Interacting particle systems and their duality theory (part II).

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The results presented in this course has been obtained in a series of works in collaboration with:

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Topics of these lectures:

- Non-equilibrium statistical mechanics.
- Exactly solvable models.

Outline

- Introduction: models (interacting particles/diffusions) & methods (algebraic approach to duality).
- Self-duality of interacting particles (independent, exclusion, inclusion). Mass transport.
- Duality between interacting diffusions and particles. Energy transport.
- Other applications: a) Population dynamics; b) Asymmetric systems & deformed algebras.
- Non-equilibrium statistical mechanics with current reservoirs. Freezing free boundary value problems.

1. Introduction:

Models

1.1: Symmetric interacting particle systems

(and their algebraic description)

Symmetric Interacting Particle Systems

For a graph $G = (V, E)$, we consider Markov processes defined on a state space $\Omega = \otimes_{i \in V} \Omega_i$, where Ω_i is a countable state space.

$$\text{Process } \big(\xi(t) \;:\; t \in \mathbb{R}_+ \cup \{0\}\big)$$

$$\text{Configuration} \quad \xi = (\xi_1, \ldots, \xi_{|V|}) \in \Omega$$

$$\text{Configuration} \quad \xi^{i,j} = (\xi_1, \ldots, \xi_i - 1, \ldots, \xi_j + 1, \ldots, \xi_{|V|})$$

Symmetric Interacting Particle Systems

Generator

$$Lf(\xi) = \sum_{(i,j)\in E} c(\xi, \xi^{i,j}) \left[f(\xi^{i,j}) - f(\xi) \right] + c(\xi, \xi^{j,i}) \left[f(\xi^{j,i}) - f(\xi) \right]$$

Symmetric means that

$$\pi_{i,j}(c(\xi, \xi^{j,i})) = c(\xi, \xi^{i,j})$$

where $\pi_{i,j}$ is the permutation of the indices (i, j).

Algebraic description of interacting particle systems

Define the column vector

$$\mathbb{P}(\xi(t) = \xi) := \mu(\xi, t)$$

The master equation reads

$$\begin{aligned}\frac{d}{dt}\mu(\xi,t) &= \sum_{\xi' \neq \xi} c(\xi',\xi)\mu(\xi',t) - \sum_{\xi' \neq \xi} c(\xi,\xi')\mu(\xi,t)\\ &= \sum_{\xi} c(\xi',\xi)\mu(\xi',t)\\ &= (L^*\mu)(\xi,t)\end{aligned}$$

where L^* denote the adjoint of the Markov generator.

The main idea is to rewrite the (adjoint of the) Markov generator using the generators of a (quantum) Lie algebra.

Lie algebra

A Lie algebra is a vector space $\mathfrak{g}$ over a field F together with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket, which satisfies:

- [Bilinearity]: for all scalars a, b in F and all elements x, y, z in $\mathfrak{g}$

$$[ax + by, z] = a[x, z] + b[y, z], \quad [z, ax + by] = a[z, x] + b[z, y]$$

- [Alternating on $\mathfrak{g}$]: for all x in $\mathfrak{g}$

$$[x, x] = 0$$

- [Jacobi identity]: for all x, y, z in $\mathfrak{g}$.

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$

Elements of a Lie algebra $\mathfrak{g}$ are said to be generators of the Lie algebra if the smallest subalgebra of $\mathfrak{g}$ containing them is $\mathfrak{g}$ itself.

Example 1.1.a: Independent walkers

Configuration $\xi = (\xi_1, \ldots, \xi_{|V|}) \in \{0, 1, 2, \ldots\}^{|V|}$

$$L^{IND} f(\xi) = \sum_{(i,j)\in E} \xi_i [f(\xi^{i,j}) - f(\xi)] + \xi_j [f(\xi^{j,i}) - f(\xi)]$$

Exercise: Show that stationary reversible measures are given by product measures with marginals Poisson(λ), i.e.

$$\mu_{stat}(\xi) = \prod_{i=1}^{|V|} \frac{\lambda^{\xi_i}}{\xi_i !} e^{-\lambda}$$

Heisenberg algebra

The Lie bracket is given by the commutator, i.e. for x, y in the algebra

$$[x, y] = xy - yx$$

The algebra is generated by the elements (a^+, a^-) that satisfy the commutation relations

$$[a^-, a^+] = \mathbf{1}$$

A representation in terms of matrices is given by **(Exercise!)**

$$\begin{cases} a^+|n\rangle = |n+1\rangle \\ \\ a^-|n\rangle = n|n-1\rangle \end{cases}$$

where, for $n \in \{0, 1, 2, \ldots\}$, $|n\rangle = e_n$ denote the orthonormal column vectors

$$(e_n)_i = \begin{cases} 1 & \text{if } i = n, \\ 0 & \text{if } i \neq n \end{cases} \qquad\qquad e_n^T \cdot e_m = \langle n|m\rangle = \delta_{n,m}$$

Heisenberg algebra on a graph $G = (V, E)$

On a graph with $|V|$ vertices we consider $|V|$ copies of the Heisenberg algebra and work with their tensor product.

Define

$$|\xi\rangle = \otimes_{i=1}^{|V|} |\xi_i\rangle$$

Then, in the previous representation,

$$\begin{cases} a_i^+ |\xi\rangle = \left(\otimes_{j\neq i} \mathbf{1} |\xi_j\rangle\right) \otimes \left(a_i^+ |\xi_i\rangle\right) = \left(\otimes_{j\neq i} |\xi_j\rangle\right) \otimes |\xi_i + 1\rangle \\ a_i^- |\xi\rangle = \left(\otimes_{j\neq i} \mathbf{1} |\xi_j\rangle\right) \otimes \left(a_i^- |\xi_i\rangle\right) = \left(\otimes_{j\neq i} |\xi_j\rangle\right) \otimes \xi_i |\xi_i - 1\rangle \end{cases}$$

The algebra generators (a_i^+, a_i^-), with $i = 1, \ldots, |V|$, satisfy

$$[a_i^-, a_j^+] = \delta_{i,j} \mathbf{1}$$

Algebraic description of independent walkers

$$L^*_{IND} = -\sum_{(i,j)\in E} \left(a_i^+ - a_j^+\right)\left(a_i^- - a_j^-\right)$$

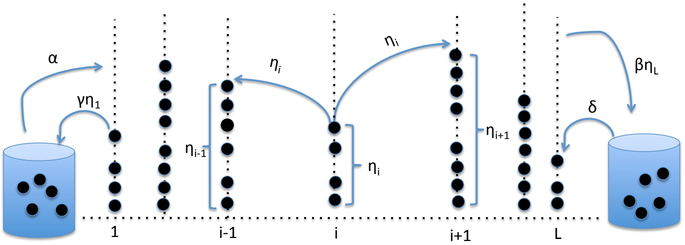
Expanding the product

$$\begin{aligned} L^*_{IND}|\xi\rangle &= \sum_{(i,j)\in E} \left(a_i^- a_j^+ + a_i^+ a_j^- - a_i^+ a_i^- - a_j^+ a_j^-\right)|\xi\rangle \\ &= \sum_{(i,j)\in E} \left(\xi_i|\xi^{i,j}\rangle + \xi_j|\xi^{j,i}\rangle - (\xi_i + \xi_j)|\xi\rangle\right) \end{aligned}$$

Rates:

$$c_{IND}(\xi,\xi') = \langle\xi|L_{IND}|\xi'\rangle = \langle\xi'|L^*_{IND}|\xi\rangle = \begin{cases} \xi_i & \text{if } \xi' = \xi^{i,j}, \\ \xi_j & \text{if } \xi' = \xi^{j,i}, \\ -(\xi_i + \xi_j) & \text{if } \xi' = \xi \end{cases}$$

Independent Walkers on a 1d chain with density reservoirs



Exercise: Show that a birth-death Markov chain with transitions: $b(n) = \alpha$ and $d(n) = \gamma n$ has stationary distribution Poisson$(\frac{\alpha}{\gamma})$.

Exercise: Check that Independent Walkers on a 1d chain with density reservoirs with parameters $\beta = \gamma = 1$ have stationary (non-reversible!) distribution $\otimes_{i=1}^{L}$ Poisson(λ_i) with $\lambda_i = \alpha + (\delta - \alpha)\frac{i}{L+1}$.

Example 1.1.b: Generalized Symmetric Exclusion Process, SEP(n)

Let $n \in \mathbb{N}$.

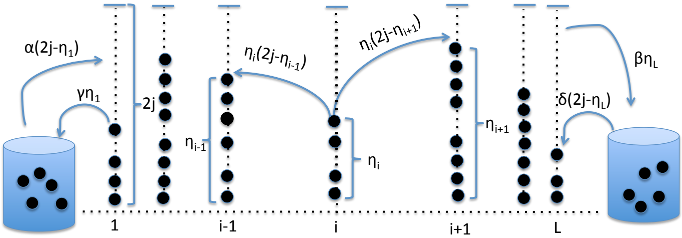
Configuration $\xi = (\xi_1, \ldots, \xi_{|V|}) \in \{0, 1, 2, \ldots, n\}^{|V|}$

$$L^*_{SEP(n)} f(\xi) = \sum_{(i,j) \in E} \xi_i (n - \xi_j)[f(\xi^{i,j}) - f(\xi)] + (n - \xi_i)\xi_j [f(\xi^{j,i}) - f(\xi)]$$

Exercise: Show that stationary reversible measures are given by product measures with marginals Binomial(n, p), i.e.

$$\mu_{stat}(\xi) = \prod_{i=1}^{|V|} \binom{n}{\xi_i} p^{\xi_i} (1-p)^{n-\xi_i}$$

SEP(n) with density reservoirs



Exercise: Show that a birth-death Markov chain with transitions: $b(n) = \alpha(2j - n)$ and $d(n) = \gamma n$ has stationary distribution given by a Binomial$(2j, \frac{\alpha}{\alpha+\gamma})$

Algebraic description of SEP(n)

$$L^*_{SEP(n)} = \sum_{(i,j)\in E} \left(J_i^+ J_j^- + J_i^- J_j^+ + 2J_i^o J_j^o - \frac{n^2}{2} \right)$$

$\{J_i^+, J_i^-, J_i^o\}$ are the generators of the $\mathfrak{su}(2)$ algebra

$$[J_i^o, J_j^\pm] = \pm\delta_{i,j} J_i^\pm \qquad [J_i^-, J_j^+] = -2\delta_{i,j} J_i^o$$

$$\begin{cases} J_i^+ |\xi_i\rangle = (n - \xi_i)\,|\xi_i + 1\rangle \\ J_i^- |\xi_i\rangle = \xi_i |\xi_i - 1\rangle \\ J_i^o |\xi_i\rangle = \left(\xi_i - \frac{n}{2}\right)|\xi_i\rangle \end{cases}$$

Example 1.1.c: Symmetric Inclusion Process, SIP(m)

Let $m \in \mathbb{R}_+$.

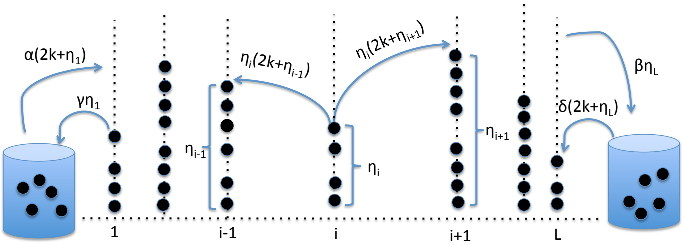
Configuration $\xi = (\xi_1, \ldots, \xi_{|V|}) \in \{0, 1, 2, \ldots\}^{|V|}$

$$L^*_{SIP(m)} f(\xi) =$$
$$\sum_{(i,j) \in E} \xi_i \left(\xi_j + \frac{m}{2} \right) [f(\xi^{i,j}) - f(\xi)] + \xi_j \left(\xi_i + \frac{m}{2} \right) [f(\xi^{j,i}) - f(\xi)]$$

Exercise: Show that stationary reversible measures are given by product measures with marginals Negative Binomial($\frac{m}{2}, p$), i.e.

$$\mu_{stat}(\xi) = \prod_{i=1}^{|V|} \frac{p^{\xi_i}(1-p)^{\frac{m}{2}}}{\xi_i!} \frac{\Gamma(\frac{m}{2} + \xi_i)}{\Gamma(\frac{m}{2})}$$

SIP(m) with density reservoirs



Exercise: Show that a birth-death Markov chain with transitions: $b(n) = \alpha(2k + n)$ and $d(n) = \gamma n$ has stationary distribution given by a Negative Binomial$(2k, \frac{\alpha}{\gamma})$

Algebraic description of SIP(m)

$$L^*_{SIP(m)} = \sum_{(i,j)\in E} \left(K_i^+ K_j^- + K_i^- K_j^+ - 2K_i^o K_j^o + \frac{m^2}{8} \right)$$

$\{K_i^+, K_i^-, K_i^o\}$ are the generators of the $\mathfrak{su}(1,1)$ algebra

$$[K_i^o, K_j^\pm] = \pm\delta_{i,j} K_i^\pm \qquad [K_i^-, K_j^+] = +2\delta_{i,j} K_i^o$$

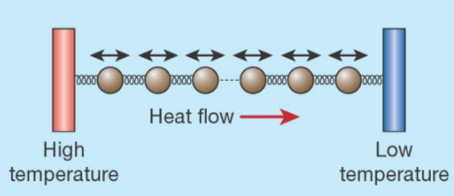
$$\begin{cases} K_i^+|\xi_i\rangle = \left(\xi_i + \frac{m}{2}\right)|\xi_i + 1\rangle \\ K_i^-|\xi_i\rangle = \xi_i|\xi_i - 1\rangle \\ K_i^o|\xi_i\rangle = \left(\xi_i + \frac{m}{4}\right)|\xi_i\rangle \end{cases}$$

1.2: Energy redistribution models,

symmetric diffusions,

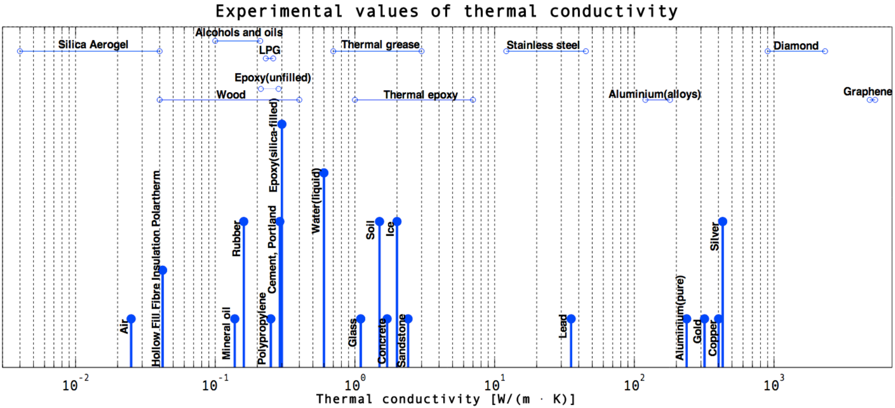
coupled map lattices.

Fourier's law

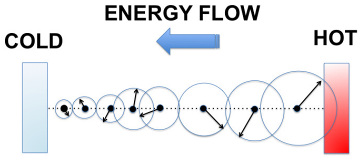


$$\langle J \rangle = \kappa \cdot \nabla T$$

$\langle J \rangle$: average heat flux	$=$	average energy transported through the unit surface per unit time
∇T : temperature gradient	$=$	spatial derivative of the temperature field
κ : thermal conductivity	$=$	constant of proportionality


Experimental values of thermal conductivity
Silica Aerogel
Alcohols and oils
LPG
Epoxy(unfilled)
Wood
Thermal grease
Thermal epoxy
Stainless steel
Aluminium(alloys)
Diamond
Graphene
Air
Hollow Fill Fibre Insulation Polartherm
Mineral oil
Rubber
Polypropylene
Cement, Portland
Epoxy(silica-filled)
Water(liquid)
Glass
Soil
Concrete
Ice
Sandstone
Lead
Aluminium(pure)
Gold
Copper
Silver
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Thermal conductivity [W/(m · K)]

Fourier law $\quad \langle J \rangle = \kappa \nabla T$



- Pure-carbon materials have extremely high thermal conductivity.
- 1D Hamiltonian models:
  - Oscillators chains (Lebowitz, Lieb, Rieder, 1967): $\kappa \sim N$.
  - Non-linear oscillators chains (Lepri, Livi, Politi, Phys. Rep. 2003): $\kappa \sim N^{\alpha}, \quad 0 < \alpha < 1$
  - Long wavelength phonons behave as ballistic heat carrier.
  - Non-linear fluctuating hydrodynamics (van Beijeren 2012, Spohn 2013)

- Example 1.2.a: KMP model

  Introduced by Kipnis, Marchioro, Presutti (1982).

  Observables: Energies at every site $z = (z_1, \ldots, z_N) \in \mathbb{R}_+^N$

  Dynamics: Select a bond at random and **uniformly** redistribute the energy under the constraint of conserving the total energy.

$$L^{KMP} f(z) =$$

$$\sum_{i=1}^{N} \int_0^1 dp \left[f(z_1, \ldots, p(z_i + z_{i+1}), (1-p)(z_i + z_{i+1}), \ldots, z_N) - f(z) \right]$$

$\rightarrow$ conductivity $0 < \kappa < \infty$; model solved by duality.

Example 1.2.b: Brownian Momentum Process (BMP)

First consider two sites, call them (i, j). Let $(x_i, x_j) \equiv$ velocities of the couple (i, j). Define

$$L_{i,j}^{BMP} f(x_i, x_j) = \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2 f(x_i, x_j)$$

- polar coordinates $$L_{i,j}^{BMP} = \frac{\partial^2}{\partial \theta_{ij}^2}$$
- Brownian motion for angle $\theta_{i,j} = \arctan(x_j / x_i)$
- total kinetic energy conserved: $r_{i,j}^2 = x_i^2 + x_j^2$

Brownian Momentum Process (BMP)

SDE description: Imagine a particle moving on the plane subject to a random space-homogeneous but time-dependent magnetic field $B(t)$ perpendicular to the plane. Let $B(t)$ be a standard Brownian motion.

Then the velocity vector $(x_i(t), x_j(t))$ evolve as [Stratonovich convention]

$$\begin{cases} dx_i(t) = dB(t)x_j(t) \\ \\ dx_j(t) = -dB(t)x_i(t) \end{cases}$$

Conservation law:

$$\begin{aligned} d(x_i^2(t) + x_j^2(t)) &= 2x_i(t)dx_i(t) + 2x_j(t)dx_j(t) \\ &= 2x_i(t)dB(t)x_j(t) - 2x_j(t)dB(t)x_i(t) \\ &= 0 \end{aligned}$$

Brownian Momentum Process (BMP)

Forward equation: Let $p(x_i, x_j, t)$ be the probability density function of the Markov process $(x_i(t), x_j(t))$. Since the process is a diffusion then

$$\frac{d}{dt}p(x_i, x_j, t) = ((L_{i,j}^{BMP})^* p)(x_i, x_j, t)$$
$$p(x_i, x_j, 0) = p_0(x_i, x_j)$$

Remark: the generator is self-adjoint w.r.t. Lebesgue measure, i.e.

$$(L_{i,j}^{BMP})^* = (L_{i,j}^{BMP})$$

Exercise: Show that stationary measures are given by product measures with marginals centered Gaussians with variance σ^2, i.e.

$$p_{stat}(x_i, x_j) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x_i^2/2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x_j^2/2\sigma^2}$$

Brownian momentum process (BMP)

For a graph $G = (V, E)$ let $\Omega = \otimes_{i \in V} \Omega_i = \mathbb{R}^{|V|}$.
Configuration $x = (x_1, \ldots, x_{|V|}) \in \Omega$

Generator BMP

$$L^{BMP} = \sum_{(i,j) \in E} L^{BMP}_{i,j} = \sum_{(i,j) \in E} \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2$$

Exercise: Show that stationary measures are Gaussian product measures with variance σ^2, i.e.

$$d\mu_{stat}(x) = \prod_{i=1}^{|V|} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x_i^2}{2\sigma^2}} \, dx_i$$

Example 1.2.c: Coupled Map Lattices

Start form the Hamiltonian

$$H(q,p) = \sum_{i=1}^{N} \frac{1}{2}\Big(p_i - A_i\Big)^2$$

$A = (A_1(q), \ldots, A_N(q))$ "vector potential" in $\mathbb{R}^N$.

$$\begin{aligned} \frac{dq_i}{dt} &= v_i \\ \frac{dv_i}{dt} &= \sum_{j=1}^{N} B_{ij} v_j \end{aligned}$$

where

$$B_{ij}(q) = \frac{\partial A_i(q)}{\partial q_j} - \frac{\partial A_j(q)}{\partial q_i}$$

antisymmetric matrix containing the "magnetic fields"

Conservation laws

- *Conservation of Energy:*
  Even if the forces depend on velocities and positions, the model conserves the total (kinetic) energy

$$\frac{d}{dt}\left(\sum_i \frac{1}{2} v_i^2\right) = \sum_{i,j} B_{ij} v_i v_j = 0$$

- *Conservation of Momentum:*
  Additional conserved quantities can be imposed by a suitable choice of the magnetic fields. E.g.: If we choose the $A_i(x)$ such that they are left invariant by the simultaneous translations $x_i \to x_i + \delta$, then the quantity $\sum_i p_i$ is conserved.

Example: discrete time dynamics with “magnetic kicks”

Let $\vec{q} = (q^{(1)}, q^{(2)})$, $\quad \vec{v} = (v^{(1)}, v^{(2)})$. Consider the map

$$\begin{aligned} \vec{q}(t+1) &= \vec{q}(t) + \vec{v}(t) \\ \vec{v}(t+1) &= R(t+1) \cdot \vec{v}(t) \end{aligned}$$

with $R(t)$ a rotation matrix

$$R(t) = \begin{pmatrix} \cos(B(\vec{q}(t))) & \sin(B(\vec{q}(t))) \\ -\sin(B(\vec{q}(t))) & \cos(B(\vec{q}(t))) \end{pmatrix}$$

Since the dynamics conserves the energy the accessible phase space is 3-dimensional.

Chaoticity properties of the map on torus $\mathbb{T}_2$ with $B(\vec{q}) = q^{(1)} + q^{(2)} - 2\pi$

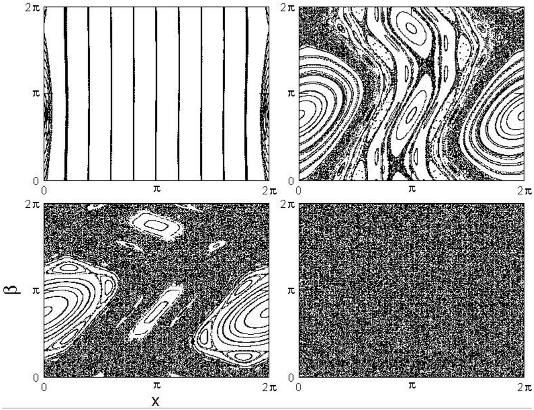


Figure : Poincare section with plane $q^{(2)} = 0$ of the map $\begin{cases} q^{(1)}_{t+1} = & q^{(1)}_t + v\cos(\beta_t) \\ q^{(2)}_{t+1} = & q^{(2)}_t + v\sin(\beta_t) \\ \beta_{t+1} = & \beta_t + B(q^{(1)}_t, q^{(2)}_t) \end{cases}$

with $v = \sqrt{v_1^2 + v_2^2}$, $\beta = \mathrm{arctanh}(v_2/v_1)$, $B(q^{(1)}, q^{(2)}) = q^{(1)} + q^{(2)} - 2\pi$.

High temperature limit

- Large velocities $\Rightarrow$ between two consecutive kicks new positions are translated by a large amount.
- Spatial coordinates are taken modulo 2π $\Rightarrow$ the sequence of positions constitutes a (quasi) random number generator.
- The position of the particle can be taken as uniformly randomly distributed.
- Because the magnetic fields are functions of the positions, this in turn means that the fields themselves are random.

Energy redistribution

Before ($e_i = \frac{1}{2} v_i^2$) and after ($e_i' = \frac{1}{2} v_i'^2$) a kick

$$\begin{aligned} e_i' &= c^2\, e_i + s^2\, e_{i+1} + 2 s\, c \sqrt{e_i e_{i+1}} \\ e_{i+1}' &= s^2\, e_i + c^2\, e_{i+1} - 2\, s\, c \sqrt{e_i e_{i+1}} \end{aligned}$$

where $s = \sin(B)$ and $c = \cos(B)$. Suppose $(q^{(1)}, q^{(2)})$ are indep. $Uni(0, 2\pi)$. Then the field B has a probability density

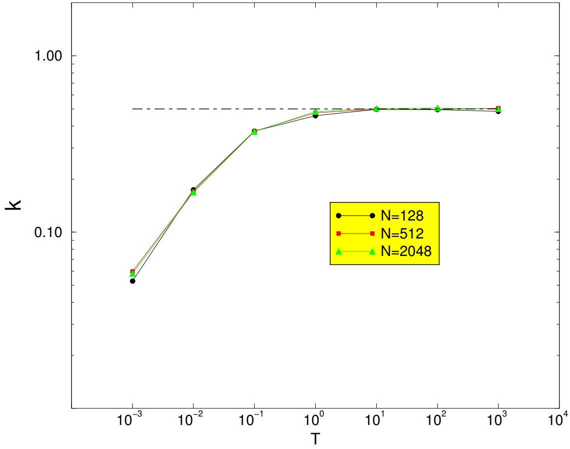
$$p_B(x) = \begin{cases} \frac{1}{2\pi}\left(1 + \frac{1}{2\pi} x\right) & x \in [-2\pi, 0] \\ \frac{1}{2\pi}\left(1 - \frac{1}{2\pi} x\right) & x \in [0, 2\pi] \end{cases}$$

Hence

$$\begin{aligned} \langle e_i' \rangle &= \frac{1}{2}(\langle e_i \rangle + \langle e_{i+1} \rangle) \\ \langle e_{i+1}' \rangle &= \frac{1}{2}(\langle e_i \rangle + \langle e_{i+1} \rangle) \end{aligned}$$

This dynamical rule, (*only as far as the means are concerned*), is equivalent to the KMP. More on this later...

Heat conductivity for a couple map lattice



Numerical results: i) finite conductivity; ii) high temperature limit approaches a constant value $\kappa = 1/2$.

2. Algebraic approach to

Stochastic Duality Theory

Duality

Definition

$(\eta_t)_{t\geq 0}$ Markov process on Ω with generator L,

$(\xi_t)_{t\geq 0}$ Markov process on Ω_{dual} with generator L_{dual}

ξ_t is dual to η_t with duality function $D : \Omega \times \Omega_{dual} \to \mathbb{R}$ if $\forall t \geq 0$

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \qquad \forall (\eta, \xi) \in \Omega \times \Omega_{dual}$$

η_t is self-dual if $L_{dual} = L$.

Duality

Condition

$$LD(\cdot,\xi)(\eta) = L_{dual}D(\eta,\cdot)(\xi)$$

Indeed

$$\begin{aligned}
\mathbb{E}_\eta(D(\eta_t,\xi)) &= e^{tL}D(\cdot,\xi)(\eta) \\
&= e^{tL_{dual}}D(\eta,\cdot)(\xi) \\
&= \mathbb{E}_\xi(D(\eta,\xi_t))
\end{aligned}$$

How to find a dual process?

1. Write the generator in abstract form , i.e. as an element of a Lie algebra, using creation and annihilation operators.

2. Self-duality is associated to symmetries, i.e. conserved quantities.

3. Duality is related to a change of representation, i.e. new operators that satisfy the same algebra.

2.1: Self-duality

Matrix formulation of self-duality for Markov chain with countable state space

The condition of self-duality ($L_{dual} = L$)

$$LD(\cdot, \xi)(\eta) = LD(\eta, \cdot)(\xi)$$

becomes

$$\mathbf{LD} = \mathbf{DL}^T$$

Indeed

$$\sum_{\eta'} \mathbf{L}(\eta, \eta')\mathbf{D}(\eta', \xi) = LD(\cdot, \xi)(\eta) = LD(\eta, \cdot)(\xi) = \sum_{\xi'} \mathbf{L}(\xi, \xi')\mathbf{D}(\eta, \xi')$$

Trivial self-duality functions from reversible measures

From a reversible measure μ, i.e.

$$\mathbf{L}(\eta,\xi)\mu(\eta) = \mathbf{L}(\xi,\eta)\mu(\xi)$$

a trivial (i.e. diagonal) self-duality function is obtained as

$$\mathbf{d}(\eta,\xi) = \frac{1}{\mu(\eta)}\delta_{\eta,\xi}$$

Indeed

$$\frac{\mathbf{L}(\eta,\xi)}{\mu(\xi)} = \sum_{\eta'} \mathbf{L}(\eta,\eta')\mathbf{d}(\eta',\xi) = \sum_{\xi'} \mathbf{L}(\xi,\xi')\mathbf{d}(\eta,\xi') = \frac{\mathbf{L}(\xi,\eta)}{\mu(\eta)}$$

Symmetries and self-duality

S: symmetry of the transposed of the generator, i.e. $[\mathbf{L^T}, \mathbf{S}] = 0$,
$\mathbf{d}$: trivial self-duality function,

$$\longrightarrow \quad \mathbf{D} = \mathbf{dS} \text{ self-duality function.}$$

Indeed

$$\mathbf{LD} = \mathbf{LdS} = \mathbf{dL}^T\mathbf{S} = \mathbf{dSL}^T = \mathbf{DL}^T$$

Self-duality is related to the action of a symmetry

Self-duality of Example 1.1.c: Symmetric Inclusion Process SIP(m)

Theorem

Let $m \in \mathbb{R}_+$. On a graph $G = (V, E)$, the SIP(m) process with state space $\mathbb{N}^{|V|}$ and generator

$$L_{SIP(m)} f(\xi) = \sum_{(i,j)\in E} \xi_i \left(\xi_j + \frac{m}{2}\right) [f(\xi^{i,j}) - f(\xi)] + \xi_j \left(\xi_i + \frac{m}{2}\right) [f(\xi^{j,i}) - f(\xi)]$$

is self-dual on functions

$$D(\eta, \xi) = \prod_{i \in V} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2} + \xi_i\right)}$$

Algebraic description of SIP(m)

$$L^*_{SIP(m)} = \sum_{(i,j)\in E} \left(K_i^+ K_j^- + K_i^- K_j^+ - 2K_i^o K_j^o + \frac{m^2}{8} \right)$$

$\{K_i^+, K_i^-, K_i^o\}$ are the generators of the $\mathfrak{su}(1,1)$ algebra

$$[K_i^o, K_j^\pm] = \pm\delta_{i,j} K_i^\pm \qquad [K_i^-, K_j^+] = +2\delta_{i,j} K_i^o$$

$$\begin{cases} K_i^+ |\xi_i\rangle = \left(\xi_i + \frac{m}{2}\right) |\xi_i + 1\rangle \\ K_i^- |\xi_i\rangle = \xi_i |\xi_i - 1\rangle \\ K_i^o |\xi_i\rangle = \left(\xi_i + \frac{m}{4}\right) |\xi_i\rangle \end{cases}$$

What is the origin of the quantum spin chain?

Let us first work with two sites ....

Casimir element

There are distinguished elements in the algebra, known as Casimir elements.

For the $\mathfrak{su}(1,1)$ algebra the Casimir is

$$C = \frac{1}{2}(K^-K^+ + K^+K^-) - (K^0)^2$$

C is in the center of the $\mathfrak{su}(1,1)$ algebra:

$$[C, K^+] = [C, K^-] = [C, K^o] = 0$$

$$\begin{aligned} C|n\rangle &= \frac{1}{2}\left((n+1)(\frac{m}{2}+n) + (\frac{m}{2}+n-1)n\right) - (n+\frac{m}{4})^2|n\rangle \\ &= n(\frac{m}{2}+n) + \frac{m}{4} - (n+\frac{m}{4})^2|n\rangle \\ &= \frac{m}{4}(1-\frac{m}{4})|n\rangle \end{aligned}$$

Co-product

The co-product is a morphism that turns the algebra into a bialgebra:

$$\Delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$$

and conserves the commutations relations

$$[\Delta(K^o), \Delta(K^{\pm})] = \pm\Delta(K^{\pm})$$
$$[\Delta(K^-), \Delta(K^+)] = 2\Delta(K^o)$$

For classical Lie-algebras the co-product is just the symmetric tensor product with the identity

$$\Delta(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x = x_1 + x_2$$

Casimir & co-product

$$\Delta(C) = \frac{1}{2}\Big(\Delta(K^-)\Delta(K^+) + \Delta(K^+)\Delta(K^-)\Big) - \Big(\Delta(K^0)\Big)^2$$

$$= \frac{1}{2}\Big((K_1^- + K_2^-)(K_1^+ + K_2^+) + (K_1^+ + K_2^+)(K_1^- + K_2^-)\Big)$$

$$- \Big(K_1^o + K_2^o\Big)^2$$

$$= K_1^- K_2^+ + K_1^+ K_2^- - 2K_1^o K_2^o + C_1 + C_2$$

$$= (L_{1,2}^{SIP(m)})^* + (C_1 + C_2 - \frac{m^2}{8}\mathbf{1} \otimes \mathbf{1})$$

$$= (L_{1,2}^{SIP(m)})^* + (\frac{m}{2}(1 - \frac{m}{2}))\mathbf{1} \otimes \mathbf{1}$$

Symmetries

Having realized that the (adjoint of the) process generator is the co-product of the Casimir, it is easy to find symmetries:

Lemma

$$[(L_{1,2}^{SIP(m)})^*, K_1^o + K_2^o] = 0$$

$$[(L_{1,2}^{SIP(m)})^*, K_1^+ + K_2^+] = [(L_{1,2}^{SIP(m)})^*, K_1^- + K_2^-] = 0$$

Proof:

$$\begin{aligned}\left[(L_{1,2}^{SIP(m)})^*, K_1^+ + K_2^+\right] &= \left[\Delta\left(C - \frac{m}{2}(1 - \frac{m}{2})\mathbf{1}\right), \Delta(K^+)\right] \\ &= \Delta([C, K^+]) \\ &= 0\end{aligned}$$

The symmetry $S_{1,2} = \exp(K_1^+ + K_2^+)$

$$
\begin{aligned}
S_{1,2}(\eta_1, \eta_2; \xi_1, \xi_2) &= \prod_{i=1}^{2} \langle \eta_i | \exp(K_i^+) | \xi_i \rangle \\
&= \prod_{i=1}^{2} \langle \eta_i | \sum_{s_i \geq 0} \frac{(K_i^+)^{s_i}}{s_i!} | \xi_i \rangle \\
&= \prod_{i=1}^{2} \langle \eta_i | \sum_{s_i \geq 0} \frac{(\frac{m}{2} + \xi_i + s_i - 1)!}{(\frac{m}{2} + \xi_i - 1)! s_i!} | \xi_i + s_i \rangle \\
&= \prod_{i=1}^{2} \frac{(\frac{m}{2} + \eta_i - 1)!}{(\frac{m}{2} + \xi_i - 1)! (\eta_i - \xi_i)!} \\
&= \prod_{i=1}^{2} \frac{\Gamma(\frac{m}{2} + \eta_i)}{\Gamma(\frac{m}{2} + \xi_i)} \frac{1}{(\eta_i - \xi_i)!}
\end{aligned}
$$

Trivial self-duality dunction $d_{1,2}$

Remember that on the graph we had

$$\mu_{rev}(\xi) = \prod_{i=1}^{|V|} \frac{p^{\xi_i}(1-p)^{\frac{m}{2}}}{\xi_i!} \frac{\Gamma(\frac{m}{2}+\xi_i)}{\Gamma(\frac{m}{2})}$$

and a trivial (i.e. diagonal) self-duality function was obtained as

$$\mathbf{d}(\eta,\xi) = \frac{1}{\mu_{rev}(\eta)}\delta_{\eta,\xi}$$

Since the total number of particles is constant, one can take

$$d_{1,2}(\eta_1,\eta_2;\xi_1,\xi_2) = \prod_{i=1}^{2} \frac{\eta_i!\,\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}+\eta_i)}\delta_{\eta_i,\xi_i}$$

The duality function $D_{1,2}$

Combining trivial self-duality and symmetry leads to

$$
\begin{aligned}
D_{1,2}(\eta_1, \eta_2; \xi_1, \xi_2) &= d_{1,2} S_{1,2}(\eta_1, \eta_2; \xi_1, \xi_2) \\
&= \prod_{i=1}^{2} \frac{\eta_i!\, \Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2} + \eta_i)} \cdot \frac{\Gamma(\frac{m}{2} + \eta_i)}{\Gamma(\frac{m}{2} + \xi_i)} \frac{1}{(\eta_i - \xi_i)!} \\
&= \prod_{i=1}^{2} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2} + \xi_i)}
\end{aligned}
$$

From two sites to a chain

Using co-associativity property of the co-product

$$(\Delta \otimes \mathbf{1})\Delta = (\mathbf{1} \otimes \Delta)\Delta$$

leads to a natural notion of the n^{th} power of the co-product

$$\begin{cases} \Delta^{(1)}(x) = \Delta(x) \\ \\ \Delta^{(n)}(x) = \Delta^{n-1}(x) \otimes \mathbf{1} + \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{n \text{ times}} \otimes x_{n+1} \end{cases}$$

Therefore

$$\left(L_N^{SIP(m)}\right)^* = \Delta^{(N-1)}\left(C - \frac{m}{2}(1 - \frac{m}{2})\mathbf{1}\right)$$

$$S_N = \Delta^{(N-1)}(J^+)$$

$$[(L_N^{SIP(m)})^*, S_N] = 0$$

Summary of self-duality for the main examples

Theorem

The INCLUSION process is self-dual on

$$D(\eta,\xi)=\prod_i \frac{\eta_i!}{(\eta_i-\xi_i)!}\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+\xi_i\right)}$$

The INDEPENDENT WALKERS process is self-dual on

$$D(\eta,\xi)=\prod_i \frac{\eta_i!}{(\eta_i-\xi_i)!}$$

The EXCLUSION process is self-dual on

$$D(\eta,\xi)=\prod_i \frac{\eta_i!}{(\eta_i-\xi_i)!}\frac{\Gamma\left(n+1-\xi_i\right)}{\Gamma\left(n+1\right)}$$

Exercise: Self-duality of independent walkers

Prove that the process with generator L^{IND} is self-dual on functions

$$D(\eta,\xi) = \prod_{i\in V} \frac{\eta_i!}{(\eta_i - \xi_i)!}$$

Hint:

$$[L^*_{IND}, \sum_i a_i^+] = [L^*_{IND}, \sum_i a_i^-] = 0$$

Self-duality fct related to the simmetry $S = e^{\sum_i a_i^+}$

Exercise: Self-duality of the SEP(n) process

Prove that the process with generator $L^{SEP(n)}$ is self-dual on functions

$$D(\eta,\xi) = \prod_{i\in V} \frac{\eta_i!}{(\eta_i-\xi_i)!} \frac{\Gamma\left(n+1-\xi_i\right)}{\Gamma\left(n+1\right)}$$

Hint:

$$[L^*_{SEP(n)}, \sum_i J_i^o] = [L^*_{SEP(n)}, \sum_i J_i^+] = [L^*_{SEP(n)}, \sum_i J_i^-] = 0$$

Self-duality corresponds to the action of the symmetry $S = e^{\sum_i J_i^+}$

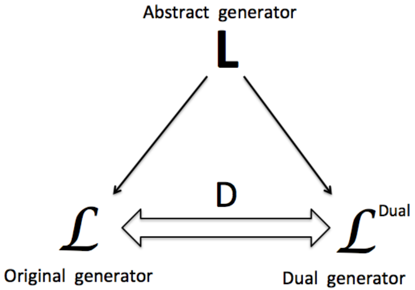
How to find a dual process?

1. Write the generator in abstract form , i.e. as an element of a Lie algebra, using creation and annihilation operators.

2. Self-duality is associated to symmetries, i.e. conserved quantities.

3. Duality is related to a change of representation, i.e. new operators that satisfy the same algebra.

2.2: Duality



Abstract generator
L
D
$\mathcal{L}$
$\mathcal{L}^{\text{Dual}}$
Original generator
Dual generator

Self-duality of Example 1.2.b:

$$\begin{array}{c}\text{Brownian momentum process}\\ \big\downarrow \\ \text{Change of representation of } \mathfrak{su}(1,1) \text{ algebra}\\ \big\downarrow \\ \text{Inclusion process with } m = 1\end{array}$$

Brownian momentum process (BMP)

For a graph $G = (V, E)$ let $\Omega = \otimes_{i \in V} \Omega_i = \mathbb{R}^{|V|}$.
Configuration $x = (x_1, \ldots, x_{|V|}) \in \Omega$

Generator BMP

$$L^{BMP} = \sum_{(i,j) \in E} L^{BMP}_{i,j} = \sum_{(i,j) \in E} \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2$$

Symmetric Inclusion Process SIP(1)

$\Omega_{dual} = \otimes_{i \in V} \Omega_i^{dual} = \{0, 1, 2, \ldots\}^{|V|}$
Configuration $\xi = (\xi_1, \ldots, \xi_{|V|}) \in \Omega_{dual}$

Generator SIP(1)

$$L^{SIP} f(\xi) = \sum_{(i,j) \in E} L_{i,j}^{SIP} f(\xi)$$

$$= \sum_{(i,j) \in E} \xi_i \left(\xi_j + \frac{1}{2} \right) [f(\xi^{i,j}) - f(\xi)] + \left(\xi_i + \frac{1}{2} \right) \xi_j \, [f(\xi^{j,i}) - f(\xi)]$$

Duality between BMP and SIP(1)

Theorem

The process $\{x(t)\}_{t\geq 0}$ with generator $L = L^{BMP}$ and the process $\{\xi(t)\}_{t\geq 0}$ with generator $L_{dual} = L^{SIP(1)}$ are dual on

$$D(x,\xi) = \prod_{i\in V} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!}$$

Proof: An explicit computation gives

$$L^{BMP} D(\cdot,\xi)(x) = L^{SIP(1)} D(x,\cdot)(\xi)$$

Duality explained

Abstract operator: $\mathfrak{su}(1,1)$ ferromagnetic quantum spin chain

$$\mathscr{L} = \sum_{(i,j)\in E} \left(K_i^+ K_j^- + K_i^- K_j^+ - 2K_i^o K_j^o + \frac{1}{8} \right)$$

with $\{K_i^+, K_i^-, K_i^o\}_{i\in V}$ satisfying $\mathfrak{su}(1,1)$ commutation relations:

$$[K_i^o, K_j^\pm] = \pm\delta_{i,j} K_i^\pm \qquad\qquad [K_i^-, K_j^+] = 2\delta_{i,j} K_i^o$$

Duality between L^{BMP} e L^{SIP} corresponds to two different representations of the operator $\mathscr{L}$.
Duality fct is the intertwiner.

Representation of $\mathfrak{su}(1,1)$ algebra in terms of differential operators

Continuous representation

$$\mathcal{K}_i^+ = \frac{1}{2}x_i^2 \qquad \mathcal{K}_i^- = \frac{1}{2}\frac{\partial^2}{\partial x_i^2}$$

$$\mathcal{K}_i^o = \frac{1}{4}\left(x_i\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i}x_i\right)$$

satisfy commutation relations **(Exercise!)**

$$[K_i^o, K_i^\pm] = \pm K_i^\pm \qquad [K_i^-, K_i^+] = 2K_i^o$$

In this representation

$$\mathscr{L} = L^{BMP} = \sum_{(i,j)\in E}\left(x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}\right)^2$$

Representation of $\mathfrak{su}(1,1)$ algebra in terms of matrices

Discrete representation

$$K_i^{+}|\xi_i\rangle = \left(\xi_i + \frac{1}{2}\right)|\xi_i + 1\rangle$$

$$K_i^{-}|\xi_i\rangle = \xi_i|\xi_i - 1\rangle$$

$$K_i^{o}|\xi_i\rangle = \left(\xi_i + \frac{1}{4}\right)|\xi_i\rangle$$

In a canonical base

$$\mathcal{K}_i^{+} = \begin{pmatrix} 0 & & & \\ \frac{1}{2} & \ddots & & \\ & \frac{3}{2} & \ddots & \\ & & \ddots & \ddots \end{pmatrix} \quad \mathcal{K}_i^{-} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & 2 & \\ & & \ddots & \ddots \\ & & & \ddots \end{pmatrix} \quad \mathcal{K}_i^{0} = \begin{pmatrix} \frac{1}{4} & 0 & & \\ & \frac{5}{4} & \ddots & \\ & & \frac{9}{4} & \ddots \\ & & & \ddots \end{pmatrix}$$

Representation of $\mathfrak{su}(1,1)$ algebra in terms of matrices

Discete representation

$$K_i^+|\xi_i\rangle = \left(\xi_i + \frac{1}{2}\right)|\xi_i + 1\rangle$$
$$K_i^-|\xi_i\rangle = \xi_i|\xi_i - 1\rangle$$
$$K_i^o|\xi_i\rangle = \left(\xi_i + \frac{1}{4}\right)|\xi_i\rangle$$

In this representation

$$\begin{aligned}\mathscr{L}^* f(\xi) &= L^{SIP(1)} f(\xi) \\ &= \sum_{(i,j)\in E} \xi_i \left(\xi_j + \frac{1}{2}\right)[f(\xi^{i,j}) - f(\xi)] + \left(\xi_i + \frac{1}{2}\right)\xi_j[f(\xi^{j,i}) - f(\xi)]\end{aligned}$$

Intertwiner as duality function

Intertwiner

$$\mathcal{K}_i^+ D_i(\cdot,\xi_i)(x_i) = K_i^+ D_i(x_i,\cdot)(\xi_i)$$
$$\mathcal{K}_i^- D_i(\cdot,\xi_i)(x_i) = K_i^- D_i(x_i,\cdot)(\xi_i)$$
$$\mathcal{K}_i^o D_i(\cdot,\xi_i)(x_i) = K_i^o D_i(x_i,\cdot)(\xi_i)$$

From the creation operators

$$\frac{x_i^2}{2} D_i(x_i,\xi_i) = \left(\xi_i + \frac{1}{2}\right) D_i(x_i,\xi_i+1)$$

Therefore

$$D_i(x_i,1) = \frac{x_i^2}{1} D_i(x_i,0)$$
$$D_i(x_i,2) = \frac{x_i^2}{3} D_i(x_i,1) = \frac{x_i^4}{3\cdot 1} D_i(x_i,0)$$
$$\vdots$$

Intertwiner as duality function

The recursion relation gives

$$D_i(x_i, \xi_i) = \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!} D_i(x_i, 0)$$

If we put $D_i(x_i, 0)$ equal to a constant, this expression solves also the relations for the annihilation operators

$$\frac{1}{2} D_i''(x_i, \xi_i) = \xi_i D_i(x_i, \xi_i - 1)$$

Indeed

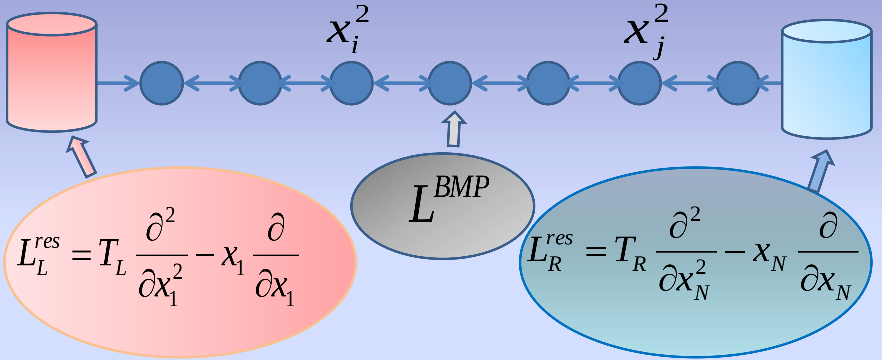
$$\frac{2\xi_i(2\xi_i - 1)x^{2\xi_i - 2}}{2(2\xi_i - 1)(2\xi_i - 3)!!} = \xi_i \frac{x_i^{2(\xi_i - 1)}}{(2(\xi_i - 1) - 1)!!}$$

Same is true for the relations with the number operator (**Exercise!**).

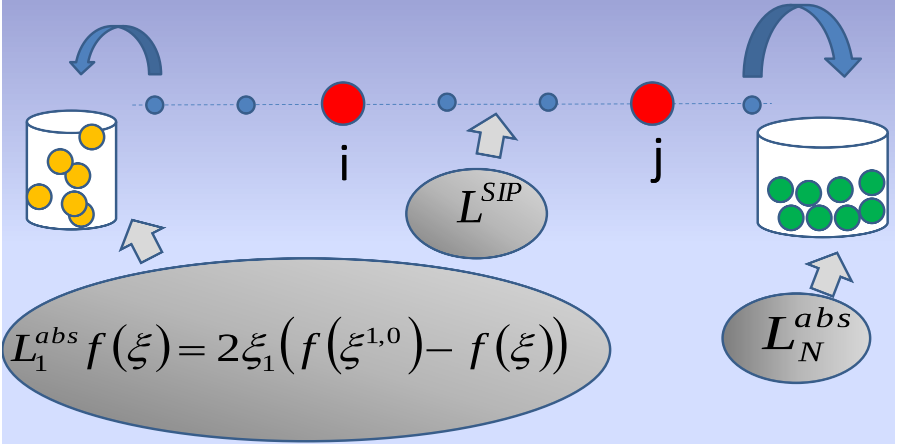
As a consequence D is a duality function between L^{BMP} and $L^{SIP(1)}$.

2.3: Boundary driven systems

Brownian Momentum Process with reservoirs



Inclusion Process with absorbing reservoirs



BMP with thermal reservoirs

Generator

$$L^{BMP,res} = L_1^{res} + \sum_{i=1}^{N-1} L_{i,i+1}^{BMP} + L_N^{res}$$

BMP with thermal reservoirs

Generator

$$L^{BMP,res} = L_1^{res} + \sum_{i=1}^{N-1} L_{i,i+1}^{BMP} + L_N^{res}$$

$$L_{i,i+1}^{BMP} = \left(x_i \frac{\partial}{\partial x_{i+1}} - x_{i+1} \frac{\partial}{\partial x_i} \right)^2 \qquad \text{Bulk}$$

BMP with thermal reservoirs

Generator

$$L^{BMP,res} = L_1^{res} + \sum_{i=1}^{N-1} L_{i,i+1}^{BMP} + L_N^{res}$$

$$L_{i,i+1}^{BMP} = \left(x_i \frac{\partial}{\partial x_{i+1}} - x_{i+1} \frac{\partial}{\partial x_i} \right)^2 \qquad \text{Bulk}$$

$$L_1^{res} = T_1 \frac{\partial^2}{\partial x_1^2} - x_1 \frac{\partial}{\partial x_1} \qquad \text{Reservoir}$$

BMP with thermal reservoirs

Generator

$$L^{BMP,res} = L_1^{res} + \sum_{i=1}^{N-1} L_{i,i+1}^{BMP} + L_N^{res}$$

$$L_{i,i+1}^{BMP} = \left(x_i \frac{\partial}{\partial x_{i+1}} - x_{i+1} \frac{\partial}{\partial x_i} \right)^2 \qquad \text{Bulk}$$

$$L_1^{res} = T_1 \frac{\partial^2}{\partial x_1^2} - x_1 \frac{\partial}{\partial x_1} \qquad \text{Reservoir}$$

$T_1 = T_N = T$ (equilibrium): Gibbs measure $\nu_T = \otimes_{i=1}^{N} \mathcal{N}(0, T)$.
$T_1 \neq T_N$ (non-equilibrium): unknown stationary measure.

SIP with absorbing boundaries

Configurations $\bar{\xi} = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

SIP with absorbing boundaries

Configurations $\bar{\xi} = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

Generator

$$L^{SIP,abs} = L_1^{abs} + \sum_{i=1}^{N-1} \mathcal{L}_{i,i+1}^{SIP} + L_N^{abs}$$

SIP with absorbing boundaries

Configurations $\bar{\xi} = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

Generator

$$L^{SIP,abs} = L_1^{abs} + \sum_{i=1}^{N-1} \mathcal{L}_{i,i+1}^{SIP} + L_N^{abs}$$

$$\mathcal{L}_{i,i+1}^{SIP} f(\xi) = \sum_{i=1}^{N-1} \xi_i(\xi_{i+1} + \frac{1}{2})[f(\xi^{i,i+1}) - f(\xi)] \qquad \text{Bulk}$$

$$+ \xi_{i+1}(\xi_i + \frac{1}{2})[f(\xi^{i+1,i}) - f(\xi)]$$

SIP with absorbing boundaries

Configurations $\bar{\xi} = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

Generator

$$L^{SIP,abs} = L_1^{abs} + \sum_{i=1}^{N-1} \mathcal{L}_{i,i+1}^{SIP} + L_N^{abs}$$

$$\mathcal{L}_{i,i+1}^{SIP} f(\xi) = \sum_{i=1}^{N-1} \xi_i(\xi_{i+1} + \frac{1}{2})[f(\xi^{i,i+1}) - f(\xi)] \qquad \text{Bulk}$$

$$+ \xi_{i+1}(\xi_i + \frac{1}{2})[f(\xi^{i+1,i}) - f(\xi)]$$

$$L_1^{abs} f(\xi) = \frac{\xi_1}{2}(f(\xi^{1,0}) - f(\xi)) \qquad \text{Reservoir}$$

Duality between BMP with reservoirs and SIP(1) with absorbing boundaries

Configurations $\bar{\xi} = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

Theorem

The process $\{x(t)\}_{t\geq 0}$ with generator $L^{BMP,res}$ is dual to the process $\{\bar{\xi}(t)\}_{t\geq 0}$ with generator $L^{SIP(1),abs}$ on

$$D(x,\bar{\xi}) = T_L^{\xi_0} \left(\prod_{i=1}^{N} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!} \right) T_R^{\xi_{N+1}}$$

Duality between BMP with reservoirs and SIP(1) with absorbing boundaries

Configurations $\bar{\xi} = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

Theorem

The process $\{x(t)\}_{t\geq 0}$ with generator $L^{BMP,res}$ is dual to the process $\{\bar{\xi}(t)\}_{t\geq 0}$ with generator $L^{SIP(1),abs}$ on

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Duality between BMP with reservoirs and SIP(1) with absorbing boundaries

Configurations $\bar{\xi} = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

Theorem

The process $\{x(t)\}_{t\geq 0}$ with generator $L^{BMP,res}$ is dual to the process $\{\bar{\xi}(t)\}_{t\geq 0}$ with generator $L^{SIP(1),abs}$ on

$$D(x, \bar{\xi}) = T_L^{\xi_0} \left(\prod_{i=1}^{N} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!} \right) T_R^{\xi_{N+1}}$$

Proof:

$$L^{BMP,res} D(\cdot, \bar{\xi})(x) = L^{SIP(1),abs} D(x, \cdot)(\bar{\xi})$$

CONSEQUENCES OF DUALITY

- From continuous to discrete:
  Interacting diffusions (BMP) studied via interacting particle systems (SIP(1)).

- From many to few:
  n-points correlation functions of N particles using n dual walkers
  Remark: $n \ll N$.

- From reservoirs to absorbing boundaries:
  Stationary state of dual process described by absorption probabilities of dual particles at the boundaries.

Expectations of duality functions in the BMP stationary state

Proposition

Let $|\xi| = \sum_{i=1}^{N} \xi_i$ be the total number of SIP dual walkers. Let $\mathbb{P}_{\bar{\xi}}(a,b) = \mathbb{P}(\xi_0(\infty) = a, \xi_{N+1}(\infty) = b \mid \xi(0) = \bar{\xi})$. Then

$$\mathbb{E}(D(x,\bar{\xi})) = \sum_{a,b:\ a+b=|\xi|} T_L^a T_R^b \ \mathbb{P}_{\bar{\xi}}(a,b)$$

Proof:

$$\mathbb{E}(D(x,\bar{\xi})) = \lim_{t\to\infty} \int \mathbb{E}_{x_0}\Big(D(x(t),\bar{\xi})\Big)\, d\nu(x_0)$$

$$= \int \lim_{t\to\infty} \mathbb{E}_{\bar{\xi}}\Big(D(x_0,\bar{\xi}(t))\Big)\, d\nu(x_0)$$

$$\textit{using} \qquad D(x,\bar{\xi}) = T_L^{\xi_0} \left(\prod_{i=1}^{N} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!} \right) T_R^{\xi_{N+1}}$$

$$= \mathbb{E}_{\bar{\xi}}\big(T_L^{\xi_0(\infty)}\, T_R^{\xi_{N+1}(\infty)}\big)$$

Temperature profile 1d linear chain

$$\vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0) \Rightarrow D(x, \vec{\xi}) = x_i^2$$

site i $\nearrow$ $\Rightarrow$ 1 SIP(1) walker $(X_t)_{t\geq 0}$ with $X_0 = i$

$$\mathbb{E}\left(x_i^2\right) = T_L\, \mathbb{P}_i(X_\infty = 0) + T_R\, \mathbb{P}_i(X_\infty = N+1)$$

$$\mathbb{E}(x_i^2) = T_L \left(1 - \frac{i}{N+1}\right) + T_R \left(\frac{i}{N+1}\right) = T_L + \left(\frac{T_R - T_L}{N+1}\right) i$$

$$\langle J \rangle = \mathbb{E}(x_{i+1}^2) - \mathbb{E}(x_i^2) = \frac{T_R - T_L}{N+1} \quad \textit{Fourier's law}$$

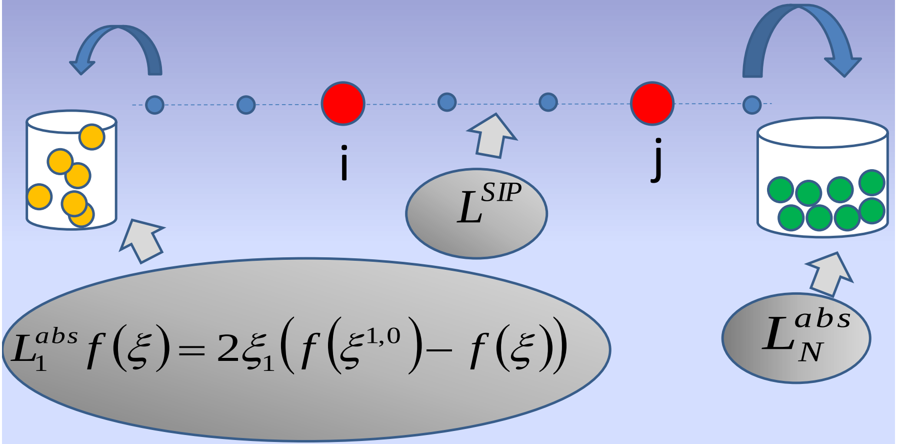
Energy covariance 1d linear chain

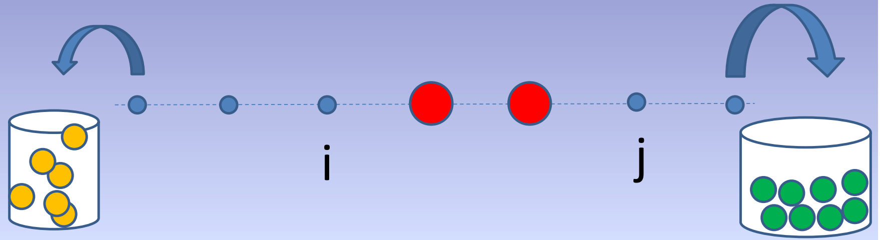
$$\text{If } \vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) \quad \Rightarrow \quad D(x, \vec{\xi}) = x_i^2 x_j^2$$

site i $\nearrow$ site j $\nearrow$

In the dual process we initialize two
SIP walkers $(X_t, Y_t)_{t \geq 0}$ with $(X_0, Y_0) = (i, j)$

Inclusion Process with absorbing reservoirs





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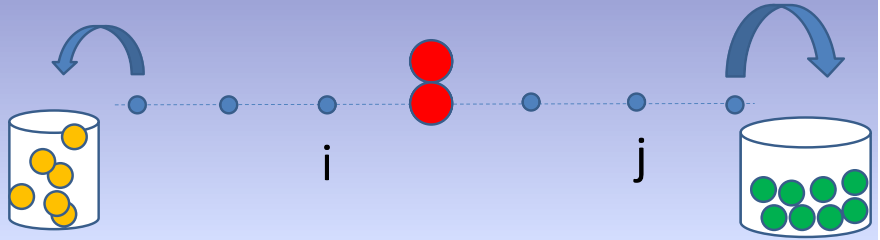

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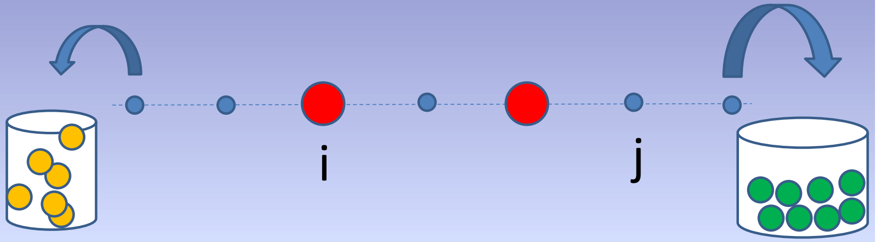
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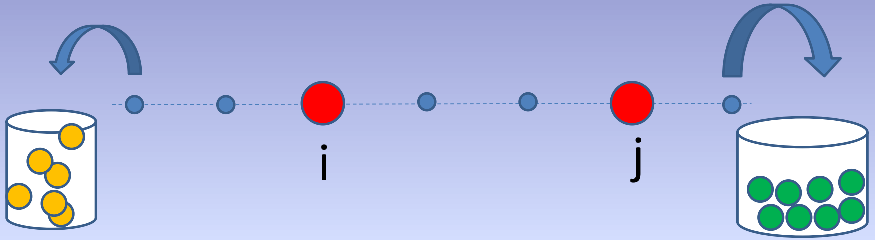
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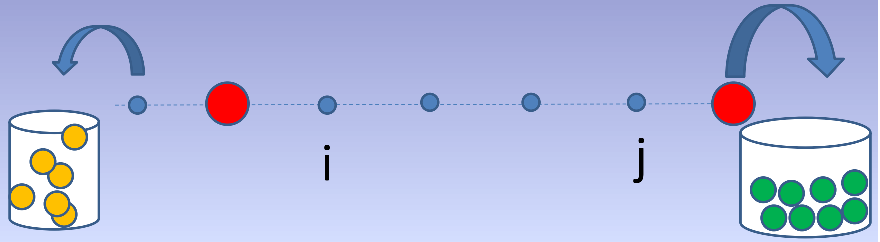
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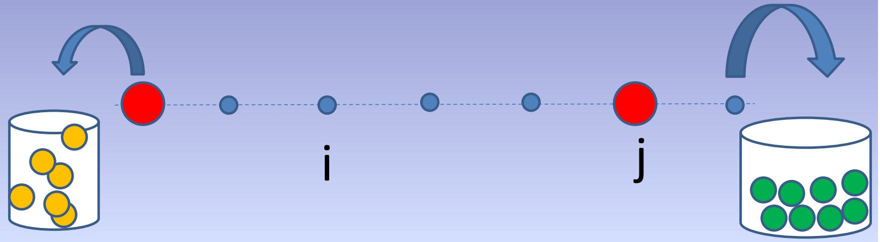
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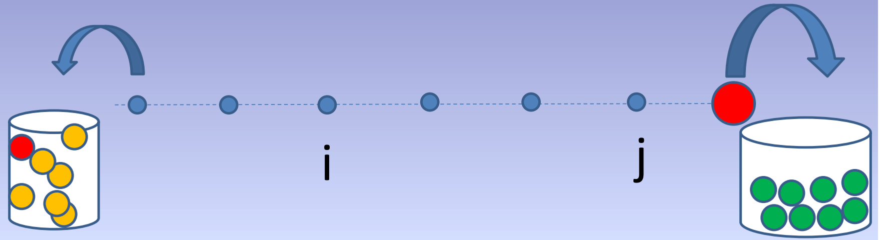

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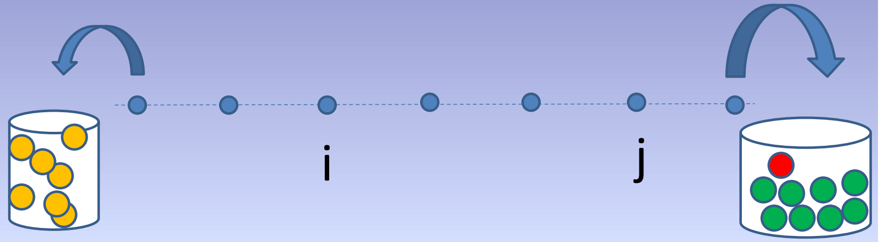
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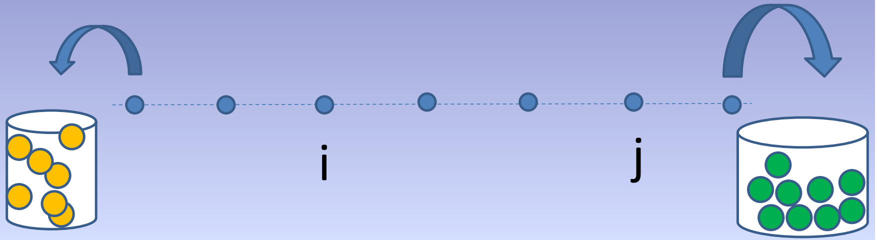


$$\mathbf{E}\left(x_i^2 x_j^2\right) = T_L^2\,\mathrm{P}\left(\substack{\bullet\\\bullet}\right) + T_R^2\,\mathrm{P}\left(\substack{\bullet\\\bullet}\right) + T_L\,T_R\left(\mathrm{P}\left(\substack{\bullet\\\bullet}\right) + \mathrm{P}\left(\substack{\bullet\\\bullet}\right)\right)$$

Energy covariance 1d linear chain

$$\mathbb{E}\left(x_i^2 x_j^2\right) - \mathbb{E}\left(x_i^2\right)\mathbb{E}\left(x_j^2\right) = \frac{2i(N+1-j)}{(N+3)(N+1)^2}(T_R - T_L)^2 \geq 0$$

Remark: up to a sign, covariance is the same in the boundary driven Exclusion Process with at most one particle per site.

Energy covariance 1d linear chain

$$\mathbb{E}\left(x_i^2 x_j^2\right) - \mathbb{E}\left(x_i^2\right)\mathbb{E}\left(x_j^2\right) = \frac{2i(N+1-j)}{(N+3)(N+1)^2}(T_R - T_L)^2 \geq 0$$

Remark: up to a sign, covariance is the same in the boundary driven Exclusion Process with at most one particle per site.

Remark: Long range correlations:

$$N\,Cov(x^2_{z_1 N}, x^2_{z_2 N}) \sim 2z_1(1-z_2)(T_R - T_L)^2$$

SIP Correlation Inequalities

Proposition

Let $\xi(t)$ be the SIP process and let ν_λ be its stationary measure. Then

$$\int \mathbb{E}_\xi \left(D\left(\xi_t, \sum_{i=1}^{n} \delta_{y_i}\right)\right) \nu_\lambda(d\xi) \geq \prod_{i=1}^{n} \int \mathbb{E}_\xi \left(D\left(\xi_t, \delta_{y_i}\right)\right) \nu_\lambda(d\xi)$$

In particular, the random variables $\{\xi_i(t)\}$ are **positively** correlated in the stationary state.

2.4: More diffusions models and redistribution models

(i). Brownian Energy Process $BEP(m)$

(ii). Instantaneous thermalization limit

(i) Brownian Energy Process: BEP

The energies of the Brownian Momentum Process

$$z_i(t) = x_i^2(t)$$

evolve with

Generator

$$L^{BEP} = \sum_{(i,j)\in E} z_i z_j \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j}\right)^2 - \frac{1}{2}(z_i - z_j)\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j}\right)$$

Stationary measures: χ-squared (1 d.f.) product measures.

Generalized Brownian Energy Process: BEP(m)

$$L^{BMP(m)} = \sum_{(i,j)\in E} \sum_{\alpha,\beta=1}^{m} \left(x_{i,\alpha} \frac{\partial}{\partial x_{j,\beta}} - x_{j,\beta} \frac{\partial}{\partial x_{i,\alpha}} \right)^2$$

The energies $z_i(t) = \sum_{\alpha=1}^{m} x_{i,\alpha}^2(t)$ evolve with

Generator

$$L^{BEP(m)} = \sum_{(i,j)\in E} z_i z_j \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)^2 - \frac{m}{2}(z_i - z_j) \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)$$

Stationary measures: product χ-squared (m d.f.) $\equiv \mathrm{Gamma}(\frac{m}{2}, \theta)$

$$\mu(dz) = \prod_{i=1}^{|V|} \frac{1}{\theta^{\frac{m}{2}} \Gamma(\frac{m}{2})} z_i^{\frac{m}{2}-1} e^{-z_i/\theta} dz_i$$

Adding-up $\mathfrak{su}(1,1)$ spins

$$\mathcal{L}^{(m)} = \sum_{(i,j)\in E} \left(\mathcal{K}_i^+ \mathcal{K}_j^- + \mathcal{K}_i^- \mathcal{K}_j^+ - 2\mathcal{K}_i^o \mathcal{K}_j^o + \frac{m^2}{8} \right)$$

$\{\mathcal{K}_i^+, \mathcal{K}_i^-, \mathcal{K}_i^o\}_{i\in V}$ satisfy $\mathfrak{su}(1,1)$

Adding-up $\mathfrak{su}(1,1)$ spins

$$\mathcal{L}^{(m)} = \sum_{(i,j)\in E} \left(\mathcal{K}_i^+ \mathcal{K}_j^- + \mathcal{K}_i^- \mathcal{K}_j^+ - 2\mathcal{K}_i^o \mathcal{K}_j^o + \frac{m^2}{8} \right)$$

$$\{\mathcal{K}_i^+, \mathcal{K}_i^-, \mathcal{K}_i^o\}_{i\in V} \qquad \text{satisfy } \mathfrak{su}(1,1)$$

$$\begin{cases} \mathcal{K}_i^+ = z_i \\ \mathcal{K}_i^- = z_i\,\partial_{z_i}^2 + \frac{m}{2}\partial_{z_i} \\ \mathcal{K}_i^0 = z_i\,\partial_{z_i} + \frac{m}{4} \end{cases} \qquad \begin{cases} \mathcal{K}_i^+|\xi_i\rangle = \left(\xi_i + \frac{m}{2}\right)|\xi_i + 1\rangle \\ \mathcal{K}_i^-|\xi_i\rangle = \xi_i|\xi_i - 1\rangle \\ \mathcal{K}_i^o|\xi_i\rangle = (\xi_i + m)\,|\xi_i\rangle \end{cases}$$

Duality between BEP(m) and SIP(m)

Theorem

The process $\{z(t)\}_{t\geq 0}$ with generator $L^{BEP(m)}$ and the process $\{\xi(t)\}_{t\geq 0}$ with generator $L^{SIP(m)}$ are dual on

$$D(z,\xi) = \prod_{i\in V} z_i^{\xi_i} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}+\xi_i)}$$

(ii) Redistribution models

Main example: KMP model. Energies $z = (z_1, \ldots, z_N) \in \mathbb{R}_+^N$

Select a bond (i, j) and **uniformly** redistribute the energy under the constraint of conserving the total energy on the bond.

Generator

$$L^{KMP} f(z) =$$

$$\sum_i \int_0^1 dp[f(z_1, \ldots, p(z_i + z_{i+1}), (1-p)(z_i + z_{i+1}), \ldots, z_N) - f(z)]$$

KMP model is an instantaneous thermalization limit of BEP(2).

Instantaneous thermalization limit

$$
\begin{aligned}
L^{IT}_{i,j} f(z_i, z_j) &:= \lim_{t\to\infty} \left(e^{tL^{BEP(m)}_{i,j}} - 1 \right) f(z_i, z_j) \\
&= \int dz_i' dz_j' \, \rho^{(m)}(z_i', z_j' \mid z_i' + z_j' = z_i + z_j)[f(z_i', z_j') - f(z_i, z_j)] \\
&= \int_0^1 dp \, \nu^{(m)}(p) \, [f(p(z_i + z_j), (1-p)(z_i + z_j)) - f(z_i, z_j)]
\end{aligned}
$$

$$
X, Y \sim \mathrm{Gamma}\left(\frac{m}{2}, \theta\right) \quad \text{i.i.d.} \quad \Longrightarrow \quad P = \frac{X}{X+Y} \sim \mathrm{Beta}\left(\frac{m}{2}, \frac{m}{2}\right)
$$

For $m = 2$: uniform redistribution

Indeed: For all bonds, BEP(m) conserves total energy. Conditioning two i.i.d. $\mathrm{Gamma}(\frac{m}{2}, \theta)$ random variables (Z_i, Z_j) to $Z_i + Z_j = E$, one has a conditional density

$$
g_{Z_i}(z | Z_i + Z_j = E) = c_m(E) \, z^{\frac{m}{2} - 1} (E - z)^{\frac{m}{2} - 1}
$$

Exercises (see JSP 152, 657-697 (2013))

- Particle redistribution models

$$L^{IT}_{i,j} f(\xi) = \sum_{s=0}^{\xi_i+\xi_j} \nu(r|\xi_i+\xi_j)\Big(f(\xi_1, \ldots, \underset{\text{site } i \nearrow}{r}, \ldots, \underset{\text{site } j \nearrow}{\xi_i+\xi_j-r}, \ldots, \xi_{|V|}) - f(\xi)\Big)$$

  Prove the following redistribution rules are obtained by taking instantaneous thermalization limits:
  - inclusion process $\to$ negative hypergeometric redistribution
  - independent walkers $\to$ binomial redistribution
  - exclusion process $\to$ hypergeometric redistribution

- Verify that particle redistribution models are self-dual. Verify that energy redistribution model (i.e. thermalization of $BEP(m)$) and the thermalization of $SIP(m)$ are dual.

3. Population dynamics

Moran model with two types

Consider a population of N individuals, each of which can be of two types (say 1 and 2). A pair of individuals are sampled uniformly at random, one dies with probability $1/2$, the other reproduces.

Define

$$K^{(N)}(t) = \text{number of individuals of type 1 at time} \quad t \geq 0$$

Then $(K^{(N)}(t))_{t\geq 0}$ is a continuos time Markov chain with state space $\Omega_N = \{0, 1, \ldots N\}$ and generator

$$L_N^{Moran} f(k) = \frac{1}{2} k(N-k)(f(k+1) + f(k-1) - 2f(k))$$

Wright-Fisher diffusion with two types

Diffusive scaling limit: consider the process $\left(X^{(N)}(t) = \frac{K^{(N)}(N^2 t)}{N}\right)_{t\geq 0}$ with state space $\Omega'_N = \{0, 1/N, \dots 1\}$. Its generator reads

$$L'_N f(\frac{k}{N}) = N^2 \frac{1}{2}\frac{k}{N}(1-\frac{k}{N})\left(f(\frac{k}{N}+\frac{1}{N}) + f(\frac{k}{N}-\frac{1}{N}) - 2f(\frac{k}{N})\right)$$

In the limit $N \to \infty$ the process $(X^{(N)}(t))_{t\geq 0}$ converges to the Wright-Fisher diffusion $(X(t))_{t\geq 0}$ with state space $[0,1]$ and generator

$$L^{WF} f(x) = \frac{1}{2}x(1-x)\frac{\partial^2 f}{\partial x^2}(x)$$

Counting blocks of the Kingman coalescence

For each $k \in \mathbb{N}$, the k-coalescence is a continuous time Markov chain on the space of equivalence relations on $\{1, 2, \dots, k\}$ with transition rates

$$c(x, y) = \begin{cases} 1 & \text{if } y \text{ is obtained by coalescing} \\ & \text{two equivalence classes of } x, \\ 0 & \text{otherwise.} \end{cases}$$

By extension the Kingman coalescent on $\mathbb{N}$ is defined by requiring that for each k its restriction to $\{1, \dots, k\}$ is a k-coalescence.

Define

$$N(t) = \text{ number of blocks in the } k\text{-coalescence at time } t \geq 0.$$

It is a death process on $\{1, \dots, k\}$ defined by the Markov generator

$$(L^{King} f)(n) = \frac{n(n-1)}{2}(f(n-1) - f(n))$$

Duality Wright-Fisher / Kingman

Theorem

The process $\{X(t)\}_{t\geq 0}$ with generator L^{WF} and the process $\{N(t)\}_{t\geq 0}$ with generator L^{King} are dual on $D(x,n) = x^n$, i.e.

$$\mathbb{E}_x(X(t)^n) = \mathbb{E}_n(x^{N(t)})$$

Indeed:

$$\begin{aligned} L^{WF} D(\cdot, n)(x) &= \frac{1}{2} x(1-x) \frac{\partial^2}{\partial x^2} x^n \\ &= \frac{n(n-1)}{2} (x^{n-1} - x^n) \\ &= \frac{n(n-1)}{2} (D(x, n-1) - D(x, n)) \\ &= L^{King} D(x, \cdot)(n) \end{aligned}$$

Extinction probability in the WF diffusion

This is defined as the probability that type (say) 1 gets extinct starting from a proportion x, i.e.

$$p_{ext}(x) = \mathbb{P}(X(\infty) = 0 | X(0) = x)$$

This is related to the asymptotics of first moment:

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}_x^{WF}(X(t)) &= 1\mathbb{P}(X(\infty) = 1 | X(0) = x) + 0\mathbb{P}(X(\infty) = 0 | X(0) = x)\\ &= 1 - p_{ext}(x)\end{aligned}$$

To compute this quantity we can use duality:

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}_x^{WF}(X(t)) &= \lim_{t\to\infty} \mathbb{E}_x^{WF}(D(X(t),1))\\ &= \lim_{t\to\infty} \mathbb{E}_1^{King}(D(x,N(t)))\\ &= x\end{aligned}$$

Therefore

$$p_{ext}(x) = 1 - x$$

Heterozygosity in the WF diffusion

This is defined as the probability that two randomly chosen individuals are of different types. To compute this quantity we can use duality:

$$
\begin{aligned}
\mathbb{E}_x^{WF}(X(t)(1-X(t))) &= \mathbb{E}_x^{WF}(D(X(t),1)-D(X(t),2)) \\
&= \mathbb{E}_1^{King}(D(x,N(t))) - \mathbb{E}_2^{King}(D(x,N(t))) \\
&= x - x^2\mathbb{P}(N(t)=2 \mid N(0)=2) \\
&\quad -x\,\mathbb{P}(N(t)=1 \mid N(0)=2) \\
&= x - x^2e^{-t} - x(1-e^{-t}) \\
&= x(1-x)e^{-t}
\end{aligned}
$$

In particular

$$
\lim_{t\to\infty} \mathbb{E}_x^{WF}(X(t)(1-X(t))) = 0
$$

Duality Wright-Fisher / Kingman : algebraic approach

In the Lie algebraic approach the duality is a consequence of a change of representation of the Heisenberg algebra:

$$\begin{cases} a^+ = x \\ \\ a^- = \frac{\partial}{\partial x} \end{cases} \qquad\qquad \begin{cases} a^+|n\rangle = |n+1\rangle \\ \\ a^-|n\rangle = n|n-1\rangle \end{cases}$$

Then the abstract element

$$L = \frac{1}{2}a^+(1-a^+)a^2$$

gives rise to the two processes and $D(x,n)$ is the intertwiner:

$L = L^{WF}$ in the representation with differential operators

$L^T = L^{King}$ in the representation with matrices

Finite population size and finite dimensional representation

Introducing well-chosen discrete derivative and discrete multiplication operators, we can also find the duality between the discrete Moran model and the Kingman's coalescent. For functions $f : \{0, \ldots, N\} \to \mathbb{R}$ define

$$\begin{aligned} \mathrm{a}_N^- f(k) &= (N-k)\, f(k+1) + (2k-N)\, f(k) - kf(k-1) \\ \mathrm{a}_N^+ f(k) &= \sum_{r=0}^{k-1} (-1)^{k-1-r} \frac{\binom{N}{r}}{\binom{N}{k}} f(r) \,, \end{aligned}$$

with the convention $f(-1) = f(N+1) = 0$.

Duality between Moran and Kingman

Consider

$$D_N(k,n) = \frac{\binom{k}{n}}{\binom{N}{n}} = \frac{k(k-1)\cdots(k-(n-1))}{N(N-1)\cdots(N-(n-1))} \; .$$

with the convention $D_N(k,0) = 1$, $D_N(k, N+1) = 0$. Let us denote by $\mathcal{W}_N$ the vector space generated by the functions $k \mapsto D_N(k,n), 0 \le n \le N$. Then we have

$$\begin{aligned}
\mathrm{a}_N^- D_N(\cdot,n)(k) &= nD_N(k,n-1), \;\forall\, 1 \le n, \forall\, k \ge n-1 \; , \\
\mathrm{a}_N^- D_N(\cdot,0)(k) &= 0 \;\forall\, 0 \le k \le N \; , \\
\mathrm{a}_N^+ D_N(\cdot,n)(k) &= D_N(k,n+1) \;\forall\; 0 \le n \le N, k \ge n \; .
\end{aligned}$$

Duality between Moran and Kingman

As a consequence, as operators on $\mathcal{W}_N$, we have

$$[\mathrm{a}_N^-, \mathrm{a}_N^+] = \mathbf{1} \ ,$$

and

$$\mathrm{a}_N^- \to^{D_N} a^- \ , \qquad \mathrm{a}_N^+ \to^{D_N} a^+$$

with a^-, a^+ the discrete representation. Moreover the generator of the Moran model in terms of $\mathrm{a}_N^-, \mathrm{a}_N^+$ reads

$$\mathrm{a}_N^+(1 - \mathrm{a}_N^+)(\mathrm{a}_N^-)^2$$

i.e., the same as the Wright Fisher generator, but now in the $\mathrm{a}_N^-, \mathrm{a}_N^+$ representation. This explains that we find the same dual generator when going to the infinite-dimensional discrete representation, but now with another duality function.

Wright Fisher diffusion with mutation.

Other "evolutionary forces" can be included. Consider the Moran model where in between reproduction events each individual of type 2 mutates to an individual of type 1 at rate θ/N. Then in the diffusive limit one has

$$\begin{aligned} L^{WF,mut} &= x(1-x)\frac{d^2}{dx^2} + \theta(1-x)\frac{d}{dx} \\ &= a^+(1-a^+)a^2 + \theta(1-a^+)a \end{aligned}$$

By changing to a discrete representation of the Heisenberg algebra this gives the dual

$$L^{King,mut} f(n) \quad = \quad n(n-1)(f(n-1) - f(n)) + \theta n(f(n-1) - f(n))$$

which corresponds to Kingman's coalescent with extra rate θn to go down from n to $n-1$, due to mutation.

Population dynamics models with $\mathfrak{su}(1,1)$ symmetry

It turns out that the diffusions of Wright-Fisher type have more structure than only the Heisenberg algebra. In the multi-type setting with parent independent mutations their generator can be written using $\mathfrak{su}(1,1)$ generators satisfying the commutation relations

$$[K^o, K^{\pm}] = \pm K^{\pm} \qquad\qquad [K^-, K^+] = 2K^o$$

$\mathfrak{su}(1,1)$ Heisenberg ferromagnet as a population model

$$\mathcal{L}_m = \frac{1}{2} \sum_{1 \le i < j \le d} \left(K_i^+ K_j^- + K_i^- K_j^+ - 2K_i^o K_j^o + \frac{m^2}{8} \right)$$

1. Written in terms of the continuous representation, $\mathcal{L}_m$ is the generator of the d-type Wright-Fisher diffusion with mutation rate $m/2$.
2. Written in terms of the discrete representation, $\mathcal{L}_m$ is the generator of the d-type Moran model with mutation rate $m/2$.
3. It commutes with

$$\sum_i K_i^{\pm}, \sum_i K_i^o$$

Multi-type Wright-Fisher diffusion with symmetric mutations

The d-types Wright-Fisher diffusion model with parent-independent mutation at rate $\theta \in \mathbb{R}$ is a diffusion process on the simplex $\sum_{i=1}^{d} x_i = 1$ with

$$\begin{aligned}
\mathscr{L}^{WF}_{d,\theta} g(x) &= \sum_{i=1}^{d-1} \frac{1}{2} x_i(1-x_i) \frac{\partial^2 g(x)}{\partial x_i^2} - \sum_{1\le i<j\le d-1} x_i x_j \frac{\partial^2 g(x)}{\partial x_i \partial x_j} \\
&+ \frac{\theta}{d-1} \sum_{i=1}^{d-1} (1-dx_i) \frac{\partial g(x)}{\partial x_i} .
\end{aligned}$$

$$\mathscr{L}^{BEP(m)}_{d} f(x_1,\dots,x_{d-1},x_d) = \mathscr{L}^{WF}_{d,\frac{m}{4}(d-1)} g(x_1,\dots,x_{d-1})$$

$$g(x_1,\dots,x_{d-1}) = f(x_1,\dots,x_{d-1}, 1-\sum_{j=1}^{d-1} x_j)$$

$$\begin{aligned}
\mathcal{L}^{BEP(m)}_{d} f(z) &= \frac{1}{2} \sum_{1\le i<j\le d} z_i z_j \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)^2 f(z) \\
&- \frac{m}{4} \sum_{1\le i<j\le d} (z_i - z_j) \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right) f(z)
\end{aligned}$$

Multi-type Moran with symmetric mutations

The d-types Moran model with N individuals and parent-independent mutation at rate $\theta \in \mathbb{R}$ is a particle process on the simplex $\sum_{i=1}^{d} k_i = N$ where pair of individuals of types i and j are sampled uniformly at random, one dies with probability 1=2 and the other reproduces. In between reproduction events each individual accumulates mutations at a constant rate θ and his type mutates to any of the others with the same probability.

$$\mathscr{L}_d^{SIP(m)} f(k_1, \ldots, k_{d-1}, k_d) = \mathscr{L}_{N,d,\frac{m}{4}(d-1)}^{Moran} g(k_1, \ldots, k_{d-1})$$

$$g(k_1, \ldots, k_{d-1}) = f(k_1, \ldots, k_{d-1}, N - \sum_{j=1}^{d-1} k_j)$$

$$\begin{aligned} \mathcal{L}_d^{SIP(m)} f(k) &= \frac{1}{2} \sum_{1 \le i < j \le d} k_i \left(k_j + \frac{m}{2}\right) (f(k + e_i - e_j) - f(k)) \\ & \frac{1}{2} \sum_{1 \le i < j \le d} k_j \left(k_i + \frac{m}{2}\right) (f(k - e_i + e_j) - f(k)) \end{aligned}$$

Dualities for multi-type Wright-Fisher / Moral models (arXiv:1302.3206)

1. The d-type Wright Fisher diffusion with mutation rate $m/2$ and the d-type discrete Moran model with mutation rate $m/2$ are dual to each other with duality function

$$D(z_1,\ldots,z_d;k_1,\ldots,k_d) = \prod_i D_i(z_i,k_i)$$

with

$$D_i(z_i,k_i) = \frac{z_i^{k_i}\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+k_i\right)}$$

2. The d-type discrete Moran model with mutation rate $m/2$ is self-dual with self-duality function

$$D(n_1,\ldots,n_d;k_1,\ldots,k_d) = \prod_i D_i(n_i,k_i)$$

with

$$D_i(n_i,k_i) = \frac{n_i!}{(n_i-k_i)!}\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+k_i\right)}$$

4. Asymmetric systems and deformed algebras

How to find a dual process?

1. Write the generator in abstract form , i.e. as an element of a Lie algebra, using creation and annihilation operators.

2. Self-duality is associated to symmetries, i.e. conserved quantities.

3. Duality is related to a change of representation, i.e. new operators that satisfy the same algebra.

Conversely, Step 1. can be turned into a constructive step.

Constructive approach to asymmetric systems via duality

i) (*Quantum Lie Algebra*): Start from the quantization $U_q(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$.

ii) (*Co-product*): Consider a co-product $\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ making the quantized enveloping algebra a bialgebra.

iii) (*Quantum Hamiltonian*): Compute the co-product $\Delta(C)$ of a Casimir element C. The quantum Hamiltonian $H_{(L)}$ is constructed by translations of $\Delta(C)$.

iv) (*Ground state transformation*): Apply a ground state transform. to $H_{(L)}$ to turn it into the generator $\mathcal{L}^{(L)}$ of a Markov process.

Symmetries of $H_{(L)}$, obtained by applying the co-product to the generators of $U_q(\mathfrak{g})$, yield self-duality, *change of representations* yield dual processes.

q-numbers

For $q \in (0,1)$ and $n \in \mathbb{N}_0$ introduce the q-number

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Remark: $\lim_{q\to 1}[n]_q = n$. The first q-number's are:

$$[0]_q = 0, \qquad [1]_q = 1, \qquad [2]_q = q+q^{-1}, \qquad [3]_q = q^2+1+q^{-2}, \quad \dots$$

Also introduce: q-factorial

$$[n]_q! := [n]_q \cdot [n-1]_q \cdot \dots \cdot [1]_q \ ,$$

q-binomial coefficient

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} \ .$$

The quantum Lie algebra $U_q(\mathfrak{sl}_2) \equiv \mathfrak{su}_q(2)$

For $q \in (0,1)$ consider the algebra with generators J^+, J^-, J^0 satisfying the commutation relations

$$[J^+, J^-] = [2J^0]_q, \qquad [J^0, J^\pm] = \pm J^\pm$$

where

$$[2J^0]_q := \frac{q^{2J^0} - q^{-2J^0}}{q - q^{-1}}$$

Its irreducible representations are $(2j+1)-$dimensional, with $j \in \mathbb{N}/2$. They are labeled by the eigenvalues of the Casimir element

$$C = J^- J^+ + [J^0]_q [J^0 + 1]_q$$

A standard representation is given by

$$\begin{cases} J^+|n\rangle & = & \sqrt{[2j-n]_q[n+1]_q}\,|n+1\rangle \\ J^-|n\rangle & = & \sqrt{[n]_q[2j-n+1]_q}\,|n-1\rangle \\ J^0|n\rangle & = & (n-j)\,|n\rangle \end{cases}$$

In this representation $C|n\rangle = [j]_q[j+1]_q|n\rangle$

Co-product

A co-product $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ is defined as

$$\begin{aligned} \Delta(J^{\pm}) &= J^{\pm} \otimes q^{-J^0} + q^{J^0} \otimes J^{\pm} \\ \Delta(J^0) &= J^0 \otimes 1 + 1 \otimes J^0 \end{aligned}$$

The co-product is an isomorphism for $U_q(\mathfrak{sl}_2)$, i.e.

$$[\Delta(J^+), \Delta(J^-)] = [2\Delta(J^0)]_q \qquad [\Delta(J^0), \Delta(J^{\pm})] = \pm\Delta(J^{\pm})$$

From the co-associativity property

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$$

one can define iteratively $\Delta^n : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)^{\otimes(n+1)}$, i.e. for $n \geq 2$

$$\begin{aligned} \Delta^n(J^{\pm}) &= \Delta^{n-1}(J^{\pm}) \otimes q^{-J^0_{n+1}} + q^{\Delta^{n-1}(J^0_i)} \otimes J^{\pm}_{n+1} \\ \Delta^n(J^0) &= \Delta^{n-1}(J^0) \otimes 1 + \underbrace{1 \otimes \ldots \otimes 1}_{n \text{ times}} \otimes J^0_{n+1} \end{aligned}$$

Quantum Spin Chain

For every $L \in \mathbb{N}$, $L \geq 2$, we consider

$$H_{(L)} := \sum_{i=1}^{L-1} \left(h_{(L)}^{i,i+1} + c_{(L)} \right) ,$$

where

$$c_{(L)} = \frac{(q^{2j} - q^{-2j})(q^{2j+1} - q^{-(2j+1)})}{(q - q^{-1})^2} \underbrace{1 \otimes \cdots \otimes 1}_{L \text{ times}}$$

$$h_{(L)}^{i,i+1} := \underbrace{1 \otimes \cdots \otimes 1}_{(i-1) \text{ times}} \otimes \Delta(C_i) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(L-i-1) \text{ times}}$$

$$\begin{aligned} \Delta(C_i) \quad = \quad & -q^{J_i^0} \Bigg\{ J_i^+ \otimes J_{i+1}^- + J_i^- \otimes J_{i+1}^+ + \frac{(q^j + q^{-j})(q^{j+1} + q^{-(j+1)})}{2} [J_i^0]_q \otimes [J_{i+1}^0]_q \\ & + \frac{[j]_q [j+1]_q}{2} \left(q^{J_i^0} + q^{-J_i^0} \right) \otimes \left(q^{J_{i+1}^0} + q^{-J_{i+1}^0} \right) \Bigg\} q^{-J_{i+1}^0} \end{aligned}$$

ASEP(q,j) process

By applying a ground state transformation (i.e. similarity transformation in this case), one obtain the Markov process ASEP(q, j) on $[1, L] \cap \mathbb{Z}$, denoted by $(\eta(t))_{t\geq 0}$, with state space $\{0, 1, \ldots, 2j\}^L$ and defined by the generator

$$(\mathcal{L}_{(L)}^{ASEP(q,j)} f)(\eta) = \sum_{i=1}^{L-1} (\mathcal{L}_{i,i+1} f)(\eta)$$

with

$$\begin{aligned}(\mathcal{L}_{i,i+1} f)(\eta) &= q^{\eta_i - \eta_{i+1} - (2j+1)} [\eta_i]_q [2j - \eta_{i+1}]_q (f(\eta^{i,i+1}) - f(\eta)) \\ &+ q^{\eta_i - \eta_{i+1} + (2j+1)} [2j - \eta_i]_q [\eta_{i+1}]_q (f(\eta^{i+1,i}) - f(\eta))\end{aligned}$$

Symmetries of $H_{(L)}$

As a consequence of the co-product structure, the elements

$$
\begin{aligned}
J^{\pm}_{(L)} &:= \Delta^{L-1}(J^{\pm}_1) = \sum_{i=1}^{L} q^{J^0_1} \otimes \cdots \otimes q^{J^0_{i-1}} \otimes J^{\pm}_i \otimes q^{-J^0_{i+1}} \otimes \ldots \otimes q^{-J^0_L} \,, \\
J^{0}_{(L)} &:= \Delta^{L-1}(J^{0}_1) = \sum_{i=1}^{L} \underbrace{1 \otimes \cdots \otimes 1}_{(i-1)\ \text{times}} \otimes J^0_i \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(L-i)\ \text{times}} \,.
\end{aligned}
$$

are symmetries of $H_{(L)}$, i.e.

$$[H_{(L)}, J^{\pm}_{(L)}] = [H_{(L)}, J^{0}_{(L)}] = 0$$

Self-duality of ASEP(q,j)

Theorem

The ASEP(q,j) on $[1,L]\cap\mathbb{Z}$ with closed boundary conditions is self-dual on

$$D_{(L)}(\eta,\xi)=\prod_{i=1}^{L}\frac{[\eta_i]_q!}{[\eta_i-\xi_i]_q!}\frac{\Gamma_q(2j+1-\xi_i)}{\Gamma_q(2j+1)}\cdot q^{(\eta_i-\xi_i)[2\sum_{k=1}^{i-1}\xi_k+\xi_i]+4ji\xi_i}$$

Remark: let $\xi^{(i)}$ be the configurations such that

$$\xi_m^{(i)}=\begin{cases}1 & \text{if } m=i\\ 0 & \text{otherwise}\end{cases}\qquad\qquad N_i(\eta):=\sum_{k\geq i}\eta_k$$

then

$$D(\eta,\xi^{(i)})=\frac{q^{4ji-1}}{q^{2j}-q^{-2j}}\cdot\left(q^{2N_i(\eta)}-q^{2N_{i+1}(\eta)}\right)$$

First q-moment of the current

Define the total current $J_i(t)$ in the time interval $[0, t]$ as the net number of particles crossing the bond $(i-1, i)$ in the right direction:

$$J_i(t) = N_i(\eta(t)) - N_i(\eta(0))$$

Theorem

$$\mathbb{E}_\eta\left[q^{2J_i(t)}\right] = q^{2(N(\eta)-N_i(\eta))} - \sum_{k=-\infty}^{i-1} q^{-4jk}\, \mathbf{E}_k\left[q^{4jx(t)}\left(1 - q^{-2\eta_{x(t)}}\right) q^{2(N_{x(t)}(\eta)-N_i(\eta))}\right]$$

where $x(t)$ denotes a continuous time asymmetric random walker on $\mathbb{Z}$ jumping left at rate $q^{2j}[2j]_q$ and jumping right at rate $q^{-2j}[2j]_q$ and $\mathbf{E}_k$ denotes the expectation with respect to the law of $x(t)$ started at site $k \in \mathbb{Z}$ at time $t = 0$.

Step initial condition

Theorem

Consider the step configurations $\eta^+ \in \{0, \ldots, 2j\}^{\mathbb{Z}}$ defined as follows

$$\eta_i^+ := \begin{cases} 0 & \text{for } \; i < 0 \\ 2j & \text{for } \; i \geq 0 \end{cases}$$

then, for the infinite volume ASEP(q, j)

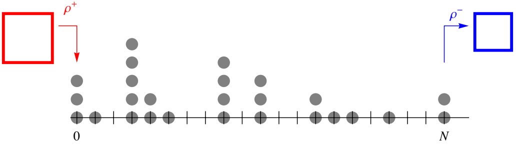
$$\mathbb{E}_{\eta^+}\left[q^{2J_k(t)}\right] = \frac{q^{4j\max\{0,k\}}}{2\pi i} \int e^{-\frac{q^{2j}[2j]_q^3 (q^{-1}-q)^2\, z}{(1+q^{4j}z)(1+z)}\, t} \left(\frac{1+z}{1+q^{4j}z}\right)^k \frac{dz}{z}$$

where the integration contour includes 0 and $-q^{-4j}$ but does not include -1.

More on arXiv:1407.3367.

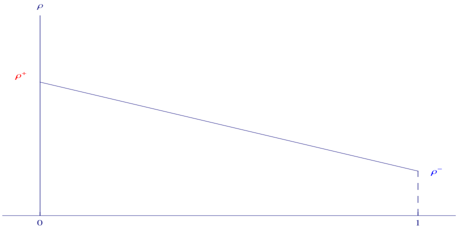
5. Random walkers with current reservoirs

Density reservoirs

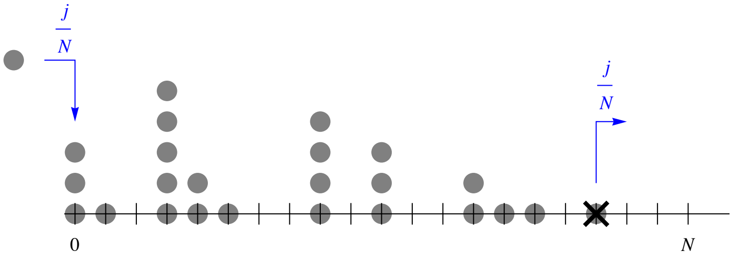


Hydrodynamic Limit
$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \dfrac{\partial^2 \rho}{\partial r^2} \\ \\ \rho(0,t) = \rho^+ \quad \rho(1,t) = \rho^- \end{cases}$$

Fick's Law



Current reservoirs



- indep. rand. walkers on $\{0, 1, \ldots, N\}$ $\rightarrow$ at rate 1
- particle created in 0 $\rightarrow$ at rate $\frac{j}{N}$
- rightmost particle deleted $\rightarrow$ at rate $\frac{j}{N}$

$\Bigg\} \longrightarrow$ current -2j

Generator

$$L = \frac{j}{N} L_a + L_0 + \frac{j}{N} L_d$$

- reflected independent walkers

$$L_0 f(\xi) = \sum_{x=0}^{N-1} \xi(x) \left[f(\xi^{x,x+1}) - f(\xi) \right] + \xi(x+1) \left[f(\xi^{x+1,x}) - f(\xi) \right]$$

$$\xi(x) = \text{number of particles at } x \qquad x \in \{0, 1, \ldots, N\}$$

- creation

$$L_a f(\xi) = f(\xi + \mathbf{1}_{\{0\}}) - f(\xi)$$

- annihilation

$$L_b f(\xi) = f(\xi - \mathbf{1}_{\{X_\xi\}}) - f(\xi)$$

$$X_\xi := \min \left\{ x \in \{0, 1, \ldots, N\} \, : \, \xi(x) > 0 \right\}$$

Remarks

- Model for Fick's law
  [**A. De Masi, E. Presutti, D. Tsagkarogiannis, M.E. Vares**]

- Topological interactions

- Microscopic model for Free Boundary Problems
  [**A. De Masi, P. Ferrari, E. Presutti**]

- Multiscale phenomena

Simulation

Hydrodynamic limit

Hydrodynamic limit: existence

$$\exists \;\; \rho_t = \rho_t(r), r \in [0,1] \quad \text{s.t.} \quad \text{``}\frac{\xi_{N^2 t}}{N} \to \rho_t\text{``} \qquad \text{as} \qquad N \to \infty$$

Hydrodynamic limit: existence

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Theorem

Let $\rho_0 \in \mathbf{L}^\infty([0,1], \mathbb{R}_+)$ and ξ_0 a discrete " approximation". Then $\exists$ $\rho_t = \rho_t(r), r \in [0,1]$, $t > 0$ continuous s.t. $\forall \zeta > 0$

$$\lim_{N\to\infty} \mathbb{P}\left[\max_{x \in \{0,\dots,N\}} \left| \frac{1}{N} F_N(x; \xi_{N^2 t}) - F(N^{-1}x; \rho_t) \right| > \zeta \right] = 0$$

where

$$F_N(x;\xi) := \sum_{y=x}^{N} \xi(y); \qquad F(r;\rho) := \int_r^1 \rho(r')dr'$$

Hydrodynamic limit: heuristic

Let ρ_t be the hydrodynamic limit and R_t its *"edge"*

$$R_t := \inf\left\{r \in [0,1] : \; F(r,\rho_t) = 0\right\} \qquad F(r,\rho) := \int_r^1 \rho(r')dr'$$

Then (R_t, ρ_t) *is the "solution" of the Free Boundary Problem*

$$\frac{\partial \rho}{\partial t} \;=\; \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2} + jD_0 - jD_{R_t}, \qquad r \in [0, R_t]$$

where D_r = Dirac delta at r

Hydrodynamic limit: strategy of the proof

Key ideas: *Barriers*, Mass transport inequalities

Let ρ_t be the hydrodynamic limit and let u_t be the FBP solution.

1. Characterization of ρ_t as the unique separating element of the *Barriers*
   - approximating processes for a discretization δ
   - limit $\delta \to 0$

2. Characterization of u_t as the unique separating element of the *Barriers*
   - FBP quasi-solutions with accuracy ϵ
   - limit $\epsilon \to 0$

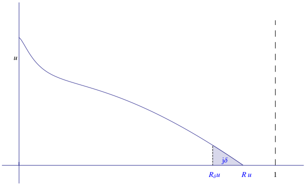
Barriers

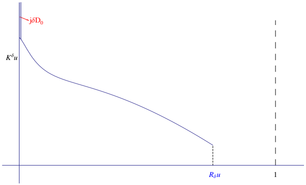
$$\mathbf{S}_{\mathbf{k}\delta}^{(\delta,+)}(\rho) := G_\delta^{\text{neum}} * K^{(\delta)} \cdots\cdots G_\delta^{\text{neum}} * K^{(\delta)}\rho \qquad (k \text{ times})$$

$$\mathbf{S}_{\mathbf{k}\delta}^{(\delta,-)}(\rho) := K^{(\delta)} G_\delta^{\text{neum}} * \cdots\cdots K^{(\delta)} G_\delta^{\text{neum}} * \rho \qquad (k \text{ times})$$

- $G_\delta^{\text{neum}}(r, r')$ = Green function of the heat equation in $[0, 1]$ with *Neumann b.c.*
-
$$K^{(\delta)}u = j\delta D_0 + u\,\mathbf{1}_{[0,R_\delta(u)]} = \textit{“the cut and paste map”}$$

$$\text{with} \qquad R_\delta(u) \qquad \text{s.t.} \qquad F(R_\delta(u), u) = \int_{R_\delta}^1 u(r)dr = j\delta$$





Mass transport inequalities

Definition (Partial order)

Let

$$F_N(x;\xi) = \sum_{y=x}^{N} \xi(y)$$

For two configurations $\xi, \xi' \in \mathbb{N}^N \times \mathbb{N}^N$ we say that

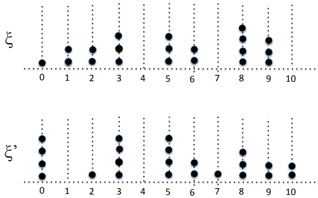
$$\xi \leq \xi'$$

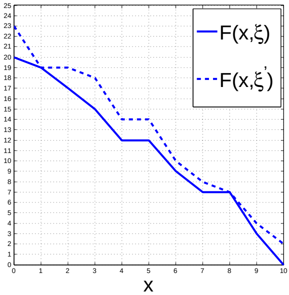
iff

$$F_N(x;\xi) \leq F_N(x;\xi') \qquad \text{for all} \quad x \in \{0, \ldots, N\}$$

If $F_N(0,\xi) = F_N(0,\xi')$ then ξ' is obtained from ξ my moving mass to the right.

Mass transport inequalities





Approximating processes

- Divide the interval $[0, N^2 T]$ into T/δ intervals of length $N^2\delta$
- Suppose in the k^{th} interval ($k = 1, \dots, T/\delta$) the process ξ_t had $B_k^{(\delta)}$ births and $D_k^{(\delta)}$ deaths.

$$\xi_t^{(\delta,-)} \longrightarrow \begin{cases} \text{evolution with } L_0 + \text{at times } kN^2\delta \text{ add } B_k^{(\delta)} \text{ particles} \\ \text{at the origin, remove the } D_k^{(\delta)} \text{ rightmost particles} \end{cases}$$

$$\xi_t^{(\delta,+)} \longrightarrow \begin{cases} \text{at times } (k-1)N^2\delta \text{ add } B_k^{(\delta)} \text{ particles at the origin,} \\ \text{remove } D_k^{(\delta)} \text{ rightmost particles + evolution with } L_0 \end{cases}$$

$$\xi_{kN^2\delta}^{(\delta,-)} \le \xi_{kN^2\delta} \le \xi_{kN^2\delta}^{(\delta,+)}$$

"stochastically": all the processes can be realized on a same space where the inequality holds pointwise almost surely.

Scheme of the proof

$$\begin{array}{ccccc}
\xi^{(\delta,-)}_{kN^2\delta} & \leq & \xi_{kN^2\delta} & \leq & \xi^{(\delta,+)}_{kN^2\delta} \\
\downarrow & & \downarrow & & \downarrow \\
S^{(\delta,-)}_{k\delta}(\rho_0) & \leq & ? & \leq & S^{(\delta,+)}_{k\delta}(\rho_0)
\end{array}$$

Hydrodynamic limit for the approximating processes

Theorem

Given any $T > 0$ for any $\delta > 0$ small enough, any $k : k\delta \le T$ and any $\zeta > 0$

$$\lim_{N\to\infty} \mathbb{P}\left[\max_{x\in\{0,\dots,N\}} \left| N^{-1} F_N(x; \xi^{(\delta,\pm)}_{kN^2\delta}) - F(N^{-1}x; S^{(\delta,\pm)}_{k\delta}(\rho_0)) \right| \le \zeta \right] = 1$$

where

$$F_N(x;\xi) := \sum_{y=x}^{N} \xi(y), \qquad F(r;\rho) := \int_r^1 \rho(r')dr'$$

and ρ_0 and ξ_0 are "close".

Barriers separating element

Definition

Let $u \in \mathbf{L}^\infty([0,1],\mathbb{R}_+)$. We say that a function u_t separates the barriers $\{S_{k\delta}^{(\delta,\pm)}(u)\}$ iff

$$S_t^{(\delta,-)}(u) \leq u_t \leq S_t^{(\delta,+)}(u) \qquad \forall \delta > 0 \text{ and } t \text{ s.t. } t = k\delta,\, k \in \mathbb{N}$$

Theorem

Let $u \in L^\infty([0,1],\mathbb{R}_+)$ and $F(0;u) > 0$. Then there exists a unique function $u(r,t)$ which separates the barriers $\{S_{k\delta}^{(\delta,\pm)}(u)\}$.

Theorem

The hydrodynamic limit ρ_t separates the barriers $\{S_{k\delta}^{(\delta,\pm)}(\rho_0)\}$.

Free Boundary Problem

Hydrodynamic limit: heuristic

Let ρ_t be the hydrodynamic limit of ξ_t and R_t its *"boundary"*

$$R_t := \inf\left\{ r \in [0,1] : \; F(r, \rho_t) = 0 \right\} \qquad F(r, \rho) := \int_r^1 \rho(r')dr'$$

Then (R_t, ρ_t) *is the "solution" of the Free Boundary Problem*

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2} + jD_0 - jD_{R_t}, \qquad r \in [0, R_t]$$

Free Boundary Problem

The pair $(X_t, u(\cdot, t))$ is a **Classical Solution** of the FBP with initial datum (X_0, u_0) in $[0, T)$ if

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial r^2} & r \in (0, X_t), \quad t \in [0, T) \\ u(X_t, t) = 0 & \qquad\qquad\quad t \in [0, T) \\ \frac{\partial u}{\partial r}(0, t) = -\frac{\partial u}{\partial r}(X_t, t) = -2j & \qquad\qquad\quad t \in [0, T) \\ u(r, 0) = u_0(r) & r \in (0, X_0), \quad X_{t=0} = X_0 \end{cases}$$

i) $X_t \in C^1([0, T), \mathbb{R}_+)$

ii) $u(\cdot, t) \in C^2((0, R_t), \mathbb{R}_+)$ and it has limits with its derivatives at 0 and X_t, $\forall t \in [0, T)$; $u(r, \cdot)$ differentiable $\forall r \in [0, X_t]$.

Connection to a Stefan problem

Define

$$v(r,t) := -\frac{1}{2}\frac{\partial u}{\partial r}(r,t) - j$$

Then

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial r^2}, \\ \\ \frac{dR_t}{dt} = -\frac{1}{2j}\frac{\partial v}{\partial r}(X_t,t) \\ \\ v(0,t) = v(X_t,0) = 0 \end{cases}$$

Local existence and uniqueness are known, then

$$u(r,t) = 2\int_r^{X_t} \Big(v(r',t) + j\Big)\,dr'$$

However global solutions are not known.

Free Boundary Problem: an equivalent formulation

The pair $(X_t, u(\cdot, t))$ is a **Classical Solution** of the FBP with initial datum (X_0, u_0) in $[0, T)$ if

$$
\begin{cases}
\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial r^2} & r \in (0, X_t), \quad t \in [0, T) \\
u(X_t, t) = 0 & t \in [0, T) \\
\frac{\partial u}{\partial r}(0, t) = -2j & t \in [0, T) \\
\int_0^{X_t} u(r,t)\, dr = \int_0^{X_0} u_0(r)\, dr & t \in [0, T) \\
u(r, 0) = u_0(r) & r \in (0, X_0), \quad X_{t=0} = X_0
\end{cases}
$$

Quasi-Solutions and Generalized Solutions

- $(X_t, u(\cdot, t), \epsilon)$ is a <u>*quasi-solution*</u> of the FBP in $[0, T)$ with <u>*accuracy*</u> ϵ if:
  - $(X_t, u(\cdot, t))$ satisfies the problem with

$$\sup_{t \le T} \left| \int_0^{X_t} u(r, t)\, dr - \int_0^{X_0} u(r, 0)\, dr \right| \le \epsilon, \qquad t \in [0, T]$$

  - $X_t > 0$ is Lipschitz and piecewise C^1; $u(r, t)$ is *"smooth"*.
- $(X_t, u(\cdot, t))$ is a <u>*generalized solution*</u> of the FBP in $[0, T)$ if it exists a sequence $(X_t^{(n)}, u^{(n)}(\cdot, t), \epsilon_n)$ of quasi-solutions s.t.

$$\lim_{n\to\infty} \epsilon_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} u^{(n)} = u \quad \text{weakly}$$

Global solution of the FBP

Theorem

For any $u_0 \in L^\infty(\mathbb{R}_+, \mathbb{R}_+) \cap L^1(\mathbb{R}_+, \mathbb{R}_+)$ *and any* $T > 0$*:*

(a) *There exists a* *<u>Generalized Solution</u>* $(X_t, u(\cdot, t))$ *of the FBP in* $[0, T)$ *with initial datum* u_0*.*

(b) *Let* $S_t(u_0)$ *be the Separating Element of the Barriers* $\{S_{k\delta}^{(\delta,\pm)}(u_0)\}$*. Then*

$$u(\cdot, t) = S_t(u_0) \qquad \textit{for all} \qquad t \in [0, T)$$

Consequence:

The Hydrodynamic limit = the FBP Generalized Solution

$$\text{“} \lim_{N \to \infty} (N^{-1} \xi_{N^2 t}, R_{\xi_{N^2 t}}) = (u(\cdot, t), X_t)\text{”}$$

Proof strategy (1)

For a given $(X_t)_{t\in[0,T]}$ the problem

$$\begin{cases} \frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial^2 w}{\partial r^2} & r \in (0, X_t), \quad t \in [0,T) \\ w(X_t, t) = 0 & t \in [0,T) \\ \frac{\partial w}{\partial r}(0,t) = -2j & t \in [0,T) \\ w(r,0) = w_0(r) & r \in (0, X_0), \quad X_{t=0} = X_0 \end{cases}$$

is solved by

$$w(r,t) := \int G^{X,\,\mathrm{neum}}_{0,t}(r', r) w_0(r')\, dr' + \int_0^t j G^{X,\,\mathrm{neum}}_{s,t}(0,r)\, ds$$

$G^{X,\,\mathrm{neum}}_{s,t}(r,\cdot) =$ probability density of Brownian motion B_t starting from r at time s, reflected at 0 and restricted to trajectories so that $B_{s'} < X_{s'}$, $\forall s' \in [s,t]$

$$\int_I G^{X,\,\mathrm{neum}}_{s,t}(r', r) dr = P_{r';s}[\tau_s^X > t\, ;\, B_t \in I], \qquad \tau_s^X = \inf\{t \geq s : B_t \geq X_t\}, \quad I \subset \mathbb{R}_+$$

Proof strategy (2)

Definition (Partial order modulo m)

For $m > 0$

$$u \leq v \quad \textit{modulo } m \quad \textit{iff} \quad \forall\, r \geq 0\text{: } F(r;u) \leq F(r;v) + m$$

We prove that:

- If $(X_t, u^{(\epsilon)}(\cdot,t), \epsilon)$ is a FBP quasi-solution with accuracy ϵ then for any $\delta > 0$, there is c so that $\forall\, k \in \mathbb{N}$ s.t. $k\delta \leq T$

$$S_{k\delta}^{(\delta,-)}(u^{(\epsilon)}(\cdot,0)) \leq u^{(\epsilon)}(\cdot,k\delta) \leq S_{k\delta}^{(\delta,+)}(u^{(\epsilon)}(\cdot,0)) \quad \text{modulo} \;\; ck\epsilon$$

- The Generalized Solution $u = \lim_{\epsilon\to 0} u^{(\epsilon)}$ of the FBP is the unique separating element between barriers

Super-hydrodynamic limit

Stationary solutions

Linear Profiles with slope $-2j$

$$\rho^{(M)}(r) := a_M - 2jr, \qquad 0 \le r \le R^{(M)} := \min\left\{\frac{a_M}{2j}, 1\right\}$$

$M :=$ *Total Mass* $\qquad \int_0^1 \rho^{(M)}(r)\,dr = M$

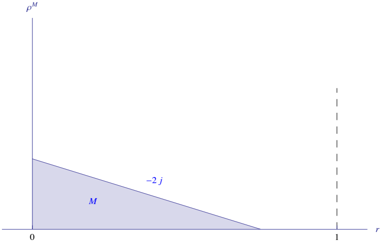


Figure : Stationary solution for $M < j$

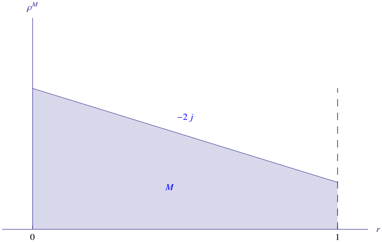


Figure : Stationary solution for $M > j$

Mass fluctuations

$|\xi_t| =$ Total number of particles at time t

$$\left.\begin{array}{l} \bullet \quad \text{particle added:} \quad |\xi| \to |\xi| + 1 \\[2em] \bullet \quad \text{particle deleted:} \quad |\xi| \to |\xi| - 1 \end{array}\right\} \longrightarrow \begin{array}{l} |\xi_t| \textit{ performs a} \\ \text{symmetric random walk} \\ \textit{with jumps } \pm 1 \textit{ at rate } \frac{j}{N} \end{array}$$

The density $\dfrac{|\xi_t|}{N}$ changes after times of the order N^3:

$$M_t^{(N)} := \frac{|\xi_{N^3 t}|}{N} \longrightarrow B_{jt} \quad \text{as} \quad N \to \infty$$

where $(B_t)_{t \geq 0}$ = Brownian Motion on $\mathbb{R}^+$ with reflecting b.c. at 0.

Super-hydrodynamic limit

Theorem

Let $\xi^{(N)}$ be a sequence such that $|\xi^{(N)}|N^{-1} \to m > 0$ as $N \to \infty$. Let t_N be an increasing divergent sequence. Then

- *If $N^{-1} t_N \to 0$, then*

$$\lim_{N\to\infty} \mathbb{P}\left[\max_{x\in\{0,\dots,N\}} \left| \frac{1}{N} F_N(x; \xi_{N^2 t_N}) - F(N^{-1}x; \rho^{(m)}) \right| \le \zeta \right] = 1$$

- *Let $t_N = Nt$ then*

$$\lim_{N\to\infty} \mathbb{P}\left[\max_{x\in\{0,\dots,N\}} \left| \frac{1}{N} F_N(x; \xi_{N^3 t}) - F(N^{-1}x; \rho^{(M_t^{(N)})}) \right| \le \zeta \right] = 1$$

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