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Modeling opinion dynamics: Theoretical analysis and continuous approximation

Juan Pablo Pinasco[a], Viktoriya Semeshenko[b], Pablo Balenzuela[c,*]

[a] Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires and IMAS UBA-CONICET, Av. Cantilo s/n, Pabellón 1, Ciudad Universitaria, 1428, Buenos Aires, Argentina

[b] Instituto Interdisciplinario de Economía Política (IIEP-BAIRES), UBA, CONICET, FCE, Av. Córdoba 2122-2do (C1120AAQ),CABA, Argentina

[c] Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires and Instituto de Física de Buenos Aires (IFIBA), CONICET, Av. Cantilo s/n, Pabellón 1, Ciudad Universitaria, 1428, Buenos Aires, Argentina

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ABSTRACT

Frequently we revise our first opinions after talking over with other individuals because we get convinced. Argumentation is a verbal and social process aimed at convincing. It includes conversation and persuasion and the agreement is reached because the new arguments are incorporated. Given the wide range of opinion formation mathematical approaches, there are however no models of opinion dynamics with nonlocal pair interactions analytically solvable. In this paper we present a novel analytical framework developed to solve the master equations with non-local kernels. For this we used a simple model of opinion formation where individuals tend to get more similar after each interactions, no matter their opinion differences, giving rise to nonlinear differential master equation with non-local terms. Simulation results show an excellent agreement with results obtained by the theoretical estimation.

1. Introduction

In group discussions individuals exchange arguments over a specific subject of conversation, and then selectively either incorporate what they have discovered or at least learn to understand one another better. That is to say, individuals may want to change their own opinions about an issue in order to get closer to or farther from others in the group. These interactions give rise to the formation of different kinds of opinions in a society. At the end of the discussion the group will be characterized either by a so called opinion consensus or coexistence of opinions (fragmentation). The processes of opinion formation and opinion change have always been under the close supervision for modeling. Until now various approaches exist and they all differ in their focus and complexity (see for instance, [1–5]). We recently published a new threshold model of opinion formation [6], in which the opinion change emerges as a consequence of a persuasion interacting dynamics between convinced agents, or between convinced and undecided agents; and a repulsion effect occurs whenever the agents belong to opposite groups. The model has been studied through simulations, and we showed that the system presents a wide spectrum of solutions, as a function of the fraction of undecided individuals and the adjustment in the individual's persuasion after interaction. We achieved to derive the masters equations that govern the process of opinion formation dynamics. These equations, a nonlinear coupled system of first order differential equations of hyperbolic type with nonlocal terms, are driven by two competitive terms representing two ubiquitous mechanisms in opinion formation: agreement and negative influence. They are of special interest for their nontrivial properties but they are very hard to being solved numerical or analytically. There are few models of this type, even for a single equation. For instance, in [3] where only agents with similar opinion can interact, the nonlocal terms involve a small neighbourhood of a given opinion and they simplify them by performing Taylor expansions. With this approach they recover local equations of Fokker-Planck type, but this is only possible in the frame of bounded confidence models and the long range interactions are lost. In [7], the authors deal with a model of opinion formation where nonlocal terms are not simplified, but they involve a coupling between each individual opinion and the mean of the opinions. As far as we know, there are no models of opinion dynamics with nonlocal pair interactions analytically solvable. A logical step then is to face this problem focusing in one of the main mechanism involved in most of the opinion models [8]: persuasion interacting dynamics and the compromise hypothesis. In order to proceed and work out the analytical framework we reduced the

* Corresponding author.
E-mail addresses: jpinasco@dm.uba.ar (J.P. Pinasco), vsemesh@econ.uba.ar (V. Semeshenko), balen@df.uba.ar (P. Balenzuela).

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original model [6] to a single population, where whenever two individuals interact, their opinions get changed by a fixed discrete quantity. We obtain a continuous approximation of the master equation that rule the evolution of the system, and in this case, it is possible to solve it explicitly using a method developed by Li and Toscani [9]. This method permits to find the exact solutions of the continuous approximation of the master equation, which then are compared with numerical simulations. Let us mention that the same idea was applied by Aletti et al. [7] to a different model of opinion dynamics, where a first order equation was derived and the mean opinion of the population appears as a coefficient of the drift. Here, we get a kind of nonlocal Porous Media type equation, which can be thought as a first order hyperbolic equation with a nonlinear, nonlocal flux. This partial differential equation develops a shock at the median of the distribution, and the median value moves toward the mean. We show that the distribution of individual's opinions converges to a Dirac's delta function concentrated at the mean opinion of the initial distribution. Let us mention that introducing the bounded confidence hypothesis and restricting the interaction to sufficiently close agents, the equation converges to a Porous Media equation backward in time similar to the ones appearing in [3]. However, in this case we obtain an ill-posed problem, lacking the continuity with respect to small perturbations of the initial data or the solutions, and this explains why the system is difficult to analyze from both the numerical and theoretical point of view. There exist few theoretical results and numerical methods for these problems, which are currently being under active research. What we observe is that we can obtain an analytical solution that can be useful to solve more complex problems where this dynamics is present, such as for instance [5,6]. The paper is organized as follows. First we present the model and derive the master equations. Then, we derive the solutions, compare them with the numerical model and present some mathematical definitions and theorems. Last, we discuss the results and conclude.

2. Models and methods

Consider the following agent-based model. Let $\{1, \cdots, N\}$ be the agents, and at time $t = 0$ we assign a real number $\sigma(i)$ (where $-\infty < \sigma < \infty$) which represents the opinion of agent i about a certain topic of discussion. The agent's opinion can only change due to pairwise interactions between agents engaged in a discussion.

Given the discrete nature of an argument exchange process, we assume that every time two agents interact, they increase or decrease their opinions by a fixed quantity h, which accounts for the influence of the new argument incorporated by the agent. We assume also that both agents are compromising to reach an agreement. So, if agents i and j interact, and $\sigma(i) < \sigma(j)$, then

$$\begin{aligned}\sigma^*(i) &= \sigma(i) + h,\\ \sigma^*(j) &= \sigma(j) - h.\end{aligned} \tag{1}$$

In this way, the persuasion dynamics is not instantaneous and could be interpreted as a discussion process in which agents get closer in opinions with time.

In order to obtain the master equations of this model, let us subdivide the real line in a family of intervals $\{I_j\}_{j\in\mathbb{Z}}$, of length h, and define:

$$s(j,t) = \frac{\#\{i : \sigma(i,t) \in I_j\}}{N}, \tag{2}$$

for $j \in \mathbb{Z}$, as the density of agents with opinion σ in the intervals I_j. Let us note that, being a finite set of agents, we have $s = 0$ outside some interval $[-M, M]$.

Let us deduce the master equation for the density s. Fixing some characteristic time τ related to the rate of interactions, we have

$$s(j,t+\tau) = s(j,t) + \frac{2}{N}(G(j,t) - L(j,t))$$

,where $G(j,\, t)$ stands for a gain term and $L(j,\, t)$ for a loss term. In a time interval of length τ only two agents change their opinions, and then the proportion of agents s_j increases or decreases by $1/N$. The factor 2 appears since we can choose an agent located at I_j as the first or the second agent in the interaction.

The gain term G is computed as the probability of an interaction between some agent located at I_{j+1} (respectively, I_{j-1}) at time t and another agent located at I_i with $i \le j$ (resp., $i \ge j$). The loss term L is computed as the probability of an interaction between some agent located at I_j and any another agent outside I_j, since in this case there are no changes.

Therefore, for each $j \in \mathbb{Z}$ we have

$$\begin{aligned}\frac{N}{2}\Big(s(j,t+\tau) - s(j,t)\Big) =& G(j,t) - L(j,t)\\ =& s(j+1,t)\sum_{i\le j} s(i,t) + s(j-1,t)\\ & \sum_{i\ge j} s(i,t) - s(j,t)\sum_{i\ne j} s(i,t)\\ =& \Big(s(j+1,t) - s(j,t)\Big)\sum_{i\le j} s(i,t) -\\ & - \Big(s(j,t) - s(j-1,t)\Big)\sum_{i\ge j} s(i,t)\\ & + 2s^2(j,t),\end{aligned} \tag{3}$$

where we have rearranged the series with the same terms in the last step. Let us recall that this equations must be complemented with the initial distribution at time $t = 0$.

The resulting system of equations is easier to study if considering the continuous version. To this end, we introduce a smooth function $u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}_{\ge 0}$ such that,

$$s(j,t) = \int_{I_j} u(x,t)dx.$$

This means that u restricted to the interval I_j behaves like $s(j,\, t)/h$.

Let us observe that, for $x \sim hj$,

$$\begin{aligned}[s(j+1,t) - s(j,t)] =& \frac{h}{h}[s(j+1,t) - s(j,t)]\\ =& h^2\left[\frac{u(x+h,t) - u(x,t)}{h}\right]\\ \approx& h^2\frac{\partial u(x,t)}{\partial x},\\ \sum_{i\le j} s(i,t) =& \frac{h}{h}\sum_{i\le j} s(i,t)\\ =& \frac{1}{h}\sum_{i\le j} s(i,t)h\\ \approx& \int_{-\infty}^{x} u(y,t)dy,\end{aligned}$$

and therefore

$$\Big(s(j+1,t) - s(j,t)\Big)\sum_{i\le j} s(i,t) \approx h^2\frac{\partial u}{\partial x}\int_{-\infty}^{x} u(y,t)dy.$$

Similar formulas hold for the other differences and sums, so for τ and h small, the equation of the continuous model reads:

$$\frac{\tau N}{2h}\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x}\Big(\int_{-\infty}^{x} u(y,t)dy - \int_{x}^{\infty} u(y,t)dy\Big) + 2u^2(x,t)$$
$$= \frac{\partial}{\partial x}\bigg(u(x,t)\Big(\int_{-\infty}^{x} u(y,t)dy - \int_{x}^{\infty} u(y,t)dy\Big)\bigg).$$

Finally, we re-scale times to get rid off the term $\tau N/2h$. We call $s = 2ht/N\tau$, and then

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial t}\frac{dt}{ds} = \frac{\tau N}{2h}\frac{\partial u}{\partial t}, \tag{4}$$

and then (renaming s as t) we obtain the following integro-differential equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\bigg(u(x,t)\Big(\int_{-\infty}^{x} u(y,t)dy - \int_{x}^{\infty} u(y,t)dy\Big)\bigg), \tag{5}$$

together with some initial distribution of opinions at $t = 0$, say

$$u(x,0) = u_0(x).$$

We can see that this equation is a nonlocal, nonlinear, first-order partial differential equation. The nonlinear drift at x involves the difference of the density of agents with opinions located at right and left of x. From the mathematical point of view, it shares some properties with the Burger equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\big(u^2(x,t)\big).$$

Here, the agents point toward the median of their distribution and they will accumulate at their mean, while in Burgers equation singularities appear when fast particles reach positions occupied by slow particles.

Indeed, Eq. (5) is closer to the Porous Media equation, a nonlocal version different from the ones obtained recently in [10], and it shares properties like mass preservation and finite time propagation.

In order to see this, we can proceed heuristically assuming that the interactions occurs only among agents with similar opinions. If we add a mechanism like the bounded confidence hypothesis, and fix some small and positive parameter d, we get the same update as in Eq. (1) only for

$$|\sigma(i) - \sigma(j)| \le d.$$

We can truncate the integrals in the drift term, since long range interactions are prohibited. Assuming that $U(y, t)$ is a primitive of $u(y, t)$ in the spatial variable, Barrow's rule together with Taylor expansions give

$$\begin{aligned}\int_{x-d}^{x} u(y,t)dy - \int_{x}^{x+d} u(y,t)dy &= U(x,t) - U(x-d,t) \\ &\quad - U(x+d) + U(x,t) \\ &\approx -d^2\frac{\partial^2 U(x,t)}{\partial x^2} \\ &= -d^2\frac{\partial u(x,t)}{\partial x}\end{aligned}$$

So, we get

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= -d^2\frac{\partial}{\partial x}\bigg(u(x,t)\frac{\partial u(x,t)}{\partial x}\bigg) \\ &= -\frac{d^2}{2}\frac{\partial^2}{\partial x^2}\big(u^2(x,t)\big)\end{aligned}$$

Moreover, we observe that this is a Porous Media equation reversed in time, and in this case we expect a finite time blow up, namely, the solution growth unboundedly in finite time. This result was proved in the 70th by Levine and Payne [11], and it is known that this equation is ill-posed, without continuous dependence on the initial data.

Let us observe that a similar equation, with a different scaling on d, was presented by Deffuant et al. in [3], for a different model where the agents opinions are updated following some weighted mean of their opinions. However, neither theoretical analysis, nor numerical approximations were provided for their model.

In the rest of the paper we study the existence of solutions for Eq. (5), and we show how to solve it explicitly following the ideas of Li and Toscani in [9] Finally, we show that the population reaches consensus in finite time, and the opinions converge to the mean opinion of the initial distribution.

2.1. Existence of solutions

By a classical solution we understand $u \in C^{1,1}(\mathbb{R}\times(0,\infty))$ satisfying the differential equation and

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = u_0(x).$$

However, let us observe that we expect a measure u as a solution, not necessarily a differentiable function. So, we need to introduce a notion of a weak solution.

Definition 1. Given the following equation,

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}[u(x,t)G(t,x,F(x,t))],$$

with initial condition $u(x,0) = u_0(x)$, where $F(x) = \int_{-\infty}^{x} u(y,t)dy$ is the cumulative distribution function associated to the density u. We say that $u \in C^1((0,\infty), L^1)$ is a weak solution if $u(x,0) = u_0(x)$, and

$$\frac{d}{dt}\int_{-\infty}^{\infty} h(x)u(x,t)dx = -\int_{-\infty}^{\infty} h'(x)u(x,t)G(t,x,F(x,t))dx \tag{6}$$

for any $h \in C_0^1(\mathbb{R})$.

Condition $u \in C^1((0,T), L^1(\mathbb{R}))$ means that, for each $t \in (0, T)$, the function $u(\,\cdot, t)$ is an integrable function on $\mathbb{R}$, and this assignation is C^1 in the variable t. Observe that weak solutions are not necessarily differentiable in the classic sense in the variable x.

In what follows we are going to solve exactly Eq. (5) following a method introduced in [9] in order to deal with granular flows. Giving $F(x) = \int_{-\infty}^{x} u(y,t)dy$, we will show that we can re-write Eq. (5) as:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\Big(u(x,t)[2F(x) - 1]\Big). \tag{7}$$

To this end, the method starts assigning a new variable for the cumulative function,

$$\rho = F(x) = \int_{-\infty}^{x} u(y,t)dy. \tag{8}$$

When $u > 0$ for any x and t, we can introduce the inverse function $\mathbb{X}(\rho,t) = F^{-1}(\rho,t)$. In other terms,

$$\rho = \int_{-\infty}^{\mathbb{X}} u(y,t)dy.$$

However, since u can be zero in some interval, it is convenient to define

$$\mathbb{X}(\rho,t) = \inf\{x : F(x,t) \ge \rho\}.$$

With this change of variables, Eq. (7) becomes an infinite system of ordinary differential equations,

$$\frac{d\mathbb{X}(\rho,t)}{dt} = 1 - 2\rho, \tag{9}$$

one for each value of ρ, which can be solved explicitly as:

$$\mathbb{X}(\rho,t) = (1-2\rho)t + \mathbb{X}(\rho,0), \tag{10}$$

where $\mathbb{X}(\rho, 0)$ is obtained from the initial datum, that is,

$$\rho = \int_{-\infty}^{\mathbb{X}(\rho,0)} u_0(y)dy.$$

We have obtained an implicit function for ρ, and for each value of t, we can obtain it in terms of $\mathbb{X}$, and since $\rho = F(\mathbb{X})$, we recover the solution u as

$$u(x,t) = \partial_{\mathbb{X}} \rho(\mathbb{X}, t).$$

Let us prove the previous claims. Let us start with the following Lemma:

Lemma 1. *Let $h \in C_0^1(\mathbb{R})$. Then*

$$\int_{-\infty}^{\infty} h(x)u(x,t)dx = \int_0^1 h(\mathbb{X}(\rho,t))d\rho.$$

Proof. Just change $x = \mathbb{X}$, and formally

$$dx = \frac{\partial \mathbb{X}}{\partial \rho} d\rho = \frac{\partial F^{-1}}{\partial \rho} d\rho = \frac{1}{u(x,t)} d\rho.$$

Hence,

$$\int_{-\infty}^{\infty} h(x)u(x,t)dx = \int_0^1 h(\mathbb{X}(\rho,t))d\rho,$$

and the proof is finished. □

We are ready to prove the main result:

Theorem 1. *Let $\mathbb{X}(\rho, t)$ be a solution of*

$$\begin{cases} \partial_t \mathbb{X}(\rho,t) = 1 - 2\rho, & t \in (0,T), \\ \mathbb{X}(\rho,0) = \inf\left\{x : \int_{-\infty}^{x} u_0(y)dy \geq \rho\right\}. \end{cases} \tag{11}$$

Then there exists a weak solution of

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \frac{\partial}{\partial x}[u(x,t)(2F(x) - 1)] & (x,t) \in \mathbb{R} \times (0,T) \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases}$$

with $\int_{\mathbb{R}} u(x,t)dx = 1$ for $0 < t < T$.

Proof. Take any function $h \in C_0^1(\mathbb{R})$, and then, using Lemma 1

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^{\infty} h(x)u(x,t)dx =& \frac{d}{dt}\int_0^1 h(\mathbb{X}(\rho,t))d\rho \\ =& \int_0^1 h'(\mathbb{X}(\rho,t))\frac{d\mathbb{X}(\rho,t)}{dt}d\rho. \end{aligned}$$

On the other hand,

$$-\int_{-\infty}^{\infty} h'(x)u(x,t)(2F(x)-1)dx = -\int_0^1 h'(\mathbb{X}(\rho,t))(2F(x)-1)d\rho.$$

We get that condition (6) is satisfied if

$$\int_0^1 h'(\mathbb{X}(\rho,t))\frac{d\mathbb{X}(\rho,t)}{dt}d\rho = -\int_0^1 h'(\mathbb{X}(\rho,t))(2\rho - 1)d\rho,$$

or, equivalently,

$$\int_0^1 h'(\mathbb{X}(\rho,t))\left[\frac{d\mathbb{X}(\rho,t)}{dt} + 2\rho - 1\right]d\rho = 0,$$

which trivially holds if $\mathbb{X}(\rho, t)$ is a solution of Eq. (11).
The proof is finished. □

With an extra effort, it could be proved also that the existence of a weak solution $u(x, t)$ implies the existence of solution $\mathbb{X}(\rho, t)$. Here, the simplicity of Eq. (11) makes unnecessary such equivalence.

2.2. Convergence to the mean

Observe that

$$\partial_t \mathbb{X}(\rho,t) = 1 - 2\rho$$

is positive for $\rho < 1/2$, and negative for $\rho > 1/2$. Therefore, since $\rho = 1/2$ gives the median of the distribution, we get that

$$\mathbb{X}(\rho,t) = \inf\left\{x : \int_{-\infty}^{x} v(y,t)dy = \rho\right\}$$

strictly increases for $0 < \rho < 1/2$, and decreases for $1/2 < \rho < 1$. Hence, there exists some c_0 such that

$$\lim_{t\to T} \mathbb{X}(\rho,t) = c_0,$$

which implies that $u(x,t) \to \delta_{c_0}$ as $t \to T$. In other words, the distribution of opinions concentrates on the mean value of the initial distribution and the population reaches consensus.

2.3. Examples with different initial conditions

Let's solve explicitly the theoretical asymptotic values of $u(x, t)$ for two different initial conditions: a symmetrical one ($u(x, 0) = cte$) and an asymmetrical ($u(x, 0) = 2x$). We will compare them with computer simulations in both cases.

2.3.1. Constant opinion's initial distribution

Let $u_0(x) = \frac{1}{2}\chi_{[-1,1]}(x)$, where $\chi_{[a,\,b]}(x)$ is equal to one if $x \in [a, b]$ and zero outside. Then,

$$\mathbb{X}(\rho,0) = \inf\left\{x : \int_{-1}^{x} dy \geq \rho\right\} = 2\rho - 1.$$

So,

$$\mathbb{X}(\rho,t) = (1-2\rho)t + \mathbb{X}(\rho,0) = (1-2\rho)t + 2\rho - 1,$$

and inverting, since $0 \leq \rho \leq 1$,

$$\rho = \left(\frac{x - t + 1}{2 - 2t}\right)\chi_{[t-1,1-t]}(x)$$

Finally, for $0 \leq t < 1$, since $u \geq 0$,

$$u(x,t) = \partial_x\left(\frac{x-t+1}{2-2t}\chi_{[t-1,1-t]}(x)\right) = (2-2t)^{-1}\chi_{[t-1,1-t]}(x).$$

Observe that the solution blows up when t reaches 1, and

$$\lim_{t\to 1} u(x,t) = \delta_0.$$

and

$$\int_{-1}^{1} xu(x,t)dx = 0,$$

2.3.2. Linear opinion's initial distribution

Let $u(x, 0) = \frac{x+1}{2}\chi_{[-1,1]}(x)$. Then,

$$\begin{aligned} F(x,0) =& \int_{-\infty}^{x} \frac{y+1}{2}\chi_{[-1,1]}(y)dy \\ =& \int_{-1}^{x} \frac{y+1}{2}\chi_{[-1,1]}(y)dy \\ =& \frac{(x+1)^2}{4}\chi_{[-1,1]}(x). \end{aligned}$$

Hence, $\mathbb{X}(\rho, t) = F^{-1}(\rho)$ and we have

$$\mathbb{X}(\rho,0) = (2\sqrt{\rho} - 1)\chi_{[0,1]}(\rho).$$

A direct computation gives

$$\mathbb{X}(\rho,t) = (1-2\rho)t + (2\sqrt{\rho} - 1), \qquad 0 \leq \rho \leq 1,$$

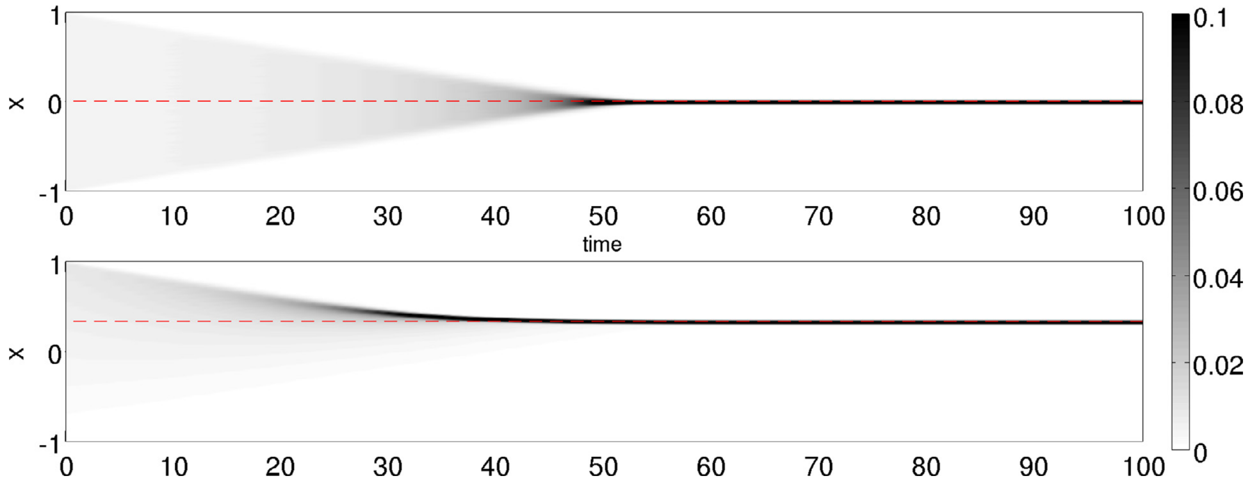

Fig. 1. Time evolution of opinion's density. Time evolution of an agent's based model governed by Eq. (1), for a population of N=10,000 agents and $h = 0.01$, and two initial distributions of opinions: Uniform (Upper panel) and Linear (Down panel).

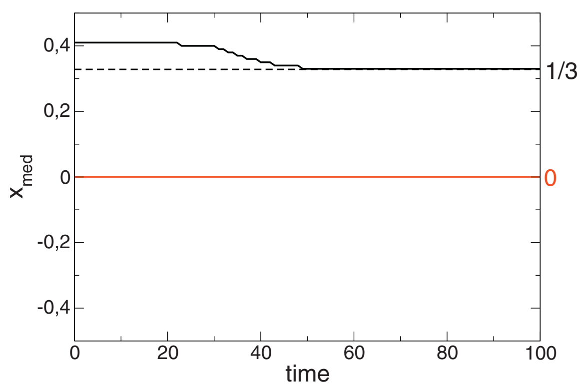

Fig. 2. Median and average value dynamics. Time evolution of the median (solid lines) and the average value of the distribution (dashed lines) for a population of N=10,000 agents and $h = 0.01$, and two initial distributions of opinions: Uniform (red lines) and Linear (black lines). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

and we recover $u(x,t) = \partial_x \mathbb{X}^{-1}(x)$ from

$$x = (1-2\rho)t + 2\sqrt{\rho} - 1, \qquad 0 \le \rho \le 1,$$

by computing the inverse function as before.

Finally, observe that, for any t,

$$\int_{-1}^{1} x u(x,t)dx = \int_0^1 \mathbb{X}(\rho,t)d\rho = \frac{1}{3},$$

and so $u(x, t) \to \delta_{1/3}$.

In order to compare these theoretical results with numerical simulations of the agent based model sketched above, we implement a system of N=10,000 agents, which follows the dynamics given by Eq. (1) with $h = 0.01$. Initial opinions are distributed in $[-1, 1]$. In this numerical implementation, one time step corresponds to N interactions.

In Fig. 1 we can observe the dynamics of $u(x, t)$ obtained for the two mentioned initial conditions $u(x, 0)$ (uniform and linear), and in Fig. 2 the dynamics of the medians. We have to take into account that the relation between the re-scaled time used to derive the analytical solution and the time used in simulations differ by a factor $(2h)/(N\tau)$, with $\tau = 1/N$. This factor makes that the theoretical re-scaled time corresponds to the time used in simulations divided by 50. We can observe how the distributions and the median converge to the predicted theoretical values 0 and 1/3, at times predicted in theory ($t = 50$, rescaled is $T = 1$) showing the perfect agreement between theory and simulations.

2.4. Final remarks

It is possible to extend the previous model including some repulsion effect, and in this case the difference between the agents opinions can increase after the interaction. To this end, we can fix two probabilities p and q determining if they tend to agree or not. However, this introduces a diffusive process, and a second order differential equation appears,

$$\frac{\partial u(x,t)}{\partial t} = a_1(p,q)\frac{\partial^2 u(x,t)}{\partial x^2} - a_2(p,q)\frac{\partial}{\partial x}\left(u(x,t)\left(\int_{-\infty}^{x} u(y,t)dy - \int_{x}^{\infty} u(y,t)dy\right)\right),$$

where a_1 and a_2 are coefficients depending on the rates of positive and negatives interactions.

This equation cannot be studied with the techniques above, and interesting phenomena appears due to the competition between p and q. This equation resembles the one obtained in [12],

$$\frac{\partial u(x,t)}{\partial t} = a\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^2 u^2(x,t)}{\partial x^2},$$

where free thinking of the agents generates the diffusive term, and prevents the formation of consensus for a big enough.

3. Conclusions

On a previous work [6] we have developed a new threshold model of opinion formation, in which the opinion change emerges as a consequence of a persuasion interacting dynamics between convinced agents, or between convinced and undecided agents; and a repulsion effect occurs whenever the agents belong to opposite groups.

We achieved to derive the masters equations that govern the process of opinion formation dynamics, but they are very hard to being solved analytically given the nonlinearities, the non-local interactions and the coupling between populations of three different groups.

In this work, we developed an analytical framework where we have reduced the complexity of the original model [6] to a single population, where whenever two individuals interact, their opinions get changed by a fixed discrete quantity. We have obtained a continuous approximation of the master equation that rule the evolution of the system, and in this case, we solved it explicitly by using a method developed in [9].

In particular, we have made the explicit computations for two kind of initial opinion distributions: a symmetrical one (uniform) and an asymmetrical (linear). In both cases, the agreement with numerical simulations of the agent based model described in Eq. (1) are remarkable: the median of the distribution converge to the average (0 and 1/3 respectively) in the time predicted in the theoretical calculations.

Given the importance of persuasion dynamics in opinion models, we believe that this theoretical framework for will help to develop analytical solutions in more complex models, as for instance those developed in [5] and [6], where no bounded confidence constraint is imposed.

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