MATH97110: Numerical Methods for Finance

Topic 1: Introduction to Lattice Methods

Imperial College London

Spring 2023

Overview

1. Construction of a lattice structure
2. European options pricing
3. Choices of model parameters and convergence results
4. Extension to American options

Motivation

- In a Black-Scholes model, there is a riskfree asset with interest rate r and a risky stock with price process $S = (S_t)_{t \geq 0}$ following a geometric Brownian motion

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t \iff S_t = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)t + \sigma B_t\right]$$

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$$e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[g(S_T)\right] \tag{1.1}$$

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- We may not be able to compute (1.1) explicitly for some complicated payoff structures
- Main idea of lattice (tree) methods: approximate the continuous price process S by a simple discrete process to facilitate the expectation computation in (1.1)

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$$S_n := S_{n-1}\xi_n \iff S_n := S_0 \prod_{i=1}^{n} \xi_i, \qquad n = 1, 2, \dots, N$$

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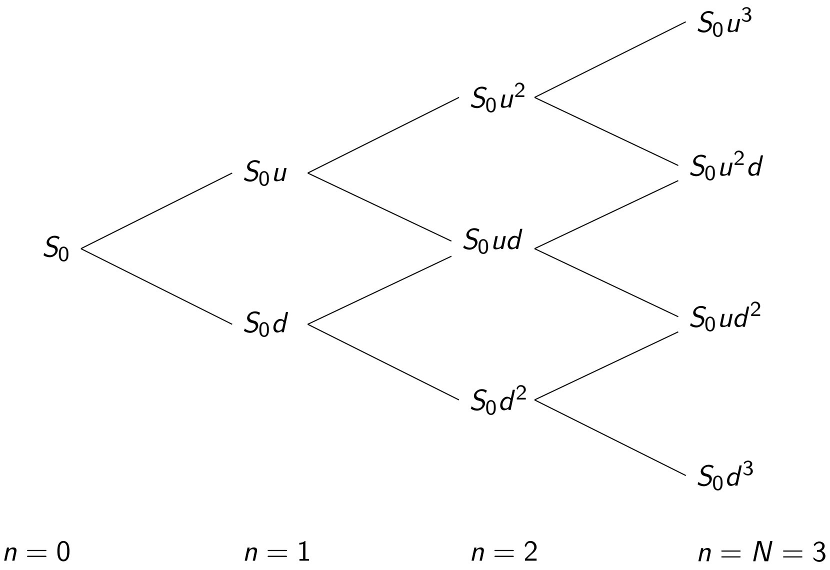
- A prime example is the binomial tree model with each ξ_i taken as a binary random variable

$$\xi_i := \begin{cases} u, & \text{with probability } q \\ d, & \text{with probability } 1-q \end{cases}$$

  with $d < 1 < u$ and $0 < q < 1$ (We will talk about how to choose u, d and q later)

Example: a three-period binomial tree ($N = 3$)



Some facts about a binomial tree

- The number of possible stock price values at time n is $n+1$ (rather than 2^n which is the total number of distinct price paths up to time n)
- At each fixed time point n, the random variable S_n takes value on the set

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$$\begin{bmatrix} S_0 & S_0 u & S_0 u^2 & S_0 u^3 & \cdots & S_0 u^N \\ & S_0 d & S_0 ud & S_0 u^2 d & \cdots & S_0 u^{N-1} d \\ & & S_0 d^2 & S_0 ud^2 & \cdots & S_0 u^{N-2} d^2 \\ & & & S_0 d^3 & \cdots & S_0 u^{N-3} d^3 \\ & & & & \ddots & \vdots \\ & & & & & S_0 d^N \end{bmatrix}$$

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- We have

$$\mathbb{Q}(S_n = s_k^n) = \mathbb{Q}(S_n = S_0 u^{n-k} d^k) = \binom{n}{k} (1-q)^k q^{n-k} \tag{1.2}$$

where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

Option pricing under a binomial tree

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- However, (1.3) is seldom used in practice because of the computational difficulty with handling $\binom{n}{k}$ for large N (eg $\binom{100}{50} \approx 10^{29}$)
- A more practical pricing approach is to employ a backward induction algorithm

The backward induction algorithm for European option

Proposition 1.1

Let $V^n := e^{-r(N-n)\triangle t}\mathbb{E}_{\mathbb{Q}}\left[g(S_N)\middle|\mathcal{F}_n\right]$ be the time-n fair value of the European option. Then the following recursive relationship holds:

$$V^n = \begin{cases} g(S_N) & \text{for } n = N; \\ e^{-r\triangle t}\mathbb{E}_{\mathbb{Q}}\left[V^{n+1}\middle|\mathcal{F}_n\right] & \text{for } n = 0, 1, ..., N-1. \end{cases} \tag{1.4}$$

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Proof. For $n = N$, we have

$$V^N = \mathbb{E}_{\mathbb{Q}}[g(S_N)|\mathcal{F}_N] = g(S_N)$$

since $g(S_N)$ is $\mathcal{F}_N$-measurable. For $n < N$, we have

$$\begin{aligned} V^n &= e^{-r(N-n)\triangle t}\mathbb{E}_{\mathbb{Q}}\left[g(S_N)\middle|\mathcal{F}_n\right] \\ &= e^{-r(N-n)\triangle t}\mathbb{E}_{\mathbb{Q}}\left\{\mathbb{E}_{\mathbb{Q}}\left[g(S_N)\middle|\mathcal{F}_{n+1}\right]\middle|\mathcal{F}_n\right\} && \text{(Tower's property)} \\ &= e^{-r\triangle t}\mathbb{E}_{\mathbb{Q}}\left\{e^{-r(N-n-1)\triangle t}\mathbb{E}_{\mathbb{Q}}\left[g(S_N)\middle|\mathcal{F}_{n+1}\right]\middle|\mathcal{F}_n\right\} \\ &= e^{-r\triangle t}\mathbb{E}_{\mathbb{Q}}\left(V^{n+1}\middle|\mathcal{F}_n\right) && \text{(definition of } V^{n+1}\text{).} \end{aligned}$$

□

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  1. Compute the option price at the terminal time N which is simply given by the payoff function, i.e. $V^N = g(S_N)$ and in particular

$$V_k^N = g(S_0 u^{N-k} d^k) \quad \text{for each } k = 0, 1, ..., N \tag{1.5}$$

  2. Loop backward in time: for $n = N-1, N-2, ..., 0$, compute

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- Note that the nature of the option (i.e. its payoff function $g(\cdot)$) only enters the pricing algorithm in the terminal condition (1.5). Thus it is very easy to incorporate new payoff structures without modifying the pricing program

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$$\mathbb{E}_{\mathbb{Q}}\left[\frac{S_{t+\triangle t}}{S_t}\right] = e^{r\triangle t}, \qquad \mathbb{E}_{\mathbb{Q}}\left[\left(\frac{S_{t+\triangle t}}{S_t}\right)^2\right] = e^{(2r+\sigma^2)\triangle t}$$

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- The first two moments of the random variable ξ are matched against the above values:

$$\begin{cases} qu + (1-q)d & = e^{r\triangle t} \\ qu^2 + (1-q)d^2 & = e^{(2r+\sigma^2)\triangle t} \end{cases} \tag{1.6}$$

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- Pin down the parameters by imposing additional constraint. The most popular choice is the Cox-Ross-Rubinstein specification in which $ud = 1$

Lemma 1.2

If we further impose $ud = 1$, the solutions to the system of equations (1.6) are given by

$$q = \frac{e^{r\triangle t} - d}{u - d}, \quad d = \frac{1}{u}, \quad u = \frac{e^{-r\triangle t}}{2}\left(1 + \nu^2 + \sqrt{(1+\nu^2)^2 - 4e^{2r\triangle t}}\right)$$

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Proof. The first equation of (1.6) gives $q = \frac{e^{r\triangle t} - d}{u-d}$. Then using the second equation we have

$$\begin{aligned}\nu^2 := e^{(2r+\sigma^2)\triangle t} = q(u^2 - d^2) + d^2 &= \frac{e^{r\triangle t} - d}{u - d}(u+d)(u-d) + d^2 \\ &= (e^{r\triangle t} - d)(u+d) + d^2 \\ &= e^{r\triangle t}u - 1 + \frac{e^{r\triangle t}}{u} \quad \text{using } ud = 1.\end{aligned}$$

Hence $e^{r\triangle t}u^2 - (1+\nu^2)u + e^{r\triangle t} = 0$ which is a quadratic equation in u. By symmetry, d will satisfy the same equation. Thus u is the larger root of this equation and its expression can be easily obtained. □

Cox-Ross-Rubinstein (CRR) binomial tree model

- If we perform Taylor expansion of u in powers of $\sqrt{\triangle t}$, we obtain

$$u = 1 + \sigma\sqrt{\triangle t} + \frac{\sigma^2}{2}\triangle t + \frac{4r^2 + 4\sigma^2 r + 3\sigma^4}{8\sigma}\triangle t^{\frac{3}{2}} + O(\triangle t^2)$$

- The above agrees with the Taylor expansion of $e^{\sigma\sqrt{\triangle t}}$ up to the $\triangle t$ term

Definition 1.3 (Cox-Ross-Rubinstein model)

The Cox-Ross-Rubinstein (CRR) model is a binomial tree with parameters:

$$q = \frac{e^{r\triangle t} - d}{u - d}, \quad u = e^{\sigma\sqrt{\triangle t}}, \quad d = e^{-\sigma\sqrt{\triangle t}}.$$

Cox-Ross-Rubinstein (CRR) binomial tree model

- If we perform Taylor expansion of u in powers of $\sqrt{\triangle t}$, we obtain

$$u = 1 + \sigma\sqrt{\triangle t} + \frac{\sigma^2}{2}\triangle t + \frac{4r^2 + 4\sigma^2 r + 3\sigma^4}{8\sigma}\triangle t^{\frac{3}{2}} + O(\triangle t^2)$$

- The above agrees with the Taylor expansion of $e^{\sigma\sqrt{\triangle t}}$ up to the $\triangle t$ term

Definition 1.3 (Cox-Ross-Rubinstein model)

The Cox-Ross-Rubinstein (CRR) model is a binomial tree with parameters:

$$q = \frac{e^{r\triangle t} - d}{u - d}, \quad u = e^{\sigma\sqrt{\triangle t}}, \quad d = e^{-\sigma\sqrt{\triangle t}}.$$

- For q to be a well-defined probability we require

$$0 < q < 1 \iff d < e^{r\triangle t} < u \iff -\sigma < r\sqrt{\triangle t} < \sigma$$

Under the typical case of positive interest rate $r > 0$, the condition becomes $\triangle t < \sigma^2/r^2$. Arbitrage will arise in the model if this condition is not satisfied

Other specifications of binomial tree's parameters

- Other specifications can be imposed on top of the system of equations (1.6) to give alternative values of (u, d, q). Examples:
  1. Jarrow-Rudd: equal probability of upward/downward move, i.e. $q = \frac{1}{2}$
  2. Tian: match the third moment of the binary random variable against that of log-normal $\left(\left(r - \frac{\sigma^2}{2}\right)\triangle t, \sigma^2\triangle t\right)$. Then

$$qu^3 + (1-q)d^3 = e^{3(r^2+\sigma^2)\triangle t}$$

  3. Modified Cox-Ross-Rubinstein: adding an arbitrary drift term to the original Cox-Ross-Rubinstein jump parameters by choosing

$$u = e^{\eta\triangle t + \sigma\sqrt{\triangle t}}, \qquad d = e^{\eta\triangle t - \sigma\sqrt{\triangle t}} \qquad \text{for some } \eta$$

- Refer to the Problem Set for some related derivations

Numerical example: European put option

Example 1.4

Use a two-period CRR binomial tree ($N = 2$) to price a European put option which payoff function is $(K - S_T)^+$ with strike price $K = 100$ and maturity $T = 1$ year. Other parameters are $S_0 = 100$, $r = 1\%$ and $\sigma = 20\%$.

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- Work out the tree parameters under CRR specifications:

$$\triangle t = \frac{T}{N} = 0.5, \quad u = e^{\sigma\sqrt{\triangle t}} = e^{0.2\times\sqrt{0.5}} = 1.15, \quad d = 1/u = 0.87, \quad q = \frac{e^{r\triangle t} - d}{u - d} = 0.48$$

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- All possible stock price values at terminal time $n = 2$ are

$$s_0^2 = S_0 u^2 = 132.69, \quad s_1^2 = S_0 ud = 100, \quad s_2^2 = S_0 d^2 = 75.36$$

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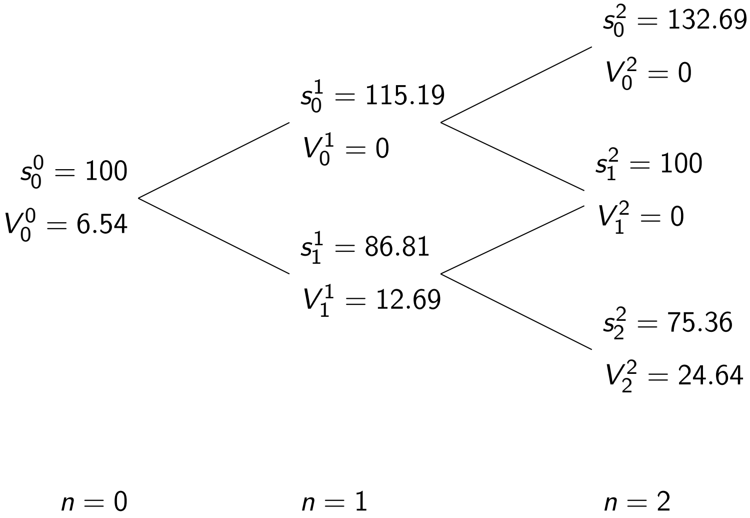
- Backward induction
  1. At terminal time $n = 2$, compute $V_k^2 = g(s_k^2) = (100 - s_k^2)^+$ for all k:
     $$V_0^2 = (100 - 132.69)^+ = 0, \quad V_1^2 = (100 - 100)^+ = 0, \quad V_2^2 = (100 - 75.36)^+ = 24.64$$
  2. At $n = 1$, $V_k^1 = e^{-r\triangle t}[qV_k^2 + (1-q)V_{k+1}^1]$. Thus
     $$V_0^1 = e^{-r\triangle t}[qV_0^2 + (1-q)V_1^2] = 0, \quad V_1^1 = e^{-r\triangle t}[qV_1^2 + (1-q)V_2^2] = 12.69$$
  3. At $n = 0$
     $$V_0^0 = e^{-r\triangle t}[qV_0^1 + (1-q)V_1^1] = 6.54$$

Numerical example: European put option (cont')



Limiting behaviour of Cox-Ross-Rubinstein model

The following proposition confirms that the CRR binomial tree model is a sensible discrete approximation of the Black-Scholes model when the number of period is sufficiently large.

Proposition 1.5 (Convergence of CRR model to Black-Scholes model)

Let $S = (S_n)_{n=1,2,\dots,N}$ be the stock price process under an N-period binomial tree with CRR parameterisation as in Definition 1.3. Fix $T > 0$ and define $\triangle t := \frac{T}{N}$. Then S_N converges in distribution to a log-normal random variable:

$$S_N \overset{dist.}{\to} S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma B_T\right] \quad \text{as} \quad N \uparrow \infty$$

where $B = (B_t)_{t \geq 0}$ is a standard Brownian motion.

Proof of Proposition 1.5

- It is equivalent to show that

$$X_N := \ln \frac{S_N}{S_0} \stackrel{\text{dist.}}{\to} \left(r - \frac{\sigma^2}{2}\right) T + \sigma B_T \sim N\left(\left(r - \frac{\sigma^2}{2}\right) T, \sigma^2 T\right) \quad \text{as } N \uparrow \infty$$

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- Recall that the distribution of a random variable Y can be fully characterised by its moment generating function (mgf) $M_Y(z) := \mathbb{E}[e^{zY}]$. Moreover, convergence of a mgf implies convergence in distribution. Hence we want to show that

$$\lim_{N\to\infty} M_{X_N}(z) = \exp\left[\left(r - \frac{\sigma^2}{2}\right) Tz + \frac{\sigma^2 T}{2} z^2\right] \tag{1.7}$$

for all z where the RHS of (1.7) is the moment generating function of a $N\left(\left(r - \frac{\sigma^2}{2}\right) T, \sigma^2 T\right)$ random variable

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- To compute the LHS of (1.7), we have

$$\begin{aligned} M_{X_N}(z) = \mathbb{E}\left[\exp\left(z \ln \frac{S_N}{S_0}\right)\right] &= \mathbb{E}\left[\exp\left(z \sum_{i=1}^{N} \ln \xi_i\right)\right] \\ &= \prod_{i=1}^{N} \mathbb{E}\left[e^{z \ln \xi_i}\right] = \left(\mathbb{E}\left[e^{z \ln \xi}\right]\right)^N \\ &= [qe^{z \ln u} + (1-q)e^{-z \ln u}]^N = [qu^z + (1-q)u^{-z}]^N \end{aligned}$$

  where we have used the i.i.d. properties of ξ_i's

Proof of Proposition 1.5 (cont')

- Using the expressions in Definition 1.3,

$$u^z = e^{z\sigma\sqrt{\triangle t}} = 1 + z\sigma\sqrt{\triangle t} + \frac{\sigma^2 z^2}{2}\triangle t + O(\triangle t^{3/2}) = 1 + \frac{z\sigma\sqrt{T}}{\sqrt{N}} + \frac{\sigma^2 z^2 T}{2N} + O(1/N^{3/2})$$

$$u^{-z} = e^{-z\sigma\sqrt{\triangle t}} = 1 - z\sigma\sqrt{\triangle t} + \frac{\sigma^2 z^2}{2}\triangle t + O(\triangle t^{3/2}) = 1 - \frac{z\sigma\sqrt{T}}{\sqrt{N}} + \frac{\sigma^2 z^2 T}{2N} + O(1/N^{3/2})$$

$$q = \frac{1}{2} + \frac{r - \frac{\sigma^2}{2}}{2\sigma}\sqrt{\triangle t} + O(\triangle t) = \frac{1}{2} + \frac{r - \frac{\sigma^2}{2}}{2\sigma}\sqrt{\frac{T}{N}} + O(1/N)$$

- Then we have

$$M_{X_N}(z) = \left\{1 + \left[\left(r - \frac{\sigma^2}{2}\right)Tz + \frac{\sigma^2}{2}Tz^2\right]\frac{1}{N} + O\left(\frac{1}{N^{3/2}}\right)\right\}^N$$

which converges to the RHS of (1.7) as $N \uparrow \infty$

$\square$

American option

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- The exercise timing strategy can be described by a stopping time τ. Examples:
  - Exercise at the end of the third period: $\tau = 3$
  - Exercise when the stock price first goes above \$100: $\tau = \inf\{n : S_n > 100\}$
- The fair time-n option price associated with a given exercise strategy τ is

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau-n)\triangle t}g(S_\tau)\middle|\mathcal{F}_n\right]$$

- However, the option seller does not know in advance what exercise strategy will be adopted by the option holder. Hence the fair time-n option price is derived as the most conservative price among all possible exercise strategies:

$$V^n := \sup_{\tau\in\mathcal{T}_{n,N}} \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau-n)\triangle t}g(S_\tau)\middle|\mathcal{F}_n\right].$$

  Here $\mathcal{T}_{n,N}$ is the set of all stopping times taking values on $\{n, n+1, ..., N\}$.

The backward induction algorithm for American option

Proposition 1.6

The time-n fair value of an American option satisfies the following recursion:

$$V^n = \begin{cases} g(S_N) & \text{for } n = N; \\ \max\left\{g(S_n), e^{-r\triangle t}\mathbb{E}_{\mathbb{Q}}\left[V^{n+1}\middle|\mathcal{F}_n\right]\right\} & \text{for } n = 0, 1, ..., N-1. \end{cases} \tag{1.8}$$

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A heuristic proof. At each time point n, there are two possibilities:

1. If it is optimal to exercise the option now, the option holder immediately receives the payoff $g(S_n)$. It is called the intrinsic value of the option at time n
2. If it is not optimal to exercise the option now, then the option continues to exist and its value in the next period will be V^{n+1} (which is random from perspective of time n). The value of this position as of time n can be computed by risk neutral pricing which is

$$\tilde{V}^n := e^{-r\triangle t}\mathbb{E}_{\mathbb{Q}}\left[V^{n+1}\middle|\mathcal{F}_n\right]$$

   We call $\tilde{V}^n$ the continuation value of the option at time n

The option value today V^n must be the larger one of the intrinsic value $g(S_n)$ and continuation value $\tilde{V}^n$, and the option is exercised if and only if $g(S_n)$ is larger than $\tilde{V}^n$. □

A formal proof of Proposition 1.6

- Let $g = (g_n)_{n=0,\ldots,N}$ be an adapted stochastic process. Define another adapted process $H = (H_n)_{n=0,\ldots,N}$ recursively via

$$H_n = \begin{cases} g_N & \text{for } n = N; \\ \max\left\{g_n, \mathbb{E}_{\mathbb{Q}}\left[H_{n+1}\middle|\mathcal{F}_n\right]\right\} & \text{for } n = 0, 1, \ldots, N-1 \end{cases}$$

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- Our goal is to show that

$$H_n = \sup_{\tau \in \mathcal{T}_{n,N}} \mathbb{E}_{\mathbb{Q}}[g_\tau|\mathcal{F}_n]. \tag{1.9}$$

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- By construction of H, for any $n < N$ we have

$$H_n = \max\left\{g_n, \mathbb{E}_{\mathbb{Q}}\left[H_{n+1}\middle|\mathcal{F}_n\right]\right\} \geq \mathbb{E}_{\mathbb{Q}}[H_{n+1}|\mathcal{F}_n]$$

  such that H is a supermartingale

A formal proof of Proposition 1.6 (cont')

- Now treat n as a fixed integer and consider an arbitrary $\tau \in \mathcal{T}_{n,N}$. We have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[g_\tau|\mathcal{F}_n] &\leq \mathbb{E}_{\mathbb{Q}}[H_\tau|\mathcal{F}_n] && (g_k \leq H_k \text{ for any } k \text{ by construction of } H)\\ &\leq H_n && (\text{by optimal stopping theorem on the supermg. } H)\end{aligned}$$

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- To prove that equality indeed holds in (1.10), we just have to identify a $\tau^* \in \mathcal{T}_{n,N}$ such that $\mathbb{E}_{\mathbb{Q}}[g_{\tau^*}|\mathcal{F}_n] = H_n$. Based on our heuristics, we try

$$\tau^* := \min\{k \geq n : g_k = H_k\}$$

which refers to the first time that the option's intrinsic value coincides with its continuation value

A formal proof of Proposition 1.6 (cont')

- We can show that for this choice of τ^*, the stopped process $M_n := H_n^{\tau^*} := H_{\tau^* \wedge n}$ is actually a martingale. Observe that for any k

$$\begin{aligned} M_{k+1} - M_k &= H_{\tau^* \wedge (k+1)} - H_{\tau^* \wedge k} \\ &= 1_{(\tau^* \geq k+1)}[H_{\tau^* \wedge (k+1)} - H_{\tau^* \wedge k}] + 1_{(\tau^* < k+1)}[H_{\tau^* \wedge (k+1)} - H_{\tau^* \wedge k}] \\ &= 1_{(\tau^* \geq k+1)}[H_{k+1} - H_k] + 1_{(\tau^* < k+1)}[H_{\tau^*} - H_{\tau^*}] \\ &= 1_{(\tau^* \geq k+1)}[H_{k+1} - H_k]. \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[M_{k+1}|\mathcal{F}_k] - M_k = \mathbb{E}_{\mathbb{Q}}[M_{k+1} - M_k|\mathcal{F}_k] &= \mathbb{E}_{\mathbb{Q}}[1_{(\tau^* \geq k+1)}(H_{k+1} - H_k)|\mathcal{F}_k] \\ &= 1_{(\tau^* \geq k+1)}\mathbb{E}_{\mathbb{Q}}[H_{k+1} - H_k|\mathcal{F}_k] \\ &= 1_{(\tau^* \geq k+1)}\left(\mathbb{E}_{\mathbb{Q}}[H_{k+1}|\mathcal{F}_k] - H_k\right) = 0. \end{aligned}$$

We have used the fact that $1_{(\tau^* \geq k+1)}$ is $\mathcal{F}_k$ measurable since τ^* is a stopping time, and by construction of τ^* we have $\mathbb{E}_{\mathbb{Q}}[H_{k+1}|\mathcal{F}_k] = H_k$ whenever $\tau^* \geq k+1$ (it refers to the event that stopping has not yet been triggered at time k)

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- We hence deduce

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[g_{\tau^*}|\mathcal{F}_n] &= \mathbb{E}_{\mathbb{Q}}[H_{\tau^*}|\mathcal{F}_n] && (H_{\tau^*} = g_{\tau^*} \text{ by definition of } \tau^*) \\ &= \mathbb{E}_{\mathbb{Q}}[H_N^{\tau^*}|\mathcal{F}_n] && (\tau^* \in T_{n,N} \implies \tau^* \leq N) \\ &= \mathbb{E}_{\mathbb{Q}}[H_n^{\tau^*}] && ((H_k^{\tau^*})_k \text{ is a martingale }) \\ &= H_n && (\tau^* \in T_{n,N} \implies \tau^* \geq n) \end{aligned}$$

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- The algorithm to compute the American option price under a binomial tree:
  1. The option price at the terminal time N is simply its intrinsic value, i.e.

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  2. Loop backward in time: for $n = N-1, N-2, ..., 0$, compute

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  3. The required time-zero option value is V_0^0
- Early exercise is optimal at node (k, n) if the intrinsic value g_k^n is greater than the continuation value $\tilde{V}_k^n := e^{-r\triangle t}[qV_k^{n+1} + (1-q)V_{k+1}^{n+1}]$

Numerical example: American put option

Example 1.7

Use a two-period CRR binomial tree ($N = 2$) to price an American put option which payoff function is $(K - S_T)^+$ with strike price $K = 100$ and maturity $T = 1$ year. Other parameters are $S_0 = 100$, $r = 1\%$ and $\sigma = 20\%$.

- Exactly the same as Example 1.4 except the option is now an American one. The backward induction equation should now be replaced by

$$V_k^n = \max\left\{g_k^n, e^{-r\triangle t}\left(qV_k^{n+1} + (1-q)V_{k+1}^{n+1}\right)\right\}$$

Numerical example: American put option

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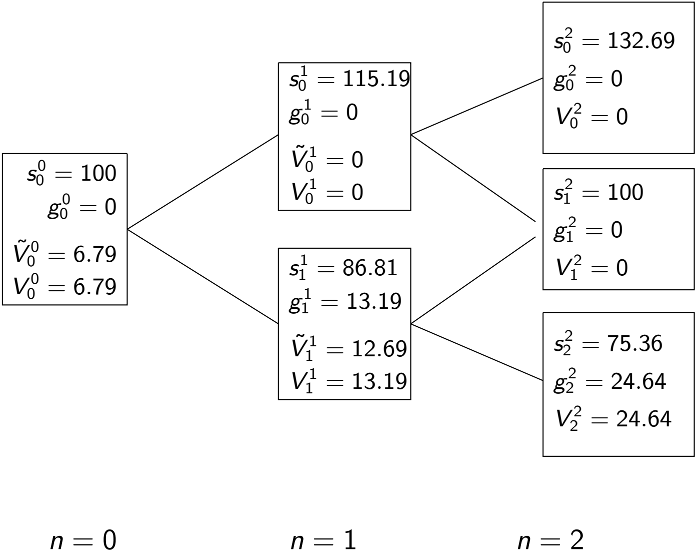
- For example, the option value at time $n = 1$ is now:

$$V_0^1 = \max(g_0^1, e^{-r\triangle t}[qV_0^2 + (1-q)V_1^2]) = \max(0, 0) = 0$$
$$V_1^1 = \max(g_1^1, e^{-r\triangle t}[qV_1^2 + (1-q)V_2^2]) = \max(13.19, 12.69) = 13.19$$

In particular, it is optimal to early exercise the option at time $n = 1$ if the stock price is at the low state of $S_1 = 86.81$

Numerical example: American put option (cont')

g_k^n, $\tilde{V}_k^n$, and V_k^n denote respectively the intrinsic value, continuation value and fair value of the American option at time n when the stock price is $s_k^n = S_0 u^{n-k} d^k$.



Optional reading

- Wilmott, P., Howson, S., Howison, S., & Dewynne, J. (1995). The Mathematics of Financial Derivatives: A Student Introduction. Chapter 10.
  - Recap of theories on binomial tree model
- Chan, J. H., Joshi, M., Tang, R., & Yang, C. (2009). Trinomial or binomial: Accelerating American put option price on trees. Journal of Futures Markets, 29(9), 826-839.
  - Comparison of pricing performance under different lattice tree specifications and implementations