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FIRST-ORDER CONTINUOUS MODELS OF OPINION FORMATION*

GIACOMO ALETTI†, GIOVANNI NALDI†, AND GIUSEPPE TOSCANI‡

Abstract. We study certain nonlinear continuous models of opinion formation derived from a kinetic description involving exchanges of opinion between individual agents. These models imply that the only possible final opinions are the extremal ones, and they are similar to models of pure drift in magnetization. Both analytical and numerical methods allow us to recover the final distribution of opinion between the two extremal ones.

Key words. nonlinear nonlocal hyperbolic equation, sociophysics, opinion formation, magnetization

AMS subject classifications. 91C20, 82B21, 60K35

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1. Introduction.

This paper is devoted to the analysis and large-time behavior of solutions of the equation

$$\frac{\partial f}{\partial t} = \gamma \frac{\partial}{\partial x}\left((1-x^2)(x-m(t))f\right), \tag{1.1}$$

where the unknown $f(x,t)$ is a time-dependent probability density which may represent the density of opinion in a community of agents. This opinion varies between the two extremal opinions represented by ± 1, so that $x \in \mathcal{I} = [-1, 1]$. The constant γ is linked to the spreading ($\gamma = -1$) or to the concentration ($\gamma = +1$) of opinions. In (1.1) $m(t)$ represents the mean value of $f(\cdot, t)$,

$$m(t) = \int_{\mathcal{I}} x\, f(x,t)\, dx, \tag{1.2}$$

and its presence introduces a nonlinear effect into its evolution.

When $\gamma = -1$, the related linear equation

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}\left(x(1-x^2)f\right) \tag{1.3}$$

has been introduced recently by Slanina in [15] to analyze the evolution of density opinions in the voter model on a complete graph. There, the equation was derived as the mean field limit of the Sznajd model [20] in the case of two opinions. Because of linearity, (1.3) allows for an analytical treatment, and it is possible (see, e.g., [1, 15]) both to obtain the exact solution and to control the rate of decay towards the equilibrium for all values of γ (see also, e.g., [3]). Equation (1.1) was introduced in [22], in

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†Department of Mathematics, University of Milano, via Saldini 50, 20133 Milano, Italy (giacomo.aletti@unimi.it, giovanni.naldi@mat.unimi.it).
‡Department of Mathematics, University of Pavia, via Ferrata 1, 27100 Pavia, Italy (giuseppe.toscani@unipv.it).

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connection with the asymptotic limit of a Boltzmann equation for the kinetic description of opinion formation involving binary exchange of opinion between individual agents. This kinetic description is based on two-body interactions involving both compromise and diffusion properties in exchanges between individuals. Compromise and diffusion were quantified in [22] by two parameters, which are mainly responsible of the behavior of the model, and allow for a rigorous asymptotic analysis in which the limiting model is a Fokker–Planck-type equation, where the second-order term is related to diffusion, while the drift term is due to compromise. In a compromise-dominated regime, the resulting equation is exactly (1.1). We point out that, contrary to (1.3), the presence of the mean value $m(t)$ in (1.1) takes into account the influence of the mean opinion on the compromise-dominated dynamics.

Microscopic models of both social and political phenomena describing collective behaviors and self-organization in a society have been recently introduced and analyzed by several authors (see, e.g., [2, 6, 7, 8, 11, 12, 14, 16, 17, 18, 20, 25, 26]). The leading idea is that collective behaviors of a society composed of a sufficiently large number of individuals (agents) can be hopefully described using the laws of statistical mechanics as it happens in a physical system composed of many interacting particles. The details of the social interactions between agents then characterize the emerging statistical phenomena.

Equation (1.3), or in general (1.1) with $\gamma = -1$, can also describe a pure drift in magnetization (see, e.g., [19] and the references therein), where the two extremal points of $\mathcal{I}$ represent the opposite attraction poles. In the kinetic picture of [22], it simply means that the compromise in the binary interaction is substituted by magnetic repulsion between agents. Moreover, the case $\gamma = +1$ is related to models of one-dimensional nonlinear friction equations considered in the study of granular flows (see, e.g., [13, 21]), in connection with the quasi-elastic limit of a Boltzmann equation for rigid spheres with dissipative collisions and variable coefficient of restitution.

The paper is organized as follows. In the next section we introduce the main properties of the model, which justify the treatment in terms of a suitable weak formulation. The qualitative analysis is given in section 3. The large-time behavior is considered in sections 4–6. It is shown that the problem can be solved in sufficiently high generality only in the case of concentration ($\gamma = +1$). This lack of generality in the analytical treatment of the large-time behavior of the solution in the spreading of opinion justifies the numerical treatment of the equation. The numerical approximation is included in section 7, where both the explicit solution of (1.3) and the knowledge of the steady state in the concentration case ($\gamma = +1$) are used as benchmark tests for the numerical scheme. Finally, we note that the mathematical methods used here are close to the recent framework considered in the context of kinetic theory of nonlinear friction equations [10] and made popular by the mass transportation community [24].

2. Main properties and weak description. As briefly described in the introduction, (1.1) describes the evolution of a probability density which represents the density of opinions in a community. For all values of the constant γ, we will show that the time-evolution driven by this equation leads the density towards a equilibrium state that is described in terms of two Dirac masses ($\gamma = -1$) or to a unique Dirac mass ($\gamma = 1$). Having in mind that the equilibrium solution to equation (1.1) is given by Dirac masses, any convergence result towards equilibrium holds in a weak*-measure sense. The recent analysis of [10] of the nonlinear friction equation introduced by McNamara and Young [13] suggests that a suitable way of treating

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(1.1) is based on a rewriting of this equation in terms of pseudoinverse functions. It is immediate to show that the drift operator on the right-hand side of (1.1) preserves positivity and mass,

$$\int_{\mathcal{I}} f(x,t)\,dx = \int_{\mathcal{I}} f_0(x)\,dx. \tag{2.1}$$

Then, given a initial datum which is a probability density (nonnegative and with unit mass), the solution remains a probability density at any subsequent time. Let $F(x)$ denote the probability distribution induced by the density $f(x)$,

$$F(x) = \int_{(-\infty,x]} f(y)\,dy \tag{2.2}$$

and let μ denote the distribution on $\mathbb{R}$ associated to F. Since $F(\cdot)$ is not decreasing, we can define its pseudoinverse function (also called quantile function) by setting, for $\rho \in (0,1)$,

$$X^\mu(\rho) = X^F(\rho) = \inf\{x : F(x) \geq \rho\}.$$

Equation (1.1) for $f(x,t)$ takes a simple form if written in terms of its pseudoinverse $X(\rho,t)$. Theorem 3.1 shows in fact that the evolution equation for $X(\rho,t)$ reads

$$\frac{\partial X(\rho,t)}{\partial t} = -\gamma\,(X(\rho,t) - m(t))\,(1 - X^2(\rho,t)), \tag{2.3}$$

where now $\rho \in (0,1)$. Note that if we assume F to be absolutely continuous with respect to x and strictly increasing, then Theorem 3.1 reduces to elementary computations. In (2.3)

$$m(t) = \int_0^1 X(\rho,t)\,d\rho. \tag{2.4}$$

Let us set $\gamma = -1$ (spreading). Then the weak form (2.3) clarifies the evolution of $X(\rho,t)$ and the role of $m(t)$. In fact, if $X(\rho,t) > m(t)$, $X(\rho,t)$ increases towards 1, while $X(\rho,t) < m(t)$ implies that $X(\rho,t)$ decreases towards -1. Hence, the mean opinion $m(t)$ represent a barrier for the density of opinions to move towards one of the two extremal opinions. The fact that the mean opinion varies with time makes the nonlinear problem harder to handle with respect to the linear problem considered in [15] where the barrier is fixed equal to zero.

Among the metrics which can be defined on the space of probability measures, which metricize the weak convergence of measures [27], one can consider the L^p-distance ($1 \leq p < \infty$) of the pseudoinverse functions

$$d_p(X,Y) = \left(\int_0^1 |X(\rho) - Y(\rho)|^p\,d\rho\right)^{1/p}. \tag{2.5}$$

In what follows, we'll use the usual identifications

$$d_p(X,Y) = d_p(f_X, f_Y) = d_p(F_X, F_Y) = d_p(\mu_X, \mu_Y)\,,$$

where μ_X (μ_Y), F_X (F_Y), and f_X (f_Y) denote the distribution, the cumulative function, and the density associated to X (Y), respectively. By this identification, as one can see [10, 23, 24], $d_2(F,G)$ is nothing but the Wasserstein metric [23].

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In addition to nonlinear friction equations arising in the modeling of granular gases [10], the strategy of passing to pseudoinverse functions has been recently applied to nonlinear diffusion equations of porous medium type [5] and to degenerate convection–diffusion equations [4]. This rewriting of nonlinear diffusion equations has been shown to be useful in order to obtain simple explicit numerical schemes that satisfy a contraction property with respect to the Wasserstein metric [9].

3. Existence, uniqueness, and well-posedness of the problem. In this section we will study the initial value problem for (1.1), with initial density

$$f(x,t=0)=f_0(x), \quad x\in\mathcal{I}. \tag{3.1}$$

As before, we will denote by $X_0(\rho)$ the quantile function corresponding to f_0, so that

$$X(\rho,t=0)=X_0(\rho)=\inf\{x : F_0(x)\geq\rho\}, \quad \rho\in[0,1]. \tag{3.2}$$

The equivalence between (1.1) and (2.3) is contained in the following.

THEOREM 3.1. *There exists a weak solution of* (1.1)–(3.1) *if and only if there exists a solution of* (2.3)–(3.2).

Proof. Suppose first that there exists $f(x,t)$ which solves (1.1)–(3.1). Then $m(t)$ is a differentiable function of time. Let $y(t)$ be the maximal C^1 solution of the Abel differential equation:

$$\begin{cases} y' = -\gamma(1-y^2)(y-m(t)), \\ y(0)=\bar{y}_0, \end{cases} \tag{3.3}$$

where, for any $y_0\in[-1,1]$, we denoted by $\bar{y}_0$ a C^1-extension of y_0 to $\mathbb{R}$. We have, in weak sense,

$$\begin{aligned} \frac{d}{dt}\int_{(-\infty,y(t)]} f(x,t)\,dx &= \int_{\mathbb{R}}\Big[\frac{\partial}{\partial t}\Big(\mathbb{1}_{(-\infty,y(t)]}(x)\Big)f(x,t)+\mathbb{1}_{(-\infty,y(t)]}(x)\frac{\partial}{\partial t}f(x,t)\Big]\,dx \\ &= \int_{\mathbb{R}}\big(y'(t)\delta_{y(t)}(x)+\gamma[\delta_{y(t)}(x)(1-x^2)(x-m(t))]\big)f(x,t)\,dx \\ &= 0\,. \end{aligned} \tag{3.4}$$

Since $X(\rho,t)\leq x \iff \rho\leq\int_{-\infty}^x f(y,t)\,dy$, the first part of the proof has been shown.

Now, let $X(\rho,t)$ be a solution of (2.3)–(3.2). As a consequence of the properties of the solution to Abel's equation (3.3), given any initial datum $X(\rho,0)$ satisfying

- $X(\rho,0)\in[-1,1]$;
- $X(\rho,0)$ is nondecreasing;
- $X(\rho,0)$ is left-continuous,

the same properties are preserved at any subsequent time $t>0$. Hence, for any t, $\{X(\rho,t),\ \rho\in(0,1)\}$ is the quantile function of a unique probability measure on $[-1,1]$. We have only to prove that (1.1) holds. This is a consequence of the change

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of variables formula. In fact, if h is a test function,

$$\begin{aligned}\int_{\mathbb{R}} h(x)\frac{\partial}{\partial t}f(x,t)\,dx &= \frac{\partial}{\partial t}\int_{\mathbb{R}} h(x)f(x,t)\,dx\\ &= \frac{\partial}{\partial t}\int_0^1 h(X_\rho(t))\,d\rho\\ &= \int_0^1 \frac{\partial}{\partial t}h(X_\rho(t))\,d\rho\\ &= \int_0^1 h'(X_\rho(t))\Big(-\gamma\,(X(\rho,t)-m(t))\,(1-X^2(\rho,t))\Big)\,d\rho\\ &= \int_{\mathbb{R}} h'(x)\big(-\gamma\,(x-m(t))\,(1-x^2)\big)f(x,t)\,dx\\ &= \int_{\mathbb{R}} h(x)\frac{\partial}{\partial x}\Big(\gamma\,(x-m(t))\,(1-x^2)f(x,t)\Big)\,dx\,. \qquad \square\end{aligned}$$

We call $(\mathcal{K},p)$ the (compact) set of probability distributions on $[-1,1]$ equipped with the p-Wasserstein distance. Note that all the p-Wasserstein distances on $(\mathcal{K},d)$ are equivalent. In fact, if $q \geq p \geq 1$,

$$\|X^{\mu_1}-X^{\mu_2}\|_p \leq \|X^{\mu_1}-X^{\mu_2}\|_q \leq 2^{1-p/q}\|X^{\mu_1}-X^{\mu_2}\|_p^{p/q}\,. \tag{3.5}$$

We will refer to $\mathcal{K}$ as the topological space of probability distributions on $[-1,1]$ induced by any of this metric: the weak*-topology. Before searching for a continuous solution of (1.1) in $\mathcal{K}$, we state the following trivial lemma.

LEMMA 3.2 (solution Abel). *Let* $\phi(x,y) = -\gamma(1-x^2)(x-y)$. *Then*

$$|\phi(x_1,y_1)-\phi(x_2,y_2)| \leq 4|x_1-x_2| + |y_1-y_2|\,.$$

Moreover, if f *is a solution of* $f'(t) = \phi(f(t),g(t))$ *with* $\sup|g(t)| \leq 1$ *and* $f(0) \in [-1,1]$,

$$|f(s)-f(t)| \leq 2|s-t|\,.$$

We call any function $\mu \in C^0(\mathbb{R},\mathcal{K})$ s.t. (1.1)–(3.1) holds a *solution of* (1.1)–(3.1). We have the following theorem.

THEOREM 3.3. *For any probability density* $f_0(x)$ *in* (3.1), *there exists a unique function* $\mu \in C^0(\mathbb{R},\mathcal{K})$ *such that if* $f(x,t)$ *denotes the weak derivative of the probability distribution* $\mu(t)$, $f(x,t)$ *satisfies* (1.1) *with initial value* (3.1). *Moreover, for any* $t \in \mathbb{R}$, *the solution depends in a continuous way on the initial datum: the problem* (1.1)–(3.1) *is well-posed in* $C^0(\mathbb{R},\mathcal{K})$.

Proof. [Existence] We prove the existence of a solution of the equivalent problem (2.3)–(3.2) (see Theorem 3.1) in a constructive way. More precisely,

(A) we construct a sequence $\{X_n,\, n \in \mathbb{N}\}$ which approximates a target solution;
(B) by compactness arguments, we find a convergent subsequence $X_{n_l} \to X$;
(C) the limit X satisfies (2.3)–(3.2).

Let $[-T,T]$ be fixed. For any $n \in \mathbb{N}$, we subdivide $[-T,T]$ into disjoint intervals of length $R/2^N$. Then we proceed as follows:

(A1) we compute $m_0^{(n)} = \int_0^1 X_0(\rho)\,d\rho$;
(A2) we solve (2.3) on $[-T/2^n, T/2^n]$ with $m^{(n)}(t) = m_0$, finding $X^{(n)}(\rho,t)$, $t \in [-T/2^n, T/2^n]$;

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(A3) for any $k = 1, \dots, 2^n - 1$,
- we compute

$$m^{(n)}_{\pm k} = \int_0^1 X^{(n)}(\rho, \pm kT/2^n)\, d\rho;$$

- we solve (2.3) on $(kT/2^n, (k+1)T/2^n]$ with $m^{(n)}(t) = m^{(n)}_k$ and initial data $X^{(n)}(\rho, kT/2^n)$, finding $X^{(n)}(\rho,t)$, $t \in (kT/2^n, (k+1)T/2^n]$;
- we solve (2.3) on $[-(k+1)T/2^n, -kT/2^n)$ with $m^{(n)}(t) = m^{(n)}_{-k}$ and initial data $X^{(n)}(\rho, -kT/2^n)$, finding $X^{(n)}(\rho,t)$, $t \in [-(k+1)T/2^n, -kT/2^n)$.

We call $\mu^{(n)} : [-T,T] \to \mathcal{K}$ the sequence of function with value in $\mathcal{K}$ associated to $X^{(n)}$.

(B) For any $n \in \mathbb{N}$, it is possible to prove (by induction on k) that for any $t \in [-T,T]$, $X^{(n)}(\rho,t) \in [-1,1]$ and $m^{(n)}(t) \in [-1,1]$. As a consequence of Lemma 3.2, we have

$$\left|X^{(n)}(\rho,s) - X^{(n)}(\rho,t)\right| \le 2|t-s|, \tag{3.6}$$

i.e., for any $\rho \in (0,1)$, $\{X^{(n)}(\rho,\cdot) : [-T,T] \to [-1,1]\}_{n\in\mathbb{N}}$ is a uniformly equicontinuous sequence. A diagonal argument together with the Ascoli–Arzelà theorem ensure the existence of a subsequence n_l s.t. $\{X^{(n_l)}(\rho,\cdot) : [-T,T] \to [-1,1]\}_{l\in\mathbb{N}}$ converges uniformly on $[-T,T]$ for each $\rho \in (0,1) \cap \mathbb{Q}$.

Now, (3.6) implies

$$\int_0^1 |X^{(n)}(\rho,s) - X^{(n)}(\rho,t)|\, d\rho \le 2|t-s|\,, \tag{3.7}$$

i.e., $\mu^{(n)}$ is a equicontinuous sequence with respect to the distance d_1 on $\mathcal{K}$. Then the Ascoli–Arzelà theorem again ensures the existence of a subsection of n_l (we call it n_l again) such that

$$\sup_{t\in[-T,T]} d_1(\mu^{(n_k)}(t), \mu^{(n_l)}(t)) \le M(k \wedge l) \underset{k\wedge l \to\infty}{\longrightarrow} 0, \tag{3.8}$$

$$\sup_{t\in[-T,T]} |X^{(n_k)}(\rho,t) - X^{(n_l)}(\rho,t)| \le N_\rho(k \wedge l) \underset{k\wedge l \to\infty}{\longrightarrow} 0 \qquad \forall \rho \in \mathbb{Q} \cap (0,1). \tag{3.9}$$

Now, let $\rho \in (0,1) \cap \mathbb{Q}$ be fixed. $\{X^{(n_l)}(\rho,\cdot)\}_{l\in\mathbb{N}}$ is a uniform convergent sequence of derivable functions converging to $X(\rho,t) = \lim_l X^{(n_l)}(\rho,t))$. Left-continuity and monotonicity of $\{X(\rho,t),\ \rho \in (0,1) \cap \mathbb{Q}\}$ extend the definition of $X(\rho,t)$ to all $\rho \in (0,1)$.

(C) What remains to prove is
- $\lim_l m^{(n_l)}(t) = \int_0^1 X(\rho,t)\, d\rho =: m(t)$;
- $X(\rho,t)$ is differentiable, and (2.3) holds.

Note that, from (3.8), it follows immediately that $|\int_0^1 X^{(n_k)}(\rho,t)\, d\rho - \int_0^1 X^{(n_l)}(\rho,t)\, d\rho| \le M(k \wedge h)$ and hence

$$\left|\int_0^1 X(\rho,t)\, d\rho - \int_0^1 X^{(n_l)}(\rho,t)\, d\rho\right| \le M(h)\,. \tag{3.10}$$

By definition of $m^{(n)}$,

$$m^{(n)}(t) = m^{(n)}([\![2^n t]\!]/2^n) = \int_0^1 X^{(n)}(\rho, [\![2^n t]\!]/2^n)\, d\rho,$$

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where $[\![\cdot]\!]$ is the integer part of $\cdot$ closer to 0. Therefore,

$$\left|m^{(n)}(t) - \int_0^1 X^{(n)}(\rho, t)\, d\rho\right| \le 2^{-n+1} \tag{3.11}$$

and hence $m^{(n_l)}(t)$ is a Cauchy sequence on $[-1, 1]$. Thus, there exists $\widehat{m}(t) = \lim_h m^{(n_l)}(t)$. By (3.10) and (3.11) it follows that $\widehat{m}(t) = m(t)$ since

$$\left|\widehat{m}(t) - \int_0^1 X(\rho, t)\, d\rho\right| \le \left|\widehat{m}(t) - m^{(n_l)}(t)\right| + 2^{-n_l+1} + M(h)\,.$$

Now, let $\rho \in (0, 1) \cap \mathbb{Q}$ be fixed. For simplicity of notation, define $H(n, \rho, t) := \frac{\partial}{\partial s} X^{(n)}(\rho, s)\big|_{s=t}$. Moreover, we define

$$H(\rho, t) := \lim_{l\to\infty} H(n_l, \rho, t) = -\gamma(1 - X(\rho, t)^2)(X(\rho, t) - m(t)).$$

The uniform convergence theorem states that $\frac{\partial}{\partial s} X(\rho, s)\big|_{s=t} = H(\rho, t)$ if $\{H(n_l, \rho, t)\}_{l\in\mathbb{N}}$ is a uniform converging sequence on $(-T, T)$. To prove this, let $k \ge l$. The triangular inequality

$$\begin{aligned}\left|H(n_k, \rho, t) - H(n_l, \rho, t)\right| &\le \left|H(n_k, \rho, t) - H(n_k, \rho, [\![2^{n_l} t]\!]/2^{n_l})\right| \\ &\quad + \left|H(n_k, \rho, [\![2^{n_l} t]\!]/2^{n_l}) - H(n_l, \rho, [\![2^{n_l} t]\!]/2^{n_l})\right| \\ &\quad + \left|H(n_l, \rho, [\![2^{n_l} t]\!]/2^{n_l}) - H(n_l, \rho, t)\right| \\ &= A_\rho(k, l, t) + B_\rho(l, k, t) + A_\rho(l, l, t)\end{aligned}$$

shows that we may prove that $\sup_{t\in[-T,T]} A_\rho(l, k, t) + B_\rho(l, k, t) + A_\rho(l, l, t) \underset{l\wedge k\to\infty}{\longrightarrow} 0$. As a consequence of (3.7) and (3.11), we have $|m^{(n)}(s) - m^{(n)}(t)| \le 2^{-n+2} + 2|t - s|$, which implies (see Lemma 3.2 and (3.6))

$$\begin{aligned}|H(n, \rho, s) - H(n, \rho, t)| &= \Big|(1 - X^{(n)}(\rho, s)^2)(X^{(n)}(\rho, s) - m^{(n)}(s)) \\ &\qquad - (1 - X^{(n)}(\rho, t)^2)(X^{(n)}(\rho, t) - m^{(n)}(t))\Big| \\ &\le 4|X^{(n)}(\rho, s) - X^{(n)}(\rho, t)| + |m^{(n)}(s) - m^{(n)}(t)| \\ &\le 10|t - s| + 2^{-n+2},\end{aligned}$$

and hence $A_\rho(k, l, t) \le 10\cdot 2^{-n_l} + 2^{-n_k+2}$. Now, let $k \ge l$ and $l \in \{-2^{n_l}+1, \ldots, 2^{n_l}-1\}$. Again, Lemma 3.2, (3.8), and (3.9) imply

$$\left|H(n_k, \rho, l/2^{n_l}) - H(n_l, \rho, l/2^{n_l})\right| \le 4N_\rho(l) + M(l),$$

and hence $B_\rho(k, l, t) \le 4N_\rho(l) + M(l)$. This completes the proof for $\rho \in (0, 1) \cap \mathbb{Q}$. Now, fixing $y_0 \in [-1, 1]$, let $y(t)$ be the maximal C^1 solution of the Abel differential equation (3.3). Since $X(\rho, t) \ge y(t) \iff X(\rho, 0) \ge y(0)$ for all $\rho \in (0, 1) \cap \mathbb{Q}$, left-continuity and monotonicity of $\{X_\rho(t),\, \rho \in (0, 1)\}$ extend the proof to $\rho \in (0, 1)$.

[Uniqueness and well-posedness] Let $Y(\rho, t), X(\rho, t)$ be two solutions of (1.1)–(1.2). Denote by $m_X(t)$ and $m_Y(t)$ the mean values of X and Y, respectively, at time

t. Since $|m_Y(t) - m_X(t)| \le d_1(X,Y)$ we have (by Lemma 3.2 and (3.5))

$$\begin{aligned}\frac{d}{dt} d_2(\mu_X(t), \mu_Y(t)) &= \frac{d}{dt}\int_0^1 \big(Y(\rho,t) - X(\rho,t)\big)^2\, d\rho \\ &\le 2\int_0^1 \big|Y(\rho,t)-X(\rho,t)\big|\Big(4\big|Y(\rho,t)-X(\rho,t)\big| + \big|m_Y(t)-m_X(t)\big|\Big)\, d\rho \\ &\le 10\cdot d_2(\mu_X(t),\mu_Y(t))\,.\end{aligned} \tag{3.12}$$

Gronwall's lemma completes the proof. □

4. Large-time behavior of solutions. Thanks to the uniqueness of solutions of Abel's equation, we obtain the following lemma.

LEMMA 4.1. *For any $t \neq s \in \mathbb{R}$ and $\rho \neq \rho' \in (0,1)$, we have*

$$X(\rho,t) = X(\rho',t) \iff X(\rho,s) = X(\rho',s)\,,$$

i.e., (1.1) *does not create or destroy delta masses in finite time.*

A direct consequence of the previous lemma is that the initial masses in $+1$, -1, and $(-1,1)$ remain unchanged in time. Let us call them p_{+1}, p_{-1}, and $1-(p_{+1}+p_{-1})$, respectively.

An important argument linked to the large-time behavior of solutions to nonlinear equations is the study both of conservation laws and of Lyapunov functionals. In addition to mass conservation, a second conserved quantity (when defined) is furnished by

$$T(t) := \int_0^1 \log\Big(\frac{1+X(\rho,t)}{1-X(\rho,t)}\Big)\, d\rho\,. \tag{4.1}$$

In addition to the conservation of both mass and $T(t)$, equation (1.1) possesses a Lyapunov functional, simply given by the variance of the solution

$$V(t) := \int_0^1 (X(\rho,t))^2\, d\rho - \Big(\int_0^1 X(\rho,t)\, d\rho\Big)^2. \tag{4.2}$$

We give below an easy-to-check condition which ensure both the the boundedness and the conservation in time of the functional (4.1).

LEMMA 4.2. *Let $\log\big((1+X(\rho,0))/(1-X(\rho,0))\big) \in L^1(0,1)$. Then for all $t \in \mathbb{R}$*

$$T(t) := \int_0^1 \log\Big(\frac{1+X(\rho,t)}{1-X(\rho,t)}\Big)\, d\rho$$

is well-defined. Moreover, $T(t)$ is differentiable and $T'(t) = 0$.

Proof. Since $\log\big((1+X(\rho,0))/(1-X(\rho,0))\big) \in L^1(0,1)$, then $X(\rho,0) \in (-1,1)$ for all $\rho \in (0,1)$. Now $(1-x^2)(x-1) \le (1-x^2)(x-m(t)) \le (1-x^2)(x-1)$, and hence the uniqueness of solutions of the Abel equations imply

$$X(\rho,t) \in (-1,1) \qquad \forall \rho \in (0,1)\,, \quad \forall t \in \mathbb{R}\,.$$

Let $G(\rho,t) = \log\big((1+X(\rho,t))/(1-X(\rho,t))\big)$. Since $|G_t(\rho,t)| \le 2$, we have

- $\exists G_t$ for all $\rho \in (-1,1)$, for all $t \in [-T,T]$;
- $|G(\rho,t)| \le |G(\rho,0)| + 2T$ and hence $T(t)$ exists;
- $|G_t(\rho,t)| \le 2$ and $2 \in L^1(0,1)$;

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and hence it is possible to differentiate under the integral sign, obtaining $T'(t) = 0$. ∎

The following lemma shows that the variance is a Lyapunov functional for (1.1).

LEMMA 4.3. *The variance $V(t)$ of $\mu(t)$ is a monotone differentiable function with values in* $[0,1]$.

Proof. $V(t)$ is clearly differentiable, and

$$V'(t) = \frac{d}{dt}\int_0^1 (X_t(\rho) - m(t))^2\, d\rho = -2\gamma \int_0^1 (1 - X_t(\rho)^2)(X_t(\rho) - m(t))^2\, d\rho\,.$$

In the case $\gamma = -1$, $V(t)$ is monotonically increasing while bounded from above by 1. In fact, the maximum value of V is attained for $\mu = (\delta_1 + \delta_{-1})/2$. If, on the contrary, $\gamma = 1$, $V(t)$ is monotonically decreasing while bounded from below by 0. In this second case, the minimum value of V is attained for $\mu = \delta_a$, $a \in [-1,1]$. ∎

The most difficult problem linked to (1.1) is the study of the evolution of the mean $m(t)$ and to the exact evaluation of its limit value $\bar{m} = \lim_{t\to\infty} m(t)$. The knowledge of $\bar{m}$ is of primary importance, since in consequence of the structure of the limit state of the solution to (2.3), this value is enough to characterize completely the steady state.

Remark. The difficulty comes out from the evolution of the mean $m(t)$, which is given by the "nonclosed" equation

$$m'(t) = -\gamma m(t) \int_0^1 X^2(\rho)\, dr + \gamma \int_0^1 X^3(\rho)\, dr. \tag{4.3}$$

In what follows, we make use of the previous results to extract information on the behavior of the mean $m(t)$.

LEMMA 4.4. *For any t, $m(t) \in [-\sqrt{1 - V(t)},\ \sqrt{1 - V(t)}]$.*

Proof. Since $V(t) = \int x^2 f(x,t)dx - (m(t))^2$, then $V(t) \le 1 - (m(t))^2$. ∎

LEMMA 4.5. *The function $m' : \mathbb{R} \to [-1,1]$ belongs to $L^2(\mathbb{R})$. Moreover, $\exists \lim_{t\to\infty} m'(t) = 0$.*

Proof. It is sufficient to note that

$$(m'(t))^2 \le \int \left(1 - X_\rho(t)^2\right)^2 (X_\rho(t) - m(t))^2\, d\rho \le V'(t),$$

and hence $m' \in L^2(\mathbb{R})$ by Lemma 4.3. Moreover, since m' is a Lipschitz function (in fact it is differentiable and $m'' \le 10$; cf. (4.3)), it follows that $\lim_{t\to\infty} m'(t) = 0$. ∎

5. Spreading of opinions. In this section we study the large behavior of (1.1) when $\gamma = -1$, leaving the opposite case $\gamma = 1$ to the following section.

Remark. If $X(\rho,0) = -X(1-\rho,0)$ (i.e., the initial distribution is symmetric), then from (4.3) it follows that $m(t) = 0$ for any subsequent time $t > 0$. If, in addition, $X(\rho_1,0) = X(\rho_1+\delta,0) = 0$, then $X(\rho_1,t) = X(\rho_1+\delta,t) = 0$ for all $t > 0$ (i.e., any initial mass in 0 is not moved away in time if the initial distribution is symmetric). In order to avoid these situations we will allow delta masses only in ± 1: $X(\rho_1,0) = X(\rho_2,0) \iff \rho_1 = \rho_2$ or $(X(\rho_1,0))^2 = 1$.

THEOREM 5.1. *Assume $X(\rho_1,0) = X(\rho_2,0) \iff \rho_1 = \rho_2$ or $(X(\rho_1,0))^2 = 1$. Then the limit distribution exists and is given by two masses located in -1 and $+1$.*

Proof. By Lemmas 4.3 and 4.4, $m(t) \in [-\sqrt{1 - V(0)},\ \sqrt{1 - V(0)}]$ for all $t \ge 0$. Thus, if $(X(\rho,t_0)^2 > 1 - V(0)$, then $(X(\rho,t)^2 > 1 - V(0)$ for all $t \ge t_0$. Since

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$X(\rho,t) < -\sqrt{1-V(0)} \Rightarrow \frac{\partial}{\partial t}X(\rho,t) \le 0$ and $X(\rho,t) > \sqrt{1-V(0)} \Rightarrow \frac{\partial}{\partial t}X(\rho,t) \ge 0$ by Lemmas 4.3 and 4.4, the two functions

$$p^h_{-1}(t) = \sup\{\rho \in (0,1)\colon X(\rho,t) < -\sqrt{1-V(0)}\} = \int_{[-1,-\sqrt{1-V(0)})} f(x,t)\,dx,$$
$$p^h_{1}(t) = \inf\{\rho \in (0,1)\colon X(\rho,t) > \sqrt{1-V(0)}\} = \int_{(\sqrt{1-V(0)},1]} f(x,t)\,dx$$

are monotone. We call $p^h_{\pm 1} = \lim_{t\to\infty} p^h_{\pm 1}(t)$. Equation (2.3) and monotonicity of Abel's solutions allow us to state that

$$\forall \rho \in [0, p^h_{-1}), \quad \lim_{t\to\infty} X(\rho,t) = -1, \tag{5.1}$$
$$\forall \rho \in (p^h_{+1}, 1], \quad \lim_{t\to\infty} X(\rho,t) = 1. \tag{5.2}$$

Hence the limit distribution has two masses in -1 and $+1$. It remains to characterize what happens for the remaining $p^h_{+1} - p^h_{-1}$ mass. Let us recall that

$$X(\rho,t) \in \big[-\sqrt{1-V(0)}, \sqrt{1-V(0)}\big] \qquad \forall \rho \in (p^h_{-1}, p^h_{+1}). \tag{5.3}$$

By Lemma 4.5, there exists $T > 0$ such that $|m'(t)| \le (V(0)/2)^2$ for all $t > T$. By contradiction, suppose that there exist $t_0 \ge T$ and $\rho \in (p^h_{-1}, p^h_1)$ such that $\big|X(\rho,t_0) - m(t_0)\big| > V(0)/4$.

Since $|\frac{\partial}{\partial t}X(\rho,t_0)| > |m'(t_0)|$, it follows that $\big|X(\rho,t) - m(t)\big| > V(0)/4$ for all $t \ge t_0$. Thus, (5.3) shows the contradiction:

- $\frac{\partial}{\partial t}X(\rho,t)$ is continuous;
- $|\frac{\partial}{\partial t}X(\rho,t)| > (V(0)/2)^2$ if $t \ge t_0$;
- $X(\rho,t)$ is bounded.

Therefore,

$$\big|X(\rho,t) - m(t)\big| \le \frac{V(0)}{4} \qquad \forall \rho \in (p^h_{-1}, p^h_{+1}), \quad t > T\,. \tag{5.4}$$

Now, let $F(x,y) = (1-x^2)(x-y)$ as in Lemma 3.2. Since F is differentiable, when $x_1 \ge x_2$, Lagrange theorem states that we can find $\xi \in (x_1,x_2)$ such that

$$F(x_1,y) - F(x_2,y) = (x_1 - x_2)\frac{\partial}{\partial x}F(x,y)\Big|_{x=\xi}.$$

Now, if $1 - x^2 \ge V(0)$ and $|x-y| \le V(0)/4$, we have

$$\frac{\partial}{\partial x}F(x,y) = 1 - 3x^2 + 2xy \ge V(0) + 2x(y-x) \ge \frac{V(0)}{2},$$

that is,

$$F(x_1,y) - F(x_2,y) \ge (x_1 - x_2)\frac{V(0)}{2}\,, \qquad x_i^2 \le 1 - V(0) \text{ and } |x_i - y| \le \frac{V(0)}{4}\,. \tag{5.5}$$

Let $p^h_{-1} < \rho_2 \le \rho_1 < p^h_{+1}$. Then both (5.3) and (5.4) are satisfied for all $t > T$, (5.5) holds, and

$$\frac{\partial}{\partial t}\Big(X(\rho_1,t) - X(\rho_2,t)\Big) \ge \big(X(\rho_1,t) - X(\rho_2,t)\big)\frac{V(0)}{2} \qquad \forall t > T\,.$$

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Since the two solutions are bounded, the only possibility is that $X(\rho_1, t) = X(\rho_2, t)$ for all $t > T$ and for all (ρ_1, ρ_2): $p^h_{-1} < \rho_2 \le \rho_1 < p^h_{+1}$, which implies $p^h_{-1} = p^h_1$ by Lemma 4.1 and hypothesis. ∎

Remark. Theorem 5.1 may be read in terms of weak*-measure convergence:

$$f(x,t) \underset{t\to\infty}{\rightharpoonup} p^h_{-1}\delta_{-1}(x) + (1 - p^h_{-1})\delta_1(x)\,.$$

In particular, since the support is compact, all the moments exist and will converge. We have the following

COROLLARY 5.2. *Assume* $X(\rho_1, 0) = X(\rho_2, 0) \iff \rho_1 = \rho_2$ *or* $(X(\rho_1, 0))^2 = 1$. *Then there exists* $\lim_{t\to+\infty} m(t) = m_\infty = 1 - 2p^h_{-1}$.

6. Concentration of opinions.

Let us recall that the stochastic partial order is naturally given on $\mathcal{K}$. Let $F(x), G(x)$ denote two probability distributions and X_F, X_G their pseudoinverse functions, respectively. We say that $F \preceq G$ if $F(x) \ge G(x)$ for all (x) or, equivalently, if $X_F(\rho) \le X_G(\rho)$ for all $\rho \in (0, 1)$.

LEMMA 6.1. *The operator*

$$\phi(X) = -(X - m(X))(1 - X^2)$$

is a monotone operator with respect to the stochastic ordering.

Proof. Assume that $X_1(\rho, s) \le X_2(\rho, s)$ for all $\rho \in (0, 1)$. Then $m_1(s) \le m_2(s)$ (they are equal if and only if the distributions coincide). Let $\rho \in (0, 1)$ be fixed. If $X_1(\rho, s) = X_2(\rho, s)$, then $X_1'(\rho, s) \le X_2'(\rho, s)$. The continuity of X' is sufficient for the remaining part of the proof. ∎

LEMMA 6.2. *Let* X_0 *in* (3.2) *be given. Then there exists* $\lim_{t\to+\infty} m(t) = m_\infty$.

Proof. Let $[a, b]$ be the class limit of $m(t)$. Suppose $a = -1$, i.e., $\liminf_t m(t) = -1$. Markov inequality then shows that the limit distribution is a mass concentrated in -1, and hence $b = -1$. Otherwise, we may assume that $m(t) \in [-1 + \delta,\, 1 - \delta]$ for all $t \ge t_0$ and let p_0 be the mass not concentrated in ± 1 at each time (recall Lemma 4.1). For all $\epsilon > 0$, $p_0 - \epsilon$ mass is in $[-1 + \epsilon,\, 1 - \epsilon]$ at $t = t_0$. Therefore, for all $\rho \in (0, 1)$, $X(\rho, t_0) \in [-1 + \epsilon,\, 1 - \epsilon]$, and $X(\rho, t)$ decays exponentially to $m(t)$ with rate not less than $(\min(\delta, \epsilon))^2$. The large behavior of this process shows three delta masses: the initial two in ± 1 and the remaining one in $m(t)$. Stationary arguments imply the existence of m_∞. ∎

The steady state of the process can now be defined by the following theorem.

THEOREM 6.3. *If* $(1 - p_1)(1 - p_{-1}) < 1$ *(i.e., if there are masses in* ± 1 *at time* $t = 0$*), then* $m_\infty = p_1 - p_{-1}$. *Otherwise, if* $\log\big((1 + X(\rho, 0))/(1 - X(\rho, 0))\big) \in L^1(0, 1)$, *then*

$$m_\infty = \frac{\exp\{T(0)\} - 1}{\exp\{T(0)\} + 1}. \tag{6.1}$$

Proof. The first part is a consequence of Lemma 6.2 and stationary arguments. The second part is a consequence of Lemma 4.2, since

$$\int_0^1 \log\Big(\frac{1 + X(\rho, 0)}{1 - X(\rho, 0)}\Big)\, d\rho = \int_0^1 \log\Big(\frac{1 + X(\rho, t)}{1 - X(\rho, t)}\Big)\, d\rho \underset{t\to\infty}{\longrightarrow} \log\Big(\frac{1 + m_\infty}{1 - m_\infty}\Big)\,,$$

the last limit being true by Lemma 6.2. ∎

Remark. Lemma 6.1 allows us to extend the previous result to cases where at least one of the two functions $\log(1 \pm X(\rho, 0))$ is integrable. If, for example,

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$\log(1+X(\rho,0)) \in L^1(0,1)$ and $\log(1-X(\rho,0)) \notin L^1(0,1)$, if we take $X^{(n)}(\rho,0) = \min\{X(\rho,0), 1-1/n\}$, then $X^{(n)}(\rho,t) \le X(\rho,t)$ for all $t \ge 0$, for all $\rho \in (0,1)$. The monotone convergence theorem states that $\lim_n T^{(n)}(0) = +\infty$, i.e., $\lim_n m_\infty^{(n)} = 1$. Thus, by the monotonicity argument of Lemma 6.1, $m_\infty = 1$.

With this remark in mind, we now show a "counterintuitive" example. Let

$$f_0(x) = \begin{cases} \frac{1-\epsilon}{\epsilon} & \text{if } -1 < x < -1+\epsilon, \\ \frac{1}{1-x}\left(\frac{\epsilon}{1-\epsilon\log(1-x)}\right)^2 & \text{if } 0 < x < 1, \end{cases}$$

and hence

$$F_0(x) = \begin{cases} 0 & \text{if } x < -1, \\ \frac{1-\epsilon}{\epsilon}(1+x) & \text{if } -1 \le x < -1+\epsilon, \\ 1-\epsilon & \text{if } -1+\epsilon \le x < 0, \\ 1 - \frac{\epsilon}{1-\epsilon\log(1-x)} & \text{if } 0 \le x < 1, \\ 1 & \text{if } 1 \le x, \end{cases}$$

which corresponds to

$$X_0(\rho) = \begin{cases} -1 + \frac{\epsilon}{1-\epsilon}\rho & \text{if } 0 < \rho \le 1-\epsilon, \\ 1 - \exp(-\frac{1}{1-\rho} + \frac{1}{\epsilon}) & \text{if } 1-\epsilon < \rho < 1. \end{cases}$$

With this data, $\log(1+X(\rho,0)) \in L^1(0,1)$ but $\log(1-X(\rho,0)) \notin L^1(0,1)$; the $1-\epsilon$ initial mass is close to -1, while the asymptotic solution is δ_1.

7. Numerical examples.

The analysis of the previous section left open the problem of the identification of the steady state in the case of the spreading of opinions. Here results can be achieved only by numerical simulation of the spreading process. To test the numerical method, we will first derive the (explicit) solution to the pure drift linear equation of spreading considered in [15] as the mean field limit of the Sznajd model [20]. This equation reads

$$\frac{\partial f}{\partial t} = \gamma \frac{\partial}{\partial x}\left(x(1-x^2)f\right), \tag{7.1}$$

namely, (1.1) without the presence of the mean $m(t)$. In terms of the quantile function $X(\rho,t)$, equation (7.1) takes the form

$$\frac{\partial X(\rho,t)}{\partial t} = -\gamma X(\rho,t)(1-X^2(\rho,t)). \tag{7.2}$$

Let us set $\gamma = -1$ (spreading), and let $X_0(\rho)$ denote the initial datum. For any given $\rho \in [0,1]$, equation (7.2) is an ordinary differential equation which can be easily integrated to give

$$X(\rho,t) = \frac{X_0(\rho)e^t}{\left(1 - X_0^2(\rho) + X_0^2(\rho)e^{2t}\right)^{1/2}}. \tag{7.3}$$

The asymptotic behavior of (7.2) can be easily deduced from the explicit solution. In fact, the solution converges exponentially in time to -1 if $X_0(\rho) < 0$, while it converges

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to $+1$ if $X_0(\rho) > 0$. Solution (7.3) can be inverted by using the definition of $X(\rho, t)$. Let $F_0(x)$ $x \in \mathcal{I}$ be the initial distribution function; then, since $X_0(F_0(x)) = x$,

$$X(F_0(x), t) = \frac{xe^t}{(1 - x^2 + x^2 e^{2t})^{1/2}}. \tag{7.4}$$

Thus, since the function on the right of (7.4) is increasing with respect to the variable x, we can invert it to obtain

$$X\left(F_0\left(\frac{y}{((1-y^2)e^{2t} + y^2)^{1/2}}\right), t\right) = y. \tag{7.5}$$

Finally, (7.5) implies

$$F(y, t) = F_0\left(\frac{y}{((1-y^2)e^{2t} + y^2)^{1/2}}\right). \tag{7.6}$$

Differentiating with respect to y, we conclude that if $f_0(x)$, $x \in \mathcal{I}$, is an initial density for (7.1), the solution in time is given by

$$f(x, t) = \frac{e^{2t}}{((1-x^2)e^{2t} + x^2)^{3/2}} f_0\left(\frac{x}{((1-x^2)e^{2t} + x^2)^{1/2}}\right). \tag{7.7}$$

The behavior of (7.7) shows the formation of two peaks in correspondence to the extremal points ± 1, while in all other points of the interval $\mathcal{I}$ there is exponential decay to zero.

Using the same procedure as above, we can easily solve the problem in the opposite case of concentration, where $\gamma = 1$. In this case, if $X_0(\rho)$ denote the initial datum,

$$X(\rho, t) = \frac{X_0(\rho)e^{-t}}{(1 - X_0^2(\rho) + X_0^2(\rho)e^{-2t})^{1/2}}. \tag{7.8}$$

The solution now converges exponentially in time to zero, except for ρ values for which $X_0(\rho) = \pm 1$, where it remains constant. In the original formulation, the solution $f(x, t)$ corresponding to the initial density $f_0(x)$, $x \in \mathcal{I}$, is

$$f(x, t) = \frac{e^{-2t}}{((1-x^2)e^{-2t} + x^2)^{3/2}} f_0\left(\frac{x}{((1-x^2)e^{-2t} + x^2)^{1/2}}\right). \tag{7.9}$$

Note that except for $x = 0$, $f(x, t)$ converges exponentially to zero. If $f_0(0) > 0$, the solution shows the formation of a peak in $x = 0$.

We perform numerical simulation for different initial data in the general nonlinear case. First, we assume an initial symmetric datum as a benchmark (see Figure 7.1), where

$$f_0(x) = \begin{cases} c_0(1-x^2)(0.64 - x^2)^{1.3} & \text{if } |x| < 0.8; \\ 0 & \text{otherwise.} \end{cases}$$

In this case we have $m(t) = 0$ for all time t. Then we know the exact solution in order to perform a comparison with numerical results. In Figure 7.1 we show the behavior

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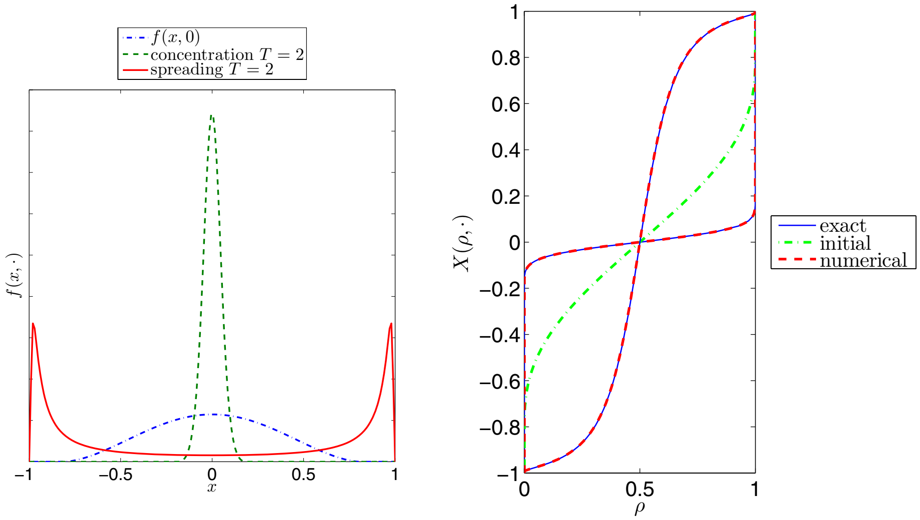

FIG. 7.1. *Benchmark case: evolution of density function (left) and comparison between analytical and numerical solutions for the quantile function (right).*

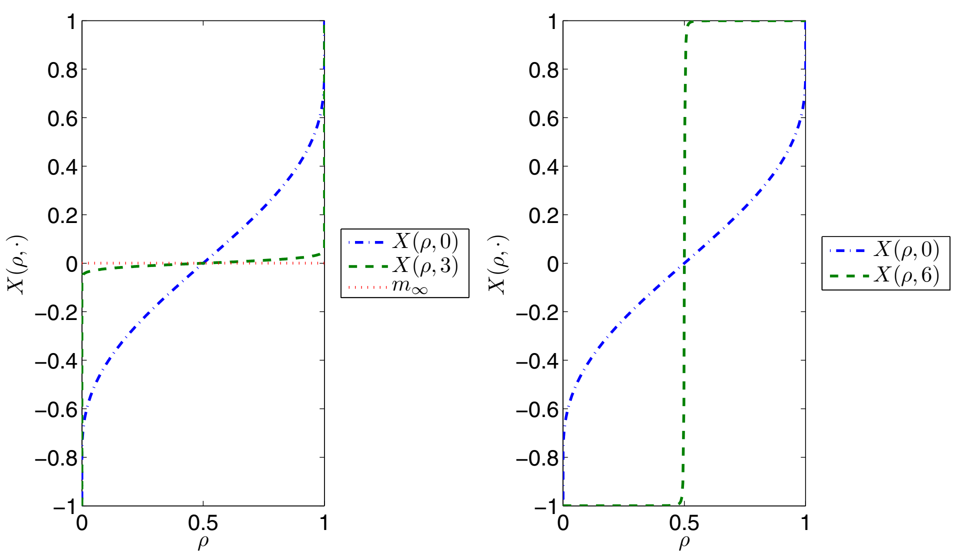

FIG. 7.2. *Plots of the behavior of a numerical solution with symmetric initial data for the concentration case (left) and for the spreading case (right).*

of an exact solution and a numerical one. The last one is obtained by using standard stiff Runge–Kutta methods, which is justified by our theoretical constructive result stated in the proof of Theorem 3.3. As one can see in Figure 7.1 we have a good agreement between the analytical and numerical solutions.

In Figure 7.2 we sketch the plot of the quantile function $X(\rho, t)$ for different times t in both the concentration and the spreading case with the same initial symmetrical data. As expected, in the concentration case, the limit values of all quantiles numerically converge to $m_\infty = 0$, while in the spreading case the quantiles converge to -1

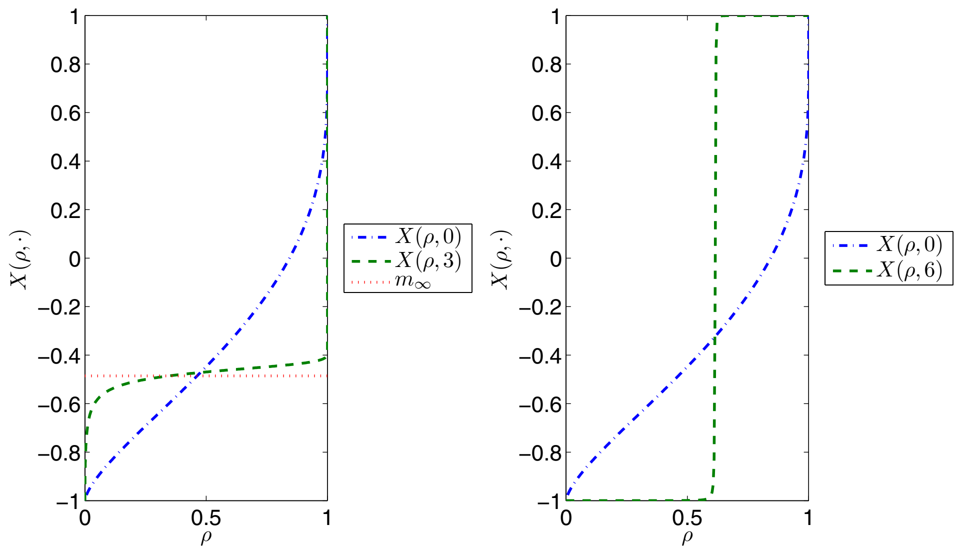

FIG. 7.3. *Plots of the behavior of a numerical solution with nonsymmetric initial data for the concentration case (left) and for the spreading case (right).*

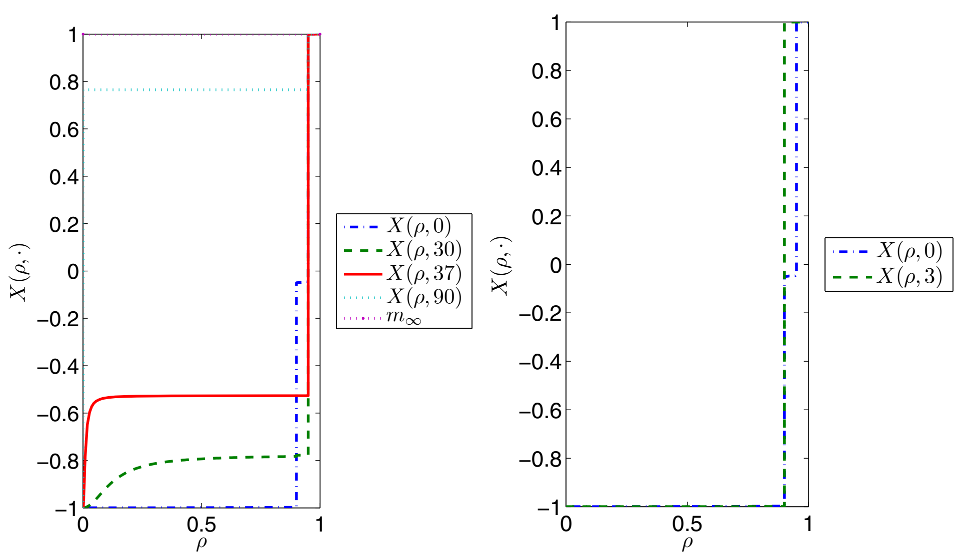

FIG. 7.4. *Plots of the behavior of a numerical "counterintuitive" solution: the concentration case tends to δ_1 (left), while the $1-2\epsilon$ mass goes to -1 in the spreading case (right).*

if $\rho < .5$, and to 1 if $\rho > .5$.

In Figure 7.3 we show an asymmetric case. Now, in the concentration case $m_\infty \neq 0$ given in (6.1) and quantiles converge to it. In the spreading case we can numerically estimate the value p^h_{-1} (see (5.1)) for which $\rho < p^h_{-1}$ implies $X(\rho,t) \to -1$, $\rho > p^h_{-1}$ implies $X(\rho,t) \to 1$.

Finally, in Figure 7.4 we show a numerical "counterintuitive example" analogous to the example given at the end of the previous section. Here, the initial datum con-

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sists of the $(1-2\epsilon)$ mass very close to -1, the ϵ mass close to 0, and the remaining ϵ mass concentrated in $+1$. In the concentration case, $\gamma = 1$, the mass goes asymptotically to $+1$, while in the spreading case, $\gamma = -1$, only 2ϵ goes asymptotically to $+1$. We point out that the spreading case is really counterintuitive. In fact, if we take $X(\rho, T)$, with $T \gg 1$, as solution of the concentration case as initial data for the spreading one, the dynamic is as follows. Initially, the $(1-\epsilon)$ mass decays; then it splits into two parts: the right one goes to $+1$, the left one goes to -1. The initial data can be chosen as close as we want to the distribution δ_1. Then, starting from this data (for which $T(t) = +\infty$; see (6.1)), a $(1-2\epsilon)$ mass will reach -1. We note that the asymptotic state for spreading phenomena seems unpredictable for such concentrated initial data.

Remark. The method we used to solve (7.1) represents in various cases a possible alternative to better known methods (like the method of characteristics) able to reckon the solution to one-dimensional first-order partial differential equations of the form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left(\phi(x) f\right). \tag{7.10}$$

Our analysis is possible in all cases where the ordinary differential equation

$$\frac{dX}{dt} = -\phi(X)$$

is explicitly solvable.

8. Conclusions. We investigated in this paper the spreading and/or the concentration of opinion in an organized society by means of a first-order nonlinear partial differential equation recently introduced in [22]. The presence of the nonlinearity renders it difficult to treat the spreading case analytically, and suitable numerical methods were discussed that are able to capture the large-time behavior of the solution in this case. This work represents a first attempt for a continuous approach to the formation of opinion in a community of agents. More complete models can be obtained by considering in addition the (linear or nonlinear) diffusion, which allows for a continuous steady state distribution function. Related problems in the presence of diffusion are presently under study.

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