Numerical Methods for Finance: Problem Set 5

1. A gamma random variable $G \sim Gamma(\alpha, \beta)$ has probability density function of

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x \geq 0$$

with parameters $\alpha, \beta > 0$, where $\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} dx$ is the gamma function.

(a) Find $m(t) := \mathbb{E}[e^{tG}]$ and state the range of t on which $m(t)$ is well-defined. This function is known as the moment generating function of a random variable/distribution.

(b) Show that $G_1 + G_2$ is a gamma random variable where $G_i \sim Gamma(\alpha_i, \beta)$. You can use the fact that a moment generating function uniquely determines the underlying distribution (i.e. if X and Y have the same moment generating function, then X is equal to Y in distribution).

2. A variance gamma random variable $X \sim VG(\sigma, \nu, \theta)$ is defined as a normal random variable with mean θG and variance $\sigma^2 G$, where G in turn is a $Gamma(\alpha = 1/\nu, \beta = 1/\nu)$ random variable.

(a) You are given that the characteristic function of a normal $N(\mu, \sigma^2)$ and gamma $G(\alpha, \beta)$ distribution are respectively

$$\phi_{normal}(u) = \exp\left(i\mu u - \frac{\sigma^2 u^2}{2}\right), \quad \phi_{gamma}(u) = \left(1 - \frac{iu}{\beta}\right)^{-\alpha}.$$

Use the above to show that the characteristic function of $VG(\sigma, \nu, \theta)$ is

$$\phi_X(u) = \left(1 - iu\theta v + \frac{\sigma^2 v u^2}{2}\right)^{-\frac{1}{v}}.$$

(b) A variance gamma process is defined as

$$X_t := \mu G_t + \sigma B_{G_t}$$

where $G_t \sim Gamma(t/\nu, 1/\nu)$. Hence X can be interpreted as a drifting Brownian motion running on a "stochastic clock" G_t. Find the characteristic function of X_t.

3. When proving Proposition 5.3, by the definition of Fourier inversion we have

$$c(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \psi(u) du$$

and then we claim $\frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \psi(u) du = \frac{e^{-\alpha k}}{\pi} Re\left\{\int_0^\infty e^{-iuk} \psi(u) du\right\}$. In this exercise, we will fill in some missing details to justify why the last equality holds.

(a) Recall that ψ is defined as the Fourier transform of $v(k) = e^{\alpha k} c(k)$, i.e.

$$\psi(u) := \int_{-\infty}^{\infty} e^{iuk} v(k) dk.$$

By consider $e^{iuk} = \cos uk + i \sin uk$ and the fact that $v(k)$ is real, show that

$$\psi(-u) = \overline{\psi(u)}$$

where $\overline{z}$ denotes the conjugate of a complex number. (Remark: if a and b are real, then the conjugate of $a + bi$ is $a - bi$.)

(b) Use (a) to show that the function ψ is odd in its imaginary part and even in its real part.

(c) Hence show that

$$\frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \psi(u) du = \frac{e^{-\alpha k}}{\pi} Re \left\{ \int_{0}^{\infty} e^{-iuk} \psi(u) du \right\}.$$

4. For the Merton's jump diffusion model introduced on slide 20 of Topic 5, prove that we must require

$$\mu = r - \frac{\sigma^2}{2} - \theta \left(\exp\left(\mu_J - \frac{\sigma_J^2}{2} \right) - 1 \right)$$

for the discount price process $e^{-rt} S_t$ to be a martingale under the risk-neutral measure.

5. One weakness of the Merton's jump diffusion model is that there is not enough flexibility to describe the jump behaviours. An alternative is the Kou's double exponential jump model where the stock price process under the risk neutral measure $\mathbb{Q}$ is modelled as

$$S_t = S_0 \exp\left(\mu t + \sigma B_t + \sum_{k=1}^{N_t} J_k \right).$$

Here B is a Brownian motion, N is a Poisson process of intensity θ and the jump size J_k's are i.i.d. with common density function of

$$f(x) = p\eta_1 e^{-\eta_1 x} 1_{(x>0)} + (1-p)\eta_2 e^{\eta_2 x} 1_{(x<0)}$$

with parameters $p \in (0,1)$, $\eta_1 > 1$ and $\eta_2 > 0$. B, N and J_k's are all independent.

The interpretation behind the density function is that when a shock arrives, there is a probability of p that the shock is "good" and that the positive jump has magnitude described by an $Exp(\eta_1)$ random variable; on the other hand there is a probability of $1-p$ that the shock is "bad" with the loss magnitude being an $Exp(\eta_2)$ random variable.

(a) Find $m(u) := \mathbb{E}[e^{uX_t}]$ the moment generating function of the logarithm of the stock price $X_t := \ln S_t$.

(b) Find $\mathbb{E}_{\mathbb{Q}}[S_t]$. Suppose interest rate is a constant r. Identify the condition on μ such that $e^{-rt} S_t$ is a martingale under $\mathbb{Q}$.

(c) Why is it important to insist that $\eta_1 > 1$ in our parameter specification?

(d) Let ϕ be the characteristic function of X_T. The Carr-Madan formula

$$C(K,T) = \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} Re \left\{ e^{-iuk} \frac{e^{-rT} \phi(u - (\alpha+1)i)}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u} du \right\}$$

is applied to price call options of strike $K := e^k$ and maturity T.

i. What is the role of the constant α in the above formula?

ii. Work out an explicit sufficient condition on α to ensure the Fourier transform above is well-defined.

6. Consider an *Ornstein-Uhlenbeck* process which is defined by the stochastic differential equation

$$dX_t = -\lambda(X_t - \theta)dt + \sigma dB_t$$

where $\lambda > 0, \sigma > 0, \theta$ are constants and B is a standard Brownian motion. We are interested in finding $\mathbb{E}[e^{iuX_T}]$ the characteristic function of X_T with $T > 0$ being a fixed constant.

(a) Consider t and u as fixed constants and let $\phi(t,x) = \mathbb{E}^{(t,x)}[e^{iuX_T}]$. Derive a PDE that should be satisfied by $\phi(t,x)$. Remember to state the terminal condition of the PDE clearly.

(b) By conjecturing a solution in form of

$$\phi(t,x) = \exp\left[a(t) + b(t)x\right]$$

to the PDE in (a), show that the functions a and b satisfy the system of ordinary differential equations (ODEs)

$$\begin{aligned} a'(t) &= -\lambda\theta b(t) - \frac{\sigma^2}{2}[b(t)]^2 \\ b'(t) &= \lambda b(t) \end{aligned}$$

with conditions $a(T) = 0$ and $b(T) = iu$. Hence show that the solution to the PDE in (a) is given by

$$\phi(t,x) = \exp\left\{\theta iu\left[1 - e^{-\lambda(T-t)}\right] - \frac{\sigma^2}{4\lambda}u^2\left[1 - e^{-2\lambda(T-t)}\right] + iuxe^{-\lambda(T-t)}\right\}.$$

(c) Hence find the characteristic function of X_T.

7. Study the code of European call option pricing under Merton jump diffusion model (available on Blackboard). Then modify the code to incorporate Heston stochastic volatility model.