Standard error propagation on the position angle calculation to show that $\sigma_\psi = \frac{\sigma_I}{2L}$

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The position angle ψ is given by

$$\psi = \frac{1}{2}\arctan\left(\frac{U}{Q}\right),$$

where U and Q are the linear Stokes parameters, and $L^2 = Q^2 + U^2$. The uncertainties on Stokes parameters are Gaussian-distributed with equal standard deviations, such that $\sigma_I = \sigma_Q = \sigma_U = \sigma_V$.

The standard Gaussian error propagation formula for a function $f(a, b, c, ...)$ is

$$\sigma_f^2 = \left(\frac{\partial f}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial f}{\partial b}\right)^2 \sigma_b^2 + \left(\frac{\partial f}{\partial c}\right)^2 \sigma_c^2 + ...$$

so applying this to the PA equation, this means that

$$\sigma_\psi^2 = \left(\frac{\partial \psi}{\partial U}\right)^2 \sigma_U^2 + \left(\frac{\partial \psi}{\partial Q}\right)^2 \sigma_Q^2. \tag{1}$$

Considering each partial derivative in Eq. (1) turn, and using $\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$:

$$\begin{aligned}\frac{\partial \psi}{\partial U} &= \frac{1}{2} \times \frac{1}{1 + (\frac{U}{Q})^2} \times \frac{1}{Q} \\ &= \frac{1}{2}\frac{Q^2}{Q^2 + U^2} \times \frac{1}{Q} \\ &= \frac{1}{2}\frac{Q}{Q^2 + U^2} \\ &= \frac{1}{2}\frac{Q}{L^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial \psi}{\partial Q} &= \frac{1}{2} \times \frac{1}{1 + (\frac{U}{Q})^2} \times -\frac{U}{Q^2} \\ &= -\frac{1}{2}\frac{Q^2}{Q^2 + U^2} \times \frac{U}{Q^2} \\ &= -\frac{1}{2}\frac{U}{Q^2 + U^2} \\ &= -\frac{1}{2}\frac{U}{L^2}\end{aligned}$$

Substituting these into Eq. (1),

$$\begin{aligned}\sigma_\psi^2 &= \left(\frac{1}{2}\frac{Q}{L^2}\right)^2 \sigma_U^2 + \left(-\frac{1}{2}\frac{U}{L^2}\right)^2 \sigma_Q^2 \\ &= \frac{1}{4}\frac{Q^2}{L^4}\sigma_U^2 + \frac{1}{4}\frac{U^2}{L^4}\sigma_Q^2.\end{aligned}$$

Therefore

$$\begin{aligned}\sigma_\psi &= \sqrt{\frac{1}{4}\frac{Q^2}{L^4}\sigma_U^2 + \frac{1}{4}\frac{U^2}{L^4}\sigma_Q^2} \\ &= \frac{1}{2L^2}\sqrt{Q^2\sigma_U^2 + U^2\sigma_Q^2},\end{aligned}$$

and if $\sigma_Q = \sigma_U = \sigma_I$, then

$$\begin{aligned}\sigma_\psi &= \frac{1}{2L^2}\sqrt{(Q^2 + U^2)\sigma_I^2} \\ &= \frac{1}{2L^2}\sqrt{L^2\sigma_I^2} \\ &= \frac{\sigma_I}{2L}.\end{aligned}$$