Resumen

Aquí va el resumen...

Abstract

Aquí inicia el abstract...

AGRADECIMIENTOS

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Introducción

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CHAPTER 2

PRELIMINARIES

Definition 2.0.1. *orthogonal complement*

Theorem 2.0.1. *For every $n \times n$ symmetric real matrix, the eigenvalues are real and the eigenvectors can be chosen real and orthonormal.*

Theorem 2.0.2 (Courant-Fisher Formula)**.** *Let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ and corresponding eigenvectors $v_1, v_2, \ldots, v_n$. Then*

$$\lambda_1 = \min_{\|x\|=1} x^T A x = \min_{x \neq 0} \frac{x^T A x}{x^T x},$$

$$\lambda_2 = \min_{\substack{\|x\|=1 \\ x \perp v_1}} x^T A x = \min_{x \neq 0} \frac{x^T A x}{x^T x},$$

$$\lambda_n = \lambda_{max} = \max_{\substack{\|x\|=1 \\ x \perp v_1}} x^T A x = \max_{\substack{x \neq 0 \\ x \perp v_1}} \frac{x^T A x}{x^T x}.$$

In general, for $1 \le k \le n$, let S_k denote the span of $v_1, v_2, \ldots, v_k$ (with $S_0 = \{0\}$). Then

$$\lambda_k = \min_{\substack{\|x\|=1 \\ x \in S_{k-1}^{\perp}}} x^T A x = \min_{\substack{x \neq 0 \\ x \in S_{k-1}^{\perp}}} \frac{x^T A x}{x^T x}.$$

Proof. Let $A = Q \Lambda Q^T$ be the eigenvalue decomposition of A, where Q is an orthogonal matrix whose columns are eigenvectors of A, and Λ is a diagonal matrix whose entries are the eigenvalues of A. ∎

2.1 Graphs and Laplacian Matrix

For the rest of the chapter, let $G = (V, E)$ be an undirected graph, where $V = \{v_1, v_2, \ldots, v_n\}$ is the non-empty set of nodes (or vertices) and E is the set of edges, composed by pairs of the form (v_i, v_j), where $v_i, v_j \in V$. Let $w : E \to R_{\geq 0}$ be a weight function and define $w_{ij} = w(v_i, v_j)$, for $1 \leq i, j \leq n$, with $w_{ij} = 0$ if there is not an edge connecting the nodes v_i and v_j.

The *weighted adjacency matrix* of the graph is the matrix defined by $W = [w_{ij}]_{n \times n}$

The *degree of a vertex* $v_i \in V$ is defined as

$$d_i = \sum_{j=1}^{n} w_{ij}.$$

The *degree matrix* D is defined as the diagonal matrix with the degrees $d_1, d_2, \ldots, d_n$ on the diagonal.

The unnormalized graph *Laplacian matrix* L is defined as

$$L = D - W$$

The normalized Laplacian matrix L_{sym} is defined as

$$L_{sym} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}}$$

Proposition 2.1.1 (Some properties of L)**.** *The matrix L, as defined above, satisfies the following properties:*

1. *For every vector* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}$ *we have*
$$x^T L x = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \left(x_i - x_j\right)^2$$
2. *L is symmetric and positive semi-definite*
3. *L has n non-negative, real-valued, eigenvalues* $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$
4. *The smallest eigenvalue L is* 0*, the corresponding eigenvector is the constant one vector* $\mathbb{1}$.

Proof. Here is your proof ∎

Definition 2.1.1. *Given a graph* $G = (V, E)$*, a partition of G is a collection of k subsets* $P_1, P_2, \ldots, P_k \subset V$ *such that:*

1. $P_i \cap P_j = \emptyset$ *for* $i \neq j$*, where* $i, j \in \{1, 2, \ldots, k\}$
2. $\cup_{i=1}^{k} P_k = V$

For two collections of vertices $A, B \subset V$ consider the following quantities related to the edges of

$$W(A, B) := \sum_{v_i \in A} \sum_{v_j \in B} w_{ij}$$

and

$$\textsc{Vol}(A) := \sum_{v_i \in A} d_i$$

In order to measure the quality of the partition we introduce the following we want to agroupe by similarity so its natural to Cut value of that partition

$$\textsc{Cut}(P_1, P_2, ..., P_k) := \frac{1}{2}\sum_{i=1}^{k} W(P_i, \overline{P_i})$$

solve the mincut problem

The next consider two different ways of measuring the size of the partitions

$$\begin{aligned}\textsc{RatioCut}(P_1, P_2, ..., P_k) &:= \frac{1}{2}\sum_{i=1}^{k}\frac{W(P_i, \overline{P_i})}{|P_i|}\\ &= \sum_{i=1}^{k}\frac{\textsc{Cut}(P_i, \overline{P_i})}{|P_i|}\end{aligned}$$

$$\begin{aligned}\textsc{NormCut}(P_1, P_2, ..., P_k) &:= \frac{1}{2}\sum_{i=1}^{k}\frac{W(P_i, \overline{P_i})}{\textsc{Vol}(P_i)}\\ &= \sum_{i=1}^{k}\frac{\textsc{Cut}(P_i, \overline{P_i})}{\textsc{Vol}(P_i)}\end{aligned}$$

The spectral method

1. Let v denote the second smallest eigenvector of $\mathscr{L}$. Sort the vertices i of G in increasing order of v_i. Let the resulting ordering be $v_1 \leq v_2 \leq \cdots v_n$
2. For each i, consider the cut induced by $\{1, 2, ..., i\}$ and its complement. Calculate its conductance.
3. Among these $n-1$ cuts, choose the one with minimum conductance.

Cheeger's inequality

For a graph $G = (V, E)$ the *conductance* or *Cheeger ratio* of a set $S \subset V$ is the ratio of the fraction of edges in the cut $(S, \overline{S})$ o the volume of S,

$$\phi(S) = \frac{E(S, \overline{S})}{\textsc{Vol}(S)}$$

The *conductance* or *Cheeger constant* of a graph G is denoted by

$$\phi(G) = \min_S \phi(S)$$

Theorem 2.1.1. *In a graph G, the Cheeger constant $\phi(G)$ and the spectral gap λ_G are related as follows:*

$$2\phi(G) \geq \lambda_G \geq \frac{\alpha_G^2}{2} \geq \frac{\phi(G)^2}{2}$$

where α_G^2 is the minimum Cheeger ratio of subsets S_i consisting of vertices with the largest i values in the eigenvector associated with $lambda_G$, over all $i \in [n]$

Generalization to many partitions

1. Perform eigenvalue decomposition to find the eigenvectors of L_{sym}.

2. Select the k largest eigenvectors $e_1, e_2, ..., e_k$ of L_{sym} associated to the largest eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$

3. Form the matrix Y from the matrix $X = [e_1, e_2, ..., e_k]$ given by

$$Y_{ij} = \frac{X_{ij}}{\left(\sum_j X_{ij}^2\right)^{\frac{1}{2}}}$$

4. Treating each row of Y as a point in $\mathbb{R}^k$, cluster them into k clusters using $K - means$

5. Finally, assign the original vertex to cluster j if and only if row i of the matrix was assigned to cluster j

CHAPTER 3

Conclusiones y trabajo a futuro

Las conclusiones y el trabajo a futuro inicia aquí...

APPENDIX A

ANALISIS

El apéndice inicia aquí.

Bibliography

[1] Datos de la publicación.