

# SOLVABLE STATIONARY NON EQUILIBRIUM STATES

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**ABSTRACT.** We consider the one dimensional boundary driven harmonic model and its continuous version, both introduced in [8]. We give an exact representation of the invariant measures which are obtained as a convex combination of inhomogeneous product of geometric distributions for the discrete model and as a convex combination of inhomogeneous product of exponential distributions for the continuous one. The mean values of the geometric and of the exponential variables are distributed according to the order statistics of i.i.d. uniform random variables on a suitable interval fixed by the boundary sources. The result is obtained solving exactly the stationary condition written in terms of the joint generating function. The method has an interest in itself and can be generalized to study other models.

## 1. INTRODUCTION

Stationary non equilibrium states (SNS) have a rich and complex structure. A natural way to generate a SNS using stochastic interacting particle systems is to put the system, that is evolving on a lattice, in contact with external sources. This is a toy model for a thermodynamic system in contact with external reservoirs. The Markov process obtained is typically non-reversible when the reservoirs are different and its invariant measure is the SNS. Due to the non reversibility, such measure is typically difficult to be computed and has long range correlations [2, 5].

For a few one dimensional solvable models it is possible to get a complete description of the fluctuations of the SNS by an exact computation of the density large deviations rate functional. This is obtained either by using combinatorial representation of the invariant measure [5] or by a variational dynamic approach [2]. Among the solvable models there are the symmetric exclusion process (SEP) and the Kipnis-Marchioro-Presutti (KMP) model [3, 10]. The solvability is related to a direct mapping between non-equilibrium and equilibrium [9, 11].

Due to the presence of long range correlations, the rate functionals are non-local and can be written in terms of the maximization (for SEP) or minimization (for KMP) of an auxiliary function. A problem of interest is the interpretation of the auxiliary function. In the case of the KMP model it has been conjectured in [1] that the auxiliary function can be interpreted as a hidden temperature and the minimization as a contraction principle. This conjecture is solved in [4] where a joint energy-temperature dynamics has been constructed; as a consequence the invariant measure of the boundary driven case is written as a convex combination of inhomogeneous product of exponential distributions whose mean values are distributed according to the invariant measure of an auxiliary opinion model.

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In this paper we consider the harmonic model with parameter  $s = 1/2$  and a suitable continuous version obtained as a scaling limit of the discrete model. The latter can be interpreted as an integrable model of heat conduction. We prove that for this pair of models the invariant measure can be written, like for the KMP model, as a mixture of products of inhomogeneous distributions. These models arise when the special (spin) value  $s = 1/2$  is chosen from a more general description, see [7,8] for the family of harmonic models labelled by  $s > 0$  and [6] for the family of integrable heat conduction models labelled by  $s > 0$ . Besides sharing the same algebraic description for any  $s > 0$ , these two models are also in a duality relation via a moment duality function [6]. Both models are of zero range type, i.e., the rate at which particles or energy exit from one site depends just on the number of particles or amount of energy present on the starting site. However, differently from the classic zero range models, here there are transitions of multiple particles and the boundary driven SNS are not of product type.

In the harmonic model of parameter  $s = 1/2$ , at each site of a graph there is a non-negative integer number of particles. When on a vertex  $x$  there are  $\eta_x$  particles, then  $k \leq \eta_x$  particles can jump across each edge exiting from  $x$  with rate  $1/k$ . We consider a one-dimensional lattice with left and right extrema coupled to reservoirs having densities  $0 < \rho_A \leq \rho_B < +\infty$ . When  $\rho_A = \rho_B$  the model is reversible and its invariant measure is of product type with each marginal being geometric with mean equal to the density of the external reservoirs. When  $\rho_A < \rho_B$  we prove that the invariant measure is a mixture of inhomogeneous product of geometric distributions. The law of the mean values of the inhomogeneous geometric distributions is the order statistics of uniform random variables in the interval  $[\rho_A, \rho_B]$ . This is somehow a natural representation, since the computation of the integral over the hidden parameters does not give a simple expression, being written in terms of hypergeometric functions. For the continuous model we have a similar representation, the heat baths attached at the end points of the bulk have temperatures  $0 < T_A \leq T_B < +\infty$  and the geometric distributions have to be substituted by the exponential ones.

Our result is proved writing the stationarity condition in terms of the joint generating function and verifying an exact telescoping property. A direct solution of the stationarity conditions seems to be much harder. In this note we apply the method just to a couple of models in order to give a direct and clear presentation. We plan to give a systematic study in the future.

## 2. THE DISCRETE HARMONIC MODEL WITH PARAMETER $s = 1/2$

**2.1. The model.** We consider a one-dimensional lattice consisting of  $N$  sites (the bulk)  $\Lambda_N := \{1, \dots, N\}$  and two ghost lattice sites (the boundaries)  $\partial\Lambda_N := \{0, N+1\}$  to which we associate two parameters  $0 < \beta_A < \beta_B < 1$ , respectively. On each lattice site we can have an arbitrarily large number of particles and we denote by  $\eta_x \in \mathbb{N}_0$  the number (possibly zero) of particles at  $x \in \Lambda_N$ . The state space is therefore  $\Omega_N = \mathbb{N}_0^{\Lambda_N}$  and we denote by  $\eta = (\eta_1, \dots, \eta_N) \in \Omega_N$  a generic configuration. The stochastic dynamics has a bulk and a boundary part which are described in terms of the generator  $L_N$  defined below. For any  $x \in \Lambda_N$ , we denote by  $\delta_x \in \Omega_N$  the configuration defined by  $\delta_x(y) = 0$  when  $y \neq x$  and  $\delta_x(x) = 1$ . We have

$$L_N := L_N^{\text{bulk}} + L_N^{\text{bound}}. \quad (2.1)$$

The bulk generator applied to bounded functions reads:

$$L_N^{\text{bulk}} f(\eta) = \sum_{\substack{x, y \in \Lambda_N \\ |x-y|=1}} \sum_{k=1}^{\eta_x} \frac{1}{k} [f(\eta - k\delta_x + k\delta_y) - f(\eta)] .$$

Furthermore, the boundary part which encodes the interaction with the reservoirs is given by:

$$\begin{aligned} L_N^{\text{bound}} f(\eta) &= \sum_{k=1}^{\eta_1} \frac{1}{k} [f(\eta - k\delta_1) - f(\eta)] + \sum_{k=1}^{\infty} \frac{\beta_A^k}{k} [f(\eta + k\delta_1) - f(\eta)] \\ &\quad + \sum_{k=1}^{\eta_N} \frac{1}{k} [f(\eta - k\delta_N) - f(\eta)] + \sum_{k=1}^{\infty} \frac{\beta_B^k}{k} [f(\eta + k\delta_N) - f(\eta)] . \end{aligned}$$

**2.2. Invariant measure.** For a generic measure  $\mu$  on  $\Omega_N$  the stationarity condition reads as follows:

$$\begin{aligned} \mu(\eta) &\left[ \sum_{k=1}^{\infty} \frac{\beta_A^k}{k} + \sum_{x=1}^N \sum_{k=1}^{\eta_x} \frac{2}{k} + \sum_{k=1}^{\infty} \frac{\beta_B^k}{k} \right] \\ &= \sum_{k=1}^{\eta_1} \mu(\eta - k\delta_1) \frac{\beta_A^k}{k} + \sum_{k=1}^{\infty} \mu(\eta + k\delta_1) \frac{1}{k} \\ &\quad + \sum_{x=1}^{N-1} \sum_{k=1}^{\eta_{x+1}} \mu(\eta + k\delta_x - k\delta_{x+1}) \frac{1}{k} + \sum_{x=2}^N \sum_{k=1}^{\eta_{x-1}} \mu(\eta + k\delta_x - k\delta_{x-1}) \frac{1}{k} \\ &\quad + \sum_{k=1}^{\eta_N} \mu(\eta - k\delta_N) \frac{\beta_B^k}{k} + \sum_{k=1}^{\infty} \mu(\eta + k\delta_N) \frac{1}{k} . \end{aligned} \tag{2.2}$$

Let  $\mathcal{G}_m(k) = \frac{1}{1+m} \left( \frac{m}{1+m} \right)^k$ ,  $k = 0, 1, \dots$ , be a geometric distribution of mean  $m$ . Given  $\underline{m} = (m_1, \dots, m_N)$  and  $\underline{k} = (k_1, \dots, k_N)$  we denote by  $\mathcal{G}_{\underline{m}}(\underline{k}) := \prod_{x=1}^N \mathcal{G}_{m_x}(k_x)$ . Given  $0 < \beta_A < \beta_B < 1$  we call  $\rho_A := \frac{\beta_A}{1-\beta_A}$ ,  $\rho_B := \frac{\beta_B}{1-\beta_B}$  and introduce  $O_N^{\rho_A, \rho_B} \subseteq [\rho_A, \rho_B]^N$  as the set defined by

$$O_N^{\rho_A, \rho_B} := \{\underline{m} : \rho_A \leq m_1 \leq \dots \leq m_N \leq \rho_B\} .$$

The Lebesgue volume is given by  $|O_N^{\rho_A, \rho_B}| = \frac{(\rho_B - \rho_A)^N}{N!}$ . Our result is the following:

**Theorem 2.1.** *The invariant measure of the process with generator (2.1) is given by*

$$\mu_N^{\rho_A, \rho_B}(\eta) = \frac{1}{|O_N^{\rho_A, \rho_B}|} \int_{O_N^{\rho_A, \rho_B}} d\underline{m} \mathcal{G}_{\underline{m}}(\eta) . \tag{2.3}$$

In the above statement we make explicit the dependence of the invariant measure on the parameters  $\rho_A, \rho_B, N$ , while in the rest of the paper we omit such dependence. For simplicity of notation we used the symbol  $\eta$  for a configuration of particles but in order to be compatible with our notation for vectors we remark that in (2.3)  $\eta \equiv \underline{\eta}$  should be interpreted as a vector.

In order to better illustrate the result we first give the proof for the case of only one site (Section 3.1) and then generalize it for the case of  $N$  sites in Section 3.2. The basic telescoping mechanism is active already in the  $N = 1$  case.

### 3. PROOF OF THEOREM 2.1

We introduce the moment generating function of the geometric distribution  $\mathcal{G}_m$ :

$$\mathcal{F}_m(\lambda) := \sum_{k=0}^{\infty} \mathcal{G}_m(k) \lambda^k = [1 + (1 - \lambda)m]^{-1}, \quad 0 \leq \lambda < \frac{1+m}{m}.$$

Like before, given  $\underline{m}$  and  $\underline{\lambda}$ , we define  $\mathcal{F}_{\underline{m}}(\underline{\lambda}) := \prod_{x=1}^N \mathcal{F}_{m_x}(\lambda_x)$ .

**3.1. The case  $N = 1$ .** In the case that our lattice is composed by one single node which is in contact with two external reservoirs, the state space of the process  $\Omega_1$  is the set of natural numbers. We denote by  $\eta_1 \in \mathbb{N}_0$  a generic element of the state space and the generator  $L_1$  (from (2.1) for  $N = 1$ ) is given by

$$L_1 f(\eta_1) = \sum_{k=1}^{\eta_1} \frac{2}{k} [f(\eta_1 - k) - f(\eta_1)] + \sum_{k=1}^{\infty} \frac{\beta_A^k + \beta_B^k}{k} [f(\eta_1 + k) - f(\eta_1)],$$

where  $0 < \beta_A < \beta_B < 1$  are the parameters associated to the two external reservoirs. The stationarity condition for the invariant measure  $\mu$  is

$$\mu(\eta_1) \left[ \sum_{k=1}^{\infty} \frac{\beta_A^k + \beta_B^k}{k} + \sum_{k=1}^{\eta_1} \frac{2}{k} \right] = \sum_{k=1}^{\infty} \mu(\eta_1 + k) \frac{2}{k} + \sum_{k=1}^{\eta_1} \mu(\eta_1 - k) \frac{\beta_A^k + \beta_B^k}{k}, \quad (3.1)$$

which must be satisfied for all  $\eta_1 \in \mathbb{N}_0$ . Theorem 2.1 says that for  $N = 1$  the invariant measure is a mixed geometric distribution, i.e.,

$$\mu(\eta_1) = \frac{1}{\rho_B - \rho_A} \int_{\rho_A}^{\rho_B} dm \mathcal{G}_m(\eta_1). \quad (3.2)$$

Note that in the limit  $\rho_A \rightarrow \rho_B$  we recover the special equilibrium case, where the invariant measure is just a geometric distribution of mean  $\rho_B$ .

Instead of checking the validity of (3.1) for each  $\eta_1 \in \mathbb{N}_0$ , it will be convenient to multiply both sides of (3.1) by  $\lambda_1^{\eta_1}$  and sum over  $\eta_1$ . In this way, we get an equality between generating functions for each value of  $\lambda$  which is equivalent to the whole set of conditions (3.1). In the sequel we will use the following elementary formulas:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}; \quad \sum_{k=1}^{\infty} \frac{x^k}{k} = \log \frac{1}{1-x}; \quad \sum_{k=0}^{+\infty} x^k \sum_{j=1}^k \frac{1}{j} = \frac{1}{1-x} \log \frac{1}{1-x} \quad |x| < 1. \quad (3.3)$$

We write separately each one of the terms that are obtained by inserting (3.2) into (3.1) and computing the generating function. The first term gives

$$\begin{aligned} \sum_{\eta_1=0}^{\infty} \lambda^{\eta_1} \mu(\eta_1) \sum_{k=1}^{\infty} \frac{\beta_A^k + \beta_B^k}{k} &= \frac{1}{\rho_B - \rho_A} \int_{\rho_A}^{\rho_B} \frac{dm}{1+m} \sum_{\eta_1=0}^{\infty} \left( \frac{m\lambda}{1+m} \right)^{\eta_1} \sum_{k=1}^{\infty} \frac{\beta_A^k + \beta_B^k}{k} \\ &= \frac{1}{\rho_B - \rho_A} \int_{\rho_A}^{\rho_B} dm \left[ \log(1 + \rho_A) + \log(1 + \rho_B) \right] \mathcal{F}_m(\lambda). \end{aligned}$$

where we used (3.3). Similarly, exchanging the order of summation, the other terms give

$$\sum_{\eta_1=0}^{\infty} \lambda^{\eta_1} \mu(\eta_1) \sum_{k=1}^{\eta_1} \frac{2}{k} = \frac{1}{\rho_B - \rho_A} \int_{\rho_A}^{\rho_B} dm \left[ 2 \log(1+m) + 2 \log \mathcal{F}_m(\lambda) \right] \mathcal{F}_m(\lambda),$$

$$\sum_{\eta_1=0}^{\infty} \lambda^{\eta_1} \sum_{k=1}^{\infty} \mu_1(\eta_1 + k) \frac{2}{k} = \frac{1}{\rho_B - \rho_A} \int_{\rho_A}^{\rho_B} dm \left[ 2 \log(1+m) \right] \mathcal{F}_m(\lambda)$$

and

$$\sum_{\eta_1=0}^{\infty} \lambda^{\eta_1} \sum_{k=1}^{\eta_1} \mu(\eta_1 - k) \frac{\beta_A^k + \beta_B^k}{k}$$

$$= \frac{1}{\rho_B - \rho_A} \int_{\rho_A}^{\rho_B} dm \left[ \log(1 + \rho_A) + \log \mathcal{F}_{\rho_A}(\lambda) + \log(1 + \rho_B) + \log \mathcal{F}_{\rho_B}(\lambda) \right] \mathcal{F}_m(\lambda).$$

All in all, by adding the terms, we get that the stationarity condition (3.1) is equivalent to

$$\int_{\rho_A}^{\rho_B} dm \left[ \log \mathcal{F}_{\rho_A}(\lambda) - 2 \log \mathcal{F}_m(\lambda) + \log \mathcal{F}_{\rho_B}(\lambda) \right] \mathcal{F}_m(\lambda) = 0. \quad (3.4)$$

By a direct computation we have the following simple relation for  $m \mapsto F_m(\lambda)$ , the antiderivative of  $m \mapsto \mathcal{F}_m(\lambda)$  :

$$F_m(\lambda) = \int_0^m dm' \mathcal{F}_{m'}(\lambda) = \frac{1}{(\lambda - 1)} \log \mathcal{F}_m(\lambda). \quad (3.5)$$

Then, in terms of the antiderivative, (3.4) is rewritten as

$$(\lambda - 1) \int_{\rho_A}^{\rho_B} dm \left[ F_{\rho_A}(\lambda) - 2F_m(\lambda) + F_{\rho_B}(\lambda) \right] F'_m(\lambda) = 0,$$

where  $F'_m(\lambda)$  denotes the derivative with respect to the parameter  $m$ . Performing the integral, apart the common  $(\lambda - 1)$  factor, we get

$$F_{\rho_A}(\lambda)(F_{\rho_B}(\lambda) - F_{\rho_A}(\lambda)) - (F_{\rho_B}^2(\lambda) - F_{\rho_A}^2(\lambda)) + F_{\rho_B}(\lambda)(F_{\rho_B}(\lambda) - F_{\rho_A}(\lambda)),$$

which is clearly zero. This concludes the proof of Theorem 2.1 for  $N = 1$ .

**3.2. The general case.** In this section we give the proof of Theorem 2.1 for general  $N$ . We now consider the full stationarity condition (2.2) which also contains the bulk terms. With computations similar to the ones done in the previous section we obtain that the stationarity condition (2.2) is equivalent to

$$\sum_{x=1}^N \int_{O_N^{\rho_A, \rho_B}} d\mathbf{m} \left[ \log \mathcal{F}_{m_{x-1}}(\lambda_x) - 2 \log \mathcal{F}_{m_x}(\lambda_x) + \log \mathcal{F}_{m_{x+1}}(\lambda_x) \right] \mathcal{F}_{\mathbf{m}}(\lambda) = 0,$$

where we have defined

$$\mathcal{F}_{m_0}(\lambda_1) \equiv \mathcal{F}_{\rho_A}(\lambda_1), \quad \text{and} \quad \mathcal{F}_{m_{N+1}}(\lambda_N) \equiv \mathcal{F}_{\rho_B}(\lambda_N).$$

Using (3.5) the above condition can be also written as

$$\sum_{x=1}^N (\lambda_x - 1) \int_{O_N^{\rho_A, \rho_B}} d\mathbf{m} \left[ F_{m_{x-1}}(\lambda_x) - 2F_{m_x}(\lambda_x) + F_{m_{x+1}}(\lambda_x) \right] F'_{\mathbf{m}}(\lambda) = 0,$$

where, as usual in this paper, we denote  $F'_{\underline{m}}(\underline{\lambda}) = \prod_{x=1}^N F'_{m_x}(\lambda_x)$ . One can check that the integrals are vanishing for each  $x \in \{1, 2, \dots, N\}$ . To verify this, let us call  $O_{N-1,x}^{\rho_A, \rho_B}$  the collection of  $N-1$  ordered variables  $m_y$ , with  $y \neq x$ , i.e., where the variable  $m_x$  is missing; we call  $\underline{m}^x$  a generic element of  $O_{N-1,x}^{\rho_A, \rho_B}$ . Then, by applying Fubini theorem, we get

$$\begin{aligned} & \int_{O_N^{\rho_A, \rho_B}} d\underline{m} \left[ F_{m_{x-1}}(\lambda_x) - 2F_{m_x}(\lambda_x) + F_{m_{x+1}}(\lambda_x) \right] F'_{\underline{m}}(\underline{\lambda}) \\ &= \int_{O_{N-1,x}^{\rho_A, \rho_B}} d\underline{m}^x \int_{m_{x-1}}^{m_{x+1}} dm_x \left[ F_{m_{x-1}}(\lambda_x) F'_{m_x}(\lambda_x) \right. \\ & \quad \left. - 2F_{m_x}(\lambda_x) F'_{m_x}(\lambda_x) \right. \\ & \quad \left. + F_{m_{x+1}}(\lambda_x) F'_{m_x}(\lambda_x) \right] F'_{\underline{m}^x}(\underline{\lambda}^x), \end{aligned}$$

where again  $\underline{\lambda}^x$  is obtained from the vector  $\underline{\lambda}$  by removing the component  $\lambda_x$ . The integral over the variable  $m_x$  on the right hand side of the above equation can now be performed and we are left with

$$\begin{aligned} & \int_{O_{N-1,x}^{\rho_A, \rho_B}} d\underline{m}^x \left[ F_{m_{x-1}}(\lambda_x) (F_{m_{x+1}}(\lambda_x) - F_{m_{x-1}}(\lambda_x)) \right. \\ & \quad \left. - (F_{m_{x+1}}^2(\lambda_x) - F_{m_{x-1}}^2(\lambda_x)) \right. \\ & \quad \left. + F_{m_{x+1}}(\lambda_x) (F_{m_{x+1}}(\lambda_x) - F_{m_{x-1}}(\lambda_x)) \right] F'_{\underline{m}^x}(\underline{\lambda}^x) \end{aligned}$$

which is clearly zero since the term inside the squared parenthesis is identically zero. This concludes the proof of Theorem 2.1.

#### 4. INTEGRABLE HEAT CONDUCTION MODEL WITH PARAMETER $s = 1/2$

**4.1. The model.** In this section we show that the same approach holds for a related model. The model was introduced in [8] as a scaling limit of the harmonic model and further generalized in [6]. The setting is as in the previous section, namely a one dimensional lattice  $\Lambda_N$  with two extra ghost sites representing the reservoirs. Here we denote by  $z_x \in \mathbb{R}_+$  the arbitrary quantity of energy at site  $x \in \Lambda_N$ , and by  $z = (z_1, \dots, z_N)$  a generic configuration in  $\Omega_N = \mathbb{R}_+^{\Lambda_N}$ , i.e., the state space. The generator of the stochastic dynamics is given as the superposition of a bulk part and a boundary part, described below:

$$L_N := L_N^{\text{bulk}} + L_N^{\text{bound}}, \quad (4.1)$$

whose action on functions  $f : \Omega_N \rightarrow \mathbb{R}$  that are bounded and  $C^1$  is

$$L_N^{\text{bulk}} f(z) = \sum_{\substack{x, y \in \Lambda_N \\ |x-y|=1}} \int_0^{z_x} \frac{d\alpha}{\alpha} [f(z - \alpha\delta_x + \alpha\delta_y) - f(z)]$$

and

$$\begin{aligned} L_N^{\text{bound}} f(z) &= \int_0^{z_1} \frac{d\alpha}{\alpha} [f(z - \alpha\delta_1) - f(z)] + \int_0^\infty \frac{d\alpha}{\alpha} e^{-\alpha/T_A} [f(z + \alpha\delta_1) - f(z)] \\ &+ \int_0^{z_N} \frac{d\alpha}{\alpha} [f(z - \alpha\delta_N) - f(z)] + \int_0^\infty \frac{d\alpha}{\alpha} e^{-\alpha/T_B} [f(z + \alpha\delta_N) - f(z)]. \end{aligned}$$

Above  $T_A$  (respectively,  $T_B$ ) is the temperature associated to the left (respectively, right) reservoir whose purpose is to destroy the conservation of energy by imposing heat conduction from one side of the chain to the other. When  $T_A = T_B = T$  there is no transport of energy, the model is reversible and its invariant measure is of product type with each marginal being exponential with mean equal to the temperature  $T$  of the external reservoirs. Note that since  $1/\alpha$  is not integrable at zero, this is a jump process with a dense set of jumps. We do not address here the delicate issues related to the definition of the process.

**4.2. Invariant measure.** The stationarity condition imposes that the density  $\mu$  of the invariant measure satisfies

$$\begin{aligned}
0 = & \int_0^{z_1} \frac{d\alpha}{\alpha} e^{-\frac{\alpha}{T_A}} [\mu(z - \alpha\delta_1) - \mu(z)] + \mu(z) \int_{z_1}^{+\infty} \frac{d\alpha}{\alpha} e^{-\frac{\alpha}{T_A}} \\
& + \int_0^{z_1} \frac{d\alpha}{\alpha} [\mu(z + \alpha\delta_1) - \mu(z)] + \int_{z_1}^{+\infty} \frac{d\alpha}{\alpha} \mu(z + \alpha\delta_1) \\
& + \int_0^{z_N} \frac{d\alpha}{\alpha} e^{-\frac{\alpha}{T_B}} [\mu(z - \alpha\delta_N) - \mu(z)] + \mu(z) \int_{z_N}^{+\infty} \frac{d\alpha}{\alpha} e^{-\frac{\alpha}{T_B}} \\
& + \int_0^{z_N} \frac{d\alpha}{\alpha} [\mu(z + \alpha\delta_N) - \mu(z)] + \int_{z_N}^{+\infty} \frac{d\alpha}{\alpha} \mu(z + \alpha\delta_N) \\
& + \sum_{\substack{x,y \in \Lambda_N \\ |x-y|=1}} \left\{ \int_0^{z_x} \frac{d\alpha}{\alpha} [\mu(z + \alpha\delta_x - \alpha\delta_y) - \mu(z)] + \int_{z_x}^{+\infty} \frac{d\alpha}{\alpha} \mu(z + \alpha\delta_x - \alpha\delta_y) \right\}.
\end{aligned} \tag{4.2}$$

Let  $\mathcal{E}_m(z) = \frac{1}{m} e^{-z/m} \mathbb{1}_{\{z \geq 0\}}$  be the density of an exponential distribution of mean  $m > 0$ . Given  $\underline{m} = (m_1, \dots, m_N)$  and  $\underline{z} = (z_1, \dots, z_N)$  we denote by  $\mathcal{E}_{\underline{m}}(\underline{z}) := \prod_{x=1}^N \mathcal{E}_{m_x}(z_x)$ . As before we introduce  $O_N^{T_A, T_B} \subseteq [T_A, T_B]^N$  as the set defined by

$$O_N^{T_A, T_B} := \{\underline{m} : T_A \leq m_1 \leq \dots \leq m_N \leq T_B\}.$$

Our result is the following:

**Theorem 4.1.** *The invariant measure of the process with generator (4.1) is given by*

$$\mu_N^{T_A, T_B}(z) = \frac{1}{|O_N^{T_A, T_B}|} \int_{O_N^{T_A, T_B}} d\underline{m} \mathcal{E}_{\underline{m}}(z). \tag{4.3}$$

Here, again, for simplicity of notation we call  $z$  a configuration of energies but in order to be compatible with our vector-notation we remark that in (4.3)  $z \equiv \underline{z}$  should be interpreted as a vector. The strategy of the proof is similar to the previous one, namely we consider  $N = 1$  first and then we show the result for a general finite chain of  $N$  sites. Below, in order to alleviate the notation for the invariant measure, we drop the dependence on the parameters  $T_A$ ,  $T_B$  and  $N$ .

## 5. PROOF OF THEOREM 4.1

We introduce the moment generating function of the exponential distribution  $\mathcal{E}_m$ :

$$\mathcal{F}_m(t) := \int_0^\infty dz \mathcal{E}_m(z) e^{tz} = \frac{1}{1 - tm}, \quad t < \frac{1}{m}$$

and we define  $\mathcal{F}_{\underline{m}}(\underline{t}) := \prod_{x=1}^N \mathcal{F}_{m_x}(t_x)$ .

5.1. **The case  $N = 1$ .** If the lattice consists of only one site then the Markov generator simplifies as

$$\begin{aligned} L_1 f(z_1) &= 2 \int_0^{z_1} \frac{d\alpha}{\alpha} [f(z - \alpha\delta_1) - f(z)] \\ &+ \int_0^\infty \frac{d\alpha}{\alpha} \left( e^{-\alpha/T_A} + e^{-\alpha/T_B} \right) [f(z + \alpha\delta_1) - f(z)]. \end{aligned}$$

The stationary condition for the invariant measure  $\mu$  reads

$$\begin{aligned} &\int_0^{z_1} \frac{d\alpha}{\alpha} \left( e^{-\alpha/T_A} + e^{-\alpha/T_B} \right) (\mu(z_1) - \mu(z_1 - \alpha)) + \int_{z_1}^\infty \frac{d\alpha}{\alpha} \left( e^{-\alpha/T_A} + e^{-\alpha/T_B} \right) \mu(z_1) \\ &= 2 \int_0^{z_1} \frac{d\alpha}{\alpha} (\mu(z_1 + \alpha) - \mu(z_1)) + 2 \int_{z_1}^\infty \frac{d\alpha}{\alpha} \mu(z_1 + \alpha), \end{aligned}$$

which must be satisfied for all  $z_1 \in \mathbb{R}_+$ . Multiplying both sides by  $e^{tz_1}$ , using the representation (4.3) for  $N = 1$  and taking the integral in  $dz_1$ , we get, as in the previous case, four different terms which can be compactly written as

$$\int_{T_A}^{T_B} dm \int_0^\infty d\alpha \frac{e^{\alpha t} - 1}{\alpha} \left[ \left( e^{-\alpha/T_A} - 2e^{-\alpha/m} + e^{-\alpha/T_B} \right) \right] \mathcal{F}_m(t) = 0. \quad (5.2)$$

The inner integrals can be computed using “Feynman’s trick” which, for  $a, b > 0$ , leads to

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \log \left( \frac{b}{a} \right),$$

so that we have

$$\int_{T_A}^{T_B} dm [\log \mathcal{F}_{T_A}(t) - 2 \log \mathcal{F}_m(t) + \log \mathcal{F}_{T_B}(t)] \mathcal{F}_m(t) = 0. \quad (5.3)$$

The key observation regarding  $F_m$ , the antiderivative of  $m \mapsto \mathcal{F}_m(t)$ , is the following

$$F_m(t) = \int_0^m dm' \mathcal{F}_{m'}(t) = -\frac{1}{t} \log(1 - tm) = \frac{1}{t} \log \mathcal{F}_m(t). \quad (5.4)$$

This allows to write (5.3) as

$$\int_{T_A}^{T_B} dm [F_{T_A}(t) - 2F_m(t) + F_{T_B}(t)] F'_m(t) = 0,$$

where  $F'_m(t)$  denotes the derivative with respect to the parameter  $m$ . As before, by inspection the left hand side of the previous equation is zero and the proof of Theorem 4.1 for  $N = 1$  is concluded.



**5.2. The general case.** For general  $N$  the stationarity condition is written in equation (4.2). As before, we multiply both sides by  $\prod_{x=1}^N e^{t_x z_x}$ , we use the representation (4.3) and take the integral. We obtain

$$\begin{aligned} \int_{O_N^{T_A, T_B}} d\mathbf{m} \left[ \int_0^\infty \frac{d\alpha}{\alpha} \left( e^{-\alpha/T_A} + \sum_{x=1}^N 2e^{\alpha t_x} e^{-\alpha/m_x} + e^{-\alpha/T_B} \right) \mathcal{F}_{\mathbf{m}}(\underline{t}) \right] = \\ \int_{O_N^{T_A, T_B}} d\mathbf{m} \int_0^\infty \frac{d\alpha}{\alpha} \left[ e^{-\alpha/m_1} + \sum_{x=1}^N e^{\alpha t_x} \left( e^{-\alpha/m_{x-1}} + e^{-\alpha/m_{x+1}} \right) + e^{-\alpha/m_N} \right] \mathcal{F}_{\mathbf{m}}(\underline{t}), \end{aligned} \quad (5.5)$$

where we have set  $m_0 := T_A$  and  $m_{N+1} := T_B$ . At this point it is enough to notice that using the telescoping cancellation

$$\begin{aligned} \int_0^\infty \frac{d\alpha}{\alpha} \left( e^{-\alpha/T_A} - e^{-\alpha/m_1} - e^{-\alpha/m_N} + e^{-\alpha/T_B} \right) \\ = \sum_{x=1}^N \int_0^\infty \frac{d\alpha}{\alpha} \left( e^{-\alpha/m_{x-1}} - 2e^{-\alpha/m_x} + e^{-\alpha/m_{x+1}} \right) \end{aligned}$$

we can rewrite (5.5) in a form analogous to (5.2):

$$\sum_{x=1}^N \int_{O_N^{T_A, T_B}} d\mathbf{m} \int_0^\infty d\alpha \left( \frac{e^{\alpha t_x} - 1}{\alpha} \right) \left[ e^{-\alpha/m_{x-1}} - 2e^{-\alpha/m_x} + e^{-\alpha/m_{x+1}} \right] \mathcal{F}_{\mathbf{m}}(\underline{t}) = 0.$$

Computing the inner integrals, we get

$$\sum_{x=1}^N \int_{O_N^{T_A, T_B}} d\mathbf{m} \left[ \log \mathcal{F}_{m_{x-1}}(t_x) - 2 \log \mathcal{F}_{m_x}(t_x) + \log \mathcal{F}_{m_{x+1}}(t_x) \right] \mathcal{F}_{\mathbf{m}}(\underline{t}) = 0,$$

which can be written in terms of the antiderivative  $F_{\mathbf{m}}$  using equation (5.4)

$$\sum_{x=1}^N \int_{O_N^{T_A, T_B}} d\mathbf{m} \left[ F_{m_{x-1}}(t_x) - 2F_{m_x}(t_x) + F_{m_{x+1}}(t_x) \right] F'_{\mathbf{m}}(\underline{t}) = 0.$$

We show that each term of the above sum is zero. To this aim we apply Fubini theorem to the  $x^{th}$  term to separate the integral in  $m_x$ , i.e.,

$$\begin{aligned} \int_{O_N^{T_A, T_B}} d\mathbf{m} \left[ F_{m_{x-1}}(t_x) - 2F_{m_x}(t_x) + F_{m_{x+1}}(t_x) \right] F'_{\mathbf{m}}(\underline{t}) = \\ \int_{O_{N-1, x}^{T_A, T_B}} d\mathbf{m}^x \int_{m_{x-1}}^{m_{x+1}} dm_x \left[ F_{m_{x-1}}(t_x) - 2F_{m_x}(t_x) + F_{m_{x+1}}(t_x) \right] F'_{m_x}(t_x) \prod_{\substack{y=1 \\ y \neq x}}^N F'_{m_y}(t_y), \end{aligned}$$

where  $O_{N-1, x}^{T_A, T_B}$  has the same meaning as before, namely the collection of  $N-1$  ordered variables  $m_y$  with  $y \neq x$ . Computing the inner integral on the right hand side we obtain zero and the proof of Theorem 4.1 is concluded.

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