Numerical Methods for Finance: Problem Set 1

For question 1 to 3, we consider a lattice model where the stock price process is modelled as

$$S_n := S_0 \prod_{i=1}^{n} \xi_i$$

with ξ_i's being some i.i.d. random variables. For an N-period model to match the first two moments of the risk-neutral stock price dynamics under Black-Scholes model over a time horizon of T, we require

$$\mathbb{E}[\xi] = e^{r\triangle t}, \qquad \mathbb{E}[\xi^2] = e^{(2r+\sigma^2)\triangle t} \tag{1}$$

where r is the riskfree rate, σ is the stock volatility and $\triangle t = \frac{T}{N}$ is the calender time represented by each period in the model.

1. In the Jarrow-Rudd model, it is assumed that

$$\xi = \begin{cases} u, & \text{with probability } \frac{1}{2} \\ d, & \text{with probability } \frac{1}{2} \end{cases}$$

with $d < 1 < u$.

(a) Derive the expressions of u and d.

(b) The actual parameters used in the Jarrow-Rudd model are

$$u = \exp\left[\left(r - \frac{\sigma^2}{2}\right)\triangle t + \sigma\sqrt{\triangle t}\right], \qquad d = \exp\left[\left(r - \frac{\sigma^2}{2}\right)\triangle t - \sigma\sqrt{\triangle t}\right].$$

Justify the above choices by comparing them against your answers in part (a) up to the $\triangle t$ term.

2. In the modified Cox-Ross-Rubinstein model, it is assumed that

$$\xi = \begin{cases} u := e^{\eta\triangle t + \sigma\sqrt{\triangle t}}, & \text{with probability } q \\ d := e^{\eta\triangle t - \sigma\sqrt{\triangle t}}, & \text{with probability } 1 - q \end{cases}$$

for a given parameter $\eta \geq 0$. The risk-neutral probability of an up-move is set to be $q := \frac{e^{r\triangle t} - d}{u - d}$. Show that the above choices of u, d, q are consistent with the required conditions in (1) up to a certain order of $\triangle t$. Does your answer depend on the value of η?

3. A trinomial tree is constructed if the discrete random variable takes three possible outcomes:

$$\xi = \begin{cases} u, & \text{with probability } q_u \\ m, & \text{with probability } q_m \\ d, & \text{with probability } q_d \end{cases}$$

where $d < m < u$ and $q_u + q_m + q_d = 1$. One possible tree parameters specification is the Kamrad-Ritchken model:

$$u = e^{\lambda\sigma\sqrt{\triangle t}}, \qquad m = 1, \qquad d = e^{-\lambda\sigma\sqrt{\triangle t}}$$

and

$$q_u = \frac{1}{2\lambda^2} + \frac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{\triangle t}}{2\lambda\sigma}, \qquad q_m = 1 - \frac{1}{\lambda^2}, \qquad q_d = \frac{1}{2\lambda^2} - \frac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{\triangle t}}{2\lambda\sigma},$$

with $\lambda \geq 1$ being a given parameter. Show that the above choices of parameters are consistent with (1) up to a certain order of $\triangle t$. Does your answer depend on the value of λ? What happens if $\lambda = 1$?

4. Revise the proof of Proposition 1.6 in Topic 1!

5. Implement the binomial option pricing model for both European and American option with a programming language of your choice. Your code should be as generic as possible such that it can easily handle different tree parameter specifications and different payoff functions.