MATH97110: Numerical Methods for Finance

Topic 4: Option Pricing with Finite Difference Methods

Imperial College London

Spring 2023

Overview

1. Path-dependent option pricing with finite difference methods
   - Barrier option
   - Asian option
   - Lookback option
2. Extension to American option
   - Variational inequality
   - Explicit scheme vs implicit scheme
   - Iterative methods for solving system of equations

Barrier option

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Payoffs of different types of barrier call option:

Down-and-out:	$(S_T - K)^+ 1_{(L_T > B)}$
Down-and-in:	$(S_T - K)^+ 1_{(L_T \le B)}$
Up-and-out:	$(S_T - K)^+ 1_{(H_T < B)}$
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- Downside: more complicated risk profiles

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- We are interested in the value of $V(t,s)$ over $(t,s) \in [0,T] \times [0, B_{ko}]$. We already have the boundary condition $V(t, B_{ko}) = 0$. Along $s = s_{min} = 0$, we expect $V(t,0) = 0$ as well since a call option is involved

PDE method for knock-in barrier option

- Now suppose $V(t, s)$ is the time-t value of an **up-and-in barrier call** option $((S_T - K)^+ 1_{(H_T \geq B_{in})})$ with knock-in level B_{in} $(> S_0)$ when the current stock price is s

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- A convenient approach is to utilise the "in-out parity":

$$(S_T - K)^+ = (S_T - K)^+ 1_{(H_T < B)} + (S_T - K)^+ 1_{(H_T \geq B)}$$

  and hence

  European call price $=$ up-and-out barrier call price $+$ up-and-in barrier call price.

  From this, the knock-in barrier option price can be deduced via the knock-out barrier option price

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  European call price = up-and-out barrier call price + up-and-in barrier call price.

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- Note that the above relationship is true for any model: it is derived from no-arbitrage principle

PDE method for general path-dependent options

- Same idea as in the analysis of path-dependent options pricing under lattice methods:

  Pick a suitable auxiliary process F such that the payoff function can be rewritten as $g(S_T, F_T)$. Then the time-t fair option value is

$$V(t,s,f) = \mathbb{E}_{\mathbb{Q}}^{(t,s,f)}[e^{-r(T-t)}g(S_T, F_T)]$$

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- An explicit but somewhat convoluted formula does exist. It is usually better to derive the PDE on spot by identifying the function $V(t,s,f)$ such that the process

  $$M_u = e^{-\int_t^u r(\theta,S_\theta,F_\theta)d\theta}V(u,S_u,F_u)$$

  is a martingale. Here t is considered as a fixed constant

PDE method for Asian option

- The payoff of an Asian option depends on the average value of stock price $\frac{1}{T}\int_0^T S_u du$.

  Let $A_t := \frac{1}{t}\int_0^t S_u du$ be the running average of the stock price up to time t. Then

$$dA_t = -\frac{1}{t^2}\left(\int_0^t S_u du\right)dt + \frac{S_t}{t}dt = \frac{S_t - A_t}{t}dt$$

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- Under Black-Scholes model, we further have $dS_t = rS_t dt + \sigma S_t dB_t$. For $M_u := e^{-r(u-t)}V(u, S_u, A_u)$ to be a martingale, we compute

$$\begin{aligned} dM_u &= -re^{-r(u-t)}Vdu + e^{-r(u-t)}\left[\dot{V}du + \frac{\partial V}{\partial s}dS_u + \frac{\partial V}{\partial a}dA_u + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}(dS_u)^2\right] \\ &= e^{-r(u-t)}\left[\left(\dot{V} + \frac{\sigma^2 S_u^2}{2}\frac{\partial^2 V}{\partial s^2} + rS_u\frac{\partial V}{\partial s} + \frac{S_u - A_u}{u}\frac{\partial V}{\partial a} - rV\right)du + \text{B.M term}\right] \end{aligned}$$

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- It is not easy to work with (4.1) directly because the term $\frac{s-a}{t}$ will explode for very small t

Transforming the PDE for pricing Asian option

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- Define a new state variable $x := \frac{K - \frac{t}{T}a}{s}$ and postulate that the value function $V(t, s, a)$ can be expressed as

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- One can show (exercise) that W satisfies the PDE

$$\frac{\partial W}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 W}{\partial x^2} - \left(\frac{1}{T} + rx\right)\frac{\partial W}{\partial x} = 0.$$

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- The terminal condition can be derived as

$$W(T, x) = W\left(T, \frac{K - a}{s}\right) = \frac{1}{s}V(T, s, a) = \frac{1}{s}(a - K)^+ = \left(\frac{a - K}{s}\right)^+ = \max(0, -x)$$

Transforming the PDE for pricing Asian option (cont')

- x can take any value in $(-\infty, \infty)$. As usual, we need to truncate the domain to work with $(t, x) \in [0, T] \times [x_{min}, x_{max}]$ for some very small and large value of x_{min} and x_{max}

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- Boundary conditions ($x := \frac{K - \frac{t}{T} a}{s}$):
  - $x \approx +\infty$ corresponds to current stock price s being small and $K - \frac{t}{T} a > 0$. Then we expect

$$A_T = \frac{1}{T} \int_0^T S_u du = \frac{1}{T} \left[\int_0^t S_u du + \int_t^T S_u du \right] \approx \frac{1}{T}(ta + 0) = \frac{t}{T} a < K$$

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  - $x \approx -\infty$ corresponds to current stock price s being small and/or current running average a being large such that $K - \frac{t}{T}a < 0$. The option is likely to expire in-the-money which value today will approximately be $e^{-r(T-t)}(\frac{t}{T}a - K)$. Thus $W(t, x_{min}) = -xe^{-r(T-t)}$

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- After we obtain $W(t, x)$, the required time-0 value of the option is

$$V(0, S_0, A_0) = S_0 W(0, K/S_0)$$

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- Remark: a similar transformation works for floating strike Asian option as well (problem set)

PDE method for lookback option

- The payoff of a lookback option depends on the maximum or minimum stock price attained over the product's lifetime.
  If we want to price a floating strike lookback call with payoff $S_T - \inf_{0 \leq u \leq T} S_u$, then a suitable choice of the auxiliary variable is $L_t := \inf_{0 \leq u \leq t} S_u$

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- How does the process L look like?
  - It is a monotonic decreasing process (and hence must be a finite variation process)
  - The process value remains unchanged for most of the time, and ticks down when S_t hits a new minimum level
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- Let the time-t value of a floating strike lookback call option be $V(t, s, \ell)$ where s and ℓ are the current values of S_t and L_t. We want to find a PDE satisfied by $V(t, s, \ell)$. Note that we only need to consider $s \ge \ell$

Deriving the PDE for lookback option

- Suppose we work under the Black-Scholes model. We want $M_u := e^{-r(u-t)}V(u, S_u, L_u)$ to be a martingale. Hence

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dM_u &= -re^{-r(u-t)}Vdu + e^{-r(u-t)}\left[\dot{V}du + \frac{\partial V}{\partial s}dS_u + \frac{\partial V}{\partial \ell}dL_u + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}(dS_u)^2\right] \\
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\end{aligned}
$$

- Why we don't need to consider the $(dL)^2$ and $(dS)(dL)$ terms in the Ito's lemma application above? It is because they are zero as L is a finite variation process

Deriving the PDE for lookback option

- Suppose we work under the Black-Scholes model. We want $M_u := e^{-r(u-t)}V(u, S_u, L_u)$ to be a martingale. Hence

$$\begin{aligned} dM_u &= -re^{-r(u-t)}Vdu + e^{-r(u-t)}\left[\dot{V}du + \frac{\partial V}{\partial s}dS_u + \frac{\partial V}{\partial \ell}dL_u + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}(dS_u)^2\right] \\ &= e^{-r(u-t)}\left[\left(\dot{V} + \frac{\sigma^2 S_u^2}{2}\frac{\partial^2 V}{\partial s^2} + rS_u\frac{\partial V}{\partial s} - rV\right)du + \frac{\partial V}{\partial \ell}dL_u + \text{B.M term}\right] \end{aligned}$$

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- Whenever $S_u = L_u$, $dL_u < 0$. We must then in additional require

$$\frac{\partial V}{\partial \ell} = 0 \tag{4.2}$$

Deriving the PDE for lookback option (cont')

- Putting everything together, the fair price of a floating strike lookback call option should solve the system

$$\begin{cases} \frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 V}{\partial s^2} - rV = 0, & s \geq \ell, \quad t < T \\ \frac{\partial V}{\partial \ell} = 0, & s = \ell, \quad t < T \\ V(T, s, \ell) = s - \ell, & t = T \end{cases} \tag{4.3}$$

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- Let $x = \ln \frac{s}{\ell}$ and we postulate that $V(t, s, \ell) = sW\left(t, \ln \frac{s}{\ell}\right) = sW(t, x)$ for some function W. Then it can be shown (exercise) that the PDE becomes

$$\begin{cases} \frac{\partial W}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 W}{\partial x^2} + \left(r + \frac{\sigma^2}{2}\right)\frac{\partial W}{\partial x} = 0, & x \geq 0, t < T \\ \frac{\partial W}{\partial x} = 0, & x = 0, t < T \\ W(T, x) = 1 - e^{-x}, & t = T \end{cases} \tag{4.4}$$

Transforming the PDE for pricing lookback option

- At $t = 0$, we have $L_0 = S_0 = s$ and hence the required option value is

$$V(0, S_0, L_0) = S_0 W(0, \ln \frac{S_0}{L_0}) = sW(0, 0)$$

  Hence the lookback option price is linear in s at $t = 0$

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- The domain of problem (4.4) is now $(t, x) \in [0, T] \times [0, \infty)$. As before, the space domain is truncated as $[x_{min}, x_{max}]$ with $x_{min} = 0$ and we take some large value of x_{max}

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- For $x = \ln \frac{s}{\ell} \approx +\infty$, the current stock price is much higher than its historical minimum and we expect $L_T = \ell$. The option price should then be $V(t, s, \ell) \approx s - e^{-r(T-t)}\ell$ and hence $W(t, x) = \frac{V}{s} \approx 1$. Hence we set $W(t, x_{max}) = 1$

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- How to incorporate the condition $\frac{\partial W}{\partial x} = 0$ at $x = x_{min} = 0$?

Neumann boundary condition within finite difference schemes

- Suppose we have reversed the time direction via $\tau := T - t$ to turn (4.4) into an initial condition problem. Explicit scheme gives us:

$$\frac{W_k^{n+1} - W_k^n}{\triangle\tau} = \frac{\sigma^2}{2}\frac{W_{k-1}^n - 2W_k^n + W_{k+1}^n}{\triangle x^2} + \left(r + \frac{\sigma^2}{2}\right)\frac{W_{k+1}^n - W_{k-1}^n}{2\triangle x}$$
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$$A_k^n := \frac{\sigma^2}{2}\frac{\triangle\tau}{\triangle x^2} - \frac{1}{2}\left(r + \frac{\sigma^2}{2}\right)\frac{\triangle\tau}{\triangle x}, \qquad B_k^n := -\sigma^2\frac{\triangle\tau}{\triangle x^2}, \qquad C_k^n := \frac{\sigma^2}{2}\frac{\triangle\tau}{\triangle x^2} + \frac{1}{2}\left(r + \frac{\sigma^2}{2}\right)$$

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- At $k = 0$, this becomes

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- Idea: approximate the condition $\frac{\partial W}{\partial x} = 0$ by central difference involving a fictitious grid point (t_n, x_{-1}) via

$$\frac{\partial W}{\partial x} \approx \frac{W_1^n - W_{-1}^n}{2\triangle x} = 0 \implies W_{-1}^n = W_1^n$$

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- Then $W_0^{n+1} = A_0^n W_1^n + (1 + B_0^n) W_0^n + C_0^n W_1^n = (1 + B_0^n) W_0^n + (A_0^n + C_0^n) W_1^n$

Modified explicit scheme for lookback option

- The recursive equations can be summarised using matrix notation

$$\underbrace{\begin{bmatrix} W_0^{n+1} \\ W_1^{n+1} \\ W_2^{n+1} \\ \vdots \\ \vdots \\ W_{M-1}^{n+1} \\ W_M^{n+1} \end{bmatrix}}_{=:W^{n+1}} = \underbrace{\begin{bmatrix} 1+B_0^n & A_0^n + C_0^n & 0 & \cdots & & & & 0 & 0 \\ A_1^n & 1+B_1^n & C_1^n & 0 & \cdots & & & 0 & 0 \\ 0 & A_2^n & 1+B_2^n & C_2^n & 0 & \cdots & & & \\ & & & & \ddots & & & & \\ \vdots & & & & & A_{M-2}^n & 1+B_{M-2}^n & C_{M-2}^n & 0 \\ & & & & & 0 & A_{M-1}^n & 1+B_{M-1}^n & C_{M-1}^n \\ 0 & 0 & \cdots & & & 0 & 0 & 0 & 1 \end{bmatrix}}_{=:\mathbb{I}+\tilde{L}^n} \underbrace{\begin{bmatrix} W_0^n \\ W_1^n \\ W_2^n \\ \vdots \\ \vdots \\ W_{M-1}^n \\ W_M^n \end{bmatrix}}_{=:W^n}$$

Here $\mathbb{I}$ is an $(M+1)\times(M+1)$ identity matrix and $\tilde{L}^n$ is an $(M+1)\times(M+1)$ matrix in form of

$$\tilde{L}^n := \begin{bmatrix} B_0^n & A_0^n + C_0^n & 0 & \cdots & & & & 0 & 0 \\ A_1^n & B_1^n & C_1^n & 0 & \cdots & & & 0 & 0 \\ 0 & A_2^n & B_2^n & C_2^n & 0 & \cdots & & & \\ & & & & \ddots & & & & \\ \vdots & & & & & A_{M-2}^n & B_{M-2}^n & C_{M-2}^n & 0 \\ & & & & & 0 & A_{M-1}^n & B_{M-1}^n & C_{M-1}^n \\ 0 & 0 & \cdots & & & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.5}$$

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- In general we need to modify the last element of W^{n+1} to make sure it satisfies the boundary condition (in this specific example, $W(t, x_{max}) = 1$). The explicit scheme is

$$W^{n+1} = B^{n+1}[(\mathbb{I} + \tilde{L}^n)W^n]$$

where $B^{n+1}(\cdot)$ is the boundary operator

Pricing American option with PDE method

- Recall that the time-t fair price of an American option is given by

$$\sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{\mathbb{Q}} \left[e^{-r(\tau - t)} g(S_\tau) \right]$$

where $\mathcal{T}_{t,T}$ is the set of stopping times taking values between t and T

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- In a discrete time lattice model, we have shown in Topic 1 that the fair option price satisfies the recursive equation

$$\begin{aligned} V^n &= \max[\underbrace{e^{-r\triangle t}\mathbb{E}(V^{n+1}|\mathcal{F}_n)}_{\text{continuation val}}, \ \underbrace{g(S_n)}_{\text{intrinsic val}} \ ] \\ \iff & \min\left(V^n - e^{-r\triangle t}\mathbb{E}(V^{n+1}|\mathcal{F}_n), V^n - g(S_n)\right) = 0 \end{aligned} \tag{4.6}$$

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$$\iff \min\left(V^n - e^{-r\triangle t}\mathbb{E}(V^{n+1}|\mathcal{F}_n), V^n - g(S_n)\right) = 0 \tag{4.6}$$

- There is a continuous-time version of (4.6) which can be expressed in terms of a PDE

American option and variational inequality

Proposition 4.1

Suppose $g(x)$ is a given function and $V = V(t,x)$ is the solution to the variational inequality

$$\begin{cases} \min\left(-\frac{\partial V}{\partial t} - b(t,x)\frac{\partial V}{\partial x} - \frac{\sigma^2(t,x)}{2}\frac{\partial^2 V}{\partial x^2} + r(t,x)V, V - g\right) = 0, & t < T; \\ V(T,x) = g(x), & t = T. \end{cases} \tag{4.7}$$

Then subject to some suitable regularity conditions on V, r, b and σ, we have

$$V(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{(t,x)}[e^{-\int_t^\tau r(u,X_u)du} g(X_\tau)]. \tag{4.8}$$

where $X = (X_s)_{s\in[t,T]}$ is the solution to the SDE

$$dX_s = b(s,X_s)ds + \sigma(s,X_s)dB_s, \quad X_t = x.$$

Moreover, the optimal stopping time τ^* associated with (4.8) is given by

$$\tau^* = \inf\{s \geq t : V(s,X_s) = g(X_s)\}. \tag{4.9}$$

Sketch of proof of Prop 4.1

- Consider t as fixed. For $s \in [t, T]$, define $M_s := e^{-\int_t^s r(u,X_u)du} V(s, X_s)$. The idea of the proof is to show that M is a supermartingale and the stopped process M^{τ^*} is a martingale

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- Write $D_s := e^{-\int_t^s r(u,X_u)du}$. Application of Ito's lemma to $M_s := D_s V(s, X_s)$ gives:

$$\begin{aligned} dM_s &= -r(s, X_s)e^{-\int_t^s r(u,X_u)du} V ds \\ &\quad + e^{-\int_t^s r(u,X_u)du} \left(\dot{V} ds + V_x dX_s + \frac{1}{2} V_{xx} (dX_s)^2 \right) \\ &= D_s \left(\dot{V} + b(s, X_s) V_x + \frac{1}{2} \sigma^2(s, X_s) V_{xx} - r(s, X_s) V \right) ds \\ &\quad + D_s V_x \sigma(s, X_s) dB_s \\ &=: D_s f(s, X_s) ds + D_s V_x \sigma(s, X_s) dB_s \end{aligned}$$

and thus

$$M_s = M_t + \int_t^s D_u f(u, X_u) du + \int_t^s D_u V_x \sigma(u, X_s) dB_u \tag{4.10}$$

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dM_s &= -r(s, X_s)e^{-\int_t^s r(u,X_u)du} V ds \\
&\quad + e^{-\int_t^s r(u,X_u)du}\left(\dot{V} ds + V_x dX_s + \frac{1}{2} V_{xx}(dX_s)^2\right) \\
&= D_s\left(\dot{V} + b(s, X_s)V_x + \frac{1}{2}\sigma^2(s, X_s)V_{xx} - r(s, X_s)V\right) ds \\
&\quad + D_s V_x \sigma(s, X_s) dB_s \\
&=: D_s f(s, X_s) ds + D_s V_x \sigma(s, X_s) dB_s
\end{aligned}
$$

and thus

$$
M_s = M_t + \int_t^s D_u f(u, X_u) du + \int_t^s D_u V_x \sigma(u, X_s) dB_u \tag{4.10}
$$

- Since V satisfies the variational inequality (4.7), $f(s, x) \le 0$. If the stochastic integral above is a true martingale, then we can deduce $\mathbb{E}[M_s | \mathcal{F}_t] \le M_t$ such that M is a supermartingale

Sketch of proof of Prop 4.1 (cont')

- Since M is a supermartingale, for any stopping time τ the stopped process $M_s^\tau := M_{s\wedge\tau}$ is also a supermartingale

Sketch of proof of Prop 4.1 (cont')

- Since M is a supermartingale, for any stopping time τ the stopped process $M_s^\tau := M_{s\wedge\tau}$ is also a supermartingale
- For any $\tau \in T_{t,T}$, we have *

$$
\begin{aligned}
\mathbb{E}^{(t,x)}[e^{-\int_t^\tau r(u,X_u)du} g(X_\tau)] &\le \mathbb{E}^{(t,x)}[e^{-\int_t^\tau r(u,X_u)du} V(\tau, X_\tau)] && (V \ge g) \\
&= \mathbb{E}^{(t,x)}[M_\tau] && (\text{definition of } M) \\
&= \mathbb{E}^{(t,x)}[M_T^\tau] && (\tau \in \mathcal{T}_{t,T} \implies \tau \le T) \\
&\le M_t^\tau = M_t = V(t,x) && (M^\tau \text{ is a supermartingale})
\end{aligned}
$$

Taking supremum on both sides gives

$$
\sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}^{(t,x)}[e^{-\int_t^\tau r(u,X_u)du} g(X_\tau)] \le V(t,x)
$$

Sketch of proof of Prop 4.1 (cont')

- Since M is a supermartingale, for any stopping time τ the stopped process $M_s^\tau := M_{s\wedge\tau}$ is also a supermartingale
- For any $\tau \in T_{t,T}$, we have *

$$\begin{aligned}
\mathbb{E}^{(t,x)}[e^{-\int_t^\tau r(u,X_u)du} g(X_\tau)] &\leq \mathbb{E}^{(t,x)}[e^{-\int_t^\tau r(u,X_u)du} V(\tau, X_\tau)] && (V \geq g) \\
&= \mathbb{E}^{(t,x)}[M_\tau] && (\text{definition of } M) \\
&= \mathbb{E}^{(t,x)}[M_T^\tau] && (\tau \in \mathcal{T}_{t,T} \implies \tau \leq T) \\
&\leq M_t^\tau = M_t = V(t,x) && (M^\tau \text{ is a supermartingale})
\end{aligned}$$

  Taking supremum on both sides gives

$$\sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{(t,x)}[e^{-\int_t^\tau r(u,X_u)du} g(X_\tau)] \leq V(t,x)$$

- To establish equality in the above, we need to demonstrate that the τ^* defined in (4.9) leads to $\mathbb{E}^{(t,x)}[e^{-\int_t^{\tau^*} r(u,X_u)du} g(X_{\tau^*})] = V(t,x)$. If we replace s by $s \wedge \tau^*$ in (4.10), then

$$\begin{aligned}
M_s^{\tau^*} := M_{s\wedge\tau^*} &= M_t + \int_t^{s\wedge\tau^*} D_u f(u, X_u) du + \int_t^{s\wedge\tau^*} D_u V_x \sigma(u, X_s) dB_u \\
&= M_t + \int_t^{s\wedge\tau^*} D_u V_x \sigma(u, X_s) dB_u
\end{aligned}$$

  because $f(u, X_u) = 0$ for $u < \tau^*$ by definition of τ^*. We can deduce the stopped process M^{τ^*} is a true martingale. If we repeat the analysis in * above, then all the inequalities there now become equalities when $\tau = \tau^*$

Comments on the proof of Prop 4.1

- The main gaps of this proof are:
  - Whether the stochastic integral against Brownian motion is a true martingale
  - Whether V is second order smooth such that Ito's lemma can be applied (actually V is only C^1 smooth in the space dimension for lot of cases!)

The PDE approach for American option pricing

- We need to design a numerical scheme to solve the variational inequality system

$$
\begin{aligned}
&\min\left(\frac{\partial V}{\partial \tau} - a(\tau,x)\frac{\partial^2 V}{\partial x^2} - b(\tau,x)\frac{\partial V}{\partial x} + c(\tau,x)V,\ V-g\right) = 0, && \tau > 0;\\
&V(0,x) = g(x), && \tau = 0.
\end{aligned}
$$

(assuming that we have reversed the time direction via $\tau := T - t$ to turn a terminal condition problem into an initial condition problem)

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$$\begin{aligned} &\min\left(\frac{\partial V}{\partial \tau} - a(\tau,x)\frac{\partial^2 V}{\partial x^2} - b(\tau,x)\frac{\partial V}{\partial x} + c(\tau,x)V,\ V-g\right) = 0, && \tau > 0; \\ &V(0,x) = g(x), && \tau = 0. \end{aligned}$$

  (assuming that we have reversed the time direction via $\tau := T - t$ to turn a terminal condition problem into an initial condition problem)

- Write

$$\mathcal{G}V := \frac{\partial V}{\partial \tau} - a(\tau,x)\frac{\partial^2 V}{\partial x^2} - b(\tau,x)\frac{\partial V}{\partial x} + c(\tau,x)V$$

  such that the system becomes $\min(\mathcal{G}V, V-g) = 0$. This means that for any $t > 0$, one of the following must hold:

  1. $\mathcal{G}V = 0$ and $V \geq g$
  2. $V = g$ and $\mathcal{G}V \geq 0$

Explicit scheme for American option pricing

- Suppose V_k^n is the approximation of $V(\tau_n, x_k)$

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- To enforce $V \geq g$, we require $V^n \geq g^n$ for all n where $g^n = [g(x_0), ..., g(x_M)]^\top$
- Under explicit scheme, the condition $\mathcal{G}V \geq 0$ can be approximated by finite difference:

$$\frac{V_k^{n+1} - V_k^n}{\triangle\tau} - a_k^n \frac{V_{k+1}^n - 2V_k^n + V_{k-1}^n}{\triangle x^2} - b_k^n \frac{V_{k+1}^n - V_{k-1}^n}{2\triangle x} + c_k^n V_k^n \geq 0$$
$$\implies V^{n+1} - B^{n+1}[(\mathbb{I} + L^n)V^n] \geq 0$$

where L^n is a tridiagonal matrix and $B^{n+1}(\cdot)$ is the boundary operator

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- Thus the system $\min(\mathcal{G}V, V - g) = 0$ can be approximated as

$$\min\left(V^{n+1} - B^{n+1}[(\mathbb{I} + L^n)V^n], V^{n+1} - g^{n+1}\right) = 0$$

from which we can deduce

$$V^{n+1} = \max\left(B^{n+1}[(\mathbb{I} + L^n)V^n], g^{n+1}\right)$$

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$$V^{n+1} = \max\left(B^{n+1}[(\mathbb{I} + L^n)V^n], g^{n+1}\right)$$

- American option pricing can be easily incorporated within an explicit scheme. All we need to do is to take the $\max(\cdot, g^n)$ into account (c.f. pricing American option with lattice method)

Implicit scheme for American option pricing

- Under implicit scheme, the condition $\mathcal{G}V \geq 0$ can be approximated by:

$$\frac{V_k^n - V_k^{n-1}}{\triangle\tau} - a_k^n \frac{V_{k+1}^n - 2V_k^n + V_{k-1}^n}{\triangle x^2} - b_k^n \frac{V_{k+1}^n - V_{k-1}^n}{2\triangle x} + c_k^n V_k^n \geq 0$$

$$\Longrightarrow [\mathbb{I} - L^n]V^n - B^n V^{n-1} \geq 0$$

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$$\frac{V_k^n - V_k^{n-1}}{\triangle\tau} - a_k^n \frac{V_{k+1}^n - 2V_k^n + V_{k-1}^n}{\triangle x^2} - b_k^n \frac{V_{k+1}^n - V_{k-1}^n}{2\triangle x} + c_k^n V_k^n \geq 0$$
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- The system $\min(\mathcal{G}V, V - g) = 0$ is now approximated as

$$\min\left([\mathbb{I} - L^n]V^n - B^n V^{n-1}, V^n - g^n\right) = 0 \tag{4.11}$$

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- To solve for V^n from (4.11), we need to find a way to solve a problem in form of $\min(Ax - b, x - g) = 0$

Numerical solution to system of equations

- We begin by looking at how an n-by-n system of equation in form of $Ax = b$ can be solved:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
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\end{aligned}
$$

- Suppose all a_{ii}'s are non-zero. Rewrite the above as

$$
\begin{aligned}
x_1 &= \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n) \\
x_2 &= \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n) \\
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x_n &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1})
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\end{aligned}
$$

- The above can be written in matrix notation as

$$x = -D^{-1}(L + U)x + D^{-1}b$$

  where $A = D + L + U$ such that:
  - D is a diagonal matrix containing the diagonal elements of A
  - L is a lower triangular matrix containing all elements of A below its diagonal
  - U is an upper triangular matrix containing all elements of A above its diagonal

Jacobi iterative method

- If x is the solution to $Ax = b$, then x must be a fixed point to the function
$$f(x) := -D^{-1}(L+U)x + D^{-1}b := Tx + c$$

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$$x^k = Tx^{k-1} + c.$$

  If the sequence of vectors convergences, then it must be the solution to the system of equations

- Algorithm of the Jacobi iterative method:
  1. Choose an initial guess vector $x^0 = [x_1^0, x_2^0, ..., x_n^0]^\top$ arbitrarily
  2. Loop through $k = 1, 2, 3, ...$

$$x_i^k = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1, j\neq i}^{n} a_{ij} x_j^{k-1} \right] \qquad \text{for } i = 1, 2, ..., n$$

  Exit the loop when $k >$ "max iterations num" or $||x^k - x^{k-1}|| <$ "error tolerance"

Gauss-Seidel iterative method

- If we write down the updating rule of Jacobi iteration method element-by-element:

$$
\begin{aligned}
x_1^k &= \frac{1}{a_{11}}(b_1 - a_{12}x_2^{k-1} - a_{13}x_3^{k-1} - \cdots - a_{1n}x_n^{k-1}) \\
x_2^k &= \frac{1}{a_{22}}(b_2 - a_{21}x_1^{k-1} - a_{23}x_3^{k-1} - \cdots - a_{2n}x_n^{k-1}) \\
x_3^k &= \frac{1}{a_{33}}(b_3 - a_{31}x_1^{k-1} - a_{32}x_2^{k-1} - \cdots - a_{2n}x_n^{k-1}) \\
&\vdots \\
x_n^k &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1^{k-1} - a_{n2}x_2^{k-1} - \cdots - a_{n,n-1}x_n^{k-1})
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\end{aligned}
$$

- The estimates obtained in the $(k-1)^{th}$ iteration remain unchanged until the entire k^{th} iteration is completed. With Gauss-Seidel method, the new values are immediately used:

$$
\begin{aligned}
x_1^k &= \frac{1}{a_{11}}(b_1 - a_{12}x_2^{k-1} - a_{13}x_3^{k-1} - \cdots - a_{1n}x_n^{k-1}) \\
x_2^k &= \frac{1}{a_{22}}(b_2 - a_{21}{\color{red}x_1^{k}} - a_{23}x_3^{k-1} - \cdots - a_{2n}x_n^{k-1}) \\
x_3^k &= \frac{1}{a_{33}}(b_3 - a_{31}{\color{red}x_1^{k}} - a_{32}{\color{red}x_2^{k}} - \cdots - a_{2n}x_n^{k-1}) \\
&\vdots
\end{aligned}
$$

Gauss-Seidel iterative method (cont')

- The updated rule can be written as

$$
\begin{aligned}
a_{11}x_1^k &= -a_{12}x_2^{k-1} - a_{13}x_3^{k-1} - a_{14}x_4^{k-1} - \cdots - a_{1n}x_n^{k-1} + b_1 \\
a_{21}x_1^k + a_{22}x_2^k &= -a_{23}x_3^{k-1} - a_{24}x_4^{k-1} - \cdots - a_{2n}x_n^{k-1} + b_2 \\
a_{31}x_1^k + a_{32}x_2^k + a_{33}x_3^k &= -a_{34}x_4^{k-1} - \cdots - a_{3n}x_n^{k-1} + b_3 \\
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and such as

$$(D+L)x^k = -Ux^{k-1} + b$$

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- Hence the iteration in matrix form is:

$$x^k = (D+L)^{-1}[-Ux^{k-1} + b]$$

Gauss-Seidel iterative method (cont')

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  1. Choose an initial guess vector $x^0 = [x_1^0, x_2^0, ..., x_n^0]^\top$ arbitrarily
  2. Loop through $k = 1, 2, 3, ...$

$$x_i^k = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=i+1}^{n} a_{ij}x_j^{k-1} \right] \qquad \text{for } i = 1, 2, ..., n$$

  Exit the loop when $k >$ "max iterations num" or $||x^k - x^{k-1}|| <$ "error tolerance"

Convergence results of the iterative methods

Lemma 4.2

Let $\rho(T)$ denote the spectral radius of a matrix T defined as

$$\rho(T) := \max(|\lambda_1|, ..., |\lambda_n|)$$

with $\lambda_1, ..., \lambda_n$ being the (real or complex) eigenvalues of the matrix T. If $\rho(T) < 1$, then

$$\lim_{n\to\infty} T^n = 0 \quad \textit{and} \; \sum_{i=0}^{\infty} T^i = (\mathbb{I} - T)^{-1}$$

Idea of proof.

- If T is diagonalisable then $T = P^{-1}\Lambda P$ for some matrix P where Λ is a diagonal matrix containing all eigenvalues of T and then it is clear that $\lim_{n\to\infty} T^n = 0$
- This can be extended to non-diagonalisable matrix using Jordan block decomposition
- Finally, $(\sum_{i=0}^{n} T^i)(\mathbb{I} - T) = \mathbb{I} - T^{n+1}$. Taking limit on both side gives

$$(\sum_{i=0}^{\infty} T^i)(\mathbb{I} - T) = \mathbb{I}$$

which gives us the expression of the geometric series

Convergence results of the iterative methods (cont')

Theorem 4.3

For an iterative method in form of $x^n = Tx^{n-1} + c$, the sequence of vectors x^n converges if and only if $\rho(T) < 1$.

Proof of the "if part". From the iterative scheme we can write

$$x^n = Tx^{n-1} + c = T^2x^{n-2} + Tc + c = T^3x^{n-3} + T^2c + Tc + c$$
$$\cdots = T^n x^0 + (T^{n-1} + T^{n-2} + \cdots + \mathbb{I})c \to (\mathbb{I} - T)^{-1}c.$$

Corollary 4.4

Suppose A is diagonally dominant (i.e. $|A_{ii}| > \sum_{j \neq i} |A_{ij}|$ for each row i of the matrix $A = [A_{ij}]$), then both the Jacobi and Gauss-Seidel methods are converging for solving $Ax = b$.

The proof is omitted here but can be found in standard numerical analysis textbooks (eg Sauer (2011)).

The variational inequality problem

- Now we look to solve $\min(Ax - b, x - g) = 0$ where g is a given vector

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  2. Loop through $k = 1, 2, 3,...$

$$x_i^k = \max\left\{ \frac{1}{a_{ii}} \left[b_i - \sum_{j=1, j\neq i}^{n} a_{ij} x_j^{k-1} \right], g_i \right\} \qquad \text{for } i = 1, 2, ..., n$$

Exit the loop when $k >$ "max iterations num" or $||x^k - x^{k-1}|| <$ "error tolerance"

The variational inequality problem

- Now we look to solve $\min(Ax - b, x - g) = 0$ where g is a given vector

- Jacobi method:
  1. Choose an initial guess vector $x^0 = [x_1^0, x_2^0, ..., x_n^0]^\top$ arbitrarily
  2. Loop through $k = 1, 2, 3, ...$

$$x_i^k = \max\left\{\frac{1}{a_{ii}}\left[b_i - \sum_{j=1, j\neq i}^{n} a_{ij}x_j^{k-1}\right], g_i\right\} \qquad \text{for } i = 1, 2, ..., n$$

  Exit the loop when $k >$ "max iterations num" or $||x^k - x^{k-1}|| <$ "error tolerance"

- Gauss-Seidel method:
  1. Choose an initial guess vector $x^0 = [x_1^0, x_2^0, ..., x_n^0]^\top$ arbitrarily
  2. Loop through $k = 1, 2, 3, ...$

$$x_i^k = \max\left\{\frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=i+1}^{n} a_{ij}x_j^{k-1}\right], g_i\right\} \qquad \text{for } i = 1, 2, ..., n$$

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- By construction of the iterative algorithm, the solution must satisfy $Ax \geq b$ and $x \geq g$ (require $a_{ii} > 0$ for all i)

American option pricing with implicit scheme

- Back to the American option pricing problem using implicit scheme which involves solving for V^n from
$$\min\left([\mathbb{I} - L^n]V^n - B^n V^{n-1}, V^n - g^n\right) = 0$$
at each time step
- If we write the unknown vector V^n to be solved as x, the known matrix $[\mathbb{I} - L^n]$ as A, the known vector $B^n V^{n-1}$ as b, the system is exactly equivalent to $\min(Ax - b, x - g) = 0$
- Under typical types of PDE and model parameters, the matrix $\mathbb{I} - L^n$ has positive diagonal entries and is diagonally dominant
- Hence Jacobi or Gauss-Seidel method can be employed to solve for V^n numerically

Optional reading

- Wilmott, P., Howson, S., Howison, S., & Dewynne, J. (1995). The Mathematics of Financial Derivatives: A Student Introduction. Chapter 9, 12, 14 & 15.
- Sauer, T. (2011). Numerical Analysis (2nd Edition). Chapter 2.