Numerical Methods for Finance: Problem Set 2

1. Suppose we want to price a fixed strike lookback put option which payoff is given by

$$\left(K - \min_{n=0,\dots,N} S_n\right)^+$$

using an N-period trinomial tree model (as defined in question 3, problem set 1). Assume further that the tree is symmetric such that $ud = 1$ and $m = 1$.

(a) Write down the expression of s_k^n which represents all possible stock price values at time n under the N-period trinomial tree model. State the range of n and k clearly.

(b) Define a suitable auxiliary variable to solve this pricing problem. With brief justifications, construct a grid for this auxiliary variable.

(c) Identify how the time $n+1$ value of your auxiliary variable is linked to its time n value as well as the time n stock price level S_n. Hence, derive the forward shooting grid function describing how the location index associated with the auxiliary variable in the trinomial tree evolves in each time time.

(d) Write down the complete algorithm which solves for the time-zero value of the fixed strike lookback put option under the trinomial tree model. Define all the variables you use clearly.

2. Barrier option is a derivative instrument which payoff is contingent on whether the underlying stock has reached a particular barrier level or not. For example, an up-and-out barrier call option ceases to exist whenever the stock price ever reaches the barrier level B (where $B > S_0$) throughout the option's lifecycle. Its payoff is given by

$$(S_N - K)^+ 1_{(H_N < B)}$$

where $H_n := \max_{i=0,1,\dots,n} S_i$ represents the running maximum of the stock price up to time n. We now want to price this up-and-out barrier option using a standard binomial tree.

(a) Take $I_n := 1_{(H_n < B)}$ as an auxiliary variable. Express I_n in terms of I_{n-1} and S_n.

(b) Let $V_{k,i}^n$ be the fair option value at time n when the current stock price is $S_n = s_k^n = S_0 u^{n-k} d^k$ and the current value of the auxiliary variable is $I_n = i$. Using (a), write down the forward shooting grid function describing how the index i evolves in each time step.

(c) What are the values of $V_{k,0}^n$ for each k and n? Hence show that

$$V_{k,1}^n = e^{-r\triangle t}[qV_{k,1}^{n+1} 1_{(s_k^{n+1} < B)} + (1-q)V_{k+1,1}^{n+1} 1_{(s_{k+1}^{n+1} < B)}].$$

3. In a similar setup as in question 2, consider a down-and-in barrier put option which payoff is given by

$$(K - S_N)^+ 1_{(L_N \leq B)}$$

where $L_n := \min_{i=0,1,\dots,n} S_i$ represents the running minimum of the stock price up to time n. Assume that $B < S_0$.

(a) Take $I_n := 1_{(L_n \leq B)}$ as an auxiliary variable. Express I_n in terms of I_{n-1} and S_n.

(b) Let $V_{k,i}^n$ be the fair option value at time n when the current stock price is $S_n = s_k^n = S_0 u^{n-k} d^k$ and the current value of the auxiliary variable is $I_n = i$. Using (a), write down the forward shooting gird function describing how the index i evolves in each time step.

(c) Write down the complete algorithm which solves for time-zero value of the down-and-in barrier put option.

(d) Write down explicitly the recursive equation for $V_{k,1}^n$. Does it depend on $V_{\tilde{k},0}^{\tilde{n}}$ at all for any $\tilde{n}$ and $\tilde{k}$? Explain your results.

4. Implement the CRR binomial tree option pricing model for Asian options (which can potentially cover all the four variations of products we mention in the class) in a programming language of your choice.