Nonconventional averages along arithmetic progressions and lattice spin systems

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Abstract

We consider nonconventional averages in the context of lattice spin systems, or equivalently random colourings of the integers. For iid colouring, we prove a large deviation principle for the number of monochromatic arithmetic progressions of size two in the box $[1, n] \cap \mathbb{Z}$ with an explicit rate function.

For more general colourings, we prove simple bounds for the number of monochromatic arithmetic progressions of arbitrary size, as well as for the maximal progression inside the box $[1, n] \cap \mathbb{Z}$.

Contents

1 Introduction

Nonconventional averages along arithmetic progressions are averages of the type

$$\frac{1}{N}\sum_{i=1}^{N} f_1(X_i)f_2(X_{2i})\cdots f_k(X_{ki}) \tag{1}$$

where (X_n) is a sequence of random variables, and $f_1,\ldots,f_k$ are bounded measurable functions.

Motivation to study such averages comes from the study of arithmetic progressions in subsets of the integers, and multiple recurrence and multiple ergodic averages. In that context, typically $X_i = T^i(x)$, with T a weakly mixing transformation, and x is distributed according to the unique invariant measure. See [5], [9] for more background on this deep and growing field.

Only recently, starting with the work of Kifer [7], and Kifer and Varadhan [8], central limit behavior of nonconventional averages was considered. These authors consider averages along progressions more general than the arithmetic ones. It is natural to consider the averages of the type (1) from a probabilistic point of view and ask questions such as whether they satisfy a large deviation principle, whether associated extremes have classical extreme value behavior, etc.

These questions are far from obvious, since even in the simplest case of f_i being all identical, the sum

$$S_N = \sum_{i=1}^{N}\prod_{j=1}^{k} f(X_{ji})$$

is quite far from an ergodic sum (a sum of shifts of a fixed function), i.e., it is highly non-translation invariant. From the point of view of statistical mechanics, large deviations of S_N/N are related to partition function and free energy associated to the "Hamiltonian" S_N. Since S_N is not translation-invariant and (extremely) long-range, even the existence of the associated free energy is not obvious.

In this paper, we restrict to random variables X_i taking values in a finite set. For some general rather obvious fluctuation properties of H_N we assume the joint distribution to be Gibbs with an exponentially decaying interaction. In the last section we explicitely compute the large deviation rate function of $\frac{1}{N}\sum_{i=1}^{N} X_iX_{2i}$ for X_i i.i.d. Bernoulli. Even if this is the absolute simplest setting, the rate function turns out to be an interesting non-trivial object.

2 The setting

We consider K-colorings of the integers and denote them as σ, η, elements of the set of configurations $\Omega = \{0, \ldots, K\}^{\mathbb{Z}}$. We assume that on Ω there is translation-invariant Gibbs measure with exponentially decaying interaction, denoted by $\mathbb{P}$. This means that, given $\alpha \in \{0, \ldots, K\}$, for the one-site conditional probability

$$\varphi_{\hat{\sigma}}(\alpha) = \mathbb{P}(\sigma_0 = \alpha | \sigma_{\mathbb{Z}\setminus\{0\}} = \hat{\sigma})$$

we assume the variation bound

$$\|\varphi_{\hat{\sigma}} - \varphi_{\hat{\eta}}\| \leq e^{-n\rho}$$

for some $\rho > 0$ whenever $\hat{\sigma}$ and $\hat{\eta}$ agree on $[-n, n] \cap \{\mathbb{Z} \setminus \{0\}\}$. This class of measures is closed under single-site transformations, i.e., if we define new spins $\sigma'_i = F(\sigma_i)$ with $F : \{0, 1, \ldots, K\} \to \{0, 1 \ldots, K'\}$, $K' < K$, then $\mathbb{P}'$, the image measure on $\{0, 1 \ldots, K'\}^{\mathbb{Z}}$, is again a Gibbs measure with exponentially decaying interaction, see e.g. [10] for a proof. In the last section, we restrict to product measures.

For the rest of the paper we consider only 2-colorings (i.e. $K = 1$). Given an integer ℓ, we are interested in the averages

$$\mathscr{A}_n^\ell = \frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{\ell} \sigma_{ji}. \tag{2}$$

The random variable $\mathscr{A}_n^\ell$ counts the number of arithmetic progressions of size ℓ with "colour" 1 (starting from one) in the block $[1, nk]$.

If we consider K-colorings and monochromatic arithmetic progressions, i.e., averages of the type

$$\frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{k} \mathbb{1}(\sigma_{ji} = \alpha)$$

for given $\alpha \in \{0, \ldots, K\}$, then we can define the new "colors" $\sigma'_i = \mathbb{1}(\sigma_i = \alpha)$ which are zero-one valued and, as stated before, are distributed according to $\mathbb{P}'$, a Gibbs measure with an exponentially decaying interaction. Therefore, if we restrict to monochromatic arithmetic progressions, there is no loss of generality if we consider 2-colorings.

Several natural questions can be asked about the average $\mathscr{A}_n^\ell$ and related quantities.

1. *Law of large numbers*: Does $\mathscr{A}_n^\ell$ converge to $(\mathbb{E}(\sigma_0))^\ell$ as $n \to \infty$ with $\mathbb{P}$ probability one ?

2. *Central limit theorem*: Does there exist some $a^2 > 0$ such that
$$\sqrt{n}\left(\mathscr{A}_n^\ell - (\mathbb{E}(\sigma_0))^\ell\right) \xrightarrow{\text{law}} \mathscr{N}(0,a^2),\ \text{as}\, n \to \infty\,?$$

3. *Large deviations*: Does the rate function
$$I(x) = \lim_{\epsilon\to 0}\lim_{n\to\infty} -\frac{1}{n}\log \mathbb{P}\left(\mathscr{A}_n^\ell \in [x-\epsilon, x+\epsilon]\right)$$
exist and have good properties ? In view of the Gärtner-Ellis theorem [4], the natural candidate for I is the Legendre transform of the "free-energy"
$$\mathscr{F}(\lambda) = \lim_{n\to\infty}\frac{1}{n}\log \mathbb{E}\left(e^{n\lambda\mathscr{A}_n}\right)$$
provided this limit exists and is differentiable. If, additionally, $\mathscr{F}$ is analytic in a neighborhood of the origin, then the central limit theorem follows [1].

4. *Statistics of nonconventional patterns.* Let
$$\mathscr{M}(n) = \max\{k \in \mathbb{N} : \exists\, 1 \le i \le n/k\, \text{such that}\, \sigma_i = 1, \sigma_{2i} = 1, \ldots, \sigma_{ki} = 1\}$$
be the maximal arithmetic progression of colour 1 starting from zero in the block $[1,n]$. One would expect
$$\mathscr{M}(n) \approx C \log n + X_n$$
where $0 < C < \infty$ and X_n is a tight sequence of random variables with an approximate Gumbel distribution, i.e.,
$$e^{-c_1 e^{-x}} \le \mathbb{P}\left(X_n \le x\right) \le e^{-c_2 e^{-x}}.$$
Related to this is the exponential law for the occurence of "rare arithmetic progressions": Let
$$\mathscr{T}(\ell) = \inf\{n \in \mathbb{N} : \exists\, 1 \le i \le n/\ell\, \text{such that}\, \sigma_i = 1, \sigma_{2i} = 1, \ldots, \sigma_{\ell i} = 1\}$$
be the smallest block $[1,n]$ in which a monochromatic arithmetic progression can be found with size k. Then one expects that $\mathscr{T}(\ell)$, appropriately normalized, has approximately (as $\ell \to \infty$) an exponential distribution. Finally, another convenient quantity is
$$\mathscr{K}(n,\ell) = \sum_{i=1}^{\lfloor n/\ell \rfloor}\prod_{j=1}^{\ell}\sigma_{ji} = \lfloor n/\ell \rfloor \mathscr{A}_{\lfloor n/\ell \rfloor}^\ell \tag{3}$$

which counts the number of monochromatic arithmetic progressions of size ℓ inside $[1, n]$.

The probability distributions of these quantities are related by the following relations:

$$\mathbb{P}(\mathscr{K}(n,\ell) = 0) = \mathbb{P}(\mathscr{M}(n) < \ell) = \mathbb{P}(\mathscr{T}(\ell) > n) \; .$$

3 Some basic probabilistic properties

In this section we prove some basic facts about the nonconventional averages considered in the previous section.

Proposition 3.1.

1. Gaussian concentration bound. *Let $\ell \geq 1$ be an integer. There exists a constant $C > 0$ such that for all $n \geq 1$ and all $t > 0$*

$$\mathbb{P}\left(\left|\mathscr{A}_n^\ell - \mathbb{E}\left(\mathscr{A}_n^\ell\right)\right| > t\right) \leq e^{-Cnt^2}. \tag{4}$$

   In particular, $\mathscr{A}_n^\ell$ converges almost surely to $(\mathbb{E}(\sigma_0))^\ell$.

2. Logarithmic upper bound for maximal monochromatic progressions. *There exists $\gamma > 0$ such that for all $c > \gamma$*

$$\mathscr{K}(n, c\log n) \to 0$$

   in probability as $n \to \infty$.

Proof.

A Gibbs measure with exponentially decaying interaction satisfies both the Gaussian concentration bound, and the Poincaré ineqaulity, see [2], [3]. For a bounded measurable function $f : \Omega \to \mathbb{R}$ let

$$\nabla_i f(\sigma) = f(\sigma^i) - f(\sigma)$$

be the discrete derivative at $i \in \mathbb{Z}$, where σ^i is the configuration obtained from $\sigma \in \Omega$ by flipping the symbol at i. Next define the variation

$$\delta_i f = \sup_\sigma \nabla_i f(\sigma)$$

and

$$\|\delta f\|_2^2 = \sum_{i\in\mathbb{Z}} (\delta_i f)^2. \tag{5}$$

Then, on the one hand, we have the Gaussian concentration inequality: there exists some $C_1 > 0$ such that

$$\mathbb{P}\left(|f - \mathbb{E}(f)| > t\right) \leq e^{-\frac{C_1}{\|\delta f\|_2^2} t^2} \tag{6}$$

for all f and $t > 0$. On the other hand, we have the Poincaré inequality: there exists some $C_2 > 0$ such that

$$\mathbb{E}\left[(f - \mathbb{E}f)^2\right] \leq C_2 \sum_{i\in\mathbb{Z}} \int (\nabla_i f)^2 \mathrm{d}\mathbb{P} \tag{7}$$

for all f. Now choosing

$$f = \mathscr{A}_n^\ell$$

we easily see that

$$\|\delta f\|_2^2 \leq \ell^2/n \ .$$

This combined with (6) gives (4). To see that this implies almost sure convergence to $\mathbb{E}(\sigma_0)^\ell$, use that one-dimensional Gibbs measures with exponentially decaying interacting are fast mixing [6], from which we conclude

$$|\mathbb{E}(\sigma_{ki}|\sigma_{ri}, r \neq k) - \mathbb{E}_(\sigma_0)| \leq Ce^{-ci}$$

which implies

$$|\mathbb{E}(\sigma_i\sigma_{2i}\ldots\sigma_{li}) - \mathbb{E}(\sigma_0)^l| \leq C_l e^{-ci}$$

This in turn implies

$$\lim_{n\to\infty} \mathbb{E}(\mathscr{A}_n^\ell) = \mathbb{E}(\sigma_0)^\ell$$

and finishes the proof of the first statement.

In order to prove the second statement, we use the bound

$$\mathbb{E}\left(\prod_{j=1}^{q} \sigma_{i_j}\right) \leq e^{-\gamma q} \tag{8}$$

for some $\gamma > 0$ and for all $i_1, \ldots, i_q \in \mathbb{Z}$. This follows immediately from the finite energy property of one-dimensional Gibbs measures, i.e., the fact that there exists $\delta \in (0,1)$ such that for all $\sigma \in \Omega, \alpha \in \{0,1\}$

$$0\delta < \mathbb{P}\left(\sigma_0 = \alpha|\sigma_{\mathbb{Z}\setminus\{0\}}\right) < 1 - \delta.$$

As a consequence,

$$\left|\nabla_j\left(\prod_{r=1}^{\ell} \sigma_{ir}\right)\right| \leq \mathbb{1}(j \in \{i, 2i, \ldots, \ell i\}) \prod_{r=1, ri\neq j}^{\ell} \sigma_{ri}$$

and hence, using the elementary inequality $(\sum_{i=1}^n a_i)^2 \le n \sum_{i=1}^n a_i^2$, we have the upper bound

$$|\nabla_j \mathscr{K}(n,\ell)|^2 \le n \sum_{i=1}^{\lfloor n/\ell \rfloor} \prod_{r=1}^{\ell} \sigma_{ir} \mathbb{1}(j \in \{i, 2i, \dots, \ell i\}).$$

Integration against $\mathbb{P}$, using (8) and summing over j yields

$$\sum_j \int (\nabla_j \mathscr{K}(n,\ell))^2 \, \mathrm{d}\mathbb{P} \le n^2 e^{-\ell\gamma} \, .$$

Choosing now

$$\ell = \ell(n) = c \log n \, ,$$

and using (7), we find

$$\mathrm{Var}(\mathscr{K}(n, \ell(n))) \le C n^{2-c\gamma} \, .$$

Hence, for $\gamma > 2/c$, the variance of $\mathscr{K}(n, c\log n)$ converges to zero. Since

$$\mathbb{E}(\mathscr{K}(n, \ell(n)) \le n e^{-\gamma k} \le n^{1-c\gamma},$$

the expectation of $\mathscr{K}(n, \ell(n))$ also converges to zero, hence we have convergence to zero in mean square sense and thus in probability. $\square$

4 Large deviations for arithmetic progressions of size two

From the point of view of functional inequalities such as the Gaussian concentration bound or the Poincaré inequality, there is hardly a difference between sums of shifts of a local function, i.e. conventional ergodic averages, and their nonconventional counterparts.

The difference becomes however manifest in the study of large deviations. If we think e.g. about $\sum_{i=1}^N \sigma_i \sigma_{i+1}$ versus $\sum_{i=1}^N \sigma_i \sigma_{2i}$ as "Hamiltonians" then the first sum is simply a nearest neighbor translation-invariant interaction, whereas the second sum is a long-range non translation invariant interaction. Therefore, from the point of view of computing partition functions, the second Hamiltonian will be much harder to deal with.

In this section we restrict to the product case, by choosing $\mathbb{P}_p$ to be product of Bernoulli with parameter p on two symbols $\{+, -\}$, and consider arithmetic progressions of size two ($k = 2$). We will show that the thermodynamic limit of the free energy function associated to the sum

$$S_N = \sum_{i=1}^{N} \sigma_i \sigma_{2i} \tag{9}$$

defined as

$$\mathscr{F}_p(\lambda) = \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}_p \left(e^{\lambda S_N}\right) \tag{10}$$

exists, is analytic as a function of λ and has an explicit expression in terms of combinations of Ising model partition functions for different volumes.

To start, assuming N to be odd (the case N even is treated similarly), we make the following useful decomposition

$$S_N = \sum_{l=1}^{\frac{N+1}{2}} S_l^N \tag{11}$$

with

$$S_l^N = \sum_{i=0}^{M_l(N)-1} \sigma_{(2l-1)2^i} \sigma_{(2l-1)2^{i+1}} \tag{12}$$

and

$$M_l(N) = \left\lfloor \log_2 \left(\frac{N}{2l-1}\right)\right\rfloor + 1 \; . \tag{13}$$

where $\lfloor x \rfloor$ denotes the integer part of x. The utility of such decomposition is that the random variable S_l^N is independent from $S_{l'}^N$ for $l \neq l'$. This implies that the partition function in the free energy (10) will factorize over different subsystems labeled by $l \in \{1, \ldots, (N+1)/2\}$, each of size $M_l(N)+1$. Therefore we can treat separately each variable S_l^N.

Furthermore, defining new spins

$$\tau_i^{(l)} = \sigma_{(2l-1)2^{i-1}} \qquad \text{for} \quad i \in \{1, \ldots, M_l(N)+1\} \; ,$$

it is easy to realize that, for a given $l \in \{1, \ldots, (N+1)/2\}$, the variable S_l^N is nothing else than the Hamiltonian of a one-dimensional nearest-neighbors Ising model, since

$$\{S_l^N\} \stackrel{\mathscr{D}}{=} \{ \sum_{i=1}^{M_l(N)} \tau_i^{(l)} \tau_{i+1}^{(l)} \} \tag{14}$$

where $\tau_i^{(l)}$ are Bernoulli random variables with parameter p, independent for different values of l and for different values of i and $\stackrel{\mathscr{D}}{=}$ denotes equality in distribution. Introduce the notation

$$\mathscr{Z}(\lambda, h, n+1) = \sum_{\tau \in \{-1,1\}^{n+1}} e^{\lambda \sum_{i=1}^{n} \tau_i \tau_{i+1} + h \sum_{i=1}^{n+1} \tau_i}$$

for the partition function of the one-dimensional Ising model with coupling strength λ and external field h in the volume $\{1, \ldots, n\}$, with *free* boundary conditions. Then we have

$$\mathbb{E}_p \left(e^{\lambda \sum_{i=1}^{n} \tau_i \tau_{i+1}} \right) = (p(1-p))^{\frac{n+1}{2}} \mathscr{Z}(\lambda, h, n+1) \tag{15}$$

with $h = \frac{1}{2} \log(p/(1-p))$. A standard computation gives

$$\mathscr{Z}(\lambda, h, n+1) = v^T M^n v = |v^T \cdot e_+|^2 \Lambda_+^n + |v^T \cdot e_-|^2 \Lambda_-^n$$

with $\Lambda_\pm$ the largest, resp. smallest eigenvalue of the transfer matrix (with elements $M_{\alpha,\beta} = e^{\lambda\alpha\beta + \frac{h}{2}(\alpha+\beta)}$), i.e.,

$$\Lambda_\pm = e^{\lambda} \left(\cosh(h) \pm \sqrt{\sinh^2(h) + e^{-4\lambda}} \right) , \tag{16}$$

v^T the vector with components $(e^{h/2}, e^{-h/2})$, $e_\pm$ the normalized eigenvectors corresponding to the eigenvalues $\Lambda_\pm$.

Using the decomposition (12), we obtain from (15)

$$\log \mathbb{E}_p \left(e^{\lambda \sum_{i=1}^{n} \tau_i \tau_{i+1}} \right) = \sum_{j=1}^{(N+1)/2} \log \left(p(1-p)^{\frac{M_l(N)+1}{2}} \mathscr{Z}(\lambda, h, M_l(N)+1) \right) .$$

Furthermore, observing that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{(N+1)/2} M_l(N) = \frac{1}{2} \int \psi(x) dx$$

with

$$\psi(x) = \left\lfloor \log_2 \left(\frac{1}{x} \right) \right\rfloor + 1 ,$$

we obtain

$$\begin{aligned} \mathscr{F}_p(\lambda) &= \frac{1}{4} \left(\int \psi(x) dx + 1 \right) \log(p(1-p)) \\ &+ \frac{1}{2} \int_0^1 \log \left(|v^T \cdot e_+|^2 \Lambda_+^{\psi(x)} + |v^T \cdot e_-|^2 \Lambda_-^{\psi(x)} \right) . \end{aligned} \tag{17}$$

To obtain a more explicit formula one can make use of the following: the normalized eigenvector corresponding to the largest eigenvalue is

$$e_+ = \frac{w_+}{\|w_+\|}$$

with

$$w_+ = \begin{pmatrix} -e^{-\lambda} \\ e^{h+\lambda} - \Lambda_+ \end{pmatrix}$$

and moreover

$$|v^T \cdot e_-|^2 = \|v\|^2 - |v^T \cdot e_+|^2 = 2\cosh(h) - |v^T \cdot e_+|^2 \ .$$

Since $\psi(x) = n+1$ for $x \in (1/2^{n+1}, 1/2^n]$, we have

$$\int \psi(x) dx = 2 \ ,$$

hence one gets

$$\mathscr{F}_p(\lambda) = \log\left([p(1-p)]^{\frac{3}{4}} |v^T \cdot e_+| \ \Lambda_+ \right) + \mathscr{G}(\lambda) \tag{18}$$

with

$$\mathscr{G}(\lambda) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^n} \log\left(1 + \left(\frac{2\cosh(h)}{|v^T \cdot e_+|^2} - 1 \right) \left(\frac{\Lambda_-}{\Lambda_+} \right)^n \right) \ .$$

In the case $p = 1/2$, we have $h = 0$, $\Lambda_+ = e^{\lambda} + e^{-\lambda}$, $|v^T \cdot e_+|^2 = \|v\|^2 = 2$ which implies $\mathscr{G}(\lambda) = 0$ and

$$\mathscr{F}_{1/2}(\lambda) = \log\left(\frac{1}{2} \left(e^{\lambda} + e^{-\lambda} \right) \right) \ . \tag{19}$$

One recognize in this case the Legendre transform of the large deviation rate function for a sum of i.i.d. bernoulli (1/2) because (only) in this case $p = 1/2$ the joint distribution of $\{\sigma_i \sigma_{2i}, i \in \mathbb{N}\}$ coincides with the joint distribution of a sequence of independent bernoulli(1/2) variables. When $p \neq 1/2$, although an explicit formula is given in (18), the expression reflects the multiscale character of the decomposition and is non-trivial.

As a consequence of the explicit formula (18), we have the following

Theorem 4.1.

1. Large deviations. *The sequence of random variables* $\frac{S_N}{N}$ *satisfies the large deviation principle with rate function*

$$I_p(x) = \sup_{\lambda \in \mathbb{R}} (\lambda x - \mathscr{F}_p(\lambda)) \tag{20}$$

where $\mathscr{F}_p(\lambda)$ *is given by* (18).
For $p = 1/2$ *we have*

$$I_{1/2}(x) = \begin{cases} \frac{1}{2}(1+x)\log(1+x) + \frac{1}{2}(1-x)\log(1-x) & \text{if } |x| \le 1\,, \\ +\infty & \text{if } |x| > 1\,. \end{cases} \tag{21}$$

2. Central limit theorem. *The sequence of random variables*

$$N^{-1/2}\left(S_N - \mathbb{E}_p(S_N)\right) \tag{22}$$

weakly converges to a Gaussian random variable with strictly positive variance $\sigma^2 = \mathscr{F}_p''(0) > 0$*. For* $p = 1/2$, $\sigma^2 = 1$*.*

Proof. The expression (18) shows that $\mathscr{F}_p$ is differentiable as a function of λ, hence the first statement follows from the Gärtner Ellis theorem [4]. The second statement follows from the fact that $\mathscr{F}_p$ is analytic in a neighborhood of the origin, which again follows directly from the explicit expression. The statements for $p = 1/2$ are immediate from the fact that in that case the joint distribution of $\{\sigma_i\sigma_{2i}, i \in \mathbb{N}\}$ coincides with the joint distribution of a sequence of independent Bernoulli(1/2) variables. $\square$

Remark 4.1. *Notice that we computed the large deviation rate function in the* ± 1 *setting. If one considers a Bernoulli measure* $\mathbb{Q}_p$ *on* $\{0,1\}^{\mathbb{Z}}$*, with* $\mathbb{Q}_p(\eta_i = 1) = p$*, then the large deviations of the sums*

$$\sum_{i=1}^{n} \eta_i\eta_{2i} \tag{23}$$

correspond to the large deviations of

$$\sum_{i=1}^{n}(1+\sigma_i)(1+\sigma_{2i}) = n + \sum_{i=1}^{n}(\sigma_i + \sigma_{2i}) + \sum_{i=1}^{n}\sigma_i\sigma_{2i}$$

where σ *is distributed according to* $\mathbb{P}_p$ *on* $\{+,-\}^{\mathbb{Z}}$*. In particular, the free energy for the large deviations of* (23) *under the measure* $\mathbb{Q}_{1/2}$ *corresponds to a free energy of the* σ *spins with non-zero magnetic field.*

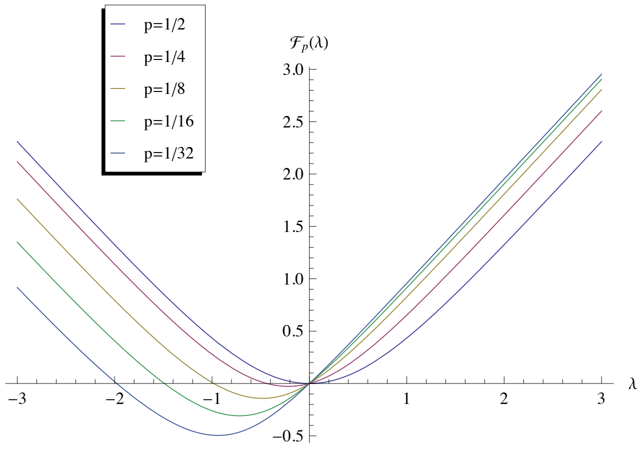

Figure 1: Plot of the free energy function for different p values. The graph has been obtained from formula (18) truncating the sum to the first 100 terms.

Remark 4.2. *A plot of the free energy for a few values of* p *is shown in Figure 1 (it is enough to analyze values in* $(0, 1/2]$ *since* $\mathscr{F}_p(\lambda) = \mathscr{F}_{1-p}(\lambda)$*). In the general case* $p \neq 1/2$ *it is interesting to compare our results to the independent case. To this aim one consider the sum* $\sum_{i=1}^{N} \xi_i \eta_i$ *where* ξ_i, η_i *are two sequences of i.i.d. Bernoulli of parameter* p*. Note that in this case the family* $\{\xi_i \eta_i\}_{i \in \{1,\ldots,N\}}$ *is made of independent Bernoulli random variables with parameter* $p^2 + (1-p)^2$*. An immediate computation of the free energy yields on this case*

$$\mathscr{F}_p^{(ind)}(\lambda) = \log\left([p^2 + (1-p)^2]e^{\lambda} + 2p(1-p)e^{-\lambda}\right) \tag{24}$$

This free energy is compared to that of formula (18) in Figure 2.

In particular one can analyze the behaviour of the minimum of the free energy functions in the two cases, corresponding to the negative value of the large deviation rate function computed at zero. This is shown in Figure 3, which suggests a general inequality between the two cases.

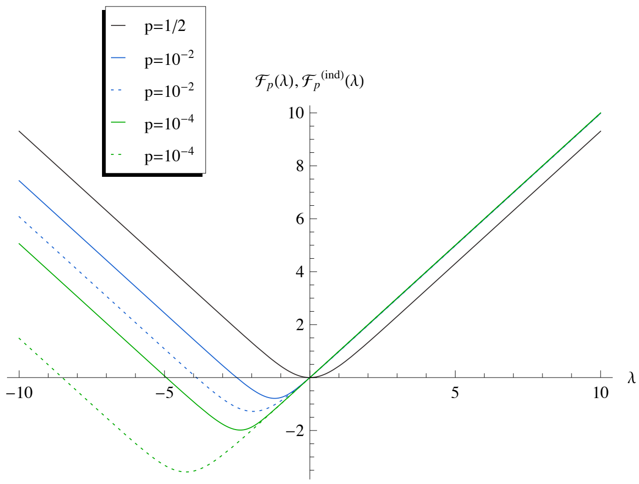

Figure 2: Plot of the free energy function (18) (continuous line) and of the free energy of the independent case (24) (dashed line).

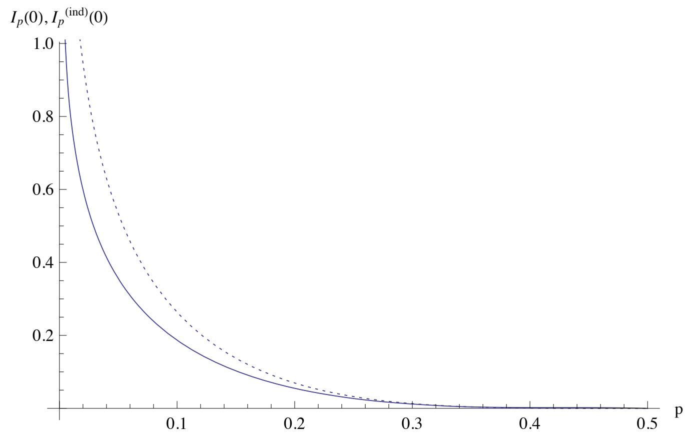

Figure 3: The negative of the minimum of the free energy as a function of p for the two cases (18) and (24).

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