of potential dangerous feedbacks, especially in illiquid markets *[…] in which stronger balance sheets feed greater demand for the asset, which in turn raises the asset's price and lead to stronger balance sheets.*

The second strand of literature concerns the effects of diversification and overlapping portfolios on systemic risk (Tasca and Battiston, 2012; Caccioli et al., 2014; Corsi et al., 2013). There is a growing consensus that a significant source of systemic risk comes from the similarity of the portfolios of financial institutions. This overlap, joined with the procyclical effect of target leveraging, can significantly increase the volatility and correlation of assets and, therefore, it can increase the systemic risk of the financial sector. This is the key-feature of the model presented in the paper by Corsi et al. (2013), where a set of financial institutions, having capital requirements in the form of VaR constraints and following standard mark-to-market and risk management rules, diversify their portfolios investing in a set of assets. Authors show that the multivariate feedback effects between investment prices and behavior of banks, induced by portfolio rebalancing in presence of asset illiquidity, leads to a significant increase of volatilities and correlations. Moreover it can trigger a transition from a stationary dynamics of price returns to a non-stationary one characterized by bubbles and bursts of market prices.

These two streams of literature are met in a recently published paper by Poledna et al. (2014). Here the authors use a detailed agent-based model to study the impact of Basel II and other regulatory frameworks on systemic risk, in an economy in which players pursue the practice of target leverage. Their model shows that the adoption of Basel II could paradoxically lead to markets that are more unstable with respect to their unregulated counter parts. Although our model shares some features (funds are target-leveraged and market is illiquid) with that of Poledna et al. (2014), the main difference with our paper is the following. While Poledna et al. (2014) adopt the standard practice in setting the haircuts (see Eq. (1)) and then they analyze the impact of different regulatory polices given these assumptions, we consider the problem of how one should set the value of the haircut in such a way of being protected by systemic risk, given the simultaneous presence of market illiquidity, target leverage, and portfolio overlap. This protection is not guaranteed when the haircuts are set according to Eq. (1). Another difference between the two papers is that in our model the presence of more than one asset is critical since, as shown below, the value of the haircut should take into account also the properties of assets different from the one used as collateral. In our settings, financiers seek protection from systemic risk and adjust Eq. (1) accordingly.[3] The financier sets the haircut conditionally on the system variables she observes at the beginning of the repo, without considering how the system is arrived in such a state.

The paper is organized as follows. In Section 2 we develop a toy-model where a fund enters in a repo with a financier who sets the haircut. In this model there are two assets (one used as collateral) and two funds (one entering the repo). The model presented is in a simplified version for analytical tractability, however we relax some assumptions and derive a more general framework. By means of Monte Carlo simulations we show that the model with more general assumptions does not significantly differ from the analytical case. The main purpose of this section is to show in detail that the joint presence of portfolio overlap, illiquidity of assets, and target leveraging can severely change the margin requirements of a financier that is properly accounting for systemic risk. In Section 3 we generalize the model to an arbitrary number of assets and funds and we show how it is still possible, under the same assumptions of the previous section, to derive a closed form expression for the haircut. As before, we use Monte Carlo simulations to explore less analytically tractable conditions. We also discuss

[3] In Poledna et al. (2014) haircuts on risky collaterals are expressed as (Poledna et al., 2014, Eq. (9), p. 203)

$$H_{col}(t) = \min[\max(H_{min}, \Phi\sigma(t)\sqrt{T} + c), 1], \tag{2}$$

where T is the duration of the repo, $\sigma(t)$ is the historical volatility of the risky collateral, Φ is a confidence interval and H_{min} is a floor on haircuts. This choice corresponds to the margin requirement in Eq. (1) of our paper, with the difference that the confidence interval is now expressed as $\Phi = \sqrt{2}\,\mathrm{erfc}^{-1}(2p)$.

which are the most important parameters the financier has to monitor in order to calculate haircuts which take into account illiquidity and systemic risk. Finally, we present the conclusions of our analysis in Section 4.

2. A simple model of systemic risk

In this section we present a toy-model of a simplified economy where a fund that needs capital enters in a repo with a lender (financier). In order to understand the model it is useful to refer to the schematic representation of Fig. 1. Suppose there are two funds, A and B, and two assets, 1 and 2. Even if we present the model in terms of two funds and two assets, one might consider instead that these terms refer to fund types (e.g. hedge funds, pension funds, banks, etc.) and asset classes (e.g. government bonds, corporate bonds, equities, etc.). A third agent, the financier, indicated by F, is providing liquidity by issuing secured loans. At time $t=0$, when the repo is started, the fund A owns a_1 shares of 1 and a_2 shares of 2. Similarly, let b_1 and b_2 be the number of shares of assets 1 and 2, respectively, owned by fund B at $t=0$. Note that the specific values of the parameters a_i and b_i could be a consequence of some portfolio optimization rule adopted by the two funds. However, in our model the financier sets the margin conditionally on the present system configuration, hence we do not need to discuss any portfolio optimization and we take the parameters a_i and b_i as model inputs. It is worth to note that, in practice, the surge of portfolio overlaps among investors could be a direct consequence of similar practices of portfolio optimization.

The asset 2 is the collateral used by the borrower B to raise capital, while asset 1 is defined as what remains of the portfolio of B once the asset 2 has been subtracted. Finally, the fund A can be thought as the “rest of the world” with respect to B. In practice, it is enough to include in A only the levered institutions that hold the collateral in their portfolios.

The key intuition behind our model is the following. Since the haircut covers the risk of a depreciation of the collateral, it should include different components. First, as currently done, for example, by clearing houses, haircuts should be related to the volatility of the asset (see Eq. (1)). Second, haircuts should be related to the liquidity of the asset. If the borrower defaults, the financier can recover the loss by selling the collateral. However the price obtained in the liquidation is typically smaller than the mark to market price. This is due to the finite liquidity of the asset and to price impact. In this paper we will assume that liquidity is exogenous and it is not affected by the events leading to and following the default of the borrower. This is clearly an unrealistic assumption and it is likely that liquidity declines (and therefore costs increase) in the presence of a massive fire sale related to the distress and default of a part of the financial system. In this sense the liquidity component can be effectively considered as a lower bound. In this paper, we want to focus mainly on a third component of the risk of holding a collateral, and this risk is related to systemic risk and overlapping portfolios. Considering again Fig. 1, if the borrower B defaults, it liquidates the existing positions depressing via market impact the price of the assets 1 and 2. Fund A will therefore experience a decrease in the value of the fund and will rebalance its portfolio, in a way that depends on its strategy and leverage management. In general, part of the assets of fund A will be liquidated and this will depress the price of assets 1 and 2. This further drop of the price of the collateral 2, due to the similarity of portfolios of funds A and B, is the systemic component of the haircut. The model we are going to present in detail describes this chain of events and will give a value of the haircut taking into account the three components of risk, namely volatility, liquidity, and systemic risk.

Details of the model: At $t=0$ the financier receives $(1-h)b_2$ shares of asset 2 from the fund B as a collateral, where $h \in [0,1)$. This means that a repo of a duration T is agreed between the fund B and the financier F. As a consequence a quantity hb_2 of asset 2 remains in the portfolio of B. At time $t=T$ either the fund B has not defaulted and can pay back the loan or it has defaulted at a certain time $t_* < T$, liquidating all its positions. The default of a fund and the consecutive liquidation of its positions may be due to several reasons, such as bad past fund's performances or competition among funds (Jagannathan et al., 2010; Getmansky, 2012). Here we do not model what causes the liquidation of the fund's position, the proposed model takes the point of view of an uniformed financier that seeks protection from the risk of insolvency of the borrower. In particular, hedge funds are one of the most risky counter-parties for financiers, essentially because they are more prone to hold illiquid assets (Getmansky et al., 2004), which hence can be severely affected by systemic risk.

It is important to stress that, if the financier seeks protection from systemic risk, she has to set the haircut at $t=0$ conditional on the default of the fund B in the time interval $(0, T]$. Hence, let us suppose that the fund B defaults at a certain

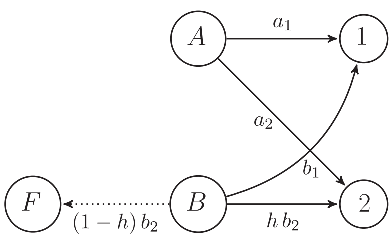

Fig. 1. The scheme of the toy-model described in Section 2. The economy is made by three players and two assets. The money borrower in the repo is indicated by B, the lender or financier is the fund F and the third player A is defined as the rest of the financial world with respect to B. The asset labeled as 2 is the collateral given by the borrower to the lender, while the asset indicated by 1 represents what remains of the portfolio of B once 2 is subtracted. The number of shares of the i-th asset owned by fund B (resp. A) is indicated by b_i (resp. a_i). Only a quantity $(1-h)b_2$, with $h \in [0,1)$, of shares of the asset 2 is given to the financier as a collateral. The remaining number of shares hb_2 is still in the portfolio of the borrower.

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time instant $t_* \in (0, T]$. It cannot sell the $(1-h)b_2$ shares given to the financier as collateral. However it will sell what remains in its portfolio in order to liquidate its position. Hence at time $t = t_*$, asset number 1 is impacted by a sell order of b_1 shares and asset 2 is impacted by a sell order of hb_2 shares.

The liquidation of fund B will decrease the value of the portfolio of the fund A. Following the empirical evidence in Adrian and Shin (2010) we assume that the fund A pursues a target leverage strategy and we indicate with λ the target leverage. Hence the total variation $\Delta V_a(t_*)$ of the value of the fund A from $t=0$ to $t = t_*$ triggers a net demand worth of $(\lambda-1)\Delta V_a(t_*)$ dollars (see Adrian and Shin, 2010, for more details). Note that the sign of the demand is included in $\Delta V_a(t_*)$, i.e. for negative (resp. positive) $\Delta V_a(t_*)$ the funds A will execute a sell (resp. buy) order. The fund value variation $\Delta V_a(t_*)$ depends on the exogenous price change of the two assets 1 and 2 plus the impact on these prices of the liquidation of fund B.

Market impact: We therefore need to model the effect of liquidation on prices, i.e. a model for market impact. Market impact is the expected price change conditioned on initiating a trade of a given size and a given sign (buy or sell). Given the time scale of interest, we are not considering here the impact of individual trades but rather of execution of large trading orders that might be split into pieces and executed incrementally (sometimes termed as metaorders). Since the seminal theoretical work of Kyle (1985) there is a growing interest toward the modeling (see Huberman and Stanzl, 2004; Tóth et al., 2011; Doyne Farmer et al., 2004, among others) and empirical estimation (see BARRA, 1997; Almgren et al., 2005; Obizhaeva, 2008; Tóth et al., 2010; Bershova and Rakhlin, 2013, for a survey) of the impact of metaorders. In the present paper we will consider two different forms of market impact. The first one is a linear function, which is justified by the model of Kyle (1985) and partly supported by the empirical work of Obizhaeva (2008), allows to obtain a closed form expression for the haircut. The second one is the square root impact law, which is more supported empirically, but for which we need to resort to numerical methods.

More specifically, consider the generic case of an order of s_i shares for asset i placed in the market. Let $P_i(t)$ be the price of asset i at instant $t \in [0, T]$ and d_i the market depth[4] for asset i . The empirical results of Obizhaeva (2008) (which are also used in the theoretical work by Cont and Wagalath, 2013) suggest an approximate linearity of this price impact at daily and intraday frequencies given by

$$\frac{V_i\sqrt{252}}{d_i \Sigma_i} \approx 1, \tag{3}$$

where Σ_i and V_i are the annual volatility and the average daily volume of asset i, respectively. Hence, indicating with $P_i(t^-)$ the price immediately before the execution of an order at time t, a simple proportion based on Eq. (3) gives the impact of an order made by a generic number s_i of shares as

$$\frac{P_i(t) - P_i(t^-)}{P_i(t^-)} = \pm \sigma_i \frac{s_i}{V_i}, \tag{4}$$

where $\sigma_i \equiv \Sigma_i/\sqrt{252}$ is the daily volatility and the sign $\pm$ depends on the direction of the order (resp. buy/sell). Note that the price $P_i(t)$ is the price immediately *after* the trade. Moreover, Eq. (4) tells us how much the price is moved as a consequence of an order worth of $Q_i \equiv s_i P_i(t^-)$ dollars, namely

$$P_i(t) - P_i(t^-) = \pm \sigma_i \frac{Q_i}{V_i}. \tag{5}$$

As said above, we assume that the linear impact function (5) is a valid approximation in order to keep analytical tractability.

A larger set of empirical studies (BARRA, 1997; Almgren et al., 2005; Tóth et al., 2011; Bershova and Rakhlin, 2013) indicates that a square-root impact function

$$\frac{P_i(t) - P_i(t^-)}{P_i(t^-)} = \pm \sigma_i \sqrt{\frac{s_i}{V_i}}, \tag{6}$$

fits the data better. Sometimes (for example in Tóth et al., 2011) a numerical prefactor close to unity is present, but in our model we neglect it. In what follows we will show that passing from (5) to (6) does not change drastically the implications of the model.

Distress sale of defaulting fund: We now describe the price dynamics following the default of fund B at time t_*. Let $P_i(t_*^-)$ be the price of asset i immediately before the default of fund B, that is[5]

$$P_i(t_*^-) \equiv \lim_{\tau \uparrow t_*} P_i(\tau) = P_i(0)(1 + \sigma_i \sqrt{t_*}\psi_i), \tag{7}$$

[4] That is a net demand of $d_i/100$ shares moves the price of asset i of 1%.

[5] We implicitly assume a simple stochastic process for the evolution of assets' returns. In fact, the return of asset i between 0 and t_*^- is assumed to be Gaussian distributed with zero mean and standard deviation $\sigma_i\sqrt{t_*}$. Although it is widely recognized that the distribution of assets' returns displays deviations from the Gaussian case, these mainly affect the dynamics of returns at high-frequencies. At daily or weekly frequencies the Gaussian approximation is reasonable and used in many cases. Hence, with the model assumption (7), we do not only obtain fully analytical tractability, but also a realistic process for the frequencies that are relevant in repurchase agreements. Indeed, the typical maturity of a repo is of one-day or even longer. Note however that, at least numerically, one could compute the haircut by assuming other distributions for price returns.

where ψ_i is a random variable with zero mean and unit standard deviation and σ_i is the volatility of asset i. In the following we will assume that ψ_i is Gaussian, but, at least numerically, one can easily extend the calculation to more general model distributions of the exogenous price dynamics. Moreover, in general there is a cross correlation between the ψ_i of different assets, describing the exogenous correlation (see below for more details).

The total price variation $\Delta P_1(t_*)$ of asset 1 from $t=0$ to $t=t_*$ is thus given by[6]

$$\Delta P_1(t_*) \equiv P_1(t_*) - P_1(0) = P_1(t_*) - P_1(t_*^-) + P_1(t_*^-) - P_1(0)$$
$$= -P_1(t_*^-)\sigma_1 \frac{b_1}{V_1} + P_1(0)\sigma_1\sqrt{t_*}\psi_1 = P_1(0)\sigma_1\left[\left(1-\sigma_1\frac{b_1}{V_1}\right)\sqrt{t_*}\psi_1 - \frac{b_1}{V_1}\right],$$

where we have used the fact that, according to the price impact function (4), $P_1(t_*)-P_1(t_*^-) = -P_1(t_*^-)\sigma_1(b_1/V_1)$, with V_1 defined as the average daily volume of asset 1. Similarly for asset 2 it is

$$\Delta P_2(t_*) \equiv P_2(t_*) - P_2(0) = P_2(0)\sigma_2\left[\left(1-\sigma_2\frac{hb_2}{V_2}\right)\sqrt{t_*}\psi_2 - \frac{hb_2}{V_2}\right], \tag{8}$$

where ψ_2 is a (Gaussian) random variable with zero mean and unit standard deviation, σ_2 and V_2 are, respectively, the volatility and the average daily volume of the asset 2. For now we assume that the two exogenous shocks ψ_1 and ψ_2 are uncorrelated. The correlated case is analyzed later in the paper.

Portfolio rebalancing of fund A: The price change of assets 1 and 2, due both to exogenous shocks and to the distress sell of the defaulting fund, modifies the value of the fund A between the two instants $t=0$ and $t=t_*$ according to

$$\begin{aligned}\Delta V_a(t_*) &\\ &= a_1\Delta P_1(t_*) + a_2\Delta P_2(t_*)\\ &= a_1P_1(0)\sigma_1\left[\left(1-\sigma_1\frac{b_1}{V_1}\right)\sqrt{t_*}\psi_1 - \frac{b_1}{V_1}\right] + a_2P_2(0)\sigma_2\left[\left(1-\sigma_2\frac{hb_2}{V_2}\right)\sqrt{t_*}\psi_2 - \frac{hb_2}{V_2}\right].\end{aligned} \tag{9}$$

As explained above, if the fund A follows a target leverage strategy, it must execute an order worth of $(\lambda-1)\Delta V_a(t_*)$ dollars to reach the target. At this point it is critical to describe how the fund decides to sell (or buy) the assets. In general, if s_i $(i=1,2)$ is the amount of shares of asset i that fund A decides to trade, it must be

$$(\lambda-1)\Delta V_a(t_*) = s_1P_1(t_*) + s_2P_2(t_*) \equiv (f_1+f_2)(\lambda-1)\Delta V_a(t_*) \tag{10}$$

where f_i is the fraction of the desired rebalancing coming from the trade of asset i. Note that, in general, the fractions f_i depend on the price $P_i(t_*)$ at the time t_* when the rebalancing is taking place.

Immediately after t_*, a time that we will indicate with t_*^+, the total variation of the price of the collateral is given by

$$\Delta P_2(t_*^+) = \Delta P_2(t_*) + \left(\sigma_2 f_2\frac{(\lambda-1)\Delta V_a(t_*)}{V_2}\right),$$

where we have used the price impact law (5). Using expressions (8) and (9) we get

$$\begin{aligned}\Delta P_2(t_*^+) = {}& P_2(0)\sigma_2\left(1-\sigma_2\frac{hb_2}{V_2}\right)\left[1+(\lambda-1)\frac{\sigma_2}{V_2}a_2f_2\right]\sqrt{t_*}\psi_2\\ &+P_1(0)\sigma_1\left(1-\sigma_1\frac{b_1}{V_1}\right)\left[(\lambda-1)\frac{\sigma_2}{V_2}a_1f_2\right]\sqrt{t_*}\psi_1\\ &+-\left[P_2(0)\sigma_2\frac{hb_2}{V_2}+\frac{\sigma_2}{V_2}f_2(\lambda-1)\left(P_1(0)\frac{\sigma_1}{V_1}a_1b_1+P_2(0)\frac{\sigma_2}{V_2}a_2\tilde{b}_2\right)\right],\end{aligned}$$

where $\tilde{b}_2 = hb_2$ is the amount of tradable (i.e not collateralized) shares of asset 2 by fund B. The last equation can be re-written as

$$\begin{aligned}\Delta P_2(t_*^+) = {}& P_2(0)\sigma_2\left(1-\sigma_2\frac{hb_2}{V_2}\right)\left[1+(\lambda-1)\frac{\sigma_2}{V_2}a_2f_2\right]\sqrt{t_*}\psi_2\\ &+P_1(0)\sigma_1\left(1-\sigma_1\frac{b_1}{V_1}\right)\left[(\lambda-1)\frac{\sigma_2}{V_2}a_1f_2\right]\sqrt{t_*}\psi_1\\ &-\left[P_2(0)\sigma_2\frac{hb_2}{V_2}+\frac{\sigma_2}{V_2}f_2(\lambda-1)\left(\ell_1v_{a,1}v_{b,1}+\ell_2v_{a,2}v_{b,2}\right)\right],\end{aligned} \tag{11}$$

where

$$\ell_i \equiv \frac{\sigma_i}{V_iP_i(0)} \tag{12}$$

[6] In what follows we indicate with $\Delta P_i(t)$ the price variation of the i-th asset between a generic instant t and zero, that is $\Delta P_i(t) = P_i(t) - P_i(0)$. An identical notation is valid for the variation of the value of fund A.

is a illiquidity parameter that corresponds to the percentage variation (in absolute value) of asset i if 1 dollar of the asset is traded[7] at time $t=0$. In the previous equation we have introduced the dollar positions of funds as

$$v_{a,1}=a_1P_1(0),\quad v_{a,2}=a_2P_2(0),\quad v_{b,1}=b_1P_1(0),\quad v_{b,2}=\tilde{b}_2P_2(0).$$

Note that, for what concerns fund B, we are considering only the non-collateralized amount (in dollars or in shares) of asset 2. The percentage price variation can be written now as

$$\begin{aligned}\frac{\Delta P_2(t_*^+)}{P_2(0)}&=\sigma_2\sqrt{t_*}\psi_2(1-\ell_2 v_{b,2})\left[1+\ell_2(\lambda-1)v_{a,2}f_2\right]\\&\quad+\sigma_1\sqrt{t_*}\psi_1(1-\ell_1 v_{b,1})\ell_2(\lambda-1)v_{a,1}f_2\\&\quad-\ell_2 v_{b,2}-\ell_2(\lambda-1)f_2\langle\mathbf{v}_a\cdot\mathbf{v}_b\rangle_{ill},\end{aligned}\tag{13}$$

where we have defined the portfolio vectors $\mathbf{v}_a=(v_{a,1},v_{a,2})'$ and $\mathbf{v}_b=(v_{b,1},v_{b,2})'$ and the illiquidity weighted scalar product between them

$$\langle\mathbf{v}_a\cdot\mathbf{v}_b\rangle_{ill}\equiv\sum_{i=1}^{K}v_{a,i}v_{b,i}\ell_i.\tag{14}$$

Note that the definition in the above equation is generalized to a number $K>2$ of assets (see also Section 3).

The expression in Eq. (13) shows that the price change of a given asset (in particular the asset given as collateral) depends on the illiquidity weighted scalar product between the portfolio of the borrower and the portfolio of the rest of the market. Clearly, this dependence holds also for the value of the haircut, which is related to the distribution of price changes, given the default.

Moreover it is worth noticing that Eq. (13) shows that the price change $\Delta P_2(t_*^+)$ depends not only on ψ_2, but also on ψ_1. Even if the exogenous shocks of the two assets ψ_1 and ψ_2 are independent, one can easily see that the portfolio rebalancing creates a statistical dependence between the observed price changes $\Delta P_1(t_*^+)$ and $\Delta P_2(t_*^+)$. If the shocks are correlated, the portfolio rebalancing will modify the correlation. This is an effect of the overlaps between the portfolio of the two funds, as discussed in a different context in Corsi et al. (2013).

In order to obtain more quantitative information on the value of the haircut we need to make some assumptions on the rebalancing strategy of fund A. This is, in general, a complicated problem, outside of the scope of the present paper. If the f_i depend on the price changes $\Delta P_i(t_*)$, they also depend on the exogenous shocks ψ_i and therefore the probability distribution of price changes of Eq. (13) will be non-Gaussian. For example, one could assume that fund A rebalances its portfolio by splitting the total order according to the relative value of assets at the moment t_* of rebalancing. This means that[8]

$$f_i=\frac{a_iP_i(t^*)}{a_1P_1(t_*)+a_2P_2(t_*)}.\tag{15}$$

If we remind that $P_i(t^*)=P_i(0)+\Delta P_i(t^*)$, we can perform a series expansion for $\Delta P_i(t^*)/P_i(0)\ll 1$ as

$$f_1\simeq f_1^0+\frac{f_2^0}{V_0}\Delta P_1(t^*)-\frac{f_1^0}{V_0}\Delta P_2(t^*),\quad f_2\simeq f_2^0+\frac{f_1^0}{V_0}\Delta P_2(t^*)-\frac{f_2^0}{V_0}\Delta P_1(t^*)\tag{16}$$

where $V_0\equiv a_1P_1(0)+a_2P_2(0)$ is the value of fund A at time $t=0$ and

$$f_i^0=\frac{a_iP_i(0)}{a_1P_1(0)+a_2P_2(0)}=\frac{v_{a,i}}{v_{a,1}+v_{a,2}}\tag{17}$$

are the fractions of value of the two assets in the portfolio of A at time $t=0$.

The simplest assumption is that $\Delta P_i(t^*)$ is so small compared to $P_i(0)$ that we can neglect the first order terms in Eq. (16) and therefore $f_i=f_i^0$. This approximation is justified especially when the duration T of the repo is not too long, since in this case prices do not change significantly. It is immediate to see that in the case where fund B uses all its shares of asset 2 as

[7] Remember that, in the linear impact assumption, we have

$$\frac{P_i(t)-P_i(t^-)}{P_i(t^-)}=\pm\sigma_i\frac{s_i}{V_i}=\pm\sigma_i\frac{s_iP_i(0)}{V_iP_i(0)}.$$

Hence if $s_iP_i(0)=1$ we have

$$\left|\frac{P_i(t)-P_i(t^-)}{P_i(t^-)}\right|=\frac{\sigma_i}{V_iP_i(0)}=\ell_i.$$

[8] There are, of course, infinitely many possible ways of choosing f_i. For example, a risk-averse fund could decide to sell assets proportionally to their volatility, setting

$$f_i=\frac{a_i\sigma_i}{a_1\sigma_1+a_2\sigma_2}.$$

However, a complete discussion on this topic is far beyond the purpose of this paper and is left for future research.

collateral ($h=0$), the expected percentage price change at t_*^+ of the asset used as collateral is

$$\mathbb{E}_0\left[\frac{\Delta P_2(t_*^+)}{P_2(0)}\right] = -\ell_2(\lambda-1)f_2\langle \mathbf{v}_a\cdot\mathbf{v}_b\rangle_{ill} \tag{18}$$

showing an interesting proportionality with the illiquidity weighted scalar product. Note that the expected value is computed conditionally on the information set at time $t=0$. Hence, the prices $P_i(0)$, the illiquidity parameters ℓ_i and the volatilities are known quantities, and therefore treated as parameters. For this reason in the following we will use the symbol $\mathbb{E}_0$ to indicate expectations conditional to the information set containing these parameters. Moreover, as we will see below, this case leads to analytically tractable expressions for the haircuts and this tractability is related to the fact that, under this approximation, the price changes *after* the portfolio rebalancing of fund A is still Gaussian. The more general case of rebalancing given by Eq. (15) must be treated numerically and this will be done later in this section. We will obtain non-Gaussian distributions for the price changes, but we will see that the main conclusions about haircuts are not substantially different from the analytical case of Eq. (17).

2.1. *Analytical case for short term repos*

Here, in order to keep analytical tractability, we assume that the liquidation is split according to weights in Eq. (17). As discussed above, this is a realistic approximation for short term repos, since the relative price variation will be small.

With this choice, after fund A has rebalanced its portfolio, i.e. at time t_*^+, the total variation of the price of the collateral is given by

$$\begin{aligned}\Delta P_2(t_*^+) = {} & \sigma_2\sqrt{t_*}\psi_2(1-\ell_2 v_{b,2})[1+\ell_2(\lambda-1)v_{a,2}f_2^0]\\ & +\sigma_1\sqrt{t_*}\psi_1(1-\ell_1 v_{b,1})\ell_2(\lambda-1)v_{a,1}f_2^0+\\ & -\ell_2 v_{b,2}-\ell_2(\lambda-1)f_2^0\langle \mathbf{v}_a\cdot\mathbf{v}_b\rangle_{ill},\end{aligned} \tag{19}$$

note that, without loss of generality we have assumed $P_2(0)=1$. This means that the value that we will derive for the haircut is expressed in units of the initial value of the collateral. It is clear that $\Delta P_2(t_*^+)$ is Gaussian distributed, being the linear combination of (possibly correlated) Gaussian random variables ψ_1 and ψ_2.

In the simplest case when ψ_1 and ψ_2 are uncorrelated, one can see that $\Delta P_2(t_*^+)\sim N(-\mu, P_2(0)^2\sigma^2 t_*)$ with

$$\mu \equiv \ell_2 v_{b,2}+\ell_2(\lambda-1)f_2^0\langle \mathbf{v}_a\cdot\mathbf{v}_b\rangle_{ill}$$

and

$$\sigma^2 \equiv \sigma_2^2(1-\ell_2 v_{b,2})^2[1+\ell_2(\lambda-1)v_{a,2}f_2^0]^2+\sigma_1^2(1-\ell_1 v_{b,1})^2[\ell_2(\lambda-1)v_{a,1}f_2^0]^2 \tag{20}$$

As in the standard approach to risk management we assume that the financier sets the haircut m according to the 99%-VaR of the price drop of the collateral (see, for example, the model by Brunnermeier and Pedersen, 2009). In formula, the margin m is chosen such that

$$p = \mathrm{Prob}_0\left(-\left(\Delta P_2(t_*^+)-\sigma_2 P_2(t_*^+)\frac{(1-h)b_2}{V_2}\right) > m\right), \tag{21}$$

with $p=0.01$, $P_2(t_*^+)=P_2(0)+\Delta P_2(t_*^+)=1+\Delta P_2(t_*^+)$ and where the subscript 0 indicates that the probability is computed conditionally on the information set at the beginning of the repo. Note that the total variation is the sum of $\Delta P_2(t_*^+)$ plus the price impact due to the sell order executed by the financier at t_*^+ in order to partially compensate for the capital lost. We are implicitly assuming that the financier is informed on the default of B after the partial liquidation of the fund A. This is exactly what the financier has to assume in order to achieve full protection from systemic, volatility and illiquidity risk. In the opposite case she is seeking protection exclusively from volatility and illiquidity risk, since the systemic component is completely neglected.

Eq. (21) can be rewritten as

$$p = \mathrm{Prob}_0\left(\Delta P_2(t_*^+)\left(1-\ell_2\frac{1-h}{h}v_{b,2}\right) < \ell_2\frac{1-h}{h}v_{b,2}-m\right).$$

Hence, the haircut is computed once we obtain the cumulative distribution function of the random variable $\Delta P_2(t_*^+)(1-\ell_2((1-h)/h)v_{b,2})$, whose distribution is Normal with mean $-\mu(1-\ell_2((1-h)/h)v_{b,2})$ and variance

$$\left(1-\ell_2\frac{1-h}{h}v_{b,2}\right)^2\sigma^2 t_*.$$

Therefore the haircut m is given by

$$m = \sqrt{2t_*}\left|1-\ell_2\frac{1-h}{h}v_{b,2}\right|\sigma\,\mathrm{erfc}^{-1}(2p)+\mu\left(1-\ell_2\frac{1-h}{h}v_{b,2}\right)+\ell_2\frac{1-h}{h}v_{b,2}. \tag{22}$$

In order to set the final value of the haircut the financier must use her prior on the probability distribution of the default time. We will consider here two cases, that can be easily generalized to the case with a more general survival function. In the first case, the financier puts herself in the most conservative setting by assuming that the default occurs at time $t^* = T$. This choice, in fact, leads to the maximal uncertainty on the value of the collateral. In the second case the financier assumes an uniform probability distribution for default time in $(0, T]$.

The first case corresponds to a default time distribution which is Dirac delta in $t^* = T$ and gives an haircut equal to

$$m_{\text{risk}} = \sqrt{2T}\left|1-\ell_2\frac{1-h}{h}v_{b,2}\right|\sigma\,\text{erfc}^{-1}(2p)+\mu\left(1-\ell_2\frac{1-h}{h}v_{b,2}\right)+\ell_2\frac{1-h}{h}v_{b,2}. \tag{23}$$

The subscript "risk" indicates that the value of the haircut is chosen by taking into account systemic risk. Note that the haircut set by a financier that is myopic to systemic risk would be the one given in Eq. (1), which in dollar units is

$$m_{\text{no-risk}} = \sqrt{2T}\sigma_2\,\text{erfc}^{-1}(2p). \tag{24}$$

In the second case, we assume that the financier will set the haircut by averaging Eq. (22) on $[0, T]$ and assuming an uniform probability on this interval. Hence the final choice for the haircut is[9]

$$m_{\text{risk}} = \sqrt{\frac{8T}{9}}\left|1-\ell_2\frac{1-h}{h}v_{b,2}\right|\sigma\,\text{erfc}^{-1}(2p)+\mu\left(1-\ell_2\frac{1-h}{h}v_{b,2}\right)+\ell_2\frac{1-h}{h}v_{b,2}. \tag{25}$$

Note that, with this second prior, the haircut set by a financier that is myopic to systemic risk would be

$$m_{\text{no-risk}} = \sqrt{\frac{8T}{9}}\sigma_2\,\text{erfc}^{-1}(2p). \tag{26}$$

The introduction of correlations between the two fundamental shocks ψ_1 and ψ_2 does not present any particular difficulty, as shown in Appendix B. Eq. (25) remains substantially unchanged, with the only modification that the volatility parameter σ that appears in Eq. (25) is now that reported in Eq. (39). In practice, a positive correlation increases the value of σ, while a negative one has the opposite effect. It is worth noting that the margin set by a financier that is myopic to systemic risk is completely unaffected by the correlation of fundamental shocks.

In conclusion, we see that the haircut set by taking into account finite liquidity and systemic risk is in general different from the naive expression for haircut given in Eq. (1). Even if we have obtained closed form solutions under the assumption of linear impact and short term repo, the expression in Eq. (25) is not immediately readable for what concerns the role of the parameters and the difference with the benchmark naive case of $m_{\text{no-risk}}$. For this reason we perform some numerical calculations and Monte Carlo simulations of the model. We are particularly interested in identifying which are the key parameters that are mainly responsible of the differences between the two cases. The idea is to better clarify which are the most important system parameters that a wise financier has to monitor in order to protect herself form systemic risk.

We performed several simulations of it in order to compute numerically the margin (25) and we compare the results with the analytical formulas (25) and (26). Fig. 2 reports, as a thick black line, the margin computed replicating 10^4 times the model (19) with the choice of the parameters reported in the first column of Table 1. Empty circles and the red dotted lines are in correspondence of the analytical formulas (25) and (26), respectively. Finally, the blue lines with empty circles plot the difference between the two expressions (25) and (26). Table 1 contains other three columns that correspond to different parameter settings and the associated results are reported, respectively, in Figs. 9–11 of Appendix C. Each of these figures is made by nine panels that, in turn, correspond to a specific level of leverage λ and a specific value of b_2, whose numeric values are reported in the titles of panels. It is quite evident that the margin required for protection from systemic risk is largely influenced by the joint values of λ and b_2. Moreover, an inspection of the results reported in Appendix C reveals that other system parameters, such as the number of shares of fund A and b_1, are not so crucial.

An inspection of the analytical formula (25) reveals that the margin m_{risk} is a linear function of the leverage λ. In order to quantify this dependence Fig. 3 plots, for two different maturities, the margin m_{risk} against the leverage λ for the same choice of the parameters reported in the first column of Table 1, with the exceptions that the leverage varies in the range reported in the horizontal axis and b_2 is now kept equal to 50% of the daily volumes, which are all set equal to 1 (see the note of Table 1). For comparison, we report as a horizontal dotted line the value of the margin $m_{\text{no-risk}}$ given by Eq. (26), which of course does not depend on λ. Moreover we report, as black dots, the margin computed assuming a square-root impact function.

The curves in Fig. 3 reveal that the dependence of the margin from the leverage inherits the shape of the impact function, i.e. it shows a linear dependence in the analytical case and it is approximately a square-root when the impact function is that in Eq. (6).

[9] Although haircut (25) has a fully analytical expression, we report in Fig. 2 the haircut computed replicating a large number of times model (19). This is done only for the purpose of a double-check of the analytical results. The replications are obtained by simply extracting the random variates ψ_i from a Gaussian distribution with zero mean and unit standard deviation. The haircut is then computed as the p-percentile of the distribution of the price drop (19) for a given value of t^* in $[0, T]$. The resulting haircut is a function of t^* that is averaged in $[0, T]$ and reported as a thick black line.

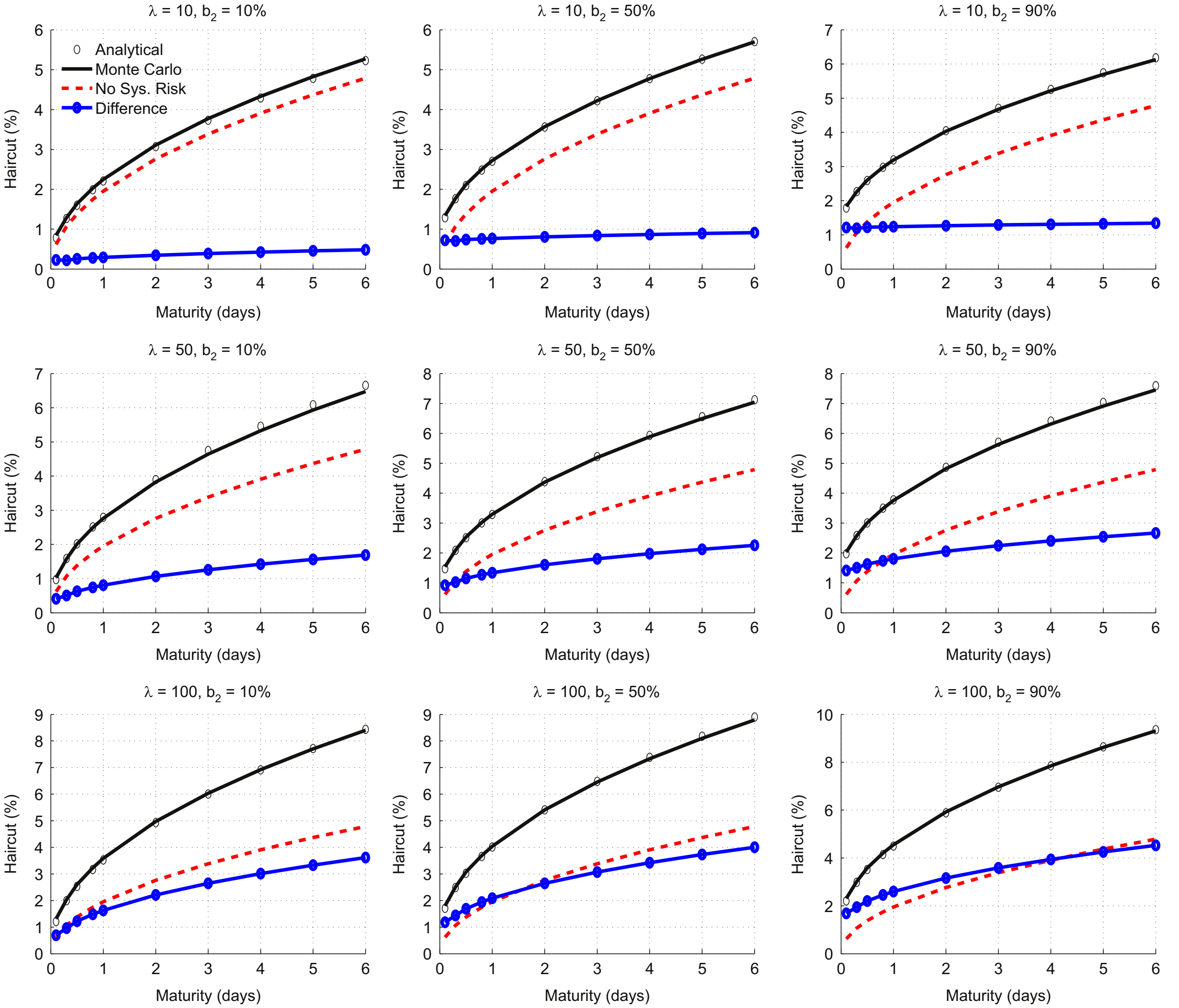

Fig. 2. Each panel shows as a thick black line the margin computed replicating 10^4 times the model (19) with the choice of the parameters reported in the first column of Table 1. Empty circles and the red dotted line are in correspondence of the analytical formulas (25) and (26), respectively. The blue line with empty circles plots the difference between the two expressions (25) and (26). Each panel corresponds to a different choice of the parameters λ and b_2, whose numeric values are displayed in the title. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

Table 1
Model parameters used as input in numerical simulations.

Parameter	Choice #1	Choice #2	Choice #3	Choice #4
σ_1	50	50	50	50
σ_2	20	20	20	20
a_1	1	90	1	1
a_2	60	60	1	60
b_1	30	30	1	30
h	0	0	0	95

Note: All quantities are expressed in percentage. Volatilities σ_1 and σ_2 are expressed in years$^{-1/2}$, hence daily volatilities are obtained dividing annual volatilities by $\sqrt{252}$. Volumes V_1 and V_2 are kept equal to 1 in all the model specifications. This corresponds to have shares of both funds A and B written as percentage of the corresponding asset volume. The number of shares b_2 and the leverage parameter λ are not reported in this table because they are specified for each panel of Fig. 2 and of the figures in Appendix C. Initial prices $P_1(0)$ and $P_2(0)$ are assumed equal to 1 in all the model specifications, nevertheless we verified that their value has no particular influence on the final result. The correlation ρ between exogenous shocks is kept equal to zero.

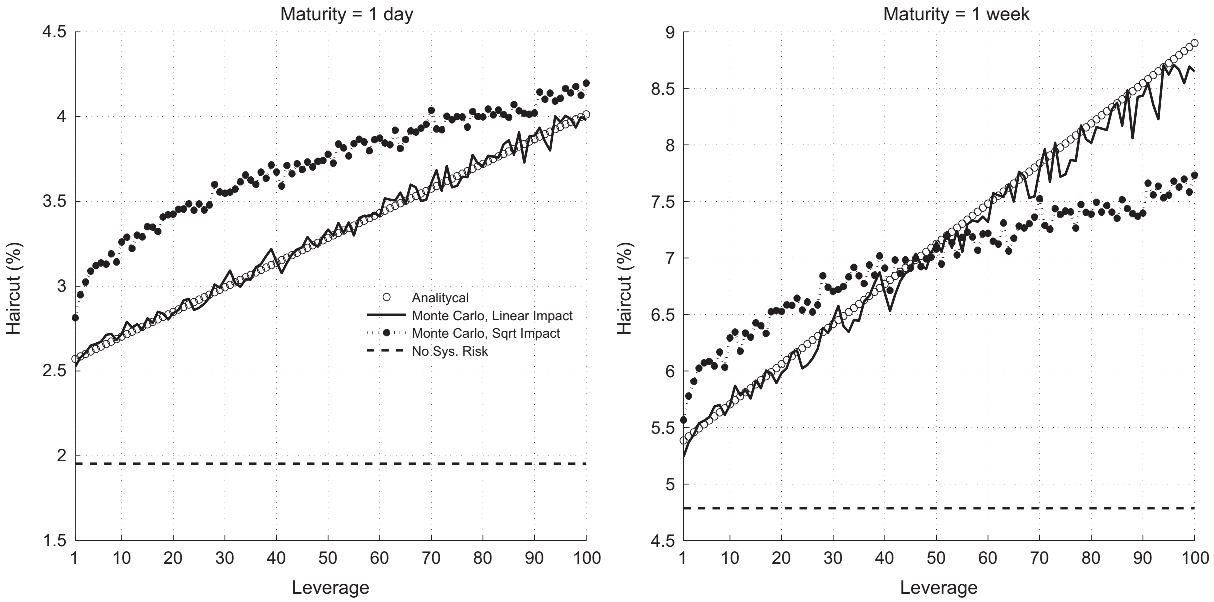

Fig. 3. The haircut as a function of the leverage λ. The thick black line is computed replicating 10^4 times the model (19) with the choice of the parameters reported in the first column of Table 1, with the exceptions that the leverage varies in the range reported in the horizontal axis and b_2 is kept equal to 50%. The left panel corresponds to a maturity of one day, while the right panel corresponds to a maturity of one week. The empty circles are in correspondence of the analytical formula (25), while the horizontal dotted line represents the value of the margin in Eq. (26), which does not depend on λ. The black circles are obtained assuming a square-root impact function as that reported in Eq. (6).

2.2. *Generalization of the model: numerical case*

We now consider the model relaxing the two assumptions: (i) that the fraction f_i of the fund A used to rebalance the portfolio is equal to f_i^0, assumption that is approximately valid for short term repos, and (ii) that market impact is a linear function of traded volume. In both cases we need to resort to numerical simulations.

Haircuts of longer term repos: Before discussing the numerical results, we note that the dependence of f_i on the price changes $\Delta P_i(t_*)$ gives non-Gaussian price changes $\Delta P_i(t_*^+)$. In fact, let us consider the expansion of Eq. (16). By plugging these first order approximations of f_i into Eq. (13), we see that the price change of asset 2 becomes

$$\Delta P_2(t_*^+) = C + A_1\psi_1 + A_2\psi_2 + B_1\psi_1^2 + B_2\psi_2^2 + B_{12}\psi_1\psi_2 \tag{27}$$

where C, A_i, and B_i are parameters which can be expressed as (complicated) functions of the model parameters. The key observation here is that even if $(\psi_1, \psi_2)'$ is a bivariate Gaussian, the last three terms lead to a non-Gaussian behavior for $\Delta P_2(t_*^+)$. In particular it is possible to show that the distribution is leptokurtic,[10] as observed in real financial data. If we replicate model (13) with the choice of the fraction f_i given by Eq. (15) and with the same parameters as those used in Fig. 2, we obtain indistinguishable results. In order to see some difference we have to input extremely high values for the volatility parameter σ_2 and, further, to explore long term maturities. This is exactly what we report in Fig. 4. A part from σ_2 that is now set at 100% years$^{-1/2}$, the other parameters are the same of those used in Fig. 2. Moreover, we extend the maximum horizon up to one year. Hence, for realistic values of volatilities and not too long repo maturities the approximation $f_i \approx f_i^0$ is a very good one.

Nonlinear market impact: Finally, we relax the assumption of linear impact in Eq. (5) changing it with the more realistic square-root impact function, which is reported in Eq. (6). We keep the approximation $f_i \approx f_i^0$ in order to solely study the difference between the two specifications of price impacts. The computation of the margin m is performed numerically with the proper modification of Eq. (21). Fig. 5 reports the results of this numerical experiment. Although weak, it is possible to note a difference between the linear (analytic) and square-root price impacts. For low levels of leverage, the margin produced by the square-root impact is always above the analytical case. When the leverage increases, the relative positions of the two specifications depend on the maturity: for short repos the analytical case is slightly below the more realistic margin produced by the square-root impact, while the converse is true at long maturities.

[10] Without entering in a detailed computation, it is possible to show that in the univariate case the random variable $x = \psi + \beta\psi^2$, where ψ is a standard normal, has a kurtosis strictly larger than 3 for any $\beta \neq 0$. In the general bivariate case of Eq. (27), the distribution of $\Delta P_2(t_*^+)$ has no analytical expression, since ψ_i^2 follows a χ^2 distribution and $\psi_1\psi_2$ as a Meijer G-function distribution, see Springer and Thompson (1970) for reference.

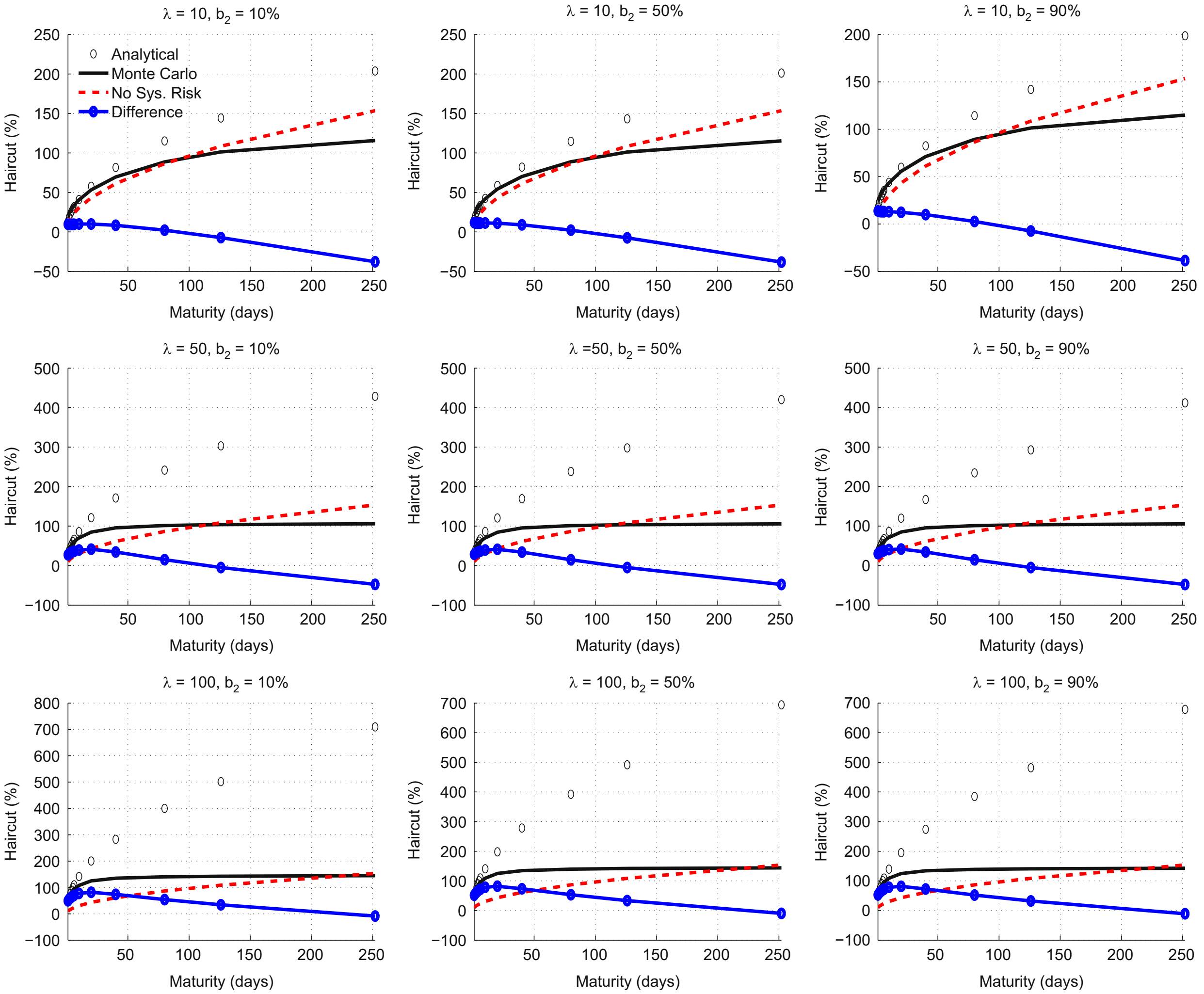

Fig. 4. Each panel reports as a thick black line the margin computed replicating 10^4 times the model in Eq. (13) with the choice of the fraction f_i given by Eq. (15). The values of the parameter are reported in the first column of Table 1, a part from the volatility σ_2 which is now set at 100%years$^{-1/2}$. Empty circles and the red dotted line are in correspondence of the analytical formulas (25) and (26), respectively. The blue line with empty circles plots the difference between the two expressions (25) and (26). Each panel corresponds to a different choice of the parameters λ and b_2, whose numeric values are displayed in the title.

3. A Model with N funds and K assets

The model presented in Section 2 is the minimal model able to capture the systemic risk effect on the haircuts of repos. In fact it is a stylized model with only two assets and two funds. The main purpose of this model is to show in details the different components of the haircut by giving their explicit analytical expressions.

The model can be easily generalized to an arbitrary number of assets and funds. In fact, let assume that there are N funds (or fund types) and K assets (or asset classes). The i-th fund, with $i = 1, \ldots, N$, is characterized by a target leverage λ_i. Without loss of generality let us assume that the financier opens a repo with the N-th fund and the asset given as a collateral is the K-th asset. For the sake of simplicity we will also assume that fund N gives all the shares of the asset K it owns to the financier (i.e. $h=0$ in the previous model). As for the simplified model, we assume that the exogenous shocks of asset prices are described by a multivariate Gaussian vector $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_K)'$, where each ψ_i has zero mean and unit variance and, for simplicity, we assume that the different components are uncorrelated. Finally, we assume that market impact is linear and follows Eq. (4), where V_i and σ_i are the average daily volume and the exogenous volatility of asset i, respectively.

It is straightforward to show that the expression of Eq. (19), which gives the price of the collateral at time t_*^+, i.e. after the default of the fund N and the portfolio rebalancing of the other $N-1$ funds, can be generalized to the following expression:

$$\Delta P_K(t_*^+) = P_K(0)\sigma_K\sqrt{t_*}\psi_K + \frac{\sigma_K}{V_K}\sum_{i=1}^{N-1} f_{i,K}(\mathbf{P}(t_*))(\lambda_i - 1)\sum_{q=1}^{K} w_{i,q}P_q(0)\sigma_q\left[\left(1-\frac{\sigma_q}{V_q}w_{N,q}\right)\sqrt{t_*}\psi_q - \frac{w_{N,q}}{V_q}\right], \quad (28)$$

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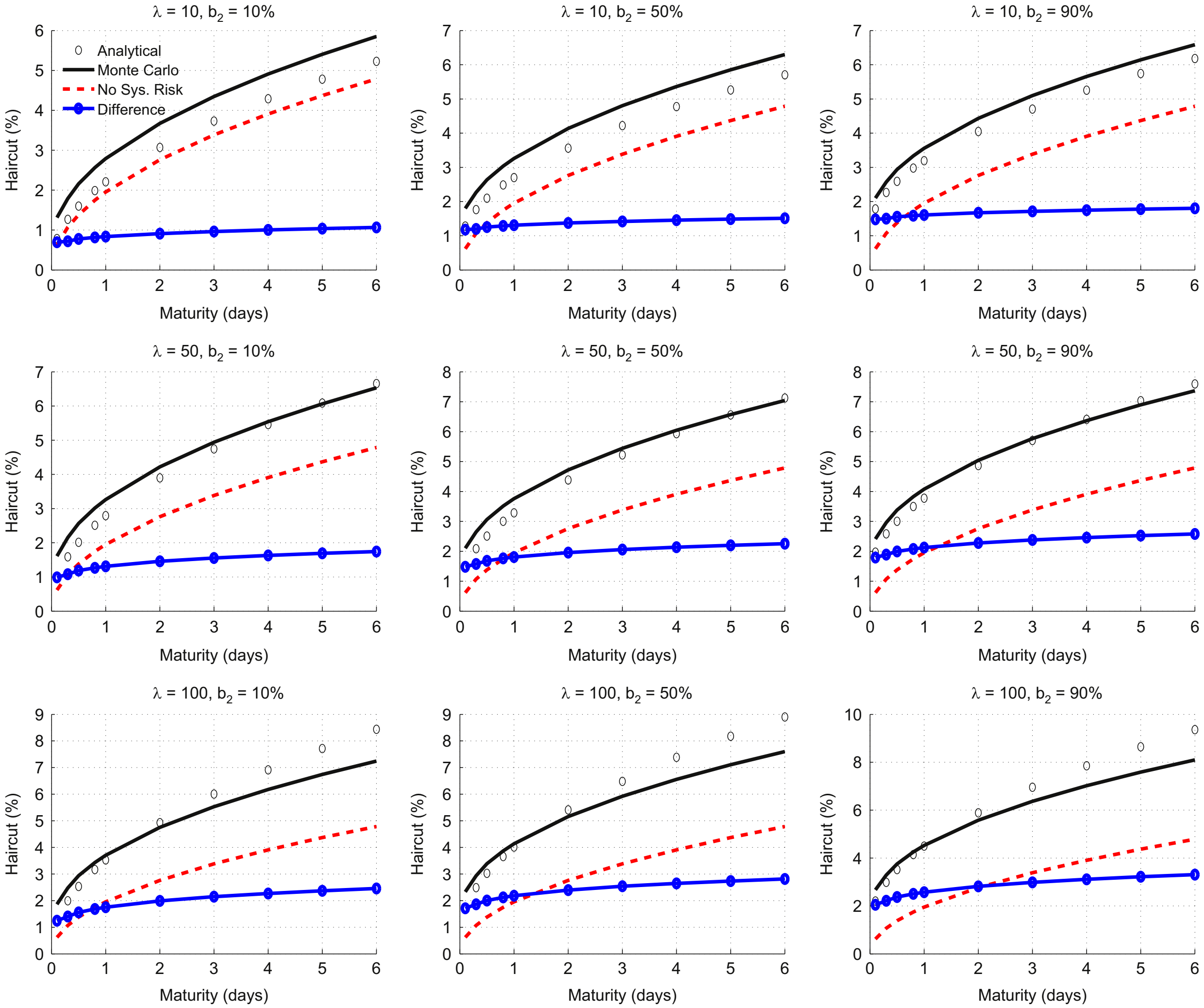

Fig. 5. Each panel reports as a thick black line the margin computed replicating 10^4 times a modified version of model (19), where the liquidation factor is that reported in Eq. (17) and the square-root impact function (6) is assumed. The values of the parameter are reported in the first column of Table 1. The red dotted line is in correspondence of the analytical formula (26). The blue line with empty circles plots the difference between the thick black the red dotted lines. Each panel corresponds to a different choice of the parameters λ and b_2, whose numeric values are displayed in the title.

where $P_q(0)$ is the price of asset q at the time the repo is opened. The quantity $w_{i,q}$ is the element of the $N\times K$ matrix W and it is defined as the number of shares of asset q owned by fund i at time $t=0$. Finally the quantity $f_{i,K}(\mathbf{P}(t_*))$ is the fraction of value of the fund i which is rebalanced by trading asset K in order to reach the target leverage. As in the simplified model, this quantity depends, in general, on the vector of prices $\mathbf{P}(t_*)\equiv(P_1(t_*),\dots,P_K(t_*))'$ at time t_*, which in turn depends on the idiosyncratic shock vector $\boldsymbol{\psi}$.

Let us consider now the short term repo approximation, where Eq. (17) gives $f_{i,K}(\mathbf{P}(t_*))\simeq f^0_{i,K}$, and we obtain

$$\Delta P_K(t_*^+) = P_K(0)\sigma_K\sqrt{t_*}\psi_K + \frac{\sigma_K}{V_K}\sum_{i=1}^{N-1} f^0_{i,K}(\lambda_i-1)\sum_{q=1}^{K} w_{i,q}P_q(0)\sigma_q\left[\left(1-\frac{\sigma_q}{V_q}w_{N,q}\right)\sqrt{t_*}\psi_q-\frac{w_{N,q}}{V_q}\right]$$

$$= P_K(0)\sigma_K\sqrt{t_*}\psi_K + \frac{\sigma_K}{V_K}\sum_{i=1}^{N-1}\frac{w_{i,K}P_K(0)}{\sum_{q=1}^{K} w_{i,q}P_q(0)}(\lambda_i-1)\sum_{q=1}^{K} w_{i,q}P_q(0)\sigma_q\left[\left(1-\frac{\sigma_q}{V_q}w_{N,q}\right)\sqrt{t_*}\psi_q-\frac{w_{N,q}}{V_q}\right].$$

Hence the percentage price variation is written as

$$\frac{\Delta P_K(t_*^+)}{P_K(0)} = \sigma_K\sqrt{t_*}\psi_K + l_K\sum_{i=1}^{N-1}\frac{v_{i,K}}{\sum_{q=1}^{K} v_{i,q}}(\lambda_i-1)\sum_{q=1}^{K} v_{i,q}\left[\sigma_q(1-l_q v_{N,q})\sqrt{t_*}\psi_q - l_q v_{N,q}\right], \tag{29}$$

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where, as in Section 2, we have introduced the dollar position of fund i in asset q at time zero as $v_{i,q}=w_{i,q}P_q(0)$ and ℓ_i is the illiquidity parameter defined in Eq. (12). By taking expectation with respect to the idiosyncratic shock vector ψ we obtain

$$\mathbb{E}_0\left[\frac{\Delta P_K(t_*^+)}{P_K(0)}\right]=-l_K\sum_{i=1}^{N-1}f_{i,K}^0(\lambda_i-1)\sum_{q=1}^{K}l_q v_{i,q}v_{N,q}=-l_K\sum_{i=1}^{N-1}f_{i,K}^0(\lambda_i-1)\langle\mathbf{v}_i\cdot\mathbf{v}_N\rangle_{ill}, \tag{30}$$

where we have defined the portfolio vector $\mathbf{v}_i=(v_{i,1},\ldots,v_{i,K})'$ of fund i and the definition of the illiquidity weighted scalar product of Eq. (14). This expression shows that the expected price change of the collateralized asset is proportional to the average illiquidity weighted scalar product of the portfolio vector of the borrower fund with the portfolio vectors of all the other levered funds, and the weighting factor depends on the leverage of the other funds.

Under the short term repo approximation, Eq. (29) leads to the exact computation of the haircut. The analytical tractability is due to the fact that, under this approximation, $\Delta P_K(t_*^+)$ is Gaussian. In the more general case of non-short repo and/or of nonlinear impact, the haircut can be calculated numerically with quantile estimation or Monte Carlo simulations.

To conclude the analytical part of this section, we show that the standard deviation of the random variable $\Delta P_K(t_*^+)$ has a simple expression and, hence, the margin can be expressed in a closed-form similar to that of Eq. (25). We start noticing that $\Delta P_K(t_*^+)$ is a linear combinations of the Gaussian shocks ψ_q. Defining

$$\begin{aligned}\alpha_i&\equiv l_K f_{i,K}^0(\lambda_i-1),\\ \beta_{i,q}&\equiv v_{i,q}\sigma_q(1-l_q v_{N,q}),\end{aligned} \tag{31}$$

we have that Eq. (29) becomes

$$\frac{\Delta P_K(t_*^+)}{P_K(0)}=\sigma_K\sqrt{t_*}\psi_K+\sqrt{t_*}\sum_{i=1}^{N}\alpha_i\sum_{q=1}^{K}\beta_{i,q}\psi_q+\mathbb{E}_0\left[\Delta P_K(t_*^+)\right].$$

Hence

$$\frac{\Delta P_K(t_*^+)}{P_K(0)}\sim N(\mu_{\Delta P_K},\sigma_{\Delta P_K}^2 t^*)$$

with

$$\mu_{\Delta P_K}=\mathbb{E}_0[\Delta P_K(t_*^+)],$$

whose explicit expression is reported in Eq. (30), and

$$\sigma_{\Delta P_K}^2=\sigma_K^2+\sum_{i,j=1}^{N}\alpha_i\alpha_j\sum_{q=1}^{K}\beta_{i,q}\beta_{j,q}.$$

Define $\boldsymbol{\alpha}$ the vector whose elements are α_i and $\mathbf{B}$ the $N\times K$ matrix whose generic element is $\beta_{i,q}$. Call $\mathbf{A}$ the $N\times N$ matrix defined as $\mathbf{A}=\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}$. After some simple computations we can write

$$\sigma_{\Delta P_K}^2=\sigma_K^2+\mathrm{Tr}(\mathbf{B}^T\mathbf{A}\mathbf{B}).$$

Finally, the margin $m_{\text{risk}}^{N,K}$ (expressed in units of $P_K(0)$) that the financier has to require in this setting, without considering the additional term due to the sell order executed after the default of the borrower, is given by

$$m_{\text{risk}}^{N,K}=\sqrt{\frac{8T}{9}}\sigma_{\Delta P_K}\,\mathrm{erfc}^{-1}(2p)+l_K\sum_{i=1}^{N-1}f_{i,K}^0(\lambda_i-1)\langle\mathbf{v}_i\cdot\mathbf{v}_N\rangle_{ill}, \tag{32}$$

where we have followed the same averaging procedure on t_* adopted in deriving expression (25).

Results of the model: Here we show how the haircut depends on the parameters of the model. We will consider the case of linear impact and short term repo. Since the model depends on many parameters, we draw them from some predefined distribution. Specifically, as a first step, for each asset $k=1,\ldots,K$, a value $P_k(0)=1$ is given to each asset in the economy. Hence, an annual volatility σ_k is assigned to asset k by extracting it from the uniform distribution in $(0,1)$, and then normalized to daily units. Moreover, for each simulation we impose $\sigma_K=20\%$ years$^{-1/2}$, i.e the volatility of the collateral is not affected by the randomization procedure (it is a quantity revealed to the financier). In the third step for each fund $i=1,\ldots,N$, an integer random number k_i is extracted from the uniform distribution in $[1,K]$ and k_i assets are randomly selected from the K available ones and assigned to fund i. This process creates a bipartite network between funds and assets in a similar way as done in Corsi et al. (2013). However, in our case, each fund invests in a different number k_i of asset classes. Moreover, a leverage λ_i is extracted in the uniform distribution $[10,80]$ and assigned randomly to the i-th fund. This assumption roughly corresponds to taking into consideration only funds having a significant levered position.

The portfolio composition of each fund is obtained with the simplifying assumption that all assets in the economy have the same market capitalization in shares ($M_k\approx\overline{M}$) and adopting it as unit of measure of $w_{i,k}$. Therefore it must be

$$\sum_{k=1}^{N}w_{i,k}=M_k\approx\overline{M}\equiv1,\quad\forall i=1,\ldots,N. \tag{33}$$

We satisfy condition (33) by extracting k_i random variables $u_{i,k}$ from the uniform distribution in [0, 1] and then normalizing them by the appropriate factor

$$u_{i,k} \sim U[0,1] \Rightarrow \mathbb{E}\left[\sum_{k=1}^{k_i} u_{i,k}\right] = \frac{K}{2}.$$

Hence the portfolio weights are defined by $w_{i,k} = \frac{2}{K} u_{i,k}$ and the condition (33) is satisfied on average. We define a parameter $g \in (0,1]$ that gives the ratio between the average daily volume (used in the market impact) and the market capitalization M_k of each asset. The turn-over of the asset is thus given by $1/g$ and represents the average number of days required to trade the entire capitalization of the asset.

As mentioned above, it is assumed that the borrower fund has given to the financier all its shares of the collateral. Since we assume that the borrower is the N-th fund and the collateral is the K-asset, we impose the condition $w_{N,K} = 0$ at each replication of the model.

After all the aforementioned random assignments have been done, a replication of the system of funds with shared assets is built and the financier is able to compute the haircut conditional on the default of the borrower using Eq. (32).

Remember that, for the sake of simplicity, when using formula (32) we are computing the haircut without considering the term due the sell order executed by the financier. Iterating this procedure for N_{rep} times allows to compute the expected margin with the corresponding 5–95% confidence bands.

The first numerical experiment we implement is designed as follows. We consider, as an example, the case with $N = 50$ and $K = 70$, fixing the number of replications at $N_{rep} = 500$ and the maturity at one day. Then we vary the value of the turn-over $1/g$ in the range displayed in the left panel of Fig. 6. For each value of the turn-over we compute the expected haircut and we plot it as a thick black line. For comparison, a red dotted line is in correspondence of the margin computed neglecting systemic risk, which is given by Eq. (26). For extremely liquid assets, that is assets with a turn-over close to one, the two margins are almost the same, while the expected value of margin (32) diverges for realistic value of g (i.e. a turn-over between 50 and 100). In summary, the left panel of Fig. 6 clearly shows how liquidity of markets affects haircuts.

As a second experiment, we take a turn-over of 20 and we vary the maturity in the same range of Fig. 2. For each maturity we compute the expected value of the margin (32) and its corresponding 5–95% confidence bands and we plot them, respectively, as thick and dotted black lines with empty circles in the right panel of Fig. 6. On the same plot we report, as a thick and dotted black lines with stars, the expected value and the corresponding 5–95% confidence bands computed by using a modified version of the model in Eq. (29). The change consists in using the liquidation factors (15) instead of (17) and the square-root impact function (6) instead of the linear one of Eq. (4). These plots serve to quantify how good are the approximations implicitly assumed in the analytical formula (32). Note that, in order to obtain the curve of the modified model, we have to perform a double Monte Carlo procedure. For each replication of the randomization procedure that gives

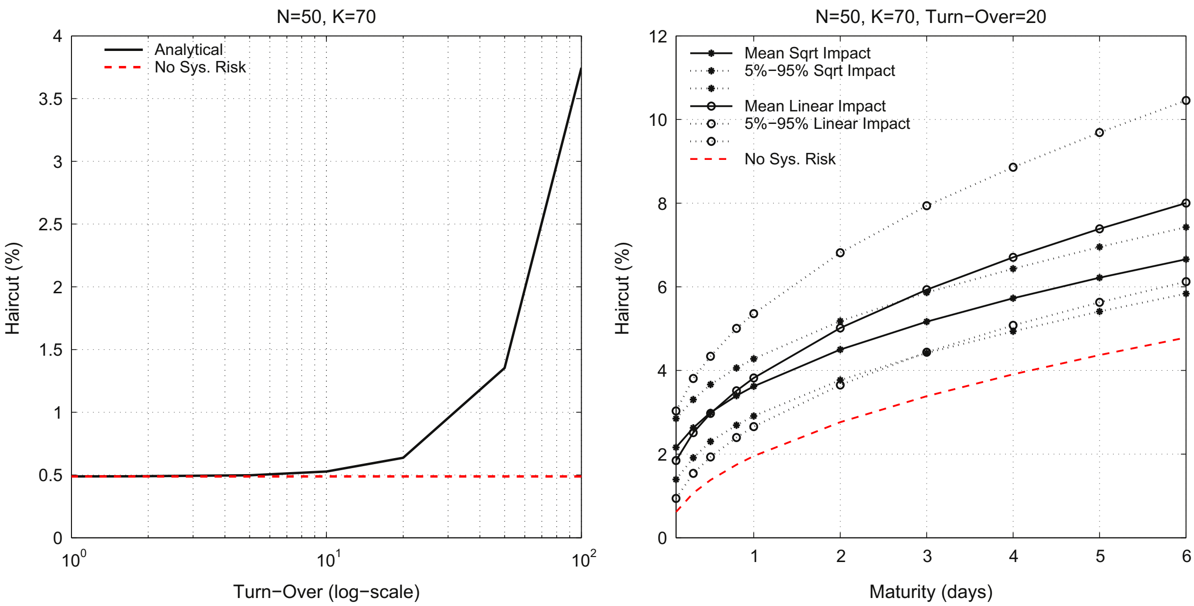

Fig. 6. The left panel reports, as a thick black line, the expected margin in Eq. (32), with a maturity of one day, as a function of the turn-over, i.e. the number of days required to trade the entire capitalization. The right panel plots, for each maturity, the expected value of the margin (32) and its corresponding 5%–95% confidence bands reported, respectively, as thick and dotted black lines with empty circles. In the same plot, the thick and dotted black lines with stars represent, respectively, the expected value and the corresponding 5%–95% confidence bands computed using a modified version of the model in Eq. (29), where the liquidation factor (15) is used instead of (17) and the square-root impact function (6) is used instead of the linear one of Eq. (4). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

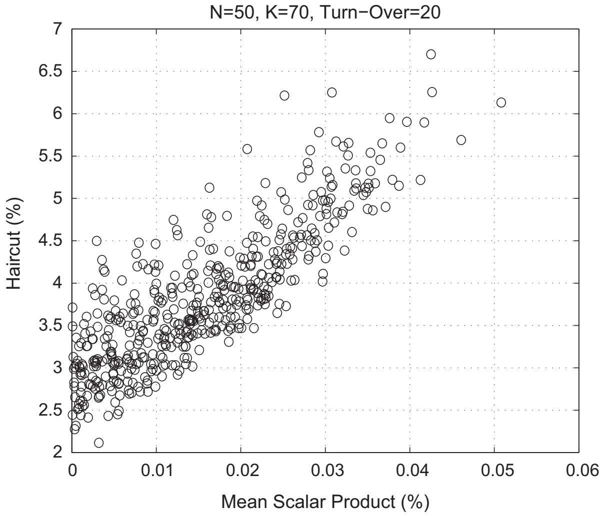

Fig. 7. Scatter plot between the margin of Eq. (32) with a maturity of one day and the average overlap of funds' portfolios defined in Eq. (34). The empty circles are in correspondence of 500 replications of the randomization procedure.

as outcome the system of funds with shared assets, we have to compute numerically the margin, since an analytical formula does not exist in this case, and this is done performing 1000 extractions of the multivariate Gaussian vector ψ and then computing the Monte Carlo distribution of the price drop after the default of the borrower. The curves of the right panel in Fig. 6 confirm the validity of the linear impact approximation implicitly assumed in the analytical formula (32), especially for short-term repo.

Finally, we investigate how the haircut depends on the average overlap of funds' portfolios. In doing this, we first compute the average illiquidity weighted scalar product defined as

$$O \equiv \frac{1}{N} \sum_{j=1}^{N} l_K \sum_{i=1}^{N} f_{i,K}^{0} (\lambda_i - 1) \langle \mathbf{v}_i \cdot \mathbf{v}_j \rangle_{ill}, \tag{34}$$

which is our proxy for the average overlap of portfolios. For each replication of the randomization procedure we record the haircut and the corresponding value for O. Eventually, we report them in the scatter plot of Fig. 7. As expected, higher haircuts are in correspondence of the system configurations associated with higher value of the portfolio overlap O.

This dependence is confirmed by the two plots of Fig. 8. Here, we fix a turn-over of 50 and perform two experiments. In the left panel we report the mean value of the margin in Eq. (32) as a function of K for different value of N. Vice versa, the right panel reports the mean value of the margin in (32) as a function of N for different values of K. It is clear that, for a given N, an increase of the number of assets reduces the mean portfolio overlap and, as a consequence, the expected haircut. On the contrary, for a given K, an higher number N of funds corresponds to an higher mean portfolio overlap and, thus, an higher haircut.

The shapes of the curves in Fig. 8 are better understood after a simple combinatorial reasoning. In fact, for a given $m = 1, \ldots, K$, we have that the probability that k_i is equal to m is given by

$$\mathrm{Prob}(k_i = m) = \frac{1}{K}.$$

Hence, the probability $p_K(m)$ that two different assets k and q appear together in the portfolio of a fund that owns exactly m assets is[11]

$$p_K(m) = \frac{\binom{K-2}{m-2}}{\binom{K}{m}} = \frac{m(m-1)}{K(K-1)}.$$

As a consequence the total probability that a fund has both assets k and q, with $k \neq q$, is given by

$$P_K = \sum_{m=1}^{K} \frac{1}{K} p_K(m) = \sum_{m=1}^{K} \frac{1}{K} \frac{m(m-1)}{K(K-1)} = \frac{K+1}{3K}.$$

[11] The combinatorics is quite simple. Given that the fund has exactly m assets, there are $\binom{K}{m}$ different ways to assign m among K assets to it. For any couple of assets, once they both are chosen, there are $m-2$ remaining assets to be assigned to the fund, and they must be chosen among the remaining $K-2$ assets, hence any couple of assets can be chosen in $\binom{K-2}{m-2}$ different ways.

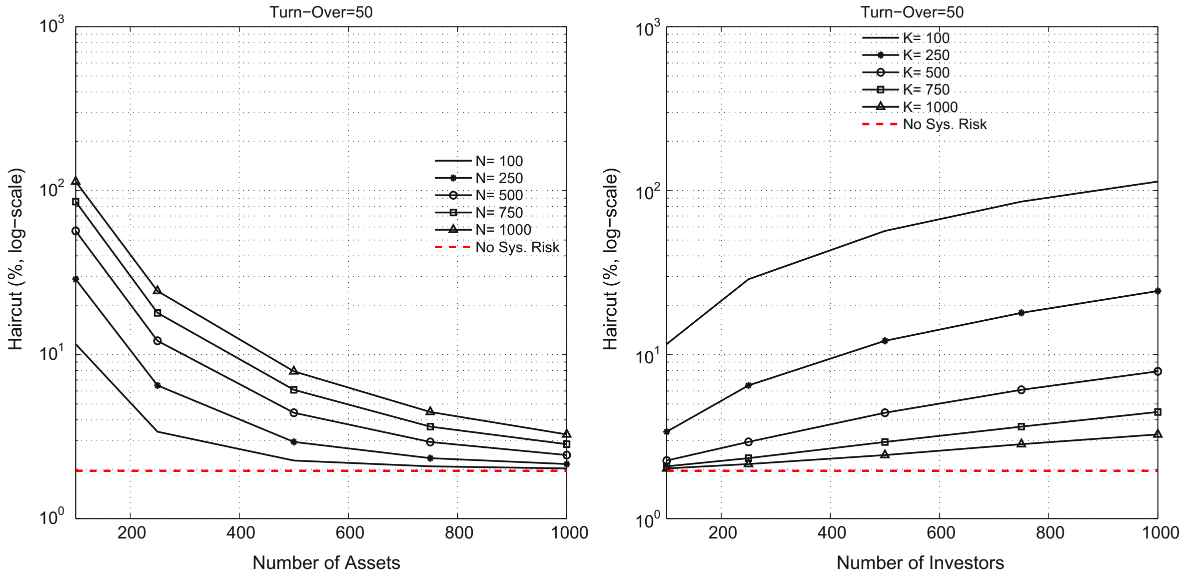

Fig. 8. The expected value of the margin (32) as a function of N for different values of K (left panel) and the expected value of the margin (32) as a function of K for different values of N (right panel). Vertical axes are in logarithmic scale. The red dotted line is in correspondence of the analytical formula (26). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

Let $\mathbf{C}$ be the $N \times K$ binary matrix whose element $C_{i,k}$ is equal to one if asset k is in the portfolio of fund i, i.e. $\mathbf{C} = sign(\mathbf{W})$. The commonality of two different assets k and q (i.e. how many times they are jointly combined within the same portfolio fund) is given by $J_{k,q} = \sum_{i=1}^{N} C_{i,k} C_{i,q}$. Form the linearity of the expectation operator

$$\mathbb{E}[J_{k,q}] = \sum_{i=1}^{N} \mathbb{E}[C_{i,k} C_{i,q}] = \sum_{i=1}^{N} 1 \cdot \text{Prob}(C_{i,k} C_{i,q} = 1) = N \frac{K+1}{3K}. \tag{35}$$

Hence, if both K and N are sent to infinity, no matter how quickly N diverges with respect to K, the average commonality will be proportional to $N/3$. The dependence on N and K observed in Fig. 8 can be attributed to the fact that the commonality of assets is, for sufficiently large K, linear in N and independent of K.

Finally, from the plots of Fig. 8, we see that the expected haircut derived from Eq. (32) is systematically larger than that implied by formula (26) reported as a dotted red line.

4. Summary and conclusions

In the forthcoming years the intensity of the economic crisis is expected to gradually decrease and the repo volumes agreed by the European Central Bank reduced accordingly. Hence, raising liquidity through risky collateralization will be affected by several risk factors. In particular, the widespread strategy of target leveraging combined with the unavoidable portfolio overlap of funds can dangerously trigger dramatic systemic events.

In this paper we analyze the impact of systemic, illiquidity and volatility risk on the margin requirements for risky collateral. We take the point of view of a financier (buyer of a repo contract) that seeks protection from these sources of risk and sets margins accordingly. Our framework keeps analytical tractability by means of a toy-model with simple assumptions on market impact and on how levered institutions rebalance their positions. We show that margins set according to standard risk management practice can be less than half of those required by a rigorous account of the systemic component.

As a striking implication of our analysis we have that a serious approach in endogenizing systemic risk in margin requirements is of paramount importance to mitigate the effect of dramatic systemic events. In fact, assets that are characterized by a low level of volatility but are shared among portfolios of highly levered institutions can be dangerously evaluated as good collaterals and, hence, improperly adopted to raise capital. As a consequence, the real total exposure of investors to risk can be much larger than that perceived by market participants, a situation that can easily translate in moral hazard.

As a final contribution, we extend the toy-model to a more generic setting in which an arbitrarily large number of funds choose among an arbitrarily large number of assets. We derive the distribution of margins under the hypothesis that all the parameters of the model follow a uniform distribution.

In our future research agenda we plan to derive estimation methods capable to produce reliable measures of the parameters that define our model and, in particular, its extensions to a generic number of assets and funds.

Further, we plan to generalize the market impact including endogenous liquidity, that is a market in which liquidity depends on the size of buy and sell orders that are executed.

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Appendix A. Standard formula for margins

Margins are set according to the standard Value at Risk rule. That is m is chosen such that

$$p = \text{Prob}[-\Delta p > m].$$

where Δp is the total price variation between the beginning of the repo at $t=0$ and its maturity at $t=T$, assuming a unitary initial price. If returns follow a Brownian motion, then Δp is a Gaussian variable with zero mean and standard deviation $\sigma\sqrt{T}$. If we denote by

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left(-\frac{t^2}{2\sigma^2 T}\right) dt,$$

we have

$$\Phi(x) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{x}{\sigma\sqrt{2T}}\right)\right],$$

and hence

$$\begin{aligned} p &= \text{Prob}\left[-\Delta p > m\right] = \text{Prob}\left[\Delta p < -m\right] \\ &= \Phi\left(-\frac{m}{\sigma\sqrt{T}}\right) \\ &= \frac{1}{2}\left[1 + \text{erf}\left(-\frac{m}{\sigma\sqrt{2T}}\right)\right] \\ &= \frac{1}{2}\left[1 - \text{erf}\left(\frac{m}{\sigma\sqrt{2T}}\right)\right] \\ &= \frac{1}{2}\text{erfc}\left(\frac{m}{\sigma\sqrt{2T}}\right). \end{aligned}$$

In summary

$$2p = \text{erfc}\left(\frac{m}{\sigma\sqrt{2T}}\right).$$

The margin m can be derived with the inversion of the previous equation, which gives

$$m = \sqrt{2T}\sigma\,\text{erfc}^{-1}(2p).$$

Appendix B. Dependent fundamental shocks

Let ψ_0 and ψ_1 be normal distributed random variables with zero mean and unit variance such that $\mathbb{E}[\psi_1\psi_0] = 0$. Define

$$\psi_2 \equiv \rho\psi_1 + \sqrt{1-\rho^2}\psi_0. \tag{36}$$

Hence

$$\mathbb{E}[\psi_2\psi_1] = \rho,$$

and

$$\mathbb{E}[\psi_2] = 0, \quad \mathbb{E}[\psi_2^2] = 1.$$

Note that the variance of

$$\gamma\psi_1 + \delta\psi_2,$$

with ψ_1 and ψ_2 independent is

$$\sigma^2 = \gamma^2 + \delta^2.$$

If now we introduce a correlation according to (36), we have to compute the variance of

$$\gamma\psi_1+\delta\left(\rho\psi_1+\sqrt{1-\rho^2}\psi_0\right),$$

which is

$$\gamma^2+\delta^2+2\rho\gamma\delta.$$

Therefore fundamental correlation increases the variance by the term $2\rho\gamma\delta$. Our specific case is that of Eq. (19). Hence, after defining

$$\gamma\equiv\sigma_2\sqrt{t_*}(1-\ell_2 v_{b,2})[1+\ell_2(\lambda-1)v_{a,2}f_2^0], \tag{37}$$

and

$$\delta\equiv\sigma_1\sqrt{t_*}(1-\ell_1 v_{b,1})\ell_2(\lambda-1)v_{a,1}f_2^0, \tag{38}$$

the margin, as a function of the fundamental correlation, is given by

$$m_{\text{risk}}(\rho)=\sqrt{\frac{8T}{9}}\left|1-\ell_2\frac{1-h}{h}v_{b,2}\right|P_2(0)\sigma_\rho\,\mathrm{erfc}^{-1}(2p)+\mu+\ell_2\frac{1-h}{h}v_{b,2},$$

with σ_ρ defined by

$$\sigma_\rho=\sqrt{\sigma^2+2\rho\gamma\delta}, \tag{39}$$

and where σ^2 is the variance defined in Eq. (20) and γ (resp. δ) is defined in Eq. (37) (resp. (38)).

Appendix C. Sensitivity analysis of model parameters

This Appendix is a brief sensitivity analysis of the toy-model described in Section 2. Figs. 9–11 report, as a thick black line, the margin computed replicating 10^4 times the model (19) with the choice of the parameters reported, respectively, in the second, third and fourth column of Table 1. Empty circles and the red dotted line are in correspondence of the analytical formulas (25) and (26), respectively.

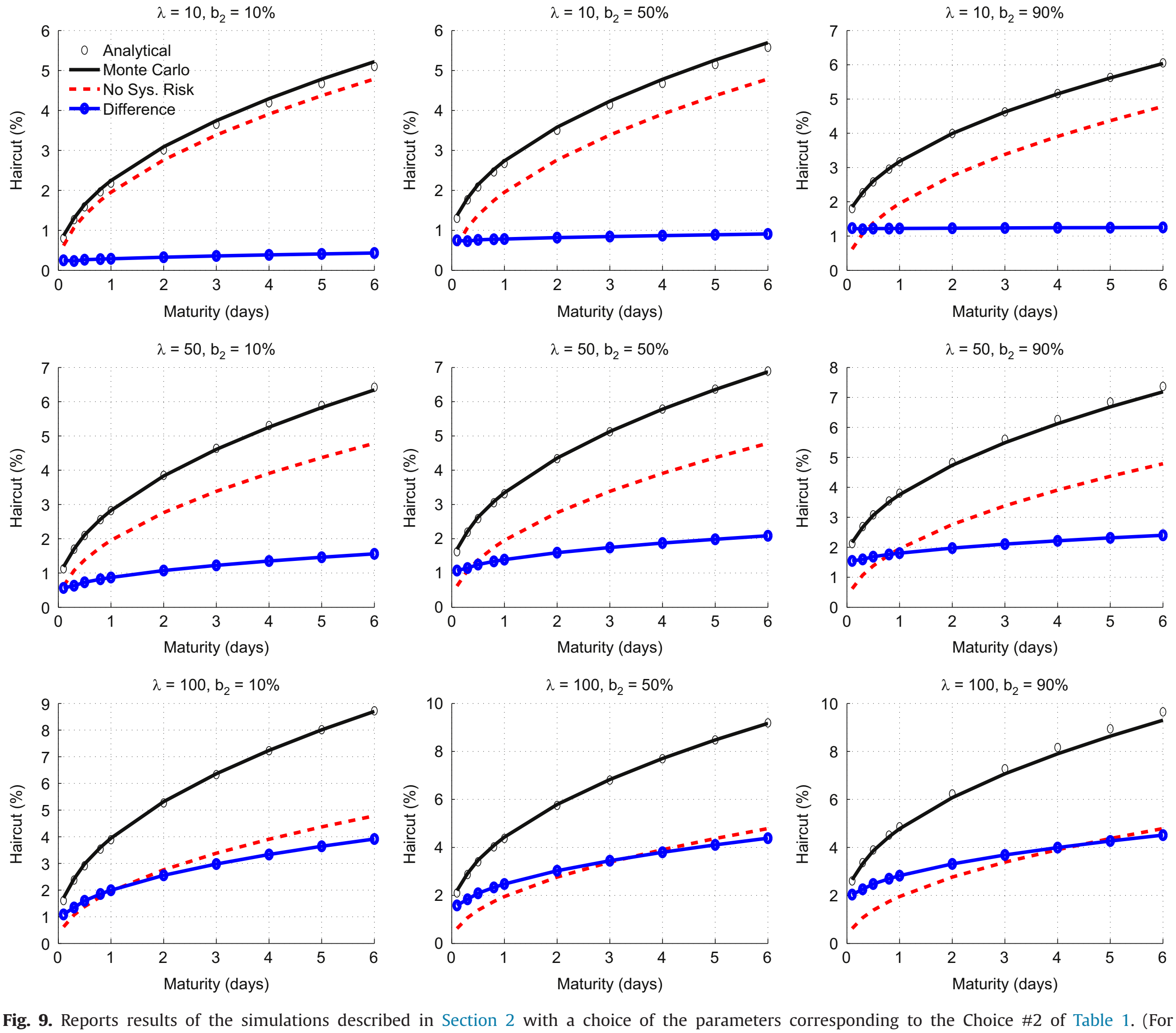

Fig. 9. Reports results of the simulations described in Section 2 with a choice of the parameters corresponding to the Choice #2 of Table 1. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

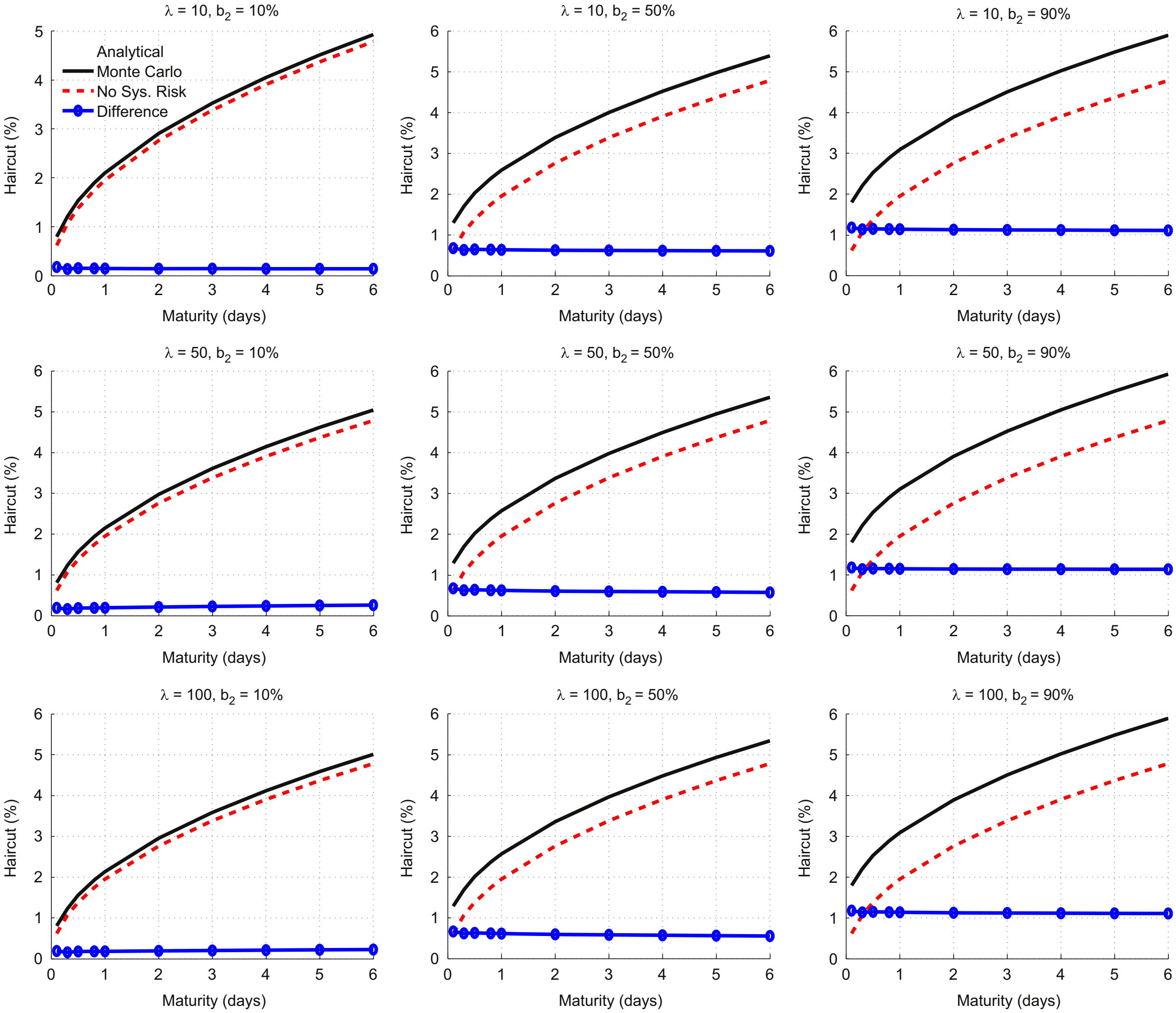

Fig. 10. Reports results of the simulations described in Section 2 with a choice of the parameters corresponding to the Choice #3 of Table 1. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

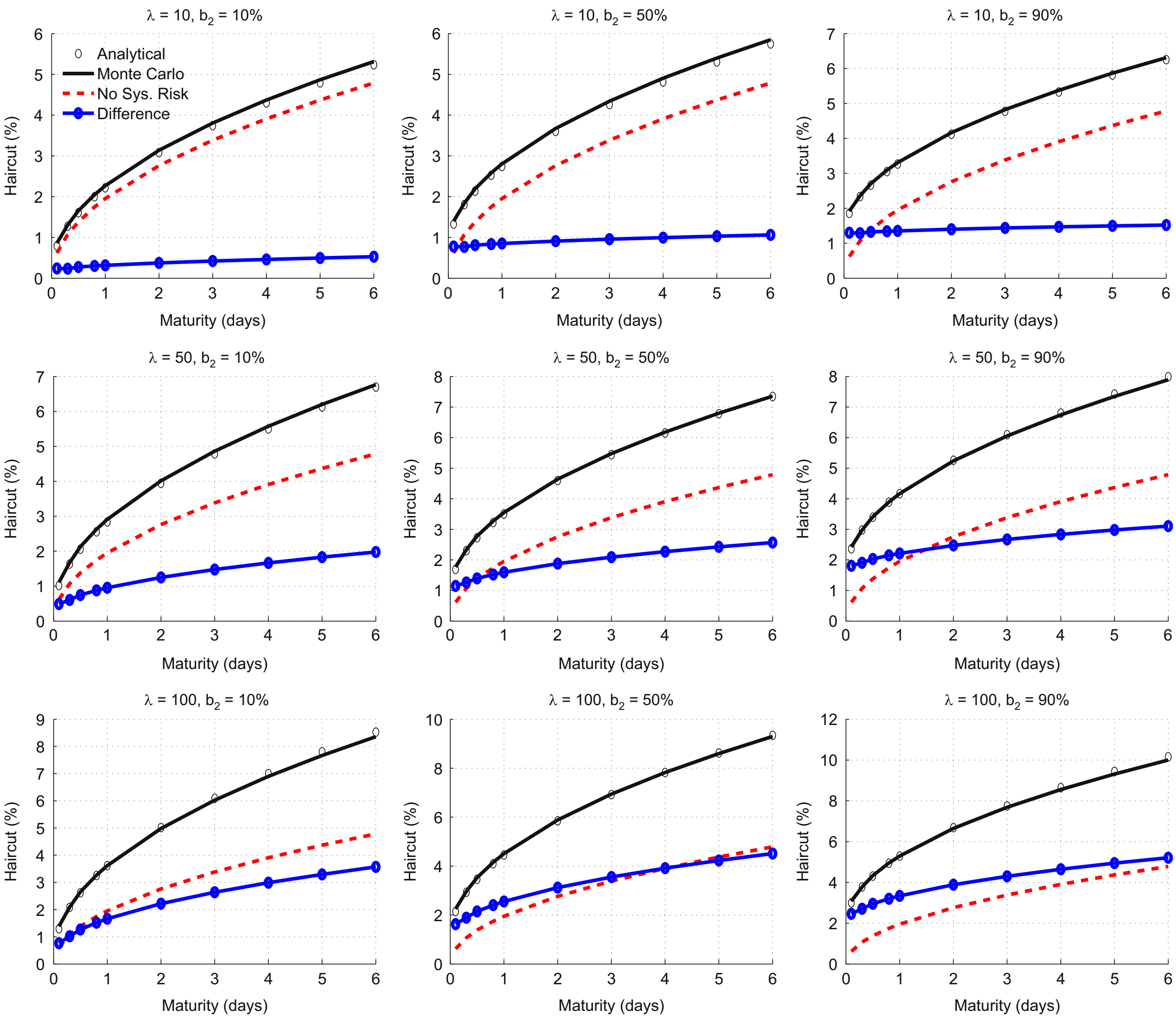

Fig. 11. Reports results of the simulations described in Section 2 with a choice of the parameters corresponding to the Choice #4 of Table 1. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

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