STATISTICAL METHODS IN FINANCE, PROBLEM SHEET 3
MSC IN MATHEMATICS AND FINANCE, 2023-2024

CRIS SALVI, WILL TURNER

Exercise 1. Let $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma} \in \mathcal{M}_n$ positive definite so that it admits a Cholesky decomposition. What is the distribution of

$$\mathcal{Q}_{\boldsymbol{\Sigma}^{-1}}(\mathbf{X} - \boldsymbol{\mu}) = (\mathbf{X} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{X} - \boldsymbol{\mu})\,?$$

Exercise 2. Prove that the marginals of $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ are independent if and only if $\boldsymbol{\Sigma} \in \mathcal{M}_n$ is diagonal.

Exercise 3 (Joint distributions)**.** The two questions below are independent.

(i) Let X_1 and X_2 two independent $\mathcal{N}(0,1)$ random variables. For $\rho \in [-1,1]$, $\mu_1, \mu_2 \in \mathbb{R}$, $\sigma_2, \sigma_2 > 0$, define

$$Y_1 := \mu_1 + \sigma_1 \left(\rho X_2 + \sqrt{1-\rho^2} X_1 \right) \qquad \text{and} \qquad Y_2 := \mu_2 + \sigma_2 X_2.$$

Determine the joint distribution of (Y_1, Y_2), the marginal distribution of Y_1 and Y_2 as well as the conditional distributions $Y_1|Y_2$ and $Y_2|Y_1$. What is the correlation between Y_1 and Y_2?

(ii) Let U_1 and U_2 denote two independent random variables with Uniform distributions on $[0,1]$, and define

$$X_1 := \sqrt{-2\log(U_1)}\ \cos(2\pi U_2) \qquad \text{and} \qquad X_2 := \sqrt{-2\log(U_1)}\ \sin(2\pi U_2).$$

Determine the joint distribution of (X_1, X_2).

Exercise 4 (Risk factor changes)**.** Suppose that the risk factor changes of a portfolio are modelled by the so-called bivariate normal variance mixture

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + \sqrt{W}\mathbf{Z},$$

where $\mathbf{Z} := \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_2)$ for $\boldsymbol{\mu} \in \mathbb{R}^2$ and $W \geq 0$ a random variable independent of $\mathbf{Z}$.

(i) Calculate the correlation $\mathrm{Corr}(X_1, X_2)$.

(ii) Are X_1 and X_2 independent? Explain.

(iii) Show that the characteristic function of $\mathbf{X}$ is

$$\phi_{\mathbf{X}}(\mathbf{u}) = \mathrm{e}^{\mathrm{i}\mathbf{u}^\top \boldsymbol{\mu}} \widehat{H}\left(\frac{1}{2}\mathbf{u}^\top \mathbf{u}\right), \quad \text{for any } \mathbf{u} \in \mathbb{R}^2,$$

where $\widehat{H}(y) := \mathbb{E}\left[\mathrm{e}^{-yW}\right]$ is the Laplace-Stieltjes transform of the distribution function of W.

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Hint: The characteristic function of $\mathrm{Y} \sim \mathcal{N}(\mathbf{b}, \mathbf{\Sigma})$ *is*

$$\phi_{\mathbf{Y}}(\mathbf{u}) = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\mathbf{u}^\top \mathbf{Y}}\right] = \exp\left[\mathrm{i}\mathbf{u}^\top \mathbf{b} - \frac{1}{2}\mathbf{u}^\top \mathbf{\Sigma}\mathbf{u}\right], \quad \textit{for any } \mathbf{u} \in \mathbb{R}^2.$$

(iv) Show that $\mathbf{a}^\top \mathbf{X} \stackrel{d}{=} \mathbf{a}^\top \boldsymbol{\mu} + R\sqrt{W\mathbf{a}^\top \mathbf{a}}$, for some $R \sim \mathcal{N}(0,1)$ independent of W.

Exercise 5 (Convergence and Central Limit Theorem)**.** Consider an iid sequence $(X_i)_{i=1,\dots,n}$ with common law a Poisson distribution with parameter $\lambda > 0$, that is such that

$$\mathbb{P}[X_1 = k] = \frac{\lambda^k \mathrm{e}^{-\lambda}}{k!}, \qquad \text{for } k = 0, 1, 2, \dots$$

(i) Compute $\mathbb{E}[X_1]$ and $\mathbb{V}[X_1]$.

(ii) Show that the the empirical average S_n converges in probability to λ as n tends to infinity, where

$$S_n := \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad \text{for } n \geq 1.$$

(iii) Define $T_n := \exp\{-S_n\}$, and show that $(T_n)_{n\geq 1}$ converges in probability to $\mathrm{e}^{-\lambda}$ as n tends to infinity.

(iv) Using the Central Limit Theorem, determine the limiting distribution of $(T_n)_{n\geq 1}$.

Exercise 6 (Convergence of random variables)**.** We consider a sequence $(X_n)_n$ of random variables on $\mathbb{R}$.

(i) Recall the second Borel-Cantelli lemma: *for a sequence of mutually independent events,* $(A_n)_{n\geq 1}$*, in some given probability space, if* $\sum_{n\geq 1} \mathbb{P}[A_n] = \infty$*, then* $\mathbb{P}(\limsup_{n\uparrow\infty} A_n) = 1$*, i.e. the probability that infinitely many events occur is one. Here, the* $\limsup$ *is defined for sequences of events as*

$$\limsup_{n\uparrow\infty} A_n := \bigcap_{n\geq 1} \bigcup_{p\geq n} A_k,$$

Consider the case where $X_n = 1$ with probability $1/n$, zero otherwise, and the $(X_n)_n$ independent. Using the Borel-Cantelli lemma, show that the sequence $(X_n)_n$ converges in probability but not almost surely.

(ii) Show that convergence in probability implies convergence in distribution. You may want to prove first that for any one-dimensional random variables X and Y and any $x \in \mathbb{R}$, $\varepsilon > 0$, we have

$$\mathbb{P}[Y \leq x] \leq \mathbb{P}[X \leq x + \varepsilon] + \mathbb{P}[|Y - X| > \varepsilon].$$

(iii) Show that convergence in distribution to a constant implies convergence in probability holds. You may want to use the following result: *the sequence* (X_n) *converges in distribution to* X *if and only if*

$$\limsup_{n\uparrow\infty} \mathbb{P}[X_n \in C] \leq \mathbb{P}[X \in C] \quad \textit{holds for any closed set } C.$$

Department of Mathematics, Imperial College London
Email address: `wft17@ic.ac.uk`