ECE183DA (Winter 2021)

Design of Robotic Systems I

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Problem set 0 solutions

1 Linear algebra

1.1 Consider the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 3 & 6 & 9 & 5 \\ 2 & 4 & 6 & 9 \end{bmatrix}$$

1(a). **What is the rank of A?**

Use the row reduction method on A:

$$r_2 \to 3r_1 - r_2 \quad : \quad \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 6 & 9 \end{bmatrix} \tag{1}$$

$$r_3 \to r_3 - 2r_1 - 5r_2 \quad : \quad \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

Thus, the rank is 2.

1(b). **For what b_3 does the equation $Ax = b = \begin{bmatrix} 1 \\ 2 \\ b_3 \end{bmatrix}$ have a solution?**

Using the same row reduction method on $[A|b]$:

$$r_2 \to 3r_1 - r_2 \quad : \quad \left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 2 & 4 & 6 & 9 & b_3 \end{array}\right] \tag{3}$$

$$r_3 \to r_3 - 2r_1 - 5r_2 \quad : \quad \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & b_3 - 7 \end{array}\right] \tag{4}$$

In order to have a solution, we need $b_3 - 7 = 0$, or $b_3 = 7$.

1(c). **What is the complete (general) solution x such that $Ax = b$, with b as above?**

From Eq. (4) with $r_1 \to r_1 - r_2$, $x = [x_1 \; x_2 \; x_3 \; x_4]^T$ satisfies

$$x_1 + 2x_2 + 3x_3 = 0 \tag{5}$$

$$x_4 = 1 \tag{6}$$

Since the rank of A is 2, there will be $4 - 2 = 2$ free parameters; we can select x_2, x_3. A particular solution is given by $x_2 = x_3 = 0 \implies x = [0 \; 0 \; 0 \; 1]^T$. The nullspace of A can be found by solving $Ax = 0$ alternately setting $(x_2, x_3) = (1, 0) \implies x_1 = -2$ and $(x_2, x_3) = (0, 1) \implies x_1 = -3$ giving a basis of $[-2 \; 1 \; 0 \; 0]^T$ and $[-3 \; 0 \; 1 \; 0]^T$, so the complete solution is given by:

$$x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{7}$$

1.2 How difficult did you find these problems (easy / needed review / hard)?

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2 Differential equations

2.1 Consider the following coupled oscillator with input $u(t)$ and output $y(t)$:

$$\begin{aligned} \dot{y}(t) &= z(t) + u(t) \quad (8) \\ \dot{z}(t) &= 2y(t) + z(t) + 3u(t) \quad (9) \end{aligned}$$

2(a). **Rewrite the system with a state-space description:**

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + Bu \\ y &= C\boldsymbol{x} + Du \end{aligned}$$

What are the values of $A, B, C, D, \boldsymbol{x}$?

This is a second order system; we can chose the two states to be y and z, so let $\boldsymbol{x} = [y\ z]^T$. Then,

$$\begin{aligned} \dot{\boldsymbol{x}} = \begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} &= \begin{bmatrix} z(t) + u(t) \\ 2y(t) + z(t) + 3u(t) \end{bmatrix} \quad (10) \\ &= \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u(t) \quad (11) \end{aligned}$$

which gives

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0.$$

2(b). **Describe the normal modes of the system and their stability.**

The normal modes of the system are given by the eigenvectors of the A matrix. Their corresponding eigenvalues dictates their stability.

$$\begin{aligned} \lambda_1 = -1; \quad \mathbf{v}_1 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{stable} \quad (12) \\ \lambda_2 = 2; \quad \mathbf{v}_1 &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{unstable} \quad (13) \end{aligned}$$

2(c). **Consider a feedback controller of the form $u(t) = ky(t)$. For what values of k is the resulting system stable?**

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + Bu \quad (14) \\ &= A\boldsymbol{x} + Bky \quad (15) \\ &= A\boldsymbol{x} + BkCx \quad (16) \\ &= (A + kBC)x \quad (17) \\ A + kBC &= \begin{bmatrix} k & 1 \\ 2 + 3k & 1 \end{bmatrix} \quad (18) \end{aligned}$$

For the resulting system to be stable, all eigenvalues of $A + kBC$ have to be in the open left half plane. The real parts of both eigenvalues are less than 0, and thus the system is stable, for $k < -1$.

2.2 How difficult did you find these problems (easy / needed review / hard)?

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3 Physics

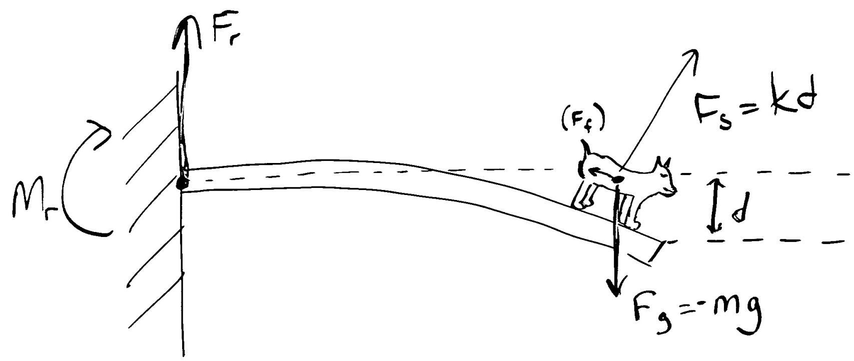
3.1 Consider a dog jumping on a diving board.

The mass of the dog is $m = 10kg$. The diving board is an ideal rectangular fiberglass beam of length $l = 2m$, width $w = 40cm$, thickness $t = 2cm$, and Young's modulus $E = 100GPa$ rigidly clamped at one end. For simplicity, you can ignore air resistance, and approximate $g = 10N/kg$.

The spring constant for a cantilevered board is given by

$$k = E\frac{wt^3}{4l^3} \tag{19}$$

3(a). **Draw a free-body diagram of the dog standing at rest on the free end of the diving board, and label all forces acting on the dog.**



3(b). **The dog drops onto the board from a height of h above the board, which then deflects down $10cm$ before coming to a momentary rest. What is h? How fast was the dog moving when it contacted the board?**

Let's put the $z = 0$ at the undeflected board. Then, the total energy is always conserved:

$$E_{tot} = \frac{1}{2}mv^2 \tag{20}$$

$$= \frac{1}{2}kd^2 - mgd \tag{21}$$

$$= mgh. \tag{22}$$

Where d is the deflection of the board and v is the velocity of the dog when it contacted the board. Eq. (20) gives the total energy (all kinetic) when the dog contacted the board; Eq. (21) gives the total energy (spring potential and gravitational potential) when the dog is at rest on the deflected board; and Eq. (22) gives the total energy (all gravitational potential) when the dog is at the top of his trajectory.

Putting in the given values yields $h = 40cm$, $v = 2.83m/s$.

3(c). **The dog walks onto a second diving board 1% thicker than the first board. It is pulled down to a deflection of $10cm$, allowed to come to rest, then released. *Approximately* how high above the board does the dog get launched?**

The spring potential energy scales as the spring constant, which scales as t^3. For a 1% increase in t, then, the spring potential energy increases by approximately 3%. The gravitational potential energy scales as h, so a 3% increase in total energy translates to a 3% increase in vertical distance traveled. The distance traveled in 3(c) was $d + h = 50cm$, so the dog now travels an additional 3% of $50cm = 1.5cm$, for a new height of $51.5cm - d = 41.5cm$.

3.2 How difficult did you find these problems (easy / needed review / hard)?

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4 Probability

4.1 Flip a coin 8 times.

Let's say you obtained the sequence: **T T T H T T H T**

4(a). **If the coin is perfectly fair, what is the probability of obtaining this sequence?**

Let's label events:

- This sequence occurs : S
- The coin is fair : F
- The coin is biased : B

$\mathrm{p}(S) = \mathrm{p}(\mathbf{T}) * \mathrm{p}(\mathbf{T}) * \mathrm{p}(\mathbf{T}) * \mathrm{p}(\mathbf{H}) * \mathrm{p}(\mathbf{T}) * \mathrm{p}(\mathbf{T}) * \mathrm{p}(\mathbf{H}) * \mathrm{p}(\mathbf{T})$.

For a fair coin, $\mathrm{p}(\mathbf{T}|F) = \mathrm{p}(\mathbf{H}|F) = 1/2$, so $\mathrm{p}(S|F) = (1/2)^8 = 1/256$

4(b). **If the coin is biased with a probability of heads p(H) $= \frac{1}{4}$, what is the probability of obtaining this sequence?**

$$\mathrm{p}(S|B) = \mathrm{p}(\mathbf{T}|B) * \mathrm{p}(\mathbf{T}|B) * \mathrm{p}(\mathbf{T}|B) * \mathrm{p}(\mathbf{H}|B) * \mathrm{p}(\mathbf{T}|B) * \mathrm{p}(\mathbf{T}|B) * \mathrm{p}(\mathbf{H}|B) * \mathrm{p}(\mathbf{T}|B)$$
$$= (3/4)^6 * (1/4)^2 = 729/65536$$

4(c). **Let's say you have 4 coins, 3 of which are fair and 1 is biased as above. You pick one coin uniformly at random from those. What is the likelihood (probability) that the coin you've picked is biased?**

$\mathrm{p}(B) =$ (#biased coins) / (#coins total) $= 1/4$.

4(d). **You flip this unknown coin 8 times to obtain the above sequence. Now what is the likelihood (probability) that the coin you've picked is biased?**

$$\begin{aligned}
p(S) &= p(S|B)p(B) + p(S|F)p(F) \\
p(B|S) &= \frac{p(S|B)p(B)}{p(S)} \\
&= \frac{p(S|B)p(B)}{p(S|B)p(B) + p(S|F)p(F)} \\
&= \frac{(3^6/4^8)(1/4)}{(3^6/4^8)(1/4) + (1/2^8)(3/4)} \\
&\approx 0.487
\end{aligned}$$

4.2 How difficult did you find these problems (easy / needed review / hard)?

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5 Programming

5.1 Consider the coins from problem 4.

We will simulate flipping a coin 100 times to try to guess which coin we've picked, refining our estimate of the likelihood after each flip's outcome.

5(a). **Write a function that takes in 2 parameters:**

- **the type of coin (fair or biased), and**
- **the number of flips;**

and returns the resulting simulated sequence of outcomes. Generate 10 sequences of 40 flips each: 5 sequences for a fair coin and 5 sequences for a biased coin.

5(b). **Modify that function to also calculate the likelihood that the coin is biased after each successive flip, given the conditions of problem 4(c).**

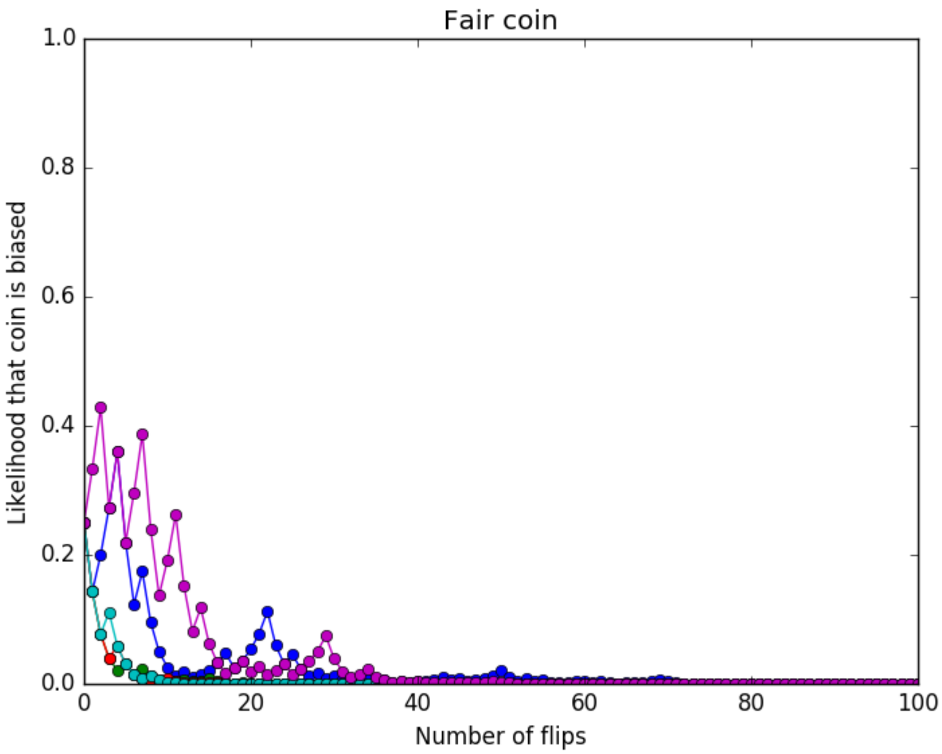
5(c). **Generate a properly labeled graph plotting the likelihood of having picked a biased coin as it evolves after each of 100 simulated flips of a fair coin. Overlay 5 independent simulations on the same graph.**

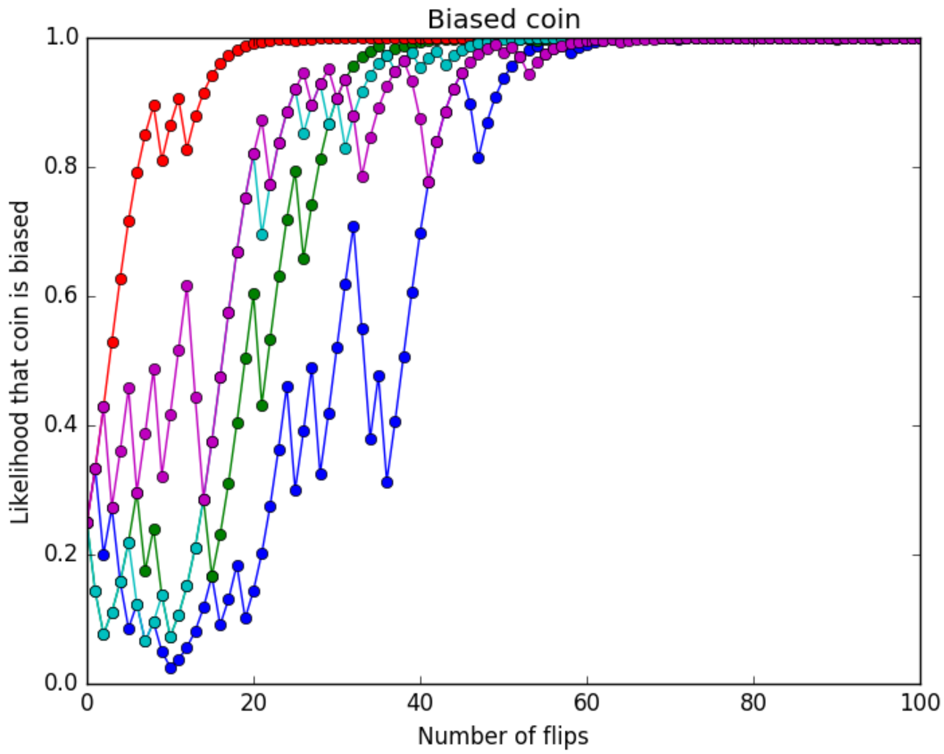
5(d). **Generate a similar graph, this time plotting 5 independent runs of simulating 100 flips of a biased coin.**

Example function in python:

```python
import matplotlib
matplotlib.use("Agg")
import matplotlib.pyplot as plt

from random import random

def flip(isBiased, numFlips):
    pHeads = isBiased and 0.25 or 0.5
    flips = range(numFlips+1)
    pBiased = range(numFlips+1)
    numTails = 0

    for flipNumber in range(numFlips+1):
        pBiased[flipNumber] = 1. / (1. + (2.**flipNumber * 3. / 3.**numTails))
        if random() < pHeads:
            flips[flipNumber] = 'H'
        else:
            numTails += 1
            flips[flipNumber] = 'T'

    return flips[:flipNumber], pBiased

numFlips = 40
for isBiased in (False, True):
    print isBiased and "Biased coin:" or "Fair coin:"
    for i in range(5):
        flips, pBiased = flip(isBiased, numFlips)
        print "".join(flips)

numFlips = 100
for isBiased in (False, True):
    for i in range(5):
        flips, pBiased = flip(isBiased, numFlips)
```

```
        plt.plot(range(numFlips+1), pBiased, 'o-')
        plt.xlabel('Number of flips')
        plt.ylabel('Likelihood that coin is biased')
        plt.ylim(0,1)
        plt.title(isBiased and "Biased coin" or "Fair coin")
    plt.savefig('%s_Flips.png' % (isBiased and "Biased" or "Fair"))
    plt.clf()
```





5.2 How difficult did you find these problems (easy / needed review / hard)?

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6 Summary

6.1 How long did this pset take you?

Please note that your answers to this question and the questions about difficulty are solely for me to assess and adapt my problem sets and lectures to better serve you. Please answer truthfully; your responses do not in any way affect your grade.