Scaling limits for continuous opinion dynamics systems

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Abstract—A class of large-scale stochastic discrete-time continuous-opinion dynamical systems is analyzed. Agents have pairwise random interactions in which their vector-valued opinions are updated to a weighted average of their current values. The intensity of the interactions is allowed to depend on the agents' opinions themselves through an interaction kernel. This class of models includes as a special case the bounded-confidence opinion dynamics models recently introduced by Deffuant et al., in which agents interact only when their opinions differ by less than a given threshold, as well as more general interaction kernels. It is shown that, in the limit as the population size increases, upon a proper rescaling of the time index, the trajectories of such stochastic processes concentrate, at an exponential rate, around the solution of a measure-valued differential equation. The asymptotic properties of the solution of such a differential equation are then studied, and convergence is proven to a convex combination of delta measures whose number depends on the interaction kernel.

I. INTRODUCTION

Opinion dynamics systems have recently attracted a considerable amount of attention from the research community. In these models, agents, belonging to a large population, are assumed to interact according to very simple local rules. The interest is in the emerging global behavior of the system. While models where the opinions are binary-, or, more generally, finite-valued, have been successfully studied within the framework of interacting particle systems [8], the last decade has witnessed an increasing interest for continuous opinion dynamics systems. This is motivated primarily by social and economic networks, in which opinions are often better modeled by continuous rather than discrete quantities, as well as by engineered multi-agent systems, where opinions usually represent positions in space or velocities. In continuous opinion dynamics it is usually assumed that each agent updates his vector-valued opinion to a convex combination of a small number of interacting agents' values.

In the present paper, we shall study a class of stochastic opinion dynamics systems in which, at each discrete time instant, a random pair of agents interact by updating their opinion to a weighted average of their current values. The probability of effective interaction between two agents will be assumed to depend on the current value of the agents' opinion through an interaction kernel. This generalizes the Deffuant-Weisbuch model of bounded confidence opinion dynamics first introduced in [5]. In the latter, interactions occur only when the agents' opinions differ by less than a certain threshold. For

this model, in scalar opinion case, it was proven in [9] that the system converges to a certain number of opinion clusters, separated by a distance not smaller than the confidence threshold itself. Similar results were observed numerically in [3], and proved analytically for an analogous deterministic model due to Krause [7] in [6], [4], [10].

The main contributions of the present paper concern the behavior of this system in the limit of large population size n. We shall show that, as n increases, a properly time-rescaled version of the discrete stochastic system concentrates -at an exponential rate- around the solution of a measure-valued differential equation. We shall prove the well-posedness (i.e. existence and uniqueness of a solution) of such equation, and then study the asymptotics of its solution, showing that it converges to a convex combination of Dirac's delta measures. Such deltas correspond to opinion clusters, and their number and mutual distances depend on the interaction kernel. While a similar differential equation for probability densities was non-rigorously introduced in [3] for the case of the Deffuant et al.'s model, no rigorous analysis of it has been proposed so far in the literature, to the best of our knowledge, and, most importantly, we are not aware of any proof of concentration of the discrete-time finite-population around its solution.

The remainder of the paper is organized as follows. After introducing the necessary notation, we shall introduce the class of discrete-time stochastic models of continuous opinion dynamics in Sect. II. In Sect. III, we shall pass from the agent-based model to the density-based one, the latter consisting in a discrete-time stochastic process in the space of probability measures over the opinion space. An intuitive interpretation of such a process as a noisy Euler discretization of a measure-valued differential equation will then be provided. In Sect. IV, we shall first prove the well-posedness of such a differential equation (Theorem 1), and then investigate the asymptotic behavior of its solution (Theorem 2). Finally, in Sect. V, we shall prove that, upon properly rescaling the time index, stochastic process corresponding to the the density-based model concentrates around the solution of the differential equation, as the population size grows.

While providing rigorous definitions and statements, we shall just sketch the proofs of our results, and address the reader interested in the details to a forthcoming full version of the paper.

II. PROBLEM FORMULATION

We start by establishing some notation, to be used throughout the paper. As usual, $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{Z}^+$ will denote the sets of reals, natural numbers, and nonnegative integers, respectively. The entries of a vector x, indexed by a finite alphabet $\mathcal{I}$, will be denoted by $x^{(i)}$, whereas $x^{(-i)}$ will stay for the vector

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of all the entries of x but the i-th. If x and y belong to $\mathbb{R}^d$, for some $d \in \mathbb{N}$, $||x - y||$ will denote their Euclidean distance. The indicator function of a set A will be denoted by $\mathbb{1}_A$, with $\mathbb{1}_A(x) = 1$ if $x \in A$, $\mathbb{1}_A(x) = 0$ if $x \notin A$. We shall denote by $\mathcal{C}_b^0(\mathbb{R}^d)$ the space of real-valued, continuous, bounded functions over $\mathbb{R}^d$, and by $\mathcal{P}(\mathbb{R}^d)$ the space of probability measures over A. For a measure $\mu \in \mathcal{P}(\mathbb{R}^d)$, and a test function $\varphi \in \mathcal{C}_b^0(\mathbb{R}^d)$, we shall write $\langle \mu, \varphi \rangle$ for the integral $\int \varphi(x) \mathrm{d}\mu(x)$, with the convention that, whenever not explicitly indicated, the domain of integration is assumed to be the entire space $\mathbb{R}^d$. For $x \in \mathbb{R}^d$, $\delta_x \in \mathcal{P}(\mathbb{R}^d)$ will be the Dirac delta measure centered in x, defined by $\langle \delta_x, \varphi \rangle = \varphi(x)$ for all $\varphi \in \mathcal{C}_b^0(\mathbb{R}^d)$. Finally, for $\mathcal{X} \subseteq \mathbb{R}^d$, the space of probability measures μ whose support is contained in $\mathcal{X}$ will be denoted by $\mathcal{P}(\mathcal{X})$.

We shall study the following family of discrete time stochastic models of continuous opinion dynamics. Agents belong to a finite population $\mathcal{A}_n$ of cardinality $|\mathcal{A}_n| = n$. Each agent $a \in \mathcal{A}_n$ starts with an initial opinion $X_0^{(a)} \in \mathbb{R}^d$. We shall assume $X_0 := \{X_0^{(a)} : a \in \mathcal{A}_n\}$ to be a family of independent and identically distributed (i.i.d.) random variables (r.v.s), the law of each $X_0^{(a)}$ being given by some probability measure $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$. The opinion profile $X_k := \{X_k^{(a)} | a \in \mathcal{A}_n\}$ is then updated according to the following stochastic rule. At each subsequent time instant $k \in \mathbb{N}$, two agents, a and b, are independently sampled from a uniform distribution over $\mathcal{A}_n$. Then, with some probability $\kappa(X_{k-1}^{(a)}, X_{k-1}^{(b)})$, possibly depending on their current opinions, agent a updates its opinion to a weighted average of its current opinion and that of agent b, by setting $X_k^{(a)} = (1 - \omega)X_{k-1}^{(a)} + \omega X_{k-1}^{(b)}$. The parameter $\omega \in [0, 1]$ has to be interpreted as a measure of the confidence that each agent puts on the opinion of other agents.

Formally, we shall assume that the stochastic process is defined on some filtrated probability space $\left(\Omega_n, \{\mathcal{F}_k^n\}_{k \in \mathbb{Z}_+}, \mathbb{P}_n\right)$, such that X_k is $\mathcal{F}_k^n$-measurable for all $k \in \mathbb{Z}_+$. Then

$$\mathbb{P}_n(X_0 \in A) = \mu_0^{\otimes n}(A),$$

for all $A \subseteq \mathbb{R}^d$ measurable, and, conditioned on the past history $\mathcal{F}_{k-1}^n$, for every $a, b \in \mathcal{A}_n$

$$X_k^{(a)} = (1 - \omega)X_{k-1}^{(a)} + \omega X_{k-1}^{(b)}, \qquad X_k^{(-a)} = X_{k-1}^{(-a)}, \quad (1)$$

with probability $\kappa(X_{k-1}^{(a)}, X_{k-1}^{(b)})$, while

$$X_k = X_{k-1} \quad \text{w.p.} \quad 1 - \sum_{a,b \in \mathcal{A}_n} \kappa(X_{k-1}^{(a)}, X_{k-1}^{(b)}). \quad (2)$$

We shall assume the interaction kernel $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to [0, 1]$ to be measurable, lower semicontinuous, and symmetric in its arguments, i.e. such that

$$\kappa(x, y) = \kappa(y, x), \qquad \forall x, y \in \mathbb{R}^d.$$

Further, we shall assume the initial probability law μ_0 to be compact supported, and denote by $\mathcal{X} \subseteq \mathbb{R}^d$ the convex closure of the support of μ_0.

Remark 1. The models considered in the cited literature usually assume the interaction to be symmetric in that, at each

$k \in \mathbb{N}$, not only does agent a updates its opinion as above, but also so does agent b by setting $X_k^{(b)} = (1 - \omega)X_{k-1}^{(b)} + \omega X_{k-1}^{(a)}$. This symmetric model may be more suitable in certain applicative contexts, the asymmetric one in some others. However, while for finite population sizes some of the properties of the two models differ (for example, in the symmetric model the average of the opinions is preserved, while this is not necessarily the case for the asymmetric model), all the results and proofs of this paper hold as well, with minor changes, for the symmetric model.

We end this section by introducing three explicit examples of interaction kernel.

Example 1. *Assume the interaction kernel $\kappa(x, y)$ is constantly equal to 1. Then, the above described system reduces to the standard asymmetric gossip on the complete graph with n agents.*

Example 2. *For some threshold value $R > 0$, let*

$$\kappa(x, y) := \mathbb{1}_{[0,R]}(||x - y||).$$

Then, our model reduces to the Deffuant-Weisbuch model of bounded confidence opinion dynamics.

Example 3. *Assume that κ is a Gaussian kernel, namely that*

$$\kappa(x, y) := \exp(-||x - y||^2 / \sigma^2),$$

for some $\sigma > 0$.

III. FROM AGENT-BASED TO DENSITY BASED MODELS

As our main interest is in the global behavior of the opinion dynamics system, rather than on that of the single agents' opinions, it turns out to be convenient to undertake an Eulerian approach, and to study the evolution of the empirical densities of the agents' opinions. Formally, this is accomplished by considering the sequence of random probability measures

$$M_k^n := \frac{1}{n} \sum_{a \in \mathcal{A}_n} \delta_{X_k^{(a)}} \in \mathcal{P}(\mathbb{R}^d), \qquad k \in \mathbb{Z}^+.$$

Observe that, for every measurable $A \subseteq \mathbb{R}^d$,

$$M_k^n(A) = \frac{1}{n} \sum_{a \in \mathcal{A}_n} \mathbb{1}_A(X_k^{(a)}),$$

is nothing but the fraction of agents whose opinion at time k belongs to A.

It turns out that the Markovian dynamics described by the updates (1) and (2), translate into a Markovian dynamics for the opinion density process $\{M_k^n\}$ which is described below. For $\mu \in \mathcal{P}(\mathbb{R}^d)$, and $\varphi \in \mathcal{C}_b^0(\mathbb{R}^d)$, define

$$\langle H(\mu), \varphi \rangle := \iint (\varphi((1 - \omega)x + \omega y) - \varphi(x)) \kappa(x, y) \mathrm{d}\mu(x) \mathrm{d}\mu(y). \quad (3)$$

Then, for all $k \in \mathbb{Z}^+$,

$$\langle M_{k+1}^n, \varphi \rangle - \langle M_k^n, \varphi \rangle = \frac{1}{n} \left(\langle H, \varphi \rangle + \langle \Delta_{k+1}^n, \varphi \rangle \right), \quad (4)$$

where the random variable $\langle \Delta_{k+1}^n, \varphi \rangle$ satisfies, for all $k \in \mathbb{Z}^+$,

$$\mathbb{E}\left[\langle \Delta_{k+1}^n, \varphi \rangle | \mathcal{F}_k^n\right] = 0, \qquad |\langle \Delta_{k+1}^n, \varphi \rangle| \leq ||\varphi||_\infty. \quad (5)$$

Equation (5) means that $\{\langle \Delta_k^n, \varphi \rangle : k \in \mathbb{N}\}$ is a sequence of bounded martingale differences, which can be thought as 'noise'. This suggests to think of the equation (4) as a noisy discretization, or Euler approximation in the numerical analysis language, of the the measure-valued ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} \mu_t = H(\mu_t). \quad (6)$$

with stepsize $1/n$. More precisely, one may conjecture that, upon rescaling the time index by $t = k/n$, the discrete time stochastic process (M_k^n) should converge, in the limit of the population size n going to infinity, to a solution $\{\mu_t : t \in [0, +\infty)\}$ of the ODE (6) with initial condition μ_0. Such an intuition lies at the basis of the so called mean field approach of statistical physics, where the differential equation (6) is usually referred to as the master equation.

The conjecture above will be formalized and proved to be true in the following sections. In Sect. IV, in particular, we shall first define what it is meant by a solution of (6), and then prove that for every compact-supported initial condition $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ there exists a unique solution $\{\mu_t : t \in [0, +\infty)\}$ of (6) with initial value μ_0. Then, we shall investigate the asymptotics of the solutions of the ODE (6). In Sect. V, we shall prove that a linearly interpolated, and properly rescaled in time, version of the discrete time stochastic process $\{M_k^n\}$, concentrates around such a solution $\{\mu_t\}$.

IV. EXISTENCE, UNIQUENESS, AND ASYMPTOTICS OF THE MEASURE-VALUED ODE PROBLEMS

In this section, we shall prove the well-posedness of the ODE (6), and then analyze the asymptotics of its solutions. To start with, we formalize what it is meant by a solution of (6).

Definition 1. *A family $\{\mu_t : t \in [0, +\infty)\}$ is a solution of ODE if, for every test function $\varphi \in \mathcal{C}_b^0(\mathbb{R}^d)$, the real-valued map*

$$t \longmapsto \langle \mu_t, \varphi \rangle, \qquad t \in [0, +\infty),$$

is absolutely continuous and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mu_t, \varphi \rangle = \langle H(\mu_t), \varphi \rangle,$$

for almost every $t \in (0, +\infty)$.

Before proving the existence of a solution of the differential equation (6), it is worth noting two conservation properties such a solution is going to enjoy. By taking $\varphi = \mathbb{1}_{\mathbb{R}^d}$ we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\mu_t(x) = 0,$$

i.e. the total mass is preserved. On the other hand, it follows from the symmetry of κ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int x \mathrm{d}\mu_t(x) = \omega \int (x - y)\kappa(x, y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) = 0,$$

i.e. the first moment is preserved.

The following result states the well-posedness of the ODE (6) with compact-supported initial value μ_0.

Theorem 1. *Let $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ be supported in a compact convex set $\mathcal{X} \subseteq \mathbb{R}^d$. Let κ be a piecewise-continuous, symmetric interaction kernel, as in Sect. II, and let H be defined as in (3). Then, there exists a unique solution of (6) with initial value μ_0. Moreover, for every $t \in [0, +\infty)$, the support of μ_t is contained in $\mathcal{X}$.*

Proof: The proof is based on a standard contraction argument in the Banach space of continuous signed-measure-valued curves. For this, the essential ingredients are the Lipschitzianity of H in the total variation distance

$$||\mu(A) - \nu(A)||_{TV} := \sup\left\{\mu(A) - \nu(A) : A \subseteq \mathbb{R}^d \text{ meas.}\right\}. \quad (7)$$

The second part of the claim follows from the easily verifiable fact that the support of $H(\mu)$ is contained in the convex closure of the support of μ. ∎

Now, we proceed to study the asymptotic properties of the solutions of the ODE (6). The following result guarantees the convergence of any such a solution to an asymptotic measure $\mu_\infty \in \mathcal{P}(\mathbb{R}^d)$. Here, convergence in $\mathcal{P}(\mathbb{R}^d)$ is intended to hold in the weak sense of probability measures, i.e. we shall say that $\lim_{t\to+\infty} \mu_t = \mu$ in $\mathcal{P}(\mathbb{R}^d)$ if

$$\lim_{t\to+\infty} \langle \mu_t, \varphi \rangle = \langle \mu, \varphi \rangle, \qquad \forall \varphi \in \mathcal{C}_b^0(\mathbb{R}^d). \quad (8)$$

Theorem 2. *Let $\{\mu_t\}$ be the solution of (6) corresponding to a compact-supported initial value μ_0. Then, there exists $\mu_\infty \in \mathcal{P}(\mathbb{R}^d)$ such that*

$$\lim_{t\to+\infty} \mu_t = \mu_\infty, \qquad \text{in } \mathcal{P}(\mathbb{R}^d).$$

Moreover, if the interaction kernel is such that, for some $R \in (0, +\infty)$ (respectively, $R = +\infty$),

$$\kappa(x, y) > 0, \qquad \forall x, y : ||x - y|| < R, \quad (9)$$

then μ_∞ is a convex combination of a finite number of Dirac's deltas centered in points whose inter-distance is not less than R (respectively $\mu_\infty = \delta_{x_0}$).

Proof: The core idea consists in studying the evolution of the second moment

$$m_t^{(2)} := \int_{\mathbb{R}^d} ||x||^2 \mathrm{d}\mu_t(x).$$

Using the fact that $\{\mu_t\}$ is a solution of (6), and the symmetry of the interaction kernel κ, one finds that

$$\frac{\mathrm{d}}{\mathrm{d}t}m_t^{(2)} = -\omega(1-\omega)\int\int ||x - y||^2 \kappa(x, y)\mathrm{d}\mu_t(x)\mathrm{d}\mu_t(y) \quad (10)$$

is always nonpositive. Hence, $m_t^{(2)}$ is nonincreasing, and therefore convergent. As in [4], this fact is used to show convergence of $\{\mu_t\}$ in the sense of distributions first, and then, by tightness, in $\mathcal{P}(\mathbb{R}^d)$.

In order to prove the second part of the claim, assume by contradiction that $x^*, y^* \in \text{supp}(\mu_\infty)$ and $||x^* - y^*|| < R$. Then $\kappa(x^*, y^*) > 0$, and, since κ is lower semicontinuous, there exists neighborhoods A and B are of x^* and y^*, respectively, such that $\kappa(x, y) > 0$ for $x \in A$ and $y \in B$. Then,

$$\int\int ||x - y||^2 \kappa(x, y) \mathrm{d}\mu(x)\mathrm{d}\mu(y)$$
$$\geq \int_A \int_B ||x - y||^2 \kappa(x, y)\mathrm{d}\mu_\infty(x)\mathrm{d}\mu_\infty(y) > 0.$$

It thus follows from (10) that $\lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} m_t^{(2)} < 0$, which is in contrast with the fact that $\lim_{t\to\infty} \mu_t = \mu_\infty$. ∎

Example 4. *Consider the case of interaction kernel $\kappa \equiv 1$. Then, Theorem 2 implies that $\lim_{t\to\infty} \mu_t = \delta_{x_0}$. Observe that the fist moment $m_t^{(1)} := \int x \mathrm{d}\mu_t(x)$, satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} m_t^{(1)} = 0,$$

so that in this case the system preserves the first moment. Assuming with no loss of generality that $m_0^{(1)} = 0$, (10) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} m_t^{(2)} = -2\omega(1 - \omega)m_t^{(2)},$$

so that $m_t^{(2)} = m_0^{(2)} e^{-2\omega(1-\omega)t}$.

Example 5. *For the case $\kappa(x, y) = \mathbb{1}_{[0,R)}(||x - y||)$, Theorem 2 guarantees convergence to a convex combination of deltas, each pair separated by a distance of at least R (opinion clusters). On the other hand, for the case $\kappa(x, y) = \exp(-||x - y||/\sigma^2)$, Theorem 2 guarantees convergence to a single delta (consensus).*

V. CONCENTRATION AROUND THE SOLUTION OF THE ODE

In this section, we finally show that, in the limit of the population size n going to infinity, the stochastic process $\{M_k^n\}$ concentrates around the solution of the ODE (6). Throughout $\mathcal{X} \subseteq \mathbb{R}^d$ will be assumed compact and convex.

In order to formalize the aforementioned notion of concentration, we need to equip the space $\mathcal{P}(\mathcal{X})$ with a suitable notion of distance. The topology induced on $\mathcal{P}(\mathcal{X})$ by the total variation distance, as defined in (7), turns out to be too strong for our purposes. Indeed, for instance, it can be immediately verified that, for every initial distribution μ_0 absolutely continuous with respect to the Lebesgue measure, $||M_0^n - \mu_0||_{TV} = 1$ for all n, so that M_0^n does not converge to μ_0 in total variation.

We shall prove our results in the so-called 1-Wasserstein distance, which is defined as follows. For two probability measures $\mu, \nu \in \mathcal{P}(\mathcal{X})$, define

$$W_1(\mu, \nu) := \min \left\{ \int_{\mathcal{X}\times\mathcal{X}} ||x_1 - x_2||\mathrm{d}\lambda(x_1, x_2) \right\}; \quad (11)$$

in the righthand side of (11), the minimization runs over all joint probability measures $\lambda \in \mathcal{P}(\mathcal{X} \times \mathcal{X})$ whose marginals

are given by μ and ν, respectively. The reason for choosing such a metric resides in the rich duality it enjoys (see, e.g. [2, Ch. 7] and [11, Ch. 6]). In particular, the so-called Kantorivich-Rubinstein duality formula allows one to rewrite the 1-Wasserstein distance as

$$W_1(\mu, \nu) = \sup \{\langle\mu, \varphi\rangle - \langle\nu, \varphi\rangle : \varphi : \mathcal{X} \to \mathbb{R}, \text{1–Lipschitz}\}$$
(12)

Furthermore, the Wasserstein distance is known to induce on $\mathcal{P}(\mathcal{X})$ a topology equivalent to the weak one, as defined by (8): see [11, Th. 6.9], and recall that $\mathcal{X}$ is compact.

We are now in the position to state our main result. This is stated in terms of the linearly interpolated, time rescaled, processes $\{\tilde{M}_t^n : t \in [0, +\infty)\}$, defined by

$$\tilde{M}_t^n := (1 + tn - \lfloor tn \rfloor)M_{\lfloor tn\rfloor}^n + (tn - \lfloor tn\rfloor)M_{\lfloor tn\rfloor+1}^n. \quad (13)$$

Theorem 3. *For $\mu_0 \in \mathcal{P}(\mathcal{X})$, let $\{\mu_t : t \in [0, +\infty)\}$ be the unique solution of the ODE (6) with initial condition μ_0. Assume that $\kappa(x, y)$ is Lipschitz. For all $n \in \mathbb{N}$, let $\{M_k^n : k \in \mathbb{Z}_+\}$ be the discrete time stochastic process introduced in Sect. II, and let $\{\tilde{M}_t^n : t \in [0, +\infty)\}$ its rescaled, interpolated version as in (13). Then, for every $\sigma, \tau \in (0, +\infty)$,*

$$\mathbb{P}_n\left(\sup_{t\in[0,\tau]}\left\{W_1(\tilde{M}_t^n, \mu_t)\right\} \geq \left(K_1\sigma + K_2\sigma^{3/2}\right)e^\tau\right)$$
$$\leq H_1\frac{\tau}{\sigma^2}\exp(H_2/\sigma)\exp(-J\sigma^3 n),$$

where K_1, K_2, H_1, H_2, and J, are positive constants depending on $\mathcal{X}$, and κ only.

Proof: The proof involves two main steps. The first one consists first in approximating the space of real-valued, 1-Lipschitz functions over $\mathcal{X}$ by a finite set of functions $\mathcal{H}$ whose cardinality is at most exponential in the inverse of the precision (evaluated in the sup norm). The second one consists in applying the Hoeffding-Azuma inequality [1, Th. 7.2.1] to the bounded martingale difference sequence $\{\langle\Delta_k^n, \varphi\rangle : k \in \mathbb{Z}^+\}$ introduced in Sect. III (recall, in particular, (5)), for any function $\varphi \in \mathcal{H}$. In particular, the second step is close in spirit to the proof of the ODE method for random graph processes and randomized algorithms [12]. Finally, the two steps are combined by means of the duality formula (12), in order to get the result. ∎

Remark 2. The additional assumption that the interaction kernel κ be Lipschitz is made for technical reasons. Indeed this guarantees global Lipschitzianity of the operator H in the Wasserstain metric over $\mathcal{P}(\mathcal{X})$. It is not hard to see that such a global Lipschitianity fails to hold true in the case of discontinous kernels as the one of Ex. 2. However, we conjecture that the result holds even for discontinuous kernels provided that the initial measure μ_0 is absolutely continuous with smooth density.

VI. CONCLUSION

We have studied a class of stochastic models of continuous opinion dynamics. The main result presented shows concentra-

tion, in the limit of increasing population size, around the solution of a measure-valued differential equation. The asymptotic properties of the solution of such differential equation have been studied as well, and convergence to a convex combination of deltas, each representing an emerged opinion cluster, has been proven. Current work involves extension of these result to more general opinion dynamics models.

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