Numerical Methods for Finance: Problem Set 4

1. Recall that the PDE for pricing a fixed strike Asian option is given by

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 V}{\partial s^2} + rs\frac{\partial V}{\partial s} + \frac{s-a}{t}\frac{\partial V}{\partial a} - rV = 0, && t < T; \\ &V(t,s,a) = (a-K)^+, && t = T. \end{aligned}$$

On defining a new state variable $x := \frac{K - \frac{t}{T}a}{s}$ and let $V(t,s,a) = sW\left(t, \frac{K-\frac{t}{T}a}{s}\right) = sW(t,x)$. Show that W satisfies the PDE

$$\begin{aligned} &\frac{\partial W}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 W}{\partial x^2} - \left(\frac{1}{T} + rx\right)\frac{\partial W}{\partial x} = 0, && t < T \\ &W(T,x) = \max(0,-x), && t = T. \end{aligned}$$

2. Under the standard Black-Scholes model. Suppose we want to price a floating strike Asian call option where the payoff is given by

$$\left(S_T - \frac{1}{T}\int_0^T S_u du\right)^+.$$

(a) Write down the PDE that is satisfied by $V(t,s,a)$ the fair option value at time t when the current stock price is $S_t = s$ and the current running average of the stock price is $A_t := \frac{1}{t}\int_0^t S_u du = a$.

(b) Perform a similar transformation as in question 1 to reduce the dimensionality of the derived PDE in (a).

3. Recall that the PDE for pricing a floating strike lookback call option is given by

$$\begin{aligned} &\frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 V}{\partial s^2} - rV = 0, && s \geq \ell, \quad t < T; \\ &\frac{\partial V}{\partial \ell} = 0, && s = \ell, \quad t < T; \\ &V(T,s,\ell) = s - \ell, && t = T. \end{aligned}$$

(a) Let $x = \ln\frac{s}{\ell}$ and $V(t,s,\ell) = sW\left(t, \ln\frac{s}{\ell}\right) = sW(t,x)$ for some function W. Show that W satisfies the PDE

$$\begin{aligned} &\frac{\partial W}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 W}{\partial x^2} + \left(r + \frac{\sigma^2}{2}\right)\frac{\partial W}{\partial x} = 0, && x \geq 0, \quad t < T; \\ &\frac{\partial W}{\partial x} = 0, && x = 0, \quad t < T; \\ &W(T,x) = 1 - e^{-x}, && t = T. \end{aligned}$$

(b) Convert the PDE in (a) into a system with an initial condition at $t = 0$, and then write down the fully implicit scheme. Explain how the Neumann condition at $x = 0$ can be handled in your scheme. Express the complete iterative scheme using matrix notation.

4. In a programming language of your choice, implement:

(a) An explicit scheme and an implicit scheme to price a floating strike lookback call option.

(b) An implicit scheme to price an American put option via Jacobi/Gauss-Seidel iterations.