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# Approximation and Parametrized Algorithms for Segment Set Cover

6

Master's thesis

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in COMPUTER SCIENCE

8

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9

June 2020

10 **Supervisor's statement**

11 Hereby I confirm that the presented thesis was prepared under my supervision and  
12 that it fulfils the requirements for the degree of Master of Computer Science.

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## **Abstract**

23 The work presents a study of different geometric set cover problems. It mostly focuses on  
24 segment set cover and its connection to the polygon set cover.

25

## **Keywords**

26 set cover, geometric set cover, FPT,  $W[1]$ -completeness, APX-completeness, PCP theorem,  
27 NP-completeness

28

## **Thesis domain (Socrates-Erasmus subject area codes)**

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30

31

## **Subject classification**

32 D. Software

33 D.127. Blabalgorithms

34 D.127.6. Numerical blabalalysis

35

## **Tytuł pracy w języku polskim**

36 Algorytmy parametryzowania i trudność aproksymacji problemu pokrywania zbiorów  
37 odcinkami na płaszczyźnie



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# Chapter 1

## Introduction

The Set Cover problem is one of the most common NP-complete problems. [tutaj referencja]  
We are given a family of sets and have to choose the smallest subfamily of these sets that cover  
all their elements. This problem naturally extends to settings where we put different weights  
on the sets and look for the subfamily of the minimal weight. This problem is NP-complete  
even without weights and if we put restrictions on what the sets can be. One of such variants  
is Vertex Cover problem, where sets have size 2 (they are edges in a graph).

In this work we focus on another such variant where the sets correspond to some geometric  
shapes and only some points of the plane have to be covered. When these shapes are rectangles  
with edges parallel to the axis, the problem can be proven to be W[1]-complete (solution of  
size  $k$  cannot be found in  $n^o(k)$  time), APX-complete (for sufficiently small  $\epsilon > 0$ , the problem  
does not admit  $1 + \epsilon$ -approximation scheme) [referencje].

Some of these settings are very easy. Set cover with lines parallel to one of the axis can  
be solved in polynomial time.

There is a notion of  $\delta$ -expansions, which loosen the restrictions on geometric set cover. We  
allow the objects to cover the points after  $\delta$ -expansion and compare the result to the original  
setting. This way we can produce both FPT and EPTAS for the rectangle set cover with  
 $\delta$ -extensions [referencje].

**Our contribution.** In this work, we prove that unweighted geometric set cover with seg-  
ments is fixed parameter tractable (FPT).

Moreover, we show that geometric set cover with segments is APX-complete for unweighted  
axis-parallel segments, even with  $1/2$ -extensions. So the problem for very thin rectangles  
also can't admit PTAS. Therefore, in the efficient polynomial-time approximation scheme  
(EPTAS) for *fat polygons* by [Har-Peled and Lee, 2009], the assumption about polygons  
being fat is necessary.

Finally, we show that geometric set cover with weighted segments in 3 directions is  
W[1]-complete. However, geometric set cover with weighted segments is FPT if we allow  
 $\delta$ -extension.

This result is especially interesting, since it's counter-intuitive that the unweighted setting  
is FPT and the weighted setting is W[1]-complete. Most of such problems (like vertex cover  
or [wiecej przykladow]) are equally hard in both weighted and unweighted settings.





## 103 Chapter 2

## 104 Definitions

### 105 2.1. Geometric Set Cover

106 In the geometric set cover problem we are given  $\mathcal{P}$  – a set of objects, which are connected  
107 subsets of the plane,  $\mathcal{C}$  – a set of points in the plane. The task is to choose  $\mathcal{R} \subseteq \mathcal{P}$  such that  
108 every point in  $\mathcal{C}$  is inside some element from  $\mathcal{R}$  and  $|\mathcal{R}|$  is minimized.

109 In the parametrized setting for a given  $k$ , we only look for a solution  $\mathcal{R}$  such that  $|\mathcal{R}| \leq k$ .

110 In the weighted setting, there is some given weight function  $f : \mathcal{P} \rightarrow \mathbb{R}^+$ , and we would  
111 like to find a solution  $\mathcal{R}$  that minimizes  $\sum_{R \in \mathcal{R}} f(R)$ .

### 112 2.2. Approximation

113 Let us recall some definitions related to optimization problems that are used in the following  
114 sections.

115 **Definition 1.** A **polynomial-time approximation scheme (PTAS)** for a minimization  
116 problem  $\Pi$  is a family of algorithms  $\mathcal{A}_\epsilon$  for every  $\epsilon > 0$  such that  $\mathcal{A}_\epsilon$  takes an instance  $I$  of  $\Pi$   
117 and in polynomial time finds a solution that is within a factor  $(1 + \epsilon)$  of being optimal. That  
118 means the reported solution has weight at most  $(1 + \epsilon)\text{opt}(I)$ , where  $\text{opt}(I)$  is the weight of  
119 an optimal solution for  $I$ .

120 **Definition 2.** A problem  $\Pi$  is **APX-hard** if assuming  $P \neq NP$ , there exists  $\epsilon > 0$  such that  
121 there is no polynomial-time  $(1 + \epsilon)$ -approximation algorithm for  $\Pi$ .

### 122 2.3. $\delta$ -extensions

123 TODO PLACEHOLDER for introductory text

124  $\delta$ -extensions is one of the modifications to a problem, that makes geometric set cover  
125 problem easier, it has been already used in literature (place some refrence here).

126 **Definition 3** ( $\delta$ -extensions for center-symmetric objects). For any  $\delta > 0$  and a center-  
127 symmetric object  $L$  with centre of symmetry  $S = (x_s, y_s)$ , the  **$\delta$ -extension** of  $L$  is the  
128 object  $L^{+\delta} = \{(1 + \delta) \cdot (x - x_s, y - y_s) + (x_s, y_s) : (x, y) \in L\}$ , that is,  $L^{+\delta}$  is the image of  $L$   
129 under homothety centered at  $S$  with scale  $(1 + \delta)$

130 The geometric set cover problem with  $\delta$ -extensions is a modified version of geometric set  
131 cover where:

- We need to cover all the points in  $\mathcal{C}$  with objects from  $\{P^{+\delta} : P \in \mathcal{P}\}$  (which always include no fewer points than the objects before  $\delta$ -extensions);

- We look for a solution that is no larger than the optimum solution for the original problem. Note that it does not need to be an optimal solution in the modified problem.

Formally, we have the following.

**Definition 4** (Geometric set cover problem with  $\delta$ -extensions). The geometric set cover problem with  $\delta$ -extensions is the problem where for an input instance  $I = (\mathcal{P}, \mathcal{C})$ , the task is to output a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that the  $\delta$ -extended set  $\{R^{+\delta} : R \in \mathcal{R}\}$  covers  $\mathcal{C}$  and is no larger than the optimal solution for the problem without extensions, i.e.  $|\mathcal{R}| \leq |\text{opt}(I)|$ .

TODO: Some text

**Definition 5** (Geometric set cover PTAS with  $\delta$ -extensions). We define a PTAS for geometric set cover with  $\delta$ -extensions as a family of algorithms  $\{\mathcal{A}_{\delta, \epsilon}\}_{\delta, \epsilon > 0}$  that each takes as an input instance  $I = (\mathcal{P}, \mathcal{C})$ , and in polynomial-time outputs a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that the  $\delta$ -extended set  $\{R^{+\delta} : R \in \mathcal{R}\}$  covers  $\mathcal{C}$  and is within a  $(1 + \epsilon)$  factor of the optimal solution for this problem without extensions, i.e.  $(1 + \epsilon)|\mathcal{R}| \leq |\text{opt}(I)|$ .

## Chapter 3

# Geometric Set Cover with segments

### 3.1. FPT for segments

#### 3.1.1. Axis-parallel segments

You can find this in Platypus book. (TODO add referece)

We show  $\mathcal{O}(2^k)$  branching algorithm. Let us take the point  $K$  which is the smallest under a lexicographic ordering on coordinates among points that are not covered yet. We need to cover  $K$  with some of the remaining segments.

We branch over choice of direction among the 2 axis-parallel directions. In this direction we greedily take the segment that covers the most points. As  $K$  was the smallest in lexicographical order, all points in  $\mathcal{C}$  colinear with  $K$  in both axis-parallel directions are only on one side of  $K$ , because their coordinates are larger. Therefore segments covering  $K$  in this direction create monotone sequence of sets and we can greedily take one segment that covers superset of all of these segments.

TODO: Maybe split it into theorem + algorithm + explanation like in section 3.1.3

#### 3.1.2. Segments in $d$ directions

The same algorithm as described in the previous section, but we branch over  $d$  directions and it runs in complexity  $\mathcal{O}(d^k)$ .

#### 3.1.3. Segments in arbitrary direction

**Theorem 1. (FPT for segment cover).** *There exists an algorithm that given a family  $\mathcal{P}$  of  $n$  segments (in any direction), a set of  $m$  points  $\mathcal{C}$  and a parameter  $k$ , runs in time  $f(k) \cdot (nm)^c$  for some computable function  $f$  and constant  $c$ , and outputs a subfamily  $\mathcal{R} \subseteq \mathcal{P}$  such that  $|\mathcal{R}| \leq k$  and  $\mathcal{R}$  covers all points in  $\mathcal{C}$  or determines that the solution of size at most  $k$  doesn't exist.*

This theorem is proved by following lemmas.

**Lemma 1. (Reduction).** *Given a family  $\mathcal{P}$  of  $n$  segments (in any direction) and a set of  $m$  points  $\mathcal{C}$  for segment cover problem, without a loss of generality we can assume that no segment covers a superset of what another segment covers. That is, for any  $A, B \in \mathcal{P}$ , we have  $A \neq B \Rightarrow A \not\subseteq B$ .*

*Proof.* Trivial. □

**Lemma 2.** *Given an instance of a problem, if there exists a line  $L$  with at least  $k + 1$  points on it, there exists a subset  $\mathcal{A} \subseteq \mathcal{P}$ ,  $|\mathcal{A}| \leq k$ , such that every solution  $\mathcal{R}$  with  $|\mathcal{R}| \leq k$  satisfies  $|\mathcal{A} \cap \mathcal{R}| \geq 1$ .*

*Proof.* First we use Lemma 1.

Let us name points from  $\mathcal{C}$  that lay on  $L$ ,  $x_1, x_2, \dots, x_t$  in the order they appear on the line.

Every segment that is not colinear with  $L$  can cover at most one of these points. Therefore in any solution of size not larger than  $k$ , among any  $k$  of these points at least one must be covered with segment colinear with  $L$ .

Therefore we need to take one of the segments colinear with  $L$  that covers any of the points  $x_1, x_2, \dots, x_k$ . After using reduction from Lemma 1, there are at most  $k$  such segments that are distinct.  $\square$

*Proof of theorem 1.*

**Algorithm.** First we use Lemma 1.

We present a recursive algorithm. Given an instance of the problem:

- (1) If there exist a line with at least  $k + 1$  points, we branch over adding to the solution one of at most  $k$  possible segments from Lemma 2, name this segment  $S$ . Then we find a solution  $\mathcal{R}$  for problem for points  $\mathcal{C} - S$ , segments  $\mathcal{P} - \{S\}$  and parameter  $k - 1$  and return  $\mathcal{R} \cup \{S\}$ .
- (2) If every line has at most  $k$  points on it and  $|\mathcal{C}| > k^2$ , then answer **NO**.
- (3) If  $|\mathcal{C}| \leq k^2$ , solve the problem by brute force algorithm.

**Correctness.** Lemma 2 proves that at least one segment that we branch over in (1) must be present in every solution  $\mathcal{R}$  with  $|\mathcal{R}| \leq k$ , therefore the recursive call can find the optimal solution.

In (2) the answer is no, because every line covers no more than  $k$  points from  $\mathcal{C}$ , which implies that every segment from  $\mathcal{P}$  covers at most  $k$ . Under this assumption we can cover only  $k^2$  points with a solution of size  $k$ , which is less than  $|\mathcal{C}|$ .

Checking all possible solutions in (3) is trivially correct.

**Complexity.** In leaves of branching (3)  $|\mathcal{C}| < k^2$ , so  $|\mathcal{P}| < k^4$ , because every segments can be uniquely identified by 2 extreme points it covers (by Lemma 1). Therefore there are  $\binom{k^4}{k}$  possible solutions to check, each can be checked in time  $O(k|\mathcal{C}|)$ . Therefore (3) takes time  $O(f(k))$ .

In this branching algorithm our parameter  $k$  is decreased with every recursive call, so we have at most  $k$  levels of recursion with branching over  $k$  possibilities. Candidates to branch over can be found on each level in time  $O(nm \log(nm))$ .

Reduction from Lemma 1 can be implemented in  $O(n^2m)$ .

Overall complexity is  $O(n^2m + nm \log(nm) \cdot f(k))$   $\square$

## 214 3.2. APX-completeness for segments parallel to axes

215 In this section we analyze whether there exists PTAS for geometric set cover for rectangles.  
 216 We show that we can restrict this problem to a very simple setting: segments parallel to axes  
 217 and allow  $(1/2)$ -extension, and the problem is still APX-hard. Note that segments are just  
 218 degenerated rectangles with one side being very narrow.

219 Our results can be summarized in the following theorem and this section aims to prove it.

220 **Theorem 2.** *(axis-parallel segment set cover with  $1/2$ -extension is APX-hard).*  
 221 *Unweighted geometric set cover with axis-parallel segments in 2D (even with  $1/2$ -extension)*  
 222 *is APX-hard. That is, assuming  $P \neq NP$ , there does not exist a PTAS for this problem.*

223 Theorem 2 implies the following.

224 **Corollary 1.** *(rectangle set cover is APX-hard).* *Unweighted geometric set cover with*  
 225 *rectangles (even with  $1/2$ -extension) is APX-hard.*

226 We prove Theorem 2 by taking a problem that is APX-hard and showing a reduction. For  
 227 this problem we choose MAX-(3,3)-SAT which we define below.

### 228 3.2.1. MAX-(3,3)-SAT and statement of reduction

229 **Definition 6.** MAX-3SAT is the following maximization problem. We are given a 3-CNF  
 230 formula, and need to find an assignment of variables that satisfies the most clauses.

231 **Definition 7.** MAX-(3,3)-SAT is a variant of MAX-3SAT with an additional restriction  
 232 that every variable appears in exactly 3 clauses. Note that thus, the number of clauses is  
 233 equal to the number of variables.

234 In our proof of Theorem 2 we use hardness of approximation of MAX-(3,3)-SAT proved  
 235 in [Håstad, 2001] and described in Theorem 3 below.

236 **Definition 8** ( $\alpha$ -satisfiable MAX-3SAT formula). MAX-3SAT formula of size  $n$  is at most  
 237  $\alpha$ -satisfiable, if every assignment of variables satisfies no more than  $\alpha n$  clauses.

238 **Theorem 3.** [Håstad, 2001]

239 *For any  $\epsilon > 0$ , it is NP-hard to distinguish satisfiable (3,3)-SAT formulas from at most*  
 240  *$(7/8 + \epsilon)$ -satisfiable (3,3)-SAT formulas.*

241 Given an instance  $I$  of MAX-(3,3)-SAT, we construct an instance  $J$  of axis-parallel seg-  
 242 ment set cover problem, such that for a sufficiently small  $\epsilon > 0$ , a polynomial time  $(1 + \epsilon)$ -  
 243 approximation algorithm for  $J$  would be able to distinguish whether an instance  $I$  of MAX-  
 244 (3,3)-SAT is fully satisfiable or is at most  $(7/8 + \epsilon)$ -satisfiable. However, according to (Theorem  
 245 3) the latter problem is NP-hard. This would imply  $P = NP$ , contradicting the assumption.

246 The following lemma encapsulates the properties of the reduction described in this section,  
 247 and it allows us to prove Theorem 2.

248 **Lemma 3.** *Given an instance  $S$  of MAX-(3,3)-SAT with  $n$  variables and optimum value*  
 249  *$opt(S)$ , we can construct an instance  $I$  of geometric set cover with axis-parallel segments in*  
 250 *2D, such that:*

251 (1) *For every solution  $X$  of instance  $I$ , there exists a solution of  $S$  that satisfies at least*  
 252  *$15n - |X|$  clauses.*

(2) For every solution of instance  $S$  that satisfies  $w$  clauses, there exists a solution of  $I$  of size  $15n - w$ .

(3) Every solution with  $1/2$ -extensions of  $I$  is also a solution to the original instance  $I$ .

Therefore, the optimum size of a solution of  $I$  is  $\text{opt}(I) = 15n - \text{opt}(S)$ .

We prove Lemma 3 in subsequent sections, but meanwhile let us prove Theorem 2 using Lemma 3 and Theorem 3.

TODO: This below can't use current template

*Proof of Theorem 2.* Consider any  $0 < \epsilon < 1/(15 \cdot 8)$ .

Let us assume that there exists a polynomial-time  $(1 + \epsilon)$ -approximation algorithm for unweighted geometric set cover with axis-parallel segments in 2D with  $(1/2)$ -extensions. We construct an algorithm that solves the problem stated in Theorem 3, thereby proving that  $P = NP$ .

Take an instance  $S$  of MAX-(3,3)-SAT to be distinguished and construct an instance of geometric set cover  $I$  using Lemma 3. We now use the  $(1 + \epsilon)$ -approximation algorithm for geometric set cover on  $I$ . Denote the size of the solution returned by this algorithm as  $\text{approx}(I)$ . We prove that if in  $S$  one can satisfy at most  $(\frac{7}{8} + \epsilon)n$  clauses, then  $\text{approx}(I) \geq 15n - (\frac{7}{8} + \epsilon)n$  and if  $S$  is satisfiable, then  $\text{approx}(I) < 15n - (\frac{7}{8} + \epsilon)n$ .

**Assume  $S$  satisfiable.** From the definition of  $S$  being satisfiable, we have:

$$\text{opt}(S) = n.$$

From Lemma 3 we have:

$$\text{opt}(I) = 14n.$$

Therefore,

$$\begin{aligned} \text{approx}(I) &\leq (1 + \epsilon)\text{opt}(I) = 14n(1 + \epsilon) = 14n + 14\epsilon \cdot n = \\ &= 14n + (15\epsilon - \epsilon)n < 14n + \left(\frac{1}{8} - \epsilon\right)n = 15n - \left(\frac{7}{8} + \epsilon\right)n \end{aligned}$$

**Assume  $S$  is at most  $(\frac{7}{8} + \epsilon)$  satisfiable.** From the definition of  $S$  being at most  $(\frac{7}{8} + \epsilon)n$  satisfiable, we have:

$$\text{opt}(S) \leq \left(\frac{7}{8} + \epsilon\right)n$$

From Lemma 3 we have:

$$\text{opt}(I) \geq 15n - \left(\frac{7}{8} + \epsilon\right)n$$

Since a solution to  $I$  with  $\frac{1}{2}$ -extensions is also a solution without extensions, by Lemma 3 (3.), we have:

$$\text{approx}(I) \geq \text{opt}(I) = 15n - \left(\frac{7}{8} + \epsilon\right)n$$

Therefore, by using the assumed  $(1 + \epsilon)$ -approximation algorithm, it is possible to distinguish the case when  $S$  is satisfiable from the case when it is at most  $(\frac{7}{8} + \epsilon)n$  satisfiable, it suffices to compute  $\text{approx}(I)$  with  $15n - (\frac{7}{8} + \epsilon)n$ . Hence, the assumed approximation algorithm cannot exist, unless  $P = NP$ .  $\square$

### 277 3.2.2. Reduction

278 We proceed to the proof of Lemma 3. That is, we show a reduction from MAX-(3,3)-SAT  
 279 problem to geometric set cover with segments parallel to axis. Moreover, the obtained instance  
 280 of geometric set cover will be robust to 1/2-extensions (have the same optimal solution after  
 281 1/2-extension).

282 The construction will be composed of 2 types of gadgets: **VARIABLE-gadgets** and  
 283 **CLAUSE-gadgets**. **CLAUSE-gadgets** would be constructed using two **OR-gadgets** con-  
 284 nected together. Every gadget consists of a point set and a segment set.

#### 285 3.2.2.1. VARIABLE-gadget

286 VARIABLE-gadget is responsible for choosing the value of a variable in a CNF formula. It  
 287 allows two minimum solutions of size 3. These two choices correspond to the two Boolean  
 288 values of the variable.

289 **Points.** Define points  $a, b, c, d, e, f, g, h$  as follows, where  $L = 12n$ :

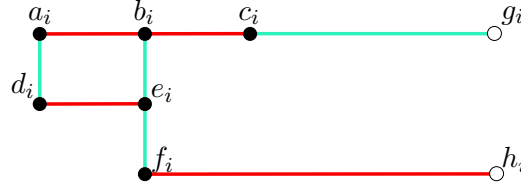


Figure 3.1: **VARIABLE-gadget**. We denote the set of points marked with black circles as  $C_{var}^i$ , and they need to be covered (are part of the set  $\mathcal{C}$ ). Note that some of the points are not marked as black dots and exists only to name segments for further reference. We denote the set of red segments as  $X_{false}^i$  and the set of blue segments as  $X_{true}^i$ .

$$290 \quad \begin{array}{llll} a = (-L, 0) & b = (-\frac{2}{3}L, 0) & c = (-\frac{1}{3}L, 0) & d = (-L, 1) \\ e = (-\frac{2}{3}L, 1) & f = (-\frac{2}{3}L, 2) & g = (L, 0) & h = (L, 2) \end{array}$$

Let us define:

$$C_{var} = \{a, b, c, d, e, f\}$$

and

$$C_{var}^i = C_{var} + (0, 4i)$$

291 We denote  $a_i = a + (0, 4i)$  etc.

292 **Segments.** Let us define:

$$\begin{aligned} X_{true}^i &= \{(a_i, d_i), (b_i, f_i), (c_i, g_i)\} \\ X_{false}^i &= \{(a_i, c_i), (d_i, e_i), (f_i, h_i)\} \end{aligned}$$

$$P_{var}^i = X_{true}^i \cup X_{false}^i$$

293 **Lemma 4.** For any  $1 \leq i \leq n$ , points in  $C_{var}^i$  can be covered using 3 segments from  $P_{var}^i$ .

294 *Proof.* We can use either set  $X_{true}^i$  or  $X_{false}^i$ . □

295 **Lemma 5.** For any  $1 \leq i \leq n$ , points in  $C_{var}^i$  can not be covered with fewer than 3 segments  
 296 from  $P_{var}^i$ .

297 *Proof.* No segment of  $P_{var}^i$  covers more than one point from  $\{d_i, f_i, c_i\}$ , therefore  $C_{var}^i$  can not  
 298 be covered with fewer than 3 segments.  $\square$

299 **Lemma 6.** For every set  $A \subseteq P_{var}^i$  such that  $A$  covers  $C_{var}^i$  and  $(c_i, g_i), (f_i, h_i) \in A$ , it holds  
 300 that  $|A| \geq 4$ .

301 *Proof.* No segment from  $P_{var}^i$  covers more than one point from  $\{a_i, e_i\}$ , therefore  $C_{var}^i$  -  
 302  $\{c_i, f_i, g_i, h_i\}$  can not be covered with fewer than 2 segments.  $\square$

### 303 3.2.2.2. OR-gadget

304 OR-gadget has 3 important segments –  $x, y, result$ .  $x$  and  $y$  don't count to the weight of  
 305 solution of OR-gadget (they are part of different gadgets). It has a minimal solution of weight  
 306  $w$  and  $result$  can be chosen only if  $x$  or  $y$  are also chosen for the solution. If none of them  
 307 are chosen, then solution choosing  $result$  segment has weight at least  $w + 1$ . Therefore the  
 308 following formula holds for a solution  $R$  assuming that  $R$  uses only  $w$  from this OR-gadget:

$$(x \in R) \vee (y \in R) \iff result \in R$$



Figure 3.2: **OR-gadget.** We denote these point as  $or\_gadget_{i,j}$ . We denote set of red segments as  $or_{i,j}^{false}$ , set of blue segments as  $or_{i,j}^{true}$ , green and yellow segments as  $or\_move\_variable_{i,j}$ .

309 **Points.**



$$\begin{array}{llll}
l_0 = (0, 0) & m_0 = (0, 1) & n_0 = (0, 2) & o_0 = (0, 3) \\
p_0 = (0, 4) & q_0 = (1, 1) & r_0 = (1, 3) & s_0 = (2, 1) \\
t_0 = (2, 2) & u_0 = (2, 3) & v_0 = (3, 2) & 
\end{array}$$

$$vec_{i,j} = (10i + 3 + 3j, 4n + 2j)$$

Define  $\{l_{i,j}, m_{i,j} \dots v_{i,j}\}$  as  $\{l_0, m_0 \dots v_0\}$  shifted by  $vec_{i,j}$

Note that  $v_{i,0} = l_{i,1}$  (see Figure 3.3)

$$C\_or\_gadget_{i,j} = \{l_{i,j}, m_{i,j}, n_{i,j}, o_{i,j}, p_{i,j}, q_{i,j}, r_{i,j}, s_{i,j}, t_{i,j}, u_{i,j}\}$$

**Segments.** We define names subsets of segments, to refer to them in lemmas.

$$or_{i,j}^{false} = \{(q_{i,j}, r_{i,j}), (s_{i,j}, u_{i,j})\}$$

$$or_{i,j}^{true} = \{(m_{i,j}, s_{i,j}), (o_{i,j}, u_{i,j}), (t_{i,j}, v_{i,j})\}$$

$$or\_move\_variable_{i,j} = \{(l_{i,j}, n_{i,j}), (n_{i,j}, p_{i,j})\}$$

Segments in OR-gadget:

$$P\_or\_gadget_{i,j} = or_{i,j}^{false} \cup or_{i,j}^{true} \cup or\_move\_variable_{i,j}$$

**Lemma 7.** For any  $1 \leq i \leq n, j \in \{0, 1\}$  and  $x \in \{l_{i,j}, p_{i,j}\}$  we can cover points in  $C\_or\_gadget_{i,j} - \{x\} \cup \{v_{i,j}\}$  with 4 segments from  $P\_or\_gadget_{i,j}$ .

*Proof.* We can do that using one segment from  $or\_move\_variable_{i,j}$  (chosen depending on the value of  $x$ ) and all segments from  $or_{i,j}^{true}$ .  $\square$

**Lemma 8.** For any  $1 \leq i \leq n, j \in \{0, 1\}$ , we can cover points in  $C\_or\_gadget_{i,j}$  with 4 segments from  $P\_or\_gadget_{i,j}$ .

*Proof.* We can do that using  $or\_move\_variable_{i,j}$  and  $or_{i,j}^{false}$ .  $\square$

### 3.2.2.3. CLAUSE-gadget

CLAUSE-gadget is responsible for calculating if choice of the variable values meets the clause in formula. It has minimal solution of weight  $w$  if at least one variable in the clause has a correct value. Otherwise it has minimal solution  $w + 1$ . This way by the minimal solution for the whole problem, we can tell how many clauses were satisfiable.

The CLAUSE-gadgets consist of two OR-gadgets. We don't want the CLAUSE-gadgets to be crammed somewhere between the very long variable segments. That's why we have a simple gadget to *pass* the value of the segment, ie. segments  $(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1})$ . Two segments and one of them is chosen if  $x$  was chosen in the solution and the other one if  $x$  wasn't.



Figure 3.3: **CLAUSE-gadget**. We denote set of these points as  $C\_clause_i$ . Every green rectangle is an OR-gadget.  $y$ -coordinates of  $x_{i,0}$ ,  $y_{i,0}$  and  $z_{i,0}$  depend on the values of variables in the  $i$ -th clause.

332 **Points.** TODO: Rephrase it

333 Assuming clause  $C_i = x_i \vee y_i \vee z_i$ , function  $idx(w)$  is returning index of the variable  $w$ ,  
 334 function  $neg(w)$  is returning whether variable  $w$  is negated in a clause.

$$\begin{aligned}
 x_{i,0} &= (10i + 1, 4 \cdot idx(x_i) + 2 \cdot neg(x_i)) & x_{i,1} &= (10i + 1, 4n) \\
 y_{i,0} &= (10i + 2, 4 \cdot idx(y_i) + 2 \cdot neg(y_i)) & y_{i,1} &= (10i + 2, 4n + 4) \\
 z_{i,0} &= (10i + 3, 4 \cdot idx(z_i) + 2 \cdot neg(z_i)) & z_{i,1} &= (10i + 3, 4n + 6)
 \end{aligned}$$

$$move\_variable_i = \{x_{i,j} : j \in \{0, 1\}\} \cup \{y_{i,j} : j \in \{0, 1\}\} \cup \{z_{i,j} : j \in \{0, 1\}\}$$

$$C\_clause_i = move\_variable_i \cup C\_or\_gadget_{i,0} \cup C\_or\_gadget_{i,1} \cup \{v_{i,1}\}$$

**Segments.**

$$\begin{aligned}
 P\_clause_i &= \{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1}), (x_{i,1}, l_{i,0}), (y_{i,1}, p_{i,0}), (z_{i,1}, p_{i,1}), \} \cup \\
 &\cup P\_or\_gadget_{i,0} \cup P\_or\_gadget_{i,1}
 \end{aligned}$$

336 **Lemma 9.** For any  $1 \leq i \leq n$  and  $a \in \{x_{i,0}, y_{i,0}, z_{i,0}\}$ , points in  $C\_clause_i - \{a\}$  can be  
 337 covered with a set of segments  $P\_true_i^a$ , a subset of  $P\_clause_i$  such that  $|P\_true_i^a| = 11$ .

338 *Proof.* For  $a = x_{i,0}$  (analogous proof for  $y_{i,0}$ ): First we use Lemma 7 twice with excluded  
 339  $x = l_{i,0}$  and  $x = l_{i,1} = v_{i,0}$ , resulting with 8 segments  $or_{i,0}^{true} \cup or_{i,1}^{true}$  which cover all required

points apart from  $x_{i,1}, y_{i,0}, y_{i,1}, z_{i,0}, z_{i,1}, l_{i,0}$ . We cover those using additional 3 segments:  
 $\{(x_{i,1}, l_{i,0}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1})\}$

For  $a = z_{0,i}$ : Using Lemma 8 and Lemma 7 with  $x = p_{i,1}$ , resulting with 8 segments  
 $or_{i,0}^{false} \cup or_{i,1}^{true}$  which cover all required points apart from  $x_{i,0}, x_{i,1}, y_{i,0}, y_{i,1}, z_{i,1}, p_{i,1}$ . We cover  
those using additional 3 segments:  $\{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,1}, p_{i,1})\}$ .  $\square$

**Lemma 10.** For any  $1 \leq i \leq n$ , points in  $C\_clause_i$  can be covered with a set of segments  
 $P\_false_i^a$ , a subset of  $P\_clause_i$  such that  $|P\_false_i^a| = 12$ .

*Proof.* Using Lemma 8 twice we can cover  $or\_gadget_{i,0}$  and  $or\_gadget_{i,1}$  with 8 segments.

To cover the remaining points we additionally use:  $\{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1}), (t_{i,1}, v_{i,1})\}$   
 $\square$

**Lemma 11.** For any  $1 \leq i \leq n$ :

(1) points in  $C\_clause_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$  can not be covered using any subset of segments  
from  $P\_clause_i$  of size less than 11;

(2) points in  $C\_clause_i$  can not be covered using any subset of segments from  $P\_clause_i$   
of size less than 12.

*Proof of no cover with fewer than 12 segments.* There is independent set of 12 points in  $C\_clause_i \supseteq$   
 $\{x_{i,0}, y_{i,0}, z_{i,0}, l_{i,0}, p_{i,0}, q_{i,0}, u_{i,0}, v_{i,0} = l_{i,1}, p_{i,1}, q_{i,1}, u_{i,1}, v_{i,1}\}$ .  $\square$

*Proof of no cover with fewer than 11 segments.* We can choose disjoint sets  $X, Y, Z$  such that  
 $X \cup Y \cup Z \subseteq C\_clause_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$  and there are no segments covering points from  
different sets. And we prove lower bounds for each of these sets.

$$X = \{x_{i,1}, y_{i,1}, z_{i,1}\}$$

Set  $X$  is an indendent set, so it must be covered with 3 segments.

$$Y = or\_gadget_{i,0} - \{l_{i,0}, p_{i,0}\}$$

$$Z = or\_gadget_{i,1} - \{l_{i,1}, p_{i,1}\}$$

For both  $Y$  and  $Z$  we can check all of the subsets of 3 segments with brutforce that none  
of them cover, so they have to be covered with 4 segments.

TODO: Funny fact, neither  $Y$  nor  $Z$  doesn't have independent set of size 4.

Therefore  $C\_clause_i$  must be covered with at least  $3 + 4 + 4 = 11$  segments.  $\square$

### 3.2.2.4. Summary

Add some smart lemmas that sets will be exclusive to each other.

**Lemma 12. Robustness to 1/2-extensions.** For every segment  $s \in \mathcal{P}$ ,  $s$  and  $s^{+1/2}$  cover  
the same points from  $\mathcal{C}$ .

*Proof.* We can just check every segment. Most of the segments  $s$  are colinear only with points  
that lay on  $s$ , so trivially  $s^{+1/2}$  cannot cover more points than  $s$  does.

TODO: list problematic segments here  $\square$



Figure 3.4: **General schema.**

General layout of VARIABLE-gadget and CLAUSE-gadget and how they interact with each other.

TODO: Rename Choose X to VARIABLE-gadget and Clause C to CLAUSE-gadget.

### 372 3.2.2.5. Summary of construction

We define:

$$\begin{aligned}\mathcal{C} &:= \bigcup_{1 \leq i \leq n} C\_variable_i \cup C\_clause_i \\ \mathcal{P} &:= \bigcup_{1 \leq i \leq n} P\_variable_i \cup P\_clause_i\end{aligned}$$

373 The subsequent sections define these sets.

374 We prove some properties of different gadgets. Every segment for a gadget will only cover  
375 points in this gadget (won't interact with any different gadget), so we can prove lemmas *locally*.

376 TODO:  $y$  axis is increasing values downward on figures (not upwards like in normal).

### 377 3.2.3. Construction lemmas and proof of Lemma 3

378 **Lemma 13.** *Given an instance  $S$  of MAX-(3,3)-SAT of size  $n$  with optimum solution satis-*  
379 *fying  $k$  clauses  $opt(S) = k$ . Instance of geometric set cover, constructed for  $S$  as described in*  
380 *Section 3.2.2, can be solved with a solution of size  $15n - k$ .*

381 *Proof.* Let us name the assignments of the variables in the optimum solution of an instance  
382  $S$  as  $y_1, y_2 \dots y_n$  and clauses as  $c_1, c_2 \dots c_n$ .

383 We cover every VARIABLE-gadget with solution described in Lemma 4, in the  $i$ -th gadget  
384 choosing the set of segments corresponding to the value of  $y_i$ . CLAUSE-gadgets that are  
385 satisfied, let us name the variable that is true in them  $a$ , are covered with set  $P\_true_i^a$   
386 described in Lemma 9 and unsatisfied with set  $P\_false_i$  described in Lemma 10.

$$\begin{aligned}R_i &= \begin{cases} X_i^{true} & \text{if } y_i \\ X_i^{false} & \text{if } \neg y_i \end{cases} \\ C_i &= \begin{cases} P\_true_i^a & \text{if } c_i \text{ satisfied} \\ P\_false_i & \text{if } c_i \text{ not satisfied} \end{cases} \\ \mathcal{R} &= \bigcup_{i=1}^n \{R_i \cup C_i : 1 \leq i \leq n\}\end{aligned}$$

387 This set covers all points from  $\mathcal{C}$ , because the smaller sets individually cover their corre-  
388 sponding gadgets (proved in respective lemmas).

389 All of these sets are disjoint, so the size of the solution is:

$$|\mathcal{R}| = \sum_{i=1}^n R_i + \sum_{i=1}^n C_i = 3n + 11k + 12(n - k) = 15n - k.$$

390

□

391 **Lemma 14.** *Given an instance  $S$  of MAX-(3,3)-SAT of size  $n$  and a solution of an instance*  
392 *of geometric set cover, as described in Section 3.2.2, that is of size  $w$ . There exists a solution*  
393 *of an instance  $S$  that satisfied at least  $15n - w$  clauses.*

394 *Proof.*

395 Given a solution  $\mathcal{R}$  of the instance of geometric set cover, we construct a solution of the  
396 instance  $S$  by constructing an assignment of variables that satisfies at least  $15n - w$  clauses in  
397  $S$ .

**Variables** We need to use at least 3 segments to cover VARIABLE-gadget (Lemma 5).  
 If we have chosen both segments  $(c_i, g_i)$  and  $(f_i, h_i)$ , then we have used at least 4 segments  
 (Lemma 6).

$$\begin{cases} |C_{var}^i \cap \mathcal{R}| \geq 4 & \text{if } (c_i, g_i) \in \mathcal{R} \wedge (f_i, h_i) \in \mathcal{R} \\ |C_{var}^i \cap \mathcal{R}| \geq 3 & \text{otherwise} \end{cases}$$

If we chose at most one of the segments  $(c_i, g_i)$  and  $(f_i, h_i)$ , choose the corresponding  
 variable value to the solution. If we chose both segments, choose the value that appears in  
 most clauses. Every variable is in exactly 3 clauses, so one value appears in at least 2 of them.  
 If we have chosen none of the segments, set value to false. Formally, we define the value of  
 the  $x_i$  variable as follows:

$$\begin{cases} x_i = \text{majority}(X_i) & \text{if } (c_i, g_i) \in \mathcal{R} \wedge (f_i, h_i) \in \mathcal{R} \\ x_i = \text{true} & \text{if } (c_i, g_i) \in \mathcal{R} \\ x_i = \text{false} & \text{if } (f_i, h_i) \in \mathcal{R} \\ x_i = \text{false} & \text{otherwise} \end{cases} \quad (3.1)$$

TODO: Maybe remove section below, because we do this calculation at the end anyway  
 To cover  $\bigcup_{1 \leq i \leq n} C_{var}^i$  we have used at least  $3n + a$  segments, where  $a$  is the number of  $i$  such  
 that we have chosen both values  $(c_i, g_i)$  and  $(f_i, h_i)$ .

**Clauses** For a clause  $C_i = x \vee y \vee z$ , we need to use at least 11 segments to cover  
 $C\_clause_i - \{x, y, z\}$  in CLAUSE-gadget (Lemma 11).

TODO: maybe put something with cases and names of sets as above

Moreover, if all of the points  $\{x_{i,0}, y_{i,0}, z_{i,0}\}$  are not covered by the segments from  $P_{var}^i$ ,  
 then we need to cover  $C\_clause_i$  with at least 12 segments by Lemma 11.

TODO: Maybe remove section below, because we do this calculation at the end anyway

We covered CLAUSE-gadget with at least 11 or at least 12 segments:  $|\bigcup_{i=1}^n P\_clause_i \cap \mathcal{R}| \geq$   
 $11n + b$ , where  $b$  is the number of clauses where none of the segments covering the points  
 $x_{i,0}, y_{i,0}, z_{i,0}$  were chosen in  $P_{var}^j$ .

**Satisfied clauses with chosen variables assignment** Clauses for which none of the  
 points  $x_{i,0}, y_{i,0}, z_{i,0}$  were covered by segments in  $P_{var}^j$ , are not satisfied in our variables assign-  
 ment, but not all clauses that cover one of these points with segment in  $P_{var}^j$  are satisfied.

Let us look at such clause  $C_i$  and of points  $x_{i,0}, y_{i,0}, z_{i,0}$  that are covered in  $P_{var}^j$ . Consider  
 the cases of choosing variable value in equation (3.1).

If only one of the segments  $(c_i, g_i)$  and  $(f_i, h_i)$  are chosen in  $P_{var}^j$ , then the value of  $x_j$  is  
 the same as the one satisfying  $C_i$  and clause is satisfied.

If we chose neither  $(c_i, g_i)$  or  $(f_i, h_i)$ , then it is impossible that this point is covered in  
 $P_{var}^j$ .

If we chose both  $(c_i, g_i)$  and  $(f_i, h_i)$ , then there are 3 clauses for which this point is covered  
 by  $P_{var}^j$ . We chose variable value in a way that only one clause using  $x_j$  is not satisfied by  
 the value of  $x_j$ . Therefore there are at most  $a$  clauses that are covered with 11 segments from  
 CLAUSE-gadget, but are not satisfied.

So in the solution to this MAX-(3,3)-SAT instance that we have shown, there are at most  
 $a + b$  unsatisfied clauses.

433 **Conclusions** We proved that given a solution of size  $w$  we have the variables assignment  
 434 that satisfies at least  $n - (a + b)$  clauses of  $S$ . At last we prove that  $n - (a + b) \geq 15n - w$ .

$$w \geq 3(n - a) + 4a + 11(n - b) + 12b = 3n + a + 11n + b = 14n + a + b$$

$$15n - w \leq 15n - 14n - a - b = n - (a + b)$$

435

□

436 *Proof of Lemma 3.* Given an instance  $S$  of MAX-(3,3)-SAT of size  $n$  with optimum solution  
 437 satisfying  $k$  clauses. Let us construct an instance of geometric set cover for  $S$  as described in  
 438 Section 3.2.2 and name it  $I$ .

Given the Lemma 13, we know that there exists a solution of  $I$  of size  $15n - k$ , so:

$$\text{opt}(I) \leq 15n - k.$$

Since the optimum solution of  $S$  satisfies  $k$  clauses, then according to Lemma 14:

$$\text{opt}(I) \geq 15n - k.$$

439 Therefore solution from Lemma 13 of size  $15n - k$  is an optimum solution for instance  
 440  $I$ . □

### 441 3.3. Weighted segments

#### 442 3.3.1. FPT for weighted segments with $\delta$ -extensions

443 **Theorem 4** (FPT for weighted segment cover with  $\delta$ -extensions). *There exists an algorithm*  
 444  *$\mathcal{A}$  that given a family  $\mathcal{P}$  of  $n$  weighted segments (in any direction), a set of  $m$  points  $\mathcal{C}$  and*  
 445 *a parameter  $k$ , runs in time  $f(k) \cdot (nm)^c$  for some computable function  $f$  and a constant  $c$ ,*  
 446 *and outputs a set  $\mathcal{R} \subseteq \mathcal{P}$  such that  $|\mathcal{R}| \leq k$  and  $\mathcal{R}^{+\delta}$  covers all points in  $\mathcal{C}$ .*

447 To solve this problem we will introduce a lemma about choosing *good* subset of points.

448 **Definition 9.** For a segment  $s$  and a set of points  $C$  that lay on  $s$ , a set  $A \subseteq C$  is  $(k, \delta)$ -**good**  
 449 if any set of segments  $R$  that covers set  $A$  such that  $|R| \leq k$ , it holds  $R^{+\delta}$  covers  $C$ .

450 **Lemma 15.** *For any segment  $s$ , set of points  $C$ ,  $\delta > 0$  and  $k \geq 1$ , there exists a  $(k, \delta)$ -good*  
 451 *set of size at most  $f(k, \delta)$  for some computable function  $f$ .*

452 *Proof.* We prove this for a fixed  $\delta < 1$  by induction over  $k$  for any segment  $s$  and set of points  
 453  $C$ .

454 **Inductive hypothesis** For any segment  $s$  and a set of points  $C$  that lay on  $s$ , there  
 455 exists a set  $A$ , that is  $(l, \delta)$ -good for every  $1 \leq l \leq k$  and  $|A| < f(\delta, k)$  for some computable  
 456 function  $f$ .

457 Set  $A$  also includes ends of segment  $s$  for convinience.

458 **Base case for  $k = 1$**  It is sufficient that  $A$  consists of 2 points, end of segment  $s$ .

459 If they are covered with one segment, it must be a segment that includes  $s$ , so it covers  
 460 set  $C$ .

**Inductive step** Assuming inductive hypothesis for any  $s$  and any set of points  $C$  that lay on  $s$  and for  $k$ , we will prove hypothesis for  $k + 1$ .

We define  $\lceil 3/\delta \rceil$  subsegments of  $s$  in the following way. We split  $s$  into  $\lceil 3/\delta \rceil$  segments  $v_i$  such that  $|v_i| = \frac{|s|\delta}{3}$ . (pewnie sufit tutaj)

$C_i$  is a subset of  $C$  such that they lay on  $v_i$ .

$t_i$  is a segment connecting leftmost and rightmost point in  $C_i$  (it might be degenerated segment if  $|C_i| = 1$  or it might be empty if  $C_i$  is empty).

TODO: Add a picture with  $v_i$  and  $t_i$  here

We use inductive hypothesis to choose  $(k, \delta)$ -good sets  $A_i$  for segments  $t_i$  and sets  $C_i$ . If  $|C_i| \leq 1$ , then  $A_i = C_i$  and it's still a  $(k, \delta)$ -good set.

Then we define  $A = \bigcup_{i=1}^{\lceil 3/\delta \rceil} A_i$ . It includes ends of  $s$ , because they are in sets  $A_1$  and  $A_{\lceil 3/\delta \rceil}$ .

**Proof that  $A$  is  $(k, \delta)$ -good for  $s$  and  $C$**  Let us take any cover of  $A$  with  $k+1$  segments and name it  $\mathcal{R}$ .

For every segment  $t_i$ , if there exists a segment  $x$  from  $\mathcal{R}$  such that it is disjoint with  $t_i$ , then we have a cover of  $A_i$  with at most  $k$  segments using  $\mathcal{R} - \{x\}$ . Since  $A_i$  is  $(k, \delta)$ -good for  $t_i$  and  $C_i$ , then  $(\mathcal{R} - \{x\})^{+\delta}$  covers  $C_i$ .

If there doesn't exist such segment, then there exists a segments  $t_i$  such that all  $k+1$  segments that cover  $A_i$  intersect with  $t_i$ . (Note: There exists only one such segment  $t_i$ ). By definition ends of  $s$  are in  $A_i$ , so there must exist segments starting in the ends of  $s$  and ending somewhere in  $t_i$ . Let us name these two segments  $y$  and  $z$ . It follows that:  $|y| + |z| + |t_i| \leq |s|$ . Since  $|t_i| \leq |v_i| = \frac{|s|\delta}{3}$ , therefore  $\max(|y|, |z|) > |s|(1 - \frac{\delta}{3})/2$ .

TODO: Add a picture with such segments here

After  $\delta$ -extension, the longer of these segments will lengthen both ways by at least:

$$|s|(1 - \frac{\delta}{3})/2\delta = |s|\delta(1/2 - \delta/6) > |s|\delta(1/2 - 1/6) = \frac{|s|\delta}{3}$$

.

Therefore the longer of segments  $y$  and  $z$  will cover all segments  $t_i$  after  $\delta$ -extension, therefore  $\mathcal{R}^{+\delta}$  covers  $C_i$ .

Since  $C = \bigcup_{i=1}^{\lceil 3/\delta \rceil} C_i$ , then  $\mathcal{R}^{+\delta}$  covers  $C$ .

□

**Lemma 16. Kernel in WSCPS.** TODO: formulate it properly

For segment cover, there is a kernel of size  $f(k)$  in WSCPS.

**Claim 1.** If there are more than  $k$  lines with at least  $k+1$  points on them, then they can't be covered with  $k$  segments.

**Claim 2.** If there is more than  $k^2$  points that don't lie on any line with more than  $k$  points on it, then they can't be covered with  $k$  segments.

**Claim 3.** For every long line  $L$  (with more than  $k$  points on them) we can choose  $f(k)$  points on them, that if we cover all of these points with at most  $k$  segments, then the rest of the points with  $\delta$ -extensions will be covered by segments in the direction of line  $L$ .



498 **Proof of Lemma 16.** After applying the previous lemmas, we have at most  $k^2 + k \cdot f(k)$   
 499 points that can be covered in any direction and for the rest of the points we can draw at most  
 500  $k \cdot f(k)$  segments along their respective long lines that have to be covered by segments after  
 501  $\delta$ -extensions.

502 Then we extend every available segment by  $\delta$ -extension and we achieve the kernel in  
 503 WSCPS for this instance of problem.

504 **Lemma 17.** *If all the points are covered with  $k$  segments and the biggest  $2(1+1/\delta)^{k+1}$  spaces*  
 505 *between points are filled, the whole segment is filled after  $\delta$ -extensions of these segments.*

506 **Proof.** Let's name the  $2(1+1/\delta)^{k+1}$ -st biggest space between points as  $y$ . We have guarantee  
 507 that all segments of length  $x > y$  are covered without  $\delta$ -extensions.

508 Let's take one space between points that is not covered before  $\delta$ -extension and we will  
 509 prove it will be covered after  $\delta$ -extensions. Let's assume it isn't.

510 This space has length  $x$ . Since it's uncovered,  $x \leq y$ .

511 Let's take side where the sum of lengths of segments covering the points is greater (left or  
 512 right). Without loss of generality, let us assume it's right.

513 There are at most  $k$  segments to the right of this space between points. Name their lengths  
 514  $l_1, l_2 \dots l_k$ . If the point is covered in the other direction, the segment is degenerated to the  
 515 point and  $l_i = 0$ . Name the space between endpoints of  $l_i$  and  $l_{i+1} - x_i$ . Of course,  $x_i$  is  
 516 uncovered space between two points, therefore  $x_i \leq y$ .

517 TUTAJ BEDZIE PEWNIE RYSUNEK Z TYMI SUPER RZECZAMI DO PRZERW

518 Let's write equations meaning that  $i$ -th segment doesn't cover space  $x$  after  $\delta$ -expansion.

$$l_1\delta < x \leq y \Rightarrow l_1 < y/\delta$$

$$l_2\delta < x + l_1 + x_1 < 2y + y/\delta \Rightarrow l_2 < 2y/\delta + y/\delta^2$$

$$l_3\delta < x + l_1 + x_1 + l_2 + x_2 < 3y + 3y/\delta + y/\delta^2 \Rightarrow l_3 < 3y/\delta + 3y/\delta^2 + y/\delta^3$$

519 From this we can "guess" induction  $l_i < y((1+1/\delta)^i - 1)$

520 Trivially for  $l_1 < y/\delta$ .

Assume that for all  $j < i$ :

$$l_j < y((1+1/\delta)^j - 1)$$

521 .

$$\begin{aligned} 522 \quad l_i\delta &< x + \sum_{j=1}^{i-1} (l_j + x_j) < iy \sum_{j=1}^{i-1} l_j < iy + \sum_{j=1}^{i-1} j = 1^{i-1} y((1+1/\delta)^j - 1) = iy - (i - \\ 523 \quad 1)y + \sum_{j=1}^{i-1} j &= 1^{i-1} y(1+1/\delta)^j = y(1 + \sum_{j=1}^{i-1} (1+1/\delta)^j) = y(2 + \sum_{j=1}^{i-1} (1+1/\delta)^j - 1) = \\ 524 \quad y(\sum_{j=0}^{i-1} (1+1/\delta)^j - 1) &= y((1+1/\delta)^i / (1 - (1+1/\delta)) - 1) = y((1+1/\delta)^i \delta - 1) < y((1+1/\delta)^i \delta - \delta) \end{aligned}$$

Of course we also know that (since we have chosen the side with greater sum of the width of segments):

$$\sum_{i=1}^k l_i \geq 1/2 \cdot y \cdot 2(1+1/\delta)^{k+1} = y \cdot (1+1/\delta)^{k+1}$$

525 But  $\sum_{i=1}^k l_i < \sum_{i=1}^k y((1+1/\delta)^i - 1) = y((1+1/\delta)^{k+1} / (1 - (1+1/\delta)) - k) = y((1 +$   
 526  $1/\delta)^{k+1} \delta - k) < y(1+1/\delta)^{k+1}$

527 Therefore the space must have been covered after  $\delta$ -expansions.

### 3.3.2. $W[1]$ -completeness for weighted segments in 3 directions

**Theorem 5.**  *$W[1]$ -completeness for weighted segments in 3 directions.* Consider the problem of covering a set  $\mathcal{C}$  of points by selecting  $k$  axis-parallel or right-diagonal weighted segments with weights from a set  $\mathcal{P}$  with minimal weight. Assuming  $ETH$ , there is no algorithm for this problem with running time  $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{o(\sqrt{k})}$  for any computable function  $f$ .

We will show reduction from grid tiling problem.

Let's have an instance of grid tiling problem – size of the grid  $k$ , number of elements available  $n$  and  $k^2$  sets of available pairs in every tile  $S_{i,j} \subseteq \{1, n\} \times \{1, n\}$ .

**Construction.** We construct a set  $\mathcal{P}$  of segments and a set  $\mathcal{C}$  of points.

First let's choose any ordering of  $n^2$  elements  $\{1, n\} \times \{1, n\}$  and name this sequence  $a_1 \dots a_{n^2}$ .

$$match_v(i, j) \iff a_i = \{x_i, y_i\} \wedge a_j = \{x_j, y_j\} \wedge x_i = x_j$$

$$match_h(i, j) \iff a_i = \{x_i, y_i\} \wedge a_j = \{x_j, y_j\} \wedge y_i = y_j$$

**Points.** Define points:

$$h_{i,j,t} = (j \cdot (n^2 + 1) + t, (n^2 + 1) \cdot i)$$

$$v_{i,j,t} = ((n^2 + 1) \cdot i, j \cdot (n^2 + 1) + t)$$

Let's define sets  $H$  and  $V$  as:

$$H = \{h_{i,j,t} : 1 \leq i, j \leq k, 1 \leq t \leq n^2\}$$

$$V = \{v_{i,j,t} : 1 \leq i, j \leq k, 1 \leq t \leq n^2\}$$

Let's define  $\epsilon = 0.1$ . For a point  $\{x, y\} = p$  we define points  $p^L = \{x - \epsilon, y\}$ ,  $p^R = \{x + \epsilon, y\}$ ,  $p^U = \{x, y - \epsilon\}$ , and  $p^D = \{x, y + \epsilon\}$ .

Then we define:

$$\mathcal{C} := H \cup \{p^L : p \in H\} \cup \{p^R : p \in H\} \cup V \cup \{p^U : p \in V\} \cup \{p^D : p \in V\}$$

**Segments.** Define horizontal segments.

$$hor_{i,j,t_1,t_2} = (h_{i,j,t_1}^R, h_{i,j+1,t_2}^L)$$

$$ver_{i,j,t_1,t_2} = (v_{i,j,t_1}^D, v_{i,j+1,t_2}^U)$$

$$horbeg_{i,t} = (h_{i,1,1}^L, h_{i,1,t}^L)$$

$$horend_{i,t} = (h_{i,n,t}^R, h_{i,n,n^2}^R)$$

$$verbeg_{i,t} = (v_{i,1,1}^U, v_{i,1,t}^U)$$

$$verend_{i,t} = (v_{i,n,t}^D, v_{i,n,n^2}^D)$$

$$\begin{aligned}
HOR &= \{hor_{i,j,t_1,t_2} : 1 \leq i \leq k, 1 \leq j < k, 1 \leq t_1, t_2 \leq n^2, match_h(t_1, t_2)\} \\
&\cup \{horbeg_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\} \\
&\cup \{horend_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\}
\end{aligned}$$

$$\begin{aligned}
VER &= \{ver_{i,j,t_1,t_2} : 1 \leq i \leq k, 1 \leq j < k, 1 \leq t_1, t_2 \leq n^2, match_v(t_1, t_2)\} \\
&\cup \{verbeg_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\} \\
&\cup \{verend_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\}
\end{aligned}$$

$$DIAG := \{(h_{i,j,t}, v_{j,i,t}) : 1 \leq i, j \leq k, 1 \leq t \leq n^2, a_t \in S_{i,j}\}$$

542 TODO: explain that these segments are in fact diagonal

$$\mathcal{P} := HOR \cup VER \cup DIAG$$

543 **Lemma 18.** *If there exists solution for grid tiling, then there exists solution for our construc-*  
544 *tion using  $2(k+1)k + k^2$  segments with weight exactly  $2k \cdot (k(n^2 + 1) - 2 - 2\epsilon(k-1))$ .*

**Claim 4.** *If there exists a solution to the grid tiling  $c_1 \dots c_k$  and  $r_1 \dots r_k$ , then there exists a solution covering all points*

$$\{h_{i,j,t} : 1 \leq i, j \leq k, t = (c_i, r_j)\} \cup \{v_{i,j,t} : 1 \leq i, j \leq k, t = (c_j, r_i)\}$$

545 *with segments in DIAG and the rest in VER or HOR and has weight  $2k \cdot (k(n^2 + 1) -$*   
546  *$2 - 2\epsilon(k-1))$ .*

547 **Proof.** TODO: jakiś prosty z definicji

548 **Lemma 19.** *If there exists solution for our construction using  $2(k+1)k + k^2$  segments with*  
549 *weight exactly  $2k \cdot (k(n^2 + 1) - 2 - 2\epsilon(k-1))$ , then there exists a solution for grid tiling*

550 **Proof.** This follows from Lemma 20, because we just take which points are covered with  
551 *DIAG.*

552 **Claim 5.** *Points  $p^L, p^R, p^U, p^D$  cannot be covered with DIAG.*

553 **Claim 6.** *Points in  $H \cup \{p^L : p \in H\} \cup \{p^R : p \in H\}$  cannot be covered with VER.*

554 *Points in  $V \cup \{p^U : p \in V\} \cup \{p^D : p \in V\}$  cannot be covered with HOR.*

555 **Claim 7.** *For given  $i, j$  if none of the points  $h_{i,j,t}$  ( $v_{i,j,t}$ ) for  $1 \leq t \leq n^2$  are covered with*  
556 *DIAG, then some spaces between neighbouring points were covered twice.*

557 **Claim 8.** *For given  $i, j$  two points  $h_{i,j,t_1}, h_{i,j,t_2}$  ( $v_{i,j,t_1}, v_{i,j,t_2}$ ) for  $1 \leq t_1 < t_2 \leq n^2$  are covered*  
558 *with DIAG, then one of them had to be also covered with a segment from HOR (VER).*

559 **Proof.** Point  $v_{i,j,t_2}^L$  had to be covered with VER from Claims 5 and 6. And every segment  
560 in VER covering  $v_{i,j,t_2}^L$ , covers also  $v_{i,j,t_1}^L$ .

561 **Lemma 20.** *If there exists solution for our construction with weight at most (exactly)  $2k \cdot$*   
562  *$(k(n^2 + 1) - 2 - 2\epsilon(k-1))$ , then for every  $i, j$  there must be exactly one  $t$  such that  $h_{i,j,t}$  ( $v_{i,j,t}$ )*  
563 *is covered with DIAG and moreover if  $h_{i,j,t_1}$  and  $h_{i,j+1,t_2}$  are uncovered, then  $match_h(t_1, t_2)$ .*  
564 *Analogically for  $v$ .*

565 **Proof.** Only  $k^2$  points can be covered only in *DIAG*, the rest has to be covered with  
566  $VER \cup HOR$ . Therefore every result must be at least  $ALL\_LINES - 2k^2\epsilon$ , because only  
567  $2k^2$  spaces of length  $\epsilon$  can be uncovered in this axis.

568 Of course if  $h_{i,j,t_1}$  and  $h_{i,j+1,t_2}$  are uncovered, then there must exist a segment in *HOR*  
569 between  $h_{i,j,t_1}^R$  and  $h_{i,j+1,t_2}^L$ , so  $math_h(t_1, t_2)$  must be true.

### 570 3.3.3. What is missing

571 We don't know FPT for axis-parallel segments without  $\delta$ -extensions.

## 572 Chapter 4

# 573 Geometric Set Cover with lines

### 574 4.1. Lines parallel to one of the axis

575 When  $\mathcal{R}$  consists only of lines parallel to one of the axis, the problem can be solved in  
576 polynomial time.

577 We create bipartial graph  $G$  with node for every line on the input split into sets:  $H$  –  
578 horizontal lines and  $V$  – vertical lines. If any two lines cover the same point from  $\mathcal{C}$ , then we  
579 add edge between them.

580 Of course there will be no edges between nodes inside  $H$ , because all of them are pararell  
581 and if they share one point, they are the same lines. Similar argument for  $V$ . So the graph is  
582 bipartial.

583 Now Geometric Set Cover can be solved with Vertex Cover on graph  $G$ . Since Vertex  
584 Cover (even in weighted setting) on bipartial graphs can be solved in polynomial time.

585 Short note for myself just to remember how to this in polynomial time:

586 Non-weighted setting - Konig theorem + max matching

587 Weighted setting - Min cut in graph of  $\neg A$  or  $\neg B$  (edges directed from  $V$  to  $H$ )

### 588 4.2. FPT for arbitrary lines

589 You can find this is Platypus book. We will show FPT kernel of size at most  $k^2$ .

590 (Maybe we need to reduce lines with one point/points with one line).

591 For every line if there is more than  $k$  points on it, you have to take it. At the end, if there  
592 is more than  $k^2$  points, return NO. Otherwise there is no more than  $k^4$  lines.

593 In weighted settings among the same lines with different weights you leave the cheapest  
594 one and use the same algorithm.

### 595 4.3. APX-completeness for arbitrary lines

596 We will show a reduction from Vertex Cover problem. Let's take an instance of the Vertex  
597 Cover problem for graph  $G$ . We will create a set of  $|V(G)|$  pairwise non-pararell lines, such  
598 that no three of them share a common point.

599 Then for every edge in  $(v, w) \in E(G)$  we put a point on crossing of lines for vertices  $v$   
600 and  $w$ . They are not pararell, so there exists exactly one such point and any other line don't  
601 cover this point (any three of them don't cross in the same point).

Solution of Geometric Set Cover for this instance would yield a sound solution of Vertex Cover for graph  $G$ . For every point (edge) we need to choose at least one of lines (vertices)  $v$  or  $w$  to cover this point.

Vertex Cover for arbitrary graph is APX-complete, so this problem is also APX-complete.

#### 4.4. 2-approximation for arbitrary lines

Vertex Cover has an easy 2-approximation algorithm, but here very many lines can cross through the same point, so we can do  $d$ -approximation, where  $d$  is the biggest number of lines crossing through the same point. So for set where any 3 lines don't cross in the same point it yields 2-approximation.

The problematic cases are where through all points cross at least  $k$  points and all lines have at least  $k$  points on them. It can be created by casting  $k$ -grid in  $k$ -D space on 2D space.

Greedy algorithm yields  $\log |\mathcal{R}|$ -approximation, but I have example for this for bipartial graph and reduction with taking all lines crossing through some point (if there are no more than  $k$ ) would solve this case. So maybe it works.

Unfortunately I haven't done this :(

I can link some papers telling it's hard to do.

#### 4.5. Connection with general set cover

Problem with finite set of lines with more dimensions is equivalent to problem in 2D, because we can project lines on the plane which is not perpendicular to any plane created by pairs of (point from  $\mathcal{C}$ , line from  $\mathcal{P}$ ).

Of course every two lines have at most one common point, so is every family of sets that have at most one point in common equivalent to some geometric set cover with lines?

No, because of Desargues's theorem. Have to write down exactly what configuration is banned.

## Chapter 5

# Geometric Set Cover with polygons

### 5.1. State of the art

Covering points with weighted discs admits PTAS [Li and Jin, 2015] and with fat polygons with  $\delta$ -extensions with unit weights admits EPTAS [Har-Peled and Lee, 2009].

Although with thin objects, even if we allow  $\delta$ -expansion, the Set Cover with rectangles is APX-complete (for  $\delta = 1/2$ ), it follows from APX-completeness for segments with  $\delta$ -expansion in Section 3.2.

Covering points with squares is W[1]-hard [Marx, 2005]. It can be proven that assuming *SETH*, there is no  $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{k-\epsilon}$  time algorithm for any computable function  $f$  and  $\epsilon > 0$  that decides if there are  $k$  polygons in  $\mathcal{P}$  that together cover  $\mathcal{C}$ , *Theorem 1.9* in [Marx and Pilipczuk, 2015].





<sup>638</sup> Chapter 6

<sup>639</sup> Conclusions



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