Numerical Methods for Finance: Problem Set 3

1. Many of our examples in the lectures are framed around pricing of options written on a stock. But it is indeed possible to adopt the PDE approach to price many other financial instruments such as interest rate products. Consider the following *short rate model* characterised by the stochastic differential equation

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dB_t$$

where $\kappa, \sigma, \theta > 0$ are constants and B is a standard Brownian motion (it is called the Cox–Ingersoll–Ross (CIR) model). The time-t fair price of a zero coupon bond with maturity T is

$$\mathbb{E}^{(t,r)}\left[\exp\left(-\int_t^T r_u du\right)\right]$$

with $\mathbb{E}^{(t,r)}[\cdot]$ being the conditional expectation given information up to time t and that $r_t = r$.

(a) Consider t as a fixed constant and define a stochastic process $M_s := \exp\left(-\int_t^s r_u du\right) V(s, r_s)$, where $V \in C^{1\times 2}$ is some function. By applying Ito's lemma to M, derive a PDE that should be satisfied by V for M to be a (local) martingale.

(b) Suppose V is the solution to the PDE you derive in (a). By specifying an appropriate terminal condition on $V(T, r)$, show that

$$V(t, r) = \mathbb{E}^{(t,r)}\left[\exp\left(-\int_t^T r_u du\right)\right].$$

You may assume any local martingale arising in your derivation is a true martingale.

2. Recall that the Black-Scholes PDE is given by

$$\frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 V}{\partial s^2} - rV = 0, \qquad t < T \tag{1}$$

with some terminal condition in form of $V(T, s) = g(s)$.

(a) Consider a transformation of $s = Ke^x$, $t = T - \frac{2}{\sigma^2}\tau$ and $V = Kv(\tau, x)$. Show that (1) can be reduced to

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv \tag{2}$$

where $k := \frac{2r}{\sigma^2}$.

(b) Consider a further transformation of $v(\tau, x) = e^{\alpha x + \beta\tau}u(\tau, x)$ for some constants α and β. Show that (2) can be further reduced to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (2\alpha + k - 1)\frac{\partial u}{\partial x} + [\alpha^2 + (k-1)\alpha - \beta - k]u. \tag{3}$$

Find the value of α and β such that the coefficients of $\frac{\partial u}{\partial x}$ and u vanish in (3).

3. Suppose we know the value of $f(x)$ at three points: x_0, $x_0 + h$ and $x_0 + 2h$.

(a) Is it possible to construct a finite difference estimate D_1 which approximates $f'(x_0)$ such that $D_1 - f'(x_0) = O(h^2)$? (Hint: try $D_1 = \alpha_1 f(x_0) + \alpha_2 f(x_0 + h) + \alpha_2 f(x_0 + 2h)$ for some α_1, α_2 and α_3)

(b) Is it possible to construct a finite difference estimate D_2 which approximates $f''(x_0)$? What is the best accuracy you can achieve?

4. Consider a heat equation in form of

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}.$$

(a) Write down the differencing equations corresponding to: explicit scheme, fully implicit scheme and θ-scheme.

(b) Show that the local truncation error associated with the fully implicit scheme is of order $O(\triangle x^2) + O(\triangle t)$

(c) Perform a von Neumann stability analysis on the θ-scheme. Under what range of θ will the scheme be unconditionally stable?

5. Implement the explicit and implicit finite difference scheme to solve the Black-Scholes PDE in a programming language of your choice. Compare your numerical results against the Black-Scholes formula when you pick the call option payoff as the terminal condition of the PDE.