STATISTICAL METHODS IN FINANCE, PROBLEM SHEET 2
MSC IN MATHEMATICS AND FINANCE, 2023-2024

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Exercise 1. A miner is lost in a tunnel with three doors. Behind the first door is a tunnel that, after three hours, leads out of the mine. Behind the second and third doors are tunnels which, after five and seven hours, respectively, lead back to the same position. Assuming all the doors are indistinguishable and that then miner always chooses randomly among them, how long on average will it take the miner get out of the mine?

Exercise 2 (Symmetric distributions)**.** Let X be a random variable supported on the whole real line, with a symmetric distribution around some point $a \in \mathbb{R}$, and admitting a smooth density f.

(i) Show that, if the n-th moment exists with n odd, then $\mathbb{E}\left[(X-a)^n\right] = 0$.

(ii) If $\mathbb{E}[X]$ exists and if $f(a) \neq 0$, show that $\mathbb{E}[X] = a$ is the unique median.

Exercise 3 (Skewness and kurtosis)**.** For a given random variable X on the real line, with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$, recall that the skewness $\mathcal{S}$ and kurtosis κ are defined as

$$\mathcal{S} := \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] \qquad \text{and} \qquad \kappa := \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right].$$

The skewness describes the asymmetry of the distribution, whereas the kurtosis is a measure of its fatness. We usually speak of the excess kurtosis, though, defined as $\kappa_+ := \kappa - 3$.

(i) Using Hölder's inequality in the notes, prove Lyapunov's inequality $\mathbb{E}\left[|X|^p\right] \leq \mathbb{E}\left[|X|^q\right]^{p/q}$, for all $0 < p < q$ for which both sides of the inequality are finite. Deduce that $\kappa_+ \geq -2$.

(ii) Compute $\mathcal{S}$ and κ for $\mathcal{N}(\mu, \sigma^2)$.

(iii) Compute $\mathcal{S}$ and κ for the Uniform random variable on $[a, b]$.

(iv) Compute $\mathcal{S}$ and κ for the Exponential random variable with intensity $\lambda > 0$, and density

$$f(x) = \partial_x \mathbb{P}[X \leq x] = \lambda \mathrm{e}^{-\lambda x}, \qquad \text{for } x \geq 0.$$

(v) A distribution with $\kappa_+ = 0$ is called mesokurtic. One with $\kappa_+ > 0$ is leptokurtic, and platykurtic if $\kappa_+ < 0$.
- Check the origin and meaning of the Greek words *meso, lepto, platy, kurtos*.
- Let X represent daily logarithmic returns of some data. Using the IPython notebooks, find leptokurtic, platykurtic, or close to mesokurtic examples, and for which $\mathcal{S} > 0$ and $\mathcal{S} < 0$.

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Exercise 4 (Log-normal distribution)**.** The two questions below are independent. Consider the standard Gaussian distribution $X \sim \mathcal{N}(0,1)$, with density

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}, \qquad \text{for all } x \in \mathbb{R}.$$

(i) Compute $\mathbb{E}[X]$, $\mathbb{V}\mathrm{ar}[X]$ and $\mathbb{E}\left[\mathrm{e}^{uX}\right]$ for all $u \in \mathbb{R}$ such that the expectation is well defined.

(ii) Define $Y := \exp\{X\}$. Compute its density, expectation, variance and moment generating function.

(iii) Does Y have a symmetric distribution? Compute its skewness to confirm your guess.

(iv) In the Black-Scholes model (the basic model in quantitative Finance), a Call option with strike $K > 0$ and maturity $t \geq 0$, is worth

$$C(t,K) := \mathrm{e}^{-rt}\mathbb{E}\left[\max(S_t - K, 0)\right],$$

where $S_t = S_0 \mathrm{e}^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma\sqrt{t}X}$, with $r, \sigma, S_0 > 0$ and $X \sim \mathcal{N}(0,1)$ as before. Determine an explicit form (in terms of the Gaussian cumulative distribution function) for $C(t,K)$.

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