STATISTICAL METHODS IN FINANCE, PROBLEM SHEET 1
MSC IN MATHEMATICS AND FINANCE, 2023-2024

CRIS SALVI, WILL TURNER

Exercise 1.

(i) Let $\mathbf{A} \in \mathcal{M}_n$ with eigenvalues $\lambda_1 \dots \lambda_n$. Which of the following statements is incorrect:

(a) $\operatorname{Tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_n$;

(b) If $\min_{i=1,\dots,n} \lambda_i > 0$ then $\mathbf{A}$ is positive definite;

(c) $\mathbf{A}$ is invertible if and only if $\min_{i=1,\dots,n} |\lambda_i| > 0$;

(d) $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i = \lambda_1 \lambda_2 \cdots \lambda_n$;

(ii) Let $\mathbf{A} \in \mathcal{M}_{m,n}$ admit a singular decomposition $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{V}^\top$, where $\mathbf{\Lambda} = \operatorname{Diag}(\sigma_1(\mathbf{A}), \dots, \sigma_r(\mathbf{A}))$, and $\mathbf{V} \in \mathcal{M}_{n,r}$ and $\mathbf{U} \in \mathcal{M}_{m,r}$. In what case are $\mathbf{V}$ and $\mathbf{U}$ of the same shape?

(a) $\mathbf{A}$ is unitary;

(b) $\sigma_i(\mathbf{A}) \geq 0$ for all $i \in \{1 \dots r\}$;

(c) $\mathbf{A} \in \mathcal{M}_n$;

(d) All the right singular vectors are orthogonal;

(iii) Let $\mathbf{x} \in \mathbb{R}^n$, then

(a) $\nabla_{\mathbf{x}}(\mathbf{A}\mathbf{x}) = \mathbf{A}$, for any $\mathbf{A} \in \mathcal{M}_{m,n}$;

(b) $\nabla_{\mathbf{x}}(\mathbf{A}\mathbf{x}) = \mathbf{A}$, for any $\mathbf{A} \in \mathcal{M}_{n,m}$;

(c) $\nabla_{\mathbf{x}}\left(\mathbf{x}^\top \mathbf{A}\right) = \mathbf{A}$, for any $\mathbf{A} \in \mathcal{M}_{n,m}$;

(d) $\nabla_{\mathbf{x}}\left(\mathbf{x}^\top \mathbf{A}\right) = \mathbf{A}^\top$, for any $\mathbf{A} \in \mathcal{M}_{m,n}$;

Exercise 2 (Weyl inequality)**.** Let $A, B \in \mathbb{R}^{n\times n}$ be two symmetric matrices. Prove that for any $i \in \{1, ..., n\}$

$$\lambda_i(A) + \lambda_n(B) \leq \lambda_i(A+B) \leq \lambda_i(A) + \lambda_1(B).$$

where the eigenvalues are ordered in decreasing order.

Exercise 3 (Courant-Fischer min-max theorem)**.** Let A be an $n \times n$ symmetric matrix. Ordering the eigenvalues of A in decreasing order as usual, show that for all $1 \leq i \leq n$

$$\lambda_i(A) = \sup_{\dim(V)=i} \inf_{v \in V : \|v\|_2 = 1} v^\top A v$$

where V ranges over all subspaces of $\mathbb{R}^n$ with the indicated dimension.

Date: May 9, 2024.

Exercise 4. Use Sterling's formula

$$n! = (1 + o(1))\sqrt{2\pi n}n^n e^{-n}$$

to show that for any $0 \leq p \leq 1$ the following holds

$$\lim_{n\to\infty} \frac{1}{n} \log \binom{n}{pn} = H(p)$$

where $H(p) = -p\log(p) - (1-p)\log(1-p)$.

Department of Mathematics, Imperial College London
Email address: `wft17@ic.ac.uk`