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Approximation and Parametrized Algorithms for Geometric Set Cover

Master's thesis
in COMPUTER SCIENCE

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Supervisor's statement

Hereby I confirm that the presented thesis was prepared under my supervision and that it fulfils the requirements for the degree of Master of Computer Science.

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Author's statement

Hereby I declare that the presented thesis was prepared by me and none of its contents was obtained by means that are against the law.

The thesis has never before been a subject of any procedure of obtaining an academic degree.

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Abstract

W pracy przedstawiono prototypową implementację blabalizatora różnicowego bazującą na teorii fetorów σ-ρ profesora Fifaka. Wykorzystanie teorii Fifaka daje wreszcie możliwość efektywnego wykonania blabalizy numerycznej. Fakt ten stanowi przełom technologiczny, którego konsekwencje trudno z góry przewidzieć.

Keywords

blabaliza różnicowa, fetory σ-ρ, fooizm, blarbarucja, blaba, fetoryka, baleronik

Thesis domain (Socrates-Erasmus subject area codes)

11.3 Informatyka

Subject classification

D. Software
D.127. Blabalgorithms
D.127.6. Numerical blabalysis

Tytuł pracy w języku polskim

Algorytmy parametryzowania i trudność aproksymacji problemu pokrywania zbiorów na płaszczyźnie

Contents

Chapter 1

Introduction

This is some very boring and really nothing on the topic introduction.

Chapter 2

Definitions

Some definitions what geometric set cover is. $\mathcal{P}$ – set of objects, $\mathcal{C}$ – set of points. Choose $\mathcal{R} \subset \mathcal{P}$ such that every point in $\mathcal{C}$ is inside some element from $\mathcal{R}$ and $|\mathcal{R}|$ is minimal.

In parametrized setting we only look among $|\mathcal{R}| \leq k$. In weighted settings there is some $f : \mathcal{P} -> \mathbb{R}$ and we minimize $\sum_{R \in \mathcal{R}} f(R)$.

Chapter 3

Geometric Set Cover with segments

3.1. FPT for segments

3.1.1. Segments parallel to one of the axis

You can find this in Platypus book.

We'll show $\mathcal{O}(2^k)$ branching algorithm. Let's take point K that hasn't been covered yet with the smallest coordinate in lexicograpical order. We need to cover K with some of the remaining segments.

We choose one of the 2 directions on which we will cover this point. In this direction we take greedly the segment that will cover the most points (there are points in $\mathcal{C}$ only on one side of K in this direction, so all segments covering K in this direction create monotone sequence of sets – zbiory zstępujące).

3.1.2. Segments in d directions

The same algorithm as before but in complexity $\mathcal{O}(d^k)$.

3.1.3. Segments in arbitrary direction

If there exist two segments a and b in $\mathcal{P}$, such that any point covered by a is also covered by b, then without loss of generality we can remove segment a from $\mathcal{P}$. We repeat this process until no such (a, b) pair exists.

Let us first assume that we reduced our instance to a kernel, where *any line* contains no more than k points.

Since any segment covers a set of colinear points, for such a kernel k segments can cover only at most k^2 points. Therefore, for the answer to be positive, the number of points has to be at most k^2. The number of segments is now bounded by k^4, since if we consider two *extreme* points covered by a given segment, then these pairs must be distinct, otherwise two segments would contain the same set of points. Since both the number of points and the number of segments is bounded by a function of k, this instance can be easily solved in time $O(f(k))$.

In remains to show how to construct the kernel.

As long as there is a line with more than k points, do branching. Let's name points on this line $x_1, x_2, \ldots x_t$ in order they appear on the line.

So we choose on which point the chosen segment on this line will start. Of course we have to take at least one segment covering at least one point among first $k + 1$ points, because covering all of them with only segments on different lines we would use exactly $k+1$ segments (any of them can't contain more than one point from this line).

Assume there exists a line l containing points $x_1, \ldots x_t$, where $t \geq k+1$. Note that a segment that does not lie on l can cover only at most one of the points x_i. Therefore, out of points x_1, ..., x_{k+1}, at least one has to be covered by a segment that lies on l, let us fix x_i to be the first such point. Then, we can greedily choose a segment that lies on l, covers x_i, and also covers the largest number of points x_j for $j > i$.

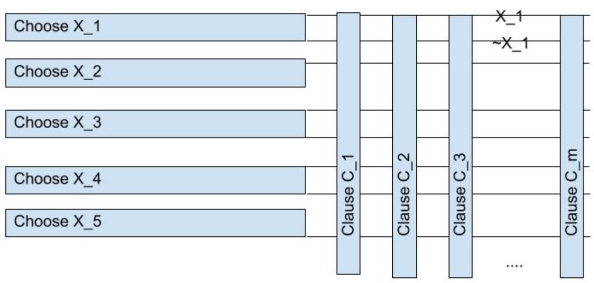
Since we have at most $k+1$ choices to branch over and each choice adds a segment to the constructed solution, we obtain an algorithm with complexity $O(k^k)$.

3.2. APX-completeness for segments pararell to axis

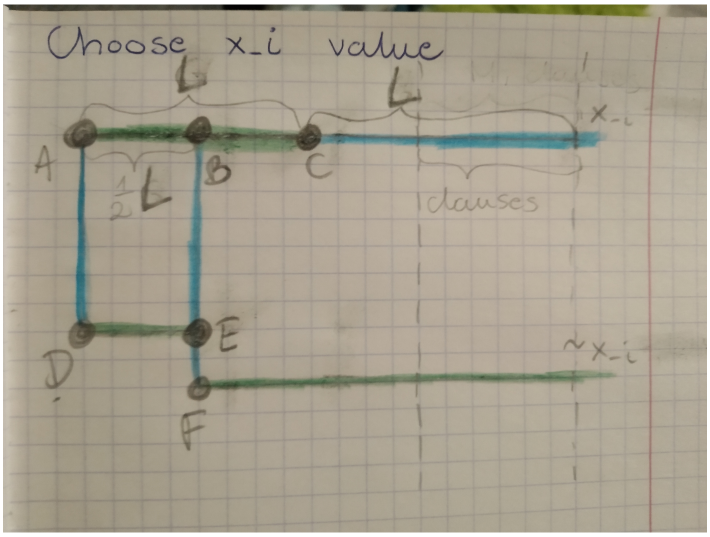
It works even with extensions for unit weights.

We will show reduction from MAXSAT to Geometric Set Cover with segments pararell to axis.

3.2.1. Reduction contruction



Choose x_i gadget



We can cover all these points with 3 segments and choose segment x_i by taking blue segments or $\neg x_i$ by taking green segments.

Lemma 3.2.1 *Points A, B, C, D, E, F can be covered using at least 3 segments (even with 1/2-extensions).*

Proof. We need to take at least one segment on line ABC, because it's the only way to cover C. All other points (D, E, F) are not colinear, so we need at least 2 other segments to cover them.

Lemma 3.2.2 *If we choose both segments x_i and $\neg x_i$, we need to use at least 4 segments to cover all points A, B, C, D, E, F (even with 1/2-extensions).*

Proof. Choosing both segments x_i and $\neg x_i$ we only cover points C (becuase B is too far away to be covered with 1/2-extension) and F.

All other points (A, B, D, E) are not colinear, so we need at least 2 other segments to cover them.

Lemma 3.2.3 *No points in **clauses** area can be covered by AC with 1/2-extension.*

3.2.2. Proof that contruction is sound

3.3. Weights

3.3.1. FPT for segments pararell with δ-extensios

3.3.2. W[1]-completeness for arbitrary segments with weights

3.3.3. What is missing

We don't know FPT for pararell segments and arbitrary lines with δ-extensions.

Chapter 4

Geometric Set Cover with lines

4.1. Lines parallel to one of the axis

When $\mathcal{R}$ consists only of lines parallel to one of the axis, the problem can be solved in polynomial time.

We create bipartial graph G with node for every line on the input split into sets: H – horizontal lines and V – vertical lines. If any two lines cover the same point from $\mathcal{C}$, then we add edge between them.

Of course there will be no edges between nodes inside H, because all of them are pararell and if they share one point, they are the same lines. Similar argument for V. So the graph is bipartial.

Now Geometric Set Cover can be solved with Vertex Cover on graph G. Since Vertex Cover (even in weighted setting) on bipartial graphs can be solved in polynomial time.

Short note for myself just to remember how to this in polynomial time:

Non-weighted setting - Konig theorem + max matching

Weighted setting - Min cut in graph of $\neg A$ or $\neg B$ (edges directed from V to H)

4.2. FPT for arbitrary lines

You can find this is Platypus book. We will show FPT kernel of size at most k^2.

(Maybe we need to reduce lines with one point/points with one line).

For every line if there is more than k points on it, you have to take it. At the end, if there is more than k^2 points, return NO. Otherwise there is no more than k^4 lines.

In weighted settings among the same lines with different weights you leave the cheapest one and use the same algorithm.

4.3. APX-completeness for arbitrary lines

We will show a reduction from Vertex Cover problem. Let's take an instance of the Vertex Cover problem for graph G. We will create a set of $|V(G)|$ pairwise non-pararell lines, such that no three of them share a common point.

Then for every edge in $(v, w) \in E(G)$ we put a point on crossing of lines for vertices v and w. They are not pararell, so there exists exactly one such point and any other line don't cover this point (any three of them don't cross in the same point).

Solution of Geometric Set Cover for this instance would yield a sound solution of Vertex Cover for graph G. For every point (edge) we need to choose at least one of lines (vertices) v or w to cover this point.

Vertex Cover for arbitrary graph is APX-complete, so this problem in also APX-complete.

4.4. 2-approximation for arbitrary lines

Vertex Cover has an easy 2-approximation algorithm, but here very many lines can cross through the same point, so we can do d-approximation, where d is the biggest number of lines crossing through the same point. So for set where any 3 lines don't cross in the same point it yields 2-approximation.

The problematic cases are where through all points cross at least k points and all lines have at least k points on them. It can be created by casting k-grid in k-D space on 2D space.

Greedy algorithm yields $\log |\mathcal{R}|$-approximation, but I have example for this for bipartial graph and reduction with taking all lines crossing through some point (if there are no more than k) would solve this case. So maybe it works.

Unfortunaly I haven't done this :(

I can link some papers telling it's hard to do.

4.5. Connection with general set cover

Problem with finite set of lines with more dimensions is equivalent to problem in 2D, because we can project lines on the plane which is not perpendicular to any plane created by pairs of (point from $\mathcal{C}$, line from $\mathcal{P}$).

Of course every two lines have at most one common point, so is every family of sets that have at most one point in common equivalent to some geometric set cover with lines?

No, because of Desargues's theorem. Have to write down exactly what configuration is banned.

Chapter 5

Geometric Set Cover with polygons

5.1. Introduction

The problem is APX-complete and W[1]-complete, so we introduce δ-expansions.

5.2. FPT – ?? I don't know :(

5.3. APX-completness for rectangles with δ-expansion without weights

It follows from APX-completeness for segments with δ-expansion.

5.4. $1+\epsilon$ approximation algorithm for weighted polygons of bounded thickness θ

This should be written.

Definition 5.4.1 ***Thickness*** *of the polygon is the ratio of the circumsribed circle's radius to the inscribed circle's radius.*

Definition 5.4.2 (MWSCP) *TODO: wstawić to jakoś wcześniej i inaczej Minimal Weight Set Cover for Polygons*

Theorem 5.4.1 (EPTAS for MWSCP with bounded thickness and δ-expansion) *There is a randomized algorithm that given a weighted family $\mathcal{P}$ of n polygons with thickness bouded by θ and set $\mathcal{C}$ of m points with total encoding size of both sets N, and parameters δ, ϵ, runs in time $f(\epsilon, \delta, \theta) \cdot (nN)^c$ for some computable functions f and constant c, and outputs a subfamily $\mathcal{S} \subseteq \mathcal{P}$ such that $\mathcal{S}^\delta$ covers the $\mathcal{C}$ and $w(\mathcal{S}) \leq (1+\epsilon)OPT(\mathcal{P})$ with probability at least 1/2.*

5.4.1. Sparsifying the family

Intuitively, we will create a new input family $\mathcal{P}'$ of polygons that can cover set of points $\mathcal{C}$ if and only if set $\mathcal{P}$ can cover set of $\mathcal{C}$ and $\mathrm{OPT}(\mathcal{P}')$ is worse only by $\mathcal{O}(\epsilon)$-fraction of $\mathrm{OPT}(\mathcal{P})$. The polygons in $\mathcal{P}'$ will be classified into groups of similar size of edge of their circumscribed squares.

Ogólnie wszystko tutaj będzie takie samo jak w paperze, ale wstawimy stałą dla siatki $1/(\delta\theta\epsilon)$ zamiast $1/\delta\epsilon$.

$L = (1/\delta\theta\epsilon)^{\ell}$ for some $\ell = \mathcal{O}(N)$ – limit for data.

Let's denote d_i as a length of edge of the circumscribed square on a polygon $P_i \in \mathcal{P}$.

Partition into layers Let's define a partition:

$$(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\ell)$$

of $\mathcal{P}$ and such reals ν_t, μ_t for $t = 1, 2, \ldots, \ell$ with the following properties satisfied for each $t \in \{1, 2, \ldots, \ell\}$:

- $\nu_t \leq d_i \leq \mu_t$ for each $P_i \in \mathcal{P}_t$
- $\nu_t = \mu_{t-1}$ (expect for $t = 1$) and $\mu_t/\nu_t = (1/\delta\theta\epsilon)^{1/\epsilon}$
- $\nu_1 \geq 1, \mu_\ell \geq L$, and all numbers ν_t and μ_t apart from ν_1 are integers.

How to divide these polygons and choose numbers is pretty straightforward, but we also use some shifting parameter $0 \leq b \leq 1/\epsilon$ to be determined later.

$\nu_t = (1/\delta\theta\epsilon)^{t/\epsilon+b}$ $\mu_t = (1/\delta\theta\epsilon)^{(t+1)/\epsilon+b}$

Hierarchical grid structure Let $a \in \{1, \ldots, L-1\}$ be an integer shift parameter, to be determined later. Given a we construct a hierarchy of grid lines in the plane.

For level t, define the *level-t unit* as $u_t = \delta\nu_t/(\theta 2\sqrt{2})$. Note that u_t is an integer.

We define a set of horiontal lines with y-coordinates from the set:

$$a + b \cdot u_t : b \in \mathbb{Z}$$

Then for every polygon $P_i \in \mathcal{P}_t$ if the lines (horizontal or vertical) from level $t+1$ cross the polygon P_i, we split it according to lines to at most 4 polygons with the same weights and add these to $\mathcal{P}'$. Otherwise $P_i \in \mathcal{P}'$.

Lemma 5.4.1 *In polynomial time one can yield a family $\mathcal{P}'$ that satisfies*

$$OPT(\mathcal{P}') \leq (1 + 16\epsilon)OPT(\mathcal{P})$$

with probability at least 3/4. Moreover one can construct the solution $S \subseteq \mathcal{P}$ back from the solution of $S' \subseteq \mathcal{P}'$ such that $w(S) \leq w(S')$.

Sketch of proof If $\nu_t \leq d_i \leq \mu_t\epsilon$, then there is at most ϵ probability that with random offset a, the line will cut this polygon on the t-th level vertically. Analogically for horizontal cuts.

If $\mu_t\epsilon < d_i < \nu_{t+1}$, then this situation happens only for one b in set $\{0, 1, 2, \ldots, 1\ \epsilon\}$.

Then for every polygon P_i in optimal solution OPT, the expected value of sum of weights for all polygons in $\mathcal{P}'$ corresponsing to the polygon P_i is at most 4ϵ.

So with Markov inequality we can prove that $\Pr(OPT(\mathcal{P}') > (1 + 16\epsilon)OPT(\mathcal{P})) \leq 1/4$

Extending polygons On every level t, for every $P_i \in \mathcal{P}'_t$, we will create a new polygon P_i' that consists of every cell in hierarchical grid on level t, that have non-empty intersection with P_i.

New polygon will fit inside P_i shifted to every dimension by $u_t\sqrt{2} = \delta\nu_t/(2\theta) \leq \delta d_i/(2\theta)$.

The larger dimension is not extended more than by δ:

$$2 \cdot \delta d_i/(2\theta) = \delta d_i/\theta \leq \delta d_i$$

The shorter dimension is at most d_i/θ, so it also wouldn't be extended by more than δ.

5.4.2. Dynamic programming

Chapter 6

Conclusions

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