Generative Models in Finance

Week 1: The Mathematics of Large Language Models

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Overview

1. Statistical Language Modelling

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2. Neural Networks as Function Approximators

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3. Recurrent Neural Networks

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4. The Transformer Architecture

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3. Recurrent Neural Networks
4. The Transformer Architecture
5. Scaling Laws

Reference: M. R. Douglas, *Large Language Models*, arXiv:2307.05782, 2023.

Part 1: Statistical Language Modelling

Goal: Build a mathematical framework for the problem of modelling natural language with probability distributions.

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- Evaluation via perplexity
- Neural language models

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- Core tasks:
  - *Estimation:* Given a sequence $(x_1, \ldots, x_T)$, compute $P(x_1, \ldots, x_T)$
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- Modern LLMs (GPT, LLaMA, Claude, etc.) are fundamentally autoregressive language models: they are trained to predict the next token given all preceding tokens

The Chain Rule Decomposition

- By the chain rule of probability, any joint distribution over a sequence can be factored as:

$$P(x_1, x_2, \ldots, x_T) = \prod_{t=1}^{T} P(x_t \mid x_1, x_2, \ldots, x_{t-1}) \tag{1.1}$$

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- Convention: $P(x_1 \mid x_0) := P(x_1)$ where x_0 is a special start-of-sequence token
- Key challenge: The context $(x_1, \ldots, x_{t-1})$ can be arbitrarily long, making direct estimation intractable for large t

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- BPE provides a practical compromise: $|\mathcal{V}| \approx 30{,}000$–$100{,}000$, handles rare words via subword decomposition, and is used by most modern LLMs (GPT uses a variant with $|\mathcal{V}| \approx 50{,}000$)

N-gram Language Models

- Idea: Approximate the conditional distribution by truncating the context to the most recent $n-1$ tokens (a Markov assumption of order $n-1$):

$$P(x_t \mid x_1, \ldots, x_{t-1}) \approx P(x_t \mid x_{t-n+1}, \ldots, x_{t-1}) \tag{1.2}$$

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  - For $|\mathcal{V}| = 50{,}000$ and $n = 5$: there are $50{,}000^5 \approx 3.1 \times 10^{23}$ possible 5-grams
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- Fundamental limitation: N-gram models cannot capture long-range dependencies in language

Evaluation: Cross-Entropy and Perplexity

- Let P be the true (unknown) distribution over sequences and Q be our model. The cross-entropy of Q relative to P is:

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- A perfect model has $\mathrm{PP} = 1$; a uniform random model over $\mathcal{V}$ has $\mathrm{PP} = |\mathcal{V}|$

Neural Language Models

- Key idea: Replace the counting-based estimation (1.3) with a neural network P_θ parameterised by θ:

$$P_\theta(x_t \mid x_1, \ldots, x_{t-1}) = \mathrm{softmax}(f_\theta(x_1, \ldots, x_{t-1}))_{x_t}$$

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- Advantages over n-grams:
  - Continuous representations allow generalisation to unseen contexts
  - No fixed context window limitation (depending on architecture)
  - The network f_θ can be an RNN, Transformer, or other architecture

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- Generalisation and double descent

Feedforward Neural Networks

Definition 1.1 (Feedforward Neural Network)

A feedforward neural network (FNN) with L layers is a function $f : \mathbb{R}^{d_0} \to \mathbb{R}^{d_L}$ defined as:

$$f(\mathbf{x}) = W_L \sigma(W_{L-1} \sigma(\cdots \sigma(W_1 \mathbf{x} + \mathbf{b}_1) \cdots) + \mathbf{b}_{L-1}) + \mathbf{b}_L \tag{1.9}$$

where:

- $W_\ell \in \mathbb{R}^{d_\ell \times d_{\ell-1}}$ *are weight matrices,* $\mathbf{b}_\ell \in \mathbb{R}^{d_\ell}$ *are bias vectors*
- $\sigma : \mathbb{R} \to \mathbb{R}$ *is a nonlinear activation function applied element-wise*
- d_0 *is the input dimension,* d_L *is the output dimension*
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- The parameters $\theta = \{(W_\ell, \mathbf{b}_\ell)\}_{\ell=1}^{L}$ are learned from data
- Depth = number of layers L; Width = $\max_\ell d_\ell$
- Total number of parameters: $\sum_{\ell=1}^{L} (d_\ell \cdot d_{\ell-1} + d_\ell)$

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  - Hyperbolic tangent: $\sigma(z) = \tanh(z) = \dfrac{e^{z}-e^{-z}}{e^{z}+e^{-z}} \in (-1,1)$
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  - Mitigates the vanishing gradient problem (gradient is 1 for $z > 0$)
- Variants: Leaky ReLU ($\max(\alpha z, z)$ for small $\alpha > 0$), GELU, SiLU/Swish ($z \cdot \sigma(z)$) — used in modern Transformers

Universal Approximation Theorem

Theorem 1.2 (Universal Approximation — Cybenko 1989, Hornik 1991)

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a continuous, non-constant, bounded activation function (e.g. sigmoid). Let $K \subset \mathbb{R}^d$ be compact and let $f \in C(K, \mathbb{R})$. Then for every $\varepsilon > 0$, there exist $N \in \mathbb{N}$, weights $W \in \mathbb{R}^{N \times d}$, $\mathbf{v} \in \mathbb{R}^N$, and biases $\mathbf{b} \in \mathbb{R}^N$ such that

$$\sup_{\mathbf{x} \in K} \left| f(\mathbf{x}) - \sum_{i=1}^{N} v_i \, \sigma(\mathbf{w}_i^\top \mathbf{x} + b_i) \right| < \varepsilon \tag{1.10}$$

where $\mathbf{w}_i^\top$ is the i-th row of W.

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- A single hidden layer network with N sufficiently large neurons can approximate any continuous function on a compact set to arbitrary accuracy

Universal Approximation Theorem

Theorem 1.2 (Universal Approximation — Cybenko 1989, Hornik 1991)

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a continuous, non-constant, bounded activation function (e.g. sigmoid). Let $K \subset \mathbb{R}^d$ be compact and let $f \in C(K, \mathbb{R})$. Then for every $\varepsilon > 0$, there exist $N \in \mathbb{N}$, weights $W \in \mathbb{R}^{N \times d}$, $\mathbf{v} \in \mathbb{R}^N$, and biases $\mathbf{b} \in \mathbb{R}^N$ such that

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- A single hidden layer network with N sufficiently large neurons can approximate any continuous function on a compact set to arbitrary accuracy
- Caveat: The theorem is an *existence* result — it says nothing about:
  - How large N needs to be (it may be exponentially large in d)
  - Whether gradient-based training will find the approximating parameters

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- Telgarsky (2016): For ReLU networks, there exist functions representable with $O(L)$ parameters at depth L that require $\Omega(2^{L/2})$ parameters at depth $O(1)$

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- Modern optimisers: Adam (Kingma & Ba, 2015) and variants, which adaptively scale learning rates per parameter

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  3. Overparameterised regime: Among all interpolating solutions, SGD finds one with *low complexity* (implicit regularisation), leading to good generalisation
- Implications for LLMs: Modern LLMs have billions of parameters (far exceeding training examples in many senses), yet they generalise remarkably well — this is consistent with the overparameterised regime of double descent

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Recurrent Neural Network (RNN) Architecture

Definition 1.3 (Elman RNN)

A recurrent neural network processes a sequence $(\mathbf{x}_1, \ldots, \mathbf{x}_T)$ *with* $\mathbf{x}_t \in \mathbb{R}^d$ *via the recurrence:*

$$\mathbf{h}_t = \sigma(W_h \mathbf{h}_{t-1} + W_x \mathbf{x}_t + \mathbf{b}_h) \tag{1.11}$$

$$\mathbf{y}_t = W_y \mathbf{h}_t + \mathbf{b}_y \tag{1.12}$$

where:

- $\mathbf{h}_t \in \mathbb{R}^{d_h}$ *is the hidden state at time* t*, with* $\mathbf{h}_0 = \mathbf{0}$
- $W_h \in \mathbb{R}^{d_h \times d_h}$, $W_x \in \mathbb{R}^{d_h \times d}$, $W_y \in \mathbb{R}^{d_y \times d_h}$ *are weight matrices*
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- For language modelling: $\mathbf{y}_t$ produces logits, and $\mathrm{softmax}(\mathbf{y}_t)$ gives $P_\theta(x_{t+1} \mid x_1, \ldots, x_t)$

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- The key quantity is the Jacobian product:

$$\prod_{j=k+1}^{t} \frac{\partial \mathbf{h}_j}{\partial \mathbf{h}_{j-1}} = \prod_{j=k+1}^{t} \operatorname{diag}(\sigma'(\mathbf{z}_j)) \cdot W_h \tag{1.14}$$

where $\mathbf{z}_j = W_h \mathbf{h}_{j-1} + W_x \mathbf{x}_j + \mathbf{b}_h$

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- More precisely (Bengio et al., 1994): if λ_1 is the largest singular value of W_h and $\gamma = \max_z |\sigma'(z)|$, then:

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- Vanishing gradients require *architectural* solutions $\Rightarrow$ LSTM, GRU

Long Short-Term Memory (LSTM)

Definition 1.4 (LSTM — Hochreiter & Schmidhuber, 1997)

An LSTM cell maintains a cell state $\mathbf{c}_t$ and hidden state $\mathbf{h}_t$ via:

$$\mathbf{f}_t = \sigma_g(W_f[\mathbf{h}_{t-1}, \mathbf{x}_t] + \mathbf{b}_f) \quad \textit{(forget gate)} \tag{1.15}$$

$$\mathbf{i}_t = \sigma_g(W_i[\mathbf{h}_{t-1}, \mathbf{x}_t] + \mathbf{b}_i) \quad \textit{(input gate)} \tag{1.16}$$

$$\tilde{\mathbf{c}}_t = \tanh(W_c[\mathbf{h}_{t-1}, \mathbf{x}_t] + \mathbf{b}_c) \quad \textit{(candidate cell state)} \tag{1.17}$$

$$\mathbf{c}_t = \mathbf{f}_t \odot \mathbf{c}_{t-1} + \mathbf{i}_t \odot \tilde{\mathbf{c}}_t \quad \textit{(cell state update)} \tag{1.18}$$

$$\mathbf{o}_t = \sigma_g(W_o[\mathbf{h}_{t-1}, \mathbf{x}_t] + \mathbf{b}_o) \quad \textit{(output gate)} \tag{1.19}$$

$$\mathbf{h}_t = \mathbf{o}_t \odot \tanh(\mathbf{c}_t) \quad \textit{(hidden state)} \tag{1.20}$$

where σ_g is the sigmoid function and $\odot$ denotes element-wise (Hadamard) product.

LSTM: Why It Addresses Vanishing Gradients

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  - Forget gate $\mathbf{f}_t$: controls what to erase from memory
  - Input gate $\mathbf{i}_t$: controls what new information to store
  - Output gate $\mathbf{o}_t$: controls what to expose as hidden state

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  - Input gate $\mathbf{i}_t$: controls what new information to store
  - Output gate $\mathbf{o}_t$: controls what to expose as hidden state
- Empirically, LSTMs can learn dependencies over sequences of length $\sim$1000, far exceeding vanilla RNNs ($\sim$10–20 steps)

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- Result: Dramatically better scalability, enabling training on much larger datasets and models (billions of parameters)

Token Embeddings and Positional Encodings

- Each token $x_t \in \mathcal{V}$ is mapped to a continuous vector via a learned embedding matrix $E \in \mathbb{R}^{|\mathcal{V}| \times d}$:

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$$\mathrm{PE}(\mathrm{pos}, 2i) = \sin\Big(\frac{\mathrm{pos}}{10000^{2i/d}}\Big) \tag{1.21}$$

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  for position $\mathrm{pos} = 1, \ldots, T$ and dimension $i = 0, \ldots, d/2 - 1$
- Property: For any fixed offset k, $\mathrm{PE}(\mathrm{pos} + k)$ is a linear function of $\mathrm{PE}(\mathrm{pos})$, allowing the model to learn relative positional relationships

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- Together, the $d/2$ frequencies give each position a *unique fingerprint*, just as d binary digits uniquely encode 2^d integers — but continuously

Relative Positions and Alternatives

- Key property: for any offset k, $\mathrm{PE}(\mathrm{pos}+k)$ is a linear function of $\mathrm{PE}(\mathrm{pos})$ via a rotation:

$$\begin{pmatrix} \sin(\omega_i(\mathrm{pos}+k)) \\ \cos(\omega_i(\mathrm{pos}+k)) \end{pmatrix} = \begin{pmatrix} \cos(\omega_i k) & \sin(\omega_i k) \\ -\sin(\omega_i k) & \cos(\omega_i k) \end{pmatrix} \begin{pmatrix} \sin(\omega_i\,\mathrm{pos}) \\ \cos(\omega_i\,\mathrm{pos}) \end{pmatrix}$$

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- Alternatives used in modern LLMs:
  - Learned embeddings (GPT-2): a trainable vector per position. Simple but limited to the maximum training length
  - RoPE (Su et al., 2021): applies the rotation above directly to Q, K. Used in LLaMA, Mistral, and most modern LLMs
  - ALiBi (Press et al., 2022): no positional encoding; adds a linear bias $-m|i-j|$ to attention logits

Scaled Dot-Product Attention

Definition 1.5 (Scaled Dot-Product Attention)

Given an input matrix $X \in \mathbb{R}^{T \times d}$ (rows are token representations), define:

$$Q = XW_Q, \quad K = XW_K, \quad V = XW_V \tag{1.23}$$

where $W_Q, W_K \in \mathbb{R}^{d \times d_k}$ and $W_V \in \mathbb{R}^{d \times d_v}$ are learned projection matrices. The scaled dot-product attention is:

$$\mathrm{Attention}(Q, K, V) = \mathrm{softmax}\left(\frac{QK^\top}{\sqrt{d_k}}\right) V \tag{1.24}$$

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- K (keys): "what do I contain?"
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- Row t of the output is a weighted average of all value vectors, where the weights are determined by the similarity of query t to all keys

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- Without scaling: Attention weights tend to concentrate on a single key (near-one-hot), reducing the model's ability to attend to multiple positions simultaneously

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where $M \in \mathbb{R}^{T \times T}$ is the mask matrix:

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- GPT-style models (decoder-only Transformers) use causal masking throughout. This allows training on all positions simultaneously while respecting the autoregressive structure

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$$\text{MultiHead}(X) = \text{Concat}(\text{head}_1, \ldots, \text{head}_h)\, W_O \tag{1.26}$$
$$\textit{where } \text{head}_i = \text{Attention}(XW_Q^{(i)}, XW_K^{(i)}, XW_V^{(i)})$$

with per-head projections $W_Q^{(i)}, W_K^{(i)} \in \mathbb{R}^{d \times d_k}$, $W_V^{(i)} \in \mathbb{R}^{d \times d_v}$, *and output projection* $W_O \in \mathbb{R}^{hd_v \times d}$.

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- Different heads can specialise: e.g. one head may attend to syntactic structure, another to semantic similarity, another to positional proximity

Position-wise Feedforward Network

- After multi-head attention, each position is independently processed by a position-wise feedforward network (the same network applied to every position):

$$\mathrm{FFN}(\mathbf{x}) = W_2 \,\mathrm{ReLU}(W_1\mathbf{x} + \mathbf{b}_1) + \mathbf{b}_2 \tag{1.27}$$

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- Interpretation (Geva et al., 2021): The FFN layers can be viewed as key-value memories, where the first layer computes pattern-matching scores and the second layer retrieves associated information
- Modern variants use SwiGLU (Shazeer, 2020) instead of ReLU:

$$\mathrm{SwiGLU}(\mathbf{x}) = (\mathrm{Swish}(W_1\mathbf{x}) \odot W_3\mathbf{x})W_2$$

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- The I term guarantees the gradient is *at least* the identity — it cannot vanish to zero, regardless of how small J_{F_1} and J_{F_2} are

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  - $\mu = \frac{1}{d}\sum_{i=1}^{d} x_i$ (mean across the d dimensions of this token)
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- After normalisation: $\mathbf{x} = (0.1, 200, -150, 0.3)^\top \;\to\; (-0.56, 1.13, -1.12, -0.55)^\top$ — all dimensions are now on a comparable scale

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- Modern variants: RMSNorm (Zhang & Sennrich, 2019) drops the mean subtraction and uses only root-mean-square normalisation — slightly faster, used in LLaMA

The Transformer Decoder Block

A single Transformer decoder block composes the following operations. Given input $X \in \mathbb{R}^{T \times d}$:

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A full decoder-only Transformer (e.g. GPT) stacks L such blocks:

$$\begin{aligned} X^{(0)} &= \mathrm{TokenEmbed}(x_1, \ldots, x_T) + \mathrm{PosEncode}(1, \ldots, T) \\ X^{(\ell)} &= \mathrm{TransformerBlock}_\ell(X^{(\ell-1)}), \quad \ell = 1, \ldots, L \\ \mathbf{y}_t &= W_{\text{out}}\, \mathrm{LayerNorm}(X_t^{(L)}) \\ P(x_{t+1} \mid x_{\le t}) &= \mathrm{softmax}(\mathbf{y}_t) \end{aligned}$$

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Example scales: GPT-3 has $L = 96$ layers, $d = 12288$, $h = 96$ heads, $d_{ff} = 49152$, totalling 175 billion parameters.

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- The $O(T^2)$ scaling with sequence length is the primary bottleneck of the Transformer architecture:
  - GPT-3: $T = 2048$; GPT-4: $T = 8192$ (or 128,000 with extensions)
  - Techniques to mitigate: FlashAttention (hardware-aware), sparse attention, linear attention approximations

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  - Naive: recompute all t dot products at each step $\rightarrow$ total cost $O(T^2 d)$
  - KV-cache: store previously computed $\mathbf{k}_j, \mathbf{v}_j$. At step t, only compute $\mathbf{q}_t$ and look up the cached keys/values $\rightarrow$ cost per step is $O(td)$, total $O(T^2 d)$ but with much better constants

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- This motivates two lines of research:
  1. Efficient attention: reduce the $O(T^2)$ cost while keeping the Transformer architecture
  2. Alternative architectures: replace attention entirely with $O(T)$ mechanisms

Linear Attention

- Key observation: standard attention computes

$$\mathrm{Attn}(Q, K, V)_t = \frac{\sum_{j=1}^{t} \exp(\mathbf{q}_t^\top \mathbf{k}_j)\, \mathbf{v}_j}{\sum_{j=1}^{t} \exp(\mathbf{q}_t^\top \mathbf{k}_j)}$$

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- Total cost: $\boxed{O(Td^2)}$ instead of $O(T^2 d)$ — linear in sequence length
- Trade-off: the feature map ϕ is an approximation; linear attention typically underperforms softmax attention on language modelling benchmarks

State Space Models (SSMs) and Mamba

- An alternative to attention: model the sequence via a continuous-time linear dynamical system:

$$\dot{\mathbf{h}}(t) = A\,\mathbf{h}(t) + B\,x(t)$$
$$y(t) = C\,\mathbf{h}(t)$$

where $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times 1}$, $C \in \mathbb{R}^{1\times N}$ are learnable, and $\mathbf{h}(t) \in \mathbb{R}^{N}$ is the hidden state

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- S4 (Gu et al., 2022): structured state space with HiPPO initialisation of A. First SSM competitive with Transformers on long-range benchmarks

State Space Models (SSMs) and Mamba

- An alternative to attention: model the sequence via a continuous-time linear dynamical system:

$$\dot{\mathbf{h}}(t) = A\,\mathbf{h}(t) + B\,x(t)$$
$$y(t) = C\,\mathbf{h}(t)$$

  where $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times 1}$, $C \in \mathbb{R}^{1\times N}$ are learnable, and $\mathbf{h}(t) \in \mathbb{R}^N$ is the hidden state

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- Open question: do SSMs scale as well as Transformers to hundreds of billions of parameters? The scaling law evidence is still limited

Attention, Linear Attention, and SSMs: Summary

- Three approaches to sequence modelling, each with different trade-offs:

	Softmax Attention	Linear Attention	SSMs (Mamba)
Training cost	$O(T^2 d)$	$O(Td^2)$	$O(T)^*$
Inference (per step)	$O(td)$ + KV-cache	$O(d^2)$	$O(d)^*$
Memory at inference	$O(T \cdot d)$ KV-cache	$O(d^2)$ fixed	$O(d)$ fixed
Content-based routing	Yes (softmax)	Approximate	Yes (selective)
Long-range	Direct (any pair)	Via recurrent state	Via recurrent state
Proven at scale	Yes (GPT-4, etc.)	Limited	Emerging

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- Hybrid architectures (Jamba, Zamba): interleave Transformer layers with Mamba layers, getting the best of both worlds

Expressivity: Universal Approximation for Transformers

Theorem 1.7 (Yun et al., 2020)

Let $1 \leq p < \infty$ and let $\mathcal{F}^p$ denote the class of continuous, permutation-equivariant functions $f : \mathbb{R}^{T \times d} \to \mathbb{R}^{T \times d}$ on compact domains. Then for any $f \in \mathcal{F}^p$ and any $\varepsilon > 0$, there exists a Transformer network g such that:

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More precisely, Transformers with $O(1)$ heads, $O(1)$ layers, and sufficient width are universal approximators for sequence-to-sequence functions.

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- Key insight: Self-attention can implement *contextual mappings* — the representation of each token can depend on the entire input sequence

Part 5: Scaling Laws

Goal: Understand the empirical relationships between model scale, data, compute, and performance, and their implications for LLM development.

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- Neural scaling laws (Kaplan et al., 2020)
- Chinchilla scaling laws (Hoffmann et al., 2022)
- Emergent abilities and their interpretation

Neural Scaling Laws (Kaplan et al., 2020)

- Setup: Train a family of Transformer LMs of different sizes on different amounts of data. Measure test loss (cross-entropy, i.e. $-\frac{1}{T}\sum_t \log P(x_t \mid x_{<t})$) as a function of three variables:
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- Empirical finding: on log-log axes, the relationship is approximately *linear* — i.e. a power law:

$$L(N) \approx \left(\frac{N_c}{N}\right)^{\alpha_N}, \quad \alpha_N \approx 0.076 \tag{1.29}$$

$$L(D) \approx \left(\frac{D_c}{D}\right)^{\alpha_D}, \quad \alpha_D \approx 0.095 \tag{1.30}$$

$$L(C) \approx \left(\frac{C_c}{C}\right)^{\alpha_C}, \quad \alpha_C \approx 0.050 \tag{1.31}$$

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- Interpretation: $\alpha_N \approx 0.076$ means that to halve the loss, you need to increase N by a factor of $2^{1/0.076} \approx 2^{13} \approx 8{,}000$. Progress is real but expensive

Scaling Laws: What Matters and What Doesn't

- Surprising finding 1: Model shape matters far less than model size
  - Varying depth vs. width, number of attention heads, FFN dimension — as long as total N is the same, loss is approximately the same
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- Surprising finding 3: Power laws hold over many orders of magnitude
  - Kaplan et al. tested models from $\sim$1K to $\sim$1B parameters — 6 orders of magnitude
  - The same power-law exponents fit the entire range with no sign of saturating
  - This is unusual — most empirical scaling relationships break down outside a narrow range

Joint Scaling and Predictability

- When both N and D are limited, Kaplan et al. proposed a joint scaling law:

$$L(N, D) \approx \left[\left(\frac{N_c}{N} \right)^{\alpha_N / \alpha_D} + \frac{D_c}{D} \right]^{\alpha_D}$$

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- Why this is practically important: you can *predict* the performance of a large model before training it
  - Train a family of small models (e.g. 10M, 50M, 100M, 500M parameters)
  - Fit the power-law exponents α_N, α_D on these small runs
  - Extrapolate to predict the loss of a 70B or 175B model
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  - This saves millions of dollars in wasted compute on bad configurations
- Real-world example: GPT-4's performance was reportedly predicted accurately from scaling laws fitted on much smaller models (OpenAI, 2023)

The Compute Budget: Where Do the FLOPs Go?

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- Where does the factor of 6 come from?
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- The central question of scaling laws: Given a budget of C FLOPs, what is the best split between N and $D = C/(6N)$?

Compute-Optimal Training (Hoffmann et al., 2022)

- Kaplan et al. (2020) answer: Scale N faster than D
  - Their analysis suggested $N_{\text{opt}} \propto C^{0.73}$, $D_{\text{opt}} \propto C^{0.27}$
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- Key insight: The three terms have clear interpretations:
  - E: the entropy of the data — no model can beat this
  - A/N^{α}: the approximation error — the model is too small to represent the true distribution
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- Minimising $L(N, D)$ subject to $C = 6ND$ via Lagrange multipliers gives:

$$N_{\text{opt}} \propto C^{0.50}, \quad D_{\text{opt}} \propto C^{0.50}$$

  Model size and data should be scaled equally

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- Optimal data-to-parameters ratio: approximately 20 tokens per parameter
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- The "Chinchilla tax": Compute-optimal training demands far more data than previously assumed. For 175B parameters, the optimal dataset is $\sim$3.5T tokens
- Beyond Chinchilla: Recent models (LLaMA-3, Gemma) train well *past* the Chinchilla-optimal point — more data than predicted optimal, because inference cost (which scales with N, not D) matters in deployment

Emergent Abilities

- Wei et al. (2022) observed that certain capabilities appear to "emerge" suddenly at specific model scales:
  - *Few-shot arithmetic*: Near-zero accuracy below 13B, then jumps to $> 50\%$
  - *Multi-step reasoning*: Appears around 100B parameters
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- This is surprising because the scaling laws predict *smooth* improvement. If the underlying loss improves smoothly, why do specific capabilities appear to jump?

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  - At $p = 0.5$: accuracy $\approx 0.5^4 = 6\%$ (looks like failure)
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- Evidence: When Schaeffer et al. re-evaluated the same tasks with continuous metrics (e.g. Brier score, token-level log-likelihood), the sharp transitions disappeared — performance improved smoothly and predictably with scale
- Current consensus: The underlying capabilities likely improve continuously with scale (consistent with smooth scaling laws), but certain *task-level metrics* can make this look like sudden emergence

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5. Scaling Laws: Test loss follows power laws in model size, data, and compute. Chinchilla-optimal training scales N and D equally. Emergent abilities are debated

Optional Reading

- Douglas, M. R. (2023). *Large Language Models*. arXiv:2307.05782.
  - Primary reference for this lecture
- Vaswani, A. et al. (2017). Attention Is All You Need. *NeurIPS*.
  - The original Transformer paper
- Kaplan, J. et al. (2020). Scaling Laws for Neural Language Models. *arXiv:2001.08361*.
- Hoffmann, J. et al. (2022). Training Compute-Optimal Large Language Models. *NeurIPS* (Chinchilla paper).
- Yun, C. et al. (2020). Are Transformers Universal Approximators of Sequence-to-Sequence Functions? *ICLR*.
- Merrill, W. & Sabharwal, A. (2023). The Parallelism Tradeoff: Limitations of Log-Precision Transformers. *TACL*.
- Siegelmann, H. & Sontag, E. (1995). On the Computational Power of Neural Nets. *JCSS*, 50(1), 132–150.
- Schaeffer, R. et al. (2023). Are Emergent Abilities of Large Language Models a Mirage? *NeurIPS*.