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# Approximation and Parametrized Algorithms for Segment Set Cover

6

Master's thesis

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in COMPUTER SCIENCE

8

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9

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10 **Supervisor's statement**

11 Hereby I confirm that the presented thesis was prepared under my supervision and  
12 that it fulfils the requirements for the degree of Master of Computer Science.

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## **Abstract**

23 The work presents a study of different geometric set cover problems. It mostly focuses on  
24 segment set cover and its connection to the polygon set cover.

25

## **Keywords**

26 set cover, geometric set cover, FPT,  $W[1]$ -completeness, APX-completeness, PCP theorem,  
27 NP-completeness

28

## **Thesis domain (Socrates-Erasmus subject area codes)**

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31

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33 D.127. Blabalgorithms

34 D.127.6. Numerical blabalysis

35

## **Tytuł pracy w języku polskim**

36 Algorytmy parametryzowania i trudność aproksymacji problemu pokrywania zbiorów  
37 odcinkami na płaszczyźnie



# Contents

39	<b>1. Introduction</b>	5
40	<b>2. Definitions</b>	7
41	2.1. Geometric Set Cover	7
42	2.2. Approximation	7
43	2.3. $\delta$ -extensions	7
44	<b>3. Geometric Set Cover with segments</b>	9
45	3.1. FPT for segments	9
46	3.1.1. Axis-parallel segments	9
47	3.1.2. Segments in $d$ directions	9
48	3.1.3. Segments in arbitrary direction	9
49	3.2. APX-completeness for segments parallel to axes	11
50	3.2.1. MAX-(3,3)-SAT and statement of reduction	11
51	3.2.2. Reduction construction	13
52	3.2.2.1. VARIABLE-gadget	13
53	3.2.2.2. OR-gadget	14
54	3.2.2.3. CLAUSE-gadget	15
55	3.2.2.4. Summary	17
56	3.2.3. Summary of construction	17
57	3.2.4. Construction lemmas and proof of Lemma 3.2.1	19
58	3.3. Weighted segments	21
59	3.3.1. FPT for weighted segments with $\delta$ -extensions	21
60	3.3.2. W[1]-completeness for weighted segments in 3 directions	23
61	3.3.3. What is missing	25
62	<b>4. Geometric Set Cover with lines</b>	27
63	4.1. Lines parallel to one of the axis	27
64	4.2. FPT for arbitrary lines	27
65	4.3. APX-completeness for arbitrary lines	27
66	4.4. 2-approximation for arbitrary lines	28
67	4.5. Connection with general set cover	28
68	<b>5. Geometric Set Cover with polygons</b>	29
69	5.1. State of the art	29
70	<b>6. Conclusions</b>	31



# Chapter 1

## Introduction

The Set Cover problem is one of the most common NP-complete problems. [tutaj referencja] We are given a family of sets and have to choose the smallest subfamily of these sets that cover all their elements. This problem naturally extends to settings where we put different weights on the sets and look for the subfamily of the minimal weight. This problem is NP-complete even without weights and if we put restrictions on what the sets can be. One of such variants is Vertex Cover problem, where sets have size 2 (they are edges in a graph).

In this work we focus on another such variant where the sets correspond to some geometric shapes and only some points of the plane have to be covered. When these shapes are rectangles with edges parallel to the axis, the problem can be proven to be W[1]-complete (solution of size  $k$  cannot be found in  $n^o(k)$  time), APX-complete (for sufficiently small  $\epsilon > 0$ , the problem does not admit  $1 + \epsilon$ -approximation scheme) [referencje].

Some of these settings are very easy. Set cover with lines parallel to one of the axis can be solved in polynomial time.

There is a notion of  $\delta$ -expansions, which loosen the restrictions on geometric set cover. We allow the objects to cover the points after  $\delta$ -expansion and compare the result to the original setting. This way we can produce both FPT and EPTAS for the rectangle set cover with  $\delta$ -extensions [referencje].

**Our contribution.** In this work, we prove that unweighted geometric set cover with segments is fixed parameter tractable (FPT).

Moreover, we show that geometric set cover with segments is APX-complete for unweighted axis-parallel segments, even with  $1/2$ -extensions. So the problem for very thin rectangles also can't admit PTAS. Therefore, in the efficient polynomial-time approximation scheme (EPTAS) for *fat polygons* by [Har-Peled and Lee, 2009], the assumption about polygons being fat is necessary.

Finally, we show that geometric set cover with weighted segments in 3 directions is W[1]-complete. However, geometric set cover with weighted segments is FPT if we allow  $\delta$ -extension.

This result is especially interesting, since it's counter-intuitive that the unweighted setting is FPT and the weighted setting is W[1]-complete. Most of such problems (like vertex cover or [wiecej przykladow]) are equally hard in both weighted and unweighted settings.





## Chapter 2

## Definitions

### 2.1. Geometric Set Cover

In the geometric set cover problem we are given  $\mathcal{P}$  – a set of objects, which are connected subsets of the plane,  $\mathcal{C}$  – a set of points in the plane. The task is to choose  $\mathcal{R} \subseteq \mathcal{P}$  such that every point in  $\mathcal{C}$  is inside some element from  $\mathcal{R}$  and  $|\mathcal{R}|$  is minimized.

In the parametrized setting for a given  $k$ , we only look for a solution  $\mathcal{R}$  such that  $|\mathcal{R}| \leq k$ .

In the weighted setting, there is some given weight function  $f : \mathcal{P} \rightarrow \mathbb{R}^+$ , and we would like to find a solution  $\mathcal{R}$  that minimizes  $\sum_{R \in \mathcal{R}} f(R)$ .

### 2.2. Approximation

Let us recall some definitions related to optimization problems that are used in the following sections.

**Definition 2.2.1.** A **polynomial-time approximation scheme (PTAS)** for a minimization problem  $\Pi$  is a family of algorithms  $\mathcal{A}_\epsilon$  for every  $\epsilon > 0$  such that  $\mathcal{A}_\epsilon$  takes an instance  $I$  of  $\Pi$  and in polynomial time finds a solution that is within a factor  $(1 + \epsilon)$  of being optimal. That means the reported solution has weight at most  $(1 + \epsilon)opt(I)$ , where  $opt(I)$  is the weight of an optimal solution for  $I$ .

**Definition 2.2.2.** A problem  $\Pi$  is **APX-hard** if assuming  $P \neq NP$ , there exists  $\epsilon > 0$  such that there is no polynomial-time  $(1 + \epsilon)$ -approximation algorithm for  $\Pi$ .

### 2.3. $\delta$ -extensions

TODO PLACEHOLDER for introductory text

$\delta$ -extensions is one of the modifications to a problem, that makes geometric set cover problem easier, it has been already used in literature (place some refrence here).

**Definition 2.3.1** ( $\delta$ -extensions for center-symmetric objects). For any  $\delta > 0$  and a center-symmetric object  $L$  with centre of symmetry  $S = (x_s, y_s)$ , the  **$\delta$ -extension** of  $L$  is the object  $L^{+\delta} = \{(1 + \delta) \cdot (x - x_s, y - y_s) + (x_s, y_s) : (x, y) \in L\}$ , that is,  $L^{+\delta}$  is the image of  $L$  under homothety centered at  $S$  with scale  $(1 + \delta)$

The geometric set cover problem with  $\delta$ -extensions is a modified version of geometric set cover where:

- We need to cover all the points in  $\mathcal{C}$  with objects from  $\{P^{+\delta} : P \in \mathcal{P}\}$  (which always include no fewer points than the objects before  $\delta$ -extensions);

- We look for a solution that is no larger than the optimum solution for the original problem. Note that it does not need to be an optimal solution in the modified problem.

Formally, we have the following.

**Definition 2.3.2** (Geometric set cover problem with  $\delta$ -extensions). The geometric set cover problem with  $\delta$ -extensions is the problem where for an input instance  $I = (\mathcal{P}, \mathcal{C})$ , the task is to output a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that the  $\delta$ -extended set  $\{R^{+\delta} : R \in \mathcal{R}\}$  covers  $\mathcal{C}$  and is no larger than the optimal solution for the problem without extensions, i.e.  $|\mathcal{R}| \leq |\text{opt}(I)|$ .

TODO: Some text

**Definition 2.3.3** (Geometric set cover PTAS with  $\delta$ -extensions). We define a PTAS for geometric set cover with  $\delta$ -extensions as a family of algorithms  $\{\mathcal{A}_{\delta, \epsilon}\}_{\delta, \epsilon > 0}$  that each takes as an input instance  $I = (\mathcal{P}, \mathcal{C})$ , and in polynomial-time outputs a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that the  $\delta$ -extended set  $\{R^{+\delta} : R \in \mathcal{R}\}$  covers  $\mathcal{C}$  and is within a  $(1 + \epsilon)$  factor of the optimal solution for this problem without extensions, i.e.  $(1 + \epsilon)|\mathcal{R}| \leq |\text{opt}(I)|$ .

## Chapter 3

# Geometric Set Cover with segments

### 3.1. FPT for segments

#### 3.1.1. Axis-parallel segments

You can find this in Platypus book. (TODO add referece)

We show  $\mathcal{O}(2^k)$  branching algorithm. Let us take the point  $K$  which is the smallest under a lexicographic ordering on coordinates among points that are not covered yet. We need to cover  $K$  with some of the remaining segments.

We branch over choice of direction among the 2 axis-parallel directions. In this direction we greedily take the segment that covers the most points. As  $K$  was the smallest in lexicographical order, all points in  $\mathcal{C}$  colinear with  $K$  in both axis-parallel directions are only on one side of  $K$ , because their coordinates are larger. Therefore segments covering  $K$  in this direction create monotone sequence of sets and we can greedily take one segment that covers superset of all of these segments.

TODO: Maybe split it into theorem + algorithm + explanation like in section 3.1.3

#### 3.1.2. Segments in $d$ directions

The same algorithm as described in the previous section, but we branch over  $d$  directions and it runs in complexity  $\mathcal{O}(d^k)$ .

#### 3.1.3. Segments in arbitrary direction

**Theorem 3.1.1. (FPT for segment cover).** *There exists an algorithm that given a family  $\mathcal{P}$  of  $n$  segments (in any direction), a set of  $m$  points  $\mathcal{C}$  and a parameter  $k$ , runs in time  $f(k) \cdot (nm)^c$  for some computable function  $f$  and constant  $c$ , and outputs a subfamily  $\mathcal{R} \subseteq \mathcal{P}$  such that  $|\mathcal{R}| \leq k$  and  $\mathcal{R}$  covers all points in  $\mathcal{C}$  or determines that the solution of size at most  $k$  doesn't exist.*

This theorem is proved by following lemmas.

**Lemma 3.1.1. (Reduction).** *Given a family  $\mathcal{P}$  of  $n$  segments (in any direction) and a set of  $m$  points  $\mathcal{C}$  for segment cover problem, without a loss of generality we can assume that no segment covers a superset of what another segment covers. That is, for any  $A, B \in \mathcal{P}$ , we have  $A \neq B \Rightarrow A \not\subseteq B$ .*

*Proof.* Trivial. □

**Lemma 3.1.2.** *Given an instance of a problem, if there exists a line  $L$  with at least  $k + 1$  points on it, there exists a subset  $\mathcal{A} \subseteq \mathcal{P}$ ,  $|\mathcal{A}| \leq k$ , such that every solution  $\mathcal{R}$  with  $|\mathcal{R}| \leq k$  satisfies  $|\mathcal{A} \cap \mathcal{R}| \geq 1$ .*

*Proof.* First we use Lemma 3.1.1.

Let us name points from  $\mathcal{C}$  that lay on  $L$ ,  $x_1, x_2, \dots, x_t$  in the order they appear on the line.

Every segment that is not colinear with  $L$  can cover at most one of these points. Therefore in any solution of size not larger than  $k$ , among any  $k$  of these points at least one must be covered with segment colinear with  $L$ .

Therefore we need to take one of the segments colinear with  $L$  that covers any of the points  $x_1, x_2, \dots, x_k$ . After using reduction from Lemma 3.1.1, there are at most  $k$  such segments that are distinct. □

*Proof of theorem 3.1.1.*

**Algorithm.** First we use Lemma 3.1.1.

We present a recursive algorithm. Given an instance of the problem:

- (1) If there exist a line with at least  $k + 1$  points, we branch over adding to the solution one of at most  $k$  possible segments from Lemma 3.1.2, name this segment  $S$ . Then we find a solution  $\mathcal{R}$  for problem for points  $\mathcal{C} - S$ , segments  $\mathcal{P} - \{S\}$  and parameter  $k - 1$  and return  $\mathcal{R} \cup \{S\}$ .
- (2) If every line has at most  $k$  points on it and  $|\mathcal{C}| > k^2$ , then answer NO.
- (3) If  $|\mathcal{C}| \leq k^2$ , solve the problem by brute force algorithm.

**Correctness.** Lemma 3.1.2 proves that at least one segment that we branch over in (1) must be present in every solution  $\mathcal{R}$  with  $|\mathcal{R}| \leq k$ , therefore the recursive call can find the optimal solution.

In (2) the answer is no, because every line covers no more than  $k$  points from  $\mathcal{C}$ , which implies that every segment from  $\mathcal{P}$  covers at most  $k$ . Under this assumption we can cover only  $k^2$  points with a solution of size  $k$ , which is less than  $|\mathcal{C}|$ .

Checking all possible solutions in (3) is trivially correct.

**Complexity.** In leaves of branching (3)  $|\mathcal{C}| < k^2$ , so  $|\mathcal{P}| < k^4$ , because every segments can be uniquely identified by 2 extreme points it covers (by Lemma 3.1.1). Therefore there are  $\binom{k^4}{k}$  possible solutions to check, each can be checked in time  $O(k|\mathcal{C}|)$ . Therefore (3) takes time  $O(f(k))$ .

In this branching algorithm our parameter  $k$  is decreased with every recursive call, so we have at most  $k$  levels of recursion with branching over  $k$  possibilities. Candidates to branch over can be found on each level in time  $O(nm \log(nm))$ .

Reduction from Lemma 3.1.1 can be implemented in  $O(n^2m)$ .

Overall complexity is  $O(n^2m + nm \log(nm) \cdot f(k))$  □

## 216 3.2. APX-completeness for segments parallel to axes

217 In this section we analyze whether there exists PTAS for geometric set cover for rectangles.  
 218 We show that we can restrict this problem to a very simple setting: segments parallel to axes  
 219 and allow  $(1/2)$ -extension, and the problem is still APX-hard. Note that segments are just  
 220 degenerated rectangles with one side being very narrow.

221 Our results can be summarized in the following theorem and this section aims to prove it.

222 **Theorem 3.2.1.** *(axis-parallel segment set cover with  $1/2$ -extension is APX-hard).*  
 223 *Unweighted geometric set cover with axis-parallel segments in 2D (even with  $1/2$ -extension)*  
 224 *is APX-hard. That is, assuming  $P \neq NP$ , there does not exist a PTAS for this problem.*

225 Theorem 3.2.1 implies the following.

226 **Corollary 3.2.1.** *(rectangle set cover is APX-hard).* *Unweighted geometric set cover*  
 227 *with rectangles (even with  $1/2$ -extension) is APX-hard.*

228 We prove Theorem 3.2.1 by taking a problem that is APX-hard and showing a reduction.  
 229 For this problem we choose MAX-(3,3)-SAT which we define below.

### 230 3.2.1. MAX-(3,3)-SAT and statement of reduction

231 **Definition 3.2.1.** MAX-3SAT is the following maximization problem. We are given a  
 232 3-CNF formula, and need to find an assignment of variables that satisfies the most clauses.

233 **Definition 3.2.2.** MAX-(3,3)-SAT is a variant of MAX-3SAT with an additional restric-  
 234 tion that every variable appears in exactly 3 clauses. Note that thus, the number of clauses  
 235 is equal to the number of variables.

236 In our proof of Theorem 3.2.1 we use hardness of approximation of MAX-(3,3)-SAT proved  
 237 in [Håstad, 2001] and described in Theorem 3.2.2 below.

238 **Definition 3.2.3** ( $\alpha$ -satisfiable MAX-3SAT formula). MAX-3SAT formula of size  $n$  is at  
 239 most  $\alpha$ -satisfiable, if every assignment of variables satisfies no more than  $\alpha n$  clauses.

240 **Theorem 3.2.2.** [Håstad, 2001]

241 *For any  $\epsilon > 0$ , it is NP-hard to distinguish satisfiable (3,3)-SAT formulas from at most*  
 242  *$(7/8 + \epsilon)$ -satisfiable (3,3)-SAT formulas.*

243 Given an instance  $I$  of MAX-(3,3)-SAT, we construct an instance  $J$  of axis-parallel seg-  
 244 ment set cover problem, such that for a sufficiently small  $\epsilon > 0$ , a polynomial time  $(1 + \epsilon)$ -  
 245 approximation algorithm for  $J$  would be able to distinguish whether an instance  $I$  of MAX-  
 246 (3,3)-SAT is fully satisfiable or is at most  $(7/8 + \epsilon)$ -satisfiable. However, according to (Theorem  
 247 3.2.2) the latter problem is NP-hard. This would imply  $P = NP$ , contradicting the assumption.

248 The following lemma encapsulates the properties of the reduction described in this section,  
 249 and it allows us to prove Theorem 3.2.1.

250 **Lemma 3.2.1.** *Given an instance  $S$  of MAX-(3,3)-SAT with  $n$  variables and optimum value*  
 251  *$opt(S)$ , we can construct an instance  $I$  of geometric set cover with axis-parallel segments in*  
 252 *2D, such that:*

253 (1) *For every solution  $X$  of instance  $I$ , there exists a solution of  $S$  that satisfies at least*  
 254  *$15n - |X|$  clauses.*

(2) For every solution of instance  $S$  that satisfies  $w$  clauses, there exists a solution of  $I$  of size  $15n - w$ .

(3) Every solution with  $1/2$ -extensions of  $I$  is also a solution to the original instance  $I$ .

Therefore, the optimum size of a solution of  $I$  is  $\text{opt}(I) = 15n - \text{opt}(S)$ .

We prove Lemma 3.2.1 in subsequent sections, but meanwhile let us prove Theorem 3.2.1 using Lemma 3.2.1 and Theorem 3.2.2.

TODO: This below can't use current template

*Proof of Theorem 3.2.1.* Consider any  $0 < \epsilon < 1/(15 \cdot 8)$ .

Let us assume that there exists a polynomial-time  $(1 + \epsilon)$ -approximation algorithm for unweighted geometric set cover with axis-parallel segments in 2D with  $(1/2)$ -extensions. We construct an algorithm that solves the problem stated in Theorem 3.2.2, thereby proving that  $P = NP$ .

Take an instance  $S$  of MAX-(3,3)-SAT to be distinguished and construct an instance of geometric set cover  $I$  using Lemma 3.2.1. We now use the  $(1 + \epsilon)$ -approximation algorithm for geometric set cover on  $I$ . Denote the size of the solution returned by this algorithm as  $\text{approx}(I)$ . We prove that if in  $S$  one can satisfy at most  $(\frac{7}{8} + \epsilon)n$  clauses, then  $\text{approx}(I) \geq 15n - (\frac{7}{8} + \epsilon)n$  and if  $S$  is satisfiable, then  $\text{approx}(I) < 15n - (\frac{7}{8} + \epsilon)n$ .

**Assume  $S$  satisfiable.** From the definition of  $S$  being satisfiable, we have:

$$\text{opt}(S) = n.$$

From Lemma 3.2.1 we have:

$$\text{opt}(I) = 14n.$$

Therefore,

$$\begin{aligned} \text{approx}(I) &\leq (1 + \epsilon)\text{opt}(I) = 14n(1 + \epsilon) = 14n + 14\epsilon \cdot n = \\ &= 14n + (15\epsilon - \epsilon)n < 14n + \left(\frac{1}{8} - \epsilon\right)n = 15n - \left(\frac{7}{8} + \epsilon\right)n \end{aligned}$$

**Assume  $S$  is at most  $(\frac{7}{8} + \epsilon)$  satisfiable.** From the definition of  $S$  being at most  $(\frac{7}{8} + \epsilon)n$  satisfiable, we have:

$$\text{opt}(S) \leq \left(\frac{7}{8} + \epsilon\right)n$$

From Lemma 3.2.1 we have:

$$\text{opt}(I) \geq 15n - \left(\frac{7}{8} + \epsilon\right)n$$

Since a solution to  $I$  with  $\frac{1}{2}$ -extensions is also a solution without extensions, by Lemma 3.2.1 (3.), we have:

$$\text{approx}(I) \geq \text{opt}(I) = 15n - \left(\frac{7}{8} + \epsilon\right)n$$

Therefore, by using the assumed  $(1 + \epsilon)$ -approximation algorithm, it is possible to distinguish the case when  $S$  is satisfiable from the case when it is at most  $(\frac{7}{8} + \epsilon)n$  satisfiable, it suffices to compute  $\text{approx}(I)$  with  $15n - (\frac{7}{8} + \epsilon)n$ . Hence, the assumed approximation algorithm cannot exist, unless  $P = NP$ . □

### 3.2.2. Reduction construction

We show reduction from MAX-(3,3)-SAT problem to geometric set cover with segments parallel to axis. Moreover the instance of geometric set cover will be robust to  $1/2$ -extensions (have the same optimal solution after  $1/2$ -extension).

The construction will be composed of 2 types of gadgets: **VARIABLE-gadgets** and **CLAUSE-gadgets**. **CLAUSE-gadgets** would be constructed using two **OR-gadgets** connected together.

#### 3.2.2.1. VARIABLE-gadget

**VARIABLE-gadget** is responsible for choosing the value of a variable in a CNF formula. It allows two minimal solutions and every minimal solution must use exactly one of the  $(c_i, g_i)$  and  $(f_i, h_i)$  segments. These two choices correspond to the two Boolean values of the variable.

**Points.** Define points:

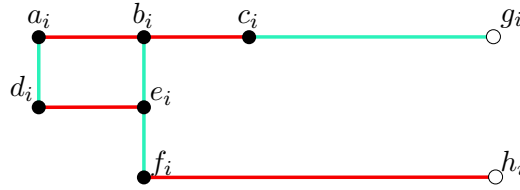


Figure 3.1: **VARIABLE-gadget**. We denote the set of points marked with black circles as  $C_{var}^i$ , and they need to be covered (are part of the set  $\mathcal{C}$ ). Note that some of the points are not marked as black dots and exists only to name segments for further reference. We denote the set of red segments as  $X_{false}^i$  and the set of green segments as  $X_{true}^i$ .

291

292 With  $L = 12n$ :

$$\begin{aligned} a &= (-L, 0) & b &= (-\frac{2}{3}L, 0) & c &= (-\frac{1}{3}L, 0) & d &= (-L, 1) \\ e &= (-\frac{2}{3}L, 1) & f &= (-\frac{2}{3}L, 2) & g &= (L, 0) & h &= (L, 2) \end{aligned}$$

293

Let us define:

$$C_{var} = \{a, b, c, d, e, f\}$$

and

$$C_{var}^i = C_{var} + (0, 4i)$$

294 **Segments.** Let us define:

$$\begin{aligned} X_{true}^i &= \{(a_i, d_i), (b_i, f_i), (c_i, g_i)\} \\ X_{false}^i &= \{(a_i, c_i), (d_i, e_i), (f_i, h_i)\} \end{aligned}$$

$$P_{var}^i = X_{true}^i \cup X_{false}^i$$

295 **Lemma 3.2.2.** For any  $1 \leq i \leq n$ , points  $C_{var}^i$  can be covered using 3 segments from  $P_{var}^i$ .

296 *Proof.* We can use either set  $X_{true}^i$  or  $X_{false}^i$ . □

297 **Lemma 3.2.3.** For any  $1 \leq i \leq n$ , points  $C_{var}^i$  can not be covered with less than 3 segments  
 298 from  $P_{var}^i$ .

299 *Proof.* No segment of  $P_{var}^i$  covers more than one point from  $\{d_i, f_i, c_i\}$ , therefore  $C_{var}^i$  can not  
 300 be covered with less than 3 segments. □

301 **Lemma 3.2.4.** If both segments  $(c_i, g_i)$  and  $(f_i, h_i)$  are chosen, then the covering the remain-  
 302 ing points from  $C_{var}^i$  requires at least 2 different segments from  $P_{var}^i$ .

303 *Proof.* No segment of  $P_{var}^i$  covers more than one point from  $\{a_i, e_i\}$ , therefore  $C_{var}^i - \{c_i, f_i, g_i, h_i\}$   
 304 can not be covered with less than 2 segments. □

### 305 3.2.2.2. OR-gadget

306 OR-gadget has 3 important segments –  $x, y, result$ .  $x$  and  $y$  don't count to the weight of  
 307 solution of OR-gadget (they are part of different gadgets). It has a minimal solution of weight  
 308  $w$  and  $result$  can be chosen only if  $x$  or  $y$  are also chosen for the solution. If none of them  
 309 are chosen, then solution choosing  $result$  segment has weight at least  $w + 1$ . Therefore the  
 310 following formula holds for a solution  $R$  assuming that  $R$  uses only  $w$  from this OR-gadget:

$$(x \in R) \vee (y \in R) \iff result \in R$$

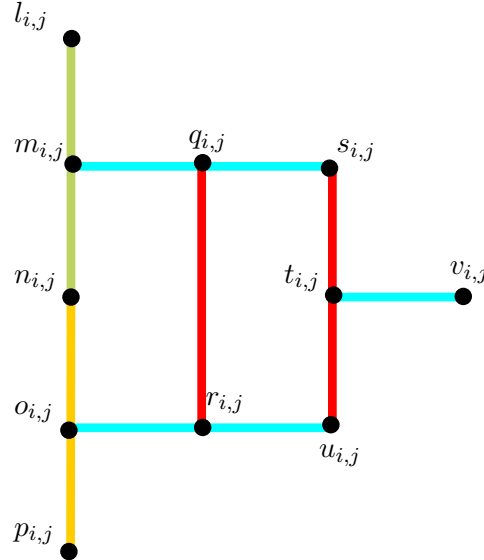


Figure 3.2: **OR-gadget.** We denote these point as  $or\_gadget_{i,j}$ . We denote set of red segments as  $or_{i,j}^{false}$ , set of blue segments as  $or_{i,j}^{true}$ , green and yellow segments as  $or\_move\_variable_{i,j}$ .



311 **Points.**

$$\begin{array}{llll}
l_0 = (0, 0) & m_0 = (0, 1) & n_0 = (0, 2) & o_0 = (0, 3) \\
p_0 = (0, 4) & q_0 = (1, 1) & r_0 = (1, 3) & s_0 = (2, 1) \\
t_0 = (2, 2) & u_0 = (2, 3) & v_0 = (3, 2) & 
\end{array}$$

$$vec_{i,j} = (10i + 3 + 3j, 4n + 2j)$$

313 Define  $\{l_{i,j}, m_{i,j} \dots v_{i,j}\}$  as  $\{l_0, m_0 \dots v_0\}$  shifted by  $vec_{i,j}$

314 Note that  $v_{i,0} = l_{i,1}$  (see Figure 3.3)

$$C\_or\_gadget_{i,j} = \{l_{i,j}, m_{i,j}, n_{i,j}, o_{i,j}, p_{i,j}, q_{i,j}, r_{i,j}, s_{i,j}, t_{i,j}, u_{i,j}\}$$

315 **Segments.** We define names subsets of segments, to refer to them in lemmas.

$$or_{i,j}^{false} = \{(q_{i,j}, r_{i,j}), (s_{i,j}, u_{i,j})\}$$

$$or_{i,j}^{true} = \{(m_{i,j}, s_{i,j}), (o_{i,j}, u_{i,j}), (t_{i,j}, v_{i,j})\}$$

$$or\_move\_variable_{i,j} = \{(l_{i,j}, n_{i,j}), (n_{i,j}, p_{i,j})\}$$

316 Segments in OR-gadget:

$$P\_or\_gadget_{i,j} = or_{i,j}^{false} \cup or_{i,j}^{true} \cup or\_move\_variable_{i,j}$$

317 **Lemma 3.2.5.** For any  $1 \leq i \leq n, j \in \{0, 1\}$  and  $x \in \{l_{i,j}, p_{i,j}\}$  we can cover points in  
318  $C\_or\_gadget_{i,j} - \{x\} \cup \{v_{i,j}\}$  with 4 segments.

319 *Proof.* We can do that using one segment from  $or\_move\_variable_{i,j}$  (chosen depending on  
320 the value of  $x$ ) and all segments from  $or_{i,j}^{true}$ .  $\square$

321 **Lemma 3.2.6.** For any  $1 \leq i \leq n, j \in \{0, 1\}$ , we can cover points in  $C\_or\_gadget_{i,j}$  with 4  
322 segments from  $P\_or\_gadget_{i,j}$ .

323 *Proof.* We can do that using  $or\_move\_variable_{i,j}$  and  $or_{i,j}^{false}$ .  $\square$

### 324 3.2.2.3. CLAUSE-gadget

325 CLAUSE-gadget is responsible for calculating if choice of the variable values meets the clause  
326 in formula. It has minimal solution of weight  $w$  if at least one variable in the clause has a  
327 correct value. Otherwise it has minimal solution  $w + 1$ . This way by the minimal solution for  
328 the whole problem, we can tell how many clauses were satisfiable.

329 The CLAUSE-gadgets consist of two OR-gadgets. We don't want the CLAUSE-gadgets  
330 to be crammed somewhere between the very long variable segments. That's why we have a  
331 simple gadget to *pass* the value of the segment, ie. segments  $(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1})$ .  
332 Two segments and one of them is chosen if  $x$  was chosen in the solution and the other one if  
333  $x$  wasn't.



Figure 3.3: **CLAUSE-gadget**. We denote set of these points as  $C\_clause_i$ . Every green rectangle is an OR-gadget.  $y$ -coordinates of  $x_{i,0}$ ,  $y_{i,0}$  and  $z_{i,0}$  depend on the values of variables in the  $i$ -th clause.

**Points.** TODO: Rephrase it

Assuming clause  $C_i = x_i \vee y_i \vee z_i$ , function  $idx(w)$  is returning index of the variable  $w$ , function  $neg(w)$  is returning whether variable  $w$  is negated in a clause.

$$\begin{aligned} x_{i,0} &= (10i + 1, 4 \cdot idx(x_i) + 2 \cdot neg(x_i)) & x_{i,1} &= (10i + 1, 4n) \\ y_{i,0} &= (10i + 2, 4 \cdot idx(y_i) + 2 \cdot neg(y_i)) & y_{i,1} &= (10i + 2, 4n + 4) \\ z_{i,0} &= (10i + 3, 4 \cdot idx(z_i) + 2 \cdot neg(z_i)) & z_{i,1} &= (10i + 3, 4n + 6) \end{aligned}$$

$$move\_variable_i = \{x_{i,j} : j \in \{0, 1\}\} \cup \{y_{i,j} : j \in \{0, 1\}\} \cup \{z_{i,j} : j \in \{0, 1\}\}$$

$$C\_clause_i = move\_variable_i \cup C\_or\_gadget_{i,0} \cup C\_or\_gadget_{i,1} \cup \{v_{i,1}\}$$

**Segments.**

$$\begin{aligned} P\_clause_i &= \{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1}), (x_{i,1}, l_{i,0}), (y_{i,1}, p_{i,0}), (z_{i,1}, p_{i,1}), \} \cup \\ &\cup P\_or\_gadget_{i,0} \cup P\_or\_gadget_{i,1} \end{aligned}$$

**Lemma 3.2.7.** For any  $1 \leq i \leq n$  and  $a \in \{x_{i,0}, y_{i,0}, z_{i,0}\}$ , points  $C\_clause_i - \{a\}$  can be covered using 11 segments from  $P\_clause_i$ .

*Proof.* For  $a = x_{i,0}$  (analogous proof for  $y_{i,0}$ ): First we use Lemma 3.2.5 twice with excluded  $x = l_{i,0}$  and  $x = l_{i,1} = v_{i,0}$ , resulting with 8 segments  $or_{i,0}^{true} \cup or_{i,1}^{true}$  which cover all required

points apart from  $x_{i,1}, y_{i,0}, y_{i,1}, z_{i,0}, z_{i,1}, l_{i,0}$ . We cover those using additional 3 segments:  
 $\{(x_{i,1}, l_{i,0}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1})\}$

For  $a = z_{0,i}$ : Using Lemma 3.2.6 and Lemma 3.2.5 with  $x = p_{i,1}$ , resulting with 8 segments  
 $or_{i,0}^{false} \cup or_{i,1}^{true}$  which cover all required points apart from  $x_{i,0}, x_{i,1}, y_{i,0}, y_{i,1}, z_{i,1}, p_{i,1}$ . We cover  
those using additional 3 segments:  $\{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,1}, p_{i,1})\}$ .  $\square$

**Lemma 3.2.8.** *Points  $C\_clause_i$  can be covered with 12 segments from  $P\_clause_i$ .*

*Proof.* Using Lemma 3.2.6 twice we can cover  $or\_gadget_{i,0}$  and  $or\_gadget_{i,1}$  with 8 segments.

To cover the remaining points we additionally use:  $\{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1}), (t_{i,1}, v_{i,1})\}$   
 $\square$

**Lemma 3.2.9.** *For any  $1 \leq i \leq n$ , points  $C\_clause_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$  can not be covered using less than 11 segments from  $P\_clause_i$ .*

*All points  $C\_clause_i$  can not be covered with less than 12 segments from  $P\_clause_i$ .*

*Proof of no cover with less than 12 segments.* There is independent set of 12 points in  $C\_clause_i \supseteq$   
 $\{x_{i,0}, y_{i,0}, z_{i,0}, l_{i,0}, p_{i,0}, q_{i,0}, u_{i,0}, v_{i,0} = l_{i,1}, p_{i,1}, q_{i,1}, u_{i,1}, v_{i,1}\}$ .  $\square$

*Proof of no cover with less than 11 segments.* We can choose disjoint sets  $X, Y, Z$  such that  
 $X \cup Y \cup Z \subseteq C\_clause_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$  and there are no segments covering points from  
different sets. And we prove lower bounds for each of these sets.

$$X = \{x_{i,1}, y_{i,1}, z_{i,1}\}$$

Set  $X$  is an indendent set, so it must be covered with 3 segments.

$$Y = or\_gadget_{i,0} - \{l_{i,0}, p_{i,0}\}$$

$$Z = or\_gadget_{i,1} - \{l_{i,1}, p_{i,1}\}$$

For both  $Y$  and  $Z$  we can check all of the subsets of 3 segments with brutforce that none  
of them cover, so they have to be covered with 4 segments.

TODO: Funny fact, neither  $Y$  nor  $Z$  doesn't have independent set of size 4.

Therefore  $C\_clause_i$  must be covered with at least  $3 + 4 + 4 = 11$  segments.  $\square$

#### 3.2.2.4. Summary

Add some smart lemmas that sets will be exclusive to each other.

**Lemma 3.2.10. Robustness to 1/2-extensions.** *For every segment  $s \in \mathcal{P}$ ,  $s$  and  $s^{+1/2}$  cover the same points from  $\mathcal{C}$ .*

#### 3.2.3. Summary of contruction

We define:

$$\mathcal{C} := \bigcup_{1 \leq i \leq n} C\_variable_i \cup C\_clause_i$$

$$\mathcal{P} := \bigcup_{1 \leq i \leq n} P\_variable_i \cup P\_clause_i$$

The subsequent sections define these sets.

We prove some properties of different gadgets. Every segment for a gadget will only cover  
points in this gadget (won't interact with any diferent gadget), so we can prove lemmas *locally*.

TODO:  $y$  axis is increasing values downward on figures (not upwards like in normal).



Figure 3.4: **General schema.**

General layout of VARIABLE-gadget and CLAUSE-gadget and how they interact with each other.

TODO: Rename Choose X to VARIABLE-gadget and Clause C to CLAUSE-gadget.

### 373 3.2.4. Construction lemmas and proof of Lemma 3.2.1

374 **Lemma 3.2.11.** *Given an instance  $S$  of MAX-(3,3)-SAT of size  $n$  with optimum solution*  
 375 *satisfying  $k$  clauses  $\text{opt}(S) = k$ . Instance of geometric cover, constructed for  $S$  according to*  
 376 *Lemma 3.2.1, can be solved with a solution of size  $15n - k$ .*

377 *Proof.* Let us name the assignments of the variables in the optimum solution of MAX-(3,3)-  
 378 SAT instance  $y_1, y_2 \dots y_n$  and clauses  $c_1, c_2 \dots c_n$ .

379 We cover every VARIABLE-gadget with solution described in Lemma 3.2.2, in the  $i$ -th  
 380 gadget choosing the set of segments corresponding to the value of  $y_i$ . CLAUSE-gadgets that  
 381 are satisfied are covered with set  $\text{clause}_i^{\text{true}}$  described in Lemma 3.2.7 and unsatisfied with  
 382 set  $\text{clause}_i^{\text{false}}$  described in Lemma 3.2.8.

$$R_i = \begin{cases} x_i^{\text{true}} & \text{if } y_i \\ x_i^{\text{false}} & \text{if } \neg y_i \end{cases}$$

$$C_i = \begin{cases} \text{clause}_i^{\text{true}} & \text{if } c_i \text{ satisfied} \\ \text{clause}_i^{\text{false}} & \text{if } c_i \text{ not satisfied} \end{cases}$$

$$\mathcal{R} = \bigcup_{i=1}^n \{R_i \cup C_i : 1 \leq i \leq n\}$$

383 This set covers all points form  $\mathcal{C}$ , because the smaller sets individually cover their corre-  
 384 sponding gadgets (proved in respective lemmas).

385 All of these sets are disjoint, so the size of the solution is:

$$|\mathcal{R}| = \sum_{i=1}^n R_i + \sum_{i=1}^n C_i = 3n + 11k + 12(n - k) = 15n - k$$

386 .

387

□

388 **Lemma 3.2.12.** *Given an instance  $S$  of MAX-(3,3)-SAT of size  $n$  and a solution of size  $w$*   
 389 *to the instance of geometric cover, constructed for  $S$  according to Lemma 3.2.1. There exists*  
 390 *a solution to MAX-(3,3)-SAT of size at least  $15n - w$ .*

391 *Proof.*

392 **Variables** We need to use at least 3 segments to cover VARIABLE-gadget (Lemma 3.2.3).  
 393 If we have chosen both segments  $(c_i, g_i)$  and  $(f_i, h_i)$ , then we have used at least 4 segments  
 394 (Lemma 3.2.4).

$$\begin{cases} |C_{\text{var}}^i \cap \mathcal{R}| \geq 4 & \text{if } (c_i, g_i) \in \mathcal{R} \wedge (f_i, h_i) \in \mathcal{R} \\ |C_{\text{var}}^i \cap \mathcal{R}| \geq 3 & \text{otherwise} \end{cases}$$

395 If we chose at most one of the segments  $(c_i, g_i)$  and  $(f_i, h_i)$ , choose the corresponding  
 396 variable value to the solution. If we chose both segments, choose the value that appears in  
 397 most clauses. Every variable is in exactly 3 clauses, so one value appears in at least 2 of them.  
 398 If we have chosen none of the segments, set value to false.

$$\begin{cases} x_i = \text{majority}(X_i) & \text{if } (c_i, g_i) \in \mathcal{R} \wedge (f_i, h_i) \in \mathcal{R} \\ x_i = \text{true} & \text{if } (c_i, g_i) \in \mathcal{R} \\ x_i = \text{false} & \text{if } (f_i, h_i) \in \mathcal{R} \\ x_i = \text{false} & \text{otherwise} \end{cases} \quad (3.1)$$

399 To cover  $\bigcup_{1 \leq i \leq n} C_{var}^i$  we have used at least  $3n + a$  segments, where  $a$  is the number of  $i$   
400 such that we have chosen both values  $(c_i, g_i)$  and  $(f_i, h_i)$ .

401 **Clauses** For a clause  $C_i = x \vee y \vee z$ , we need to use at least 11 segments to cover  
402  $C_{clause_i} - \{x, y, z\}$  in CLAUSE-gadget (Lemma 3.2.9).

403 TODO: maybe put something with cases and names of sets as above

404 Moreover, if all of the points  $\{x, y, z\}$  are not covered by the segments from  $P_{var}^i$ , with at  
405 least 12 segments by Lemma 3.2.9.

406 We covered CLAUSE-gadget with at least 11 or at least 12 segments:  $|\bigcap_{i=1}^n P_{clause_i} \cap$   
407  $\mathcal{R}| \geq 11n + b$ , where  $b$  is the number of clauses where none of the segments covering the points  
408  $x, y, z$  were chosen in  $P_{var}^j$ .

409 **Satisfied clauses in chosen variables assignment** Clauses for which none of the  
410 segments covering the points  $x, y, z$  were chosen in  $P_{var}^j$ , are not satisfied in our variables  
411 assignment, but not all clauses that cover one of these points with segment in  $P_{var}^j$  are satisfied.

412 Let us look at such equation and cases of choosing variable value in equation (3.1).

413 If only one of the segments  $(c_i, g_i)$  and  $(f_i, h_i)$  are chosen in  $P_{var}^i$ , then every clause that  
414 uses variable  $x_i$  with value that we chose is satisfied.

415 If we chose neither  $(c_i, g_i)$  or  $(f_i, h_i)$ , then every clause that uses variable  $x_i$  as *false* is  
416 satisfied. If the CLAUSE-gadget was covered by 11 segments instead of 12, then some other  
417 variable had to have its variable value segment covered.

418 If we chose both  $(c_i, g_i)$  and  $(f_i, h_i)$ , then there might exist one clause that was covered by  
419 11 segments instead of 12, but the clause is not satisfied with the value that we set to  $x_i$ . This  
420 happens because adding both  $(c_i, g_i)$  and  $(f_i, h_i)$  to the solution makes CLAUSE-gadgets, that  
421 use  $x_i$  with both values, possible to cover with 11 segments instead of 12, but we set value of  
422  $x_i$  to the one used in majority of clauses, so there exists at most one that is covered with 11  
423 segments and not satisfied. We had  $a$  such variables, so there are at most  $a$  clauses that are  
424 covered with 11 segments and not satisfied.

425 So in the solution to this MAX-(3,3)-SAT instance that we have shown, there are at most  
426  $a + b$  unsatisfied clauses.

427 **Conclusions** We proved that given a solution of size  $w$  we have the variables assignment  
428 that satisfies at least  $n - (a + b)$  clauses of  $S$ . At last we prove that  $n - (a + b) \geq 15n - w$ .

$$w \geq 3(n - a) + 4a + 11(n - b) + 12b = 3n + a + 11n + b = 14n + a + b$$

$$15n - w \leq 15n - 14n - a - b = n - (a + b)$$

429

□

430 *Proof of Lemma 3.2.1.* Given an instance  $S$  of MAX-(3,3)-SAT of size  $n$  with optimum solu-  
 431 tion satisfying  $k$  clauses. Let us construct an instance of geometric cover for  $S$ , constructed  
 432 in aforementioned manner and name it  $I$ .

433 Given the Lemma 3.2.11, we know that there exists a solution for instance  $I$  of size  $15n - k$ .  
 434 This is an upper bound for optimum solution for  $I$ . Since  $k$  is the optimum solution for  $S$ ,  
 435 then according to Lemma 3.2.12 there is no solution for instance  $I$  of size less than  $15n - k$ .  
 436 Therefore  $15 - k$  is an optimum solution for instance  $I$ .  $\square$

### 437 3.3. Weighted segments

#### 438 3.3.1. FPT for weighted segments with $\delta$ -extensions

439 **Theorem 3.3.1. (*FPT for weighted segment cover with  $\delta$ -extensions*).** *There exists*  
 440 *an algorithm that given a family  $\mathcal{P}$  of  $n$  weighted segments (in any direction), a set of  $m$*   
 441 *points  $\mathcal{C}$  and a parameter  $k$ , runs in time  $f(k) \cdot (nm)^c$  for some computable function  $f$  and*  
 442 *constant  $c$ , and outputs a subfamily  $\mathcal{R} \subseteq \mathcal{P}$  such that  $|\mathcal{R}| \leq k$  and  $\mathcal{R}^{+\delta}$  covers all points in  $\mathcal{C}$ .*

443 To solve this problem we will introduce kernel for slightly different problem: Weighted  
 444 segment cover of points and segments. In shortcut: WSCPS.

445 **Lemma 3.3.1. (*Algorithm for kernel of WSCPS*).** *There exists an algorithm that given*  
 446 *a family  $\mathcal{P}$  of  $n$  weighted segments (in any direction), a set of  $m_1$  points  $\mathcal{C}_1$  and  $m_2$  segments*  
 447  *$\mathcal{C}_2$  and a parameter  $k$ , runs in time  $f(k) \cdot g(m_1, m_2) \cdot n^c$  for some computable functions  $f, g$*   
 448 *and constant  $c$ , and outputs a subfamily  $\text{sol} \subseteq \mathcal{P}$  such that  $|\mathcal{R}| \leq k$  and  $\mathcal{R}$  covers all points in*  
 449  *$\mathcal{C}_1$  and all segments in  $\mathcal{C}_2$ .*

450 **Proof** Only sketch for now.

451 We can compute dynamic programming  $dp(A, B, z)$  – the best cost to cover at least whole  
 452 segment  $A, B$  using at most  $z$  segments.  $A, B$  are all interesting points – ends of any segment  
 453 given on the input or points given on the input. We can compute it in polynomial time.

454 Then we can create a new double weighted set (original weight, number of used segments  
 455 from  $\mathcal{P}$ ) –  $\mathcal{P}_2$  that has only segments which never cover partially any segment from  $\mathcal{C}_2$  (covers  
 456 the whole segment or doesn't cover at all). In such  $\mathcal{P}_2$  we can find solution  $\mathcal{R}$  where any  
 457 2 segments have empty intersection (don't cover each other and don't meet at the ends).  
 458 Because if we had such solution, we can merge these two segments and such segment there's  
 459 also in  $\mathcal{P}_2$ .

460 In that case we can find kernel of  $\mathcal{P}_2$  of size  $k \cdot (m_1 + 2m_2)^2$ , because we only need to take  
 461 the best weight covering some subset of  $\mathcal{C}_1 \cup \mathcal{C}_2$ .

462 **Lemma 3.3.2. *Kernel in WSCPS.*** *TODO: formulate it properly*

463 *For segment cover, there is a kernel of size  $f(k)$  in WSCPS.*

464 **Claim 3.3.1.** *If there are more than  $k$  lines with at least  $k + 1$  points on them, then they can't*  
 465 *be covered with  $k$  segments.*

466 **Claim 3.3.2.** *If there is more than  $k^2$  points that don't lie on any line with more than  $k$*   
 467 *points on it, then they can't be covered with  $k$  segments.*

468 **Claim 3.3.3.** *For every long line  $L$  (with more than  $k$  points on them) we can choose  $f(k)$*   
 469 *points on them, that if we cover all of these points with at most  $k$  segments, then the rest of*  
 470 *the points with  $\delta$ -extensions will be covered by segments in the direction of line  $L$ .*

**Proof of Lemma 3.3.2.** After applying the previous lemmas, we have at most  $k^2 + k \cdot f(k)$  points that can be covered in any direction and for the rest of the points we can draw at most  $k \cdot f(k)$  segments along their respective long lines that have to be covered by segments after  $\delta$ -extensions.

Then we extend every available segment by  $\delta$ -extension and we achieve the kernel in WSCPS for this instance of problem.

**Lemma 3.3.3.** *If all the points are covered with  $k$  segments and the biggest  $2(1 + 1/\delta)^{k+1}$  spaces between points are filled, the whole segment is filled after  $\delta$ -extensions of these segments.*

**Proof.** Let's name the  $2(1+1/\delta)^{k+1}$ -st biggest space between points as  $y$ . We have guarantee that all segments of length  $x > y$  are covered without  $\delta$ -extensions.

Let's take one space between points that is not covered before  $\delta$ -extension and we will prove it will be covered after  $\delta$ -extensions. Let's assume it isn't.

This space has length  $x$ . Since it's uncovered,  $x \leq y$ .

Let's take side where the sum of lengths of segments covering the points is greater (left or right). Without loss of generality, let us assume it's right.

There are at most  $k$  segments to the right of this space between points. Name their lengths  $l_1, l_2 \dots l_k$ . If the point is covered in the other direction, the segment is degenerated to the point and  $l_i = 0$ . Name the space between endpoints of  $l_i$  and  $l_{i+1} - x_i$ . Of course,  $x_i$  is uncovered space between two points, therefore  $x_i \leq y$ .

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Let's write equations meaning that  $i$ -th segment doesn't cover space  $x$  after  $\delta$ -expansion.

$$l_1\delta < x \leq y \Rightarrow l_1 < y/\delta$$

$$l_2\delta < x + l_1 + x_1 < 2y + y/\delta \Rightarrow l_2 < 2y/\delta + y/\delta^2$$

$$l_3\delta < x + l_1 + x_1 + l_2 + x_2 < 3y + 3y/\delta + y/\delta^2 \Rightarrow l_3 < 3y/\delta + 3y/\delta^2 + y/\delta^3$$

From this we can "guess" induction  $l_i < y((1 + 1/\delta)^i - 1)$

Trivially for  $l_1 < y/\delta$ .

Assume that for all  $j < i$ :

$$l_j < y((1 + 1/\delta)^j - 1)$$

$$l_i\delta < x + \sum_{j=1}^{i-1}(l_j + x_j) < iy \sum_{j=1}^{i-1} l_j < iy + \sum j = 1^{i-1}y((1 + 1/\delta)^j - 1) = iy - (i - 1)y + \sum j = 1^{i-1}y(1 + 1/\delta)^j = y(1 + \sum_{j=1}^{i-1}(1 + 1/\delta)^j) = y(2 + \sum_{j=1}^{i-1}(1 + 1/\delta)^j - 1) = y(\sum_{j=0}^{i-1}(1 + 1/\delta)^j - 1) = y((1 + 1/\delta)^i / (1 - (1 + 1/\delta)) - 1) = y((1 + 1/\delta)^i \delta - 1) < y((1 + 1/\delta)^i \delta - \delta)$$

Of course we also know that (since we have chosen the side with greater sum of the width of segments):

$$\sum_{i=1}^k l_i \geq 1/2 \cdot y \cdot 2(1 + 1/\delta)^{k+1} = y \cdot (1 + 1/\delta)^{k+1}$$

$$\text{But } \sum_{i=1}^k l_i < \sum_{i=1}^k y((1 + 1/\delta)^i - 1) = y((1 + 1/\delta)^{k+1} / (1 - (1 + 1/\delta)) - k) = y((1 + 1/\delta)^{k+1} \delta - k) < y(1 + 1/\delta)^{k+1}$$

Therefore the space must have been covered after  $\delta$ -expansions.



### 501 3.3.2. $W[1]$ -completeness for weighted segments in 3 directions

502 **Theorem 3.3.2.**  *$W[1]$ -completeness for weighted segments in 3 directions.* Consider  
 503 the problem of covering a set  $\mathcal{C}$  of points by selecting  $k$  axis-parallel or right-diagonal weighted  
 504 segments with weights from a set  $\mathcal{P}$  with minimal weight. Assuming  $ETH$ , there is no algorithm  
 505 for this problem with running time  $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{o(\sqrt{k})}$  for any computable function  $f$ .

506 We will show reduction from grid tiling problem.

507 Let's have an instance of grid tiling problem – size of the grid  $k$ , number of elements  
 508 available  $n$  and  $k^2$  sets of available pairs in every tile  $S_{i,j} \subseteq \{1, n\} \times \{1, n\}$ .

509 **Construction.** We construct a set  $\mathcal{P}$  of segments and a set  $\mathcal{C}$  of points.

510 First let's choose any ordering of  $n^2$  elements  $\{1, n\} \times \{1, n\}$  and name this sequence  
 511  $a_1 \dots a_{n^2}$ .

$$match_v(i, j) \iff a_i = \{x_i, y_i\} \wedge a_j = \{x_j, y_j\} \wedge x_i = x_j$$

$$match_h(i, j) \iff a_i = \{x_i, y_i\} \wedge a_j = \{x_j, y_j\} \wedge y_i = y_j$$

**Points.** Define points:

$$h_{i,j,t} = (j \cdot (n^2 + 1) + t, (n^2 + 1) \cdot i)$$

$$v_{i,j,t} = ((n^2 + 1) \cdot i, j \cdot (n^2 + 1) + t)$$

Let's define sets  $H$  and  $V$  as:

$$H = \{h_{i,j,t} : 1 \leq i, j \leq k, 1 \leq t \leq n^2\}$$

$$V = \{v_{i,j,t} : 1 \leq i, j \leq k, 1 \leq t \leq n^2\}$$

512 Let's define  $\epsilon = 0.1$ . For a point  $\{x, y\} = p$  we define points  $p^L = \{x - \epsilon, y\}$ ,  $p^R = \{x + \epsilon, y\}$ ,  
 513  $p^U = \{x, y - \epsilon\}$ , and  $p^D = \{x, y + \epsilon\}$ .

Then we define:

$$\mathcal{C} := H \cup \{p^L : p \in H\} \cup \{p^R : p \in H\} \cup V \cup \{p^U : p \in V\} \cup \{p^D : p \in V\}$$

514 **Segments.** Define horizontal segments.

$$hor_{i,j,t_1,t_2} = (h_{i,j,t_1}^R, h_{i,j+1,t_2}^L)$$

$$ver_{i,j,t_1,t_2} = (v_{i,j,t_1}^D, v_{i,j+1,t_2}^U)$$

$$hor_{beg_{i,t}} = (h_{i,1,1}^L, h_{i,1,t}^L)$$

$$hor_{end_{i,t}} = (h_{i,n,t}^R, h_{i,n,n^2}^R)$$

$$ver_{beg_{i,t}} = (v_{i,1,1}^U, v_{i,1,t}^U)$$

$$ver_{end_{i,t}} = (v_{i,n,t}^D, v_{i,n,n^2}^D)$$

$$\begin{aligned}
HOR &= \{hor_{i,j,t_1,t_2} : 1 \leq i \leq k, 1 \leq j < k, 1 \leq t_1, t_2 \leq n^2, match_h(t_1, t_2)\} \\
&\cup \{horbeg_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\} \\
&\cup \{horend_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\}
\end{aligned}$$

$$\begin{aligned}
VER &= \{ver_{i,j,t_1,t_2} : 1 \leq i \leq k, 1 \leq j < k, 1 \leq t_1, t_2 \leq n^2, match_v(t_1, t_2)\} \\
&\cup \{verbeg_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\} \\
&\cup \{verend_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\}
\end{aligned}$$

$$DIAG := \{(h_{i,j,t}, v_{j,i,t}) : 1 \leq i, j \leq k, 1 \leq t \leq n^2, a_t \in S_{i,j}\}$$

515 TODO: explain that these segments are in fact diagonal

$$\mathcal{P} := HOR \cup VER \cup DIAG$$

516 **Lemma 3.3.4.** *If there exists solution for grid tiling, then there exists solution for our con-*  
517 *struction using  $2(k+1)k + k^2$  segments with weight exactly  $2k \cdot (k(n^2 + 1) - 2 - 2\epsilon(k-1))$ .*

**Claim 3.3.4.** *If there exists a solution to the grid tiling  $c_1 \dots c_k$  and  $r_1 \dots r_k$ , then there exists a solution covering all points*

$$\{h_{i,j,t} : 1 \leq i, j \leq k, t = (c_i, r_j)\} \cup \{v_{i,j,t} : 1 \leq i, j \leq k, t = (c_j, r_i)\}$$

518 *with segments in DIAG and the rest in VER or HOR and has weight  $2k \cdot (k(n^2 + 1) -$*   
519  *$2 - 2\epsilon(k-1))$ .*

520 **Proof.** TODO: jakiś prosty z definicji

521 **Lemma 3.3.5.** *If there exists solution for our construction using  $2(k+1)k + k^2$  segments*  
522 *with weight exactly  $2k \cdot (k(n^2 + 1) - 2 - 2\epsilon(k-1))$ , then there exists a solution for grid tiling*

523 **Proof.** This follows from Lemma 3.3.6, because we just take which points are covered with  
524 *DIAG.*

525 **Claim 3.3.5.** *Points  $p^L, p^R, p^U, p^D$  cannot be covered with DIAG.*

526 **Claim 3.3.6.** *Points in  $H \cup \{p^L : p \in H\} \cup \{p^R : p \in H\}$  cannot be covered with VER.*  
527 *Points in  $V \cup \{p^U : p \in V\} \cup \{p^D : p \in V\}$  cannot be covered with HOR.*

528 **Claim 3.3.7.** *For given  $i, j$  if none of the points  $h_{i,j,t}$  ( $v_{i,j,t}$ ) for  $1 \leq t \leq n^2$  are covered with*  
529 *DIAG, then some spaces between neighbouring points were covered twice.*

530 **Claim 3.3.8.** *For given  $i, j$  two points  $h_{i,j,t_1}, h_{i,j,t_2}$  ( $v_{i,j,t_1}, v_{i,j,t_2}$ ) for  $1 \leq t_1 < t_2 \leq n^2$  are*  
531 *covered with DIAG, then one of them had to be also covered with a segment from HOR*  
532 *(VER).*

533 **Proof.** Point  $v_{i,j,t_2}^L$  had to be covered with  $VER$  from Claims 3.3.5 and 3.3.6. And every  
534 segment in  $VER$  covering  $v_{i,j,t_2}^L$ , covers also  $v_{i,j,t_1}^L$ .

535 **Lemma 3.3.6.** *If there exists solution for our construction with weight at most (exactly)*  
536  *$2k \cdot (k(n^2 + 1) - 2 - 2\epsilon(k - 1))$ , then for every  $i, j$  there must be exactly one  $t$  such that*  
537  *$h_{i,j,t}$  ( $v_{i,j,t}$ ) is covered with  $DIAG$  and moreover if  $h_{i,j,t_1}$  and  $h_{i,j+1,t_2}$  are uncovered, then*  
538  *$math_h(t_1, t_2)$ . Analogically for  $v$ .*

539 **Proof.** Only  $k^2$  points can be covered only in  $DIAG$ , the rest has to be covered with  
540  $VER \cup HOR$ . Therefore every result must be at least  $ALL\_LINES - 2k^2\epsilon$ , because only  
541  $2k^2$  spaces of length  $\epsilon$  can be uncovered in this axis.

542 Of course if  $h_{i,j,t_1}$  and  $h_{i,j+1,t_2}$  are uncovered, then there must exist a segment in  $HOR$   
543 between  $h_{i,j,t_1}^R$  and  $h_{i,j+1,t_2}^L$ , so  $math_h(t_1, t_2)$  must be true.

### 544 3.3.3. What is missing

545 We don't know FPT for axis-pararell segments without  $\delta$ -extensions.



## 546 Chapter 4

# 547 Geometric Set Cover with lines

### 548 4.1. Lines parallel to one of the axis

549 When  $\mathcal{R}$  consists only of lines parallel to one of the axis, the problem can be solved in  
550 polynomial time.

551 We create bipartial graph  $G$  with node for every line on the input split into sets:  $H$  –  
552 horizontal lines and  $V$  – vertical lines. If any two lines cover the same point from  $\mathcal{C}$ , then we  
553 add edge between them.

554 Of course there will be no edges between nodes inside  $H$ , because all of them are pararell  
555 and if they share one point, they are the same lines. Similar argument for  $V$ . So the graph is  
556 bipartial.

557 Now Geometric Set Cover can be solved with Vertex Cover on graph  $G$ . Since Vertex  
558 Cover (even in weighted setting) on bipartial graphs can be solved in polynomial time.

559 Short note for myself just to remember how to this in polynomial time:

560 Non-weighted setting - Konig theorem + max matching

561 Weighted setting - Min cut in graph of  $\neg A$  or  $\neg B$  (edges directed from  $V$  to  $H$ )

### 562 4.2. FPT for arbitrary lines

563 You can find this is Platypus book. We will show FPT kernel of size at most  $k^2$ .

564 (Maybe we need to reduce lines with one point/points with one line).

565 For every line if there is more than  $k$  points on it, you have to take it. At the end, if there  
566 is more than  $k^2$  points, return NO. Otherwise there is no more than  $k^4$  lines.

567 In weighted settings among the same lines with different weights you leave the cheapest  
568 one and use the same algorithm.

### 569 4.3. APX-completeness for arbitrary lines

570 We will show a reduction from Vertex Cover problem. Let's take an instance of the Vertex  
571 Cover problem for graph  $G$ . We will create a set of  $|V(G)|$  pairwise non-pararell lines, such  
572 that no three of them share a common point.

573 Then for every edge in  $(v, w) \in E(G)$  we put a point on crossing of lines for vertices  $v$   
574 and  $w$ . They are not pararell, so there exists exactly one such point and any other line don't  
575 cover this point (any three of them don't cross in the same point).

Solution of Geometric Set Cover for this instance would yield a sound solution of Vertex Cover for graph  $G$ . For every point (edge) we need to choose at least one of lines (vertices)  $v$  or  $w$  to cover this point.

Vertex Cover for arbitrary graph is APX-complete, so this problem is also APX-complete.

#### 4.4. 2-approximation for arbitrary lines

Vertex Cover has an easy 2-approximation algorithm, but here very many lines can cross through the same point, so we can do  $d$ -approximation, where  $d$  is the biggest number of lines crossing through the same point. So for set where any 3 lines don't cross in the same point it yields 2-approximation.

The problematic cases are where through all points cross at least  $k$  points and all lines have at least  $k$  points on them. It can be created by casting  $k$ -grid in  $k$ -D space on 2D space.

Greedy algorithm yields  $\log |\mathcal{R}|$ -approximation, but I have example for this for bipartial graph and reduction with taking all lines crossing through some point (if there are no more than  $k$ ) would solve this case. So maybe it works.

Unfortunately I haven't done this :(

I can link some papers telling it's hard to do.

#### 4.5. Connection with general set cover

Problem with finite set of lines with more dimensions is equivalent to problem in 2D, because we can project lines on the plane which is not perpendicular to any plane created by pairs of (point from  $\mathcal{C}$ , line from  $\mathcal{P}$ ).

Of course every two lines have at most one common point, so is every family of sets that have at most one point in common equivalent to some geometric set cover with lines?

No, because of Desargues's theorem. Have to write down exactly what configuration is banned.

## 600 Chapter 5

# 601 Geometric Set Cover with polygons

### 602 5.1. State of the art

603 Covering points with weighted discs admits PTAS [Li and Jin, 2015] and with fat polygons  
604 with  $\delta$ -extensions with unit weights admits EPTAS [Har-Peled and Lee, 2009].

605 Although with thin objects, even if we allow  $\delta$ -expansion, the Set Cover with rectangles is  
606 APX-complete (for  $\delta = 1/2$ ), it follows from APX-completeness for segments with  $\delta$ -expansion  
607 in Section 3.2.

608 Covering points with squares is W[1]-hard [Marx, 2005]. It can be proven that assuming  
609 *SETH*, there is no  $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{k-\epsilon}$  time algorithm for any computable function  $f$  and  
610  $\epsilon > 0$  that decides if there are  $k$  polygons in  $\mathcal{P}$  that together cover  $\mathcal{C}$ , *Theorem 1.9* in [Marx  
611 and Pilipczuk, 2015].





<sup>612</sup> Chapter 6

<sup>613</sup> Conclusions



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