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T E S I S

**A GRAPH NEURAL NETWORK APPROACH FOR
LARGE-SCALE GRAPH PARTITIONING**

QUE PARA OBTENER EL TÍTULO DE:

MAESTRÍA EN CIENCIAS DE LA COMPUTACIÓN

PRESENTA:

JOSÉ CRISPÍN ALVARADO CALDERÓN

DIRECTORES DE TESIS:

DR. RICARDO MENCHACA MÉNDEZ

DR. ROLANDO MENCHACA MÉNDEZ

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RESUMEN

El diseño de algoritmos para resolver problemas de Optimización Combinatoria es una tarea desafiante debido a la naturaleza de intratabilidad que poseen los mismos. Uno de los problemas fundamentales, y también uno de los más estudiados en el área, es el problema del *Particionado de Grafos*. En años recientes, se han desarrollado distintos algoritmos basados en Deep Learning para lidiar con éste y otros problemas de Optimización Combinatoria, los cuales han mostrado resultados satisfactorios. Uno de los algoritmos más famosos, entre los antes mencionados, es *Generalizable Approximate Partitioning (GAP) framework*. El objetivo de este trabajo es presentar una extensión de este *framework* que funcione para grafos mas generales, i.e. grafos sin atributos, y generalice bien para grafos grandes.

Una de las primeras observaciones hechas en esta investigación es que debido al hecho que el framework GAP usa una arquitectura basada en *Graph Convolutional Networks*, éste tiende a desempeñarse lentamente en grafos grandes. Por lo tanto, se llevaron a cabo algunas modificaciones significantes que mantienen las ventajas que ofrecen las Graph Convolutional Networks y al mismo tiempo reducen el tiempo de cómputo sin perder la potencia del framework original.

Además, se observó que el framework GAP funciona únicamente con grafos con características en sus nodos como consecuencia de la arquitectura seleccionada por sus autores. Esta dependencia fue eliminada a través de un enfoque basado en caminatas aleatorias para generar las características de los nodos. Esta se puede considerar como una modificación importante porque el algoritmo ahora depende únicamente de la estructura del grafo lo cuál es suficiente para el problema del Particionado de Grafos.

La última parte de esta investigación muestra cómo se adaptó una técnica de *negative sampling* al algoritmo propuesto para acelerar y hacer más efectivo el proceso de encontrar particiones balanceadas sin sacrificar la calidad de las particiones antes mencionadas. El algoritmo propuesto mostró resultados comparables con algoritmos ampliamente usados para el particionado de grafos como lo son METIS o SCOTCH.

ABSTRACT

The design of algorithms that solve Combinatorial Optimization problems is a challenging task due to their intractable nature. One of the fundamental and most studied problems in the area is the Graph Partitioning problem. In recent years, several Deep Learning algorithms have been developed to deal with this and other Combinatorial Optimization problems, which have shown satisfactory results. One of the famous ones is the Generalizable Approximate Partitioning (GAP) framework. This work aims to present an extension of this framework that works for more general graphs and generalizes well to large ones.

One of the first observations made in this research is that the GAP framework uses an architecture based on Graph Convolutional Networks (GCN's) which tends to be slow for big graphs. Therefore, significant modifications were carried out that preserve the advantages that Graph Neural Networks offer and at the same time reduce the computation time without losing the power of the original framework.

In addition, it was noted that as a consequence of the model architecture chosen by its authors, GAP requires graphs with node features. That dependency was eliminated by using a random walk approach to generate node features. This was an important modification because the algorithm now relies purely on the graph's structure which is enough for the Graph Partitioning Problem.

The last part of this research shows how to use a negative sampling technique to accelerate and improve the process of finding balanced partitions without sacrificing the quality of said partitions. The proposed algorithm shows results comparable with widely used partitioning algorithms like METIS or SCOTCH.

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INTRODUCTION

1.1 Presentation

Combinatorial Optimization is a prominent field that results from the intersection of mathematics and theoretical computer science, specifically from areas such as combinatorics, algorithm theory, and operations research. Combinatorial Optimization problems are characterized by their solution space that consists of a finite collection of objects. This collection of objects typically grows exponentially in size, so going through all the objects and selecting the optimal one is not feasible [33].

The computational complexity behind solving Combinatorial Optimization (CO) problems is a very well-known concern, but the main motivation to solving them is the number of real-life problems that can be modeled using this framework. Thus, developing new techniques that provide Combinatorial Optimization algorithms which perform well has been in the spotlight for more than 50 years [25], and it is the subject matter of this research.

Considering the aforementioned, it is important to note what the essence of a CO algorithm is. As explained eloquently by Maltby and Ross [26], a CO algorithm uses mathematical methods either to make the search of possible solutions faster or to reduce the size of the set of feasible solutions. Some of the state-of-the-art techniques that have been proposed to produce those algorithms can be summarized as:

- Heuristic methods: a series of strategies where experience-based techniques are followed to solve the problem. Generally used when classical methods are too slow and

when an approximate solution is enough for the implicit purposes.

- Branch and bound method: consists of a systematic division of the solution space where the core elements are called *branches*. Then, branches are recursively explored and compared against estimated bounds of the optimal solution. [28]
- Mixed/Integer Programming: some of the decision variables in the problem are constrained to be integer values at the optimal solution, and then, the problem can be solved by one or several methods before mentioned.

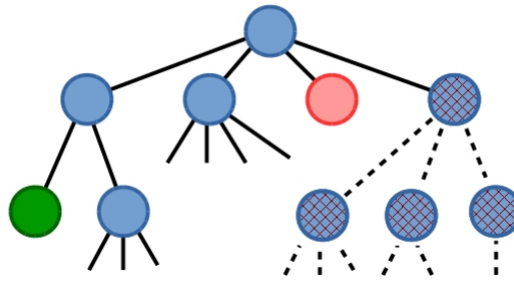


Figure 1.1: Illustration of the process of solving a CO algorithm with the branch and bound method [20].

For further information about the most popular strategies or an exhaustive explanation of those techniques, those considered to be good resources are [10] and [21].

It is well known that most CO problems can be modeled with mathematical graphs [25]. Hence, most of the efforts in developing CO algorithms are dedicated to solving graph problems. These efforts have incremented in the last decade on account of the rapid growth in digital technology. Along with this change, the computing needs and the usual assumptions that algorithms used to make have become more demanding in terms of resource allocation.

On one hand, there is a need to research new theoretical proposals and to further investigate the new constraints that modern problems imply. Then, one could use this knowledge to influence the way new strategies approach those problems. On the other hand, there is a pragmatic need to solve those problems in real life and come up with practical and efficient solutions in a considerably small amount of time. In short, the theoretical and practical aspects need to be considered as part of the equation in contemporary applications where strong optimization strategies, like heuristic methods and Integer Programming, are just not enough.

The ability to address the underlying issues that emerge in those kinds of problems has been improved drastically during the last years. Those huge steps are directed to obtain fast

approximations for CO problems and have given rise to new ways to find algorithms that provide empirically-efficient solutions.

Some recent approaches are being designed to take advantage of Machine Learning capabilities. In particular, Deep Learning strategies are providing alternative ways to deal with some of the underlying issues that arise with those kinds of problems. For specific problems, those approaches have shown to be the most successful in terms of running time efficiency, less human intervention, and the quality of the solutions.

Balanced graph partitioning is one of the fundamental Combinatorial Optimization problems and one which has gotten the best from Deep Learning. Stated in simple words, the Balanced Graph Partitioning Problem is the task of finding a partition of a given graph into mutually exclusive sub-graphs, i.e. they do not share nodes, and the partitions have the same size relative to a given measure. Finding a partition with the desired characteristics is a hard task, but the importance of solving it has incremented in recent years due to its numerous applications.

1.2 Objectives

1.2.1 General objective

The main target of this research is to develop a machine learning algorithm that solves the graph partitioning problem which is capable of performing on abstract graphs and large-scale graph instances and provide a more robust algorithm.

1.2.2 Specific objectives

The following objectives will help accomplish the general objective:

- To understand and analyze the Generalizable Approximate Partitioning (GAP) framework and to use it for solving the graph partitioning problem.
- To design an algorithm based on GAP that works for general (non-attributed) graphs and at the same time relies completely on their structure.
- To build a framework that is easy to modify in order to use different objective functions in the partitioning stage.

- To explore different types of sampling and study the impact they have in the efficiency of the algorithm.

1.3 Justification

Graphs are a mathematical representation of what is colloquially known as networks. They are used to describe the relationships between objects and allows one to model an extensive amount of real life problems. Due to their modeling capacity and their power of abstraction, they are widely studied in different areas of computer science and applied mathematics.

The Graph Partitioning (GP) problem is relevant in solving a big number of graph-related tasks. Frequently, GP is employed as a pre-processing subroutine to solve a distinct graph-related problem. If the edges that cross between the partition groups are relatively small compared to the original graph, then the partitioned graph may be more appropriate for analysis and problem-solving than the original [23].

Finding good-quality partitions of a graph is generally the first step of distributed graph computing tasks. The next paragraphs are dedicated to describe some of the most distinguished applications of GP. Applications of GP to solve graph-related and other abstract computer science problems are presented first followed by some applications to other areas and real-life situations.

One of the first and most useful applications of GP that one can find in literature is graph compression. A popular method for graph compression is the reordering method, which bisects a graph into two sets of equal cardinality aiming to minimize a compression-related objective function. *Graph bisection* is a special case of GP where the objective number of partitions is two. For instance, Bouritsas et. al. [2] showed a way to use a Machine Learning approach combined with a parametric GP algorithm to solve this task.

Another important application of GP is to implement *graph sparsification*. Graph sparsification is the task of representin big graphs in a way that takes less space in memory but preserves most of the relevant properties. Spielman and Teng [35] proposed a novel nearly-linear time partitioning algorithm that is used to produce spectral sparsifiers. The same partitioning algorithm is used by their authors to efficiently solve linear systems [37]. In an more recent work, Gatti et. al. [12] presented a way to solve sparse symmetric systems of linear equations using the *nested dissection ordering* algorithm. This algorithm is a divide and conquer heuristic based on GP. Previous work in this algorithms was made by Gupta [17].

Graph partitioning also plays an essential role in paralleling computations and the design

of new algorithms on large graphs. For example, in the *device placement* problem one aims to distribute work accross multiple devices. This is a relevant problem in Deep Learning in situations where it is important to train Neural Networks accross multiple devices [27].

In other applications, Sun et. al. [38] proposed a solution for the problem "intentional islanding in power systems considering load generation balance" where the main component is GP. For their part, Grady and Schwartz [16] made use of the isoperimetric constant to generate a GP algorithm that segmentate images.

Undoubtedly, the most recent direction where the GP problem has gained importance is in its application for clustering in complex networks. Those types of networks include but are not limited to social networks, transportation networks, web graphs, and biological networks. For an example, see Figure 1.2.



Figure 1.2: The global mobility network. Representation of 4069 airports worldwide where gray lines between them are direct connections [3].

An extensive report about the relationship between clustering in complex networks and GP can be found in [9]. That book is a collection of the classical approaches that involves this two related problems. The reader can also look over [36] for a more recently-related research.

The approaches followed by most of the mentioned research applications use state-of-the-art GP algorithms. Some of the same algorithms for those tasks are unexplored in modern large-scale graphs, and they do not consider the growth in size of the emerging graph data in the following years. The possibility that they become obsolete in the near future needs to be studied carefully alongside with the development of new strategies that overcome the impending challenges.

Something important to highlight is that the target is not only load-balance but some-

times the minimization of communication volume. This is a consideration not all of the existing approaches have and something worth researching.

In this context, the project "A Graph Neural Network Approach for Large-Scale Graph Partitioning" was created as a proposal to address some of those challenges. It took one of the most successful algorithms in taking all those needs that the mentioned applications demand

talk about social relevance and theoretical value.

1.4 Project scope and limitations

During the development of this project the following considerations were taken:

- The algorithm has not been tested on real-world graphs. Most of the graphs where the experiments took place were taken from a famous dataset where some of the most successful partitioning algorithms have been tested. Even though this offers a good comparison point, the size of those graphs is pretty small compared to massive graphs like the Amazon or Pinterest networks. The largest graph in the data set contains 500,000 nodes.
- In general, the graphs used for running the experiments are not dense. Some of the graphs that are indeed dense, belong to the category of small-size graphs with less than 50,000 nodes. Although a number of real-world graphs are sparse, it is suspected that special considerations should be contemplated during the training of the algorithm in the future.
- Most of the parameter values are based on previous work where they were shown to have a good performance or they were recommended by the original authors. Nonetheless, more experiments in different scenarios should be run to determine better values for the GP specific task.
- The machines where the experiments were executed have very limited power computing relative to the ambitions of this project. Though hardware specifications in those machines were enough to run experiments on the dataset, it would be impossible to process complex networks with this technology.
- The proposed algorithm was not designed to run in a distributed environment. Experiments were run in single thread and the algorithm was trained only in GPU. To modify

the algorithm so it can run in parallel and to distribute the training load between GPU and CPU are interesting research topics.

- The proposed framework only accepts certain input formats for the graphs, more specific JOSTLE and METIS formats.

1.5 Research Problem

The state-of-the-art algorithms for graph partitioning Graph partitioning algorithms based on DL

<https://arxiv.org/pdf/2104.03546.pdf>

1.6 Hypothesis

When analyzing GAP in depth some observations were made.

1.7 Project organization

The organization of the remaining parts of the thesis is as follows:

- In Chapter 2, it is presented the basic mathematical terminology that is going to be used during the next chapters. Next, it provides a glance to the spectral methods used to solve the GP problem. Then it finishes with a review of some of the most popular Deep Learning techniques for GP.
- In Chapter 3 the proposed graph partitioning algorithm for large-scale graphs is presented. It provides a description of the main components in the modules of the framework. Then it shows how those modules can be used by the algorithm to effectively perform in abstract graphs. Finally, it is shown how to incorporate a sampling technique that improves the training process.
- In Chapter 4 the experiments carried out are described along with the dataset and the comparison metrics used in them. The proposed solution is compared to state-of-the-art methods according to the proposed metrics. At the end, the obtained results are summarized.

- In Chapter 5, a summary of the contributions of this project is given. The advantages of the proposed algorithm over other existing approaches are mentioned as well as further research areas found during its development.

THEORY AND CONCEPTUAL FRAMEWORK

2.1 Preliminaries

Description of concepts

Definition 2.1.1. *orthogonal complement*

Theorem 2.1.1. *For every $n \times n$ symmetric real matrix, the eigenvalues are real and the eigenvectors can be chosen real and orthonormal.*

Theorem 2.1.2 (Courant-Fisher Formula). *Let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ and corresponding eigenvectors v_1, v_2, \dots, v_n . Then*

$$\begin{aligned}\lambda_1 &= \min_{\|x\|=1} x^T A x = \min_{x \neq 0} \frac{x^T A x}{x^T x}, \\ \lambda_2 &= \min_{\substack{\|x\|=1 \\ x \perp v_1}} x^T A x = \min_{x \neq 0} \frac{x^T A x}{x^T x}, \\ \lambda_n &= \lambda_{\max} = \max_{\substack{\|x\|=1 \\ x \perp v_1}} x^T A x = \max_{\substack{x \neq 0 \\ x \perp v_1}} \frac{x^T A x}{x^T x}.\end{aligned}$$

In general, for $1 \leq k \leq n$, let S_k denote the span of v_1, v_2, \dots, v_k (with $S_0 = \{0\}$). Then

$$\lambda_k = \min_{\substack{\|x\|=1 \\ x \in S_{k-1}^\perp}} x^T A x = \min_{\substack{x \neq 0 \\ x \in S_{k-1}^\perp}} \frac{x^T A x}{x^T x}.$$

2.2 Graphs and Laplacian Matrices

For the rest of the chapter, let $G = (V, E)$ be a graph, where $V = \{v_1, v_2, \dots, v_n\}$ is the non-empty set of nodes (or vertices) and E is the set of edges, composed by pairs of the form (v_i, v_j) , where $v_i, v_j \in V$.

It is assumed that all graphs are undirected, meaning that if $(v_i, v_j) \in E$, then $(v_j, v_i) \in E$, for every $v_i, v_j \in V$. For that reason, the edge (v_i, v_j) will be represented as the unordered set $\{v_i, v_j\}$.

A convenient way to represent a graph is through an *adjacency matrix* $A \in \mathbb{R}^{|V| \times |V|}$. Giving a specific order to the graph nodes, one can represent the edges as binary entries in this matrix:

$$A[v_i, v_j] = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathcal{N}(v_i)$ denote the neighborhood of node v_i , i.e., the set of the adjacent nodes to it. The quantity that represents the number of nodes in $\mathcal{N}(v_i)$ is called the *degree of the vertex* v_i . This is one of the most obvious and informative feature for the structure of the graph and is denoted by

$$d_i = \sum_{j=1}^n A[v_j, v_i].$$

Finally, we can summarize that graph's information in the *degree matrix* D which is defined as the diagonal matrix with the degrees d_1, d_2, \dots, d_n on the diagonal.

2.2.1 The Unnormalized Laplacian

The unnormalized graph *Laplacian matrix* L is defined as

$$L = D - A$$

Proposition 2.2.1 (Some properties of L). *The matrix L , as defined above, satisfies the following properties:*

1. For every vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}$ we have

$$x^T L x = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (x_i - x_j)^2$$

2. L is symmetric and positive semi-definite

3. L has n non-negative, real-valued, eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$
4. The smallest eigenvalue of L is 0, the corresponding eigenvector is the constant one vector $\mathbb{1}$.

2.2.2 Normalized Laplacians

The symmetric normalized Laplacian matrix L_{sym} is defined as

$$L_{sym} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}}$$

while the random walk Laplacian is defined as

$$L_{rw} = D^{-1} L$$

2.2.3 The graph partitioning problem

In order to introduce the graph partitioning problem in its different settings, the mathematical definition of the concepts involved are presented below.

Definition 2.2.1. Given a graph $G = (V, E)$ and an integer K , a partition of G is a collection of K subsets $P_1, P_2, \dots, P_K \subset V$ such that:

1. $P_i \cap P_j = \emptyset$ for $i \neq j$, where $i, j \in \{1, 2, \dots, K\}$
2. $\cup_{k=1}^K P_k = V$

A partition of a graph can be seen as simply removing edges from the original graph in such way the obtained partitions are subgraphs. There are many ways a graph can be partitioned into subgraphs and the way it gets done depends completely on the application of interest. However, independently of the problem to solve, the objective relies on minimizing the connections between the partitions in the original graph. The following concepts provides a useful notation to turn the problem into an optimization one.

For a collection $S \subset V$ of vertices, we define the *edge boundary* $\partial(S)$ to consist of all edges in E with exactly one endpoint in S , that is,

$$\partial(S) := \{\{u, v\} \in E \mid u \notin S \text{ and } v \in S\}$$

Now the problem turns into finding a partition P_1, P_2, \dots, P_K such that minimizes the *cut value* of the partition, usually called just *cut*, which is defined as

$$\text{CUT}(P_1, P_2, \dots, P_K) := \frac{1}{2} \sum_{k=1}^K |\partial(P_k)| \quad (2.1)$$

The notion of cut allows to measure the quality of any partition, nevertheless solving the min cut problem ...

For a collection of vertices $S \subset V$, consider the following quantities related to the edges of

$$\text{VOL}(S) := \sum_{v_i \in S} d_i$$

we want to agroupe by similarity so its natural to Cut value of that partition
solve the mincut problem

The next consider two different ways of measuring the size of the partitions

$$\begin{aligned} \text{RATIOCUT}(P_1, P_2, \dots, P_K) &:= \frac{1}{2} \sum_{k=1}^K \frac{|\partial(P_k)|}{|P_k|} \\ &= \sum_{k=1}^K \frac{\text{CUT}(P_k, \overline{P_k})}{|P_k|} \end{aligned}$$

$$\begin{aligned} \text{NORMCUT}(P_1, P_2, \dots, P_K) &:= \frac{1}{2} \sum_{k=1}^K \frac{|\partial(P_k)|}{\text{VOL}(P_k)} \\ &= \sum_{k=1}^K \frac{\text{CUT}(P_k, \overline{P_k})}{\text{VOL}(P_k)} \end{aligned}$$

2.2.4 Spectral partitioning and Normalized Cut

Here will be presented the derivation of the normalized cut as relaxation of the problem to solve the spectral partitioning problem

2.3 Literature review

B-GRAP: BALANCED GRAPH PARTITIONING ALGORITHM FOR LARGE GRAPHS: <https://www.rintonpre2/116-135.pdf>

Trading Quality for Efficiency of Graph Partitioning: An Inductive Method across Graphs
<https://openreview.net/forum?id=e6MWIbNeW1>

Application Driven Graph Partitioning <https://dl.acm.org/doi/abs/10.1145/3318464.3389745>

Graph Partitioning and Sparse Matrix Ordering using Reinforcement Learning [12]

2.3.1 Generalizable Approximate Graph Partitioning (GAP) Framework

From all the Deep Learning approaches that solve the Graph Partitioning problem, one of the most notorious not only for its simplicity but for its exceptional results, is the *Generalizable Approximate Graph Partitioning* framework better known as GAP [15].

GAP is a Graph Neural Network approach that proposes a continuous relaxation of the problem using a differentiable loss function that is based on the normalized cut. According to Nazi et al. [29], it is an unsupervised learning algorithm that is capable of generalization, meaning that it can be trained in small graphs, which allows it to generalize into unseen much larger ones. This section describes the model described in the original paper which consists of two modules: the Graph Embedding Module and the Graph Partitioning Module. See Figure 2.1.

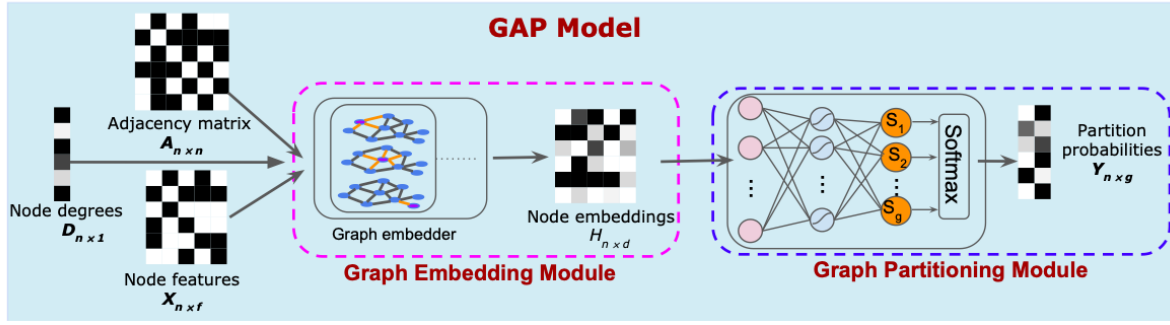


Figure 2.1: Architecture of the Generalizable Approximate Partitioning framework [29].

In the following subsections, it is assumed that the framework takes a graph $G = (V, E)$ as input, where $V = \{v_1, v_2, \dots, v_n\}$, and outputs the probabilities tensor $Y \in \mathbb{R}^{n \times K}$, where Y_{ik} represents the probability that node v_i belongs to partition P_k . Before going into the model description, the deduction of the loss function is as follows.

2.3.1.1 Expected Normalized Cut Loss Function

Recall the normalized cut given by

$$\text{NORMCUT}(P_1, P_2, \dots, P_K) = \sum_{k=1}^K \frac{\text{CUT}(P_k, \overline{P_k})}{\text{VOL}(P_k)} \quad (2.2)$$

In order to calculate the normalized cut expected value, one needs to compute the expected value of $\text{CUT}(P_k, \overline{P_k})$ and $\text{VOL}(P_k)$ from Equation 2.2. For the deduction of those quantities, an approach similar to the one presented in [13] will be followed.

Since Y_{ik} represents the probability that node $v_i \in P_k$, $1 - Y_{ik}$ is the probability that $v_i \notin P_k$, hence

$$\mathbb{E}[\text{CUT}(P_k, \overline{P_k})] = \sum_{i=1}^{|V|} \sum_{v_j \in \mathcal{N}(v_i)} Y_{ik}(1 - Y_{jk}) \quad (2.3)$$

as can it can be deduced from Figure 2.2.

Due to the fact that for a given node the adjacent nodes can be retrieved from the adjacency matrix A , Equation 2.3 can be rewritten as follows:

$$\mathbb{E}[\text{CUT}(P_k, \overline{P_k})] = \sum_{i=1}^{|V|} \sum_{j=1}^{|V|} Y_{ik}(1 - Y_{kj}^T) A_{ij} \quad (2.4)$$

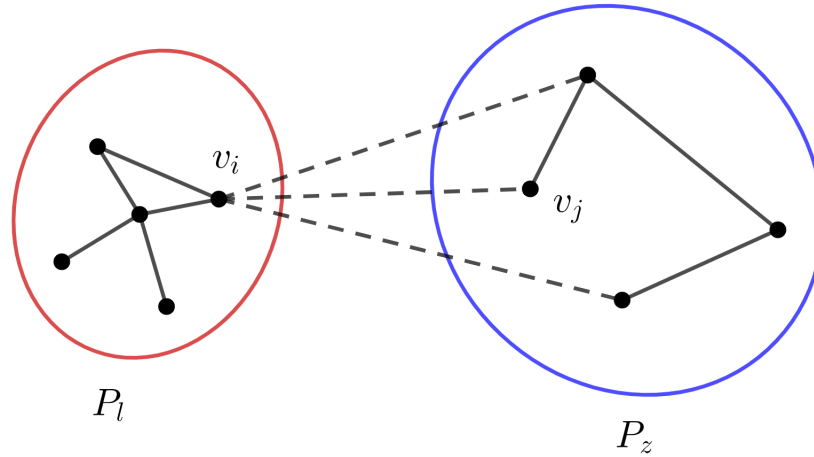


Figure 2.2: The expected value of the number of edges in the boundary of a determined partition P_l is given by the sum of the probabilities that each node $v_i \in V$ belongs to P_l , multiplied by the probability that each of its neighbors does not belong to P_l .

Keeping in mind that $\text{VOL}(P_k)$ is the sum of the degrees of the nodes in P_k , Δ is defined to be the column tensor where Δ_i is the degree of the node $v_i \in V$. Then, given Y , one can calculate the expected value of $\text{VOL}(P_k)$ as follows:

$$\begin{aligned} \Gamma &= Y^T \Delta \\ \mathbb{E}[\text{VOL}(P_k)] &= \Gamma_k \end{aligned} \quad (2.5)$$

From the results obtained in Equation 2.4 and Equation 2.5, a way to calculate the expected value of $\text{NORMCUT}(P_1, P_2, \dots, P_K)$ is given by:

$$\mathbb{E}[\text{NORMCUT}(P_1, P_2, \dots, P_K)] = \sum_{k=1}^K \sum_{i=1}^{|V|} \sum_{j=1}^{|V|} \frac{Y_{ik}(1 - Y_{kj}^T) A_{ij}}{\Gamma_k} \quad (2.6)$$

Nazi et al. [29] also showed that given the probability tensor Y , one can evaluate how balanced those partitions are. Note that the sum of the columns in Y is the expected number of nodes in each partition, i.e., $\mathbb{E}[|P_k|] = \sum_{i=1}^{|V|} Y_{ik}$. On the other hand, in order to have balanced partitions, the number of nodes in each one should be $\frac{|V|}{K}$. As a consequence, the quantity $\left| \sum_{i=1}^{|V|} Y_{ik} - \frac{|V|}{K} \right|$ measures how balanced the partition P_k is.

Using the last result, and replacing the absolute value by the squared function, one can derive the loss function from Equation 2.6. This is the one originally used in GAP that intends to minimize the expected value of the normalized cut and at the same time balances the cardinalities of the partitions:

$$\mathcal{L} = \sum_{k=1}^K \sum_{i=1}^{|V|} \sum_{j=1}^{|V|} \frac{Y_{ik}(1 - Y_{kj}^T)A_{ij}}{\Gamma_k} + \sum_{k=1}^K \left(\sum_{i=1}^{|V|} Y_{ik} - \frac{|V|}{K} \right)^2 \quad (2.7)$$

2.3.1.2 The Embedding Module

In the graph embedding module, the algorithm learns node embeddings by encoding local structure information and the node features. The embeddings are calculated using Graph Neural Networks (GNN) which have become very popular during the recent years. To ensure generalization, the GAP authors opted for an inductive GNN approach by leveraging GraphSAGE with a Graph Convolutional Network (GCN) based approach.

In the paper where GAP was presented, the authors used a 3-layer GCN using Xavier initialization that can be found in [14]

$$Z = \tanh(\hat{A} \tanh(\hat{A} \tanh(\hat{A} X \mathbf{W}^{(0)}) \mathbf{W}^{(1)}) \mathbf{W}^{(2)})$$

where $\hat{A} = (D + I)^{-\frac{1}{2}}(A + I)(D + I)^{-\frac{1}{2}}$ is a normalized variant of the adjacency matrix with self loops, X is the feature matrix, and $\mathbf{W}^{(l)}$ is a learnable parameter matrix.

2.3.1.3 The Partitioning Module

The second module of GAP is composed of a fully connected layer that takes as input a node embedding vector \mathbf{z}_u generated in the embedding module. This fully connected layer is then followed by a softmax layer trained to minimize the expected normalized loss function given by Equation 2.7.

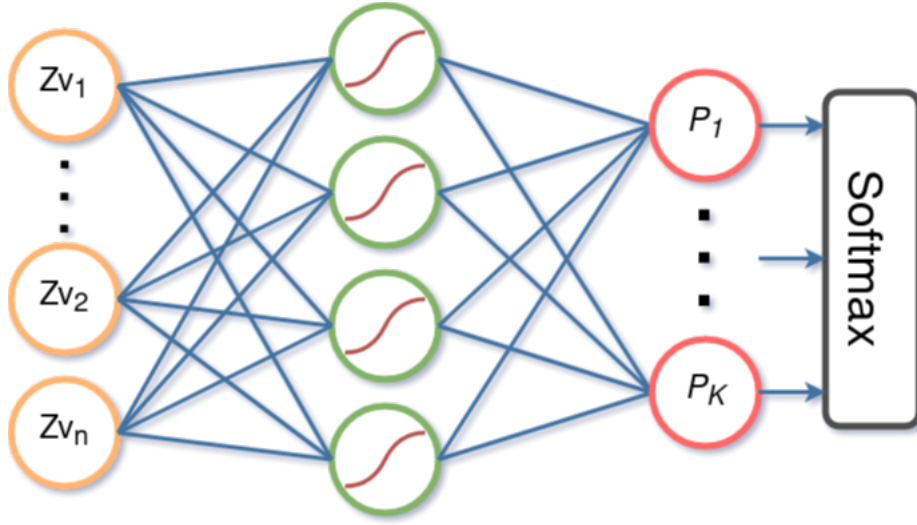


Figure 2.3: The partitioning module. A dense neural network that processes a node embedding and outputs the probability of the corresponding node to belong to each partition.

This module is the responsible for partitioning the graph by returning $Y \in \mathbb{R}^{|V| \times K}$, the probabilities matrix that each node belongs to each of the partitions P_1, P_2, \dots, P_K . At the same time, it ensures that for a given node, the sum of the probabilities of belonging each of the partitions is 1

$$\sum_{k=0}^K Y_{ik} = 1$$

GRAPH PARTITIONING FOR LARGE GRAPHS

This chapter is going to develop all the concepts related to the modifications made to the GAP framework. It contains the proposed solution and the principles to understand the theory behind it.

3.1 Node Embeddings

As mentioned by Zhang et. al. [44], GCN's and its variants have become a very hot topic in Machine Learning and they are being extremely used to solve plenty of problems. Indeed, the use of Graph Convolutional Networks (GCN) to generate *node embeddings* was a key factor that made the GAP framework very successful in producing good quality partitions. Recall that node embeddings are dense vector representations for the nodes in the involved graph.

Graph Convolutional Networks can solve the following limitations found in traditional encoders [41]:

- Traditional encoders do not scale, they generate unique embeddings for each node.
- Traditional approaches can only generate embeddings for a single fixed graph.
- They only use the graph structure and they do not consider node features
- Those encoders focus on a specific architecture and cannot be adapted to train with different loss functions.

3.1.1 Graph Convolutional Networks

Graph Convolutional Networks were originally proposed by Kipf et. al. [24]. When they first came up with its algorithm, they proposed a model where a single layer follows the following propagation rule:

$$H^{(l+1)} = f(H^{(l)}, A) = \sigma \left(\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} H^{(l)} W^{(l)} \right) \quad (3.1)$$

$$H^{(0)} = X, \quad (3.2)$$

where $W^{(l)}$ is a the learnable parameter weight matrix for the l -th network layer, $\hat{A} = A + I$ is the adjacency matrix with self loops, \hat{D} is the diagonal node degree matrix of \hat{A} , X is the feature matrix, and σ is a non-linear activation function. Also note that the term $\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}}$ is a symmetric normalized version of the graph Laplacian.

Although GCN's are extremely powerful to generate node embeddings, in practice they have the following downsides:

- GCN's are slow to train. As it can be seen in Equation 3.1, they multiply feature matrices by powers of the normalized graph's Laplacian, i.e., they are computationally very expensive.
- GCN's consume a lot of memory when training. They are not suitable for either dense graphs or large graphs.
- They present over-smoothing. After stacking multiple rounds of message passing, GCN's contain similar information in all nodes embedding representations [18].

To solve this issues in GCN's, several solutions have been proposed that deal with the downsides mentioned without loosing the expressibility of the GCN. Some of the most relevant solutions for this work are FastGCN, GraphSAGE and PinSAGE.

FastGCN [6] introduces sampling to the GCN framework and reformulate the loss function to provide a mechanism for inductive learning. I address the efficiency problems found in training GCN's by adding a batching training scheme using Monte Carlo techniques. Finally FastGCN shows better generalization than its preceding framework.

Similar to FastGCN, GraphSAGE [19] is an inductive version of GCN's. It introduces sampling techniques that remove the dependency of having all nodes during the training stage improving the performance and running time of GCN's. GraphSAGE naturally generalizes to unseen nodes and can learn about the local graph structure.

From all the mentioned solution, PinSAGE [42] is perhaps the most complete and successful. This is the architecture used by Pinterest for their recommendation system. PinSAGE operates on an enormous graph with 3 billion nodes and 18 billion edges. It uses a combination of random walks and a MapReduce-based inference to compute the node embeddings improving dramatically the scalability of GCN's.

For this research, GraphSAGE was chosen due to its flexibility and its superior performance on many graph-related tasks. As its authors suggested [19], new sampling techniques can be easily adapted as well as an application-driven loss function. Alternative approaches that can be also adapted to work in the problem in question are described in [45] or [18].

3.1.2 GraphSAGE

As claimed by the authors of the paper [19] where it was first proposed, GraphSAGE is an inductive framework that generate node embeddings for previously unseen data. It can be seen as an extension of the GCN framework to the inductive setting but it results advantageous in this process as it:

- Performs localized convolutions without needing to use the whole normalized Laplacian resulting in a reduction of the computing time. Due to the nature of the GCN's, they make use of the symmetrically normalized graph Laplacian localized convolutions.
- Does not require that all nodes are present during training of the embeddings which turns out in a highly scalable framework.
- Naturally generalize to unseen nodes and even complete sub-graphs.
- Is capable of recognizing local and global structural properties of nodes based only on its neighborhood even though it still depends on node features.
- Uses an unsupervised loss function that tries to preserve graph structure without being specific on the task.

The basic idea of this framework to compute an embedding for node v can be summarized by the following three steps:

1. Sample a set of nodes from the neighborhood of v uniformly at random
2. Aggregate feature information from neighbors

Having this idea in mind, the simplest form of a layer-wise propagation rule for GraphSAGE is given by

$$\begin{aligned} \mathbf{h}_v^{(l)} &= \sigma \left(\mathbf{W}^{(l)} \cdot \text{CONCAT} \left(\mathbf{h}_v^{(l-1)}, \mathbf{h}_{N_l(v)}^{(l)} \right) \right) \\ \mathbf{h}_{N_l(v)}^{(l)} &= \text{AGGREGATE}_l \left(\left\{ \mathbf{h}_u^{(l-1)} \mid u \in N_l(v) \right\} \right) \\ \mathbf{h}_v^{(0)} &= \mathbf{X}_v \end{aligned} \quad (3.3)$$

where as usual $\mathbf{W}^{(l)}$ is the learnable parameter weight matrix for the l -th network layer and σ is a non-linear activation function, $\{\mathbf{X}_v \mid v \in V\}$ are the input features, AGGREGATE_l are differentiable aggregator functions, and $N_l: v \rightarrow 2^V$ are neighborhood sampling functions.

After computing the node embedding for the l -th layer, the algorithm also performs a normalization step by dividing by the vector norm. This step is a common technique to prevent gradient explosion.

To learn the GraphSAGE parameters and get useful predictive node representations, the model can be trained in a fully unsupervised manner by using the following graph-based loss function:

$$J_G(\mathbf{z}_u) = -\log(\sigma(\mathbf{z}_u^T \mathbf{z}_v)) - Q \cdot \mathbb{E}_{v_n \sim P_n(v)} \log(\sigma(-\mathbf{z}_u^T \mathbf{z}_{v_n})) \quad (3.4)$$

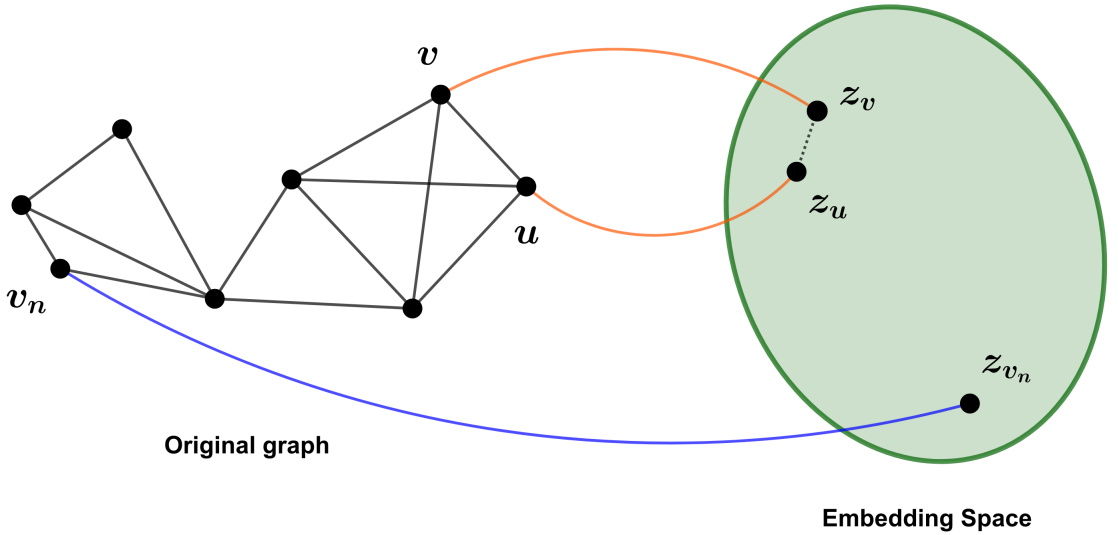


Figure 3.1: Graphical representation of the embeddings generated by GraphSAGE

As explained by Abraham [1], the supervised loss function given by Equation 3.4 is designed in a way that if nodes u and v are close in the original graph, their node embeddings should be similar. See Figure 3.1.

On one hand, if u and v are close, we expect the inner product of their representations z_u and z_v to be large and the first term of Equation 3.4 will be close to zero. On the other hand, if u and v are distant to each other, it is expected that their inner product is negative so the value of the second term will be close to zero. Since all distant nodes, also known as negative nodes, cannot be used to minimize the loss function, only Q of them are sampled from the distribution of negative nodes $P_n(v)$.

3.2 Node Features

For a graph $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_n\}$, the feature matrix X can be described as the $n \times F$ matrix where the rows X_{v_i} are node features that depends on the graph. For example, in the context of social networks, node features can be gender, city or any other user profile information.

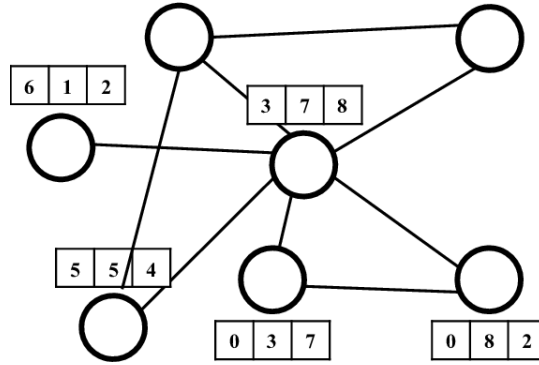


Figure 3.2: Graph with 7 nodes, five of them with 3 node features [11].

Many graphs from different applications come with very rich node feature information that can be very representative in the node embedding generation process. The key idea of Graph Neural Networks is to generate representations of nodes that depends not only on the structure of the graph but from any feature information of the graph.

As mentioned by Chen Wang et. al. [40], Graph Neural Networks (GNN's) aim at learning node representations by learning the similarities shared between connected nodes. However, the expressive ability of a GNN is highly dependent on the quality of node features.

In the version of the GP problem presented here, node features are irrelevant. The information that nodes or edges could contain does not determine if a node will belong to a specific partition or if an edge will be removed in the partitioned graph. Due to this fact and the inherent characteristic of GCN's before mentioned, one of the main limitations found in

the GAP framework is that it requires node features. Furthermore, in specific important GP applications, graphs do not have node features.

Learning inductive representations on graphs without node features it is still an open problem [45]. Nonetheless, for specific applications some efforts have been accomplished which generate new research opportunities. The next paragraphs are dedicated to study the impact of considering different sources of information as node features.

Two of the most common approaches where no node features are available are to use *identity features* and to use random feature initialization. However, those approaches have shown to be equivalent and to make the model incapable of generalizing to unseen nodes [18].

Another option is to use graph statistics like: node degree, node centrality, number of closed triangles or the clustering coefficient. Cai and Wang [4] proposed a degree-based approach. They used some statistics of the node's degree to generate the node features. Specifically, they used the feature vectors given by $(degree(v), \min(DN(v)), \max(DN(v)), \text{mean}(DN(v)), \text{std}(DN(v)))$ where $DN(v) = \{degree(u) \mid (u, v) \in E\}$. This is an interesting research that showed prominent results.

Other methods propose using applying methods such as PCA to the adjacency matrix to extract the top k-eigenvalues [5, 22] but those approaches are computationally very expensive, which is infeasible for large graphs.

In this work, to help capturing local information related to the graph's structure, a random walk approach was preferred. Given the the existent research, this type of approach looks very promising in terms of scalability to large graphs and structural representation capabilities. In experiments carried out independently by Duong et. al. [8] and Cui et. al. [7], who compared some of the already mentioned approaches, they concluded that matrix-decomposition-based methods are very powerful to extract positional node information. DeepWalk [30] produces a low-rank tranformation of the graph's normalized Laplacian matrix [31], i.e., DeepWalk is implicitly factorizing and reducing dimensionality. For that reason, DeepWalk is going to be used for this task.

3.2.1 DeepWalk as feature extraction

playing with the parameters

in particular Deep Walk to extract features about the topology of the network

According to the original paper [30], the DeepWalk algorithm consists of two main

components: a random walk generator and an update procedure.

In the first component, a random node v_i is taken uniformly at random to be the root of a random walk \mathcal{W}_{v_i} which in its turn samples recursively from the neighbors of the last visited vertex until the maximum length γ is reached.

Traditional approaches like METIS or SCOTCH implements different versions of the algorithm according to the balancedness measure, either the cardinality or the volume of the partitions. One of the great innovations presented in [29] is the introduction of a loss function derived from the expectation of the normalized cut. Even though they

Ratio cut it is still used and relevant to modern research cite here <https://proceedings.neurips.cc/paper/2020/file/aa108f56a10e75c1f20f27723ecac85f-Paper.pdf>

or here <https://www.springerprofessional.de/en/metaheuristic-approaches-for-ratio-cut-and-normalized-cut-graph-/20360832>

the authors of [13] proposed some modifications to GAP so it can be used for graphs without features. However the presented

for future - Study the weighted problem Study better features related to the problem other ways to extract useful features for non-attributed graphs

METIS was used from the networkx interface

a table for commonly used notation

3.3 Markov Chain Negative Sampling (MCNS)

EXPERIMENTAL RESULTS

4.1 Technical details and implementation

As specified by Perozzi et. al. [30], their experiments suggest that the number of walks started per vertex should be greater or equal than $\gamma = 30$, the latent dimension greater or equal than $d = 64$, and they fixed the sensible values of $w = 10$ for the window size, and $t = 40$ for the walk length. Based on those recommendations, in the experiments carried out in [43], and the computational needs of the problem to be solved, it was found convenient to set $\gamma = 60$, $d = 64$, $w = 15$, and $t = 80$.

Saving and manipulating only the adjacency lists instead of the whole adjacency matrix, is an important

For the algorithm that is proposed here, the implementation by the Karate Club API [32] was used. The number of partitions was set to four differently from the original paper where three partitions were used. It was used this number of partitions to have a comparison point with the result in The Graph Partitioning archive. There, the number of partitions is always a power of 2.

4.2 Dataset

For the dataset used to train and test the algorithm "The Graph Partitioning Archive" [34] The graphs are in the standard format used by JOSTLE and METIS. A description of the

format and the requirements can be found in [39].

For the purposes of this research, the graphs contained in "The Graph Partitioning Archive" were categorized in one of the following categories according to their vertex cardinality:

- Tiny graphs: those with less than 10,000 nodes,
- Small graphs: those with node size between 45,000 and 75,000 and,
- Medium graphs: those with at least 99,000 nodes but no more than 500,000.

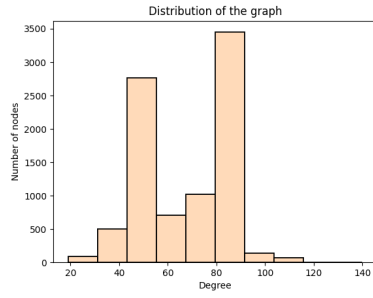
Ask Luis: is there a word for graphs with size more than big

Computation Graphs		
Name	Nodes	Edges
data	2851	15093
add32	4960	9462
bcsstk33	8738	291583
crack	10240	30380
fe_body	45087	163734
t60k	60005	89440
wing	62032	121544
finan512	74752	261120
fe_rotor	99617	662431
598a	110971	741934
m14b	214765	1679018
auto	448695	3314611

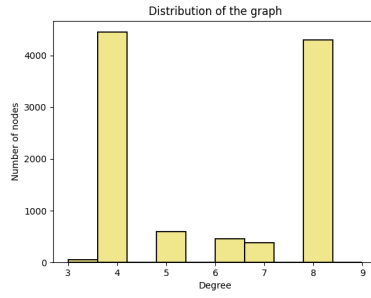
Table 4.1: Summary of the graphs characteristics. Taken from "The Graph Partitioning Archive" [34]

4.3 Results

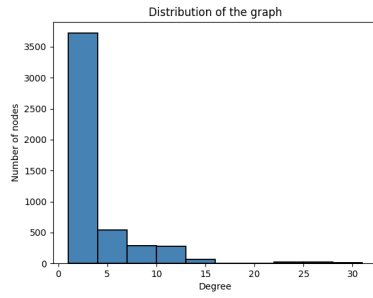
In table



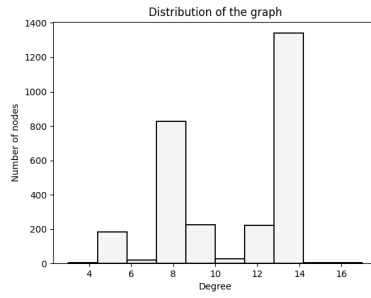
(a) bcsstk33



(b) crack



(c) add32



(d) data

Figure 4.1: Degree histogram of small graphs: bcsstk33, crack, add32, and data. Graphs are not so dense.

Graphs	METIS		Modified GAP	
	Edge Cut	Balancedness	Edge Cut	Balancedness
data	0	0	0	0
add32	0	0	0	0
bcsstk33	0	0	0	0
crack	0	0	0	0
fe_body	0	0	0	0
t60k	0	0	0	0
wing	0	0	0	0
finan512	0	0	0	0
fe_rotor	0	0	0	0
598a	0	0	0	0
m14b	0	0	0	0
auto	0	0	0	0

Table 4.2: Comparison of the results obtained by different algorithms in the computation graphs

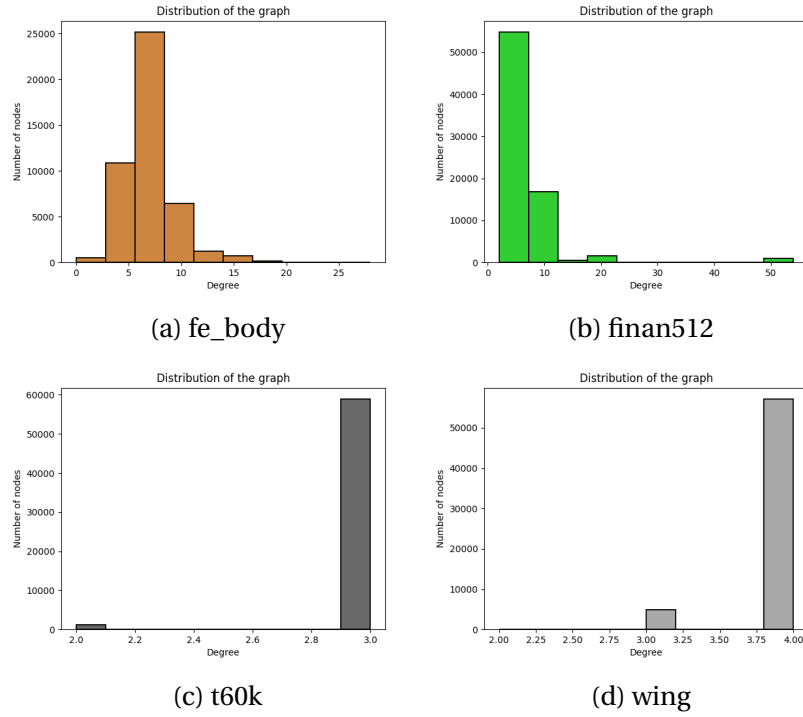


Figure 4.2: Degree histogram of medium graphs. Graphs are not so dense.

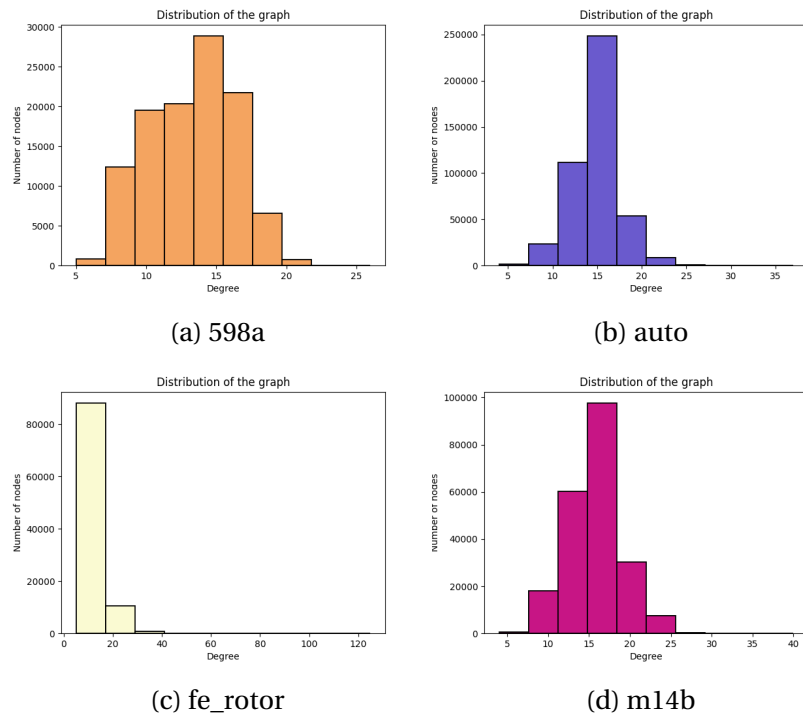


Figure 4.3: Degree histogram of large graphs. Graphs are not so dense.

CONCLUSION

Las conclusiones y el trabajo a futuro inicia aquí...

Future work extend to different problem statements, such as balanced graph partitioning with tolerance or weighted graph partitioning. Study the real impact on choosing the way to extract the features and try different algorithms look for new ways the algorithm relies only on topological graph's structure To generalize the algorithm and code to construct graphs from the features Explore new techniques for sampling Build a generic graph construction Training the algorithm with different graphs, look for useful training sets that allow to extend the recognition of different structural properties in graphs. For example, training the graph on only sparse or complete graphs Explore improvements in the training of the algorithm, parallel tasks or mix between CPU and GPU GraphSAGE still takes a lot of time to run on really large graphs Extend the accepted formats Study another way to extract features

5.1 Contributions

while it remains an open problem and it still very unexplored, the problem of feature generation...

5.2 Recommendations and future work

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