

Integrable heat conduction model

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Abstract

We consider a stochastic process of heat conduction where energy is redistributed along a chain between nearest neighbor sites via an improper beta distribution. Similar to the well-known Kipnis-Marchioro-Presutti (KMP) model, the finite chain is coupled at its ends with two reservoirs that break the conservation of energy when working at different temperatures. At variance with KMP, the model considered here is integrable and one can write in a closed form the n -point correlation functions of the non-equilibrium steady state. As a consequence of the exact solution one can directly prove that the system is in a ‘local equilibrium’ and described at the macro-scale by a product measure. Integrability manifests itself through the description of the model via the open Heisenberg chain with non-compact spins. The algebraic formulation of the model allows to interpret its duality relation with a purely absorbing particle system as a change of representation.

1 Introduction

In their paper from 1982 [1], Kipnis, Marchioro and Presutti (KMP) introduced a solvable model of heat conduction. They considered a chain of energy-exchanging harmonic oscillators, that is coupled to Gibbs reservoirs of different temperatures at the two ends of the chain. They successfully implemented the idea of modelling the non-harmonic effects by a stochastic process which redistributes ‘microcanonically’ the energy between nearest-neighbor oscillators, i.e. keeping constant their total energy. The pioneering idea behind the KMP model was substantially expanded in several later works and found applications in different contexts such as fluctuations and large deviations of the temperature profile and of the current [2, 3, 4, 5], dynamical phase transitions [6], anomalous heat conduction of momentum conserving systems [7, 8, 9, 10], duality theory of Markov processes and hidden symmetries [11, 12].

The KMP model was thus the first mathematical model of energy-transport where it was possible to rigorously prove that the non-equilibrium steady state has the local equilibrium property and furthermore Fourier’s law with a constant conductivity holds. The main reason behind the solvability of the KMP process is “duality”, i.e. the existence of an associated dual process that is simpler to study. By duality, the computation of the n -point correlations functions in the non-equilibrium steady state is reduced to the problem of the exit distribution of n dual particles. Propagation of chaos and local equilibrium were then obtained by showing that, in the macroscopic limit, the dual particles essentially behave like independent particles [1].

However, the full microscopic description of the non-equilibrium stationary state remained inaccessible in the KMP model, as the absorption probabilities of the dual particles have not been computed in closed-form. In this paper we consider an energy-redistribution model which, while being similar to the one of KMP, allows a full description *at microscopic level* of the non-equilibrium steady state. More precisely, for the integrable heat conduction model considered in this paper we characterize the stationary measure (non-equilibrium steady state) by computing its multivariate moments. From the exact solution we establish that when the reservoir parameters are equal (equilibrium) we have a Gibbs measure corresponding to a product of Gamma distributions. If instead the reservoir parameters are different (non-equilibrium) we find long-range correlations. Local equilibrium, i.e. a product measure around each *macroscopic* point with inhomogeneous parameters interpolating between the reservoir parameters, is then recovered as a consequence of the exact solution.

The closed form expression for the multivariate moments paves the way to the rigorous study of density fluctuations (both typical fluctuations as well as large deviations) for non-equilibrium steady states [13], with the possibility of a direct comparison to the predictions of the Macroscopic Fluctuation Theory (MFT) [5] developed for boundary driven diffusive systems.

Added note. In the progress of this work, we became aware of the interesting article [14] that relates to the model studied in this paper. More precisely, the authors considered the most general model of energy redistribution with a redistribution rule that is of “zero-range” type, namely the rate at which the energy is moved between two sites is just a function of the energy at the site where it is taken off. Remarkably, within this large class of models, the authors show that the model with generator (2.2) studied in this paper, as well as the model with generator (2.13) studied in [15, 16], emerge as the only two models of “zero-range” type where the following three properties are fulfilled: *i)* they are of gradient type; *ii)* they have a product measure at equilibrium; *iii)* they have constant diffusivity and quadratic mobility, which in turn guarantees the possibility of computing large deviation functions using the approach of Macroscopic Fluctuation Theory [5]. Thus, within the class of “zero-range”

energy redistribution models, the explicit writing of the microscopic correlation functions in the non-equilibrium steady state (a fact that is rooted in the duality and integrability of the models) seems to be related to the solvability of the variational problem emerging in the study of large deviation functions via MFT.

Paper organization. The paper is organized as follows. In Section 2.1 we define our model and state our main results, first in the equilibrium set-up (Section 2.3), then for the non-equilibrium setting (Section 2.4). We also comment on the relation of our model to other energy and/or particle redistribution models previously considered in the literature in Section 2.2. The proof of our results are contained in Section 3, which contains the verification of duality via a direct computation and the proof of the main results. Finally in Section 4 duality is derived from the algebraic structure of the model, which is essentially given by the open XXX Heisenberg chain with non-compact spins. In Appendix A a direct proof of reversibility at equilibrium is given, while Appendix B contains some formulas used in the proof of the algebraic structure of the model.

2 The model and the main result

2.1 Model definition

We consider a one-dimensional lattice system of interacting particles. For $i \in \{1, \dots, N\}$ we denote by $y_i \in \mathbb{R}_+ := (0, \infty)$ the energy of the i^{th} particle and we collect all the energies into the vector $y = (y_1, \dots, y_N) \in \mathbb{R}_+^N$.

The system undergoes a stochastic evolution defined by a Markov process $\{y(t), t \geq 0\}$ with generator working on function $f : \mathbb{R}_+^N \rightarrow \mathbb{R}$ given by

$$\mathcal{L} = \mathcal{L}_1 + \sum_{i=1}^{N-1} \mathcal{L}_{i,i+1} + \mathcal{L}_N . \quad (2.1)$$

The bulk term $\mathcal{L}_{i,i+1}$, which depends on a parameter $s > 0$, describes the exchange of energy between the particles on the bond $(i, i+1)$; it will be convenient to further split it as

$$\mathcal{L}_{i,i+1} = \mathcal{L}_{i,i+1}^{\rightarrow} + \mathcal{L}_{i,i+1}^{\leftarrow} \quad (2.2)$$

where

$$\mathcal{L}_{i,i+1}^{\rightarrow} f(y) = \int_0^{y_i} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_i}\right)^{2s-1} [f(y_1, \dots, y_i - \alpha, y_{i+1} + \alpha, \dots, y_N) - f(y)] , \quad (2.3)$$

and

$$\mathcal{L}_{i,i+1}^{\leftarrow} f(y) = \int_0^{y_{i+1}} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_{i+1}}\right)^{2s-1} [f(y_1, \dots, y_i + \alpha, y_{i+1} - \alpha, \dots, y_N) - f(y)] . \quad (2.4)$$

Thus $\mathcal{L}_{i,i+1}^{\rightarrow}$ describes the stochastic movement of energy towards the right, i.e. from site i to site $i+1$, and similarly $\mathcal{L}_{i,i+1}^{\leftarrow}$ describes the movement of energy towards the left. Obviously, the process generated by the bulk generator preserves the total energy $\sum_{i=1}^N y_i$. For a discussion about the origin and interpretation of such generator see Section 2.2.

The boundary terms, which depend on additional parameters $\lambda_L > 0, \lambda_R > 0$, are given by

$$\begin{aligned}\mathcal{L}_1 f(y) &= \int_0^{y_1} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_1}\right)^{2s-1} [f(y_1 - \alpha, \dots, y_N) - f(y)] \\ &\quad + \int_0^{+\infty} \frac{d\alpha}{\alpha} e^{-\lambda_L \alpha} [f(y_1 + \alpha, \dots, y_N) - f(y)] ,\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}\mathcal{L}_N f(y) &= \int_0^{y_N} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_N}\right)^{2s-1} [f(y_1, \dots, y_N - \alpha) - f(y)] \\ &\quad + \int_0^{+\infty} \frac{d\alpha}{\alpha} e^{-\lambda_R \alpha} [f(y_1, \dots, y_N + \alpha) - f(y)] .\end{aligned}\tag{2.6}$$

They model the reservoirs which insert and remove energy at the left and right ends of the chain.

Thus, the full process generated by (2.1) takes value on \mathbb{R}_+^N and it does not conserve the total energy. In fact, it is a model for heat conduction (or energy transport) from one side of the chain to the other.

Remark. *In the following we will consider the action of the generator (2.1) on polynomial functions. We do not address here the problem of characterizing the domain of the generator.*

2.2 Comparison to other energy-redistribution models

The stochastic energy redistribution rule encoded in the bulk generator (2.2) can be described as follows. Fix a couple $(i, i+1)$ and consider the movement of energy to the right described by $\mathcal{L}_{i,i+1}^{\rightarrow}$ (a similar reasoning can be made for the movement of energy to the left described by $\mathcal{L}_{i,i+1}^{\leftarrow}$). For $a, b > 0$ let U be a random variable with Beta(a, b) distribution, i.e. having probability density

$$\sigma_{a,b}(u) = \frac{u^{a-1} (1-u)^{b-1}}{B(a,b)} ,$$

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the Beta function. Consider the process where the energy at site i evolves as follows: at rate 1 energy is exchanged between the pair $(i, i+1)$ by drawing a number $0 \leq u \leq 1$, independently of everything else, according to the Beta(a, b) distribution and a fraction uy_i of the energy at site i is moved to site $i+1$ and all other energies remain unchanged. This is described by the generator $L_{i,i+1}^{\rightarrow, a, b}$ defined as

$$\begin{aligned}L_{i,i+1}^{\rightarrow, a, b} f(y) &= \int_0^1 \sigma_{a,b}(u) [f(y_1, \dots, y_i - y_i u, y_{i+1} + y_i u, \dots, y_N) - f(y)] du \\ &= \mathbb{E} [f(y_1, \dots, y_i - y_i U, y_{i+1} + y_i U, \dots, y_N) - f(y)] .\end{aligned}\tag{2.7}$$

Speeding up the time by a factor $B(a, b)$ and applying the change of variable $u = \alpha/y_i$, then one can see that

$$\mathcal{L}_{i,i+1}^{\rightarrow} = \lim_{\substack{a \rightarrow 0 \\ b \rightarrow 2s}} B(a, b) \cdot L_{i,i+1}^{\rightarrow, a, b} .\tag{2.8}$$

Thus one gets that $\mathcal{L}_{i,i+1}^{\rightarrow}$ corresponds to a process with infinite intensity, which is indeed a pure jump Levy process with Levy measure $\nu(d\alpha) = \frac{1}{\alpha} \left(1 - \frac{\alpha}{y_i}\right)^{2s-1} d\alpha$.

The description above of the process allows its comparison to other models of energy redistribution.

KMP model. The original KMP model [1] was defined by a uniform redistribution rule for the sum of the energies of nearest neighbour sites and by instantaneous thermalization with reservoirs imposing an exponential (Gibbs) distribution at the boundaries. Here we consider its generalized version, introduced in [12], where the redistribution rule is given in terms of a Beta($2s, 2s$) random variable and the reservoirs thermalize with Gamma distributions of shape parameter $2s > 0$, so that the original KMP is recovered for $s = 1/2$. Namely, we consider the process with generator

$$\mathcal{L}^{\text{KMP}} = \mathcal{L}_1^{\text{KMP}} + \sum_{i=1}^{N-1} \mathcal{L}_{i,i+1}^{\text{KMP}} + \mathcal{L}_N^{\text{KMP}} \quad (2.9)$$

where we have

$$\mathcal{L}_{i,i+1}^{\text{KMP}} f(y) = \int_0^1 \sigma_{2s,2s}(w) [f(y_1, \dots, w(y_i + y_{i+1}), (1-w)(y_i + y_{i+1}), \dots, y_N) - f(y)] dw,$$

while

$$\mathcal{L}_1^{\text{KMP}} f(y) = \int_0^{+\infty} dy'_1 \frac{(\lambda_L)^{2s}}{\Gamma(2s)} (y'_1)^{2s-1} e^{-\lambda_L y'_1} [f(y'_1, \dots, y_N) - f(y)] \quad (2.10)$$

and

$$\mathcal{L}_N^{\text{KMP}} f(y) = \int_0^{+\infty} dy'_N \frac{(\lambda_R)^{2s}}{\Gamma(2s)} (y'_N)^{2s-1} e^{-\lambda_R y'_N} [f(y_1, \dots, y'_N) - f(y)] . \quad (2.11)$$

Comparing the bulk generators of our process to the one of the KMP process we see that the generator (2.1) is associated to a redistribution rule of the type

$$(y_i, y_{i+1}) \longrightarrow \begin{cases} \left(y_i - y_i U, y_{i+1} + y_i U \right) & \text{with probability } \frac{1}{2}, \\ \left(y_i + y_{i+1} V, y_{i+1} - V y_{i+1} \right) & \text{with probability } \frac{1}{2}, \end{cases}$$

whereas the KMP process is associated to

$$(y_i, y_{i+1}) \longrightarrow \left((y_i + y_{i+1})W, (y_i + y_{i+1})(1-W) \right).$$

Here U denotes an “improper” Beta($0, 2s$) distribution, V denotes an independent copy of U , and W denotes a Beta($2s, 2s$) distribution.

As for the reservoirs, we remark that the boundary generators of our integrable heat transport (2.5, 2.6) as well as those of the generalized KMP model (2.10, 2.11) are both reversible with respect to the Gamma distribution, although they are substantially different. A proof of this can be found in Appendix A.

Immediate exchange model. The heat conduction of this paper gets its integrable structure from the improper Beta random variables used in the energy distribution rule. The process constructed using the generator (2.7) with standard Beta random variables is known as the “Immediate Exchange Model”, and it has been studied in the economics literature as a model of wealth redistribution. See [17, 18] and references therein for the study of its duality properties and [19] for the study of its spectral gap problem.

“Harmonic” model. Our model arises as the scaling limit of an interacting particle systems, the “harmonic” model introduced in [15], so-called because it involves harmonic numbers. The harmonic model is integrable and was solved in [16]. Infact, as we shall prove later, our model is also integrable and its solution relies on the fact that the boundary-driven harmonic model and our model have *the same absorbing dual process*.

To see the scaling limit, we recall that the harmonic process is an interacting particle system defined by the generator

$$\mathcal{L}^{\text{Har}} = \mathcal{L}_1^{\text{Har}} + \sum_{i=1}^{N-1} \mathcal{L}_{i,i+1}^{\text{Har}} + \mathcal{L}_N^{\text{Har}}, \quad (2.12)$$

where the bulk term reads

$$\mathcal{L}_{i,i+1}^{\text{Har}} = \mathcal{L}_{i,i+1}^{\text{Har},\rightarrow} + \mathcal{L}_{i,i+1}^{\text{Har},\leftarrow} \quad (2.13)$$

with

$$(\mathcal{L}_{i,i+1}^{\text{Har},\rightarrow} f)(\eta) = \sum_{k=1}^{\eta_i} \varphi_s(k, \eta_i) \left[f(\eta - k\delta_i + k\delta_{i+1}) - f(\eta) \right]$$

and

$$(\mathcal{L}_{i,i+1}^{\text{Har},\leftarrow} f)(\eta) = \sum_{k=1}^{\eta_{i+1}} \varphi_s(k, \eta_{i+1}) \left[f(\eta + k\delta_i - k\delta_{i+1}) - f(\eta) \right].$$

Here $\eta \in \mathbb{N}^N$ denotes a particle configuration, with η_i being the number of particles at site i , and δ_i denotes the configuration with only particle placed at site i . The rates $\varphi_s : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}_+$ are defined by

$$\varphi_s(k, n) := \frac{1}{k} \frac{\Gamma(n+1)\Gamma(n-k+2s)}{\Gamma(n-k+1)\Gamma(n+2s)} \mathbb{1}_{\{1,2,\dots,n\}}(k) \quad (2.14)$$

where $\mathbb{1}_A$ is the indicator function of the set A . For the boundary terms we have

$$(\mathcal{L}_1^{\text{Har}} f)(\eta) = \sum_{k=1}^{\eta_1} \varphi_s(k, \eta_1) \left[f(\eta - k\delta_1) - f(\eta) \right] + \sum_{k=1}^{\infty} \frac{\beta_L^k}{k} \left[f(\eta + k\delta_1) - f(\eta) \right] \quad (2.15)$$

and

$$(\mathcal{L}_N^{\text{Har}} f)(\eta) = \sum_{k=1}^{\eta_N} \varphi_s(k, \eta_N) \left[f(\eta - k\delta_N) - f(\eta) \right] + \sum_{k=1}^{\infty} \frac{\beta_R^k}{k} \left[f(\eta + k\delta_i) - f(\eta) \right]. \quad (2.16)$$

Denote by $\{\eta(t), t \geq 0\}$ the boundary-driven “harmonic” process with generator (2.12). Introducing the scaling parameter $\epsilon > 0$, one can show that the process $\{\epsilon\eta(t), t \geq 0\}$ with β_L, β_R such that $\epsilon^{-1}\beta_L = \lambda_L(1 + o(1))$ and $\epsilon^{-1}\beta_R = \lambda_R(1 + o(1))$ converges (weakly) to the process $\{y(t), t \geq 0\}$ with generator (2.1), as $\epsilon \rightarrow 0$, see the Appendix A of [15] for details. Both for the bulk and for the boundary, the scaling is obtained by using the asymptotics

$$\frac{\Gamma[z + \gamma_1]}{\Gamma[z + \gamma_2]} \simeq z^{\gamma_1 - \gamma_2} \left(1 + O\left(\frac{1}{z}\right) \right) \quad \text{as } z \rightarrow \infty. \quad (2.17)$$

2.3 Equilibrium

For all system sizes N and all choices of the reservoirs parameters $\lambda_L, \lambda_R > 0$, the Markov process $\{y(t), t \geq 0\}$ with generator (2.1) has a unique stationary measure, supported on \mathbb{R}_+^N , that we shall denote μ_N . We avoid writing the dependence on λ_L, λ_R to not burden the notation.

When the system is in an equilibrium set-up, i.e when $\lambda_L = \lambda_R = \lambda$, the heat conduction model has an invariant Gibbs product measure.

Theorem 2.1 (Equilibrium reversible measure). *If $\lambda_L = \lambda_R = \lambda$ then the product probability measure with marginal the Gamma distribution with shape parameter $2s > 0$ and rate parameter $\lambda > 0$, i.e.*

$$\mu_N(dy) = \prod_{i=1}^N \frac{\lambda^{2s}}{\Gamma(2s)} y_i^{2s-1} e^{-\lambda y_i} dy_i, \quad (2.18)$$

is stationary. Furthermore it is also reversible.

Stationarity will be deduced as a consequence of duality in Section 3. Reversibility is proven in Appendix A by a direct computation showing that the generator (2.1) is self-adjoint in the Hilbert space $L_2(\mathbb{R}_+^N, \mu_N)$.

2.4 Non-equilibrium steady state

In a non-equilibrium set-up, i.e. when $\lambda_L \neq \lambda_R$, reversibility is lost and the stationary measure μ_N is non-product. Our main result is the characterization of the stationary measure in non-equilibrium through the computation of its multivariate moments.

Theorem 2.2 (Non-equilibrium steady state). *Define the left and right “reservoir temperatures” as*

$$T_L = \frac{1}{\lambda_L} \quad \text{and} \quad T_R = \frac{1}{\lambda_R}. \quad (2.19)$$

Then we have:

- *For a multi-index $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{N}_0^N$, the multivariate moments of the stationary measure of the process generated by (2.1) are given by*

$$\int_{\mathbb{R}_+^N} \left[\prod_{i=1}^N y_i^{\xi_i} \frac{\Gamma(2s)}{\Gamma(2s + \xi_i)} \right] \mu_N(dy) = \sum_{n=0}^{|\xi|} T_R^{|\xi|-n} (T_L - T_R)^n g_\xi(n) \quad (2.20)$$

where $|\xi| = \sum_{i=1}^N \xi_i$ and

$$g_\xi(n) = \sum_{\substack{(n_1, \dots, n_N) \in \mathbb{N}_0^N \\ n_1 + \dots + n_N = n}} \binom{\xi_i}{n_i} \prod_{j=1}^{n_i} \frac{2s(N+1-i) - j + \sum_{k=i}^N n_k}{2s(N+1) - j + \sum_{k=i}^N n_k}. \quad (2.21)$$

- *Equivalently, interpreting the multi-index $\xi = \sum_{i=1}^N \xi_i \delta_i$ as a configuration of (dual) particles located at positions $(x_k)_{1 \leq k \leq |\xi|}$ with $x_1 \leq x_2 \leq \dots \leq x_{|\xi|}$, we have*

$$\int_{\mathbb{R}_+^N} \left[\prod_{i=1}^N y_i^{\xi_i} \frac{\Gamma(2s)}{\Gamma(2s + \xi_i)} \right] \mu_N(dy) = \sum_{n=0}^{|\xi|} T_R^{|\xi|-n} (T_L - T_R)^n g_x(n) \quad (2.22)$$

with

$$g_x(n) = \sum_{1 \leq i_1 < \dots < i_n \leq |\xi|} \prod_{\alpha=1}^n \frac{n - \alpha + 2s(N+1 - x_{i_\alpha})}{n - \alpha + 2s(N+1)}. \quad (2.23)$$

Remark (Recovering the equilibrium case). *If one chooses equal reservoir parameters in Theorem 2.2 then one finds the moments of the equilibrium stationary measure of Theorem 2.1. Indeed, setting $T_L = T_R = T$, only the term $n = 0$ survives in the right hand side of (2.20) yielding*

$$\int_{\mathbb{R}_+^N} \left[\prod_{i=1}^N y_i^{\xi_i} \frac{\Gamma(2s)}{\Gamma(2s + \xi_i)} \right] \mu_N(dy) = T^{|\xi|}.$$

Remark (Long-range correlations). *Considering the dual configuration with just one dual particle at site i one deduces a microscopic linear profile*

$$\int_{\mathbb{R}_+^N} y_i \mu_N(dy) = 2s \left(T_L + \frac{i}{N+1} (T_R - T_L) \right). \quad (2.24)$$

The expression for the multivariate moments given in (2.22), (2.23) implies long-range correlations. For instance the covariance between the energies at two sites $i < j$ read

$$\begin{aligned} \int_{\mathbb{R}_+^N} y_i y_j \mu_N(dy) - \left(\int_{\mathbb{R}_+^N} y_i \mu_N(dy) \right) \left(\int_{\mathbb{R}_+^N} y_j \mu_N(dy) \right) \\ = (2s)^2 (T_L - T_R)^2 \frac{i}{(N+1)^2} \frac{N+1-j}{(2s(N+1)+1)} \end{aligned} \quad (2.25)$$

More generally, considering the cumulants κ_n of $n \geq 2$, the energies at the microscopic points $i_1 < \dots < i_n$, for large N behave as

$$\lim_{N \rightarrow \infty} N^{n-1} \kappa_n = f_n(u_1, \dots, u_n) (T_R - T_L)^n, \quad (2.26)$$

where $i_k = \lfloor Nu_k \rfloor$ and appropriate functions f_n (the first being $f_2(u_1, u_2) = (2s)u_1(1 - u_2)$).

Although Theorem 2.2 shows that in non-equilibrium the stationary measure is not a Gibbs measure, it is possible to show that in the thermodynamic limit $N \rightarrow \infty$ the non-equilibrium stationary measure approaches *locally* a Gibbs distribution, i.e. the integrable heat conduction model satisfies *local equilibrium*. Furthermore, transport of energy across the system satisfies Fourier's law.

To formalize this, let \mathcal{O} be the algebra of cylindrical bounded functions on \mathbb{R}_+^N and denote by τ_i the translation by i , i.e. for all function $f \in \mathcal{O}$ define $(\tau_i f)(j) = f(i+j)$. For a measure μ , we denote by $\mathbb{E}_\mu(\cdot)$ the expectation under this measure.

Theorem 2.3 (Local equilibrium & Fourier's law). *Let T_L, T_R be defined by (2.19) and let μ_N be the unique invariant measure of the process generated by (2.1). Then the following holds:*

(i) *For $u \in (0, 1)$*

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}(\tau_{\lfloor uN \rfloor} f) = \mathbb{E}_{\nu_\lambda(u)}(f) \quad \forall f \in \mathcal{O} \quad (2.27)$$

where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$, $\lambda(u) = 1/T(u)$ and

$$T(u) = T_L + (T_R - T_L)u, \quad (2.28)$$

and ν_λ is the product measure supported on \mathbb{R}_+^N with marginals given by a Gamma distributions with shape parameter $2s > 0$ and rate parameter $\lambda > 0$

$$\nu_\lambda(dy) = \prod_{i=1}^{\infty} \frac{\lambda^{2s}}{\Gamma(2s)} y_i^{2s-1} e^{-\lambda y_i} dy_i. \quad (2.29)$$

(ii) Define the stationary energy flux between two neighbor sites $i, i+1$ by

$$J_{i,i+1} = \int_{R_+^N} \mu_N(dy) (y_i - y_{i+1}), \quad (2.30)$$

and the total stationary current in the thermodynamic limit as

$$J = \lim_{N \rightarrow \infty} \sum_{i=1}^{N-1} J_{i,i+1}. \quad (2.31)$$

Then Fourier's law holds, namely

$$J = -K_s \frac{dT(u)}{du}, \quad u \in (0, 1), \quad (2.32)$$

where the conductivity $K_s = 2s$ and the macroscopic temperature profile $T(\cdot)$ is defined by (2.28).

3 Proofs of the results

In this section we prove the theorems stated previously in Section 2. The main ingredient used is the duality relation which is discussed below. More precisely, we will show that the open heat conduction model is dual to a purely absorbing particle system. This dual process is the same dual process obtained for the boundary-driven ‘‘harmonic’’ model in [16] but, as we will see, with a different duality function.

The absorbing dual process and the associated duality function are identified in Section 3.1. Once the duality result is established, we can use the expression for the absorption probabilities of the dual particles that were obtained in [16] to prove Theorem 2.2, see Section 3.3. The case of equilibrium, i.e. Theorem 2.1, does not require the explicit form of the absorption probabilities and is treated separately in Section 3.2.

Theorem 2.3 about local equilibrium and Fourier's law will be proved in Section 3.4. Contrary to the usual situation of non-integrable models where one needs a coupling argument with independent particles, here local equilibrium is obtained by a direct computation that uses the explicit form of the multi-point correlation function of Theorem 2.2.

The fact that the heat conduction model studied in this paper and the boundary-driven ‘harmonic’ model considered in [16] have the same dual process can be explained by the fact that the two models both arise from the integrable open XXX Heisenberg spin chain when considering different representations of the non-compact $\mathfrak{sl}(2)$ Lie algebra. This is discussed in Section 4.

3.1 The associated dual process

We introduce the Markov process obtained in [16] which is related via duality to the one of the heat conduction model in (2.1).

Definition 3.1 (Dual absorbing process). *For $s > 0$, let the function $\varphi_s : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}_+$*

$$\varphi_s(k, n) := \frac{1}{k} \frac{\Gamma(n+1)\Gamma(n-k+2s)}{\Gamma(n-k+1)\Gamma(n+2s)} \mathbb{1}_{\{1,2,\dots,n\}}(k) \quad (3.1)$$

where $\mathbb{1}_A$ is the indicator function of the set A . We consider the Markov chain $\{\xi(t) : t \geq 0\}$ on \mathbb{N}^{N+2} whose generator $\mathcal{L}^{\text{dual}}$, acting on functions $f : \mathbb{N}^{N+2} \rightarrow \mathbb{R}$, is given by

$$\mathcal{L}^{\text{dual}} = \mathcal{L}_{0,1}^{\text{dual}} + \sum_{i=1}^{N-1} \mathcal{L}_{i,i+1}^{\text{dual}} + \mathcal{L}_{N,N+1}^{\text{dual}}, \quad (3.2)$$

where for $i \in \{1, 2, \dots, N-1\}$

$$\begin{aligned} (\mathcal{L}_{i,i+1}^{\text{dual}} f)(\xi) &= \sum_{k=1}^{\xi_i} \varphi_s(k, \xi_i) \left[f(\xi - k\delta_i + k\delta_{i+1}) - f(\xi) \right] \\ &+ \sum_{k=1}^{\xi_{i+1}} \varphi_s(k, \xi_{i+1}) \left[f(\xi + k\delta_i - k\delta_{i+1}) - f(\xi) \right], \end{aligned} \quad (3.3)$$

while

$$\mathcal{L}_{0,1}^{\text{dual}} f(\xi) = \sum_{k=1}^{\xi_1} \varphi_s(k, \xi_1) \left[f(\xi - k\delta_1 + k\delta_0) - f(\xi) \right], \quad (3.4)$$

and

$$\mathcal{L}_{N,N+1}^{\text{dual}} f(\xi) = \sum_{k=1}^{\xi_N} \varphi_s(k, \xi_N) \left[f(\xi - k\delta_N + k\delta_{N+1}) - f(\xi) \right]. \quad (3.5)$$

The Markov process $\{\xi(t) : t \geq 0\}$, initialized from a configuration $\xi \in \mathbb{N}^{N+2}$ describes the motion of $|\xi| = \xi_0 + \dots + \xi_{N+1}$ indistinguishable particles which move on the lattice sites $\{0, \dots, N+1\}$ and are absorbed at the boundary sites 0 and $N+1$, i.e. they cannot reenter the chain with the lattice sites $\{1, \dots, N\}$. Eventually, all the $|\xi|$ particles get absorbed at the boundary sites.

Theorem 3.2 (Duality). *Let $D : \mathbb{R}_+^N \times \mathbb{N}^{N+2}$ be the duality function defined by*

$$D(y, \xi) = \left(\frac{1}{\lambda_L} \right)^{\xi_0} \left(\prod_{i=1}^N d(y_i, \xi_i) \right) \left(\frac{1}{\lambda_R} \right)^{\xi_{N+1}} \quad (3.6)$$

with

$$d(y_i, \xi_i) = y_i^{\xi_i} \frac{\Gamma(2s)}{\Gamma(2s + \xi_i)}. \quad (3.7)$$

Then for every $t \geq 0$ and for all $(y, \xi) \in \mathbb{R}_+^N \times \mathbb{N}^{N+2}$ one has

$$\mathbb{E}_y \left[D(y(t), \xi) \right] = \mathbb{E}_\xi \left[D(y, \xi(t)) \right], \quad (3.8)$$

where on the left hand side the expectation is w.r.t. the process $\{y(t), t \geq 0\}$ initialized from y and on the right hand side the expectation is w.r.t. the process $\{\xi(t), t \geq 0\}$ initialized from ξ .

Proof. It is enough to prove that, for all $y \in \mathbb{R}_+^N$ and $\xi \in \mathbb{N}^{N+2}$, it holds

$$\left(\mathcal{L} D(\cdot, \xi) \right)(y) = \left(\mathcal{L}^{\text{dual}} D(y, \cdot) \right)(\xi). \quad (3.9)$$

To prove this we treat each term appearing in the sum defining the generators separately. Let's start with the bulk part and consider the local generator acting on the bond $(i, i+1)$. Its right action on the duality function (3.6) is

$$\begin{aligned} \left(\mathcal{L}_{i,i+1}^{\rightarrow} D(\cdot, \xi) \right)(y) &= T_L^{\xi_0} \left(\prod_{j \neq \{i, i+1\}} d(y_j, \xi_j) \right) T_R^{\xi_{N+1}}. \\ &\int_0^{y_i} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_i} \right)^{2s-1} \left[(y_i - \alpha)^{\xi_i} (y_{i+1} + \alpha)^{\xi_{i+1}} - y_i^{\xi_i} y_{i+1}^{\xi_{i+1}} \right] \frac{\Gamma(2s)\Gamma(2s)}{\Gamma(\xi_i + 2s)\Gamma(\xi_{i+1} + 2s)}. \end{aligned} \quad (3.10)$$

We focus on the integral on the right hand side of the above display. Using the change of variable $\alpha \rightarrow \alpha/y_i$, the integral can be rewritten as

$$\int_0^1 \frac{d\alpha}{\alpha} (1-\alpha)^{2s-1} \left[(y_i - \alpha y_i)^{\xi_i} (y_{i+1} + \alpha y_i)^{\xi_{i+1}} - y_i^{\xi_i} y_{i+1}^{\xi_i} \right].$$

Next, expanding the term $(y_{i+1} + \alpha)^{\xi_{i+1}}$ with the Newton binomial and exchanging sum and integration we get

$$\left(\sum_{\ell=0}^{\xi_{i+1}} \binom{\xi_{i+1}}{\ell} y_i^{\xi_i+\ell} y_{i+1}^{\xi_{i+1}-\ell} \int_0^1 d\alpha \alpha^{\ell-1} (1-\alpha)^{2s-1+\xi_i} \right) - y_i^{\xi_i} y_{i+1}^{\xi_{i+1}} \int_0^1 d\alpha \alpha^{-1} (1-\alpha)^{2s-1}.$$

Furthermore, using the integral representation of the Beta function

$$B(a, b) = \int_0^1 \alpha^{a-1} (y_i - \alpha)^{b-1} d\alpha \quad (3.11)$$

and isolating the term $\ell = 0$ in the summation we arrive to

$$\begin{aligned} \int_0^{y_i} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_i} \right)^{2s-1} \left[(y_i - \alpha)^{\xi_i} (y_{i+1} + \alpha)^{\xi_{i+1}} - y_i^{\xi_i} y_{i+1}^{\xi_i} \right] = \\ \left(\sum_{\ell=1}^{\xi_{i+1}} \binom{\xi_{i+1}}{\ell} y_i^{\xi_i+\ell} y_{i+1}^{\xi_{i+1}-\ell} B(\ell, \xi_i + 2s) \right) + y_i^{\xi_i} y_{i+1}^{\xi_{i+1}} \int_0^1 d\alpha \alpha^{-1} (1-\alpha)^{2s-1} \left[(1-\alpha)^{\xi_i} - 1 \right]. \end{aligned}$$

Inserting this into (3.10) we get

$$\left(\mathcal{L}_{i,i+1}^{\rightarrow} D(\cdot, \xi) \right)(y) = \text{I} + \text{II} \quad (3.12)$$

where

$$\text{I} = T_L^{\xi_0} \left(\prod_{j \neq \{i, i+1\}} d(y_j, \xi_j) \right) T_R^{\xi_{N+1}} \cdot \left(\sum_{\ell=1}^{\xi_{i+1}} \binom{\xi_{i+1}}{\ell} y_i^{\xi_i+\ell} y_{i+1}^{\xi_{i+1}-\ell} B(\ell, \xi_i + 2s) \right) \frac{\Gamma(2s)\Gamma(2s)}{\Gamma(\xi_i + 2s)\Gamma(\xi_{i+1} + 2s)}$$

and

$$\text{II} = T_L^{\xi_0} \left(\prod_{j \neq \{i, i+1\}} d(y_j, \xi_j) \right) T_R^{\xi_{N+1}} \cdot \left(y_i^{\xi_i} y_{i+1}^{\xi_{i+1}} \int_0^1 d\alpha \alpha^{-1} (1-\alpha)^{2s-1} \left[(1-\alpha)^{\xi_i} - 1 \right] \right) \frac{\Gamma(2s)\Gamma(2s)}{\Gamma(\xi_i + 2s)\Gamma(\xi_{i+1} + 2s)}.$$

The first term I can be written in terms of Gamma functions as

$$\begin{aligned} \text{I} = T_L^{\xi_0} \left(\prod_{j \neq \{i, i+1\}} d(y_j, \xi_j) \right) T_R^{\xi_{N+1}} \cdot \\ \sum_{\ell=1}^{\xi_{i+1}} \frac{\Gamma(\xi_{i+1} + 1)}{\Gamma(\ell + 1)\Gamma(\xi_{i+1} - \ell + 1)} y_i^{\xi_i+\ell} y_{i+1}^{\xi_{i+1}-\ell} \frac{\Gamma(\ell)\Gamma(\xi_i + 2s)}{\Gamma(\xi_i + \ell + 2s)} \frac{\Gamma(2s)\Gamma(2s)}{\Gamma(\xi_i + 2s)\Gamma(\xi_{i+1} + 2s)}. \end{aligned}$$

Recalling the definition (3.6) of the duality function and the definition (3.1) of the function $\varphi_s(k, n)$ one finds

$$\text{I} = \sum_{\ell=1}^{\xi_{i+1}} \varphi_s(\ell, \xi_{i+1}) D(y; \xi + \ell \delta_i - \ell \delta_{i+1}). \quad (3.13)$$

For the second term Π , we observe that

$$\int_0^1 d\alpha \alpha^{-1} (1-\alpha)^{2s-1} \left[(1-\alpha)^{\xi_i} - 1 \right] = \psi(2s) - \psi(\xi_i + 2s) \quad (3.14)$$

where ψ indicates the digamma function (i.e. the logarithmic derivative of the Gamma function). Here we used integral representation of the digamma function

$$\psi(z+1) = -\gamma_e + \int_0^1 d\beta \frac{1-\beta^z}{1-\beta}, \quad (3.15)$$

with γ_e the Euler–Mascheroni constant. Therefore, recalling again the definition (3.6) of the duality function, we get

$$\Pi = \left(\psi(2s) - \psi(\xi_i + 2s) \right) D(y; \xi). \quad (3.16)$$

We then proceed by observing that

$$\psi(2s) - \psi(\xi_i + 2s) = - \sum_{\ell=1}^{\xi_i} \varphi_s(\ell, \xi_i). \quad (3.17)$$

This identity can be shown by writing the rates (3.1) in terms of the Beta function

$$\varphi_s(k, n) = \binom{n}{k} B(k, n-k+2s) \quad (3.18)$$

and using the integral representation of the Beta function (3.11) and of the digamma function (3.15) we find

$$\sum_{k=1}^n \varphi_s(k, n) = \int_0^1 dt \frac{t^{n+2s-1}}{1-t} \sum_{k=1}^n \binom{n}{k} \left(\frac{1-t}{t} \right)^k = \int_0^1 dt \frac{t^{2s-1}}{1-t} (1-t^n) = \psi(n+2s) - \psi(2s).$$

Thus, combining together (3.16) and (3.17) we arrive to

$$\Pi = - \sum_{\ell=1}^{\xi_i} \varphi_s(\ell, \xi_i) D(y; \xi). \quad (3.19)$$

Inserting the expressions (3.13) and (3.19) into the right hand side of (3.12), we get

$$\left(\mathcal{L}_{i,i+1}^{\rightarrow} D(\cdot, \xi) \right)(y) = \sum_{\ell=1}^{\xi_{i+1}} \varphi_s(\ell, \xi_{i+1}) D(y; \xi + \ell \delta_i - \ell \delta_{i+1}) - \sum_{\ell=1}^{\xi_i} \varphi_s(\ell, \xi_i) D(y; \xi).$$

If we now define

$$h_s(n) = \sum_{k=1}^n \varphi_s(k, n), \quad (3.20)$$

and we recall Definition 3.1 of the dual process, then we have shown that

$$\left(\mathcal{L}_{i,i+1}^{\rightarrow} D(\cdot, \xi) \right)(y) = \left(\mathcal{L}_{i,i+1}^{\leftarrow, \text{dual}} D(y, \cdot) \right)(\xi) + \left(h_s(\xi_{i+1}) - h_s(\xi_i) \right) D(y, \xi).$$

Repeating the same computation for the left part of the bulk generator leads to

$$\left(\mathcal{L}_{i,i+1}^{\leftarrow} D(\cdot, \xi)\right)(y) = \left(\mathcal{L}_{i,i+1}^{\rightarrow, \text{dual}} D(y, \cdot)\right)(\xi) + \left(h_s(\xi_i) - h_s(\xi_{i+1})\right) D(y, \xi).$$

All in all, adding up the last two expressions one arrives to

$$\left(\mathcal{L}_{i,i+1} D(\cdot, \xi)\right)(y) = \left(\mathcal{L}_{i,i+1}^{\text{dual}} D(y, \cdot)\right)(\xi), \quad (3.21)$$

which proves bulk duality.

We now consider the left reservoir. Spelling out the action of the generator \mathcal{L}_1 on the duality function D we get

$$\left(\mathcal{L}_1 D(\cdot, \xi)\right)(y) = \text{III} + \text{IV} \quad (3.22)$$

where

$$\text{III} = T_L^{\xi_0} \int_0^{y_1} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_1}\right)^{2s-1} \left[(y_1 - \alpha)^{\xi_1} - y_1^{\xi_1}\right] \frac{\Gamma(2s)}{\Gamma(2s + \xi_1)} \left(\prod_{j=2}^N d(y_j, \xi_j)\right) T_R^{\xi_{N+1}}$$

and

$$\text{IV} = T_L^{\xi_0} \int_0^\infty \frac{d\alpha}{\alpha} e^{-\lambda_L \alpha} \left[(y_1 + \alpha)^{\xi_1} - y_1^{\xi_1}\right] \frac{\Gamma(2s)}{\Gamma(2s + \xi_1)} \left(\prod_{j=2}^N d(y_j, \xi_j)\right) T_R^{\xi_{N+1}}.$$

For the term III we change variable $\alpha \rightarrow \alpha/y_1$ so that we get

$$\text{III} = T_L^{\xi_0} \int_0^1 d\alpha \alpha^{-1} (1 - \alpha)^{2s-1} \left[(1 - \alpha)^{\xi_1} - 1\right] y_1^{\xi_1} \frac{\Gamma(2s)}{\Gamma(2s + \xi_1)} \left(\prod_{j=2}^N d(y_j, \xi_j)\right) T_R^{\xi_{N+1}}.$$

By using (3.14) and (3.17) we recognize that

$$\text{III} = - \sum_{\ell=1}^{\xi_1} \varphi_s(\ell, \xi_1) D(y, \xi). \quad (3.23)$$

For the term IV we write the Newton binomial of $(y_1 + \alpha)^{\xi_1}$ and we note that the first term of the sum cancel to get

$$\text{IV} = T_L^{\xi_0} \sum_{\ell=1}^{\xi_1} \binom{\xi_1}{\ell} y_1^{\xi_1 - \ell} \frac{\Gamma(2s)}{\Gamma(2s + \xi_1)} \int_0^\infty d\alpha \alpha^{\ell-1} e^{-\lambda_L \alpha} \left(\prod_{j=2}^N d(y_j, \xi_j)\right) T_R^{\xi_{N+1}}.$$

The integral gives $(T_L)^\ell \Gamma(\ell)$ so that we can reconstruct the function $\varphi_s(\ell, \xi_1)$ and get

$$\text{IV} = \sum_{\ell=1}^{\xi_1} \varphi_s(\ell, \xi_1) T_L^{\xi_0 + \ell} y_1^{\xi_1 - \ell} \frac{\Gamma(2s)}{\Gamma(2s + \xi_1 - \ell)} \left(\prod_{j=2}^N d(y_j, \xi_j)\right) T_R^{\xi_{N+1}}.$$

Upon recalling the definition of the duality function (3.6), this gives

$$\text{IV} = \sum_{\ell=1}^{\xi_1} \varphi_s(\ell, \xi_1) D(y, \xi - \ell \delta_1 + \ell \delta_0). \quad (3.24)$$

Inserting the expressions (3.23) and (3.24) into the right hand side of (3.22) we then find

$$\left(\mathcal{L}_1 D(\cdot, \xi)\right)(y) = \left(\mathcal{L}_{0,1}^{\text{dual}} D(y, \cdot)\right)(\xi), \quad (3.25)$$

which proves the left boundary duality. Similarly one gets

$$\left(\mathcal{L}_N D(\cdot, \xi)\right)(y) = \left(\mathcal{L}_{N,N+1}^{\text{dual}} D(y, \cdot)\right)(\xi), \quad (3.26)$$

which establishes the right boundary duality. The combination of (3.21), (3.25) and (3.26) implies (3.9) and the proof of the theorem is completed. \square

3.2 Proof of Theorem 2.1

The unique stationary measure can be characterized in terms of its moments via the duality relation. The following statement (see Proposition 1 and 2 of [20]) is a classical consequence of the duality relation: it holds in and out of equilibrium and it allows to write the expectation (with respect to the stationary measure) of the duality function in terms of the dual process. The stationary expectation of the duality function can be written in terms of the absorption probabilities of the dual process, namely

$$\mathbb{E}_{\mu_N} [D(y, \xi)] = \sum_{k=0}^{|\xi|} \left(\frac{1}{\lambda_L}\right)^k \left(\frac{1}{\lambda_R}\right)^{|\xi|-k} p_\xi(k). \quad (3.27)$$

Above $p_\xi(k) = \mathbb{P}_\xi(\xi(\infty) = k\delta_0 + (|\xi| - k)\delta_{N+1})$ denotes the probability of k dual particles being eventually absorbed at site 0 and the remaining $|\xi| - k$ particles being eventually absorbed at site $N + 1$, when the dual process is initialized from the configuration $\xi \in \mathbb{N}^{N+2}$ with $|\xi| = \sum_{i=1}^N \xi_i$ particles and no particles at the sites $\{0, N + 1\}$.

Under equilibrium $\lambda_L = \lambda_R = \lambda$ we get

$$\mathbb{E}_{\mu_N} [D(y, \xi)] = \left(\frac{1}{\lambda}\right)^{|\xi|}.$$

Using the explicit form of the duality function in equation (3.6), the above display becomes

$$\mathbb{E}_{\mu_N} \left[\prod_{i=1}^N y_i^{\xi_i} \right] = \prod_{i=1}^N \left(\frac{1}{\lambda}\right)^{\xi_i} \frac{\Gamma(2s + \xi_i)}{\Gamma(2s)},$$

which are recognized as the multivariate moments of the product probability measure in equation (2.18).

In Appendix A we show that μ_N is also reversible by verifying with an explicit computation that the generator \mathcal{L} in (2.1) is self-adjoint in the Hilbert space $L_2(\mathbb{R}_+^N, \mu_N)$.

3.3 Proof of Theorem 2.2

The proof starts again from equation (3.27), which expresses the expectation of the duality function computed in a configuration ξ with n dual particles as a polynomial of order n in the two temperatures T_L and T_R , whose coefficient are the absorption probabilities of the n dual particles. As our dual process is the same of [16] (cf. Definition 3.1 of this paper to Definition 2.3 of [16]), we can just use the expression for the absorption probabilities found there (see equation (2.50) in [16]). Thus we obtain the multivariate moments given in (2.20), (2.21) when the configuration having n dual particles is described by the occupation numbers, or the expression given in (2.22), (2.23) when the configuration is described by assigning the ordered positions of the n dual particles.

3.4 Proof of Theorem 2.3

Let $u \in (0, 1)$. To prove the first item (local equilibrium) it is enough to study the convergence of the moments in a macroscopic point $\lfloor uN \rfloor$. Considering a configuration $(\xi_1, \dots, \xi_N) \in \mathbb{N}^N$ having $|\xi|$ dual particles located at positions $1 \leq x_1 \leq x_2 \leq \dots \leq x_{|\xi|} \leq N$, we define the shifted configuration $(\xi_1^u, \dots, \xi_N^u) \in \mathbb{N}^N$ having the particles located at positions $x_1^u \leq x_2^u \leq \dots \leq x_{|\xi|}^u$ with $x_k^u = x_k + \lfloor uN \rfloor$. Then we would like to prove that

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}_+^N} \left[\prod_{i=1}^N y_i^{\xi_i^u} \frac{\Gamma(2s)}{\Gamma(2s + \xi_i^u)} \right] = [T(u)]^{|\xi|} \quad (3.28)$$

where $T(u)$ denotes the macroscopic linear profile (2.28). Formula (2.23) gives that

$$\lim_{N \rightarrow \infty} g_{x^u}(n) = \lim_{N \rightarrow \infty} \sum_{1 \leq i_1 < \dots < i_n \leq |\xi|} \prod_{\alpha=1}^n \frac{n - \alpha + 2s(N + 1 - x_{i_\alpha} - \lfloor uN \rfloor)}{n - \alpha + 2s(N + 1)} = \binom{|\xi|}{n} (1 - u)^n. \quad (3.29)$$

Using then (2.22) we get

$$\begin{aligned} \lim_{N \rightarrow \infty} \int_{\mathbb{R}_+^N} \left[\prod_{i=1}^N y_i^{\xi_i^u} \frac{\Gamma(2s)}{\Gamma(2s + \xi_i^u)} \right] &= \sum_{n=0}^{|\xi|} T_R^{|\xi|-n} (T_L - T_R)^n \binom{|\xi|}{n} (1 - u)^n \\ &= [T_R + (T_L - T_R)(1 - u)]^{|\xi|}. \end{aligned} \quad (3.30)$$

This proves (3.28), which in turn implies (2.27).

To prove the second item (Fourier's law) we observe that, as a consequence of the linear microscopic profile (2.24), the average current is the same for all bonds and is given by

$$J_{i,i+1} = -2s \frac{T_R - T_L}{N + 1}. \quad (3.31)$$

From this, (2.32) immediately follows.

4 Algebraic description

In this section we show how the integrable heat conduction model (2.1) can be expressed algebraically in terms of the generators of the $\mathfrak{sl}(2)$ Lie algebra. More precisely, we establish a relation between the Markov generator of the integrable heat conduction model and the Hamiltonian of the open XXX Heisenberg spin chain, in a particular representation, by using the algebra generators as building blocks.

Once the algebraic description of the Levy generator of the heat conduction model via the $\mathfrak{sl}(2)$ algebra generators is obtained, the proof of duality is a consequence of a basic intertwining relation between two representations of the underlying algebra. In other words, the dual absorbing harmonic process also arises from the open XXX chain with non-compact spins, using a different representation. Duality is essentially the statement that the two representations (one leading to the boundary-driven Levy process and the other leading to the absorbing harmonic process) are indeed equivalent representations.

The identification of the algebraic expression of the Levy generator closely follows the techniques used in the work of Derkachov [21] in relation to Baxter Q-operators for the XXX Heisenberg spin chain.

4.1 Levy generator

For the description of the Levy generator we consider the following representation (labeled by the parameter $s > 0$) of the $\mathfrak{sl}(2)$ Lie algebra

$$\mathcal{K}^+ = y, \quad \mathcal{K}^- = (y\partial_y + 2s)\partial_y, \quad \mathcal{K}^0 = y\partial_y + s. \quad (4.1)$$

The operators $\mathcal{K}^+, \mathcal{K}^-, \mathcal{K}^0$, acting on polynomial functions, generate highest weight state representations of the $\mathfrak{sl}(2)$ Lie algebra and satisfy the commutation relations:

$$[\mathcal{K}^0, \mathcal{K}^\pm] = \pm \mathcal{K}^\pm \quad \text{and} \quad [\mathcal{K}^+, \mathcal{K}^-] = -2\mathcal{K}^0. \quad (4.2)$$

At each lattice site $i \in \{1, \dots, N\}$ we consider a copy of the $\mathfrak{sl}(2)$ algebra. We write the site in the subscript of the generator \mathcal{K}_i^a with $a \in \{+, -, 0\}$. Generators at different sites commute.

We treat the bulk and boundary parts of the Markov generator of our model in the two subsections below. We will show that the local generators in equation (2.3) and (2.4) can be written as

$$\mathcal{L}_{i,i+1}^\rightarrow = -e^{\mathcal{K}_i^+(\mathcal{K}_{i+1}^0+s)^{-1}\mathcal{K}_{i+1}^-}(\psi(\mathcal{K}_i^0+s) - \psi(2s))e^{-\mathcal{K}_i^+(\mathcal{K}_{i+1}^0+s)^{-1}\mathcal{K}_{i+1}^-} \quad (4.3)$$

and similarly

$$\mathcal{L}_{i,i+1}^\leftarrow = -e^{\mathcal{K}_{i+1}^+(\mathcal{K}_i^0+s)^{-1}\mathcal{K}_i^-}(\psi(\mathcal{K}_{i+1}^0+s) - \psi(2s))e^{-\mathcal{K}_{i+1}^+(\mathcal{K}_i^0+s)^{-1}\mathcal{K}_i^-}. \quad (4.4)$$

For the boundary terms we will need to consider an additional representation of the $\mathfrak{sl}(2)$ Lie algebra, which will be associated to fictitious extra-sites 0 and $N+1$

$$\mathcal{S}_0^+ = T_L(T_L\partial_{T_L} + 2s), \quad \mathcal{S}_0^- = \partial_{T_L}, \quad \mathcal{S}_0^0 = T_L\partial_{T_L} + s. \quad (4.5)$$

$$\mathcal{S}_{N+1}^+ = T_R(T_R\partial_{T_R} + 2s), \quad \mathcal{S}_{N+1}^- = \partial_{T_R}, \quad \mathcal{S}_{N+1}^0 = T_R\partial_{T_R} + s. \quad (4.6)$$

The operators $\mathcal{S}_i^+, \mathcal{S}_i^-, \mathcal{S}_i^0$, with $i \in \{0, N+1\}$, acting on polynomial functions of variable T_L when $i = 0$ and of variable T_R when $i = N+1$, also satisfy the $\mathfrak{sl}(2)$ commutation relations:

$$[\mathcal{S}_i^0, \mathcal{S}_i^\pm] = \pm \mathcal{S}_i^\pm \quad \text{and} \quad [\mathcal{S}_i^+, \mathcal{S}_i^-] = -2\mathcal{S}_i^0. \quad (4.7)$$

Then, for the boundary terms (2.5) and (2.6), we will show that they can be written as

$$\mathcal{L}_1 = -e^{-\mathcal{S}_0^+(\mathcal{S}_0^0+s)^{-1}\mathcal{K}_1^-}(\psi(\mathcal{K}_1^0+s) - \psi(2s))e^{\mathcal{S}_0^+(\mathcal{S}_0^0+s)^{-1}\mathcal{K}_1^-}, \quad (4.8)$$

and

$$\mathcal{L}_N = -e^{-\mathcal{K}_N^-\mathcal{S}_{N+1}^+(\mathcal{S}_{N+1}^0+s)^{-1}}(\psi(\mathcal{K}_N^0+s) - \psi(2s))e^{\mathcal{K}_N^-\mathcal{S}_{N+1}^+(\mathcal{S}_{N+1}^0+s)^{-1}} \quad (4.9)$$

where $\mathcal{S}_0^+(\mathcal{S}_0^0+s)^{-1} = T_L$ and $\mathcal{S}_{N+1}^+(\mathcal{S}_{N+1}^0+s)^{-1} = T_R$.

4.1.1 Bulk generator

To show the equivalence of the bulk Levy generator as given in (2.3) and (2.4) and the algebraic expressions (4.3) and (4.4) we first insert the algebra generators (4.1) into the latter and obtain

$$\mathcal{L}_{i,i+1}^{\rightarrow} = -e^{y_i \partial_{i+1}} (\psi(y_i \partial_i + 2s) - \psi(2s)) e^{-y_i \partial_{i+1}} \quad (4.10)$$

and

$$\mathcal{L}_{i,i+1}^{\leftarrow} = -e^{y_{i+1} \partial_i} (\psi(y_{i+1} \partial_{i+1} + 2s) - \psi(2s)) e^{-y_{i+1} \partial_i}. \quad (4.11)$$

Here and in the following, we write in shorthand ∂_i for the partial derivative ∂_{y_i} . Focussing on (4.10) and proceeding formally, using the integral representation (3.15) of the digamma function, it can be written

$$\mathcal{L}_{i,i+1}^{\rightarrow} = - \int_0^1 d\beta \frac{\beta^{2s-1}}{\beta-1} \left[e^{y_i \partial_{i+1}} \beta^{y_i \partial_i} e^{-y_i \partial_{i+1}} - 1 \right]. \quad (4.12)$$

We can now determine the action of this operator on functions. Noting that the exponential $e^{\pm y_i \partial_{i+1}}$ induces the shift $y_{i+1} \rightarrow y_{i+1} \pm y_i$ (see formula (B.1)) and the operator $\beta^{y_i \partial_i}$ induces the rescaling $y_i \rightarrow \beta y_i$ (see formula (B.2)) we obtain

$$\mathcal{L}_{i,i+1}^{\rightarrow} f(y) = - \int_0^1 d\beta \frac{\beta^{2s-1}}{\beta-1} [f(y_1, \dots, \beta y_i, y_{i+1} + (1-\beta)y_i, \dots, y_N) - f(y)]. \quad (4.13)$$

Finally, after a change of variables $\beta = 1 - \alpha y_i^{-1}$ we find

$$\mathcal{L}_{i,i+1}^{\rightarrow} f(y) = \int_0^{y_i} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_i} \right)^{2s-1} [f(y_1, \dots, y_i - \alpha, y_{i+1} + \alpha, \dots, y_N) - f(y)] \quad (4.14)$$

which coincides with (2.3). In the same way, for $\mathcal{L}_{i,i+1}^{\leftarrow}$ we derive (2.4) from (4.11).

4.1.2 Boundary generator

We now turn to the boundary terms. We treat the left reservoir only, as the the right reservoir is treated analogously. To obtain (2.5) from the algebraic expressions (4.8) we again insert the algebra generators (4.1) for site 1 and we use the algebra generators (4.5) for the extra site 0. Then we can write \mathcal{L}_1 as

$$\mathcal{L}_1 = -e^{-T_L(y_1 \partial_1 + 2s) \partial_1} (\psi(y_1 \partial_1 + 2s) - \psi(2s)) e^{T_L(y_1 \partial_1 + 2s) \partial_1}. \quad (4.15)$$

When expanding the exponentials, we obtain

$$\mathcal{L}_1 = - \sum_{k,l=0}^{\infty} (-1)^l (\psi(y_1 \partial_1 + k + 2s) - \psi(2s)) \frac{\left(-T_L(y_1 \partial_1 + 2s) \partial_1 \right)^{k+l}}{k!l!} \quad (4.16)$$

where we used the identity

$$[(y_1 \partial_1 + 2s) \partial_1]^k f(y_1 \partial_1 + 2s) = f(y_1 \partial_1 + k + 2s) [(y_1 \partial_1 + 2s) \partial_1]^k$$

Moving all derivatives to the right of the number operators we further get

$$\begin{aligned}
\mathcal{L}_1 &= - \sum_{k,l=0}^{\infty} (-1)^l (\psi(y_1 \partial_1 + k + 2s) - \psi(2s)) \frac{\Gamma(y_1 \partial_1 + 2s + k + l)}{\Gamma(y_1 \partial_1 + 2s)} \frac{(-T_L \partial_1)^{k+l}}{k!l!} \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^m (-1)^{m-k} (\psi(y_1 \partial_1 + k + 2s) - \psi(2s)) \frac{\Gamma(y_1 \partial_1 + 2s + m)}{\Gamma(y_1 \partial_1 + 2s)} \frac{(-T_L \partial_1)^m}{k!(m-k)!} \\
&= \psi(y_1 \partial_1 + 2s) - \psi(2s) + \log(1 - T_L \partial_1),
\end{aligned} \tag{4.17}$$

where in the final equality we used that

$$\sum_{k=0}^m \frac{(-1)^{-k}}{k!(m-k)!} (\psi(y_1 \partial_1 + k + 2s) - \psi(2s)) = \begin{cases} \psi(y_1 \partial_1 + 2s) - \psi(2s) & \text{for } m = 0 \\ -\frac{1}{m} \frac{\Gamma(y_1 \partial_1 + 2s)}{\Gamma(y_1 \partial_1 + 2s + m)} & \text{for } m > 0 \end{cases}, \tag{4.18}$$

cf. [16, (3.22)].

As before for the case of the bulk terms, we find that the first two terms in (4.17) yield

$$(\psi(y_1 \partial_1 + 2s) - \psi(2s)) f(y) = - \int_0^{y_1} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_1}\right)^{2s-1} (f(y_1 - \alpha, \dots, y_N) - f(y)). \tag{4.19}$$

The third term in the last line of (4.17) yields

$$\log(1 - T_L \partial_1) f(y) = - \int_0^{\infty} \frac{d\alpha}{\alpha} \exp[-T_L^{-1} \alpha] (f(y_1 + \alpha, \dots, y_N) - f(y)). \tag{4.20}$$

This can be seen by using the shift formula (B.1) and by writing

$$\log(1 - T_L \partial_1) = -\log(T_L^{-1}) + \log(T_L^{-1} - \partial_1) = - \int_0^{\infty} \frac{d\alpha}{\alpha} \exp[-T_L^{-1} \alpha] (\exp[\alpha \partial_1] - 1). \tag{4.21}$$

Here we used the integral representation for $x > 0$

$$\log(x) = \int_0^{\infty} \frac{d\alpha}{\alpha} (e^{-\alpha} - e^{-x\alpha}). \tag{4.22}$$

Combining together (4.17), (4.19) and (4.20) and using that $T_L = \lambda_L^{-1}$ we have thus shown that the algebraic expression (4.8) produces the boundary generator (2.5).

Similar boundary operators appeared in the study of high energy QCD and $\mathcal{N} = 4$ super Yang-Mills theory [22, 23, 24].

4.2 Absorbing harmonic process

For the description of the dual absorbing process we consider another representation, labelled by $s > 0$, acting on functions $g : \mathbb{N}_0 \rightarrow \mathbb{R}$ as

$$\bar{K}^+ g(n) = (2s + n)g(n+1), \quad \bar{K}^- g(n) = ng(n-1), \quad \bar{K}^0 g(n) = (n+s)g(n), \tag{4.23}$$

where $g(-1) = 0$. They satisfy the so-called “dual” (or “conjugate”) $\mathfrak{sl}(2)$ Lie algebra with commutation relations

$$[\bar{K}^0, \bar{K}^{\pm}] = \mp \bar{K}^{\pm} \quad \text{and} \quad [\bar{K}^+, \bar{K}^-] = 2\bar{K}^0. \tag{4.24}$$

As before, treating the bulk and boundary generators separately we will show that the generator of the dual absorbing process in (3.2) can be written in more abstract form using the representation (4.23). We note that this representation is closely related to (although different from) the one used in cf. [16, (3.2)], see the remark below.

For the bulk part it will be convenient to split the action of the local generator in left and right part. As for the Levy generator we will show the following four identities: on one hand for the bulk terms we have

$$\begin{aligned} \mathcal{L}_{i,i+1}^{\rightarrow,\text{dual}} = & -e^{-\bar{K}_i^-(\bar{K}_i^0+s)^{-1}\bar{K}_{i+1}^+}(\psi(\bar{K}_{i+1}^0+s) - \psi(2s))e^{\bar{K}_i^-(\bar{K}_i^0+s)^{-1}\bar{K}_{i+1}^+} \\ & - h_s(\bar{K}_i^0 - s) + h_s(\bar{K}_{i+1}^0 - s), \end{aligned} \quad (4.25)$$

and similarly

$$\begin{aligned} \mathcal{L}_{i,i+1}^{\leftarrow,\text{dual}} = & -e^{-\bar{K}_i^+\bar{K}_{i+1}^-(\bar{K}_{i+1}^0+s)^{-1}}(\psi(\bar{K}_i^0+s) - \psi(2s))e^{\bar{K}_i^+\bar{K}_{i+1}^-(\bar{K}_{i+1}^0+s)^{-1}} \\ & + h_s(\bar{K}_i^0 - s) - h_s(\bar{K}_{i+1}^0 - s). \end{aligned} \quad (4.26)$$

Obviously, when taking the sum $\mathcal{L}_{i,i+1}^{\text{dual}} = \mathcal{L}_{i,i+1}^{\rightarrow,\text{dual}} + \mathcal{L}_{i,i+1}^{\leftarrow,\text{dual}}$ in (3.3) the diagonal terms related to the function h_s disappear. On the other hand, for the boundary terms, we will have

$$\mathcal{L}_{0,1}^{\text{dual}} = -e^{(\bar{K}_0^0+s)^{-1}\bar{K}_0^+\bar{K}_1^-}(\psi(\bar{K}_1^0+s) - \psi(2s))e^{-(\bar{K}_0^0+s)^{-1}\bar{K}_0^+\bar{K}_1^-}, \quad (4.27)$$

and

$$\mathcal{L}_{N,N+1}^{\text{dual}} = -e^{\bar{K}_N^-(\bar{K}_{N+1}^0+s)^{-1}\bar{K}_{N+1}^+}(\psi(\bar{K}_N^0+s) - \psi(2s))e^{-\bar{K}_N^-(\bar{K}_{N+1}^0+s)^{-1}\bar{K}_{N+1}^+}. \quad (4.28)$$

Remark. Consider the representation of the $\mathfrak{sl}(2)$ algebra given by

$$K^+g(n) = (2s+n-1)g(n-1), \quad K^-g(n) = (n+1)g(n+1), \quad K^0g(n) = (n+s)g(n),$$

which satisfies the commutation relations

$$[K^0, K^\pm] = \pm K^\pm \quad \text{and} \quad [K^+, K^-] = -2K^0.$$

This representation is just the transpose of the one introduced in (4.23), i.e.

$$(\bar{K}^a)^t = K^a \quad a \in \{+, -, 0\}. \quad (4.29)$$

The Hamiltonian of the open Heisenberg XXX spin chain with absorbing boundaries [16, (4.17)] is then recovered from the negative transpose of the Markov generator $\mathcal{L}_{i,i+1}^{\text{dual}}$ after using the identity

$$\begin{aligned} e^{-\bar{K}_i^-(\bar{K}_i^0+s)^{-1}\bar{K}_{i+1}^+}(\psi(\bar{K}_{i+1}^0+s) - \psi(2s))e^{\bar{K}_i^-(\bar{K}_i^0+s)^{-1}\bar{K}_{i+1}^+} - h_s(\bar{K}_i^0 - s) + h_s(\bar{K}_{i+1}^0 - s) \\ = e^{\bar{K}_i^-(\bar{K}_{i+1}^0+s)^{-1}\bar{K}_{i+1}^+}(\psi(\bar{K}_i^0+s) - \psi(2s))e^{-\bar{K}_i^-(\bar{K}_{i+1}^0+s)^{-1}\bar{K}_{i+1}^+}. \end{aligned} \quad (4.30)$$

The identity above follows from the equivalence of the action of the bulk generator (4.25) for $i = N$ and the boundary generator (4.28) (that is indeed verified by direct computation in (4.42) and (4.46) respectively).

4.2.1 Bulk

We now compute the action of the generator (4.25) on functions $g(n)$. The action the generator (4.26) then follows. For this purpose we consider the following object

$$O_+(\gamma) = -e^{-\gamma\bar{K}^+}(\psi(\bar{K}^0 + s) - \psi(2s))e^{\gamma\bar{K}^+}. \quad (4.31)$$

When computed on site i and for $\gamma = \bar{K}_{i+1}^-(\bar{K}_{i+1}^0 + s)^{-1}$ it matches the right hand side of equation (4.26), up to the diagonal terms.

We proceed now by an explicit computation of the action of (4.31) on function $g : \mathbb{N}_0 \rightarrow \mathbb{R}$. The inner part gives that

$$e^{\gamma\bar{K}^+}g(n) = \sum_{j=0}^{\infty} \frac{\gamma^j}{j!} (\bar{K}^+)^j g(n) = \sum_{j=0}^{\infty} \frac{\gamma^j}{j!} \frac{\Gamma(n+2s+j)}{\Gamma(n+2s)} g(n+j).$$

Furthermore, since \bar{K}^0 acts diagonally, recalling (3.17) we have

$$(\psi(\bar{K}^0 + s) - \psi(2s))g(n) = h_s(n)g(n). \quad (4.32)$$

Thus we get

$$e^{-\gamma\bar{K}^+}(\psi(\bar{K}^0 + s) - \psi(2s))e^{\gamma\bar{K}^+}g(n) = \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} (-1)^l \frac{\gamma^{j+l}}{j!l!} \frac{\Gamma(n+2s+j+l)}{\Gamma(n+2s)} h_s(n+l)g(n+j+l). \quad (4.33)$$

Performing a change of variables and then changing the order of summation we have

$$e^{-\gamma\bar{K}^+}(\psi(\bar{K}^0 + s) - \psi(2s))e^{\gamma\bar{K}^+}g(n) = \sum_{k=0}^{\infty} \gamma^k \sum_{l=0}^k \frac{(-1)^l}{l!(k-l)!} \frac{\Gamma(n+2s+k)}{\Gamma(n+2s)} h_s(n+l)g(n+k). \quad (4.34)$$

Separating the term for $k = 0$ and using the identity (4.18) for $k > 0$, i.e.

$$\sum_{l=0}^k \frac{(-1)^l}{l!(k-l)!} \frac{\Gamma(n+2s+k)}{\Gamma(n+2s)} h_s(n+l) = -\frac{1}{k}, \quad (4.35)$$

we find that

$$O_+(\gamma)g(n) = -h_s(n)g(n) + \sum_{k=1}^{\infty} \frac{\gamma^k}{k} g(n+k). \quad (4.36)$$

Now we consider the operator $O_+(\gamma)$ above on site i , we set $\gamma = \bar{K}_{i+1}^-(\bar{K}_{i+1}^0 + s)^{-1}$ and we act on a function $f : \mathbb{N}^{N+2} \rightarrow \mathbb{R}$ to get

$$-e^{-\bar{K}_i^+ \bar{K}_{i+1}^-(\bar{K}_{i+1}^0 + s)^{-1}}(\psi(\bar{K}_i^0 + s) - \psi(2s))e^{\bar{K}_i^+ \bar{K}_{i+1}^-(\bar{K}_{i+1}^0 + s)^{-1}}f(\xi) = \quad (4.37)$$

$$\sum_{k=1}^{\infty} \frac{(\bar{K}_{i+1}^-(\bar{K}_{i+1}^0 + s)^{-1})^k}{k} f(\xi + k\delta_i) - h_s(\xi_i)f(\xi) = \quad (4.38)$$

$$\sum_{k=1}^{n_{i+1}} \frac{n_{i+1}!}{k(n_{i+1} - k)!} \frac{\Gamma(n_{i+1} + 2s + k)}{\Gamma(n_{i+1} + 2s)} f(\xi + k\delta_i - k\delta_{i+1}) - h_s(\xi_i)f(\xi) = \quad (4.39)$$

$$\sum_{k=1}^{n_{i+1}} \varphi_s(k, n_{i+1}) f(\xi + k\delta_i - k\delta_{i+1}) - h_s(\xi_i)f(\xi) = \quad (4.40)$$

$$(\mathcal{L}_{i,i+1}^{\leftarrow, \text{dual}} + h_s(\xi_{i+1}) - h_s(\xi_i))f(\xi). \quad (4.41)$$

Thus (4.26) is proved. Similarly,

$$-e^{-\bar{K}_i^-(\bar{K}_i^0+s)^{-1}\bar{K}_{i+1}^+}(\psi(\bar{K}_{i+1}^0+s)-\psi(2s))e^{\bar{K}_i^-(\bar{K}_i^0+s)^{-1}\bar{K}_{i+1}^+}f(\xi) = \quad (4.42)$$

$$\left(\mathcal{L}_{i,i+1}^{\rightarrow,\text{dual}} + h_s(\xi_i) - h_s(\xi_{i+1})\right)f(\xi), \quad (4.43)$$

from which (4.25) is established.

4.2.2 Boundary

As before, consider the following object

$$O_-(\gamma) = -e^{\gamma\bar{K}^-}(\psi(\bar{K}^0+s)-\psi(2s))e^{-\gamma\bar{K}^-}. \quad (4.44)$$

When computed on site 1 and for $\gamma = (\bar{K}_0^0+s)^{-1}\bar{K}_0^+$ it matches the right hand side of equation (4.27). We start by computing the action of the inner element:

$$e^{-\gamma\bar{K}^-}g(n) = \sum_{j=0}^{\infty} \frac{(-\gamma)^j}{j!} (\bar{K}^-)^j g(n) = \sum_{j=0}^n (-1)^j \frac{\gamma^j}{j!} \frac{n!}{(n-j)!} g(n-j).$$

Next, using again the identity (4.32), we have

$$(\psi(\bar{K}^0+s)-\psi(2s))e^{-\gamma\bar{K}^-}g(n) = h_s(n) \sum_{j=0}^n (-1)^j \frac{\gamma^j}{j!} \frac{n!}{(n-j)!} g(n-j).$$

Thus we arrive to

$$e^{\gamma\bar{K}^-}(\psi(\bar{K}^0+s)-\psi(2s))e^{-\gamma\bar{K}^-}g(n) = \sum_{j=0}^n \sum_{l=0}^{n-j} (-1)^j \frac{\gamma^{l+j}}{l!j!} \frac{n!}{(n-l-j)!} h_s(n-l) g(n-l-j)$$

Performing the change of variable $k = l + j$ and changing the order of summation we find that the above corresponds to

$$\sum_{k=0}^n \gamma^k g(n-k) \sum_{l=0}^k (-1)^{k-l} \binom{n}{l} \binom{n-l}{k-l} h_s(n-l).$$

Separating the case $k = 0$ together with the identity cf. [16, (3.13)]

$$\sum_{l=0}^k (-1)^{k-l} \binom{n}{l} \binom{n-l}{k-l} h_s(n-l) = -\varphi_s(k, n),$$

we find that

$$O_-(\gamma) = \sum_{k=1}^n \gamma^k \varphi_s(k, n) g(n-k) - h_s(n) g(n) \quad (4.45)$$

Setting $\gamma = (\bar{K}_0^0+s)^{-1}\bar{K}_0^+$ and evaluating the above expression in site $i = 1$ on functions $f : \mathbb{N}^{N+2} \rightarrow \mathbb{R}$ we find

$$\begin{aligned} & -e^{(\bar{K}_0^0+s)^{-1}\bar{K}_0^+\bar{K}_1^-} \left(\psi(\bar{K}_1^0+s) - \psi(2s) \right) e^{-(\bar{K}_0^0+s)^{-1}\bar{K}_0^+\bar{K}_1^-} f(\xi) \\ &= \sum_{k=1}^{\xi_1} \varphi_s(k, \xi_1) f(\xi - k\delta_1 + k\delta_0) - h_s(\xi_1) f(\xi) = \mathcal{L}_{0,1}^{\text{dual}} f(\xi). \end{aligned} \quad (4.46)$$

4.3 Duality and change of representation

In Section 3.1 we proved the duality relation between the heat conduction model and the absorbing particle process through a direct computation. Here we argue that duality can elegantly be obtained from the algebraic descriptions of the previous sections. The idea behind the algebraic approach is that one can deduce duality relations for Markov processes as a consequence of a change of representation of the underlying algebra. For the case of interest, the argument is described in the following.

First we note that the two representations (4.1) and (4.23) satisfy the duality relation

$$\mathcal{K}^a d(\cdot, n)(z) = \bar{K}^a d(z, \cdot)(n) \quad a \in \{+, -, 0\} \quad (4.47)$$

for the single-site duality function $d : \mathbb{R}^+ \times \mathbb{N}_0 \rightarrow \mathbb{R}$ given by

$$d(z, n) = z^n \frac{\Gamma(2s)}{\Gamma(2s + n)}. \quad (4.48)$$

Furthermore, the two representations (4.5) and (4.23) satisfy the duality relation

$$\mathcal{S}^a b(\cdot, n)(z) = \bar{K}^a b(z, \cdot)(n) \quad a \in \{+, -, 0\} \quad (4.49)$$

for the single-site duality function

$$b(z, n) = z^n. \quad (4.50)$$

After having expressed the Markov processes algebraically, the duality relations of the Lie algebra generators above can be used to infer a duality relation between the processes when regarding the Markov generator as an expansion in terms of elements of the universal enveloping algebra of $\mathfrak{sl}(2)$. The key idea is contained in the following general theorem (for a proof see Theorem 2.1 of [25], see also [26] in the context of self-duality):

Theorem 4.1. *Let \mathfrak{g} be a Lie algebra generated by $\{A_i\}_{i=1}^n$ and let $\{B_i\}_{i=1}^n$ be generators of the conjugate Lie algebra. If, for every $i = 1, \dots, n$, A_i is dual to B_i with duality function D then the element $(A_{i_1})^{n_1} \dots (A_{i_k})^{n_k}$ is dual with the same duality function D to the element $(B_{i_k})^{n_k} \dots (B_{i_1})^{n_1}$ for all $k \in \mathbb{N}$ and for all $n_1, \dots, n_k \in \mathbb{N}$.*

The rule of thumb of the theorem is that a given sequence of A operators is dual to a sequence of B operators but in reversed order. Combining together the above theorem with the algebraic expression of the Markov generators found in Section 4.1 and in Section 4.2 one deduces that the algebra dualities (4.47) and (4.49) imply the duality of Theorem 3.2 with

$$D(y, \xi) = b_0(T_L, \xi_0) \left[\prod_{i=1}^N d(y_i, \xi_i) \right] b(T_R, \xi_{N+1}), \quad (4.51)$$

for $y \in \mathbb{R}_+^N$ and $\xi \in \mathbb{N}_0^{N+2}$.

Remark. *The duality function (4.51) thus emerges from the representation theory of $\mathfrak{sl}(2)$ and is indeed the duality function of several other models with the same underlying Lie algebra [25, 27]. Via the algebraic approach one can obtain other duality functions as well, e.g. classical orthogonal polynomials [28, 29], or extend the dualities to asymmetric systems, see for instance [30, 31] and references therein. For the asymmetric version of the harmonic model studied in this paper we refer the reader to [32, 33, 34, 35].*

A Reversibility at equilibrium

Here we show that, in equilibrium setting $\lambda_L = \lambda_R = \lambda$, the generator \mathcal{L} in (2.1) is self-adjoint in the Hilbert space $L_2(\mathbb{R}_+^N, \mu_N)$ with scalar product

$$\langle f, g \rangle = \int_{\mathbb{R}_+^N} f(y)g(y)\mu_N(dy).$$

Here μ_N is the probability measure in (2.18). In order to show that $\mathcal{L}^* = \mathcal{L}$, with $*$ denoting the adjoint operator, we prove that $(\mathcal{L}_{i,i+1})^* = \mathcal{L}_{i,i+1}$ and similarly $(\mathcal{L}_1)^* = \mathcal{L}_1$ and $(\mathcal{L}_N)^* = \mathcal{L}_N$. Using (2.1) the claim then follows.

To alleviate notation, when considering the action of $\mathcal{L}_{i,i+1}$ on a function g we write only the variables y_i and y_{i+1} and skip the dependence of the functions on all the other variables, which stay untouched. Similarly we do not write all the integrals involving variables other than y_i and y_{i+1} . Thus, recalling the decomposition $\mathcal{L}_{i,i+1} = \mathcal{L}_{i,i+1}^{\rightarrow} + \mathcal{L}_{i,i+1}^{\leftarrow}$ and modulo the abuse of notation, we may write

$$\begin{aligned} & \left(\frac{\Gamma(2s)}{\lambda^{2s}} \right)^2 \cdot \langle \mathcal{L}_{i,i+1} f, g \rangle = \\ & \int_{\mathbb{R}_+} dy_i \int_{\mathbb{R}_+} dy_{i+1} \int_0^{y_i} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_i} \right)^{2s-1} f(y_i - \alpha, y_{i+1} + \alpha) g(y_i, y_{i+1}) (y_i y_{i+1})^{2s-1} e^{-\lambda(y_i + y_{i+1})} \\ & \quad - \int_{\mathbb{R}_+} dy_i \int_{\mathbb{R}_+} dy_{i+1} \int_0^{y_i} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_i} \right)^{2s-1} f(y_i, y_{i+1}) g(y_i, y_{i+1}) (y_i y_{i+1})^{2s-1} e^{-\lambda(y_i + y_{i+1})} \\ & + \int_{\mathbb{R}_+} dy_i \int_{\mathbb{R}_+} dy_{i+1} \int_0^{y_{i+1}} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_{i+1}} \right)^{2s-1} f(y_i + \alpha, y_{i+1} - \alpha) g(y_i, y_{i+1}) (y_i y_{i+1})^{2s-1} e^{-\lambda(y_i + y_{i+1})} \\ & \quad - \int_{\mathbb{R}_+} dy_i \int_{\mathbb{R}_+} dy_{i+1} \int_0^{y_{i+1}} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_{i+1}} \right)^{2s-1} f(y_i, y_{i+1}) g(y_i, y_{i+1}) (y_i y_{i+1})^{2s-1} e^{-\lambda(y_i + y_{i+1})}. \end{aligned}$$

We proceed formally and consider the first multiple integral on the right hand side of the above display. Applying (twice) Fubini's theorem and changing to new variables $z_i = y_i - \alpha$ and $z_{i+1} = y_{i+1} + \alpha$ this can be rewritten as

$$\int_{\mathbb{R}_+} dz_i \int_{\mathbb{R}_+} dz_{i+1} \int_0^{z_{i+1}} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{z_{i+1}} \right)^{2s-1} f(z_i, z_{i+1}) g(z_i + \alpha, z_{i+1} - \alpha) (z_i z_{i+1})^{2s-1} e^{-\lambda(z_i + z_{i+1})}.$$

Similarly, applying Fubini's theorem and using the change of variables $z_i = y_i + \alpha$ and $z_{i+1} = y_{i+1} - \alpha$, the third multiple integral on the right hand side can be rewritten as

$$\int_{\mathbb{R}_+} dz_i \int_{\mathbb{R}_+} dz_{i+1} \int_0^{z_i} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{z_i} \right)^{2s-1} f(z_i, z_{i+1}) g(z_i - \alpha, z_{i+1} + \alpha) (z_i z_{i+1})^{2s-1} e^{-\lambda(z_i + z_{i+1})}.$$

All in all, one gets

$$\begin{aligned}
& \left(\frac{\Gamma(2s)}{\lambda^{2s}} \right)^2 \cdot \langle \mathcal{L}_{i,i+1} h, g \rangle = \\
& \int_{\mathbb{R}_+} dz_i \int_{\mathbb{R}_+} dz_{i+1} \int_0^{z_{i+1}} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{z_{i+1}} \right)^{2s-1} f(z_i, z_{i+1}) g(z_i + \alpha, z_{i+1} - \alpha) (z_i z_{i+1})^{2s-1} e^{-\lambda(z_i + z_{i+1})} \\
& - \int_{\mathbb{R}_+} dy_i \int_{\mathbb{R}_+} dy_{i+1} \int_0^{y_i} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_i} \right)^{2s-1} f(y_i, y_{i+1}) g(y_i, y_{i+1}) (y_i y_{i+1})^{2s-1} e^{-\lambda(y_i + y_{i+1})} \\
& + \int_{\mathbb{R}^+} dz_i \int_{\mathbb{R}_+} dz_{i+1} \int_0^{z_i} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{z_i} \right)^{2s-1} f(z_i, z_{i+1}) g(z_i - \alpha, z_{i+1} + \alpha) (z_i z_{i+1})^{2s-1} e^{-\lambda(z_i + z_{i+1})} \\
& - \int_{\mathbb{R}_+} dy_i \int_{\mathbb{R}_+} dy_{i+1} \int_0^{y_{i+1}} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_{i+1}} \right)^{2s-1} f(y_i, y_{i+1}) g(y_i, y_{i+1}) (y_i y_{i+1})^{2s-1} e^{-\lambda(y_i + y_{i+1})}.
\end{aligned}$$

Combining together the first and the fourth lines of the r.h.s. of the above display, we recognize the action of $\mathcal{L}_{i,i+1}^{\leftarrow}$ on the function g . Similarly, combining together the second and the third lines, we recognize the action of $\mathcal{L}_{i,i+1}^{\rightarrow}$ on the function g . So we have proved that $\langle \mathcal{L}_{i,i+1} f, g \rangle = \langle f, \mathcal{L}_{i,i+1} g \rangle$.

For the boundary terms a similar computation holds. Let's consider the generator of the left reservoirs as the right one is similar. We have

$$\begin{aligned}
\langle \mathcal{L}_1 h, g \rangle &= \int_0^{+\infty} dy_1 \int_0^{y_1} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_1} \right)^{2s-1} h(y_1 - \alpha) g(y_1) \frac{\lambda^{2s}}{\Gamma(2s)} y_1^{2s-1} e^{-\lambda y_1} \\
& - \int_0^{+\infty} dy_1 \int_0^{y_1} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_1} \right)^{2s-1} h(y_1) g(y_1) \frac{\lambda^{2s}}{\Gamma(2s)} y_1^{2s-1} e^{-\lambda y_1} \\
& + \int_0^{+\infty} dy_1 \int_0^{+\infty} \frac{d\alpha}{\alpha} e^{-\lambda \alpha} h(y_1 + \alpha) g(y_1) \frac{\lambda^{2s}}{\Gamma(2s)} y_1^{2s-1} e^{-\lambda y_1} \\
& - \int_0^{+\infty} dy_1 \int_0^{+\infty} \frac{d\alpha}{\alpha} e^{-\lambda \alpha} h(y_1) g(y_1) \frac{\lambda^{2s}}{\Gamma(2s)} y_1^{2s-1} e^{-\lambda y_1}.
\end{aligned}$$

As before, let's start by considering the first double integral of the right hand side. By Fubini and the change of variable $y_1 = z_1 + \alpha$ we find

$$\begin{aligned}
& \int_0^{+\infty} dy_1 \int_0^{y_1} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_1} \right)^{2s-1} h(y_1 - \alpha) g(y_1) \frac{\lambda^{2s}}{\Gamma(2s)} y_1^{2s-1} e^{-\lambda y_1} \\
& = \int_0^{+\infty} dz_1 \int_0^{+\infty} \frac{d\alpha}{\alpha} e^{-\lambda \alpha} h(z_1) g(z_1 + \alpha) \frac{\lambda^{2s}}{\Gamma(2s)} z_1^{2s-1} e^{-\lambda z_1}.
\end{aligned}$$

Similarly, applying Fubini and changing variable $y_1 = z_1 - \alpha$ to the third double integral we get

$$\begin{aligned}
& \int_0^{+\infty} dy_1 \int_0^{+\infty} \frac{d\alpha}{\alpha} e^{-\lambda \alpha} h(y_1 + \alpha) g(y_1) \frac{\lambda^{2s}}{\Gamma(2s)} y_1^{2s-1} e^{-\lambda y_1} \\
& = \int_0^{+\infty} dz_1 \int_0^{z_1} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{z_1} \right)^{2s-1} h(z_1) g(z_1 - \alpha) \frac{\lambda^{2s}}{\Gamma(2s)} z_1^{2s-1} e^{-\lambda z_1}.
\end{aligned}$$

Substituting above, this shows that $\langle \mathcal{L}_1 h, g \rangle = \langle h, \mathcal{L}_1 g \rangle$.

B Shift formulas

In this appendix we collect two useful formulas. First we note that the Taylor series of the function $f(x + \alpha)$ around $\alpha = 0$ can be written as

$$f(x + \alpha) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} f^{(k)}(x) = e^{\alpha \partial_x} f(x). \quad (\text{B.1})$$

Further, the Taylor series of the function $f(\alpha x)$ around $\alpha = 1$ gives

$$\begin{aligned} f(\alpha x) &= \sum_{k=0}^{\infty} \frac{(\alpha - 1)^k}{k!} x^k f^{(k)}(x) \\ &= \sum_{k=0}^{\infty} \frac{(\alpha - 1)^k}{k!} \frac{\Gamma(x \partial_x + 1)}{\Gamma(x \partial_x + 1 - k)} f(x) \\ &= \alpha^{x \partial_x} f(x), \end{aligned} \quad (\text{B.2})$$

where it has been used that

$$x^k \partial_x^k = \frac{\Gamma(x \partial_x + 1)}{\Gamma(x \partial_x + 1 - k)}. \quad (\text{B.3})$$

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