STATISTICAL METHODS IN FINANCE, PROBLEM SHEET 5
MSC IN MATHEMATICS AND FINANCE, 2023-2024

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Exercise 1 (Method of moments)**.** We consider the model $Z = c + \mathrm{e}^X$, where $X \sim \mathcal{N}(\mu, \sigma^2)$.

(i) Compute $\mathbb{E}[Z]$, $\mathbb{E}[Z^2]$ and $\mathbb{E}[Z^3]$.

(ii) Deduce a Method of Moments estimator for the triplet (c, μ, σ^2). **(challenging!)**.

Exercise 2 (Degenerate Likelihood)**.** Consider the function

$$f(x) = \frac{1}{6}\left(\frac{\mathbb{1}_{(0,1]}(|x|)}{\sqrt{|x|}} + \frac{\mathbb{1}_{(1,\infty)}(|x|)}{x^2}\right). \tag{0.1}$$

Show that f is a genuine density function on $\mathbb{R}$. Consider now the family $(f_\theta)_{\theta\in\mathbb{R}}$ obtained by translation $f_\theta(x) := f(x - \theta)$ for all $x \neq \theta$, which corresponds to the common distribution of an iid sample $(X_1, \ldots, X_n)$.

(i) Compute the log-likelihood function, and discuss the existence of any maximum likelihood estimator.

(ii) What about an estimator obtained by the method of moments?

(iii) Compute the cumulative distribution function corresponding to f, and show that the median is null.

(iv) By translation, determine an estimator for θ.

Exercise 3 (Maximum Likelihood)**.** Let $\theta \in (0, 1)$ be the unknown parameter and consider the random variable X satisfying, for any non-negative integer n,

$$\mathbb{P}_\theta[X = n] = (n + 1)(1 - \theta)^2\theta^n.$$

(i) Show that, for any $\theta \in (0, 1)$,

$$\mathbb{E}[X] = \frac{2\theta}{1-\theta} \qquad \text{and} \qquad \mathbb{V}[X] = \frac{2\theta}{(1-\theta)^2}.$$

(ii) Give a method of moment estimator $\widehat{\theta}_n$ for θ given an iid sample $\mathcal{X} = (X_1, \ldots, X_n)$.

(iii) Is the maximum likelihood estimator of θ given the sample $\mathcal{X}$ well defined?

(iv) Check whether $\widehat{\theta}_n$ is consistent and compute its limiting distribution when n tends to infinity.

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Exercise 4 (Exponential distribution)**.** Let $(X_n)_{n\in\mathbb{N}}$ denote an iid sequence with common distribution given by the shifted Exponential distribution with density

$$f_X(x) = b\mathrm{e}^{-b(x-a)}\mathbb{1}_{(a,\infty)}(x), \qquad \text{for some } b > 0, a \in \mathbb{R}.$$

(i) Compute $\mathbb{E}[X]$
(ii) Compute the cumulative distribution of X.
(iii) Derive the maximum likelihood estimators $\hat{a}_{\mathrm{ML}}$ and $\hat{b}_{\mathrm{ML}}$.
(iv) Compute $\mathbb{E}\left[\hat{a}_{\mathrm{ML}}\right]$.
(v) For a random variable Y, a zero-order approximation to $\mathbb{E}[1/Y]$ is given by $1/\mathbb{E}[Y]$. Compute a zero-order approximation to $\mathbb{E}\left[\hat{b}_{\mathrm{ML}}\right]$.

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