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Self-criticality and stochasticity of an S&P 500 index time series

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Abstract

In this paper, three time series representative of the daily high, low and closing prices of S&P 500 index time series, as from 1 December 1988 to 1 April 1998 are studied. The hypothesis advanced by Osborne that the stock market time series satisfy a log-normal distribution is rejected. The self-critical behavior of these time series is investigated. A fractional Brownian motion model for such time series is supported. Arguments are directed towards a negation of a chaotic explanation of these time series.

1. Introduction

In the last decades there appeared books, conference proceedings and papers relating the theory of Brownian motion [1–4], fractional Brownian notion [5–7], nonlinearity [8–11], chaos and fractals [12–23], scaling behavior [24–31], and self-criticality [32–35] to the study of financial time series.

We notice that the main hope of economists relating chaos to economic time series was guided by the hope of forecasting such developments [22]. However, as it is reported by Brock et al. [9], the usual method of Farmer and Sidorowich [36] does not provide better results for forecasting economic time series as the statistical ones. Most of tests developed in the area of economic theory provide evidence of nonlinear dynamics which is a necessary but not sufficient condition for chaos [17]. This nonlinearity may be deterministic or non-deterministic. Here, we support further that the nonlinearity for economic time series may associated to a stochastic model for them.

We study three time series emanating from the S&P 500 index time series. In the first paragraph, we present the statistical properties of these time series. Then using a relative cumulative frequency diagram, we show that in contrast with the hypothesis advanced by Osborne [1,2], that these time series do not satisfy a log-normal distribution. Hence, we provide support to the criticism by Laurent [3] addressed to the work of Osborne [1], that the financial time series are mainly stochastic time series.

Afterwards, we study the scaling behavior of these time series. We support further the results of Mantenga and Stanley [26] that these time series do not obey a turbulent behavior scenario for the economic time series as it has been proposed in [28].

Next, we show the existence of a factor of $1/f^{\alpha}$ for the power spectrum of these times series, which is an indice of a self-organized critical behavior [32]. Then using a result of Voss [37], we indicate that these time series behave approximately as fractional Brownian motions [5,6]. We support further these results by using the structure function test of Provenzale et al. [38]. However, the logarithmic prices of the S&P 500 index time series posses a $1/f^2$ spectrum and their increments behave like a white noise.

We report previous work on the investigation of chaotic features for the S&P 500 index time series. We discuss also the results of Peters [15] announcing a strange attractor on the S&P 500 index time series. We

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support further the indication by Hsieh [14] of the non-appearance of chaotic manifestation for the S&P 500 index time series.

2. Description of the S&P 500 index time series and their statistical treatment

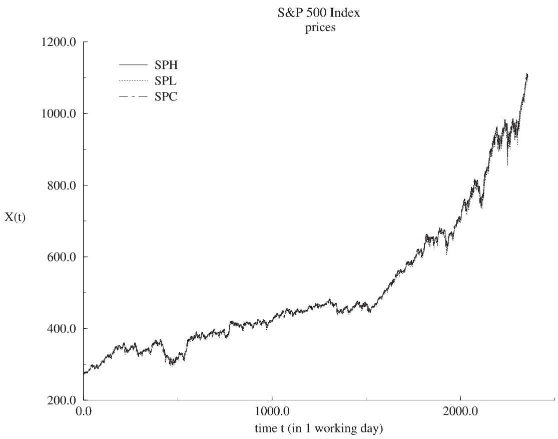
The S&P 500 index is composed of 500 large America's company common stocks, chosen for market size, liquidity and industry group representation. It is a market value weighted index (stock price times numbers of shares outstanding) which each stocks weighted in the index proportional to its market value. It is composed by key industries transportation utilities, industrial and financial companies. These stocks represent approximately 70% of the market value of all common stocks traded in the United States.

We consider three time series emanating from the S&P 500 index time series. They correspond to high (denoted SPH), low (denoted SPL) and closing (denoted SPC) prices of the S&P 500 index time series, for the period from 1 December 1988 to 1 April 1998. Counting only business days, these data contain 2356 individual prices for the S&P 500 index time series.

In Fig. 1, we present the specific time series of the S&P 500 index. It is also argued that the logarithm of the price is a better measure than the price itself [25,39,7]. This suggestion gains support from empirical data that the distribution of log-price in common stock looks like a normal distribution [1]. Hence, we represent in Fig. 2, the logarithmic values, of these time series denoted as LSPH, LSPL and LSPC, respectively. Subsequently, we reproduce the time series of the increments by the rate, $X(t+1) - X(t)$, where $X(t)$ denotes a time series. These time series are denoted as DSPH, DSPL and DSPC, respectively. Finally, we consider the increments of their logarithmic values defined by $\log(X(t+1)) - \log(X(t))$. These time series has been considered in [1] for the NYSE index, in [7] for the IBM stock closing prices, in [10] concerning the Athens, Greece, stock and in [31] the BUD, the Budapest, Hungary, stock. We denote the corresponding time series by DLSPH, DLSPL and DLSPC. (See Figs. 1–4.)

Using the KaleidaGraph program [40] on a Macintosch power PC computer, we have calculated the following statistical indices: mean, median, standard deviation, variance, skewness and kurtosis [41] for the SPH, SPL and SPC time series. It is worth noting the positive value of the kurtosis of these time series. In all the time series emanating from the S&P 500 index [14,29,30] a positive number for the kurtosis has been obtained, which is in accordance with [29] a constant feature of high frequency data. (See Table 1.)

Let us now check the hypothesis advanced by Osborne [1,2] that the above prices satisfy a log-normal distribution which means that their logarithmic values, i.e., LSPH, LSPL and LSPC satisfy a normal distribution. As we observe, from their relative cumulative frequency diagrams (see [41]), we do not deduce



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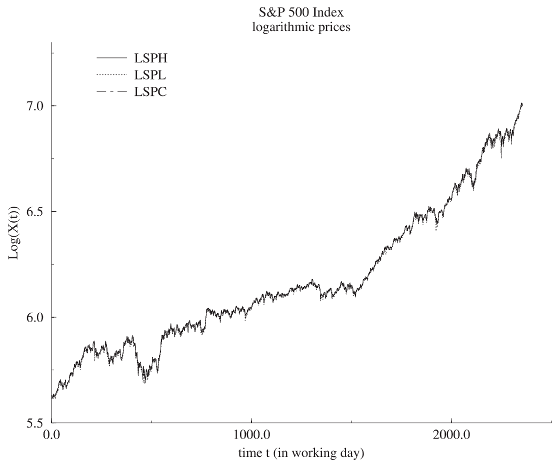

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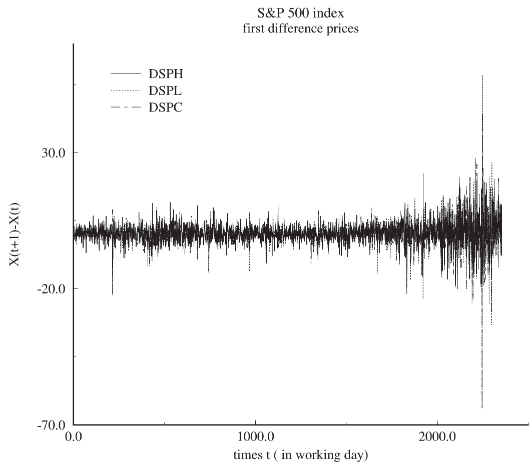
Fig. 1. The S&P 500 index time series, SPH, SPL and SPC.

Fig. 2. The logarithmic values of the S&P 500 index time series, LSPH, LSPL and LSPC.



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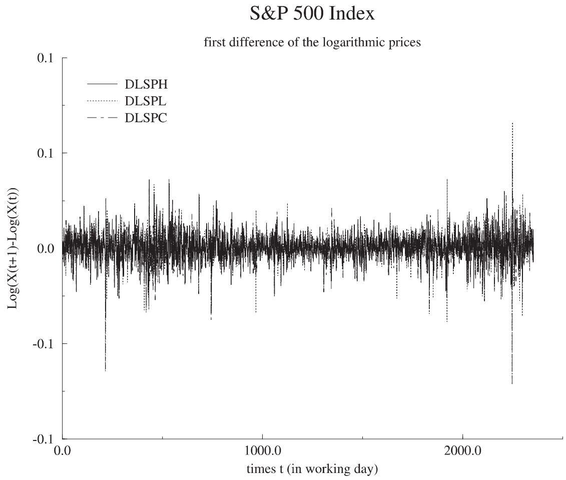

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Fig. 3. The increments of the S&P 500 index time series, DSPH, DSPL and DSPC.

Fig. 4. The increments of the logarithmic values of the S&P 500 index time series, DLSPH, DLSPL and DLSPC.

Table 1
The statistical treatment of the S&P 500 index time series

Statistical indices	SPH	SPL	SPC
Mean	508.77524	503.6129	506,54393
Median	450.245	447.235	448.86501
St. deviation	191.44632	180.70222	190.30597
Variance	36651.693	35608.528	36216.361
Skewness	1.252758	1.2406287	1.2473211
Kurtosis	0.69755018	0.6834991	0.69203644

a straight line. Hence, we do not have a normal distribution. Therefore, we support further the necessity of a deeper analysis of these data. (See Fig. 5.)

We notice also that in his paper, Osborne [1] used already in 1959 the term "Financial Chaos" and related it to the "molecular chaos" in statistical physics.

3. Scaling behavior in the S&P 500 index time series

In [24,26] the S&P 500 index recorded over a six year period from January 1984 to December 1989 and corresponding to 1 447 514 points with various time selection steps has been studied. They provide a critical foundation for a rejection of the hypothesis advanced by Ghashghaie et al. [28] supporting the behavior of the economic time series to turbulence. Here, we report also a scaling behavior of the SPH, SPL and SPC time series, which however does not agree with the values proposed by the Traditional Stock Market Theory (TSMT) [42] and the turbulence theory [28].

The return over n time steps is defined as

$$Z_n(t) = X(t + n\,\Delta t) - X(t), \tag{1}$$

where $X(t)$ is an entry of the S&P 500 index time series and $\Delta t = 1$ day the sampling time.

We have found that the moments of the distributions of Z_n possessing a scaling behavior as a function of n may be expressed as

$$\langle |\, Z_n(t)\, |^q \rangle_t \sim n^{\xi_q}, \tag{2}$$

where ξ_q is the self-affinity exponent.

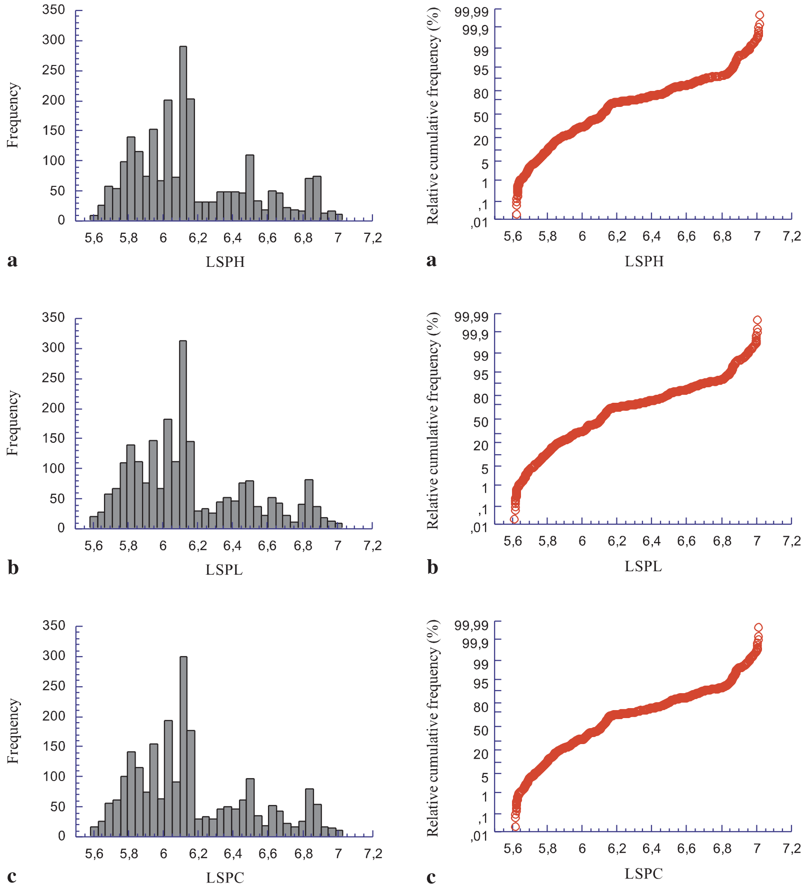


Fig. 5. The histograms and the relative cumulative frequency diagram for the: (a) LSPH; (b) LSPL; (c) LSPC time series.

In Fig. 6 we show the scaling behavior of the $q=1$ moment for $n=1,\ldots,500$ for the SPH, SPL and SPC time series. In Table 2, we indicated the values obtained for $q=1$–5 and compare them with the theoretical values $\xi_q = q/2$ derived from TSMT [42] and $\xi_q = q/3$ for turbulent flows [28]. We show that the behavior of the SPH, SPL and SPC time series is not in agreement with a theory which predicts Brownian motion for the stock prices and also a turbulent behavior.

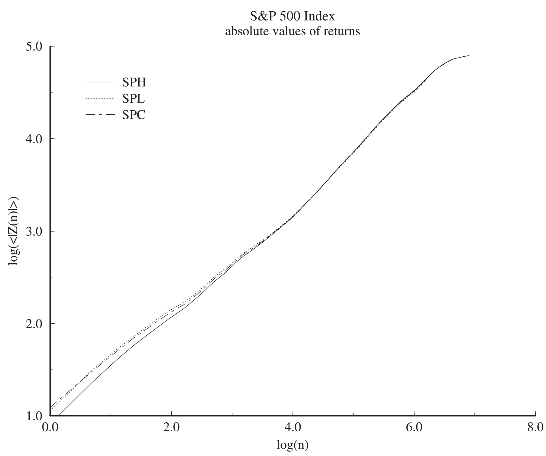


Fig. 6. Scaling behavior of the absolute values returns for the S&P 500 index time series, SPH, SPL, and SPC.

Table 2
The scaling behavior of the *p*-momentum of the S&P 500 index time series

p-Momentum	TSMT	Turbulence	SPH	SPL	SPC
1-Momentum	0.5	0.33	0.64	0.62	0.62
2-Momentum	1	0.66	1.29	1.25	1.27
3-Momentum	1.5	1	1.9	1.81	1.84
4-Momentum	2	1.33	2.45	2.3	2.36
5-Momentum	2.5	1.66	2.98	2.83	2.83

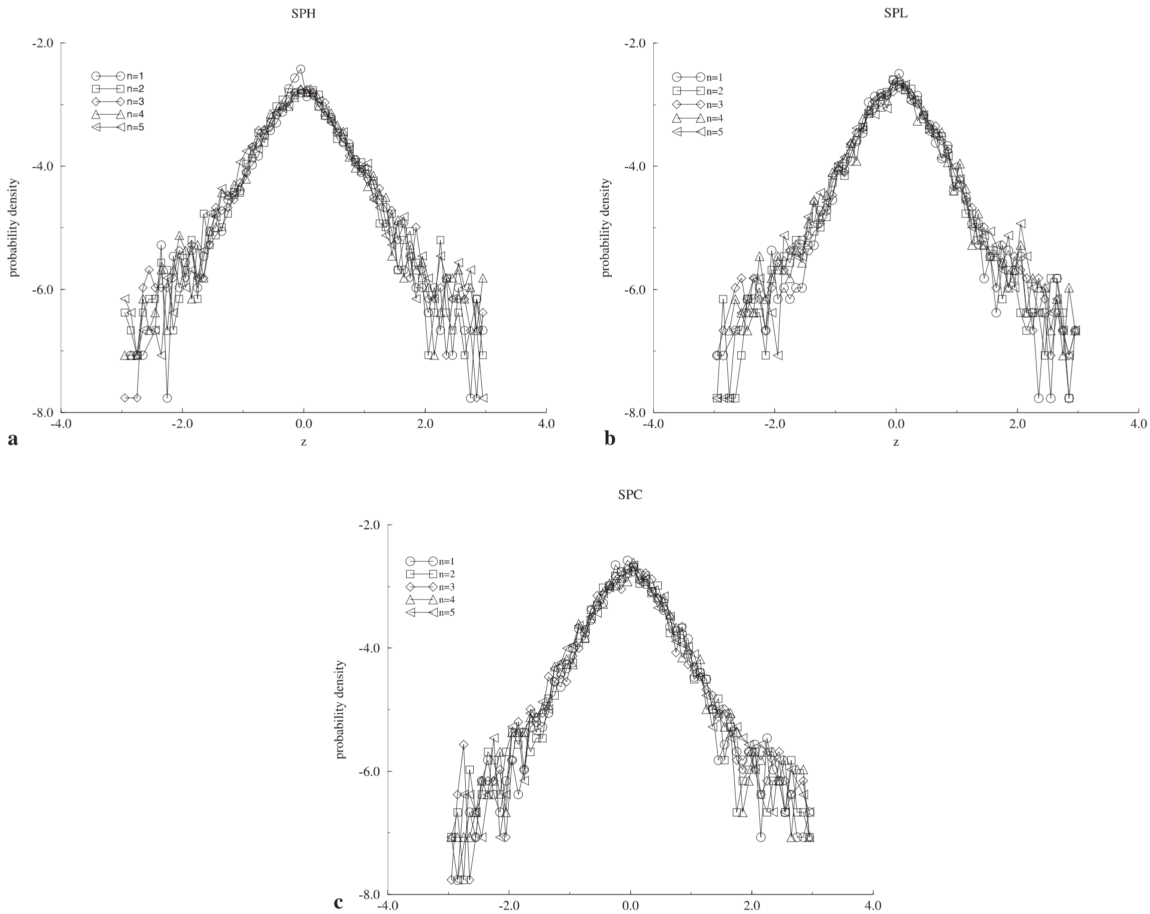


Fig. 7. The probability density $P_n(z)$ for the log-linear plot for the increments Z_n, with $n = 1$–5 for the: (a) SPH; (b) SPL; (c) SPC series.

Afterwards we study the distribution $P_n(Z)$ for the returns $Z_n(t)$ shown in Fig. 7, for $n = 1$–5. These curves are similar to those obtained in [26,29].

4. Self-criticality of the S&P 500 index time series

The power spectrum of the SPH, SPL and SPC time series displays a behavior of the form $1/f^{\alpha}$, where α is positive real. We note also in [24] a power spectrum of the form $1/f^2$ has been reported for an S&P 500 index time series.

This behavior is strictly related with the self-critical phenomena reported by Bak and his coworkers [32,33] and the concept of strange kinetics [35]. A possible explanation of the $1/f^{\alpha}$ may be found, within the framework of a fractional Brownian motion theory of these time series.

However, we show that there is no reason to associate economic time series with time series of earthquakes based only on the behavior of the power spectrum [34]. In fact, we recall the construction by Tsonis [43], of a stochastic time series whose power spectrum has a behavior of the form $1/f^{\alpha}$, for every positive real α.

Furthermore, the appearance of a scaling behavior in the power spectrum of economic time series support further, according to Theiler [44], the existence of a self-organization with many degrees of freedom for these economic time series. This supports also, according to Shaw [45] the appearance of a driving dynamic for economic time series.

4.1. Power spectrum

Let us consider our time series which possess 2356 points. This yields time series with a finite number of points and we can always proceed to the calculation of their power spectrum $P(f)$ by applying the following discrete Fourier transform [34],

$$P(f) = N\|A(f)\|^2, \tag{3}$$

where $\|A(f)\|$ is the module of the complex number

$$A(f) = \frac{1}{N}\sum_{j=1}^{N} x_j \mathrm{e}^{\mathrm{i}2\pi fj/N}. \tag{4}$$

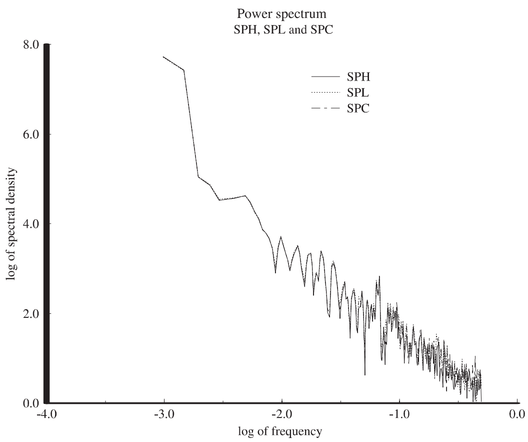
In Fig. 8, we observe a power spectrum of the SPH, SPL, SPC series. For all of them, we have found a behavior of the type

$$P(f) \propto \frac{1}{f^{\alpha}}, \tag{5}$$

where α is a positive real number.

The power spectrum of the DSPH and DSPL and DSPC does not obey this behavior, see Fig. 9.

However the power spectrum of the logarithmic prices (Fig. 10) LSPH, LSPL, LSPC displays a behavior as in (5) with $\alpha = 2$. These time series behave like a random walk. We observe that the power spectrum of their increments displays also a behavior similar to a white noise (Fig. 11). Here we are close to traditional behavior of the stock markets, see [7].



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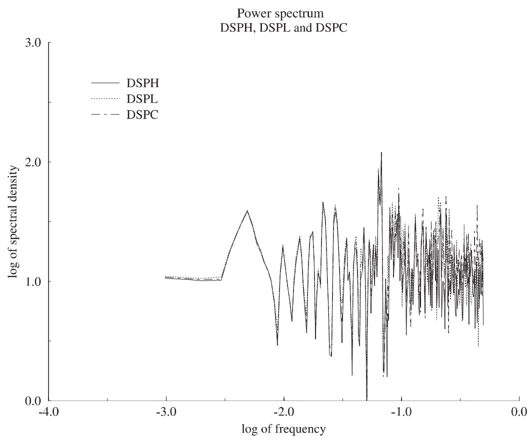

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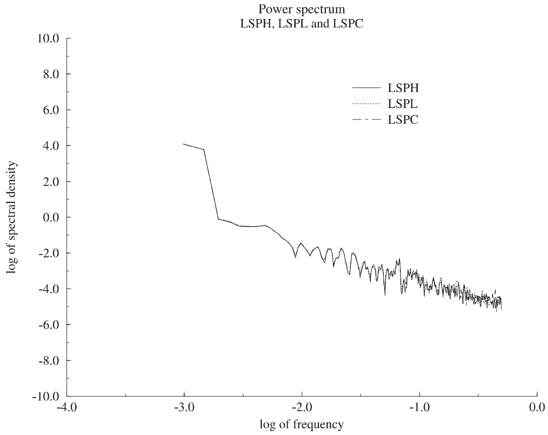
Fig. 8. The power spectrum (log–log scale) of the S&P 500 index time series, SPH, SPL and SPC.

Fig. 9. The power spectrum (log–log scale) of the increments values of the S&P 500 index time series, DSPH, DSPL and DSPC.



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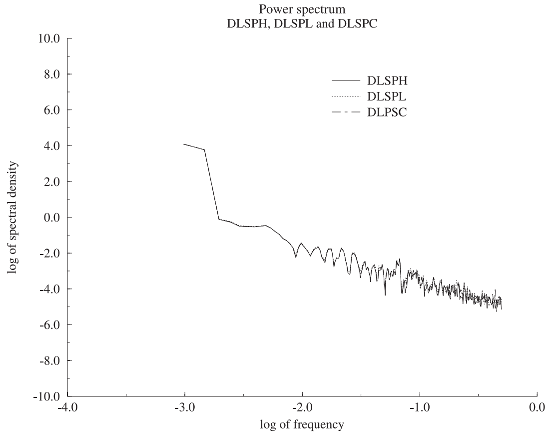

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Fig. 10. The power spectrum (log–log scale) of the logarithmic values of the S&P 500 index time series, LSPH, LSPL and LSPC.
Fig. 11. The power spectrum (log–log scale) of the increments of the logarithmic values of the S&P 500 index time series, DLSPH, DLSPL and DLSPC.

4.1.1. Earthquakes

In the theory of economic time series, it has been proposed by Bachelier [42], and Malkiel [4], that the economic time series fluctuate as random walks. An efficient market assumption supports the fact that the price fluctuations around its value and new information concerning the stock will be quickly incorporated in the stock market. This up and down of the prices have been interpreted by Malkiel as corresponding to similar phenomena arising in earthquakes.

These analogies have been discussed in detail by Li [34], and related to the fact that earthquake time series demonstrate a power low behavior in their spectrum [46]. This appears also in the so-called self-organized sand piles [47].

However this explanation is not adequate as it is possible to associate for every time series a stochastic time series with the same power law of spectrum, see [43,44]. Indeed, for every positive real a, a time series $X(i)$, with $i = 1, 2, \ldots, N$, whose power spectrum satisfies the relation (5), with $\alpha = a$, can be constructed via the formula:

$$X(i) = \sum_{k=1}^{N/2} \left[Ck^{-a} \left(\frac{2\pi}{N} \right)^{1-a} \right]^{1/2} \cos\left(\frac{2\pi i k}{N} + \phi_k \right), \tag{6}$$

where C is a constant and ϕ_k are random phases randomly distributed over the range $[0, 2\pi]$ [43].

In Fig. 12, we represent three time series obtained by using (6) for different values of the constant $C = 0.5, 1$ and 1.5, in case of $\alpha = 2.32$ the numerical value of the power spectrum of the SPH time series.

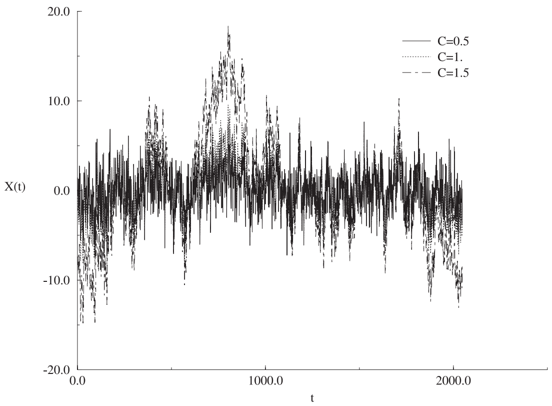
4.2. Fractional Brownian motions

In this sequel we discuss the relation of the SPH, SPL and SPC time series to the fractional Brownian motion.

Let us now consider a time series T with N points, whose power spectrum satisfies (5). The number α in (5) is related with the box dimension of the graph $G(T) = \{(n, T(n)) \setminus n = 1, \ldots, N\}$ of the corresponding time series T. We recall first the definition of the box dimension of the $G(T)$ [48]. We denote by C_δ the smallest number of the sets of diameters δ covering the $G(T)$. If the following limit exists

$$\lim_{\delta \to 0} \frac{\log C_\delta}{-\log \delta}, \tag{7}$$

it is called the box dimension of the $G(T)$ and it is denoted by $D_{\mathrm{b}}(T)$.



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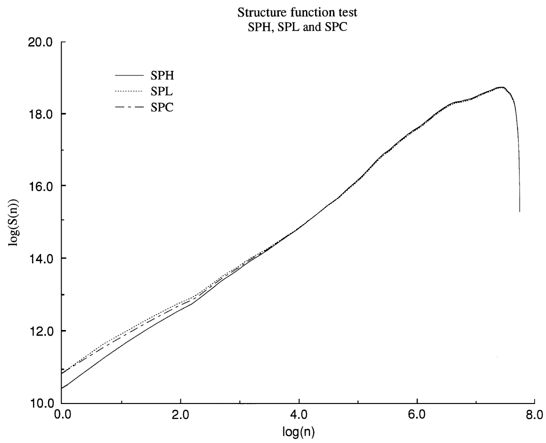

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Fig. 12. Three random generated time series with a power law behavior $P(f) = 1/f^a$ with $a = 2.34$.

Fig. 13. A log–log plot of the Structure Function Test for the S&P 500 index time series, SPH, SPL and SPC.

According to Falconer [48], if a time series T has a box dimension $D_b(T)$ and its power spectrum have a behavior given by (5) then the following formula should hold

$$D_b(T) = \frac{1}{2}(5 - \alpha). \tag{8}$$

Eq. (8) has been rigorously proved by Voss [37] in the case of a fractional Brownian motion and

$$1 \leqslant D_b(T) \leqslant 3. \tag{9}$$

A fractional Brownian motion $V_H(t)$ is a single valued function of one variable, the increments of which $V_H(t_2) - V_H(t_1)$ obey a Gaussian distribution with a variance

$$\langle | V_H(t_2) - V_H(t_1) | \rangle \propto | t_2 - t_1 |^{2H}, \tag{10}$$

where the brackets $\langle\ \rangle$ denote an ensemble of averages over many samples of $V_H(t)$ and the power H has a value $0 < H < 1$.

Applying the formula (8), we obtain the following values described in Table 3.

Applying now the results of Voss [37], we conclude that the SPH, SPL and SPC may be explained within the framework of fractional Brownian motion as their box dimension satisfying relation (9).

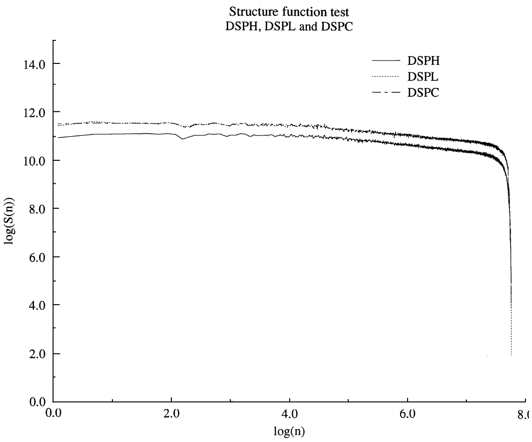
4.3. The structure function test

In the sequel we apply the structure function test developed by Provenzale et al. [38] to our economic time series in order to support and extend the results obtained in the previous section. This test was developed as a tool for distinguishing between a deterministic and a stochastic origin of time series whose power spectrum displays a scaling behavior.

We consider a time series T with a finite length equal to N. For every $n, 1 \leqslant n \leqslant N$, the structure function associated with T is defined by:

Table 3
The scaling behavior of the power spectrum of the S&P 500 index time series

Time series	Power exponent	Box dimension
SPH	2.32	1.84
SPL	2.27	1.825
SPC	2.25	1.86



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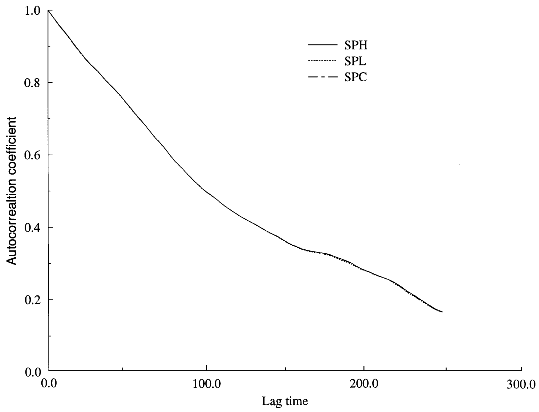

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Fig. 14. A log–log plot of the Structure Function Test for the S&P 500 index time series, DSPH, DSPL and DSPC.
Fig. 15. Autocorrelation function for the S&P 500 index time series, SPH, SPL and SPC.

$$S(n) = \sum_{i=1}^{N-n} (T(i + n\Delta t) - T(i))^2, \tag{11}$$

where Δt denotes the sample rate of the time series T. For a time series T with a power law spectrum

$$P(f) \propto f^{-\alpha}, \tag{12}$$

where α is positive real, one can expect a scaling behavior of the form [5]

$$S(n) \propto n^{2H} \tag{13}$$

for small values of n. Here H is called the scaling exponent.

In the case of a fractional Brownian motion there holds [38]

$$\alpha = 2H + 1. \tag{14}$$

When the signal is a fractal noise, the graph of $\log(S(n))$ versus $\log(n)$ displays an extended scaling regime and its graph, as in the previous plot, is closely approximated by a straight line.

On the other hand, if the time series correspond to the motion of a strange attractor whose fractal structure is due to close returns in phase space, the graph of $\log(S(n))$ versus $\log(n)$ is closely approximated at small values of n, by a straight line with slope $2H = 1$. Then for an intermediate value of n, $S(n)$ displays an oscillatory behavior, due to an orbit occurrence in phase space, and approach finally a constant value. This is supported by the fact that a limited phase space is visited by the system.

We display in Fig. 13 the graphs of $\log(S(n))$ versus $\log(n)$, for the SPH, SPL and SPC time series as follows. In our time series, the sampling rate Δt in (3) is equal to 1 day. We find a behavior like a fractional noise. This fact is supported further, by the Structure Function Test for the first difference time series, DSPH, DSPL and DSPC (Fig. 14) which appear as straight lines, as it is expected for a fractional noise behavior [38].

5. Nonlinearity and chaos in the S&P 500 index time series

Previous papers on the study of the S&P 500 index, has been reported in [11]. These authors observe the existence of a nonlinear structure of these studies. However, they exclude the appearance of a low deterministic chaos.

Peters [16], used monthly returns for S&P 500, over a period starting from January 1950 until July 1989. Hence he constructs a time series with 463 observations. He reports the appearance of a finite correlation dimension of 2.3 and a positive largest Lyapunov exponent of 0.024. However the numerical time series on which he reports the above results is not the time series of S&P 500 time series S_i defined on the month i as follows:

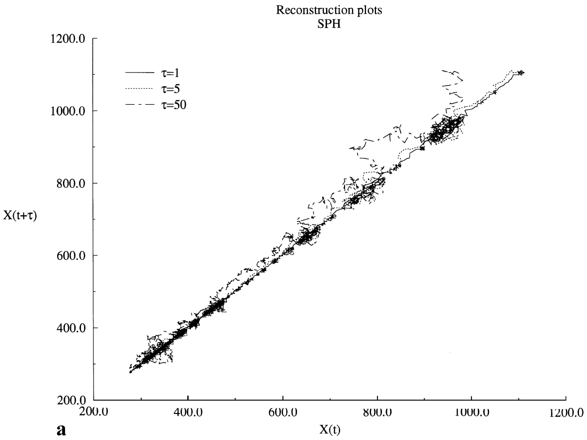
$$S_i = \log_e(P_i) - (a \log_e(\mathrm{CPI}_i) + \text{constant}), \tag{15}$$

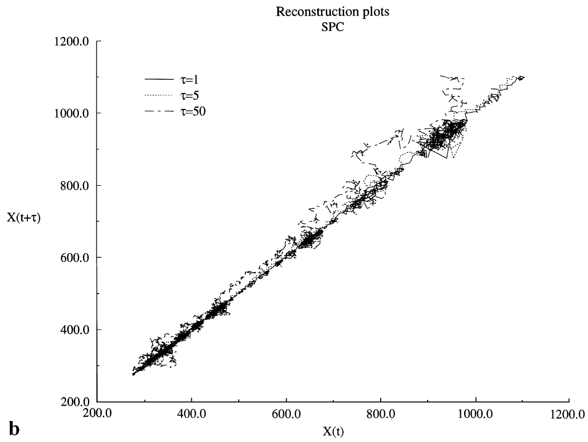
where a is a constant, P_i the S&P 500 price for the month i and the CPI_i the Consumer Price Index for the month i.

The numerical values obtained for the correlation dimension is in accordance with the rule proposed by Ruelle [49] and Barnett and Serletis [23], as for a correlation of order 2, we need at least 460 points. However, the criticism addressed by LeBaron [21] to the previous paper by Peters concern mainly the non availability of his data.

As we see for Eq. (15), Peters does not specify an attractor for the original S&P 500 index time series. A possible explanation of his findings may be due to the coupling of the S&P 500 with the CPI index. This is in accordance with previous work by Tsonis and Elsner [50] who have proposed an explanation of the low dimensionality on the climate chaos as being due to the coexistence of subsystems. These idea has obtained further support by Lorenz [51]. A mathematical model of this coexistence has recently been proposed by Argyris and Andreadis [52].

Hsieh [14] considers 3 time series according to S&P 500 index. A weekly returns from 1962 to 1989, daily return form 1983 to 1989 and 15-minute returns during 1988 divided into 4 approximately equal sub-





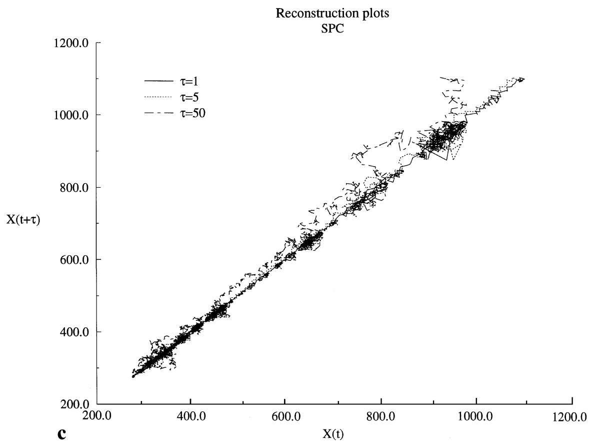


Fig. 16. Two-dimensional phase space plot for the S&P 500 index: (a) SPH; (b) SPL; (c) SPC time series with $\tau = 1{,}5$ and 50. No correlation is present.

samples. He reports that these time series reject the Independent and Identical Distribution (i.i.d.) hypothesis but shown no evidence for a low-dimensional chaos.

Here, we intend to apply the Grassberger–Procaccia algorithm [53], in order to search for a correlation dimension. Hence, we calculated the autocorrelation coefficient of the time series whose first zero point is an evidence of a finite delay time [54,55].

We recall that the autocorrelation coefficient of a time series T as defined by Jenkins and Watts [56] is given by

$$A(l) = \frac{\sum_{i=1}^{N-l} \bar{T}(t_i)\bar{T}(t_{i+l})}{\sum_{i=1}^{N} \bar{T}(t_i)^2}, \quad \bar{T}(t_i) = T(t_i) - \frac{1}{N}\sum_{j=1}^{N} T(t_j). \tag{16}$$

The corresponding autocorrelation coefficients for the SPH, SPL and SPC time series are presented in Fig. 15.

We observe that the autocorrelation coefficients never become zero which proves a lack of correlation in our data. In all the well-known chaotic systems, Lorenz, Henon, [55] a zero point of the autocorrelation coefficient is reported. This provides an evidence for the non-existence of a deterministic chaos for the SPH, SPL and SPC time series.

These evidences are extended further by constructing the reconstruction plots according to the method of Takens [57] and Packard et al. [58].

In Figs. 16 a, b and c, we calculate the reconstruction plots $(X(t), X(t+\tau))$ for $\tau = 1,\ 5$ and 50. In all the figures the motion is concentrated on the first diagonal and no correlation is observed.

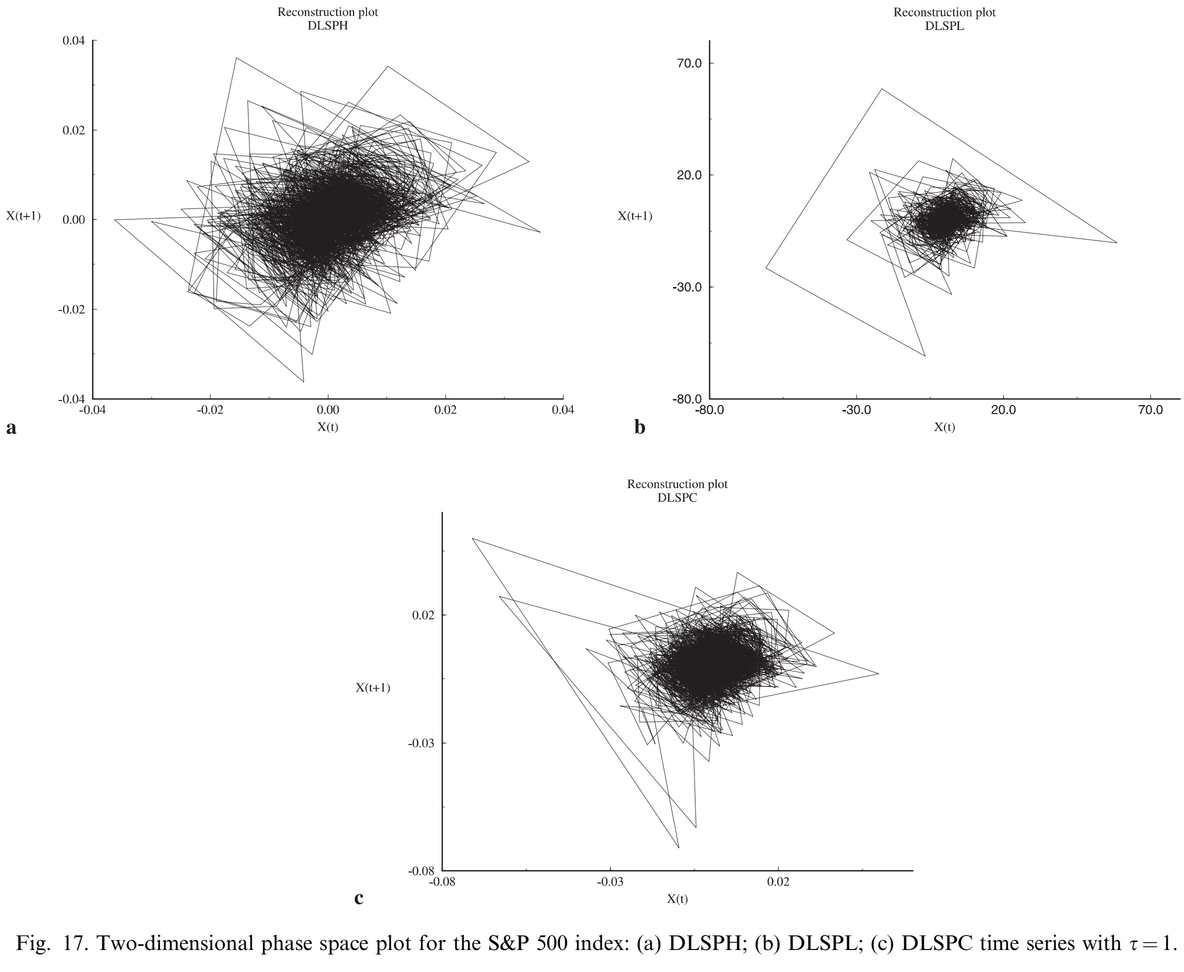


Fig. 17. Two-dimensional phase space plot for the S&P 500 index: (a) DLSPH; (b) DLSPL; (c) DLSPC time series with $\tau = 1$.

However, in the work by Papaioannou and Karytinos [10] the reconstruction plot even for a τ as low as 1 shows a decorrelation for the diagonal. This is because the authors use the first difference of the logarithmic values of the Athens stock index. Effectively, if we consider the DLSPH, DLSPL and DLSPC similar behavior are obtained, see Figs. 17 a, b, and c.

Nevertheless, we do not pursue further the study of the DLSPH, DLSPL and DLSPC for the following reason. Let us suppose that we have found a chaotic behavior and we can predict a value of $\log(X(T+1)/X(T)) = M$ at time T. Then, we obtain $X(T+1) = X(T)\exp(M)$, which provides no prediction for $X(T+1)$ as we do not have previous information for the value of $X(T)$.

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