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Approximation and Parametrized Algorithms for Segment Set Cover

6

Master's thesis

7

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8

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9

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10 **Supervisor's statement**

11 Hereby I confirm that the presented thesis was prepared under my supervision and
12 that it fulfils the requirements for the degree of Master of Computer Science.

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15 Hereby I declare that the presented thesis was prepared by me and none of its contents
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Abstract

23 The work presents a study of different geometric set cover problems. It mostly focuses on
24 segment set cover and its connection to the polygon set cover.

25

Keywords

26 set cover, geometric set cover, FPT, $W[1]$ -completeness, APX-completeness, PCP theorem,
27 NP-completeness

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36 Algorytmy parametryzowania i trudność aproksymacji problemu pokrywania zbiorów
37 odcinkami na płaszczyźnie

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Chapter 1

Introduction

The Set Cover problem is one of the most common NP-complete problems. [tutaj referencja]
We are given a family of sets and have to choose the smallest subfamily of these sets that cover
all their elements. This problem naturally extends to settings where we put different weights
on the sets and look for the subfamily of the minimal weight. This problem is NP-complete
even without weights and if we put restrictions on what the sets can be. One of such variants
is Vertex Cover problem, where sets have size 2 (they are edges in a graph).

In this work we focus on another such variant where the sets correspond to some geometric
shapes and only some points of the plane have to be covered. When these shapes are rectangles
with edges parallel to the axis, the problem can be proven to be W[1]-complete (solution of
size k cannot be found in $n^o(k)$ time), APX-complete (for sufficiently small $\epsilon > 0$, the problem
does not admit $1 + \epsilon$ -approximation scheme) [referencje].

Some of these settings are very easy. Set cover with lines parallel to one of the axis can
be solved in polynomial time.

There is a notion of δ -expansions, which loosen the restrictions on geometric set cover. We
allow the objects to cover the points after δ -expansion and compare the result to the original
setting. This way we can produce both FPT and EPTAS for the rectangle set cover with
 δ -extensions [referencje].

Our contribution. In this work, we prove that unweighted geometric set cover with seg-
ments is fixed parameter tractable (FPT).

Moreover, we show that geometric set cover with segments is APX-complete for unweighted
axis-parallel segments, even with $1/2$ -extensions. So the problem for very thin rectangles
also can't admit PTAS. Therefore, in the efficient polynomial-time approximation scheme
(EPTAS) for *fat polygons* by [Har-Peled and Lee, 2009], the assumption about polygons
being fat is necessary.

Finally, we show that geometric set cover with weighted segments in 3 directions is
W[1]-complete. However, geometric set cover with weighted segments is FPT if we allow
 δ -extension.

This result is especially interesting, since it's counter-intuitive that the unweighted setting
is FPT and the weighted setting is W[1]-complete. Most of such problems (like vertex cover
or [wiecej przykladow]) are equally hard in both weighted and unweighted settings.

Chapter 2

Definitions

2.1. Geometric Set Cover

In the geometric set cover problem we are given \mathcal{P} – a set of objects, which are connected subsets of the plane, \mathcal{C} – a set of points in the plane. The task is to choose $\mathcal{R} \subseteq \mathcal{P}$ such that every point in \mathcal{C} is inside some element from \mathcal{R} and $|\mathcal{R}|$ is minimized.

In the parametrized setting for a given k , we only look for a solution \mathcal{R} such that $|\mathcal{R}| \leq k$.

In the weighted setting, there is some given weight function $f : \mathcal{P} \rightarrow \mathbb{R}^+$, and we would like to find a solution \mathcal{R} that minimizes $\sum_{R \in \mathcal{R}} f(R)$.

2.2. Approximation

Let us recall some definitions related to optimization problems that are used in the following sections.

Definition 1. A **polynomial-time approximation scheme (PTAS)** for a minimization problem Π is a family of algorithms \mathcal{A}_ϵ for every $\epsilon > 0$ such that \mathcal{A}_ϵ takes an instance I of Π and in polynomial time finds a solution that is within a factor $(1 + \epsilon)$ of being optimal. That means the reported solution has weight at most $(1 + \epsilon)\text{opt}(I)$, where $\text{opt}(I)$ is the weight of an optimal solution for I .

Definition 2. A problem Π is **APX-hard** if assuming $P \neq NP$, there exists $\epsilon > 0$ such that there is no polynomial-time $(1 + \epsilon)$ -approximation algorithm for Π .

2.3. δ -extensions

TODO PLACEHOLDER for introductory text

δ -extensions is one of the modifications to a problem, that makes geometric set cover problem easier, it has been already used in literature (place some refrence here).

Definition 3 (δ -extensions for center-symmetric objects). For any $\delta > 0$ and a center-symmetric object L with centre of symmetry $S = (x_s, y_s)$, the δ -**extension** of L is the object $L^{+\delta} = \{(1 + \delta) \cdot (x - x_s, y - y_s) + (x_s, y_s) : (x, y) \in L\}$, that is, $L^{+\delta}$ is the image of L under homothety centered at S with scale $(1 + \delta)$

The geometric set cover problem with δ -extensions is a modified version of geometric set cover where:

- We need to cover all the points in \mathcal{C} with objects from $\{P^{+\delta} : P \in \mathcal{P}\}$ (which always include no fewer points than the objects before δ -extensions);

- We look for a solution that is no larger than the optimum solution for the original problem. Note that it does not need to be an optimal solution in the modified problem.

Formally, we have the following.

Definition 4 (Geometric set cover problem with δ -extensions). The geometric set cover problem with δ -extensions is the problem where for an input instance $I = (\mathcal{P}, \mathcal{C})$, the task is to output a solution $\mathcal{R} \subseteq \mathcal{P}$ such that the δ -extended set $\{R^{+\delta} : R \in \mathcal{R}\}$ covers \mathcal{C} and is no larger than the optimal solution for the problem without extensions, i.e. $|\mathcal{R}| \leq |\text{opt}(I)|$.

TODO: Some text

Definition 5 (Geometric set cover PTAS with δ -extensions). We define a PTAS for geometric set cover with δ -extensions as a family of algorithms $\{\mathcal{A}_{\delta, \epsilon}\}_{\delta, \epsilon > 0}$ that each takes as an input instance $I = (\mathcal{P}, \mathcal{C})$, and in polynomial-time outputs a solution $\mathcal{R} \subseteq \mathcal{P}$ such that the δ -extended set $\{R^{+\delta} : R \in \mathcal{R}\}$ covers \mathcal{C} and is within a $(1 + \epsilon)$ factor of the optimal solution for this problem without extensions, i.e. $(1 + \epsilon)|\mathcal{R}| \leq |\text{opt}(I)|$.

Chapter 3

Geometric Set Cover with segments

3.1. FPT for segments

3.1.1. Axis-parallel segments

You can find this in Platypus book. (TODO add referece)

We show $\mathcal{O}(2^k)$ branching algorithm. Let us take the point K which is the smallest under a lexicographic ordering on coordinates among points that are not covered yet. We need to cover K with some of the remaining segments.

We branch over choice of direction among the 2 axis-parallel directions. In this direction we greedily take the segment that covers the most points. As K was the smallest in lexicographical order, all points in \mathcal{C} colinear with K in both axis-parallel directions are only on one side of K , because their coordinates are larger. Therefore segments covering K in this direction create monotone sequence of sets and we can greedily take one segment that covers superset of all of these segments.

TODO: Maybe split it into theorem + algorithm + explanation like in section 3.1.3

3.1.2. Segments in d directions

The same algorithm as described in the previous section, but we branch over d directions and it runs in complexity $\mathcal{O}(d^k)$.

3.1.3. Segments in arbitrary direction

Theorem 1. (FPT for segment cover). *There exists an algorithm that given a family \mathcal{P} of n segments (in any direction), a set of m points \mathcal{C} and a parameter k , runs in time $f(k) \cdot (nm)^c$ for some computable function f and constant c , and outputs a subfamily $\mathcal{R} \subseteq \mathcal{P}$ such that $|\mathcal{R}| \leq k$ and \mathcal{R} covers all points in \mathcal{C} or determines that the solution of size at most k doesn't exist.*

This theorem is proved by following lemmas.

Lemma 1. (Reduction). *Given a family \mathcal{P} of n segments (in any direction) and a set of m points \mathcal{C} for segment cover problem, without a loss of generality we can assume that no segment covers a superset of what another segment covers. That is, for any $A, B \in \mathcal{P}$, we have $A \neq B \Rightarrow A \not\subseteq B$.*

Proof. Trivial. □

Lemma 2. *Given an instance of a problem, if there exists a line L with at least $k + 1$ points on it, there exists a subset $\mathcal{A} \subseteq \mathcal{P}$, $|\mathcal{A}| \leq k$, such that every solution \mathcal{R} with $|\mathcal{R}| \leq k$ satisfies $|\mathcal{A} \cap \mathcal{R}| \geq 1$.*

Proof. First we use Lemma 1.

Let us name points from \mathcal{C} that lay on L , x_1, x_2, \dots, x_t in the order they appear on the line.

Every segment that is not colinear with L can cover at most one of these points. Therefore in any solution of size not larger than k , among any k of these points at least one must be covered with segment colinear with L .

Therefore we need to take one of the segments colinear with L that covers any of the points x_1, x_2, \dots, x_k . After using reduction from Lemma 1, there are at most k such segments that are distinct. \square

Proof of theorem 1.

Algorithm. First we use Lemma 1.

We present a recursive algorithm. Given an instance of the problem:

(1) If there exist a line with at least $k + 1$ points, we branch over adding to the solution one of at most k possible segments from Lemma 2, name this segment S . Then we find a solution \mathcal{R} for problem for points $\mathcal{C} - S$, segments $\mathcal{P} - \{S\}$ and parameter $k - 1$ and return $\mathcal{R} \cup \{S\}$.

(2) If every line has at most k points on it and $|\mathcal{C}| > k^2$, then answer **NO**.

(3) If $|\mathcal{C}| \leq k^2$, solve the problem by brute force algorithm.

Correctness. Lemma 2 proves that at least one segment that we branch over in (1) must be present in every solution \mathcal{R} with $|\mathcal{R}| \leq k$, therefore the recursive call can find the optimal solution.

In (2) the answer is no, because every line covers no more than k points from \mathcal{C} , which implies that every segment from \mathcal{P} covers at most k . Under this assumption we can cover only k^2 points with a solution of size k , which is less than $|\mathcal{C}|$.

Checking all possible solutions in (3) is trivially correct.

Complexity. In leaves of branching (3) $|\mathcal{C}| < k^2$, so $|\mathcal{P}| < k^4$, because every segments can be uniquely identified by 2 extreme points it covers (by Lemma 1). Therefore there are $\binom{k^4}{k}$ possible solutions to check, each can be checked in time $O(k|\mathcal{C}|)$. Therefore (3) takes time $O(f(k))$.

In this branching algorithm our parameter k is decreased with every recursive call, so we have at most k levels of recursion with branching over k possibilities. Candidates to branch over can be found on each level in time $O(nm \log(nm))$.

Reduction from Lemma 1 can be implemented in $O(n^2m)$.

Overall complexity is $O(n^2m + nm \log(nm) \cdot f(k))$ \square

213 3.2. APX-completeness for segments parallel to axes

214 In this section we analyze whether there exists PTAS for geometric set cover for rectangles.
 215 We show that we can restrict this problem to a very simple setting: segments parallel to axes
 216 and allow $(1/2)$ -extension, and the problem is still APX-hard. Note that segments are just
 217 degenerated rectangles with one side being very narrow.

218 Our results can be summarized in the following theorem and this section aims to prove it.

219 **Theorem 2.** *(axis-parallel segment set cover with $1/2$ -extension is APX-hard).*
 220 *Unweighted geometric set cover with axis-parallel segments in 2D (even with $1/2$ -extension)*
 221 *is APX-hard. That is, assuming $P \neq NP$, there does not exist a PTAS for this problem.*

222 Theorem 2 implies the following.

223 **Corollary 1.** *(rectangle set cover is APX-hard).* *Unweighted geometric set cover with*
 224 *rectangles (even with $1/2$ -extension) is APX-hard.*

225 We prove Theorem 2 by taking a problem that is APX-hard and showing a reduction. For
 226 this problem we choose MAX-(3,3)-SAT which we define below.

227 3.2.1. MAX-(3,3)-SAT and statement of reduction

228 **Definition 6.** MAX-3SAT is the following maximization problem. We are given a 3-CNF
 229 formula, and need to find an assignment of variables that satisfies the most clauses.

230 **Definition 7.** MAX-(3,3)-SAT is a variant of MAX-3SAT with an additional restriction
 231 that every variable appears in exactly 3 clauses. Note that thus, the number of clauses is
 232 equal to the number of variables.

233 In our proof of Theorem 2 we use hardness of approximation of MAX-(3,3)-SAT proved
 234 in [Håstad, 2001] and described in Theorem 3 below.

235 **Definition 8** (α -satisfiable MAX-3SAT formula). MAX-3SAT formula of size n is at most
 236 α -satisfiable, if every assignment of variables satisfies no more than αn clauses.

237 **Theorem 3.** [Håstad, 2001]

238 *For any $\epsilon > 0$, it is NP-hard to distinguish satisfiable (3,3)-SAT formulas from at most*
 239 *$(7/8 + \epsilon)$ -satisfiable (3,3)-SAT formulas.*

240 Given an instance I of MAX-(3,3)-SAT, we construct an instance J of axis-parallel seg-
 241 ment set cover problem, such that for a sufficiently small $\epsilon > 0$, a polynomial time $(1 + \epsilon)$ -
 242 approximation algorithm for J would be able to distinguish whether an instance I of MAX-
 243 (3,3)-SAT is fully satisfiable or is at most $(7/8 + \epsilon)$ -satisfiable. However, according to (Theorem
 244 3) the latter problem is NP-hard. This would imply $P = NP$, contradicting the assumption.

245 The following lemma encapsulates the properties of the reduction described in this section,
 246 and it allows us to prove Theorem 2.

247 **Lemma 3.** *Given an instance S of MAX-(3,3)-SAT with n variables and optimum value*
 248 *$opt(S)$, we can construct an instance I of geometric set cover with axis-parallel segments in*
 249 *2D, such that:*

250 (1) *For every solution X of instance I , there exists a solution of S that satisfies at least*
 251 *$15n - |X|$ clauses.*

(2) For every solution of instance S that satisfies w clauses, there exists a solution of I of size $15n - w$.

(3) Every solution with $1/2$ -extensions of I is also a solution to the original instance I .

Therefore, the optimum size of a solution of I is $\text{opt}(I) = 15n - \text{opt}(S)$.

We prove Lemma 3 in subsequent sections, but meanwhile let us prove Theorem 2 using Lemma 3 and Theorem 3.

TODO: This below can't use current template

Proof of Theorem 2. Consider any $0 < \epsilon < 1/(15 \cdot 8)$.

Let us assume that there exists a polynomial-time $(1 + \epsilon)$ -approximation algorithm for unweighted geometric set cover with axis-parallel segments in 2D with $(1/2)$ -extensions. We construct an algorithm that solves the problem stated in Theorem 3, thereby proving that $P = NP$.

Take an instance S of MAX-(3,3)-SAT to be distinguished and construct an instance of geometric set cover I using Lemma 3. We now use the $(1 + \epsilon)$ -approximation algorithm for geometric set cover on I . Denote the size of the solution returned by this algorithm as $\text{approx}(I)$. We prove that if in S one can satisfy at most $(\frac{7}{8} + \epsilon)n$ clauses, then $\text{approx}(I) \geq 15n - (\frac{7}{8} + \epsilon)n$ and if S is satisfiable, then $\text{approx}(I) < 15n - (\frac{7}{8} + \epsilon)n$.

Assume S satisfiable. From the definition of S being satisfiable, we have:

$$\text{opt}(S) = n.$$

From Lemma 3 we have:

$$\text{opt}(I) = 14n.$$

Therefore,

$$\begin{aligned} \text{approx}(I) &\leq (1 + \epsilon)\text{opt}(I) = 14n(1 + \epsilon) = 14n + 14\epsilon \cdot n = \\ &= 14n + (15\epsilon - \epsilon)n < 14n + \left(\frac{1}{8} - \epsilon\right)n = 15n - \left(\frac{7}{8} + \epsilon\right)n \end{aligned}$$

Assume S is at most $(\frac{7}{8} + \epsilon)$ satisfiable. From the definition of S being at most $(\frac{7}{8} + \epsilon)n$ satisfiable, we have:

$$\text{opt}(S) \leq \left(\frac{7}{8} + \epsilon\right)n$$

From Lemma 3 we have:

$$\text{opt}(I) \geq 15n - \left(\frac{7}{8} + \epsilon\right)n$$

Since a solution to I with $\frac{1}{2}$ -extensions is also a solution without extensions, by Lemma 3 (3.), we have:

$$\text{approx}(I) \geq \text{opt}(I) = 15n - \left(\frac{7}{8} + \epsilon\right)n$$

Therefore, by using the assumed $(1 + \epsilon)$ -approximation algorithm, it is possible to distinguish the case when S is satisfiable from the case when it is at most $(\frac{7}{8} + \epsilon)n$ satisfiable, it suffices to compute $\text{approx}(I)$ with $15n - (\frac{7}{8} + \epsilon)n$. Hence, the assumed approximation algorithm cannot exist, unless $P = NP$. \square

276 3.2.2. Reduction

277 We proceed to the proof of Lemma 3. That is, we show a reduction from MAX-(3,3)-SAT
 278 problem to geometric set cover with segments parallel to axis. Moreover, the obtained instance
 279 of geometric set cover will be robust to 1/2-extensions (have the same optimal solution after
 280 1/2-extension).

281 The construction will be composed of 2 types of gadgets: **VARIABLE-gadgets** and
 282 **CLAUSE-gadgets**. **CLAUSE-gadgets** would be constructed using two **OR-gadgets** con-
 283 nected together. Every gadget consists of a point set and a segment set.

284 3.2.2.1. VARIABLE-gadget

285 VARIABLE-gadget is responsible for choosing the value of a variable in a CNF formula. It
 286 allows two minimum solutions of size 3. These two choices correspond to the two Boolean
 287 values of the variable.

288 **Points.** Define points a, b, c, d, e, f, g, h as follows, where $L = 12n$:

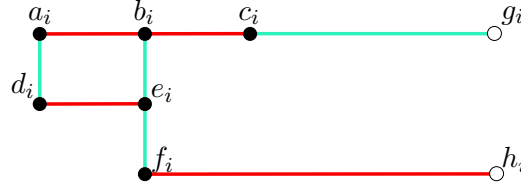


Figure 3.1: **VARIABLE-gadget**. We denote the set of points marked with black circles as C_{var}^i , and they need to be covered (are part of the set \mathcal{C}). Note that some of the points are not marked as black dots and exists only to name segments for further reference. We denote the set of red segments as X_{false}^i and the set of blue segments as X_{true}^i .

$$289 \quad \begin{array}{llll} a = (-L, 0) & b = (-\frac{2}{3}L, 0) & c = (-\frac{1}{3}L, 0) & d = (-L, 1) \\ e = (-\frac{2}{3}L, 1) & f = (-\frac{2}{3}L, 2) & g = (L, 0) & h = (L, 2) \end{array}$$

Let us define:

$$C_{var} = \{a, b, c, d, e, f\}$$

and

$$C_{var}^i = C_{var} + (0, 4i)$$

290 We denote $a_i = a + (0, 4i)$ etc.

291 **Segments.** Let us define:

$$\begin{aligned} X_{true}^i &= \{(a_i, d_i), (b_i, f_i), (c_i, g_i)\} \\ X_{false}^i &= \{(a_i, c_i), (d_i, e_i), (f_i, h_i)\} \end{aligned}$$

$$P_{var}^i = X_{true}^i \cup X_{false}^i$$

292 **Lemma 4.** For any $1 \leq i \leq n$, points in C_{var}^i can be covered using 3 segments from P_{var}^i .

293 *Proof.* We can use either set X_{true}^i or X_{false}^i . □

294 **Lemma 5.** For any $1 \leq i \leq n$, points in C_{var}^i can not be covered with fewer than 3 segments
 295 from P_{var}^i .

296 *Proof.* No segment of P_{var}^i covers more than one point from $\{d_i, f_i, c_i\}$, therefore C_{var}^i can not
 297 be covered with fewer than 3 segments. \square

298 **Lemma 6.** For every set $A \subseteq P_{var}^i$ such that A covers C_{var}^i and $(c_i, g_i), (f_i, h_i) \in A$, it holds
 299 that $|A| \geq 4$.

300 *Proof.* No segment from P_{var}^i covers more than one point from $\{a_i, e_i\}$, therefore C_{var}^i -
 301 $\{c_i, f_i, g_i, h_i\}$ can not be covered with fewer than 2 segments. \square

302 3.2.2.2. OR-gadget

303 OR-gadget has 3 important segments – $x, y, result$. x and y don't count to the weight of
 304 solution of OR-gadget (they are part of different gadgets). It has a minimal solution of weight
 305 w and $result$ can be chosen only if x or y are also chosen for the solution. If none of them
 306 are chosen, then solution choosing $result$ segment has weight at least $w + 1$. Therefore the
 307 following formula holds for a solution R assuming that R uses only w from this OR-gadget:

$$(x \in R) \vee (y \in R) \iff result \in R$$



Figure 3.2: **OR-gadget.** We denote these point as $or_gadget_{i,j}$. We denote set of red segments as $or_{i,j}^{false}$, set of blue segments as $or_{i,j}^{true}$, green and yellow segments as $or_move_variable_{i,j}$.

308 **Points.**

$$\begin{array}{llll}
l_0 = (0, 0) & m_0 = (0, 1) & n_0 = (0, 2) & o_0 = (0, 3) \\
p_0 = (0, 4) & q_0 = (1, 1) & r_0 = (1, 3) & s_0 = (2, 1) \\
t_0 = (2, 2) & u_0 = (2, 3) & v_0 = (3, 2) &
\end{array}$$

$$vec_{i,j} = (10i + 3 + 3j, 4n + 2j)$$

Define $\{l_{i,j}, m_{i,j} \dots v_{i,j}\}$ as $\{l_0, m_0 \dots v_0\}$ shifted by $vec_{i,j}$
Note that $v_{i,0} = l_{i,1}$ (see Figure 3.3)

$$C_or_gadget_{i,j} = \{l_{i,j}, m_{i,j}, n_{i,j}, o_{i,j}, p_{i,j}, q_{i,j}, r_{i,j}, s_{i,j}, t_{i,j}, u_{i,j}\}$$

Segments. We define names subsets of segments, to refer to them in lemmas.

$$or_{i,j}^{false} = \{(q_{i,j}, r_{i,j}), (s_{i,j}, u_{i,j})\}$$

$$or_{i,j}^{true} = \{(m_{i,j}, s_{i,j}), (o_{i,j}, u_{i,j}), (t_{i,j}, v_{i,j})\}$$

$$or_move_variable_{i,j} = \{(l_{i,j}, n_{i,j}), (n_{i,j}, p_{i,j})\}$$

Segments in OR-gadget:

$$P_or_gadget_{i,j} = or_{i,j}^{false} \cup or_{i,j}^{true} \cup or_move_variable_{i,j}$$

Lemma 7. For any $1 \leq i \leq n, j \in \{0, 1\}$ and $x \in \{l_{i,j}, p_{i,j}\}$ we can cover points in $C_or_gadget_{i,j} - \{x\} \cup \{v_{i,j}\}$ with 4 segments from $P_or_gadget_{i,j}$.

Proof. We can do that using one segment from $or_move_variable_{i,j}$ (chosen depending on the value of x) and all segments from $or_{i,j}^{true}$. \square

Lemma 8. For any $1 \leq i \leq n, j \in \{0, 1\}$, we can cover points in $C_or_gadget_{i,j}$ with 4 segments from $P_or_gadget_{i,j}$.

Proof. We can do that using $or_move_variable_{i,j}$ and $or_{i,j}^{false}$. \square

3.2.2.3. CLAUSE-gadget

CLAUSE-gadget is responsible for calculating if choice of the variable values meets the clause in formula. It has minimal solution of weight w if at least one variable in the clause has a correct value. Otherwise it has minimal solution $w + 1$. This way by the minimal solution for the whole problem, we can tell how many clauses were satisfiable.

The CLAUSE-gadgets consist of two OR-gadgets. We don't want the CLAUSE-gadgets to be crammed somewhere between the very long variable segments. That's why we have a simple gadget to *pass* the value of the segment, ie. segments $(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1})$. Two segments and one of them is chosen if x was chosen in the solution and the other one if x wasn't.



Figure 3.3: **CLAUSE-gadget**. We denote set of these points as C_clause_i . Every green rectangle is an OR-gadget. y -coordinates of $x_{i,0}$, $y_{i,0}$ and $z_{i,0}$ depend on the values of variables in the i -th clause.

331 **Points.** TODO: Rephrase it

332 Assuming clause $C_i = x_i \vee y_i \vee z_i$, function $idx(w)$ is returning index of the variable w ,
 333 function $neg(w)$ is returning whether variable w is negated in a clause.

$$\begin{aligned}
 x_{i,0} &= (10i + 1, 4 \cdot idx(x_i) + 2 \cdot neg(x_i)) & x_{i,1} &= (10i + 1, 4n) \\
 y_{i,0} &= (10i + 2, 4 \cdot idx(y_i) + 2 \cdot neg(y_i)) & y_{i,1} &= (10i + 2, 4n + 4) \\
 z_{i,0} &= (10i + 3, 4 \cdot idx(z_i) + 2 \cdot neg(z_i)) & z_{i,1} &= (10i + 3, 4n + 6)
 \end{aligned}$$

$$move_variable_i = \{x_{i,j} : j \in \{0, 1\}\} \cup \{y_{i,j} : j \in \{0, 1\}\} \cup \{z_{i,j} : j \in \{0, 1\}\}$$

$$C_clause_i = move_variable_i \cup C_or_gadget_{i,0} \cup C_or_gadget_{i,1} \cup \{v_{i,1}\}$$

Segments.

$$\begin{aligned}
 P_clause_i &= \{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1}), (x_{i,1}, l_{i,0}), (y_{i,1}, p_{i,0}), (z_{i,1}, p_{i,1}), \} \cup \\
 &\cup P_or_gadget_{i,0} \cup P_or_gadget_{i,1}
 \end{aligned}$$

335 **Lemma 9.** For any $1 \leq i \leq n$ and $a \in \{x_{i,0}, y_{i,0}, z_{i,0}\}$, points in $C_clause_i - \{a\}$ can be
 336 covered with a set of segments $P_true_i^a$, a subset of P_clause_i such that $|P_true_i^a| = 11$.

337 *Proof.* For $a = x_{i,0}$ (analogous proof for $y_{i,0}$): First we use Lemma 7 twice with excluded
 338 $x = l_{i,0}$ and $x = l_{i,1} = v_{i,0}$, resulting with 8 segments $or_{i,0}^{true} \cup or_{i,1}^{true}$ which cover all required

points apart from $x_{i,1}, y_{i,0}, y_{i,1}, z_{i,0}, z_{i,1}, l_{i,0}$. We cover those using additional 3 segments:
 $\{(x_{i,1}, l_{i,0}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1})\}$

For $a = z_{0,i}$: Using Lemma 8 and Lemma 7 with $x = p_{i,1}$, resulting with 8 segments
 $or_{i,0}^{false} \cup or_{i,1}^{true}$ which cover all required points apart from $x_{i,0}, x_{i,1}, y_{i,0}, y_{i,1}, z_{i,1}, p_{i,1}$. We cover
those using additional 3 segments: $\{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,1}, p_{i,1})\}$. \square

Lemma 10. For any $1 \leq i \leq n$, points in C_clause_i can be covered with a set of segments
 $P_false_i^a$, a subset of P_clause_i such that $|P_false_i^a| = 12$.

Proof. Using Lemma 8 twice we can cover $or_gadget_{i,0}$ and $or_gadget_{i,1}$ with 8 segments.

To cover the remaining points we additionally use: $\{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1}), (t_{i,1}, v_{i,1})\}$
 \square

Lemma 11. For any $1 \leq i \leq n$:

(1) points in $C_clause_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$ can not be covered using any subset of segments
from P_clause_i of size less than 11;

(2) points in C_clause_i can not be covered using any subset of segments from P_clause_i
of size less than 12.

Proof of no cover with fewer than 12 segments. There is independent set of 12 points in $C_clause_i \supseteq$
 $\{x_{i,0}, y_{i,0}, z_{i,0}, l_{i,0}, p_{i,0}, q_{i,0}, u_{i,0}, v_{i,0} = l_{i,1}, p_{i,1}, q_{i,1}, u_{i,1}, v_{i,1}\}$. \square

Proof of no cover with fewer than 11 segments. We can choose disjoint sets X, Y, Z such that
 $X \cup Y \cup Z \subseteq C_clause_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$ and there are no segments covering points from
different sets. And we prove lower bounds for each of these sets.

$$X = \{x_{i,1}, y_{i,1}, z_{i,1}\}$$

Set X is an independent set, so it must be covered with 3 segments.

$$Y = or_gadget_{i,0} - \{l_{i,0}, p_{i,0}\}$$

$$Z = or_gadget_{i,1} - \{l_{i,1}, p_{i,1}\}$$

For both Y and Z we can check all of the subsets of 3 segments with brutforce that none
of them cover, so they have to be covered with 4 segments.

TODO: Funny fact, neither Y nor Z doesn't have independent set of size 4.

Therefore C_clause_i must be covered with at least $3 + 4 + 4 = 11$ segments. \square

3.2.2.4. Summary

Add some smart lemmas that sets will be exclusive to each other.

Lemma 12. Robustness to 1/2-extensions. For every segment $s \in \mathcal{P}$, s and $s^{+1/2}$ cover
the same points from \mathcal{C} .

Proof. We can just check every segment. Most of the segments s are colinear only with points
that lay on s , so trivially $s^{+1/2}$ cannot cover more points than s does.

TODO: list problematic segments here \square



Figure 3.4: **General schema.**

General layout of VARIABLE-gadget and CLAUSE-gadget and how they interact with each other.

TODO: Rename Choose X to VARIABLE-gadget and Clause C to CLAUSE-gadget.

371 3.2.2.5. Summary of construction

We define:

$$\begin{aligned}\mathcal{C} &:= \bigcup_{1 \leq i \leq n} C_variable_i \cup C_clause_i \\ \mathcal{P} &:= \bigcup_{1 \leq i \leq n} P_variable_i \cup P_clause_i\end{aligned}$$

372 The subsequent sections define these sets.

373 We prove some properties of different gadgets. Every segment for a gadget will only cover
374 points in this gadget (won't interact with any different gadget), so we can prove lemmas *locally*.

375 TODO: y axis is increasing values downward on figures (not upwards like in normal).

376 3.2.3. Construction lemmas and proof of Lemma 3

377 **Lemma 13.** *Given an instance S of MAX-(3,3)-SAT of size n with optimum solution satis-*
378 *fying k clauses $opt(S) = k$. Instance of geometric set cover, constructed for S as described in*
379 *Section 3.2.2, can be solved with a solution of size $15n - k$.*

380 *Proof.* Let us name the assignments of the variables in the optimum solution of an instance
381 S as $y_1, y_2 \dots y_n$ and clauses as $c_1, c_2 \dots c_n$.

382 We cover every VARIABLE-gadget with solution described in Lemma 4, in the i -th gadget
383 choosing the set of segments corresponding to the value of y_i . CLAUSE-gadgets that are
384 satisfied, let us name the variable that is true in them a , are covered with set $P_true_i^a$
385 described in Lemma 9 and unsatisfied with set P_false_i described in Lemma 10.

$$\begin{aligned}R_i &= \begin{cases} X_i^{true} & \text{if } y_i \\ X_i^{false} & \text{if } \neg y_i \end{cases} \\ C_i &= \begin{cases} P_true_i^a & \text{if } c_i \text{ satisfied} \\ P_false_i & \text{if } c_i \text{ not satisfied} \end{cases} \\ \mathcal{R} &= \bigcup_{i=1}^n \{R_i \cup C_i : 1 \leq i \leq n\}\end{aligned}$$

386 This set covers all points from \mathcal{C} , because the smaller sets individually cover their corre-
387 sponding gadgets (proved in respective lemmas).

388 All of these sets are disjoint, so the size of the solution is:

$$|\mathcal{R}| = \sum_{i=1}^n R_i + \sum_{i=1}^n C_i = 3n + 11k + 12(n - k) = 15n - k.$$

389

□

390 **Lemma 14.** *Given an instance S of MAX-(3,3)-SAT of size n and a solution of an instance*
391 *of geometric set cover, as described in Section 3.2.2, that is of size w . There exists a solution*
392 *of an instance S that satisfied at least $15n - w$ clauses.*

393 *Proof.*

394 Given a solution \mathcal{R} of the instance of geometric set cover, we construct a solution of the
395 instance S by constructing an assignment of variables that satisfies at least $15n - w$ clauses in
396 S .

397 **Variables** We need to use at least 3 segments to cover VARIABLE-gadget (Lemma 5).
 398 If we have chosen both segments (c_i, g_i) and (f_i, h_i) , then we have used at least 4 segments
 399 (Lemma 6).

$$\begin{cases} |C_{var}^i \cap \mathcal{R}| \geq 4 & \text{if } (c_i, g_i) \in \mathcal{R} \wedge (f_i, h_i) \in \mathcal{R} \\ |C_{var}^i \cap \mathcal{R}| \geq 3 & \text{otherwise} \end{cases}$$

400 If we chose at most one of the segments (c_i, g_i) and (f_i, h_i) , choose the corresponding
 401 variable value to the solution. If we chose both segments, choose the value that appears in
 402 most clauses. Every variable is in exactly 3 clauses, so one value appears in at least 2 of them.
 403 If we have chosen none of the segments, set value to false. Formally, we define the value of
 404 the x_i variable as follows:

$$\begin{cases} x_i = \text{majority}(X_i) & \text{if } (c_i, g_i) \in \mathcal{R} \wedge (f_i, h_i) \in \mathcal{R} \\ x_i = \text{true} & \text{if } (c_i, g_i) \in \mathcal{R} \\ x_i = \text{false} & \text{if } (f_i, h_i) \in \mathcal{R} \\ x_i = \text{false} & \text{otherwise} \end{cases} \quad (3.1)$$

405 **TODO:** Maybe remove section below, because we do this calculation at the end anyway
 406 To cover $\bigcup_{1 \leq i \leq n} C_{var}^i$ we have used at least $3n + a$ segments, where a is the number of i such
 407 that we have chosen both values (c_i, g_i) and (f_i, h_i) .

408 **Clauses** For a clause $C_i = x \vee y \vee z$, we need to use at least 11 segments to cover
 409 $C_{\text{clause}_i} - \{x, y, z\}$ in CLAUSE-gadget (Lemma 11).

410 **TODO:** maybe put something with cases and names of sets as above

411 Moreover, if all of the points $\{x_{i,0}, y_{i,0}, z_{i,0}\}$ are not covered by the segments from P_{var}^i ,
 412 then we need to cover C_{clause_i} with at least 12 segments by Lemma 11.

413 **TODO:** Maybe remove section below, because we do this calculation at the end anyway

414 We covered CLAUSE-gadget with at least 11 or at least 12 segments: $|\bigcup_{i=1}^n P_{\text{clause}_i} \cap \mathcal{R}| \geq$
 415 $11n + b$, where b is the number of clauses where none of the segments covering the points
 416 $x_{i,0}, y_{i,0}, z_{i,0}$ were chosen in P_{var}^j .

417 **Satisfied clauses with chosen variables assignment** Clauses for which none of the
 418 points $x_{i,0}, y_{i,0}, z_{i,0}$ were covered by segments in P_{var}^j , are not satisfied in our variables assign-
 419 ment, but not all clauses that cover one of these points with segment in P_{var}^j are satisfied.

420 Let us look at such clause C_i and of points $x_{i,0}, y_{i,0}, z_{i,0}$ that are covered in P_{var}^j . Consider
 421 the cases of choosing variable value in equation (3.1).

422 If only one of the segments (c_i, g_i) and (f_i, h_i) are chosen in P_{var}^j , then the value of x_j is
 423 the same as the one satisfying C_i and clause is satisfied.

424 If we chose neither (c_i, g_i) or (f_i, h_i) , then it is impossible that this point is covered in
 425 P_{var}^j .

426 If we chose both (c_i, g_i) and (f_i, h_i) , then there are 3 clauses for which this point is covered
 427 by P_{var}^j . We chose variable value in a way that only one clause using x_j is not satisfied by
 428 the value of x_j . Therefore there are at most a clauses that are covered with 11 segments from
 429 CLAUSE-gadget, but are not satisfied.

430 So in the solution to this MAX-(3,3)-SAT instance that we have shown, there are at most
 431 $a + b$ unsatisfied clauses.

432 **Conclusions** We proved that given a solution of size w we have the variables assignment
 433 that satisfies at least $n - (a + b)$ clauses of S . At last we prove that $n - (a + b) \geq 15n - w$.

$$w \geq 3(n - a) + 4a + 11(n - b) + 12b = 3n + a + 11n + b = 14n + a + b$$

$$15n - w \leq 15n - 14n - a - b = n - (a + b)$$

434 □

435 *Proof of Lemma 3.* Given an instance S of MAX-(3,3)-SAT of size n with optimum solution
 436 satisfying k clauses. Let us construct an instance of geometric set cover for S as described in
 437 Section 3.2.2 and name it I .

Given the Lemma 13, we know that there exists a solution of I of size $15n - k$, so:

$$\text{opt}(I) \leq 15n - k.$$

Since the optimum solution of S satisfies k clauses, then according to Lemma 14:

$$\text{opt}(I) \geq 15n - k.$$

438 Therefore solution from Lemma 13 of size $15n - k$ is an optimum solution for instance
 439 I . □

440 3.3. FPT for weighted segments with δ -extensions

441 **Theorem 4** (FPT for weighted segment cover with δ -extensions). *There exists an algorithm \mathcal{A}*
 442 *that given a family \mathcal{P} of n weighted segments (in any direction), a set of m points \mathcal{C} parameters*
 443 *k and δ , runs in time $f(k, \delta) \cdot (nm)^c$ for some computable function f and a constant c , and*
 444 *outputs a set $\mathcal{R} \subseteq \mathcal{P}$ such that $|\mathcal{R}| \leq k$ and $\mathcal{R}^{+\delta}$ covers all points in \mathcal{C} .*

445 To solve this problem we will introduce a lemma about choosing *good* subset of points.

446 **Definition 9.** For a set of colinear points C , a subset $A \subseteq C$ is (k, δ) -**good** if for any set of
 447 segments R that covers set A such that $|R| \leq k$, it holds $R^{+\delta}$ covers C .

448 **Lemma 15.** *There exists an algorithm that for any set of colinear points C , $\delta > 0$ and $k \geq 1$,*
 449 *there exists a (k, δ) -good set of size at most $f(k, \delta)$ for some computable function f . This*
 450 *algorithm runs in time $O(|C| \cdot f(k, \delta))$.*

451 *Proof.* We prove this for a fixed δ by induction over k for any set of colinear points C .

452 **Inductive hypothesis** For any set of colinear points C , there exists an algorithm that:

- 453 • finds a set A that is (l, δ) – *good* for every $1 \leq l \leq k$,
- 454 • A has size $|A| < f(\delta, k)$ for some computable function f ,
- 455 • extreme points from C are in A ,
- 456 • runs in time $O(|C|f(k, \delta))$ for some computable function f .

457 **Base case for $k = 1$** It is sufficient that A consists of 2 points: extreme points from C
 458 or a single point if $|C| = 1$.

459 If they are covered with one segment, it must be a segment that includes the extreme
 460 points from C , so it covers whole set C .

Inductive step Assuming inductive hypothesis for any set of colinear points C and for k , we will prove hypothesis for $k + 1$.

Let us name s the minimal segment that includes all points from C .

We define $M = \lceil 1 + \frac{2}{\delta} \rceil$ subsegments of s in the following way. We split s into M parts v_i of equal length, that is $|v_i| = \frac{|s|}{M}$ for any $1 \leq i \leq M$.

C_i is a subset of C such that they lay on v_i .

t_i is a segment connecting leftmost and rightmost point in C_i (it might be degenerated segment if $|C_i| = 1$ or it might be empty if C_i is empty).

TODO: Add a picture with v_i and t_i here

We use inductive hypothesis to choose (k, δ) -good sets A_i for sets C_i . If $|C_i| \leq 1$, then $A_i = C_i$ and it's still a (k, δ) -good set.

Then we define $A = \bigcup_{i=1}^M A_i$. It includes ends of s , because they are in sets A_1 and A_M .

Proof that A is (k, δ) -good for s and C Let us take any cover of A with $k+1$ segments and name it \mathcal{R} .

For every segment t_i , if there exists a segment x from \mathcal{R} such that it is disjoint with t_i , then we have a cover of A_i with at most k segments using $\mathcal{R} - \{x\}$. Since A_i is (k, δ) -good for t_i and C_i , then $(\mathcal{R} - \{x\})^{+\delta}$ covers C_i .

If there exists a segment t_i for which a segment x as defined above does not exist, then all $k+1$ segments that cover A_i intersect with t_i . (Note: There exists only one such segment t_i). From the inductive hypothesis ends of s are in A_1 and A_M respectively, so \mathcal{R} must cover them. Hence there must exist segments starting in the ends of s and ending somewhere in t_i . Let us name these two segments y and z . It follows that: $|y| + |z| + |t_i| \geq |s|$. Since $|t_i| \leq |v_i| = \frac{|s|}{M} \leq \frac{|s|}{1 + \frac{2}{\delta}} = \frac{|s|(\delta)}{\delta+2}$, therefore $\max(|y|, |z|) > |s|(1 - \frac{\delta}{\delta+2})/2 = \frac{|s|}{\delta+2}$.

TODO: Add a picture with such segments here

After δ -extension, the longer of these segments will lengthen both ways by at least:

$$\frac{|s|\delta}{\delta+2} = \frac{|s|}{1 + \frac{2}{\delta}} > \frac{|s|}{M} = v_i > t_i.$$

Therefore the longer of segments y and z will cover the segment t_i after δ -extension, therefore $\mathcal{R}^{+\delta}$ covers C_i .

Since $C = \bigcup_{i=1}^M C_i$, then $\mathcal{R}^{+\delta}$ covers C .

Complexity We use the recursive algorithm for subsets C_i . Every point from C belongs to at most 2 sets C_i .

Apart from recursive algorithm we perform operations linear in size of $|C| + M$ to calculate the sets C_i .

Therefore it has complexity:

$$O(|C| + M) + \sum_i^M O(|C_i|f(k, \delta)) = O(|C|f(k, \delta)) + O((\sum_i^M |C_i|)f(k, \delta)) \leq O(|C|f(k, \delta)).$$

□

Proof of Theorem 4. To construct an algorithm for this problem let us formulate some claims about the problem first.

Definition 10. Line is **long** if there are at least $k+1$ points from \mathcal{C} on it.

496 **Claim 1.** *If there are more than k long lines, then they can not be covered with k segments.*

497 **Claim 2.** *If there is more than k^2 points from \mathcal{C} that do not lie on any long line, then they*
 498 *can not be covered with k segments.*

499 Applying the above claims, if we have more than k long lines or more than k^2 points from
 500 \mathcal{C} that do not lie on any long line, then we answer that there is no solution of size at most k .

501 Otherwise, we can split \mathcal{C} into at most $k + 1$ sets: D , at most k^2 points that do not lie on
 502 any long line and C_i – points that lay on i -th long line. Sets C_i do not need to be disjoint.

503 Then for every set C_i , we can use Lemma 15 to get (k, δ) -good set A_i for set of points C_i
 504 and segment between two extreme points from C_i .

505 Then we have set $D \cup \bigcup A_i$ of size at most $f(k, \delta)$ for some computable function f , that
 506 if we have a solution \mathcal{R} of size at most k that covers $D \cup \bigcup A_i$, then $\mathcal{R}^{+\delta}$ covers \mathcal{C} .

507 \mathcal{R} already covers points D , they cover C_i , because they cover (k, δ) -good set A_i with at
 508 most k segments, so $\mathcal{R}^{+\delta}$ covers C_i .

509 After that we shrunk down size of \mathcal{C} to size of $f(k, \delta)$ for some computable function f .
 510 Then we would like to shrink down size of \mathcal{P} . For every colinear subset of D , we can choose
 511 one segment from \mathcal{P} that covers these points and have the lowest weight or decide there is
 512 no segment that cover them. There are at most $|D|^2$ different segments, because we can
 513 distinguish these colinear sets by their extreme points.

514 This has complexity $O(|D|^2|\mathcal{P}|)$ and produce shrunk down set \mathcal{P} of size $f(k, \delta)$ for some
 515 computable functions f .

516 Then we can iterate over all subsets of shrunk down set \mathcal{P} and choose the set with the
 517 lowest sum of weights that cover D . This solution would have weight not larger than optimal
 518 solution for the problem without extension, because we iterate over all possibilities of covering
 519 the subset of \mathcal{C} .

520 □

521 3.4. $W[1]$ -completeness for weighted segments in 3 directions

522 **Theorem 5.** *$W[1]$ -completeness for weighted segments in 3 directions. Consider the*
 523 *problem of covering a set \mathcal{C} of points by selecting k axis-parallel or right-diagonal weighted*
 524 *segments with weights from a set \mathcal{P} with minimal weight. Assuming ETH, there is no algorithm*
 525 *for this problem with running time $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{o(\sqrt{k})}$ for any computable function f .*

526 We will show reduction from grid tiling problem.

527 Let's have an instance of grid tiling problem – size of the grid k , number of elements
 528 available n and k^2 sets of available pairs in every tile $S_{i,j} \subseteq \{1, n\} \times \{1, n\}$.

529 **Construction.** We construct a set \mathcal{P} of segments and a set \mathcal{C} of points.

530 First let's choose any ordering of n^2 elements $\{1, n\} \times \{1, n\}$ and name this sequence
 531 $a_1 \dots a_{n^2}$.

$$match_v(i, j) \iff a_i = \{x_i, y_i\} \wedge a_j = \{x_j, y_j\} \wedge x_i = x_j$$

$$match_h(i, j) \iff a_i = \{x_i, y_i\} \wedge a_j = \{x_j, y_j\} \wedge y_i = y_j$$

Points. Define points:

$$h_{i,j,t} = (j \cdot (n^2 + 1) + t, (n^2 + 1) \cdot i)$$

$$v_{i,j,t} = ((n^2 + 1) \cdot i, j \cdot (n^2 + 1) + t)$$

Let's define sets H and V as:

$$H = \{h_{i,j,t} : 1 \leq i, j \leq k, 1 \leq t \leq n^2\}$$

$$V = \{v_{i,j,t} : 1 \leq i, j \leq k, 1 \leq t \leq n^2\}$$

532 Let's define $\epsilon = 0.1$. For a point $\{x, y\} = p$ we define points $p^L = \{x - \epsilon, y\}$, $p^R = \{x + \epsilon, y\}$,
 533 $p^U = \{x, y - \epsilon\}$, and $p^D = \{x, y + \epsilon\}$.

Then we define:

$$\mathcal{C} := H \cup \{p^L : p \in H\} \cup \{p^R : p \in H\} \cup V \cup \{p^U : p \in V\} \cup \{p^D : p \in V\}$$

534 **Segments.** Define horizontal segments.

$$hor_{i,j,t_1,t_2} = (h_{i,j,t_1}^R, h_{i,j+1,t_2}^L)$$

$$ver_{i,j,t_1,t_2} = (v_{i,j,t_1}^D, v_{i,j+1,t_2}^U)$$

$$horbeg_{i,t} = (h_{i,1,1}^L, h_{i,1,t}^L)$$

$$horend_{i,t} = (h_{i,n,t}^R, h_{i,n,n^2}^R)$$

$$verbeg_{i,t} = (v_{i,1,1}^U, v_{i,1,t}^U)$$

$$verend_{i,t} = (v_{i,n,t}^D, v_{i,n,n^2}^D)$$

$$\begin{aligned} HOR &= \{hor_{i,j,t_1,t_2} : 1 \leq i \leq k, 1 \leq j < k, 1 \leq t_1, t_2 \leq n^2, match_h(t_1, t_2)\} \\ &\cup \{horbeg_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\} \\ &\cup \{horend_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\} \end{aligned}$$

$$\begin{aligned} VER &= \{ver_{i,j,t_1,t_2} : 1 \leq i \leq k, 1 \leq j < k, 1 \leq t_1, t_2 \leq n^2, match_v(t_1, t_2)\} \\ &\cup \{verbeg_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\} \\ &\cup \{verend_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\} \end{aligned}$$

$$DIAG := \{(h_{i,j,t}, v_{j,i,t}) : 1 \leq i, j \leq k, 1 \leq t \leq n^2, a_t \in S_{i,j}\}$$

535 TODO: explain that these segments are in fact diagonal

$$\mathcal{P} := HOR \cup VER \cup DIAG$$

536 **Lemma 16.** *If there exists solution for grid tiling, then there exists solution for our construc-*
 537 *tion using $2(k+1)k + k^2$ segments with weight exactly $2k \cdot (k(n^2 + 1) - 2 - 2\epsilon(k-1))$.*

Claim 3. *If there exists a solution to the grid tiling $c_1 \dots c_k$ and $r_1 \dots r_k$, then there exists a solution covering all points*

$$\{h_{i,j,t} : 1 \leq i, j \leq k, t = (c_i, r_j)\} \cup \{v_{i,j,t} : 1 \leq i, j \leq k, t = (c_j, r_i)\}$$

538 *with segments in DIAG and the rest in VER or HOR and has weight $2k \cdot (k(n^2 + 1) -$*
 539 *$2 - 2\epsilon(k - 1))$.*

540 **Proof.** TODO: jakiś prosty z definicji

541 **Lemma 17.** *If there exists solution for our construction using $2(k + 1)k + k^2$ segments with*
 542 *weight exactly $2k \cdot (k(n^2 + 1) - 2 - 2\epsilon(k - 1))$, then there exists a solution for grid tiling*

543 **Proof.** This follows from Lemma 18, because we just take which points are covered with
 544 *DIAG.*

545 **Claim 4.** *Points p^L, p^R, p^U, p^D cannot be covered with DIAG.*

546 **Claim 5.** *Points in $H \cup \{p^L : p \in H\} \cup \{p^R : p \in H\}$ cannot be covered with VER.*

547 *Points in $V \cup \{p^U : p \in V\} \cup \{p^D : p \in V\}$ cannot be covered with HOR.*

548 **Claim 6.** *For given i, j if none of the points $h_{i,j,t}$ ($v_{i,j,t}$) for $1 \leq t \leq n^2$ are covered with*
 549 *DIAG, then some spaces between neighbouring points were covered twice.*

550 **Claim 7.** *For given i, j two points h_{i,j,t_1}, h_{i,j,t_2} (v_{i,j,t_1}, v_{i,j,t_2}) for $1 \leq t_1 < t_2 \leq n^2$ are covered*
 551 *with DIAG, then one of them had to be also covered with a segment from HOR (VER).*

552 **Proof.** Point v_{i,j,t_2}^L had to be covered with VER from Claims 4 and 5. And every segment
 553 in VER covering v_{i,j,t_2}^L , covers also v_{i,j,t_1}^L .

554 **Lemma 18.** *If there exists solution for our construction with weight at most (exactly) $2k \cdot$*
 555 *$(k(n^2 + 1) - 2 - 2\epsilon(k - 1))$, then for every i, j there must be exactly one t such that $h_{i,j,t}$ ($v_{i,j,t}$)*
 556 *is covered with DIAG and moreover if h_{i,j,t_1} and $h_{i,j+1,t_2}$ are uncovered, then $\text{math}_h(t_1, t_2)$.*
 557 *Analogically for v .*

558 **Proof.** Only k^2 points can be covered only in DIAG, the rest has to be covered with
 559 *VER \cup HOR.* Therefore every result must be at least *ALL_LINES* - $2k^2\epsilon$, because only
 560 $2k^2$ spaces of length ϵ can be uncovered in this axis.

561 Of course if h_{i,j,t_1} and $h_{i,j+1,t_2}$ are uncovered, then there must exist a segment in HOR
 562 between h_{i,j,t_1}^R and $h_{i,j+1,t_2}^L$, so $\text{math}_h(t_1, t_2)$ must be true.

563 3.5. What is missing

564 We don't know FPT for axis-parallel segments without δ -extensions.

565 Chapter 4

566 Geometric Set Cover with lines

567 4.1. Lines parallel to one of the axis

568 When \mathcal{R} consists only of lines parallel to one of the axis, the problem can be solved in
569 polynomial time.

570 We create bipartial graph G with node for every line on the input split into sets: H –
571 horizontal lines and V – vertical lines. If any two lines cover the same point from \mathcal{C} , then we
572 add edge between them.

573 Of course there will be no edges between nodes inside H , because all of them are pararell
574 and if they share one point, they are the same lines. Similar argument for V . So the graph is
575 bipartial.

576 Now Geometric Set Cover can be solved with Vertex Cover on graph G . Since Vertex
577 Cover (even in weighted setting) on bipartial graphs can be solved in polynomial time.

578 Short note for myself just to remember how to this in polynomial time:

579 Non-weighted setting - Konig theorem + max matching

580 Weighted setting - Min cut in graph of $\neg A$ or $\neg B$ (edges directed from V to H)

581 4.2. FPT for arbitrary lines

582 You can find this is Platypus book. We will show FPT kernel of size at most k^2 .

583 (Maybe we need to reduce lines with one point/points with one line).

584 For every line if there is more than k points on it, you have to take it. At the end, if there
585 is more than k^2 points, return NO. Otherwise there is no more than k^4 lines.

586 In weighted settings among the same lines with different weights you leave the cheapest
587 one and use the same algorithm.

588 4.3. APX-completeness for arbitrary lines

589 We will show a reduction from Vertex Cover problem. Let's take an instance of the Vertex
590 Cover problem for graph G . We will create a set of $|V(G)|$ pairwise non-pararell lines, such
591 that no three of them share a common point.

592 Then for every edge in $(v, w) \in E(G)$ we put a point on crossing of lines for vertices v
593 and w . They are not pararell, so there exists exactly one such point and any other line don't
594 cover this point (any three of them don't cross in the same point).

Solution of Geometric Set Cover for this instance would yield a sound solution of Vertex Cover for graph G . For every point (edge) we need to choose at least one of lines (vertices) v or w to cover this point.

Vertex Cover for arbitrary graph is APX-complete, so this problem is also APX-complete.

4.4. 2-approximation for arbitrary lines

Vertex Cover has an easy 2-approximation algorithm, but here very many lines can cross through the same point, so we can do d -approximation, where d is the biggest number of lines crossing through the same point. So for set where any 3 lines don't cross in the same point it yields 2-approximation.

The problematic cases are where through all points cross at least k points and all lines have at least k points on them. It can be created by casting k -grid in k -D space on 2D space.

Greedy algorithm yields $\log |\mathcal{R}|$ -approximation, but I have example for this for bipartial graph and reduction with taking all lines crossing through some point (if there are no more than k) would solve this case. So maybe it works.

Unfortunately I haven't done this :(

I can link some papers telling it's hard to do.

4.5. Connection with general set cover

Problem with finite set of lines with more dimensions is equivalent to problem in 2D, because we can project lines on the plane which is not perpendicular to any plane created by pairs of (point from \mathcal{C} , line from \mathcal{P}).

Of course every two lines have at most one common point, so is every family of sets that have at most one point in common equivalent to some geometric set cover with lines?

No, because of Desargues's theorem. Have to write down exactly what configuration is banned.

Chapter 5

Geometric Set Cover with polygons

5.1. State of the art

Covering points with weighted discs admits PTAS [Li and Jin, 2015] and with fat polygons with δ -extensions with unit weights admits EPTAS [Har-Peled and Lee, 2009].

Although with thin objects, even if we allow δ -expansion, the Set Cover with rectangles is APX-complete (for $\delta = 1/2$), it follows from APX-completeness for segments with δ -expansion in Section 3.2.

Covering points with squares is W[1]-hard [Marx, 2005]. It can be proven that assuming *SETH*, there is no $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{k-\epsilon}$ time algorithm for any computable function f and $\epsilon > 0$ that decides if there are k polygons in \mathcal{P} that together cover \mathcal{C} , *Theorem 1.9* in [Marx and Pilipczuk, 2015].

⁶³¹ Chapter 6

⁶³² Conclusions

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