MATH97110: Numerical Methods in Finance

Topic 5: Option Pricing with Fourier Transform Methods

Imperial College London

Spring 2023

Overview

1. Fourier transformation of option pricing function
2. Numerical implementation via fast Fourier transform
3. Applications to models with jumps and stochastic volatility

Motivation

- Consider an asset price process $S_t = \exp(X_t)$ for some X, such that (as usual) under the risk neutral measure $\mathbb{Q}$ the discounted price process $e^{-rt}S_t$ is a martingale

- We know that the time-zero price of a European call option with strike K and maturity T is given by (suppress the argument T)

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$$C(k) = e^{-rT}\mathbb{E}_{\mathbb{Q}}[(e^{X_T} - e^k)^+] = e^{-rT}\int_k^{\infty}(e^x - e^k)q(x)dx$$

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- The challenge is that $q(x)$ is not available in general and not much can be said about the above integral

- However, the characteristic function of X_T could be available in closed-form even for some very advanced models

Characteristic function of a random variable

- The characteristic function ϕ_X of a real-valued random variable X is defined as

$$\phi_X(u) := \mathbb{E}\left[e^{iuX}\right] = \int_{-\infty}^{\infty} e^{iux} f(x) dx$$

where the second equality holds provided that X admits a probability density function f

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- If X and Y are two independent r.v's, then $X + Y$ has characteristic function of

$$\phi_{X+Y}(u) = \mathbb{E}\left[e^{iu(X+Y)}\right] = \mathbb{E}\left[e^{iuX}\right]\mathbb{E}\left[e^{iuY}\right] = \phi_X(u)\phi_Y(u)$$

Example of characteristic function

- Characteristic function of a standard normal random variable $Z \sim N(0,1)$ is

$$\phi_Z(u) = \mathbb{E}\left[e^{iuZ}\right] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{iux} dx = e^{-\frac{u^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-iu)^2}{2}} dx = e^{-\frac{u^2}{2}}$$

(Note: the above result is not as trivial as it seems. Computation of this kind in general requires contour integration)

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- Under the Black-Scholes model, stock price at time T is given by

$$S_T = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right) T + \sigma B_T\right] \overset{\text{dist.}}{=} S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right) T + \sigma\sqrt{T} Z\right]$$

- Then the characteristic function of $X_T := \ln S_T = \ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T + \sigma\sqrt{T} Z$ is

$$\begin{aligned}
\phi_X(u) = \mathbb{E}\left[e^{iuX_T}\right] &= \exp\left[iu \ln S_0 + \left(r - \frac{\sigma^2}{2}\right) iuT\right] \mathbb{E}\left[\exp(iu\sigma\sqrt{T}Z)\right] \\
&= \exp\left[iu \ln S_0 + \left(r - \frac{\sigma^2}{2}\right) iuT\right] \phi_Z(\sigma\sqrt{T}u) \\
&= \exp\left[iu\left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T\right) - \frac{u^2\sigma^2 T}{2}\right]
\end{aligned}$$

Fourier transform of option pricing function

Proposition 5.1 (Fourier inversion)

Let F be the Fourier transform of f defined as $F(u) = \int_{-\infty}^{\infty} e^{iuk} f(k)dk$. If f and F are integrable functions, then

$$f(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} F(u)du.$$

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- Note that the option pricing function $C(k)$ is not integrable since

$$\lim_{k\to-\infty} C(k) = \lim_{k\to-\infty} e^{-rT}\mathbb{E}_{\mathbb{Q}}[(S_T - e^k)^+] = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\lim_{k\to-\infty}(S_T - e^k)^+\right] = S_0 > 0$$

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  (The swap of lim and $\mathbb{E}[\cdot]$ can be justified by monotone convergence theorem). Hence the standard conditions of Fourier inversion do not hold
- However, one trick to get around this is to consider

$$v(k) := e^{\alpha k} C(k)$$

  for some dampening factor $\alpha > 0$ to ensure the function decays to zero when $k \to -\infty$
- Note that α cannot be too large or else there could be explosion for large k

Fourier transform of option pricing function (cont')

Now we work out the Fourier transform of the adjusted option price:

$$
\begin{aligned}
\psi(u) := \int_{-\infty}^{\infty} e^{iuk} v(k) dk &= \int_{k=-\infty}^{k=\infty} e^{iuk} e^{\alpha k} C(k) dk \\
&= \int_{k=-\infty}^{k=\infty} e^{iuk} e^{\alpha k} \left(e^{-rT} \int_{x=k}^{x=\infty} (e^x - e^k) q(x) dx \right) dk \\
&= e^{-rT} \int_{k=-\infty}^{k=\infty} \int_{x=k}^{x=\infty} (e^{x+\alpha k+iuk} - e^{k+\alpha k+iuk}) q(x) dx dk \\
&= e^{-rT} \int_{x=-\infty}^{x=\infty} \int_{k=-\infty}^{k=x} (e^{x+\alpha k+iuk} - e^{k+\alpha k+iuk}) q(x) dk dx \\
&= e^{-rT} \int_{x=-\infty}^{x=\infty} \left(\frac{e^{(1+\alpha+iu)x}}{\alpha+iu} - \frac{e^{(1+\alpha+iu)x}}{\alpha+1+iu} \right) q(x) dx \\
&= e^{-rT} \int_{x=-\infty}^{x=\infty} \frac{e^{(1+\alpha+iu)x} q(x)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u} dx \\
&= \frac{e^{-rT} \phi(u-(1+\alpha)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}
\end{aligned}
$$

where ϕ is the characteristic function of X_T, and recall that $q(x)$ is the pdf of X_T

Existence of the Fourier transform of option pricing function

- If we choose $\alpha = 0$, then

$$\psi(u) = \frac{e^{-rT}\phi(u-i)}{-u^2 + iu}$$

and hence $\psi(u)$ explodes at $u = 0$ because the denominator goes to zero but the numerator is strictly positive and finite

$$\phi(-i) = \mathbb{E}(e^{-i^2 X_T}) = \mathbb{E}[e^{\ln S_T}] = \mathbb{E}(S_T) = e^{rT} S_0 > 0$$

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Lemma 5.2

The Fourier transform of $v(k) := e^{\alpha k} C(k)$ is well-defined (i.e. v is integrable) if $\alpha > 0$ and $\mathbb{E}(S_T^{1+\alpha}) < \infty$.

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Proof. By definition, $\psi(u) := \int_{-\infty}^{\infty} e^{iuk} v(k)dk$ and hence

$$\int_{-\infty}^{\infty} v(k)dk = \psi(0) = \frac{e^{-rT}\phi[-(1+\alpha)i]}{\alpha^2 + \alpha}$$

which is guaranteed to be finite if $\alpha > 0$ and $\phi[-(1+\alpha)i] < \infty$. But

$$\phi[-(1+\alpha)i] = \mathbb{E}\left[e^{-(1+\alpha)i^2 X_T}\right] = \mathbb{E}\left[e^{(1+\alpha)\ln S_T}\right] = \mathbb{E}[S_T^{1+\alpha}].$$

The result then follows. $\square$

Carr-Madan formula

Proposition 5.3

The price of a European call option with maturity T and strike $K = e^k$ is given by

$$C(k) = \frac{e^{-\alpha k}}{\pi} Re\left(\int_0^\infty e^{-iuk}\psi(u)du\right) \tag{5.1}$$

$$\text{where} \quad \psi(u) = \frac{e^{-rT}\phi(u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}$$

with $Re(\cdot)$ denoting the real part of a complex number and ϕ being the characteristic function of $X_T = \ln S_T$ and, α is any strictly positive constant such that $\mathbb{E}[S_T^{1+\alpha}] < \infty$.

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Proof. We already know the Fourier transform of $v(k) = e^{\alpha k}C(k)$ is $\psi(u)$. Inverting the transformation using Prop 5.1 leads to

$$C(k) = e^{-\alpha k}v(k) = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\psi(u)du = \frac{e^{-\alpha k}}{\pi} Re\left(\int_0^\infty e^{-iuk}\psi(u)\right)du.$$

The last equality holds because $C(k)$ is a real function, which implies the function $\psi(u)$ is odd in its imaginary part and even in its real part. (See problem set) $\square$

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- An integral in form of $I := \int_a^b f(x)dx$ can be approximated by certain weighted sum of the function values

$$I \approx \sum_{j=1}^{n} f(x_j)\omega_j \triangle x$$

for some weighting vector ω_j's. Examples are:

Left rectangular rule: $[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_{n-1})] \times \triangle x$

Trapezoidal rule: $[f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n)] \times \frac{\triangle x}{2}$

Simpson's rule: $[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)] \times \frac{\triangle x}{3}$

where

$$\triangle x := \frac{b-a}{n-1}, \qquad x_j := a + (j-1)\triangle x \text{ for } j = 1, ..., n.$$

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- The integral (5.1) can be approximated over a range of strike levels k very efficiently using *Fast Fourier transform*

Fast Fourier transform (FFT)

- A discrete Fourier transform involves converting a vector $(\chi_j : j = 1, 2, ..., N)$ into another vector $(w_n : n = 1, 2, ..., N)$ via

$$w_n = \sum_{j=1}^{N} \exp\left(-\frac{2i\pi(j-1)(n-1)}{N}\right)\chi_j, \quad n = 1, 2, ..., N. \tag{5.2}$$

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- The algorithm works by exploiting the periodic behaviour of e^{ix} that

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- The algorithm is universally available and has been implemented in many programming languages such as Python

Ideas behind the FFT algorithm

- Split the sum in (5.2) into the odd and even terms:

$$
\begin{aligned}
w_n &= \sum_{j=1}^{N} \exp\left(-\frac{2i\pi(j-1)(n-1)}{N}\right)\chi_j \\
&= \sum_{m=0}^{\frac{N}{2}-1} \exp\left(-\frac{2i\pi(2m)(n-1)}{N}\right)\chi_{2m+1} + \sum_{m=1}^{N/2} \exp\left(-\frac{2i\pi(2m-1)(n-1)}{N}\right)\chi_{2m} \\
&= \sum_{m=0}^{\frac{N}{2}-1} \exp\left(-\frac{2i\pi m(n-1)}{N/2}\right)\chi_{2m+1} + \sum_{m=1}^{N/2} \exp\left(-\frac{2i\pi(m-1)(n-1)}{N/2}\right)\underbrace{\exp\left(-\frac{2i\pi(n-1)}{N}\right)\chi_{2m}}_{=:a_m}.
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Hence $(w_n : n = 1, ..., N/2)$ can be computed by discrete Fourier transforms applied on $(\chi_{2m+1} : m = 0, ..., N/2-1)$ and $(a_m : m = 1, ..., N/2)$ respectively

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- Moreover, using the periodic properties in (5.3) we can show that

$$
\begin{aligned}
w_{n+\frac{N}{2}} = &\sum_{m=0}^{\frac{N}{2}-1} \exp\left(-\frac{2i\pi m(n-1)}{N/2}\right) \chi_{2m+1} \\
&- \sum_{m=1}^{N/2} \exp\left(-\frac{2i\pi(m-1)(n-1)}{N/2}\right) \exp\left(-\frac{2i\pi(n-1)}{N}\right) \chi_{2m}
\end{aligned}
$$

Hence the two discrete Fourier transforms performed previously can give us the complete vector $(w_n : n = 1, ..., N)$

Ideas behind the FFT algorithm (cont')

- We have shown that the splitting above reduces the whole computation to two discrete Fourier transforms on vectors of size $N/2$. The number of operations required is roughly

$$2 \times \left(\frac{N}{2}\right)^2 + N = N^2/2 + N$$

(2 Fourier transformation, where each of them takes $(N/2)^2$ operations. Summing the two vectors up takes another N operations). Hence the complexity is $O(N^2/2 + N)$

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- What if we split again those vectors of size $N/2$? Then each Fourier transform no longer has complexity of $O((N/2)^2)$. After the second splitting, the total number of operations required now becomes

$$2 \times \left(2 \times \left(\frac{N}{4}\right)^2 + \frac{N}{2}\right) + N = N^2/4 + 2N$$

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- What if we split again those vectors of size $N/2$? Then each Fourier transform no longer has complexity of $O((N/2)^2)$. After the second splitting, the total number of operations required now becomes

$$2 \times \left(2 \times \left(\frac{N}{4}\right)^2 + \frac{N}{2}\right) + N = N^2/4 + 2N$$

- It is possible to show inductively that after p splitting, the total number of operations required is of order $\frac{N^2}{2^p} + pN$

Ideas behind the FFT algorithm (cont')

- We have shown that the splitting above reduces the whole computation to two discrete Fourier transforms on vectors of size $N/2$. The number of operations required is roughly

$$2 \times \left(\frac{N}{2}\right)^2 + N = N^2/2 + N$$

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- It is possible to show inductively that after p splitting, the total number of operations required is of order $\frac{N^2}{2^p} + pN$

- Thus if $N = 2^K$ for some K, we can perform the splitting K times in total. The total number of operations is then

$$\frac{N^2}{2^K} + KN = \frac{N^2}{N} + N \ln_2 N = N + N \ln_2 N$$

which justifies the complexity of $O(N \ln_2 N)$

FFT and option pricing

- For example, if the integral in (5.1) is approximated by left rectangular rule, then

$$C(k) \approx \frac{e^{-\alpha k}}{\pi} Re \left\{ \sum_{j=1}^{N} \exp\left(-iu_j k\right) \psi(u_j)\eta \right\}, \quad u_j := \eta(j-1)$$

  where N is some large number representing the number of points used and η is the step size of the grid

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where N is some large number representing the number of points used and η is the step size of the grid

- In practice, we often want to obtain option prices for a range of strikes. Consider log-strikes in range of $k \in [-b, b)$ (where $b > 0$ is some large number):

$$k_n = -b + \lambda(n-1), \quad n = 1, ..., N \quad \text{with} \quad \lambda = 2b/N$$

where λ is the spacing between the log-strikes. Then

$$\begin{aligned} C(k_n) &\approx \frac{e^{-\alpha k_n}}{\pi} Re \left\{ \sum_{j=1}^{N} \exp\left(-iu_j k_n\right) \psi(u_j)\eta \right\} \\ &= \frac{e^{-\alpha k_n}}{\pi} Re \left\{ \sum_{j=1}^{N} \exp\left[-i\eta(j-1)(-b+\lambda(n-1))\right] \psi(u_j)\eta \right\} \\ &= \frac{e^{-\alpha k_n}}{\pi} Re \left\{ \sum_{j=1}^{N} \exp\left[-i\eta\lambda(j-1)(n-1)\right] e^{ib\eta(j-1)} \psi(u_j)\eta \right\} \end{aligned}$$

FFT and option pricing (cont')

- If we choose the spacing parameters λ and η such that $\lambda\eta = 2\pi/N$, then

$$C(k_n) \approx \frac{e^{-\alpha k_n}}{\pi} Re \left\{ \sum_{j=1}^{N} \exp\left[-\frac{2i\pi(j-1)(n-1)}{N} \right] e^{ib\eta(j-1)} \psi(\eta(j-1))\eta \right\}$$

FFT and option pricing (cont')

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- The summation term above is exactly a discrete Fourier transform applied to the vector

$$\chi_j := e^{ib\eta(j-1)} \psi(\eta(j-1))\eta, \quad j = 1, 2, ..., N$$

and thus FFT can be used

FFT: higher order approximation

- A higher order approximation can be adopted when evaluating the integration to give much more accurate results. For example, if Simpson's rule is used then

$$C(k_n) \approx \frac{e^{-\alpha k_n}}{\pi} Re\left\{ \sum_{j=1}^{N} \exp\left[-\frac{2i\pi(j-1)(n-1)}{N}\right] e^{ib\eta(j-1)}\psi(\eta(j-1))\omega_j\eta \right\}$$

where ω_j is the Simpson's rule weighting defined by

$$\omega_j = \begin{cases} \frac{1}{3}, & j = 1, N \\ \frac{4}{3}, & 1 < j < N \text{ and } j \text{ is even} \\ \frac{2}{3}, & 1 < j < N \text{ and } j \text{ is odd} \end{cases}$$

- The summation term above can be estimated by applying FFT to the vector

$$\chi_j := e^{ib\eta(j-1)}\psi(\eta(j-1))\omega_j\eta, \quad j = 1, 2, ..., N$$

Summary of European call option pricing using FFT

1. Given a stock price model $S = (S_t)_{t \geq 0}$, work out ϕ the characteristic function of $X_T := \ln S_T$

2. Define the function ψ based on Prop 5.3

3. Supply the dampening parameter α, the number of points for discretisation of the Fourier space N (must be a power of 2) and the width of the sub-interval η. Then compute $\lambda = \frac{2\pi}{\eta N}$ and $b = \frac{\lambda N}{2}$

4. Compute the vector

$$\chi_j := e^{ib\eta(j-1)}\psi(\eta(j-1))\omega_j\eta, \quad j = 1, 2, ..., N$$

   where ω_j is the weighting vector depending on what integration method is used

5. Apply FFT to χ to obtain $w = FFT(\chi)$. Then

$$C(k_n) = \frac{e^{-\alpha k_n}}{\pi} Re(w_n), \quad n = 1, 2, ..., N$$

   which refers to the call option value when the logarithm of strike price is $k_n = -b + \lambda(n-1)$

6. (Use interpolation to estimate the option value of any arbitrary strike level)

Introduction to models with jumps: Poisson process

- Poisson process is an important building block of any stock price model with jumps
- A Poisson process $N = (N_t)_{t \geq 0}$ with intensity $\lambda > 0$ is a pure jump process with unit increment satisfying the following properties:
  1. $N_0 = 0$
  2. N has independent increments: for any $0 \leq s \leq t$, $N_t - N_s$ is independent of $\{N_u : u \leq s\}$
  3. N is stationary: for any $0 \leq s \leq t$, $N_t - N_s \sim Poi(\lambda(t-s))$

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  3. N is stationary: for any $0 \leq s \leq t$, $N_t - N_s \sim Poi(\lambda(t-s))$
- Reminder: a Poisson random variable $X \sim Poi(\theta)$ has probability mass function of

$$\mathbb{P}(X = k) = \frac{e^{-\theta}\theta^k}{k!}, \qquad k = 0, 1, 2...$$

Thus λ reflects the rate of jump occurrence in a small time interval $(t, t + \triangle t)$:

$$\mathbb{P}(N_{t+\triangle t} - N_t = 0) = e^{-\lambda \triangle t} \approx 1 - \lambda \triangle t$$
$$\mathbb{P}(N_{t+\triangle t} - N_t = 1) = e^{-\lambda \triangle t}\lambda \triangle t \approx \lambda \triangle t$$

Compound Poisson process

- A compound Poisson process has the form of

$$X_t = \sum_{j=1}^{N_t} Y_j.$$

$N = (N_t)_{t \geq 0}$ is a Poisson process with intensity λ and Y_j's are i.i.d random variables with common characteristic function ϕ_Y. N and the Y_j's are independent

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- A model of randomly arriving jumps of random size. Examples:
  - Cumulative value of claims faced by an insurance company
  - Total write-off of a credit portfolio consisting of multiple loans

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- A model of randomly arriving jumps of random size. Examples:
  - Cumulative value of claims faced by an insurance company
  - Total write-off of a credit portfolio consisting of multiple loans

- Characteristic function of X_t is

$$\phi_X(u) := \mathbb{E}\left[e^{iuX_t}\right] = \mathbb{E}\left[e^{iu\sum_{j=1}^{N_t} Y_j}\right] = \mathbb{E}\left[\mathbb{E}\left(e^{iu\sum_{j=1}^{N_t} Y_j}\middle| N_t\right)\right].$$

But $\mathbb{E}\left(e^{iu\sum_{j=1}^{N_t} Y_j}\middle| N_t\right) = (\mathbb{E}[e^{iuY_j}])^{N_t} = [\phi_Y(u)]^{N_t}$ since Y_j's are i.i.d. Hence

$$\begin{aligned}\phi_X(u) = \mathbb{E}\left([\phi_Y(u)]^{N_t}\right) = \sum_{k=0}^{\infty}[\phi_Y(u)]^k \frac{e^{-\lambda t}(\lambda t)^k}{k!} &= e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t\phi_Y(u))^k}{k!} \\ &= \exp\left[\lambda t\left(\phi_Y(u) - 1\right)\right]\end{aligned}$$

Example: Merton's jump diffusion model

- Under the risk-neutral measure, stock price process S is modelled as

$$S_t = S_0 \exp\left(\mu t + \sigma B_t + \sum_{k=1}^{N_t} J_k\right) \tag{5.4}$$

where B is a standard Brownian motion, N is a Poisson process with intensity λ and $J_k \sim N(\mu_J, \sigma_J^2)$. These three random components are all independent

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  where B is a standard Brownian motion, N is a Poisson process with intensity λ and $J_k \sim N(\mu_J, \sigma_J^2)$. These three random components are all independent

- Logarithm return is a mixture of a drifting Brownian motion and a compound Poisson process (small fluctuations plus occasional discrete shocks of stochastic size)

- Characteristic function of logarithm of stock price $X_t := \ln S_t$ is

$$\begin{aligned}
\phi_t(u) = \mathbb{E}\left[e^{iu \ln S_t}\right] &= \mathbb{E}\left\{\exp\left[\left(\ln S_0 + \mu t + \sigma B_t + \sum_{k=1}^{N_t} J_k\right) iu\right]\right\} \\
&= e^{(\mu t + \ln S_0) iu}\mathbb{E}\left[\exp\left(\sigma iu B_t\right)\right]\mathbb{E}\left[\exp\left(\sum_{k=1}^{N_t} J_k\right) iu\right] \\
&= e^{(\mu t + \ln S_0) iu} e^{-\frac{\sigma^2 t u^2}{2}} \exp\left[\lambda t\left(\phi_J(u) - 1\right)\right] \\
&= \exp\left\{iu \ln S_0 + \left[i\mu u - \frac{\sigma^2 u^2}{2} + \lambda\left(e^{i\mu_J u - \sigma_J^2 u^2/2} - 1\right)\right] t\right\}
\end{aligned}$$

Heston stochastic volatility model

- It is widely accepted that volatility of an asset is not constant (in contrast to Black-Scholes model) but rather varies stochastically over time

Heston stochastic volatility model

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- A popular stochastic volatility model is the Heston model, where stock price under risk-neutral measure $\mathbb{Q}$ has the dynamics

$$\frac{dS_t}{S_t} = rdt + \sqrt{V_t}dB_t, \quad S_0 > 0$$

and the variance process V_t is also stochastic with SDE

$$dV_t = \kappa(\eta - V_t)dt + \theta\sqrt{V_t}dW_t, \quad V_0 = \sigma_0^2$$

where B, W are two correlated $\mathbb{Q}$ Brownian motions with correlation ρ such that $dB_t dW_t = \rho dt$

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- Variance follows a CIR model which is commonly used for interest rate modelling:
  - σ_0^2: initial variance level
  - η: long-term variance level
  - κ: mean reversion speed of variance
  - θ: volatility of variance
  - ρ: correlation between asset price and its variance

- The variance process is guaranteed to be positive if $2\kappa\eta > \theta^2$

Heston stochastic volatility model (cont')

- By Ito's lemma, logarithm of asset price $X_t := \ln S_t$ has SDE of

$$dX_t = \left(r - \frac{V_t}{2}\right) dt + \sqrt{V_t} dB_t, \quad X_0 = \ln S_0$$

$$dV_t = \kappa(\eta - V_t) dt + \theta \sqrt{V_t} dW_t, \quad V_0 = \sigma_0^2$$

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- Unlike Black-Scholes model, there is no analytic solution to the above SDE

- But it turns out that the closed-form expression of the characteristic function of X_t indeed exists
  - It is exactly the reason why the Heston model is so popular!

- Efficient calibration of model parameters to given market prices is possible

Characteristic function of Heston model

Proposition 5.4

The characteristic function of logarithm of stock price $X_t := \ln S_t$ under Heston model is given by

$$\begin{aligned}\phi_t(u) := \mathbb{E}_{\mathbb{Q}}[e^{iuX_t}] = \exp\left[(\ln S_0 + rt)\, iu\right] \\ \times \exp\left\{\eta\kappa\theta^{-2}\left[(\kappa - \rho\theta ui - d)t - 2\ln\left(\frac{1-ge^{-dt}}{1-g}\right)\right]\right\} \\ \times \exp\left\{\sigma_0^2\theta^{-2}\left[\frac{(\kappa-\rho\theta ui - d)(1-e^{-dt})}{1-ge^{-dt}}\right]\right\},\end{aligned}$$

where

$$d := \left[(\rho\theta ui - \kappa)^2 + \theta^2(iu + u^2)\right]^{\frac{1}{2}}, \qquad g := \frac{\kappa - \rho\theta ui - d}{\kappa - \rho\theta ui + d}.$$

- Warning: there is another equivalent version of the above formula (e.g. in the original paper of Heston) which is prone to numerical instabilities

Finding the characteristic function of a process given its SDE

- It is relatively easy to obtain the characteristic function of a process if we have its closed-form expression
  - Example: the Merton's jump diffusion model in form of

$$S_t = S_0 \exp\left(\mu t + \sigma B_t + \sum_{k=1}^{N_t} J_k\right)$$

- What if we only have the SDE of the stock price process (eg in the case of Heston) but not a closed-form expression of S (or $X := \ln S$)?

- A common technique is the PDE formulation via Feynman-Kac (c.f. Prop 3.1).

Solving for characteristic function from a SDE: a simple example

- Black-Scholes model of stock price

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t \iff dX_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dB_t$$

where we know $X_t := \ln S_t = \ln S_0 + \left(r - \frac{\sigma^2}{2}\right) t + \sigma B_t$ and thus

$$\mathbb{E}_{\mathbb{Q}}[e^{iuX_T}] = \exp\left[iu\left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T\right) - \frac{u^2\sigma^2 T}{2}\right]$$

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- Let's pretend we don't know the answer above. Let $M_t = \phi(t, X_t)$ for some function $\phi(t, x)$. We want to find the condition on $\phi(t, x)$ such that M is a martingale.

Solving for characteristic function from a SDE: a simple example

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- Let's pretend we don't know the answer above. Let $M_t = \phi(t, X_t)$ for some function $\phi(t,x)$. We want to find the condition on $\phi(t,x)$ such that M is a martingale.

- Applying Ito's lemma to $M_t = \phi(t, X_t)$ gives

$$\begin{aligned} dM_t &= \frac{\partial \phi}{\partial t}dt + \frac{\partial \phi}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}(dX_t)^2 \\ &= \frac{\partial \phi}{\partial t}dt + \frac{\partial \phi}{\partial x}\left(\left(r - \frac{\sigma^2}{2}\right)dt + \sigma dB_t\right) + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}\sigma^2 dt \\ &= \left[\frac{\partial \phi}{\partial t} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial \phi}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 \phi}{\partial x^2}\right]dt + \text{local mg} \end{aligned}$$

Solving for characteristic function from a SDE: a simple example

- The drift term must be zero for M to be a martingale. Moreover, impose the terminal condition $\phi(T,x)=e^{iux}$ such that we have the PDE

$$\begin{cases} \frac{\partial \phi}{\partial t} + \left(r-\frac{\sigma^2}{2}\right)\frac{\partial \phi}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 \phi}{\partial x^2} = 0, & t<T; \\ \phi(T,x)=e^{iux}, & t=T. \end{cases}$$

  Then $\mathbb{E}_{\mathbb{Q}}[e^{iuX_T}] = \mathbb{E}_{\mathbb{Q}}[\phi(T,X_T)] = \mathbb{E}_{\mathbb{Q}}[M_T] = M_0 = \phi(0,X_0)$

- To solve for $\phi(t,x)$, if we conjecture a solution of the form $\phi(t,x)=\exp(iux+C(t))$ then the PDE becomes

$$C'(t) + \left(r-\frac{\sigma^2}{2}\right)iu - \frac{\sigma^2}{2}u^2 = 0$$

$$\implies C(t) = C + \left[\frac{\sigma^2}{2}u^2 - \left(r-\frac{\sigma^2}{2}\right)iu\right]t \quad \text{for some (possibly complex) constant } C$$

- The terminal condition $\phi(T,x)=e^{iux}$ would imply $C = -\left[\frac{\sigma^2}{2}u^2 - \left(r-\frac{\sigma^2}{2}\right)iu\right]T$ and hence the solution to the PDE is

$$\phi(t,x) = \exp\left[iux + \left[\left(r-\frac{\sigma^2}{2}\right)iu - \frac{\sigma^2}{2}u^2\right](T-t)\right]$$

- We thus have $\mathbb{E}[e^{iuX_T}] = \phi(0,X_0) = \phi(0,\ln S_0) = \exp\left[iu\left(\ln S_0 + \left(r-\frac{\sigma^2}{2}\right)T\right) - \frac{u^2\sigma^2 T}{2}\right]$

Deriving the Heston characteristic function: sketch of idea

- Let $M_t = \phi(t, X_t, V_t)$
- Applying Ito's lemma to $M_t = \phi(t, X_t, V_t)$ as a function of X_t and V_t gives

$$
\begin{aligned}
dM_t &= \frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial x} dX_t + \frac{\partial \phi}{\partial v} dV_t + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}(dX_t)^2 + \frac{1}{2}\frac{\partial^2 \phi}{\partial v^2}(dV_t)^2 + \frac{\partial^2 \phi}{\partial x \partial v}(dX_t)(dV_t) \\
&= \frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial x}\left(\left(r - \frac{1}{2}V_t\right) dt + \sqrt{V_t} dB_t\right) + \frac{\partial \phi}{\partial v}(\kappa(\eta - V_t)dt + \theta\sqrt{V_t} dW_t) \\
&\quad + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2} V_t dt + \frac{1}{2}\frac{\partial^2 \phi}{\partial v^2}(\theta^2 V_t) dt + \frac{\partial^2 \phi}{\partial x \partial v}(\theta \rho V_t dt) \\
&= \left[\frac{\partial \phi}{\partial t} + \left(r - \frac{V_t}{2}\right)\frac{\partial \phi}{\partial x} + \kappa(\eta - V_t)\frac{\partial \phi}{\partial v} + \frac{V_t}{2}\frac{\partial^2 \phi}{\partial x^2} + \frac{\theta^2 V_t}{2}\frac{\partial^2 \phi}{\partial v^2} + \theta\rho V_t \frac{\partial^2 \phi}{\partial x \partial v}\right] dt \\
&\quad + \text{local mg}
\end{aligned}
$$

- After setting the drift to be zero and imposing the terminal condition, we have

$$
\begin{cases}
\frac{\partial \phi}{\partial t} + \left(r - \frac{v}{2}\right)\frac{\partial \phi}{\partial x} + \kappa(\eta - v)\frac{\partial \phi}{\partial v} + \frac{v}{2}\frac{\partial^2 \phi}{\partial x^2} + \frac{\theta^2 v}{2}\frac{\partial^2 \phi}{\partial v^2} + \theta\rho v \frac{\partial^2 \phi}{\partial x \partial v} = 0, & t < T; \\
\phi(t, x, v) = e^{iuX}, & t = T
\end{cases}
$$

- The PDE can be reduced to a system of Riccati ODEs on substitution of $\phi(t, x, v) = \exp(C(t) + vD(t) + iux)$ for some functions C and D

Optional reading

- Carr, P., and Madan, D. (1999). Option valuation using the fast Fourier transform. Journal of computational finance, 2(4), 61-73.
- Schoutens, W. (2003). Lévy processes in finance: pricing financial derivatives. Chapter 5.
- Kwok, Y.K., Leung, K.S., and Wong, H.Y. (2012). Efficient options pricing using the fast Fourier transform. In: Handbook of Computational Finance, 579-604.