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# Approximation and Parametrized Algorithms for Segment Set Cover

6

Master's thesis

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in COMPUTER SCIENCE

8

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9

June 2020

10 **Supervisor's statement**

11 Hereby I confirm that the presented thesis was prepared under my supervision and  
12 that it fulfils the requirements for the degree of Master of Computer Science.

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## **Abstract**

23 The work presents a study of different geometric set cover problems. It mostly focuses on  
24 segment set cover and its connection to the polygon set cover.

25

## **Keywords**

26 set cover, geometric set cover, FPT,  $W[1]$ -completeness, APX-completeness, PCP theorem,  
27 NP-completeness

28

## **Thesis domain (Socrates-Erasmus subject area codes)**

29 11.3 Informatyka

30

31

## **Subject classification**

32 D. Software

33 D.127. Blabalgorithms

34 D.127.6. Numerical blabalysis

35

## **Tytuł pracy w języku polskim**

36 Algorytmy parametryzowania i trudność aproksymacji problemu pokrywania zbiorów  
37 odcinkami na płaszczyźnie



# Contents

39	<b>1. Introduction</b>	5
40	<b>2. Definitions</b>	7
41	2.1. Geometric Set Cover	7
42	2.2. Approximation	7
43	2.3. $\delta$ -extensions	7
44	<b>3. Geometric Set Cover with segments</b>	9
45	3.1. FPT for segments	9
46	3.1.1. Segments parallel to one of the axis	9
47	3.1.2. Segments in $d$ directions	9
48	3.1.3. Segments in arbitrary direction	9
49	3.2. APX-completeness for segments parallel to axes	10
50	3.2.1. MAX-(3,3)-SAT problem	10
51	3.2.2. Reduction construction	12
52	3.2.2.1. VARIABLE-gadget	12
53	3.2.2.2. OR-gadget	13
54	3.2.2.3. CLAUSE-gadget	15
55	3.2.2.4. Summary	16
56	3.2.3. Summary of construction	17
57	3.2.4. Proofs of construction Lemma 3.2.1	17
58	3.2.4.1. Proof of Lemma 3.2.1	18
59	3.3. Weighted segments	18
60	3.3.1. FPT for weighted segments with $\delta$ -extensions	18
61	3.3.2. W[1]-completeness for weighted segments in 3 directions	20
62	3.3.3. What is missing	22
63	<b>4. Geometric Set Cover with lines</b>	23
64	4.1. Lines parallel to one of the axis	23
65	4.2. FPT for arbitrary lines	23
66	4.3. APX-completeness for arbitrary lines	23
67	4.4. 2-approximation for arbitrary lines	24
68	4.5. Connection with general set cover	24
69	<b>5. Geometric Set Cover with polygons</b>	25
70	5.1. State of the art	25
71	<b>6. Conclusions</b>	27



## Chapter 1

# Introduction

The Set Cover problem is one of the most common NP-complete problems. [tutaj referencja]  
We are given a family of sets and have to choose the smallest subfamily of these sets that cover  
all their elements. This problem naturally extends to settings where we put different weights  
on the sets and look for the subfamily of the minimal weight. This problem is NP-complete  
even without weights and if we put restrictions on what the sets can be. One of such variants  
is Vertex Cover problem, where sets have size 2 (they are edges in a graph).

In this work we focus on another such variant where the sets correspond to some geometric  
shapes and only some points of the plane have to be covered. When these shapes are rectangles  
with edges parallel to the axis, the problem can be proven to be W[1]-complete (solution of  
size  $k$  cannot be found in  $n^o(k)$  time), APX-complete (for sufficiently small  $\epsilon > 0$ , the problem  
does not admit  $1 + \epsilon$ -approximation scheme) [referencje].

Some of these settings are very easy. Set cover with lines parallel to one of the axis can  
be solved in polynomial time.

There is a notion of  $\delta$ -expansions, which loosen the restrictions on geometric set cover. We  
allow the objects to cover the points after  $\delta$ -expansion and compare the result to the original  
setting. This way we can produce both FPT and EPTAS for the rectangle set cover with  
 $\delta$ -extensions [referencje].

**Our contribution.** In this work, we prove that unweighted geometric set cover with seg-  
ments is fixed parameter tractable (FPT).

Moreover, we show that geometric set cover with segments is APX-complete for unweighted  
axis-parallel segments, even with  $1/2$ -extensions. So the problem for very thin rectangles  
also can't admit PTAS. Therefore, in the efficient polynomial-time approximation scheme  
(EPTAS) for *fat polygons* by [Har-Peled and Lee, 2009], the assumption about polygons  
being fat is necessary.

Finally, we show that geometric set cover with weighted segments in 3 directions is  
W[1]-complete. However, geometric set cover with weighted segments is FPT if we allow  
 $\delta$ -extension.

This result is especially interesting, since it's counter-intuitive that the unweighted setting  
is FPT and the weighted setting is W[1]-complete. Most of such problems (like vertex cover  
or [wiecej przykladow]) are equally hard in both weighted and unweighted settings.





## Chapter 2

## Definitions

### 2.1. Geometric Set Cover

In the geometric set cover problem we are given  $\mathcal{P}$  – set of objects,  $\mathcal{C}$  – set of points. The task is to choose  $\mathcal{R} \subseteq \mathcal{P}$  such that every point in  $\mathcal{C}$  is inside some element from  $\mathcal{R}$  and  $|\mathcal{R}|$  is minimal.

In the parametrized setting for a given  $k$ , we only look for a solution  $\mathcal{R}$  such that  $|\mathcal{R}| \leq k$ .

In the weighted setting, there is some given weight function  $f : \mathcal{P} \rightarrow \mathbb{R}^+$ , and we would like to minimize  $\sum_{R \in \mathcal{R}} f(R)$ .

### 2.2. Approximation

Let us recall some of the definitions related to approximation problems that will be used in the following sections.

**Definition 2.2.1.** A *polynomial-time approximation scheme (PTAS)* for a minimization problem  $\Pi$  is a family of algorithms  $\mathcal{A}_\epsilon$  for every  $\epsilon > 0$ , such that  $\mathcal{A}_\epsilon$  takes an instance  $I$  of  $\Pi$  and in polynomial time finds a solution that is within a factor  $(1 + \epsilon)$  of being optimal. That means the reported solution has weight at most  $(1 + \epsilon)\text{opt}(I)$ , where  $\text{opt}(I)$  is the weight of an optimal solution for  $I$ .

**Definition 2.2.2.** Problem is *APX-hard* if assuming  $P \neq NP$ , there exists such  $\epsilon > 0$  such that there is no polynomial time  $(1 + \epsilon)$ -approximation algorithm.

### 2.3. $\delta$ -extensions

TODO PLACEHOLDER for introductory text

$\delta$ -extensions is one of the modifications to a problem, that makes geometric set cover problem easier, it has been already used in literature (place some reference here).

**Definition 2.3.1.  $\delta$ -extensions for center-symmetric objects** For any  $\delta > 0$  and center-symmetric geometric object  $L$  with centre of symmetry  $S = (x_s, y_s)$ , a  $\delta$ -*extension* of this object,  $L^{+\delta}$ , is an object without border, but with all vertices extended by  $\delta d$ , i.e.  $L^{+\delta} = \{(1 + \delta) \cdot (x - x_s, y - y_s) + (x_s, y_s) : (x, y) \in L\}$ .

A relaxed cover problem with  $\delta$ -extensions is a modified version of a problem where:

- We need to cover all the points in  $\mathcal{C}$  with objects from  $\{P^{+\delta} : P \in \mathcal{P}\}$  (which always include no less points than the objects before  $\delta$ -extensions);
- We look for a solution that is no larger than the optimal solution of the original problem. It doesn't need to be an optimal solution in the modified problem.

**Definition 2.3.2. Geometric set cover problem with  $\delta$ -extensions** We define an optimization cover problem  $\mathcal{P}$  with  $\delta$ -extensions as the problem where for an input instance  $I$ , the task is to output a solution  $\mathcal{R}$ , such that the  $\delta$ -extended set  $\{R^{+\delta} : R \in \mathcal{R}\}$  covers  $I$  and is no worse than the optimal solution for the problem without extensions, i.e.  $|\mathcal{R}| \leq |\mathcal{R}^{opt}|$ .

**Definition 2.3.3. Geometric set cover PTAS with  $\delta$ -extensions** We define a PTAS for cover problem  $\mathcal{P}$  with  $\delta$ -extensions as an algorithm that takes as an input instance  $I$ , and outputs a solution  $\mathcal{R}$ , such that the  $\delta$ -extended set  $\{R^{+\delta} : R \in \mathcal{R}\}$  covers  $I$  and is within a  $(1 + \epsilon)$  factor of the optimal solution for this problem without extensions, i.e.  $(1 + \epsilon)|\mathcal{R}| \leq |\mathcal{R}^{opt}|$ .

## Chapter 3

# Geometric Set Cover with segments

### 3.1. FPT for segments

#### 3.1.1. Segments parallel to one of the axis

You can find this in Platypus book.

We'll show  $\mathcal{O}(2^k)$  branching algorithm. Let's take point  $K$  that hasn't been covered yet with the smallest coordinate in lexicographical order. We need to cover  $K$  with some of the remaining segments.

We choose one of the 2 directions on which we cover this point. In this direction we take greedily the segment that will cover the most points (there are points in  $\mathcal{C}$  only on one side of  $K$  in this direction, so all segments covering  $K$  in this direction create monotone sequence of sets – zbiory zstępujące).

#### 3.1.2. Segments in $d$ directions

The same algorithm as before but in complexity  $\mathcal{O}(d^k)$ .

#### 3.1.3. Segments in arbitrary direction

**Theorem 3.1.1. (FPT for segment cover).** *There exists an algorithm that given a family  $\mathcal{P}$  of  $n$  segments (in any direction), a set of  $m$  points  $\mathcal{C}$  and a parameter  $k$ , runs in time  $f(k) \cdot (nm)^c$  for some computable function  $f$  and constant  $c$ , and outputs a subfamily  $\mathcal{R} \subseteq \mathcal{P}$  such that  $|\mathcal{R}| \leq k$  and  $\mathcal{R}$  covers all points in  $\mathcal{C}$  or determines that the solution of size at most  $k$  doesn't exist.*

This theorem is proved by following lemmas.

**Lemma 3.1.1. (Reduction).** *Given a family  $\mathcal{P}$  of  $n$  segments (in any direction), a set of  $m$  points  $\mathcal{C}$  for segment cover problem, without a loss of generality we can assume that no two segments cover the same set of points.*

**Proof.** Trivial.

**Lemma 3.1.2. (Kernel for segment cover).** *For a problem of segment cover that given a family  $\mathcal{P}$  of  $n$  segments (in any direction), a set of  $m$  points  $\mathcal{C}$  and a parameter  $k$ , there exists a family of kernels  $K \subset \mathcal{P}$ , where any line contains no more than  $k$  points and there exists optimal solution, that is subset of  $K$ . We can find all such kernels in complexity  $\mathcal{O}(k^k)$  using branching technique and there are  $\mathcal{O}(k^k)$  of them.*

**Proof.** First we use Lemma 3.1.1.

Assume there exists a line  $l$  containing points  $x_1, \dots, x_t$ , where  $t \geq k + 1$ . Note that a segment that does not lie on  $l$  can cover only at most one of the points  $x_i$ . Therefore, out of points  $x_1, \dots, x_{k+1}$ , at least one has to be covered by a segment that lies on  $l$ , let us fix  $x_i$  to be the first such point (We branch over  $i \in \{1 \dots k + 1\}$ ). Then, we can greedily choose a segment that lies on  $l$ , covers  $x_i$ , and also covers the largest number of points  $x_j$  for  $j > i$ .

Since we have at most  $k + 1$  choices to branch over and each choice adds a segment to the constructed solution, we obtain an algorithm with complexity  $O(k^k)$ .

**Proof of theorem 3.1.1.** Assuming Lemma 3.1.2.

First we use Lemma 3.1.1.

Since any segment covers a set of colinear points, for such a kernel  $k$  segments can cover only at most  $k^2$  points. Therefore, for the answer to be positive, the number of points has to be at most  $k^2$ . The number of segments is now bounded by  $k^4$ , since if we consider two *extreme* points covered by a given segment, then these pairs must be distinct, otherwise two segments would contain the same set of points. Since both the number of points and the number of segments is bounded by a function of  $k$ , this instance can be easily solved in time  $O(f(k))$ . Since there are  $O(k^k)$  possible kernels, the final algorithm work in  $f(k) \cdot (nm)$ .

## 3.2. APX-completeness for segments parallel to axes

In this section we analyze whether there exists an  $(1 + \epsilon)$ -approximation scheme for geometric set cover for rectangles. We show that we can restrict this problem to a very simple setting: segments parallel to axes and allow  $(1/2)$ -extension, and the problem is still APX-hard. Note that segments are just degenerated rectangles with one side being very narrow.

Our results can be summarized in the following theorem and this section aims to prove it.

**Theorem 3.2.1.** (*axis-parallel segment set cover with  $1/2$ -extension is APX-hard*). *Unweighted geometric set cover with axis-parallel segments in 2D (even with  $1/2$ -extension) is APX-hard, that is, assuming  $P \neq NP$ , there does not exist a PTAS for this problem.*

Theorem 3.2.1 implies the following.

**Corollary 3.2.1.** (*rectangle set cover is APX-hard*). *Unweighted geometric set cover with rectangles (even with  $1/2$ -extension) is APX-hard.*

We prove Theorem 3.2.1 by taking a problem that is APX-hard and showing a reduction. For this problem we choose MAX-(3,3)-SAT which we define below.

Given an instance  $I$  of MAX-(3,3)-SAT, we construct an instance  $J$  of axis-parallel segment set cover problem, such that for a sufficiently small  $\epsilon > 0$ , a polynomial  $(1 + \epsilon)$ -approximation algorithm for  $J$  would be able to distinguish whether an instance  $I$  of MAX-(3,3)-SAT is fully satisfiable or  $(7/8 + \epsilon)$ -satisfiable. However according to (Theorem 3.2.2) that problem is NP-hard. Then it concludes that  $P = NP$ , which is a contradiction with assumption.

### 3.2.1. MAX-(3,3)-SAT problem

**Definition 3.2.1.** *MAX-3SAT is a maximization problem. We are given a 3-CNF formula, and need to find an assignment of variables that satisfies the most clauses.*

214 **Definition 3.2.2.** *MAX-(3,3)-SAT is MAX-3SAT with an additional restriction that every*  
 215 *variable appears in exactly 3 clauses. Note that thus, the number of clauses is equal to number*  
 216 *of variables.*

217 In the lemmas above we use a property of MAX-(3,3)-SAT proved in [Håstad, 2001] and  
 218 described in Theorem 3.2.2.

219 **Theorem 3.2.2.** *[Håstad, 2001]*

220 *For any  $\epsilon > 0$ , it is NP-hard to distinguish satisfiable (3,3)-SAT formulas from at most*  
 221  *$(7/8 + \epsilon)$ -satisfiable (3,3)-SAT formulas.*

222 The following lemma encapsulates the properties of the reduction described in this section,  
 223 and it allows us to prove Theorem 3.2.1.

224 **Lemma 3.2.1.** *Given an instance  $S$  of MAX-(3,3)-SAT with  $n$  variables and optimum value*  
 225  *$OPT(S)$ , we can construct an instance  $I$  of geometric set cover with axis-parallel segments in*  
 226 *2D with  $1/2$ -extensions, such that:*

- 227 (1) *For every solution  $X$  of instance  $I$ , there exists a solution of  $S$  of size at least  $15n - |X|$ .*
- 228 (2) *For every solution  $X$  of instance  $S$ , there exists a solution of  $I$  of size  $15n - |X|$ .*
- 229 (3) *Every solution with  $1/2$ -extensions for  $I$  is also a solution to the original instance  $I$ .*

230 *Therefore, the optimal solution of  $I$  is  $OPT(I) = 15n - OPT(S)$ .*

231 We prove Lemma 3.2.1 in subsequent sections, but meanwhile let us prove Theorem 3.2.1  
 232 using Lemma 3.2.1 and Theorem 3.2.2.

233 TODO: This below can't use current template

234 **Proof of Theorem 3.2.1** . Consider any  $0 < \epsilon < 1/(15 \cdot 8)$ .

235 Let us assume that there exists a polynomial-time  $(1 + \epsilon)$ -approximation scheme for un-  
 236 weighted geometric set cover with axis-parallel segments in 2D with  $(1/2)$ -extensions. We  
 237 construct an algorithm that solves the problem stated in Theorem 3.2.2, thereby proving that  
 238  $P = NP$ .

239 Take an instance  $S$  of MAX-(3,3)-SAT to be distinguished and construct an instance of  
 240 geometric set cover  $I$  using Lemma 3.2.1.

241 We now use the  $(1 + \epsilon)$ -approximation algorithm for geometric set cover on  $I$ , denote the  
 242 cost of the result of applying this algorithm as  $approx(I)$ .

243 We prove that if in  $S$  one can satisfy at most  $(\frac{7}{8} + \epsilon)n$  clauses, then  $approx(I) \geq 15n -$   
 244  $(\frac{7}{8} + \epsilon)n$  and if  $S$  is satisfiable, then  $approx(I) < 15n - (\frac{7}{8} + \epsilon)n$ .

**Assume  $S$  satisfiable.** From the definition of  $S$  being satisfiable, we have:

$$OPT(S) = n.$$

245 From Lemma 3.2.1 we have:

$$OPT(I) = 14n.$$

Therefore,

$$\begin{aligned} approx(I) &\leq (1 + \epsilon)OPT(I) = 14n(1 + \epsilon) = 14n + 14\epsilon \cdot n = \\ &= 14n + (15\epsilon - \epsilon)n < 14n + \left(\frac{1}{8} - \epsilon\right)n = 15n - \left(\frac{7}{8} + \epsilon\right)n \end{aligned}$$

**Assume  $S$  is at most  $(\frac{7}{8} + \epsilon)$  satisfiable** . From the definition of  $S$  being at most  $(\frac{7}{8} + \epsilon)n$  satisfiable, we have:

$$OPT(S) \leq \left(\frac{7}{8} + \epsilon\right)n$$

From Lemma 3.2.1 we have:

$$OPT(I) \geq 15n - \left(\frac{7}{8} + \epsilon\right)n$$

Since a solution to  $I$  with extensions is also a solution without extensions, by Lemma 3.2.1 (3.), we have:

$$approx(I) \geq OPT(I) = 15n - \left(\frac{7}{8} + \epsilon\right)n$$

Therefore, by using the assumed  $(1 + \epsilon)$ -approximation algorithm, it is possible to distinguish the case when  $S$  is satisfiable from the case when it is at most  $(\frac{7}{8} + \epsilon)n$  satisfiable, since there exists a threshold on the approximation result in segment set cover that distinguishes these two cases. Hence, the assumed approximation algorithm cannot exist, unless  $P = NP$ .

### 3.2.2. Reduction construction

We show reduction from MAX-(3,3)-SAT problem to geometric set cover with segments parallel to axis. Moreover the instance of geometric set cover will be robust to 1/2-extensions (have the same optimal solution after 1/2-extension).

The construction will be composed of 2 types of gadgets: **VARIABLE-gadgets** and **CLAUSE-gadgets**. **CLAUSE-gadgets** would be constructed using two **OR-gadgets** connected together.

#### 3.2.2.1. VARIABLE-gadget

**VARIABLE-gadget** is responsible for choosing the value of a variable in a CNF formula. It allows two minimal solutions and every minimal solution must use exactly one of the  $(c_i, g_i)$  and  $(f_i, h_i)$  segments. These two choices correspond to the two Boolean values of the variable.

**Points.** Define points:

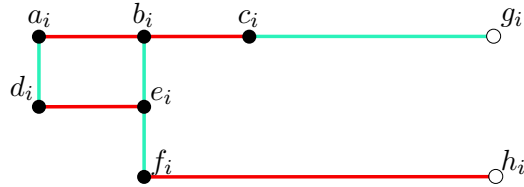


Figure 3.1: **VARIABLE-gadget**. We denote the set of points marked with black circles as  $C_{var}^i$ , and they need to be covered (are part of the set  $\mathcal{C}$ ). Note that some of the points are not marked as black dots and exists only to name segments for further reference. We denote the set of red segments as  $X_{false}^i$  and the set of green segments as  $X_{true}^i$ .

With  $L = 12n$ :

$$\begin{aligned} a &= (-L, 0) & b &= (-\frac{2}{3}L, 0) & c &= (-\frac{1}{3}L, 0) & d &= (-L, 1) \\ e &= (-\frac{2}{3}L, 1) & f &= (-\frac{2}{3}L, 2) & g &= (L, 0) & h &= (L, 2) \end{aligned}$$

Let us define:

$$C_{var} = \{a, b, c, d, e, f\}$$

and

$$C_{var}^i = C_{var} + (0, 4i)$$

266 **Segments.** Let us define:

$$X_{true}^i = \{(a_i, d_i), (b_i, f_i), (c_i, g_i)\}$$

$$X_{false}^i = \{(a_i, c_i), (d_i, e_i), (f_i, h_i)\}$$

$$P_{var}^i = X_{true}^i \cup X_{false}^i$$

267 **Lemma 3.2.2.** *For any  $1 \leq i \leq n$ , points  $C_{var}^i$  can be covered using 3 segments from  $P_{var}^i$ .*

268 *Proof.* We can use either set  $X_{true}^i$  or  $X_{false}^i$ . □

269 **Lemma 3.2.3.** *For any  $1 \leq i \leq n$ , points  $C_{var}^i$  can not be covered with less than 3 segments*  
 270 *from  $P_{var}^i$ .*

271 *Proof.* No segment of  $P_{var}^i$  covers more than one point from  $\{d_i, f_i, c_i\}$ , therefore  $C_{var}^i$  can not  
 272 be covered with less than segments. □

273 **Lemma 3.2.4.** *If both segments  $(c_i, g_i)$  and  $(f_i, h_i)$  are chosen, then the covering the remain-*  
 274 *ing points from  $C_{var}^i$  requires at least 2 different segments from  $P_{var}^i$ .*

275 *Proof.* No segment of  $P_{var}^i$  covers more than one point from  $\{a_i, e_i\}$ , therefore  $C_{var}^i - \{c_i, f_i, g_i, h_i\}$   
 276 can not be covered with less than 2 segments. □

### 277 3.2.2.2. OR-gadget

278 OR-gadget has 3 important segments –  $x, y, result$ .  $x$  and  $y$  don't count to the weight of  
 279 solution of OR-gadget (they are part of different gadgets). It has a minimal solution of weight  
 280  $w$  and  $result$  can be chosen only if  $x$  or  $y$  are also chosen for the solution. If none of them  
 281 are chosen, then solution choosing  $result$  segment has weight at least  $w + 1$ . Therefore the  
 282 following formula holds for a solution  $R$  assuming that  $R$  uses only  $w$  from this OR-gadget:

$$(x \in R) \vee (y \in R) \iff result \in R$$

283 **Points.**

$$\begin{array}{llll} l_0 = (0, 0) & m_0 = (0, 1) & n_0 = (0, 2) & o_0 = (0, 3) \\ p_0 = (0, 4) & q_0 = (1, 1) & r_0 = (1, 3) & s_0 = (2, 1) \\ t_0 = (2, 2) & u_0 = (2, 3) & v_0 = (3, 2) & \end{array}$$

$$vec_{i,j} = (10i + 3 + 3j, 4n + 2j)$$

285 Define  $\{l_{i,j}, m_{i,j} \dots v_{i,j}\}$  as  $\{l_0, m_0 \dots v_0\}$  shifted by  $vec_{i,j}$

286 Note that  $v_{i,0} = l_{i,1}$  (see Figure 3.3)

$$C\_or\_gadget_{i,j} = \{l_{i,j}, m_{i,j}, n_{i,j}, o_{i,j}, p_{i,j}, q_{i,j}, r_{i,j}, s_{i,j}, t_{i,j}, u_{i,j}\}$$



Figure 3.2: **OR-gadget**. We denote these point as  $or\_gadget_{i,j}$ . We denote set of red segments as  $or_{i,j}^{false}$ , set of blue segments as  $or_{i,j}^{true}$ , green and yellow segments as  $or\_move\_variable_{i,j}$ .

287 **Segments.** We define names subsets of segments, to refer to them in lemmas.

$$or_{i,j}^{false} = \{(q_{i,j}, r_{i,j}), (s_{i,j}, u_{i,j})\}$$

$$or_{i,j}^{true} = \{(m_{i,j}, s_{i,j}), (o_{i,j}, u_{i,j}), (t_{i,j}, v_{i,j})\}$$

$$or\_move\_variable_{i,j} = \{(l_{i,j}, n_{i,j}), (n_{i,j}, p_{i,j})\}$$

288 Segments in OR-gadget:

$$P\_or\_gadget_{i,j} = or_{i,j}^{false} \cup or_{i,j}^{true} \cup or\_move\_variable_{i,j}$$

289 **Lemma 3.2.5.** For any  $1 \leq i \leq n, j \in \{0, 1\}$  and  $x \in \{l_{i,j}, p_{i,j}\}$  we can cover points in  
 290  $C\_or\_gadget_{i,j} - \{x\} \cup \{v_{i,j}\}$  with 4 segments.

291 **Proof.** We can do that using one segment from  $or\_move\_variable_{i,j}$  (chosen depending on  
 292 the value of  $x$ ) and all segments from  $or_{i,j}^{true}$ .

293 **Lemma 3.2.6.** For any  $1 \leq i \leq n, j \in \{0, 1\}$ , we can cover points in  $C\_or\_gadget_{i,j}$  with 4  
 294 segments from  $P\_or\_gadget_{i,j}$ .

295 **Proof.** We can do that using  $or\_move\_variable_{i,j}$  and  $or_{i,j}^{false}$ .



### 296 3.2.2.3. CLAUSE-gadget

297 CLAUSE-gadget is responsible for calculating if choice of the variable values meets the clause  
 298 in formula. It has minimal solution of weight  $w$  if at least one variable in the clause has a  
 299 correct value. Otherwise it has minimal solution  $w + 1$ . This way by the minimal solution for  
 300 the whole problem, we can tell how many clauses were satisfiable.

301 The CLAUSE-gadgets consist of two OR-gadgets. We don't want the CLAUSE-gadgets  
 302 to be crammed somewhere between the very long variable segments. That's why we have a  
 303 simple gadget to *pass* the value of the segment, ie. segments  $(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1})$ .  
 304 Two segments and one of them is chosen if  $x$  was chosen in the solution and the other one if  
 305  $x$  wasn't.

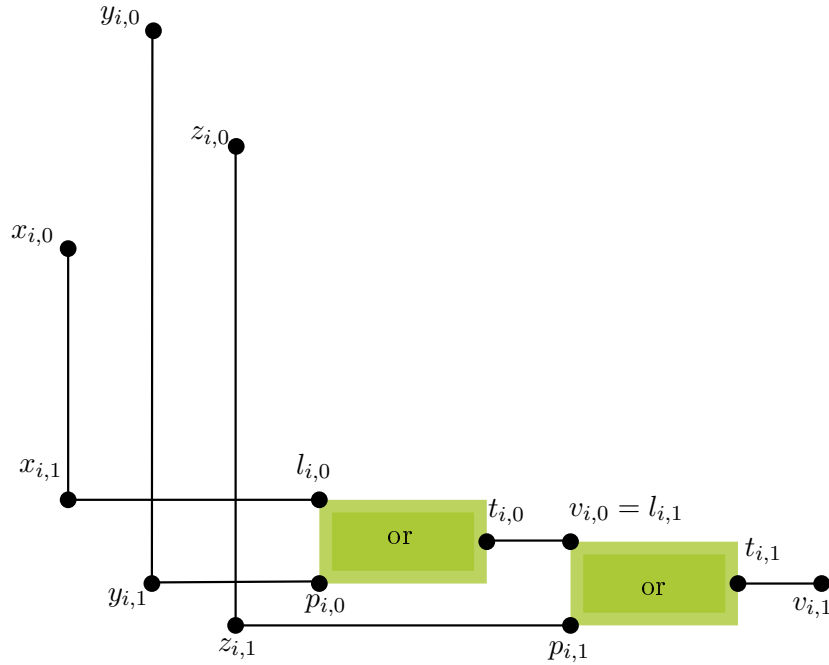


Figure 3.3: **CLAUSE-gadget**. We denote set of these points as  $C\_clause_i$ . Every green rectangle is an OR-gadget.  $y$ -coordinates of  $x_{i,0}$ ,  $y_{i,0}$  and  $z_{i,0}$  depend on the values of variables in the  $i$ -th clause.

306 **Points.** TODO: Rephrase it

307 Assuming clause  $C_i = x_i \vee y_i \vee z_i$ , function  $idx(w)$  is returning index of the variable  $w$ ,  
 308 function  $neg(w)$  is returning whether variable  $w$  is negated in a clause.

$$\begin{aligned}
 x_{i,0} &= (10i + 1, 4 \cdot idx(x_i) + 2 \cdot neg(x_i)) & x_{i,1} &= (10i + 1, 4n) \\
 y_{i,0} &= (10i + 2, 4 \cdot idx(y_i) + 2 \cdot neg(y_i)) & y_{i,1} &= (10i + 2, 4n + 4) \\
 z_{i,0} &= (10i + 3, 4 \cdot idx(z_i) + 2 \cdot neg(z_i)) & z_{i,1} &= (10i + 3, 4n + 6)
 \end{aligned}$$

$$move\_variable_i = \{x_{i,j} : j \in \{0, 1\}\} \cup \{y_{i,j} : j \in \{0, 1\}\} \cup \{z_{i,j} : j \in \{0, 1\}\}$$

$$C\_clause_i = move\_variable_i \cup C\_or\_gadget_{i,0} \cup C\_or\_gadget_{i,1} \cup \{v_{i,1}\}$$

**Segments.**

$$P\_clause_i = \{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1}), (x_{i,1}, l_{i,0}), (y_{i,1}, p_{i,0}), (z_{i,1}, p_{i,1}), \} \cup \\ \cup P\_or\_gadget_{i,0} \cup P\_or\_gadget_{i,1}$$

**Lemma 3.2.7.** *For any  $1 \leq i \leq n$  and  $a \in \{x_{i,0}, y_{i,0}, z_{i,0}\}$ , points  $C\_clause_i - \{a\}$  can be covered using 11 segments from  $P\_clause_i$ .*

**Proof.** For  $a = x_{i,0}$  (analogous proof for  $y_{i,0}$ ): First we use Lemma 3.2.5 twice with excluded  $x = l_{i,0}$  and  $x = l_{i,1} = v_{i,0}$ , resulting with 8 segments  $or_{i,0}^{true} \cup or_{i,1}^{true}$  which cover all required points apart from  $x_{i,1}, y_{i,0}, y_{i,1}, z_{i,0}, z_{i,1}, l_{i,0}$ . We cover those using additional 3 segments:  $\{(x_{i,1}, l_{i,0}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1})\}$

For  $a = z_{i,0}$ : Using Lemma 3.2.6 and Lemma 3.2.5 with  $x = p_{i,1}$ , resulting with 8 segments  $or_{i,0}^{false} \cup or_{i,1}^{true}$  which cover all required points apart from  $x_{i,0}, x_{i,1}, y_{i,0}, y_{i,1}, z_{i,1}, p_{i,1}$ . We cover those using additional 3 segments:  $\{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,1}, p_{i,1})\}$ .

**Lemma 3.2.8.** *Points  $C\_clause_i$  can be covered with 12 segments from  $P\_clause_i$ .*

**Proof.** Using Lemma 3.2.6 twice we can cover  $or\_gadget_{i,0}$  and  $or\_gadget_{i,1}$  with 8 segments.

To cover the remaining points we additionally use:  $\{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1}), (t_{i,1}, v_{i,1})\}$

**Lemma 3.2.9.** *For any  $1 \leq i \leq n$ , points  $C\_clause_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$  can not be covered using less than 11 segments from  $P\_clause_i$ .*

*All points  $C\_clause_i$  can not be covered with less than 12 segments from  $P\_clause_i$ .*

**Proof of no cover with less than 12 segments.** There is independent set of 12 points in  $C\_clause_i \supseteq \{x_{i,0}, y_{i,0}, z_{i,0}, l_{i,0}, p_{i,0}, q_{i,0}, u_{i,0}, v_{i,0} = l_{i,1}, p_{i,1}, q_{i,1}, u_{i,1}, v_{i,1}\}$ .

**Proof of no cover with less than 11 segments.** We can choose disjoint sets  $X, Y, Z$  such that  $X \cup Y \cup Z \subseteq C\_clause_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$  and there are no segments covering points from different sets. And we prove lower bounds for each of these sets.

$$X = \{x_{i,1}, y_{i,1}, z_{i,1}\}$$

Set  $X$  is an indendent set, so it must be covered with 3 segments.

$$Y = or\_gadget_{i,0} - \{l_{i,0}, p_{i,0}\}$$

$$Z = or\_gadget_{i,1} - \{l_{i,1}, p_{i,1}\}$$

For both  $Y$  and  $Z$  we can check all of the subsets of 3 segments with brutforce that none of them cover, so they have to be covered with 4 segments.

TODO: Funny fact, neither  $Y$  nor  $Z$  doesn't have independent set of size 4.

Therefore  $C\_clause_i$  must be covered with at least  $3 + 4 + 4 = 11$  segments.

#### 3.2.2.4. Summary

Add some smart lemmas that sets will be exclusive to each other.

**Lemma 3.2.10. Robustness to 1/2-extensions.** *For every segment  $s \in \mathcal{P}$ ,  $s$  and  $s^{+1/2}$  cover the same points from  $\mathcal{C}$ .*

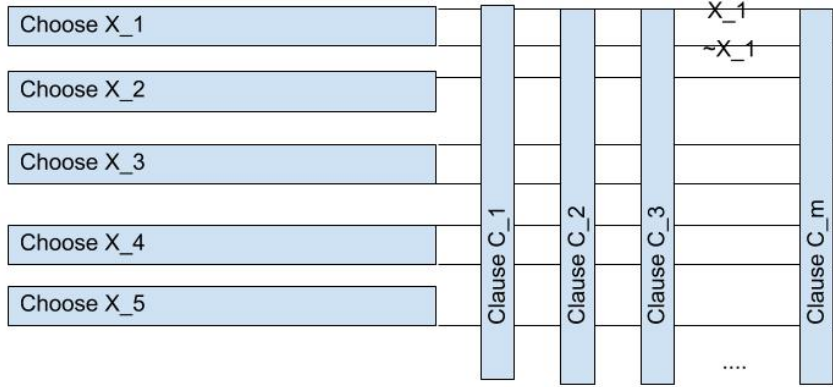


Figure 3.4: **General schema.**

General layout of VARIABLE-gadget and CLAUSE-gadget and how they interact with each other.

TODO: Rename Choose X to VARIABLE-gadget and Clause C to CLAUSE-gadget.

### 3.2.3. Summary of contruction

We define:

$$\mathcal{C} := \bigcup_{1 \leq i \leq n} C\_variable_i \cup C\_clause_i$$

$$\mathcal{P} := \bigcup_{1 \leq i \leq n} P\_variable_i \cup P\_clause_i$$

The subsequent sections define these sets.

We prove some properties of different gadgets. Every segment for a gadget will only cover points in this gadget (won't interact with any diferent gadget), so we can prove lemmas *locally*.

TODO:  $y$  axis is increasing values downward on figures (not upwards like in normal).

### 3.2.4. Proofs of construction Lemma 3.2.1

**Lemma 3.2.11.** *Given an instance of MAX-(3,3)-SAT of size  $n$  with optimal solution  $k$ . For instance of geometric cover, constructed according to Lemma 3.2.1, there exists a solution of weight  $15n - k$ .*

**Proof.** Let's name the assignments of the variables in MAX-(3,3)-SAT instance, that achieve the optimal solution,  $y_1, y_2 \dots y_n$ , Let's cover every VARIABLE-gadget with solution described in Lemma 3.2.2, in the  $i$ -th gadget choosing the set of segments responsible for the value of  $y_i$  (true –  $x_i^{true}$  or false –  $x_i^{false}$ ).

Cover every satisfied CLAUSE-gadget with solution described in Lemma 3.2.7 and unsatisfied CLAUSE-gadget with solution from Lemma 3.2.8.

This solution uses  $3n + (11m + (m - k)) = 15n - k$  segments.

**Lemma 3.2.12.** *Given an instance of MAX-(3,3)-SAT of size  $n$ , and solution of size  $w$  to the instance of geometric cover, constructed according to Lemma 3.2.1, there exists a solution to MAX-(3,3)-SAT of size at least  $15n - w$ .*

**Proof.** Among  $x_i^{true} \cup x_i^{false}$ , we need to use at least 3 segments (Lemma 3.2.3). If we have chosen both segments  $(c_i, g_i)$  and  $(f_i, h_i)$ , then we have used at least 4 segments (Lemma 3.2.4).

If we chose at most one of the segments  $(c_i, g_i)$  and  $(f_i, h_i)$ , choose the corresponding variable value to the solution. If we chose both segments, choose the value that appears in most (at least 2) clauses. If we have chosen none of the segments, choose any value.

To cover  $\bigcup_{1 \leq i \leq n} C\_variable_i$  we have used at least  $3n + a$  segments, where  $a$  is the number of  $i$  such that we have chosen both values  $(c_i, g_i)$  and  $(f_i, h_i)$ .

Among the segments responsible for the clause  $C_i = x \vee y \vee z$  we need to use at least 11 segments (Lemma 3.2.9) and if we can cover it with 11 segments, then we have earlier chosen segment responsible for the value of variable  $x, y$  or  $z$  that satisfies  $C_i$ .

So we have at least 11 segments for satisfied clauses and at least 12 segments for unsatisfied clauses, so we cover it with at least  $11n + b$  segments, where  $b$  is number of clauses where none of the variables  $x, y, z$  were chosen. If the segment responsible for value of  $x$  was taken, but this variable is set to have different value, then we have chosen segments for both  $x$  and  $\neg x$  for this variable, so "we cheated" and this maybe clause is not met, but we assigned the value for this  $x_i$  that meets the most clauses, so for each of such "cheated" variables, at most one of the clauses isn't met.

So there are at most  $a + b$  unsatisfied clauses in this instance, so we have shown the assignment with at least  $n - (a + b)$  satisfied clauses.

$$w \geq 3n + a + 11n + b = 14n + a + b$$

$$15n - w \leq 15n - 14n - a - b = n - (a + b)$$

#### 3.2.4.1. Proof of Lemma 3.2.1

Given an instance of MAX-(3,3)-SAT of size  $n$  with optimal result  $k$ . Let's construct an instance of geometric cover, constructed in aforementioned manner.

Given the Lemma 3.2.11, we know the optimal solution for the constructed geometric cover is at most  $15n - k$  and since the  $k$  is optimal solution for MAX-(3,3)-SAT, then according to Lemma 3.2.12 there doesn't exist a solution with cost less than  $15n - k$ .

### 3.3. Weighted segments

#### 3.3.1. FPT for weighted segments with $\delta$ -extensions

**Theorem 3.3.1. (FPT for weighted segment cover with  $\delta$ -extensions).** *There exists an algorithm that given a family  $\mathcal{P}$  of  $n$  weighted segments (in any direction), a set of  $m$*

points  $\mathcal{C}$  and a parameter  $k$ , runs in time  $f(k) \cdot (nm)^c$  for some computable function  $f$  and constant  $c$ , and outputs a subfamily  $\mathcal{R} \subseteq \mathcal{P}$  such that  $|\mathcal{R}| \leq k$  and  $\mathcal{R}^{+\delta}$  covers all points in  $\mathcal{C}$ .

To solve this problem we will introduce kernel for slightly different problem: Weighted segment cover of points and segments. In shortcut: WSCPS.

**Lemma 3.3.1. (Algorithm for kernel of WSCPS).** *There exists an algorithm that given a family  $\mathcal{P}$  of  $n$  weighted segments (in any direction), a set of  $m_1$  points  $\mathcal{C}_1$  and  $m_2$  segments  $\mathcal{C}_2$  and a parameter  $k$ , runs in time  $f(k) \cdot g(m_1, m_2) \cdot n^c$  for some computable functions  $f, g$  and constant  $c$ , and outputs a subfamily  $\text{sol} \subseteq \mathcal{P}$  such that  $|\text{sol}| \leq k$  and  $\text{sol}$  covers all points in  $\mathcal{C}_1$  and all segments in  $\mathcal{C}_2$ .*

**Proof** Only sketch for now.

We can compute dynamic programming  $dp(A, B, z)$  – the best cost to cover at least whole segment  $A, B$  using at most  $z$  segments.  $A, B$  are all interesting points – ends of any segment given on the input or points given on the input. We can compute it in polynomial time.

Then we can create a new double weighted set (original weight, number of used segments from  $\mathcal{P}$ ) –  $\mathcal{P}_2$  that has only segments which never cover partially any segment from  $\mathcal{C}_2$  (covers the whole segment or doesn't cover at all). In such  $\mathcal{P}_2$  we can find solution  $\mathcal{R}$  where any 2 segments have empty intersection (don't cover each other and don't meet at the ends). Because if we had such solution, we can merge these two segments and such segment there's also in  $\mathcal{P}_2$ .

In that case we can find kernel of  $\mathcal{P}_2$  of size  $k \cdot (m_1 + 2m_2)^2$ , because we only need to take the best weight covering some subset of  $\mathcal{C}_1 \cup \mathcal{C}_2$ .

**Lemma 3.3.2. Kernel in WSCPS. TODO: formulate it properly**

*For segment cover, there is a kernel of size  $f(k)$  in WSCPS.*

**Claim 3.3.1.** *If there are more than  $k$  lines with at least  $k+1$  points on them, then they can't be covered with  $k$  segments.*

**Claim 3.3.2.** *If there is more than  $k^2$  points that don't lie on any line with more than  $k$  points on it, then they can't be covered with  $k$  segments.*

**Claim 3.3.3.** *For every long line  $L$  (with more than  $k$  points on them) we can choose  $f(k)$  points on them, that if we cover all of these points with at most  $k$  segments, then the rest of the points with  $\delta$ -extensions will be covered by segments in the direction of line  $L$ .*

**Proof of Lemma 3.3.2.** After applying the previous lemmas, we have at most  $k^2 + k \cdot f(k)$  points that can be covered in any direction and for the rest of the points we can draw at most  $k \cdot f(k)$  segments along their respective long lines that have to be covered by segments after  $\delta$ -extensions.

Then we extend every available segment by  $\delta$ -extension and we achieve the kernel in WSCPS for this instance of problem.

**Lemma 3.3.3.** *If all the points are covered with  $k$  segments and the biggest  $2(1 + 1/\delta)^{k+1}$  spaces between points are filled, the whole segment is filled after  $\delta$ -extensions of these segments.*

**Proof.** Let's name the  $2(1+1/\delta)^{k+1}$ -st biggest space between points as  $y$ . We have guarantee that all segments of length  $x > y$  are covered without  $\delta$ -extensions.

Let's take one space between points that is not covered before  $\delta$ -extension and we will prove it will be covered after  $\delta$ -extensions. Let's assume it isn't.

This space has length  $x$ . Since it's uncovered,  $x \leq y$ .

Let's take side where the sum of lengths of segments covering the points is greater (left or right). Without loss of generality, let us assume it's right.

There are at most  $k$  segments to the right of this space between points. Name their lengths  $l_1, l_2 \dots l_k$ . If the point is covered in the other direction, the segment is degenerated to the point and  $l_i = 0$ . Name the space between endpoints of  $l_i$  and  $l_{i+1} - x_i$ . Of course,  $x_i$  is uncovered space between two points, therefore  $x_i \leq y$ .

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Let's write equations meaning that  $i$ -th segment doesn't cover space  $x$  after  $\delta$ -expansion.

$$l_1\delta < x \leq y \Rightarrow l_1 < y/\delta$$

$$l_2\delta < x + l_1 + x_1 < 2y + y/\delta \Rightarrow l_2 < 2y/\delta + y/\delta^2$$

$$l_3\delta < x + l_1 + x_1 + l_2 + x_2 < 3y + 3y/\delta + y/\delta^2 \Rightarrow l_3 < 3y/\delta + 3y/\delta^2 + y/\delta^3$$

From this we can "guess" induction  $l_i < y((1 + 1/\delta)^i - 1)$

Trivially for  $l_1 < y/\delta$ .

Assume that for all  $j < i$ :

$$l_j < y((1 + 1/\delta)^j - 1)$$

$$\begin{aligned} l_i\delta &< x + \sum_{j=1}^{i-1}(l_j + x_j) < iy \sum_{j=1}^{i-1} l_j < iy + \sum j = 1^{i-1}y((1 + 1/\delta)^j - 1) = iy - (i - 1)y + \sum j = 1^{i-1}y(1 + 1/\delta)^j = y(1 + \sum_{j=1}^{i-1}(1 + 1/\delta)^j) = y(2 + \sum_{j=1}^{i-1}(1 + 1/\delta)^j - 1) = \\ &y(\sum_{j=0}^{i-1}(1 + 1/\delta)^j - 1) = y((1 + 1/\delta)^i / (1 - (1 + 1/\delta)) - 1) = y((1 + 1/\delta)^i \delta - 1) < y((1 + 1/\delta)^i \delta - \delta) \end{aligned}$$

Of course we also know that (since we have chosen the side with greater sum of the width of segments):

$$\sum_{i=1}^k l_i \geq 1/2 \cdot y \cdot 2(1 + 1/\delta)^{k+1} = y \cdot (1 + 1/\delta)^{k+1}$$

$$\text{But } \sum_{i=1}^k l_i < \sum_{i=1}^k y((1 + 1/\delta)^i - 1) = y((1 + 1/\delta)^{k+1} / (1 - (1 + 1/\delta)) - k) = y((1 + 1/\delta)^{k+1} \delta - k) < y(1 + 1/\delta)^{k+1}$$

Therefore the space must have been covered after  $\delta$ -expansions.

### 3.3.2. W[1]-completeness for weighted segments in 3 directions

**Theorem 3.3.2.** *W[1]-completeness for weighted segments in 3 directions.* Consider the problem of covering a set  $\mathcal{C}$  of points by selecting  $k$  axis-parallel or right-diagonal weighted segments with weights from a set  $\mathcal{P}$  with minimal weight. Assuming ETH, there is no algorithm for this problem with running time  $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{o(\sqrt{k})}$  for any computable function  $f$ .

We will show reduction from grid tiling problem.

Let's have an instance of grid tiling problem – size of the grid  $k$ , number of elements available  $n$  and  $k^2$  sets of available pairs in every tile  $S_{i,j} \subseteq \{1, n\} \times \{1, n\}$ .

456 **Construction.** We construct a set  $\mathcal{P}$  of segments and a set  $\mathcal{C}$  of points.

457 First let's choose any ordering of  $n^2$  elements  $\{1, n\} \times \{1, n\}$  and name this sequence

458  $a_1 \dots a_{n^2}$ .

$$match_v(i, j) \iff a_i = \{x_i, y_i\} \wedge a_j = \{x_j, y_j\} \wedge x_i = x_j$$

$$match_h(i, j) \iff a_i = \{x_i, y_i\} \wedge a_j = \{x_j, y_j\} \wedge y_i = y_j$$

**Points.** Define points:

$$h_{i,j,t} = (j \cdot (n^2 + 1) + t, (n^2 + 1) \cdot i)$$

$$v_{i,j,t} = ((n^2 + 1) \cdot i, j \cdot (n^2 + 1) + t)$$

Let's define sets  $H$  and  $V$  as:

$$H = \{h_{i,j,t} : 1 \leq i, j, \leq k, 1 \leq t \leq n^2\}$$

$$V = \{v_{i,j,t} : 1 \leq i, j, \leq k, 1 \leq t \leq n^2\}$$

459 Let's define  $\epsilon = 0.1$ . For a point  $\{x, y\} = p$  we define points  $p^L = \{x - \epsilon, y\}$ ,  $p^R = \{x + \epsilon, y\}$ ,  
460  $p^U = \{x, y - \epsilon\}$ , and  $p^D = \{x, y + \epsilon\}$ .

Then we define:

$$\mathcal{C} := H \cup \{p^L : p \in H\} \cup \{p^R : p \in H\} \cup V \cup \{p^U : p \in V\} \cup \{p^D : p \in V\}$$

461 **Segments.** Define horizontal segments.

$$hor_{i,j,t_1,t_2} = (h_{i,j,t_1}^R, h_{i,j+1,t_2}^L)$$

$$ver_{i,j,t_1,t_2} = (v_{i,j,t_1}^D, v_{i,j+1,t_2}^U)$$

$$horbeg_{i,t} = (h_{i,1,1}^L, h_{i,1,t}^L)$$

$$horend_{i,t} = (h_{i,n,t}^R, h_{i,n,n^2}^R)$$

$$verbeg_{i,t} = (v_{i,1,1}^U, v_{i,1,t}^U)$$

$$verend_{i,t} = (v_{i,n,t}^D, v_{i,n,n^2}^D)$$

$$\begin{aligned} HOR &= \{hor_{i,j,t_1,t_2} : 1 \leq i \leq k, 1 \leq j < k, 1 \leq t_1, t_2 \leq n^2, match_h(t_1, t_2)\} \\ &\cup \{horbeg_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\} \\ &\cup \{horend_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\} \end{aligned}$$

$$\begin{aligned} VER &= \{ver_{i,j,t_1,t_2} : 1 \leq i \leq k, 1 \leq j < k, 1 \leq t_1, t_2 \leq n^2, match_v(t_1, t_2)\} \\ &\cup \{verbeg_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\} \\ &\cup \{verend_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\} \end{aligned}$$

$$DIAG := \{(h_{i,j,t}, v_{j,i,t}) : 1 \leq i, j \leq k, 1 \leq t \leq n^2, a_t \in S_{i,j}\}$$

TODO: explain that these segments are in fact diagonal

$$\mathcal{P} := HOR \cup VER \cup DIAG$$

**Lemma 3.3.4.** *If there exists solution for grid tiling, then there exists solution for our construction using  $2(k+1)k + k^2$  segments with weight exactly  $2k \cdot (k(n^2 + 1) - 2 - 2\epsilon(k-1))$ .*

**Claim 3.3.4.** *If there exists a solution to the grid tiling  $c_1 \dots c_k$  and  $r_1 \dots r_k$ , then there exists a solution covering all points*

$$\{h_{i,j,t} : 1 \leq i, j \leq k, t = (c_i, r_j)\} \cup \{v_{i,j,t} : 1 \leq i, j \leq k, t = (c_j, r_i)\}$$

*with segments in DIAG and the rest in VER or HOR and has weight  $2k \cdot (k(n^2 + 1) - 2 - 2\epsilon(k-1))$ .*

**Proof.** TODO: jakiś prosty z definicji

**Lemma 3.3.5.** *If there exists solution for our construction using  $2(k+1)k + k^2$  segments with weight exactly  $2k \cdot (k(n^2 + 1) - 2 - 2\epsilon(k-1))$ , then there exists a solution for grid tiling*

**Proof.** This follows from Lemma 3.3.6, because we just take which points are covered with DIAG.

**Claim 3.3.5.** *Points  $p^L, p^R, p^U, p^D$  cannot be covered with DIAG.*

**Claim 3.3.6.** *Points in  $H \cup \{p^L : p \in H\} \cup \{p^R : p \in H\}$  cannot be covered with VER.*

*Points in  $V \cup \{p^U : p \in V\} \cup \{p^D : p \in V\}$  cannot be covered with HOR.*

**Claim 3.3.7.** *For given  $i, j$  if none of the points  $h_{i,j,t}$  ( $v_{i,j,t}$ ) for  $1 \leq t \leq n^2$  are covered with DIAG, then some spaces between neighbouring points were covered twice.*

**Claim 3.3.8.** *For given  $i, j$  two points  $h_{i,j,t_1}, h_{i,j,t_2}$  ( $v_{i,j,t_1}, v_{i,j,t_2}$ ) for  $1 \leq t_1 < t_2 \leq n^2$  are covered with DIAG, then one of them had to be also covered with a segment from HOR (VER).*

**Proof.** Point  $v_{i,j,t_2}^L$  had to be covered with VER from Claims 3.3.5 and 3.3.6. And every segment in VER covering  $v_{i,j,t_2}^L$ , covers also  $v_{i,j,t_1}^L$ .

**Lemma 3.3.6.** *If there exists solution for our construction with weight at most (exactly)  $2k \cdot (k(n^2 + 1) - 2 - 2\epsilon(k-1))$ , then for every  $i, j$  there must be exactly one  $t$  such that  $h_{i,j,t}$  ( $v_{i,j,t}$ ) is covered with DIAG and moreover if  $h_{i,j,t_1}$  and  $h_{i,j+1,t_2}$  are uncovered, then  $\text{math}_h(t_1, t_2)$ . Analogically for  $v$ .*

**Proof.** Only  $k^2$  points can be covered only in DIAG, the rest has to be covered with  $VER \cup HOR$ . Therefore every result must be at least  $ALL\_LINES - 2k^2\epsilon$ , because only  $2k^2$  spaces of length  $\epsilon$  can be uncovered in this axis.

Of course if  $h_{i,j,t_1}$  and  $h_{i,j+1,t_2}$  are uncovered, then there must exist a segment in HOR between  $h_{i,j,t_1}^R$  and  $h_{i,j+1,t_2}^L$ , so  $\text{math}_h(t_1, t_2)$  must be true.

### 3.3.3. What is missing

We don't know FPT for axis-pararell segments without  $\delta$ -extensions.



## 493 Chapter 4

# 494 Geometric Set Cover with lines

### 495 4.1. Lines parallel to one of the axis

496 When  $\mathcal{R}$  consists only of lines parallel to one of the axis, the problem can be solved in  
497 polynomial time.

498 We create bipartial graph  $G$  with node for every line on the input split into sets:  $H$  –  
499 horizontal lines and  $V$  – vertical lines. If any two lines cover the same point from  $\mathcal{C}$ , then we  
500 add edge between them.

501 Of course there will be no edges between nodes inside  $H$ , because all of them are pararell  
502 and if they share one point, they are the same lines. Similar argument for  $V$ . So the graph is  
503 bipartial.

504 Now Geometric Set Cover can be solved with Vertex Cover on graph  $G$ . Since Vertex  
505 Cover (even in weighted setting) on bipartial graphs can be solved in polynomial time.

506 Short note for myself just to remember how to this in polynomial time:

507 Non-weighted setting - Konig theorem + max matching

508 Weighted setting - Min cut in graph of  $\neg A$  or  $\neg B$  (edges directed from  $V$  to  $H$ )

### 509 4.2. FPT for arbitrary lines

510 You can find this is Platypus book. We will show FPT kernel of size at most  $k^2$ .

511 (Maybe we need to reduce lines with one point/points with one line).

512 For every line if there is more than  $k$  points on it, you have to take it. At the end, if there  
513 is more than  $k^2$  points, return NO. Otherwise there is no more than  $k^4$  lines.

514 In weighted settings among the same lines with different weights you leave the cheapest  
515 one and use the same algorithm.

### 516 4.3. APX-completeness for arbitrary lines

517 We will show a reduction from Vertex Cover problem. Let's take an instance of the Vertex  
518 Cover problem for graph  $G$ . We will create a set of  $|V(G)|$  pairwise non-pararell lines, such  
519 that no three of them share a common point.

520 Then for every edge in  $(v, w) \in E(G)$  we put a point on crossing of lines for vertices  $v$   
521 and  $w$ . They are not pararell, so there exists exactly one such point and any other line don't  
522 cover this point (any three of them don't cross in the same point).

Solution of Geometric Set Cover for this instance would yield a sound solution of Vertex Cover for graph  $G$ . For every point (edge) we need to choose at least one of lines (vertices)  $v$  or  $w$  to cover this point.

Vertex Cover for arbitrary graph is APX-complete, so this problem is also APX-complete.

#### 4.4. 2-approximation for arbitrary lines

Vertex Cover has an easy 2-approximation algorithm, but here very many lines can cross through the same point, so we can do  $d$ -approximation, where  $d$  is the biggest number of lines crossing through the same point. So for set where any 3 lines don't cross in the same point it yields 2-approximation.

The problematic cases are where through all points cross at least  $k$  points and all lines have at least  $k$  points on them. It can be created by casting  $k$ -grid in  $k$ -D space on 2D space.

Greedy algorithm yields  $\log |\mathcal{R}|$ -approximation, but I have example for this for bipartial graph and reduction with taking all lines crossing through some point (if there are no more than  $k$ ) would solve this case. So maybe it works.

Unfortunately I haven't done this :(

I can link some papers telling it's hard to do.

#### 4.5. Connection with general set cover

Problem with finite set of lines with more dimensions is equivalent to problem in 2D, because we can project lines on the plane which is not perpendicular to any plane created by pairs of (point from  $\mathcal{C}$ , line from  $\mathcal{P}$ ).

Of course every two lines have at most one common point, so is every family of sets that have at most one point in common equivalent to some geometric set cover with lines?

No, because of Desargues's theorem. Have to write down exactly what configuration is banned.

## 547 Chapter 5

# 548 Geometric Set Cover with polygons

### 549 5.1. State of the art

550 Covering points with weighted discs admits PTAS [Li and Jin, 2015] and with fat polygons  
551 with  $\delta$ -extensions with unit weights admits EPTAS [Har-Peled and Lee, 2009].

552 Although with thin objects, even if we allow  $\delta$ -expansion, the Set Cover with rectangles is  
553 APX-complete (for  $\delta = 1/2$ ), it follows from APX-completeness for segments with  $\delta$ -expansion  
554 in Section 3.2.

555 Covering points with squares is W[1]-hard [Marx, 2005]. It can be proven that assuming  
556 *SETH*, there is no  $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{k-\epsilon}$  time algorithm for any computable function  $f$  and  
557  $\epsilon > 0$  that decides if there are  $k$  polygons in  $\mathcal{P}$  that together cover  $\mathcal{C}$ , *Theorem 1.9* in [Marx  
558 and Pilipczuk, 2015].



<sup>559</sup> Chapter 6

<sup>560</sup> Conclusions



## 561 Bibliography

- 562 [Har-Peled and Lee, 2009] Har-Peled, S. and Lee, M. (2009). Weighted geometric set cover  
563 problems revisited. *Journal of Computational Geometry*, 3.
- 564 [Håstad, 2001] Håstad, J. (2001). Some optimal inapproximability results. *J. ACM*,  
565 48(4):798–859.
- 566 [Li and Jin, 2015] Li, J. and Jin, Y. (2015). A PTAS for the weighted unit disk cover problem.  
567 *CoRR*, abs/1502.04918.
- 568 [Marx, 2005] Marx, D. (2005). Efficient approximation schemes for geometric problems? In  
569 Brodal, G. S. and Leonardi, S., editors, *Algorithms – ESA 2005*, pages 448–459, Berlin,  
570 Heidelberg. Springer Berlin Heidelberg.
- 571 [Marx and Pilipczuk, 2015] Marx, D. and Pilipczuk, M. (2015). Optimal parameterized algo-  
572 rithms for planar facility location problems using voronoi diagrams. *CoRR*, abs/1504.05476.