COMP 527 - Interoperability between Ordered Logic and Linear Logic
Extended abstract

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Abstract

We attempt a translation between two substructural logics, namely linear logic (a logic that only allows exchange as its structural rule) and ordered logic (a logic that allows none of the three structural rules). We introduce the translation for contexts and connectives used in linear logic to ordered logic. Later on, this paper extends on the translation of proofs in linear logic to proofs in ordered logic.

1 Introduction

Substructural logics, such as linear, affine, or relevant logic, offer an alternative, powerful framework that classical logic may fail to capture. The logic mentioned above use at least exchange as its structural rule, but the existence of exchange might not be suited for some frameworks. An interesting framework is linguistics, where the word orders do matter in creating a meaningful sentence.

In modern program logic, reasoning often involves juggling multiple systems, each with distinct characteristics and restrictions. Some systems support precise reasoning but are more complicated, while others are easier to use but yield only broader and weaker conclusions. However, proofs rarely live in isolation; assumptions from one logic must sometimes be transferred to another. This logical interoperability problem becomes critical when the logic differ in structural rules, complicating interactions between their frameworks.

Ordered logic treats the order of elements as strict and does not allow exchange (i.e. reordering assumptions). In contrast, linear logic permits exchange but forbids weakening or contraction (discarding or duplicating assumptions). These logic model proofs in very different ways: ordered logic tracks rigid ordering constraints, while linear logic handles unordered collections. Understanding how these logics interact is important because many systems rely on both kind of reasoning. In this project, we study how to translate between them, connecting ordered and linear reasoning in a principled way.

2 Proposed approach

Our approach is more focused on translating derivations from linear logic to ordered logic, while trying to accommodate the differences in structural rules, especially the absence of exchange in ordered logic. In ordered logic, the context is written as Ω where the order of assumptions matters and cannot be permuted. In contrast, linear logic's context is written as Δ and is an unordered collection of assumptions, i.e. the sequence in which propositions appear does not affect the validity of the derivation.

In our translation attempt, we introduce a mobile modality $\clubsuit A$. Mobile propositions can be placed anywhere in the ordered context, with rules as defined below.

$$\frac{\Omega_L, A, \Omega_R \vdash C}{\clubsuit A, \Omega_L, \Omega_R \vdash C}\ \clubsuit L$$

$$\frac{\Omega_L, A, \Omega_R \vdash C}{\Omega_L, \Omega_R, \clubsuit A \vdash C}\ \clubsuit L$$

$$\frac{\Omega \vdash C}{\Omega \vdash \clubsuit C} \; \clubsuit R$$

To translate a derivation $\Delta \vdash C$ in linear logic to an equivalent one in ordered logic, we first define the following translation from linear context to ordered context.

$$|\cdot| = \cdot$$
$$|\Delta| = \Omega$$
$$|\Delta, A| = \Omega_L, [A], \Omega_R$$

The following is the translation from linear connectives to ordered connectives.

$$\begin{aligned}
[A \otimes B] \quad &= [A] \bullet [B] \\
[A \& B] \quad &= [A] \& [B] \\
[A \multimap B] \quad &= \clubsuit[A] \twoheadrightarrow [B] \\
&\quad \text{if only } \Omega_L \text{ on the left of } [A \multimap B] \\
[A \multimap B] \quad &= \clubsuit[A] \rightarrowtail [B] \\
&\quad \text{if only } \Omega_R \text{ on the right of } [A \multimap B]
\end{aligned}$$

The reader might notice the existence of two $\clubsuit L$ rules as arising from the fact that a mobile preposition $\clubsuit$C can move to the left or right of its current position (for this paper, we do not differentiate the naming of both rules). There are also two translations of the $\multimap$ connective since we have two equivalent rules in ordered logic, whose location in the ordered context determines which of the two implication rules we can use.

If Δ contains no additional proposition, we may translate it as Ω_L, $\Omega_R \vdash$ C, which we simplify for ease of notation to $\Omega \vdash C$ (i.e. $\Omega = \Omega_L, \Omega_R$). However, when Δ is followed by a specific arbitrary proposition A, we preserve its exact position in the ordering by translating the linear context as $\Omega_L, [A], \Omega_R$. This embedding ensures that the structure of ordered logic is respected within the linear logic framework, since $[A]$ might be located anywhere inside the ordered logic context.

This then allows us to have the theorem:

Let $\Delta \vdash C$ be a derivation in linear logic, where $\Delta = A_1, ..., A_n$ is a multiset of unordered assumptions, and C is a conclusion. Let $|\Delta|$ be the translation of the linear context as defined above accordingly.

Then the following holds:

If $\Delta \vdash C$, then $|\Delta| \vdash [C]$.

3 Related Work

Our main sources in this research emphasized that order matters in Ordered Logic (OL) and provide strategies for adapting systems like Linear Logic (LL) that allow reordering into one that strictly prohibits it. The CMU lecture notes on OL rules grounded our understanding of the core inference rules and structural restrictions of OL, which we follow closely in our translation (Lecture Notes on Ordered Logic, 2012). The LICS 2009 paper on Substructural Operational Semantics (SSOS) inspired our use of OL as a framework to enforce execution order and guide the flow of proof steps, similarly to how it models computation (Pfenning Simmons, n.d.). Finally, the lecture on Ordered Abstract Machines informed our treatment of implication and subsingleton contexts, helping us refine our rule translation and ensure only one assumption is used at a time (Lecture Notes on from Subsingleton to Ordered Logic, 2016). Our framework formalizes the adaptation of LL derivations into OL proofs in the same spirit that these works reframe semantics or rule naming to respect ordered constraints.

4 Motivation

Understanding how to translate between Linear Logic (LL) and Ordered Logic (OL) is crucial because both systems are used in different reasoning tasks. LL allows reordering (exchange), while OL does not. OL's stricter structure makes it suitable for applications where the order of operations matters, such as modeling computation steps or linguistic structures. Since OL is more restrictive, any derivation valid in OL is also valid in LL. This makes the reverse direction (translating from LL to OL) more interesting and challenging. Our work focuses on this translation to bridge the gap between these logics and ensure that flexible LL proofs can be interpreted within the stricter OL framework.

5 Limitation

At first we were struggling with how to come up with the proper translation and how to work in a logic without exchange, thus we tried several times different versions of translation before we arrived at the current translation. We were unable to do the proof terms for the connectives, but this can be a good potential research for further exploring and understanding ordered logic.

Framework - Artifact

Translation from Linear Logic (LL) to Ordered Logic (OL)

Cut Rule

$$\dfrac{\overset{D_1}{\Delta_1 \vdash A} \quad \overset{D_2}{\Delta_2, A \vdash C}}{\Delta_1, \Delta_2 \vdash C}\ \text{cut}$$

$$\begin{array}{rl} \Omega_1 \vdash [A] & \text{by IH on } D_1 \\ \Omega_L, [A], \Omega_R \vdash [C] & \text{by IH on } D_2 \\ \Omega_L, \Omega_1, \Omega_R \vdash [C] & \text{by cut rule in OL} \end{array}$$

Tensor Right $(\otimes R)$

$$\dfrac{\overset{D_1}{\Delta_1 \vdash A} \quad \overset{D_2}{\Delta_2 \vdash B}}{\Delta_1, \Delta_2 \vdash A \otimes B}\ \otimes R$$

$$\begin{array}{rl} \Omega_1 \vdash [A] & \text{by IH on } D_1 \\ \Omega_2 \vdash [B] & \text{by IH on } D_2 \\ \Omega_1, \Omega_2 \vdash [A] \bullet [B] & \text{by } \bullet R \\ \Omega_1, \Omega_2 \vdash [A \bullet B] & \text{by translation of } \bullet \end{array}$$

Tensor Left $(\otimes L)$

$$\dfrac{\Delta, A, B \vdash C}{\Delta, A \otimes B \vdash [C]}\ \otimes L$$

$$\begin{array}{rl} \Omega_L, [A], [B], \Omega_R \vdash [C] & \text{by IH} \\ \Omega_L, [A] \bullet [B], \Omega_R \vdash [C] & \text{by } \bullet L \\ \Omega_L, [A \bullet B], \Omega_R \vdash [C] & \text{by translation of } \bullet \\ |\Delta, A \otimes B] \vdash [C] & \text{by context translation} \end{array}$$

Implication Right $(\multimap R)$

$$\dfrac{\Delta, A \vdash B}{\Delta \vdash A \multimap B}\ \multimap R$$

Implication Right (right-focus)

$$\begin{array}{rl} \Omega_L, [A], \Omega_R \vdash [B] & \text{by IH} \\ \Omega_L, \Omega_R, \clubsuit[A] \vdash [B] & \text{by } \clubsuit L \\ \Omega_L, \Omega_R \vdash \clubsuit[A] \twoheadrightarrow B & \text{by } \twoheadrightarrow R \end{array}$$

$$\dfrac{\Delta, A \vdash B}{\Delta \vdash A \multimap B}\ \multimap R$$

Implication Right (left-focus)

$$\begin{array}{rl} \Omega_L, [A], \Omega_R \vdash [B] & \text{by IH} \\ \clubsuit[A], \Omega_L, \Omega_R \vdash [B] & \text{by } \clubsuit L \\ \Omega_L, \Omega_R \vdash \clubsuit[A] \rightarrowtail [B] & \text{by } \rightarrowtail R \end{array}$$

Implication Left $(\multimap L)$

$$\dfrac{\overset{D_1}{\Delta_1 \vdash A} \quad \overset{D_2}{\Delta_2, B \vdash C}}{\Delta_1, \Delta_2, A \multimap B \vdash C}\ \multimap L$$

Implication Left (right-focus)

$$\begin{array}{rl} \Omega_1 \vdash [A] & \text{by IH on } D_1 \\ \Omega_1 \vdash \clubsuit[A] & \text{by } \clubsuit R \\ \Omega_L, [B], \Omega_R \vdash [C] & \text{by IH on } D_2 \\ \Omega_L, \clubsuit[A] \twoheadrightarrow [B], \Omega_1, \Omega_R \vdash [C] & \text{by } \twoheadrightarrow L \\ \Omega_L, [A \multimap B], \Omega_1, \Omega_R \vdash [C] & \text{by the first translation of } \multimap \end{array}$$

$$\dfrac{\overset{D_1}{\Delta_1 \vdash A} \quad \overset{D_2}{\Delta_2, B \vdash C}}{\Delta_1, \Delta_2, A \multimap B \vdash C}\ \multimap L$$

Implication Left (left-focus)

$$\begin{array}{rl} \Omega_1 \vdash [A] & \text{by IH on } D_1 \\ \Omega_1 \vdash \clubsuit[A] & \text{by } \clubsuit R \\ \Omega_L, [B], \Omega_R \vdash [C] & \text{by IH on } D_2 \\ \Omega_L, \Omega_1, \clubsuit[A] \rightarrowtail [B], \Omega_R \vdash [C] & \text{by } \rightarrowtail L \\ \Omega_L, \Omega_1, [A \multimap B], \Omega_R \vdash [C] & \text{by the second translation of } \multimap \end{array}$$

With Right $(\& R)$

$$\dfrac{\overset{D_1}{\Delta \vdash A} \quad \overset{D_2}{\Delta \vdash B}}{\Delta \vdash A \& B}\ \& R$$

$$\begin{array}{rl} \Omega \vdash [A] & \text{by IH on } D_1 \\ \Omega \vdash [B] & \text{by IH on } D_2 \\ \Omega \vdash [A] \& [B] & \text{by } \& R \\ \Omega \vdash [A \& B] & \text{by translation of } \& \end{array}$$

With Left $(\& L_1)$

$$\dfrac{\Delta, A \vdash C}{\Delta, A \& B \vdash C}\ \& L_1$$

$$\begin{array}{rl} \Omega_L, [A], \Omega_R \vdash C & \text{by IH} \\ \Omega_L, [A] \& [B], \Omega_R \vdash C & \text{by } \& L_1 \\ \Omega_L, [A \& B], \Omega_R \vdash C & \text{by translation of } \& \end{array}$$

With Left $(\& L_2)$

$$\dfrac{\Delta, B \vdash C}{\Delta, A \& B \vdash C}\ \& L_2$$

$$\begin{array}{rl} \Omega_L, [B], \Omega_R \vdash C & \text{by IH} \\ \Omega_L, [A] \& [B], \Omega_R \vdash C & \text{by } \& L_2 \\ \Omega_L, [A \& B], \Omega_R \vdash C & \text{by translation of } \& \end{array}$$

Annex

$$\frac{\Omega_1 \vdash A \quad \Omega_L, A, \Omega_R \vdash C}{\Omega_L, \Omega_1, \Omega_R \vdash C} \text{ cut}$$

$$\frac{\Omega_L, A, B, \Omega_R \vdash C}{\Omega_L, A \bullet B, \Omega_R \vdash C} \bullet L \qquad \frac{\Omega_1 \vdash A \quad \Omega_2 \vdash B}{\Omega_1, \Omega_2 \vdash A \bullet B} \bullet R$$

$$\frac{\Omega_1 \vdash A \quad \Omega_L, B, \Omega_R \vdash C}{\Omega_L, A \twoheadrightarrow B, \Omega_1, \Omega_R \vdash C} \twoheadrightarrow L \qquad \frac{\Omega, A \vdash B}{\Omega \vdash A \twoheadrightarrow B} \twoheadrightarrow R$$

$$\frac{\Omega_1 \vdash A \quad \Omega_L, B, \Omega_R \vdash C}{\Omega_L, \Omega_1, A \rightarrowtail B, \Omega_R \vdash C} \rightarrowtail L \qquad \frac{A, \Omega \vdash B}{\Omega \vdash A \rightarrowtail B} \rightarrowtail R$$

$$\frac{\Omega_L, A, \Omega_R \vdash C}{\Omega_L, A \& B, \Omega_R \vdash C} \& L_1 \qquad \frac{\Omega_L, B, \Omega_R \vdash C}{\Omega_L, A \& B, \Omega_R \vdash C} \& L_2$$

$$\frac{\Omega \vdash A \quad \Omega \vdash B}{\Omega \vdash A \& B} \& R$$

References

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