Duality Theory of Interacting Particle Systems

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Work in progress:
monograph (Carinci, Giardinà, Redig) with tentative title
"Duality and Hidden Symmetries of Interacting Particle Systems"

Outline

1. Introduction to duality via simple examples from mathematical population genetics.

2. Key ideas of a (Lie) algebraic approach to duality.

3. Self-duality of interacting particles (independent, exclusion, inclusion). Application to boundary-driven systems.

4. Duality between interacting diffusions and particles. Application to Fourier's law.

5. More on asymmetric systems and multi-type population models.

1. Introduction

1.1 – What is duality?

Markov processes

An Ω-valued process $(\eta_t)_{t\geq 0}$ is Markov if

$$\mathbb{E}[f(\eta_{t+s}) \,|\, \mathscr{F}_t] = \mathbb{E}[f(\eta_{t+s}) \,|\, \eta_t], \qquad f \in B(\Omega)$$

Markov semigroup $(\mathcal{S}_t)_{t\geq 0}$ gives the expectation

$$\mathcal{S}_s f(\eta_t) := \mathbb{E}[f(\eta_{t+s}) \,|\, \mathscr{F}_t]$$

Markov generator $\mathscr{L}$ gives the infinitesimal change of distribution

$$\mathbb{E}[f(\eta_{t+\Delta t}) \,|\, \mathscr{F}_t] = f(\eta_t) + \mathscr{L} f(\eta_t)\Delta t + o(\Delta t)$$

Duality of Markov processes

Definition

$(\eta_t)_{t\geq 0}$ Markov process on Ω

$(\xi_t)_{t\geq 0}$ Markov process on Ω_{dual}

ξ_t is dual to η_t with duality function $D : \Omega \times \Omega_{dual} \to \mathbb{R}$ if $\forall t \geq 0$

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \qquad \forall(\eta, \xi) \in \Omega \times \Omega_{dual}$$

Furthermore, η_t is self-dual if ξ_t is a copy of η_t.

$(\mathcal{S}_t)_{t\geq 0}$ and $(\mathcal{S}_t^{dual})_{t\geq 0}$ Markov semigroups

$$(\mathcal{S}_t D(\cdot, \xi))(\eta) = (\mathcal{S}_t^{dual} D(\eta, \cdot))(\xi)$$

Duality of generators

Definition

$(\eta_t)_{t\geq 0}$ Markov process on Ω with generator $\mathscr{L}$,

$(\xi_t)_{t\geq 0}$ Markov process on Ω_{dual} with generator $\mathscr{L}_{dual}$

$\mathscr{L}$ is dual to $\mathscr{L}_{dual}$ with duality function $D : \Omega \times \Omega_{dual} \to \mathbb{R}$ if

$$(\mathscr{L} D(\cdot,\xi))(\eta) = (\mathscr{L}_{dual} D(\eta,\cdot))(\xi) \qquad \forall(\eta,\xi) \in \Omega \times \Omega_{dual}$$

Furthermore, $\mathscr{L}$ is self-dual if $\mathscr{L}_{dual} = \mathscr{L}$.

Equivalence of definitions

Under appropriate conditions, the two notions of duality are equivalent.

E.g.: in the context of Markov chains with finite state space

$$\begin{aligned}
\mathbb{E}_\eta(D(\eta_t, \xi)) &= e^{t\mathscr{L}} D(\cdot, \xi)(\eta) \\
&= e^{t\mathscr{L}_{dual}} D(\eta, \cdot)(\xi) \\
&= \mathbb{E}_\xi(D(\eta, \xi_t))
\end{aligned}$$

Equivalence of definitions

Assume that

$$D(\cdot, y) \in \mathcal{D}(\mathscr{L}) \quad \text{and} \quad \mathcal{S}_t D(\cdot, y) \in \mathcal{D}(\mathscr{L}) \quad \forall\, y \in \Omega_{dual}$$

$$D(x, \cdot) \in \mathcal{D}(\mathscr{L}_{dual}) \quad \text{and} \quad \mathcal{S}_t^{dual} D(x, \cdot) \in \mathcal{D}(\mathscr{L}_{dual}) \quad \forall\, x \in \Omega\,,$$

with $\mathcal{D}(\mathscr{L})$ and $\mathcal{D}(\mathscr{L}_{dual})$ the generators domain.

Then Markov process duality is equivalent to generators duality.

In the following, we always assume equivalence of definitions.

Duality for continuos-time Markov chains

If state spaces Ω, Ω_{dual} are countable sets, then
the Markov generator $\mathscr{L}$ acts as multiplication by a matrix $L(\eta, \eta')$ s.t.

$$L(\eta, \eta') \geq 0 \quad \text{if } \eta \neq \eta', \qquad \sum_{\eta' \in \Omega} L(\eta, \eta') = 0$$

$$\mathscr{L} D(\cdot, \xi)(\eta) = \mathscr{L}_{dual} D(\eta, \cdot)(\xi)$$

amounts to

$$LD = DL_{dual}^{T}$$

Indeed

$$\sum_{\eta' \in \Omega} L(\eta, \eta') D(\eta', \xi) = \sum_{\xi' \in \Omega_{dual}} L_{dual}(\xi, \xi') D(\eta, \xi')$$

Composing “building blocks” duality

Suppose we have two ”building blocks” dualities
(with the same duality function)

$$A^{(1)}D = D(A^{(1)}_{dual})^T \qquad A^{(2)}D = D(A^{(2)}_{dual})^T$$

Then for the composition we get

$$A^{(1)}A^{(2)}D = D(A^{(2)}_{dual}A^{(1)}_{dual})^T$$

Indeed

$$A^{(1)}A^{(2)}D = A^{(1)}D(A^{(2)}_{dual})^T = D(A^{(1)}_{dual})^T(A^{(2)}_{dual})^T = D(A^{(2)}_{dual}A^{(1)}_{dual})^T$$

Why duality?

- A useful tool for:
  - – interacting particle systems [Spitzer, Ligget]
  - – hydrodynamic limit [Presutti, De Masi]
  - – integrable probability [Schütz, Spohn, Corwin, Sasamoto]
  - – population genetics [Mhole]
  - ...
- Simplifications due to duality:
  - – "from many to few"
  - – "from continuous to discrete"
  - – "from forward to backward"
  - – "from fluctuating to absorbing boundaries"
  - ...
- Questions
  - how to find a dual process and a duality function?
  - how to construct processes with a duality property?

1.2 – Simple examples

Moran model with two types

Consider a population of N individuals, each of which can be of two types (say 1 and 2). A pair of individuals are sampled uniformly at random, one dies with probability $1/2$, the other reproduces.

Define

$$K^{(N)}(t) = \text{ number of individuals of type 1 at time } t$$

Then $(K^{(N)}(t))_{t\geq 0}$ is a continuos time Markov chain with state space $\Omega_N = \{0, 1, \ldots N\}$ and generator

$$\mathscr{L}_N^{Moran} f(k) = \frac{1}{2} k(N-k)(f(k+1) + f(k-1) - 2f(k))$$

Wright-Fisher diffusion with two types

Infinite-population limit: consider the process $\left(X^{(N)}(t) = \frac{K^{(N)}(t)}{N}\right)_{t\geq 0}$ with state space $\Omega'_N = \{0, 1/N, \ldots 1\}$. Its generator reads

$$\mathscr{L}_N f(\frac{k}{N}) = N^2 \frac{1}{2}\frac{k}{N}(1-\frac{k}{N})\left(f(\frac{k}{N}+\frac{1}{N}) + f(\frac{k}{N}-\frac{1}{N}) - 2f(\frac{k}{N})\right)$$

In the limit $N \to \infty$ the process $(X^{(N)}(t))_{t\geq 0}$ converges to the Wright-Fisher diffusion $(X(t))_{t\geq 0}$ with state space $[0, 1]$ and generator

$$\mathscr{L}^{WF} f(x) = \frac{1}{2}x(1-x)\frac{\partial^2 f}{\partial x^2}(x)$$

Counting blocks of the Kingman coalescence

For each $k \in \mathbb{N}$, the k-coalescence is a continuous time Markov chain on the space of equivalence relations on $\{1, 2, \ldots, k\}$ with transition rates

$$c(x, y) = \begin{cases} 1 & \text{if } y \text{ is obtained by coalescing} \\ & \text{two equivalence classes of } x, \\ 0 & \text{otherwise.} \end{cases}$$

By extension the Kingman coalescent on $\mathbb{N}$ is defined by requiring that for each k its restriction to $\{1, \ldots, k\}$ is a k-coalescence.

Define

$$N(t) = \text{ number of blocks in the } k\text{-coalescence at time } t \geq 0.$$

It is a death process on $\{1, \ldots, k\}$ defined by the Markov generator

$$\mathscr{L}^{King} f(n) = \frac{n(n-1)}{2}(f(n-1) - f(n))$$

Theorem

The process $(X(t))_{t\geq 0}$ with generator $\mathscr{L}^{WF}$ and the process $(N(t))_{t\geq 0}$ with generator $\mathscr{L}^{King}$ are dual on $D(x,n) = x^n$, i.e.

$$\mathbb{E}_x(X(t)^n) = \mathbb{E}_n(x^{N(t)})$$

Indeed:

$$\begin{aligned}
\mathscr{L}^{WF} D(\cdot,n)(x) &= \frac{1}{2}x(1-x)\frac{\partial^2}{\partial x^2}x^n \\
&= \frac{n(n-1)}{2}(x^{n-1} - x^n) \\
&= \frac{n(n-1)}{2}(D(x,n-1) - D(x,n)) \\
&= \mathscr{L}^{King} D(x,\cdot)(n)
\end{aligned}$$

Extinction probability in the WF diffusion

Probability that type (say) 1 gets extinct starting from a proportion x, i.e.

$$p_{ext}(x) = \mathbb{P}(X(\infty) = 0 | X(0) = x)$$

is related to the asymptotics of first moment:

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}_x^{WF}(X(t)) &= 1\mathbb{P}(X(\infty) = 1|X(0) = x) + 0\mathbb{P}(X(\infty) = 0|X(0) = x)\\ &= 1 - p_{ext}(x)\end{aligned}$$

To compute this quantity we can use duality:

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}_x^{WF}(X(t)) &= \lim_{t\to\infty} \mathbb{E}_x^{WF}(D(X(t),1))\\ &= \lim_{t\to\infty} \mathbb{E}_1^{King}(D(x,N(t)))\\ &= x\end{aligned}$$

Therefore

$$p_{ext}(x) = 1 - x$$

Heterozygosity in the WF diffusion

Probability that two randomly chosen individuals are of different types. To compute this quantity we can use duality:

$$
\begin{aligned}
\mathbb{E}_x^{WF}(X(t)(1-X(t))) &= \mathbb{E}_x^{WF}(D(X(t),1) - D(X(t),2)) \\
&= \mathbb{E}_1^{King}(D(x,N(t))) - \mathbb{E}_2^{King}(D(x,N(t))) \\
&= x - x^2\mathbb{P}(N(t)=2 \mid N(0)=2) \\
&\quad -x\,\mathbb{P}(N(t)=1 \mid N(0)=2) \\
&= x - x^2 e^{-t} - x(1-e^{-t}) \\
&= x(1-x)e^{-t}
\end{aligned}
$$

In particular

$$\lim_{t\to\infty} \mathbb{E}_x^{WF}(X(t)(1-X(t))) = 0$$

Duality Wright-Fisher / Kingman : algebraic approach

In the Lie algebraic approach the duality is a consequence of a change of representation of the Heisenberg algebra $[\mathtt{A}, \mathtt{A}^+] = 1$

$$\begin{cases} af(x) = \frac{\partial f}{\partial x}(x) \\ \\ a^+ f(x) = xf(x) \end{cases} \qquad \begin{cases} \mathscr{A} f(n) = (n+1)f(n+1) \\ \\ \mathscr{A}^+ f(n) = f(n-1) \end{cases}$$

$f : \mathbb{R} \to \mathbb{R}$, smooth $\qquad f : \mathbb{N} \to \mathbb{R}$

The abstract element

$$\mathtt{L} = \frac{1}{2}\mathtt{A}^+(1 - \mathtt{A}^+)\mathtt{A}^2$$

gives rise to the two processes

$\mathtt{L} = \mathscr{L}^{WF}$ in the representation with differential operators

$\mathtt{L} = (\mathscr{L}^{King})^*$ in the representation with matrices

$D(x, n) = x^n$ is the intertwiner between (a, a^+) and $(\mathscr{A}^*, (\mathscr{A}^+)^*)$

Finite population size and finite dimensional representation

Introducing well-chosen discrete derivative and discrete multiplication operators, we can also find the duality between the discrete Moran model and the Kingman's coalescent. For functions $f : \{0, \ldots, N\} \to \mathbb{R}$ define

$$\begin{aligned} \mathrm{a}_N f(k) &= (N-k)\, f(k+1) + (2k-N)\, f(k) - k f(k-1) \\ \mathrm{a}_N^+ f(k) &= \sum_{r=0}^{k-1} (-1)^{k-1-r} \frac{\binom{N}{r}}{\binom{N}{k}} f(r)\,, \end{aligned}$$

with the convention $f(-1) = f(N+1) = 0$.

Duality between Moran and Kingman

Consider

$$D_N(k,n) = \frac{\binom{k}{n}}{\binom{N}{n}} = \frac{k(k-1)\cdots(k-(n-1))}{N(N-1)\cdots(N-(n-1))}$$

with the convention $D_N(k,0)=1$, $D_N(k,N+1)=0$.

Let us denote by $\mathcal{W}_N$ the vector space generated by the functions $k \mapsto D_N(k,n), 0 \le n \le N$. Then

$$\begin{aligned}
\mathrm{a}_N D_N(\cdot,n)(k) &= nD_N(k,n-1), && \forall\, 1 \le n, \forall\, k \ge n-1\,,\\
\mathrm{a}_N D_N(\cdot,0)(k) &= 0 && \forall\, 0 \le k \le N\,,\\
\mathrm{a}_N^{+} D_N(\cdot,n)(k) &= D_N(k,n+1) && \forall\;\, 0 \le n \le N, k \ge n\,.
\end{aligned}$$

Duality between Moran and Kingman

As a consequence, as operators on $\mathcal{W}_N$, we have

$$[\mathrm{a}_N, \mathrm{a}_N^+] = 1 \ ,$$

and

$$\mathrm{a}_N \to^{D_N} \mathscr{A}^*, \qquad \mathrm{a}_N^+ \to^{D_N} (\mathscr{A}^+)^*$$

Moreover the generator of the Moran model in terms of $\mathrm{a}_N^-, \mathrm{a}_N^+$ reads

$$\mathscr{L}_N^{Moran} = \mathrm{a}_N^+(1 - \mathrm{a}_N^+)(\mathrm{a}_N)^2$$

i.e., the same as the Wright Fisher generator, but now in the $\mathrm{a}_N, \mathrm{a}_N^+$ representation. This explains that we find the same dual generator when going to the infinite-dimensional discrete representation, but now with another duality function.

Wright Fisher diffusion with mutation.

Other "evolutionary forces" can be included. Consider the Moran model where in between reproduction events each individual of type 2 mutates to an individual of type 1 at rate θ. Then in the diffusive limit one has

$$\begin{aligned}\mathscr{L}^{WF,mut} &= x(1-x)\frac{d^2}{dx^2} + \theta(1-x)\frac{d}{dx} \\ &= a^+(1-a^+)a^2 + \theta(1-a^+)a\end{aligned}$$

By changing to a discrete representation of the Heisenberg algebra this gives the dual

$$\mathscr{L}^{King,mut} f(n) \quad = \quad n(n-1)(f(n-1)-f(n)) + \theta n(f(n-1)-f(n))$$

which corresponds to Kingman's coalescent with extra rate θn to go down from n to $n-1$, due to mutation.

2. Lie algebraic approach to duality theory

Lie algebra

A Lie algebra is a vector space $\mathfrak{g}$ over a field F with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (Lie bracket)

- $\forall$ a, b in F and $\forall$ u, v, w in $\mathfrak{g}$

  $$[au + bv, w] = a[u, w] + b[v, w], \quad [w, au + bv] = a[w, u] + b[w, v]$$

- $\forall$ u in $\mathfrak{g}$: $[u, u] = 0$
- [Jacobi identity]: $\forall$ u, v, w in $\mathfrak{g}$

  $$[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$$

Elements of a Lie algebra $\mathfrak{g}$ are said to be generators if the smallest subalgebra of $\mathfrak{g}$ containing them is $\mathfrak{g}$ itself.

2.1 – Two basic principles

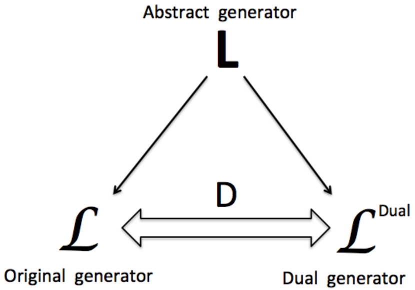
Lie algebraic approach to duality

$\star$ Key idea: Markov generator in abstract form, i.e. as an element of (a universal enveloping algebra of) a Lie algebra.

I. Duality is related to a change of representation.
Duality functions are the intertwiners.

II. Dualities are associated to symmetries.
Acting with a symmetry on a duality fct. yields another duality fct.

I. Change of representation



Example of Wright-Fisher / Kingman duality

Two representations of the Heisenberg algebra: $[\mathtt{A}, \mathtt{A}^+] = \mathtt{1}$

$$\begin{cases} a = \frac{\partial}{\partial x} \\ a^+ = x \end{cases} \qquad \begin{cases} A\, e_n = n\, e_{n-1} \\ A^+ e_n = e_{n+1} \end{cases}$$

Abstract element $\mathtt{L} = \dfrac{1}{2}\mathtt{A}^+(\mathtt{1} - \mathtt{A}^+)(\mathtt{A})^2$

$\mathtt{L} = \mathscr{L}^{WF}$ in the first representation

$\mathtt{L} = (\mathscr{L}^{King})^*$ in the second representation

Duality fct. $D(x,n) = x^n$ is the intertwiner:

$$xD(x,n) = D(x,n+1) \qquad \frac{\partial}{\partial x} D(x,n) = nD(x,n-1)$$

II. Symmetries

Continuos time Markov chains $(\eta_t)_{t\geq 0}, (\xi_t)_{t\geq 0}$ taking values in Ω and Ω_{dual} countable, with generators $\mathscr{L}$ and $\mathscr{L}_{dual}$.

Symmetry $\mathscr{S}$ of the original Markov generator, i.e. $[L, S] = 0$

Duality function $d : \Omega \times \Omega_{dual} \to \mathbb{R}$, i.e. $Ld = dL_{dual}^T$

$\longrightarrow \quad D = Sd \quad$ is also duality function

Indeed

$$LD = LSd = SLd = SdL_{dual}^T = DL_{dual}^T$$

“Cheap” self-duality

Let $\mu : \Omega \to [0,1]$ a reversible measure: $\mu(\eta)L(\eta,\xi) = \mu(\xi)L(\xi,\eta)$

A cheap (i.e. diagonal) self-duality is

$$d(\eta,\xi) = \frac{1}{\mu(\eta)}\delta_{\eta,\xi}$$

Indeed

$$\frac{L(\eta,\xi)}{\mu(\xi)} = \sum_{\eta'} L(\eta,\eta')d(\eta',\xi) = \sum_{\xi'} L(\xi,\xi')d(\eta,\xi') = \frac{L(\xi,\eta)}{\mu(\eta)}$$

“Cheap” duality

Let $\mu : \Omega \to [0,1]$ an invariant measure: $\sum_\eta \mu(\eta)L(\eta,\xi) = 0$

Let the dual process $(\xi_t)_{t\geq 0}$ be the time-reversed process of $(\eta_t)_{t\geq 0}$

$$L_{dual}(\xi,\xi') = \mu(\xi)^{-1}L(\xi',\xi)\mu(\xi')$$

A cheap (i.e. diagonal) duality is

$$d(\eta,\xi) = \frac{1}{\mu(\eta)}\delta_{\eta,\xi}$$

Indeed

$$\frac{L(\eta,\xi)}{\mu(\xi)} = \sum_{\eta'} L(\eta,\eta')d(\eta',\xi) = \sum_{\xi'} L_{dual}(\xi,\xi')d(\eta,\xi') = \frac{L_{dual}(\xi,\eta)}{\mu(\eta)} = \frac{L(\eta,\xi)}{\mu(\xi)}$$

2.2 – Constructive approach

Lie algebraic approach to duality

- ⋆ Key idea: Markov generator in abstract form, i.e. as an element of a universal enveloping algebra of a Lie algebra.

I. Duality is related to a change of representation.
Duality functions are the intertwiners.

II. Dualities are associated to symmetries.
Acting with a symmetry on a duality fct. yields another duality fct.

Conversely, the approach can be turned into a constructive method.

Construction of Markov generators with algebraic structure

i) (*Lie Algebra*): Start from a Lie algebra $\mathfrak{g}$.

ii) (*Casimir*): Pick an element in the center of $\mathfrak{g}$, e.g. the Casimir C.

iii) (*Co-product*): Consider a co-product $\Delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ making the algebra a bialgebra and conserving the commutation relations.

iv) (*Quantum Hamiltonian*): Compute $H = \Delta(C)$.

v) (*Symmetries*): $S = \Delta(X)$ with $X \in \mathfrak{g}$ is a symmetry of H:

$$[H, S] = [\Delta(C), \Delta(X)] = \Delta([C, X]) = \Delta(0) = 0.$$

vi) (*Markov generator*): Apply a "ground state transformation" to turn H into a Markov generator L.

3. Interacting particle systems

3.1 – Independent random walkers

3.2 – Inclusion process

3.3 – Generalized exclusion process

3.4 – Self-dualities

3.5 – Stationary state of boundary driven IRW

Interacting Particle Systems

For a graph $G = (V, E)$, we consider Markov processes defined on a state space $\Omega = \otimes_{i \in V} \Omega_i$, where Ω_i is a countable state space

Process $\{\xi(t) \ : \ t \geq 0\}$

Configuration $\quad \xi = (\xi_1, \ldots, \xi_{|V|}) \in \Omega$

Rates $\quad L(\xi, \xi') > 0 \qquad L(\xi, \xi) = -\sum_{\xi' \neq \xi} L(\xi, \xi')$

Generator
$$\mathscr{L} f(\xi) = \sum_{\xi' \neq \xi} L(\xi, \xi') \Big(f(\xi') - f(\xi) \Big)$$
$$= \sum_{\xi} L(\xi, \xi') f(\xi')$$

Algebraic description of interacting particle systems

Define the column vector

$$\mu_t(\xi) := \mathbb{P}(\xi(t) = \xi)$$

The master equation reads

$$\begin{aligned}\frac{d}{dt}\mu_t(\xi) &= \sum_{\xi' \neq \xi} L(\xi', \xi)\mu_t(\xi') - \sum_{\xi' \neq \xi} L(\xi, \xi')\mu_t(\xi) \\ &= \sum_{\xi'} L(\xi', \xi)\mu_t(\xi') \\ &= (\mathscr{L}^* \mu_t)(\xi)\end{aligned}$$

where $\mathscr{L}^*$ denote the adjoint of the Markov generator.

The main idea is to rewrite the (adjoint of the) Markov generator using the generators of a Lie algebra.

3.1 – Independent symmetric random walkers

Independent symmetric random walkers

Configuration $\xi = (\xi_1, \ldots, \xi_{|V|}) \in \{0, 1, 2, \ldots\}^{|V|}$

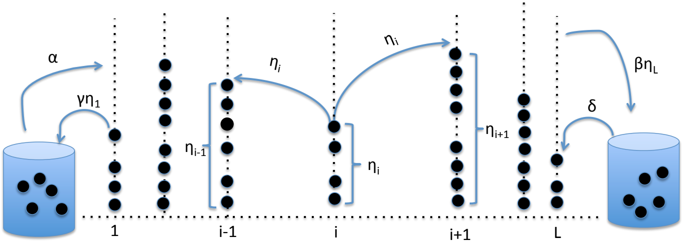
$$\mathscr{L}^{IND} f(\xi) = \sum_{(i,j) \in E} \xi_i [f(\xi^{i,j}) - f(\xi)] + \xi_j [f(\xi^{j,i}) - f(\xi)]$$

$$\xi^{i,j} = (\xi_1, \ldots, \xi_i - 1, \ldots, \xi_j + 1, \ldots, \xi_{|V|})$$

Exercise: Show that stationary reversible measures are given by product measures with marginals Poisson(λ), i.e.

$$\mu_{rev}(\xi) = \prod_{i=1}^{|V|} \frac{\lambda^{\xi_i}}{\xi_i!} e^{-\lambda}$$

Independent Walkers on a 1d chain with reservoirs



Exercise: Show that a birth-death Markov chain $(N_t)_{t \geq 0}$ with birth rates $b(n) = \alpha$ and death rate $d(n) = \gamma n$ has stationary distribution Poisson$(\frac{\alpha}{\gamma})$.

Can you tell if IRW on 1d chain with reservoirs have reversible measure? What about a stationary measure?

Heisenberg algebra

The Lie bracket is given by the commutator, i.e. for $\mathtt{x}, \mathtt{y}$ in the algebra

$$[\mathtt{x}, \mathtt{y}] = \mathtt{xy} - \mathtt{yx}$$

The algebra is generated by the elements $\mathtt{A}, \mathtt{A}^+$ that satisfy

$$[\mathtt{A}, \mathtt{A}^+] = \mathtt{1}$$

A representation in terms of matrices is given by

$$\begin{cases} A|n\rangle = n|n-1\rangle \\ \\ A^+|n\rangle = |n+1\rangle \end{cases}$$

where, for $n \in \{0, 1, 2, \ldots\}$, $|n\rangle = e_n$ denote the orthonormal column vectors

$$(e_n)_i = \begin{cases} 1 & \text{if } i = n, \\ 0 & \text{if } i \neq n \end{cases} \qquad e_n^T \cdot e_m = \langle n|m\rangle = \delta_{n,m}$$

Heisenberg algebra on a graph $G = (V, E)$

On a graph with $|V|$ vertices we consider $|V|$ copies of the Heisenberg algebra and work with their tensor product.

$$|\xi\rangle = \otimes_{i=1}^{|V|} |\xi_i\rangle$$

Then

$$\begin{cases} A_i|\xi\rangle = \left(\otimes_{j\neq i} 1|\xi_j\rangle\right) \otimes (A_i|\xi_i\rangle) = \left(\otimes_{j\neq i}|\xi_j\rangle\right) \otimes \xi_i|\xi_i - 1\rangle \\ A_i^+|\xi\rangle = \left(\otimes_{j\neq i} 1|\xi_j\rangle\right) \otimes \left(A_i^+|\xi_i\rangle\right) = \left(\otimes_{j\neq i}|\xi_j\rangle\right) \otimes |\xi_i + 1\rangle \end{cases}$$

The algebra generators (A_i, A_i^+), with $i = 1, \ldots, |V|$, satisfy

$$[A_i, A_j^+] = \delta_{i,j} 1$$

Algebraic description of independent walkers

$$L^*_{IND} = -\sum_{(i,j)\in E} \left(A_i^+ - A_j^+\right)\left(A_i - A_j\right)$$

Expanding the product

$$\begin{aligned} L^*_{IND}|\xi\rangle &= \sum_{(i,j)\in E} \left(A_i A_j^+ + A_i^+ A_j - A_i^+ A_i - A_j^+ A_j\right)|\xi\rangle \\ &= \sum_{(i,j)\in E} \left(\xi_i|\xi^{i,j}\rangle + \xi_j|\xi^{j,i}\rangle - (\xi_i + \xi_j)|\xi\rangle\right) \end{aligned}$$

Rates:

$$L_{IND}(\xi,\xi') = \langle \xi | L_{IND} | \xi' \rangle = \langle \xi' | L^*_{IND} | \xi \rangle = \begin{cases} \xi_i & \text{if } \xi' = \xi^{i,j}, \\ \xi_j & \text{if } \xi' = \xi^{j,i}, \\ -(\xi_i + \xi_j) & \text{if } \xi' = \xi \end{cases}$$

3.2 – Symmetric Inclusion Process

Symmetric Inclusion Process, SIP(m)

Let $m \in \mathbb{R}_+$

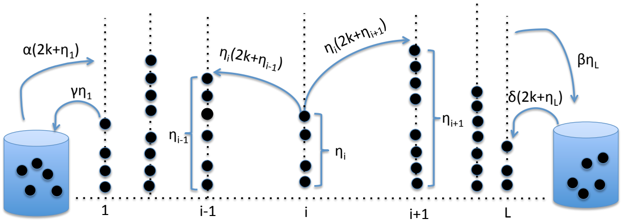
Configuration $\xi = (\xi_1, \ldots, \xi_{|V|}) \in \{0, 1, 2, \ldots\}^{|V|}$

$$\mathscr{L}_{SIP(m)} f(\xi) = \sum_{(i,j) \in E} \xi_i \left(\xi_j + \frac{m}{2}\right) [f(\xi^{i,j}) - f(\xi)] + \xi_j \left(\xi_i + \frac{m}{2}\right) [f(\xi^{j,i}) - f(\xi)]$$

Exercise: Show that stationary reversible measures are given by product measures with marginals Negative Binomial($\frac{m}{2}, p$), i.e.

$$\mu_{rev}(\xi) = \prod_{i=1}^{|V|} \frac{p^{\xi_i}}{\xi_i!} \frac{\Gamma(\frac{m}{2} + \xi_i)}{\Gamma(\frac{m}{2})} (1-p)^{\frac{m}{2}}$$

SIP(m) on a 1d chain with reservoirs



Exercise: Show that a birth-death Markov chain $(N_t)_{t\geq 0}$ with birth rate $b(n) = \alpha(\frac{m}{2} + n)$ and death rate $d(n) = \gamma n$ has stationary distribution given by a Negative Binomial$(\frac{m}{2}, \frac{\alpha}{\gamma})$.

What about the invariant measure of SIP(m) on a 1d chain with reservoirs?

Algebraic description of SIP(m)

Heisenberg spin chain

$$L^*_{SIP(m)} = \sum_{(i,j)\in E} \left(K_i^+ K_j^- + K_i^- K_j^+ - 2K_i^o K_j^o + \frac{m^2}{8} \right)$$

$\{K_i^+, K_i^-, K_i^o\}$ are the generators of the $\mathfrak{su}(1,1)$ algebra

$$[K_i^o, K_j^\pm] = \pm\delta_{i,j} K_i^\pm \qquad [K_i^-, K_j^+] = +2\delta_{i,j} K_i^o$$

$$\begin{cases} K_i^+|\xi_i\rangle = \left(\xi_i + \frac{m}{2}\right)|\xi_i + 1\rangle \\ K_i^-|\xi_i\rangle = \xi_i|\xi_i - 1\rangle \\ K_i^o|\xi_i\rangle = \left(\xi_i + \frac{m}{4}\right)|\xi_i\rangle \end{cases}$$

3.3 – Generalized Symmetric Exclusion Process

Generalized Symmetric Exclusion Process, SEP(n)

Let $n \in \mathbb{N}$

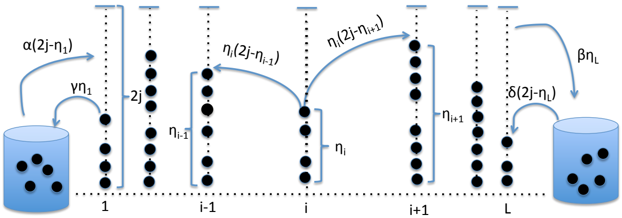
Configuration $\xi = (\xi_1, \ldots, \xi_{|V|}) \in \{0, 1, 2, \ldots, n\}^{|V|}$

$$\mathscr{L}_{SEP(n)} f(\xi) = \sum_{(i,j) \in E} \xi_i (n - \xi_j)[f(\xi^{i,j}) - f(\xi)] + (n - \xi_i)\xi_j [f(\xi^{j,i}) - f(\xi)]$$

Exercise: Show that stationary reversible measures are given by product measures with marginals Binomial(n, p), i.e.

$$\mu_{rev}(\xi) = \prod_{i=1}^{|V|} \binom{n}{\xi_i} p^{\xi_i} (1 - p)^{n - \xi_i}$$

SEP(n) on a 1d chain with reservoirs



Exercise: Show that a birth-death Markov chain $(M_t)_{t\geq 0}$ with birth rate $b(m) = \alpha(n-m)$ and death rate $d(m) = \gamma m$ has stationary distribution given by a Binomial$(n, \frac{\alpha}{\alpha+\gamma})$

What about the invariant measure of SIP(m) on a 1d chain with reservoirs?

Algebraic description of SEP(n)

Heisenberg spin chain

$$L^*_{SEP(n)} = \sum_{(i,j)\in E} \left(J_i^+ J_j^- + J_i^- J_j^+ + 2J_i^o J_j^o - \frac{n^2}{2} \right)$$

$\{J_i^+, J_i^-, J_i^o\}$ are the generators of the $\mathfrak{su}(2)$ algebra

$$[J_i^o, J_j^\pm] = \pm\delta_{i,j} J_i^\pm \qquad\qquad [J_i^-, J_j^+] = -2\delta_{i,j} J_i^o$$

$$\begin{cases} J_i^+ |\xi_i\rangle = (n-\xi_i)\,|\xi_i+1\rangle \\[2ex] J_i^- |\xi_i\rangle = \xi_i|\xi_i-1\rangle \\[2ex] J_i^o|\xi_i\rangle = \left(\xi_i - \frac{n}{2}\right)|\xi_i\rangle \end{cases}$$

3.4 – Self-dualities

Self-duality of Symmetric Inclusion Process SIP(m)

Theorem

Let $m \in \mathbb{R}_+$. On a graph $G = (V, E)$, the SIP(m) process with state space $\mathbb{N}^{|V|}$ and generator

$$\mathscr{L}_{SIP(m)} f(\xi) = \sum_{(i,j)\in E} \xi_i \left(\xi_j + \frac{m}{2}\right) [f(\xi^{i,j}) - f(\xi)] + \xi_j \left(\xi_i + \frac{m}{2}\right) [f(\xi^{j,i}) - f(\xi)]$$

is self-dual on functions

$$D(\eta, \xi) = \prod_{i \in V} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2} + \xi_i\right)}$$

Algebraic description of SIP(m)

$$L^*_{SIP(m)} = \sum_{(i,j)\in E} \left(K_i^+ K_j^- + K_i^- K_j^+ - 2K_i^o K_j^o + \frac{m^2}{8} \right)$$

$\{K_i^+, K_i^-, K_i^o\}$ are the generators of the $\mathfrak{su}(1,1)$ algebra

$$[K_i^o, K_j^{\pm}] = \pm\delta_{i,j} K_i^{\pm} \qquad\qquad [K_i^-, K_j^+] = +2\delta_{i,j} K_i^o$$

$$\begin{cases} K_i^+ |\xi_i\rangle = \left(\xi_i + \frac{m}{2}\right) |\xi_i + 1\rangle \\ \\ K_i^- |\xi_i\rangle = \xi_i |\xi_i - 1\rangle \\ \\ K_i^o |\xi_i\rangle = \left(\xi_i + \frac{m}{4}\right) |\xi_i\rangle \end{cases}$$

What is the origin of the quantum spin chain?

Let us first work with two sites ....

Casimir element

There are distinguished elements in the algebra, known as Casimir elements.

For the $\mathfrak{su}(1,1)$ algebra the Casimir is

$$C = \frac{1}{2}(K^-K^+ + K^+K^-) - (K^0)^2$$

C is in the center of the $\mathfrak{su}(1,1)$ algebra:

$$[C, K^+] = [C, K^-] = [C, K^o] = 0$$

$$\begin{aligned}
C|n\rangle &= \frac{1}{2}\left((n+1)(\frac{m}{2}+n) + (\frac{m}{2}+n-1)n\right) - (n+\frac{m}{4})^2|n\rangle \\
&= n(\frac{m}{2}+n) + \frac{m}{4} - (n+\frac{m}{4})^2|n\rangle \\
&= \frac{m}{4}(1-\frac{m}{4})|n\rangle
\end{aligned}$$

Co-product

The co-product is a morphism that turns the algebra into a bialgebra:

$$\Delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$$

and conserves the commutations relations

$$[\Delta(K^o), \Delta(K^\pm)] = \pm\Delta(K^\pm)$$
$$[\Delta(K^-), \Delta(K^+)] = 2\Delta(K^o)$$

For classical Lie-algebras the co-product is just the symmetric tensor product with the identity, i.e. for $x \in \{K^+, K^-, K^o\}$

$$\Delta(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x = x_1 + x_2$$

Casimir & co-product

$$
\begin{aligned}
\Delta(C) =& \frac{1}{2}\Big(\Delta(K^-)\Delta(K^+) + \Delta(K^+)\Delta(K^-)\Big) - \Big(\Delta(K^0)\Big)^2 \\
=& \frac{1}{2}\Big((K_1^- + K_2^-)(K_1^+ + K_2^+) + (K_1^+ + K_2^+)(K_1^- + K_2^-)\Big) \\
& - \Big(K_1^o + K_2^o\Big)^2 \\
=& K_1^- K_2^+ + K_1^+ K_2^- - 2K_1^o K_2^o + C_1 + C_2 \\
=& (L_{1,2}^{SIP(m)})^* + (C_1 + C_2 - \frac{m^2}{8}) \\
=& (L_{1,2}^{SIP(m)})^* + (\frac{m}{2}(1 - \frac{m}{2}))
\end{aligned}
$$

Symmetries

Having realized that the (adjoint of the) process generator is the co-product of the Casimir, it is easy to find symmetries:

Lemma

$$[(L_{1,2}^{SIP(m)})^*, K_1^o + K_2^o] = 0$$

$$[(L_{1,2}^{SIP(m)})^*, K_1^+ + K_2^+] = [(L_{1,2}^{SIP(m)})^*, K_1^- + K_2^-] = 0$$

Proof:

$$\begin{aligned}\left[(L_{1,2}^{SIP(m)})^*, K_1^+ + K_2^+\right] &= \left[\Delta\Big(C - \frac{m}{2}(1 - \frac{m}{2})\Big), \Delta(K^+)\right] \\ &= \Delta([C, K^+]) \\ &= 0\end{aligned}$$

The symmetry $S_{1,2} = \exp(K_1^+ + K_2^+)$

$$
\begin{aligned}
S_{1,2}(\eta_1, \eta_2; \xi_1, \xi_2) &= \prod_{i=1}^{2} \langle \eta_i | \exp(K_i^+) | \xi_i \rangle \\
&= \prod_{i=1}^{2} \langle \eta_i | \sum_{s_i \geq 0} \frac{(K_i^+)^{s_i}}{s_i!} | \xi_i \rangle \\
&= \prod_{i=1}^{2} \langle \eta_i | \sum_{s_i \geq 0} \frac{(\frac{m}{2} + \xi_i + s_i - 1)!}{(\frac{m}{2} + \xi_i - 1)! s_i!} | \xi_i + s_i \rangle \\
&= \prod_{i=1}^{2} \frac{(\frac{m}{2} + \eta_i - 1)!}{(\frac{m}{2} + \xi_i - 1)!(\eta_i - \xi_i)!} \\
&= \prod_{i=1}^{2} \frac{\Gamma(\frac{m}{2} + \eta_i)}{\Gamma(\frac{m}{2} + \xi_i)} \frac{1}{(\eta_i - \xi_i)!}
\end{aligned}
$$

Trivial self-duality dunction $d_{1,2}$

Remember that on the graph we had

$$\mu_{rev}(\xi) = \prod_{i=1}^{|V|} \frac{p^{\xi_i}}{\xi_i!} \frac{\Gamma(\frac{m}{2}+\xi_i)}{\Gamma(\frac{m}{2})}(1-p)^{\frac{m}{2}}$$

and a trivial (i.e. diagonal) self-duality function was obtained as

$$d(\eta,\xi) = \frac{1}{\mu_{rev}(\eta)}\delta_{\eta,\xi}$$

Since the total number of particles is constant

$$d_{1,2}(\eta_1,\eta_2;\xi_1,\xi_2) = \prod_{i=1}^{2} \frac{\eta_i!\,\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}+\eta_i)}\delta_{\eta_i,\xi_i}$$

The duality function $D_{1,2}$

Combining trivial self-duality and symmetry leads to

$$
\begin{aligned}
D_{1,2}(\eta_1,\eta_2;\xi_1,\xi_2) &= d_{1,2} S_{1,2}(\eta_1,\eta_2;\xi_1,\xi_2) \\
&= \prod_{i=1}^{2} \sum_{\zeta_i} \frac{\eta_i!\,\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}+\eta_i)} \delta_{\eta_i,\zeta_i} \cdot \frac{\Gamma(\frac{m}{2}+\zeta_i)}{\Gamma(\frac{m}{2}+\xi_i)} \frac{1}{(\zeta_i-\xi_i)!} \\
&= \prod_{i=1}^{2} \frac{\eta_i!\,\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}+\eta_i)} \cdot \frac{\Gamma(\frac{m}{2}+\eta_i)}{\Gamma(\frac{m}{2}+\xi_i)} \frac{1}{(\eta_i-\xi_i)!} \\
&= \prod_{i=1}^{2} \frac{\eta_i!}{(\eta_i-\xi_i)!} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}+\xi_i)}
\end{aligned}
$$

Summary of self-dualities for the three IPS

Theorem

The INCLUSION process is self-dual on

$$D(\eta,\xi)=\prod_i \frac{\eta_i!}{(\eta_i-\xi_i)!}\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+\xi_i\right)}$$

The INDEPENDENT WALKERS process is self-dual on

$$D(\eta,\xi)=\prod_i \frac{\eta_i!}{(\eta_i-\xi_i)!}$$

The EXCLUSION process is self-dual on

$$D(\eta,\xi)=\prod_i \frac{\eta_i!}{(\eta_i-\xi_i)!}\frac{\Gamma\left(n+1-\xi_i\right)}{\Gamma\left(n+1\right)}$$

Exercise: Self-duality of independent walkers

Prove that the process with generator L^{IND} is self-dual on functions

$$D(\eta,\xi) = \prod_{i\in V} \frac{\eta_i!}{(\eta_i - \xi_i)!}$$

Hint:

$$[L^*_{IND}, \sum_i A_i^+] = [L^*_{IND}, \sum_i A_i] = 0$$

Self-duality fct related to the simmetry $S = e^{\sum_i A_i^+}$

Exercise: Self-duality of the SEP(n) process

Prove that the process with generator $L^{SEP(n)}$ is self-dual on functions

$$D(\eta,\xi) = \prod_{i\in V} \frac{\eta_i!}{(\eta_i-\xi_i)!} \frac{\Gamma(n+1-\xi_i)}{\Gamma(n+1)}$$

Hint:

$$[L^*_{SEP(n)}, \sum_i J^o_i] = [L^*_{SEP(n)}, \sum_i J^+_i] = [L^*_{SEP(n)}, \sum_i J^-_i] = 0$$

Self-duality corresponds to the action of the symmetry $S = e^{\sum_i J^+_i}$

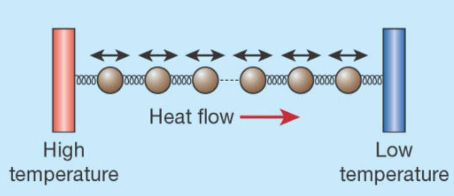
3.5 – A simple application of duality: stationary state of boundary driven Independent Random Walkers

Blackboard!

4. Interacting diffusions

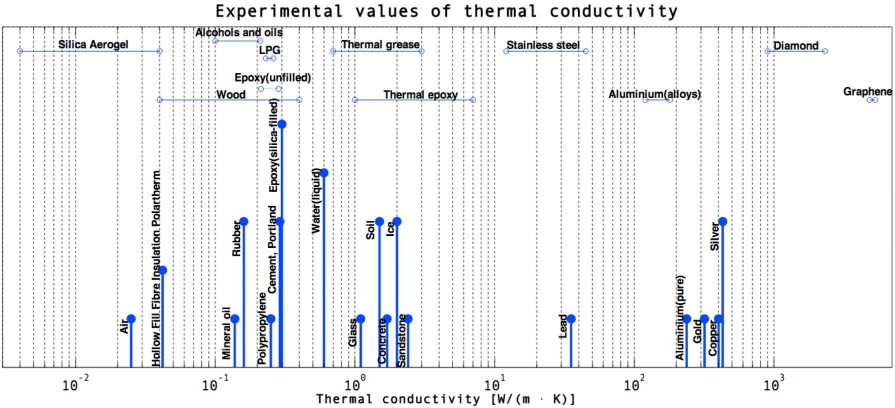
4.1 – Fourier's law and energy redistribution models

Fourier's law



$$J = \kappa \nabla T$$

J : average heat flux	$=$	average energy transported through the unit surface per unit time
∇T : temperature gradient	$=$	spatial derivative of the temperature
κ : thermal conductivity	$=$	constant of proportionality



Experimental values of thermal conductivity
Silica Aerogel
Alcohols and oils
LPG
Thermal grease
Stainless steel
Diamond
Epoxy(unfilled)
Wood
Thermal epoxy
Aluminium(alloys)
Graphene
Epoxy(silica-filled)
Hollow Fill Fibre Insulation Polartherm
Rubber
Cement, Portland
Water(liquid)
Soil
Ice
Silver
Air
Mineral oil
Polypropylene
Glass
Concrete
Sandstone
Lead
Aluminium(pure)
Gold
Copper
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Thermal conductivity [W/(m · K)]

Mathematical models

- Macroscopic (PDE's)

$$\frac{\partial T}{\partial t} = \nabla \cdot (k(T)\nabla T)$$

- Microscopic (interacting particle systems)
  - in generic 1D Hamiltonian models Fourier's law fails:

    – oscillators chains $\kappa \sim N$

    – non-linear oscillators chains $\kappa \sim N^{\alpha}$ $\quad 0 < \alpha < 1$

    – non-linear fluctuating hydrodynamics
  - use stochastic models ...

Energy redistribution models

KMP model (introduced by Kipnis, Marchioro, Presutti in 1982)

Observables: Energies at every site $z = (z_1, \ldots, z_N) \in \mathbb{R}_+^N$

Dynamics: Select a bond at random and uniformly redistribute the energy under the constraint of conserving the total energy.

$$\mathscr{L}^{KMP} f(z) =$$

$$\sum_{i=1}^{N} \int_0^1 dp \left[f(z_1, \ldots, p(z_i + z_{i+1}), (1-p)(z_i + z_{i+1}), \ldots, z_N) - f(z) \right]$$

conductivity $0 < \kappa < \infty$, model solved by duality.

4.2 – Brownian momentum process

Brownian Momentum Process (BMP) on two sites

First consider two sites, call them (i,j). Let $(x_i(t), x_j(t))$ be the diffusion process on $\mathbb{R}^2$ with generator

$$\mathscr{L}_{i,j}^{BMP} f(x_i, x_j) = \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2 f(x_i, x_j)$$

- polar coordinates: $x_i = r_{ij} \cos(\theta_{ij}) \qquad y_i = r_{ij} \sin(\theta_{ij})$

$$\mathscr{L}_{i,j}^{BMP} = \frac{\partial^2}{\partial \theta_{ij}^2}$$

  Brownian motion for angle $\theta_{ij}(t) = \arctan(x_j(t)/x_i(t))$

- total kinetic energy conserved: $\quad r_{ij}^2(t) = x_i^2(t) + x_j^2(t) = r_{ij}(0)$

Brownian Momentum Process (BMP) on two sites

SDE description: Imagine a particle moving on the plane subject to a random space-homogeneous but time-dependent magnetic field $B(t)$ perpendicular to the plane. Let $B(t)$ be a standard Brownian motion.

Then the velocity vector $(x_i(t), x_j(t))$ evolve as [Stratonovich]

$$\begin{cases} dx_i(t) = x_j(t)dB(t) \\ \\ dx_j(t) = -x_i(t)dB(t) \end{cases}$$

Conservation law:

$$\begin{aligned} d(x_i^2(t) + x_j^2(t)) &= 2x_i(t)dx_i(t) + 2x_j(t)dx_j(t) \\ &= 2x_i(t)x_j(t)dB(t) - 2x_j(t)x_i(t)dB(t) \\ &= 0 \end{aligned}$$

Brownian Momentum Process (BMP) on two sites

Forward equation: Let $p_t(x_i, x_j)$ be the probability density function of the diffusion $(x_i(t), x_j(t))$. Then

$$\frac{d}{dt} p_t(x_i, x_j) = (\mathscr{L}_{i,j}^{BMP})^* p_t(x_i, x_j)$$
$$p_0(x_i, x_j) = p(x_i, x_j)$$

Remark: the generator is self-adjoint w.r.t. Lebesgue measure, i.e.

$$(\mathscr{L}_{i,j}^{BMP})^* = \mathscr{L}_{i,j}^{BMP}$$

Exercise: Show that stationary measures are given by product measures with marginals centered Gaussians with variance σ^2, i.e.

$$p_{stat}(x_i, x_j) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x_i^2/2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x_j^2/2\sigma^2}$$

Brownian momentum process (BMP)

For an irreducibile graph $G = (V, E)$ let $\Omega = \otimes_{i \in V} \Omega_i = \mathbb{R}^{|V|}$.

Definition

The Brownian momentum process is the diffusion process $x(t) = (x_1(t), \ldots x_{|V|}(t))$ on Ω with generator

$$\mathscr{L}^{BMP} = \sum_{(i,j) \in E} \mathscr{L}_{i,j}^{BMP} = \sum_{(i,j) \in E} \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2$$

Exercise: Show that stationary measures are Gaussian product measures with variance σ^2, i.e.

$$d\mu_{stat}(x) = \prod_{i=1}^{|V|} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x_i^2}{2\sigma^2}} \, dx_i$$

4.3 – On the origin of BMP

On the origin of BMP

BMP process should arise as the "high-temperature" limit of an Hamiltonian dynamics. Start form the Hamiltonian

$$H(q,p) = \sum_{i=1}^{N} \frac{1}{2}\Big(p_i - A_i\Big)^2$$

positions $q = (q_1, \ldots, q_N)$, momenta $p = (p_1, \ldots, p_N)$, "vector potential" $A(q) = (A_1(q), \ldots, A_N(q))$.

Hamilton equations
$$\begin{cases} \frac{dq_i}{dt} & = v_i \\ \frac{dv_i}{dt} & = \sum_{j=1}^{N} B_{ij}(q) v_j \end{cases}$$

where

$$B_{ij}(q) = \frac{\partial A_i(q)}{\partial q_j} - \frac{\partial A_j(q)}{\partial q_i}$$

antisymmetric matrix containing the "magnetic fields"

Conservation laws

- *Conservation of Energy:*
  Even if the forces depend on velocities and positions, the model conserves the total (kinetic) energy

$$\frac{d}{dt}\left(\sum_i \frac{1}{2} v_i^2\right) = \sum_{i,j} B_{ij}(q) v_i v_j = 0$$

- *Conservation of Momentum:*
  If we choose the $A_i(x)$ such that they are left invariant by the simultaneous translations $x_i \to x_i + \delta$, then the quantity $\sum_i p_i$ is conserved.

Example: discrete time dynamics with "magnetic kicks"

For $n \in \mathbb{N}$, let $q_n = (q_n^{(1)}, q_n^{(2)}), \quad v_n = (v_n^{(1)}, v_n^{(2)})$.
Consider the map

$$\begin{cases} q_{n+1} & = q_n + v_n \\ \\ v_{n+1} & = R_{n+1} \cdot v_n \end{cases}$$

with R_n a 2×2 rotation matrix

$$R_n = \begin{pmatrix} \cos(B(q_n)) & \sin(B(q_n)) \\ -\sin(B(q_n)) & \cos(B(q_n)) \end{pmatrix}$$

Since the dynamics conserves the energy $(v^{(1)})^2 + (v^{(2)})^2$, then the accessible phase space is 3-dimensional.

Chaoticity properties of the map on torus $\mathbb{T}_2$ with $B(q) = q^{(1)} + q^{(2)} - 2\pi$

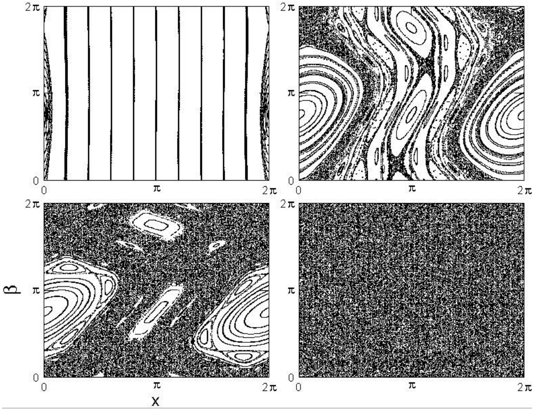


Figure : Poincare section with plane $q^{(2)} = 0$ of the map
$$\begin{cases} q^{(1)}_{n+1} &= \quad q^{(1)}_n + v\cos(\beta_n) \\ q^{(2)}_{n+1} &= \quad q^{(2)}_n + v\sin(\beta_n) \\ \beta_{n+1} &= \quad \beta_n + B(q_n) \end{cases}$$

with $v = \sqrt{(v^{(1)})^2 + (v^{(2)})^2}, \quad \beta = \operatorname{arctanh}(v^{(2)}/v^{(1)})$

High temperature limit

- Large velocities $\Rightarrow$ between two consecutive kicks new positions are translated by a large amount.
- Spatial coordinates are taken modulo $2\pi \Rightarrow$ the sequence of positions constitutes a (quasi) random number generator.
- The position of the particle can be taken as uniformly randomly distributed on $[0, 2\pi)^2$.
- Because the magnetic fields are functions of the positions, this in turn means that the fields themselves are random.

Energy redistribution

Energies before ($e_i = \frac{1}{2} v_i^2$) and after ($e_i' = \frac{1}{2} v_i'^2$) a kick

$$
\begin{aligned}
e_i' &= c^2\, e_i + s^2\, e_{i+1} + 2 s\, c \sqrt{e_i e_{i+1}} \\
e_{i+1}' &= s^2\, e_i + c^2\, e_{i+1} - 2\, s\, c \sqrt{e_i e_{i+1}}
\end{aligned}
$$

where $s = \sin(B)$ and $c = \cos(B)$. Suppose $(q^{(1)}, q^{(2)})$ are indep. $Uni(0, 2\pi)$. Then the field B has a probability density

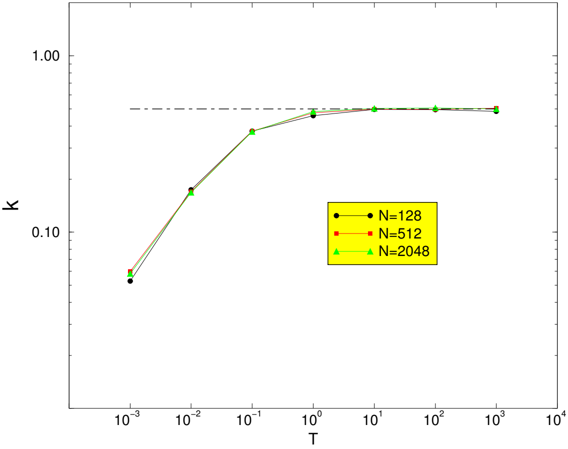
$$
p_B(x) = \begin{cases} \frac{1}{2\pi}\left(1 + \frac{1}{2\pi} x\right) & x \in [-2\pi, 0] \\ \frac{1}{2\pi}\left(1 - \frac{1}{2\pi} x\right) & x \in [0, 2\pi] \end{cases}
$$

Hence

$$
\begin{aligned}
\langle e_i' \rangle &= \frac{1}{2}(\langle e_i \rangle + \langle e_{i+1} \rangle) \\
\langle e_{i+1}' \rangle &= \frac{1}{2}(\langle e_i \rangle + \langle e_{i+1} \rangle)
\end{aligned}
$$

This dynamical rule, *only as far as the means are concerned*, is equivalent to the KMP.

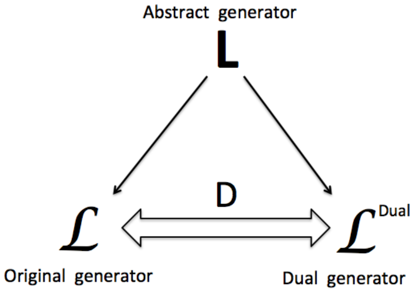
Heat conductivity for a couple map lattice with N sites



Numerical results: i) finite conductivity; ii) high temperature limit approaches a constant value $\kappa = 1/2$.

4.4 – Duality between BMP and SIP(1)

Duality as change of representation



Duality BMP/SIP(1)

Brownian momentum process

$\downarrow$

Change of representation of $\mathfrak{su}(1,1)$ algebra

$\downarrow$

Inclusion process with $m = 1$

Brownian momentum process (BMP)

For a graph $G = (V, E)$ let $\Omega = \otimes_{i \in V} \Omega_i = \mathbb{R}^{|V|}$.

The BMP process $x(t) = (x_1(t), \ldots, x_{|V|}(t))$ on Ω has generator

$$\mathscr{L}^{BMP} = \sum_{(i,j) \in E} \mathscr{L}_{i,j}^{BMP} = \sum_{(i,j) \in E} \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2$$

Symmetric Inclusion Process SIP(1)

For a graph $G = (V, E)$ let $\Omega_{dual} = \otimes_{i \in V} \Omega_i^{dual} = \{0, 1, 2, \ldots\}^{|V|}$.

The SIP process with parameter $m = 1$ on Ω_{dual} has generator

$$\begin{aligned}
\mathscr{L}^{SIP} f(\xi) &= \sum_{(i,j) \in E} \mathscr{L}^{SIP}_{i,j} f(\xi) \\
&= \sum_{(i,j) \in E} \xi_i \left(\xi_j + \frac{1}{2} \right) [f(\xi^{i,j}) - f(\xi)] \\
&+ \sum_{(i,j) \in E} \left(\xi_i + \frac{1}{2} \right) \xi_j \, [f(\xi^{j,i}) - f(\xi)]
\end{aligned}$$

Duality between BMP and SIP(1)

Theorem

The process $(x(t))_{t\geq 0}$ with generator $\mathscr{L}^{BMP}$ and the process $(\xi(t))_{t\geq 0}$ with generator $\mathscr{L}^{SIP(1)}$ are dual on

$$D(x,\xi) = \prod_{i\in V} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!}$$

Proof: An explicit computation gives

$$\mathscr{L}^{BMP} D(\cdot,\xi)(x) = \mathscr{L}^{SIP(1)} D(x,\cdot)(\xi)$$

Duality explained

Abstract operator: $\mathfrak{su}(1,1)$ ferromagnetic quantum spin chain

$$\mathrm{L} = \sum_{(i,j)\in E} \left(\mathrm{K}_i^+ \mathrm{K}_j^- + \mathrm{K}_i^- \mathrm{K}_j^+ - 2\mathrm{K}_i^o \mathrm{K}_j^o + \frac{1}{8} \right)$$

with $\{\mathrm{K}_i^+, \mathrm{K}_i^-, \mathrm{K}_i^o\}_{i\in V}$ satisfying $\mathfrak{su}(1,1)$ commutation relations:

$$[\mathrm{K}_i^o, \mathrm{K}_j^\pm] = \pm\delta_{i,j}\mathrm{K}_i^\pm \qquad [\mathrm{K}_i^-, \mathrm{K}_j^+] = 2\delta_{i,j}\mathrm{K}_i^o$$

Duality between $\mathscr{L}^{BMP}$ e $\mathscr{L}^{SIP(1)}$ corresponds to two different representations of the abstract operator L.

Duality fct is the intertwiner.

Representation of $\mathfrak{su}(1,1)$ algebra in terms of differential operators

The operators

$$\mathrm{k}_i^+ = \frac{1}{2}x_i^2 \qquad \mathrm{k}_i^- = \frac{1}{2}\frac{\partial^2}{\partial x_i^2} \qquad \mathrm{k}_i^o = \frac{1}{4}\left(x_i\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i}x_i\right)$$

satisfy commutation relations **(Exercise!)**

$$[\mathrm{k}_i^o, \mathrm{k}_i^\pm] = \pm\mathrm{k}_i^\pm \qquad [\mathrm{k}_i^-, \mathrm{k}_i^+] = 2\mathrm{k}_i^o$$

In this representation

$$\mathbb{L} = \mathscr{L}^{BMP}$$

Representation of $\mathfrak{su}(1,1)$ algebra in terms of matrices

The (infinite dimensional) matrices

$$K_i^+|\xi_i\rangle = \left(\xi_i + \frac{1}{2}\right)|\xi_i + 1\rangle$$

$$K_i^-|\xi_i\rangle = \xi_i|\xi_i - 1\rangle$$

$$K_i^o|\xi_i\rangle = \left(\xi_i + \frac{1}{4}\right)|\xi_i\rangle$$

satisfy commutation relations **(Exercise!)**

$$[K_i^o, K_i^\pm] = \pm K_i^\pm \qquad [K_i^-, K_i^+] = 2K_i^o$$

In a canonical base

$$K_i^+ = \begin{pmatrix} 0 & & & \\ \frac{1}{2} & \ddots & & \\ & \frac{3}{2} & \ddots & \\ & & \ddots & \ddots \end{pmatrix} \quad K_i^- = \begin{pmatrix} 0 & 1 & & \\ & \ddots & 2 & \\ & & \ddots & \ddots \\ & & & \ddots \end{pmatrix} \quad K_i^0 = \begin{pmatrix} \frac{1}{4} & 0 & & \\ & \frac{5}{4} & \ddots & \\ & & \frac{9}{4} & \ddots \\ & & & \ddots \end{pmatrix}$$

Representation of $\mathfrak{su}(1,1)$ algebra in terms of matrices

The (infinite dimensional) matrices

$$K_i^+|\xi_i\rangle = \left(\xi_i + \frac{1}{2}\right)|\xi_i + 1\rangle$$
$$K_i^-|\xi_i\rangle = \xi_i|\xi_i - 1\rangle$$
$$K_i^o|\xi_i\rangle = \left(\xi_i + \frac{1}{4}\right)|\xi_i\rangle$$

satisfy commutation relations **(Exercise!)**

$$[K_i^o, K_i^\pm] = \pm K_i^\pm \qquad [K_i^-, K_i^+] = 2K_i^o$$

In this representation

$$\mathbb{L} = (\mathscr{L}^{SIP(1)})^*$$

Intertwiner as duality function

Intertwiner

$$\mathrm{k}_i^+ D_i(\cdot,\xi_i)(x_i) = (K_i^+)^* D_i(x_i,\cdot)(\xi_i)$$
$$\mathrm{k}_i^- D_i(\cdot,\xi_i)(x_i) = (K_i^-)^* D_i(x_i,\cdot)(\xi_i)$$
$$\mathrm{k}_i^o D_i(\cdot,\xi_i)(x_i) = (K_i^o)^* D_i(x_i,\cdot)(\xi_i)$$

From the intertwining of creation operators one gets

$$\frac{x_i^2}{2} D_i(x_i,\xi_i) = \left(\xi_i + \frac{1}{2}\right) D_i(x_i,\xi_i+1)$$

Therefore

$$D_i(x_i,1) = \frac{x_i^2}{1} D_i(x_i,0)$$

$$D_i(x_i,2) = \frac{x_i^2}{3} D_i(x_i,1) = \frac{x_i^4}{3\cdot 1} D_i(x_i,0)$$

$$\vdots$$

Intertwiner as duality function

The recursion relation gives

$$D_i(x_i, \xi_i) = \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!} D_i(x_i, 0)$$

If we put $D_i(x_i, 0)$ equal to a constant, this expression solves also the intertwining for the annihilation operators

$$\frac{1}{2} D_i''(x_i, \xi_i) = \xi_i D_i(x_i, \xi_i - 1)$$

Indeed

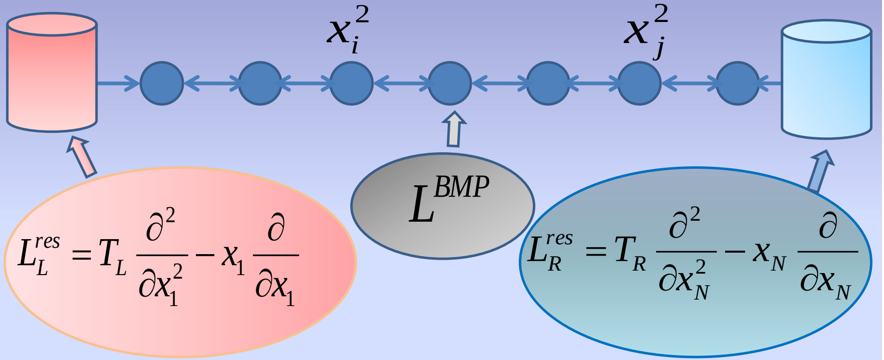
$$\frac{2\xi_i(2\xi_i - 1)x^{2\xi_i - 2}}{2(2\xi_i - 1)!!} = \xi_i \frac{x_i^{2(\xi_i - 1)}}{(2(\xi_i - 1) - 1)!!}$$

Same is true for the relations with the number operator (**Exercise!**).

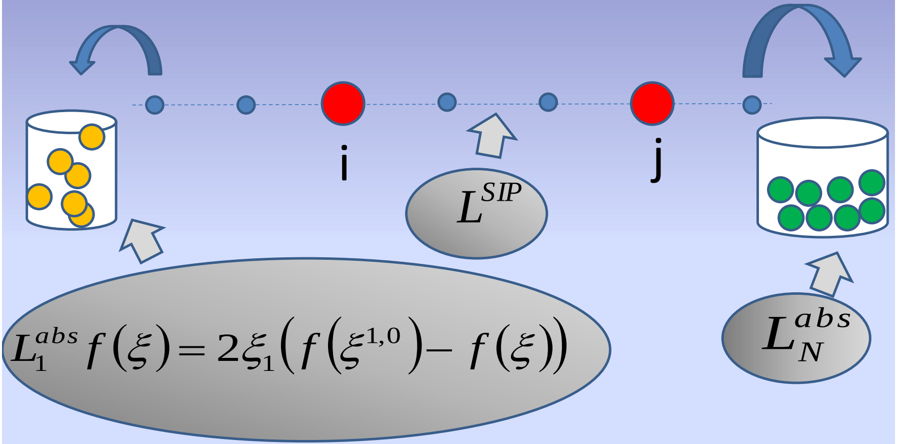
Thus $D = \prod_i D_i$ is a duality fct. between $\mathscr{L}^{BMP}$ and $\mathscr{L}^{SIP(1)}$

4.5 – BMP as a model of heat conduction

Brownian Momentum Process with reservoirs



Inclusion Process with absorbing reservoirs



BMP with thermal reservoirs

Generator

$$\mathscr{L}^{BMP,res} = \mathscr{L}_1^{res} + \sum_{i=1}^{N-1} \mathscr{L}_{i,i+1}^{BMP} + \mathscr{L}_N^{res}$$

BMP with thermal reservoirs

Generator

$$\mathscr{L}^{BMP,res} = \mathscr{L}_1^{res} + \sum_{i=1}^{N-1} \mathscr{L}_{i,i+1}^{BMP} + \mathscr{L}_N^{res}$$

$$\mathscr{L}_{i,i+1}^{BMP} = \left(x_i \frac{\partial}{\partial x_{i+1}} - x_{i+1} \frac{\partial}{\partial x_i} \right)^2 \qquad \text{Bulk}$$

BMP with thermal reservoirs

Generator

$$\mathscr{L}^{BMP,res} = \mathscr{L}_1^{res} + \sum_{i=1}^{N-1} \mathscr{L}_{i,i+1}^{BMP} + \mathscr{L}_N^{res}$$

$$\mathscr{L}_{i,i+1}^{BMP} = \left(x_i \frac{\partial}{\partial x_{i+1}} - x_{i+1} \frac{\partial}{\partial x_i} \right)^2$$

Bulk

$$\mathscr{L}_i^{res} = T_i \frac{\partial^2}{\partial x_1^2} - x_1 \frac{\partial}{\partial x_1}$$

Reservoirs

BMP with thermal reservoirs

Generator

$$\mathscr{L}^{BMP,res} = \mathscr{L}_1^{res} + \sum_{i=1}^{N-1} \mathscr{L}_{i,i+1}^{BMP} + \mathscr{L}_N^{res}$$

$$\mathscr{L}_{i,i+1}^{BMP} = \left(x_i \frac{\partial}{\partial x_{i+1}} - x_{i+1} \frac{\partial}{\partial x_i} \right)^2 \qquad \text{Bulk}$$

$$\mathscr{L}_i^{res} = T_i \frac{\partial^2}{\partial x_1^2} - x_1 \frac{\partial}{\partial x_1} \qquad \text{Reservoirs}$$

$T_1 = T_N = T$ (equilibrium): Gibbs measure $\nu_T = \otimes_{i=1}^N \mathcal{N}(0, T)$.
$T_1 \neq T_N$ (non-equilibrium): unknown stationary measure.

SIP with absorbing boundaries

Configurations $\xi = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

SIP with absorbing boundaries

Configurations $\xi = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

Generator

$$\mathscr{L}^{SIP,abs} = \mathscr{L}_1^{abs} + \sum_{i=1}^{N-1} \mathscr{L}_{i,i+1}^{SIP(1)} + \mathscr{L}_N^{abs}$$

SIP with absorbing boundaries

Configurations $\xi = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

Generator

$$\mathscr{L}^{SIP,abs} = \mathscr{L}_1^{abs} + \sum_{i=1}^{N-1} \mathscr{L}_{i,i+1}^{SIP(1)} + \mathscr{L}_N^{abs}$$

$$\mathscr{L}_{i,i+1}^{SIP(1)} f(\xi) = \sum_{i=1}^{N-1} \xi_i(\xi_{i+1} + \frac{1}{2})[f(\xi^{i,i+1}) - f(\xi)] \qquad \text{Bulk}$$

$$+ \xi_{i+1}(\xi_i + \frac{1}{2})[f(\xi^{i+1,i}) - f(\xi)]$$

SIP with absorbing boundaries

Configurations $\xi = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

Generator

$$\mathscr{L}^{SIP,abs} = \mathscr{L}_1^{abs} + \sum_{i=1}^{N-1} \mathscr{L}_{i,i+1}^{SIP(1)} + \mathscr{L}_N^{abs}$$

$$\mathscr{L}_{i,i+1}^{SIP(1)} f(\xi) = \sum_{i=1}^{N-1} \xi_i(\xi_{i+1} + \frac{1}{2})[f(\xi^{i,i+1}) - f(\xi)] \qquad \text{Bulk}$$

$$+ \xi_{i+1}(\xi_i + \frac{1}{2})[f(\xi^{i+1,i}) - f(\xi)]$$

$$\mathscr{L}_1^{abs} f(\xi) = \frac{\xi_1}{2}(f(\xi^{1,0}) - f(\xi)) \qquad \text{Absorbing}$$

Duality between BMP with reservoirs and SIP(1) with absorbing boundaries

Theorem

The process $\{x(t)\}_{t\geq 0}$ with generator $\mathscr{L}^{BMP,res}$ is dual to the process $\{\bar{\xi}(t)\}_{t\geq 0}$ with generator $\mathscr{L}^{SIP(1),abs}$ on

$$D(x,\xi) = T_L^{\xi_0} \left(\prod_{i=1}^{N} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!} \right) T_R^{\xi_{N+1}}$$

Duality between BMP with reservoirs and SIP(1) with absorbing boundaries

Theorem

The process $\{x(t)\}_{t\geq 0}$ with generator $\mathscr{L}^{BMP,res}$ is dual to the process $\{\bar{\xi}(t)\}_{t\geq 0}$ with generator $\mathscr{L}^{SIP(1),abs}$ on

$$D(x,\xi) = T_L^{\xi_0}\left(\prod_{i=1}^{N} \frac{x_i^{2\xi_i}}{(2\xi_i-1)!!}\right) T_R^{\xi_{N+1}}$$

Proof: It is enough to check that

$$\mathscr{L}^{BMP,res} D(\cdot,\xi)(x) = \mathscr{L}^{SIP(1),abs} D(x,\cdot)(\xi)$$

Simplifications of duality

- From continuous to discrete:
  Interacting diffusions (BMP) studied via interacting particle systems (SIP(1)).

- From many to few:
  n-points correlation functions of N particles using n dual walkers
  Remark: $n \ll N$.

- From reservoirs to absorbing boundaries:
  Stationary state of dual process described by absorption probabilities of dual particles at the boundaries.

Expectations of duality functions in the BMP stationary state

Proposition

Let $|\xi| = \sum_{i=1}^{N} \xi_i$ be the total number of SIP dual walkers. Let $\mathbb{P}_\xi(a,b) = \mathbb{P}(\xi_0(\infty) = a, \xi_{N+1}(\infty) = b \mid \xi(0) = \xi)$. Then

$$\mathbb{E}(D(x,\xi)) = \sum_{a,b:\ a+b=|\xi|} T_L^a T_R^b \ \mathbb{P}_\xi(a,b)$$

Proof:

$$\mathbb{E}(D(x,\xi)) = \lim_{t\to\infty} \int \mathbb{E}_{x_0}\Big(D(x(t),\xi)\Big)\, d\nu(x_0)$$

$$= \int \lim_{t\to\infty} \mathbb{E}_\xi\Big(D(x_0,\xi(t))\Big)\, d\nu(x_0)$$

$$\text{using} \qquad D(x,\xi) = T_L^{\xi_0} \left(\prod_{i=1}^{N} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!}\right) T_R^{\xi_{N+1}}$$

$$= \mathbb{E}_\xi\big(T_L^{\xi_0(\infty)} T_R^{\xi_{N+1}(\infty)}\big)$$

Temperature profile 1d linear chain

$$\vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0) \Rightarrow D(x, \vec{\xi}) = x_i^2$$

site i $\nearrow$ $\Rightarrow$ 1 SIP(1) walker $(X_t)_{t \geq 0}$ with $X_0 = i$

$$\mathbb{E}\left(x_i^2\right) = T_L\, \mathbb{P}_i(X_\infty = 0) + T_R\, \mathbb{P}_i(X_\infty = N+1)$$

$$\mathbb{E}(x_i^2) = T_L\left(1 - \frac{i}{N+1}\right) + T_R\left(\frac{i}{N+1}\right) = T_L + \left(\frac{T_R - T_L}{N+1}\right) i$$

$$J = \mathbb{E}(x_{i+1}^2) - \mathbb{E}(x_i^2) = \frac{T_R - T_L}{N+1} \quad \textit{Fourier's law}$$

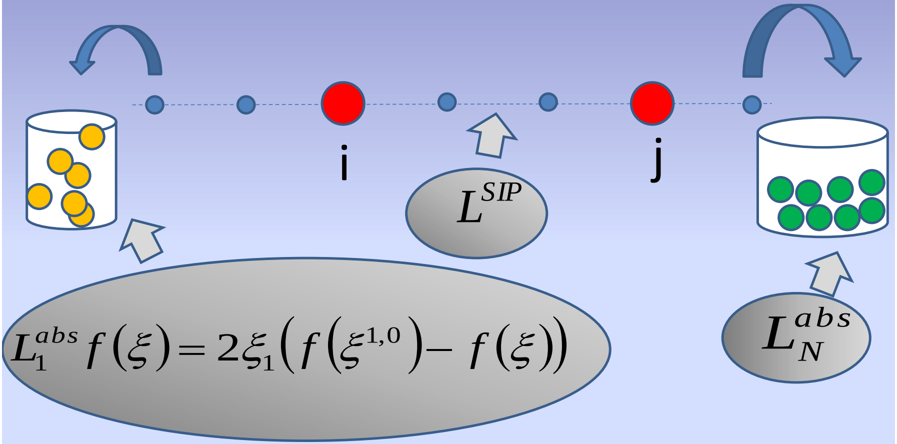
Energy covariance 1d linear chain

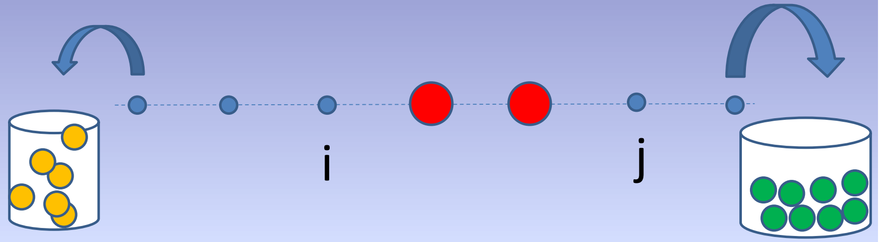
$$\text{If } \xi = (0,\ldots,0,1,0,\ldots,0,1,0,\ldots,0) \quad \Rightarrow \quad D(x,\xi) = x_i^2 x_j^2$$

site $i \nearrow$ site $j \nearrow$

In the dual process we initialize two SIP walkers $(X_t, Y_t)_{t\geq 0}$ with $(X_0, Y_0) = (i,j)$

Inclusion Process with absorbing reservoirs

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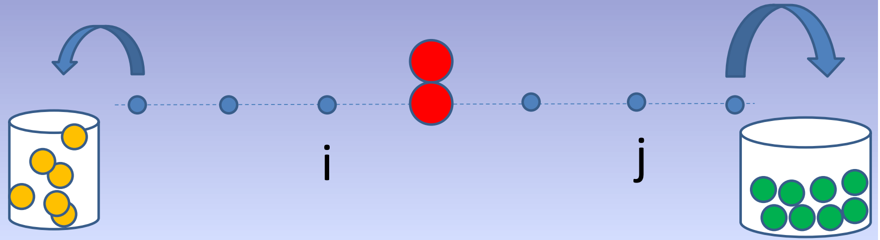
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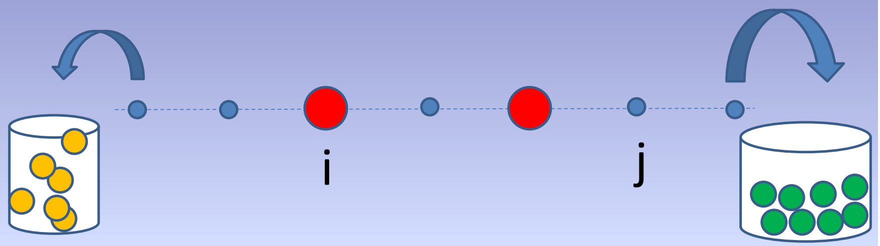
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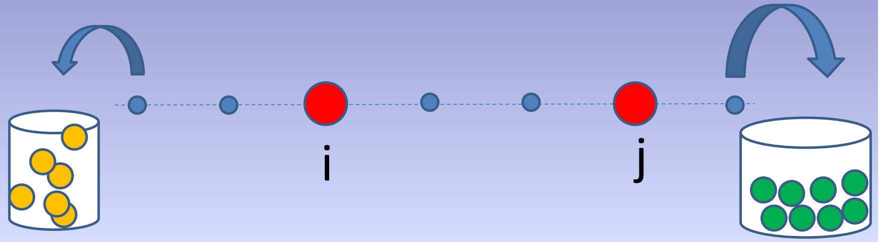

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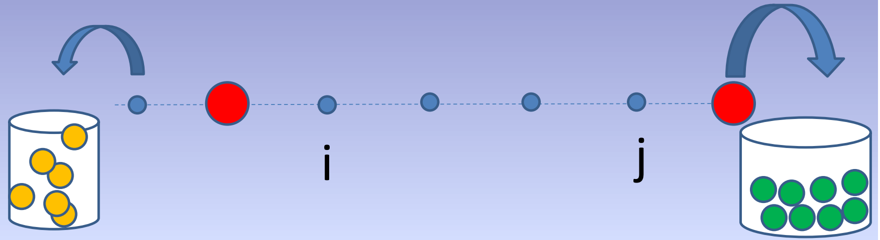
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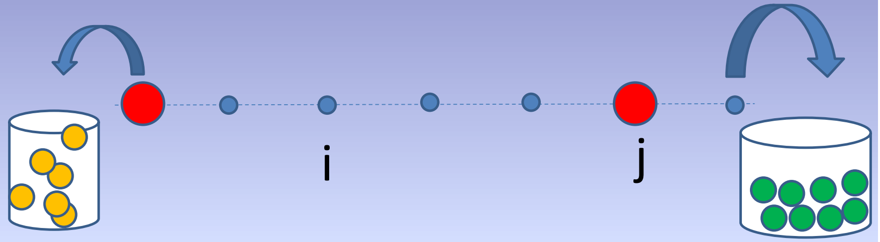
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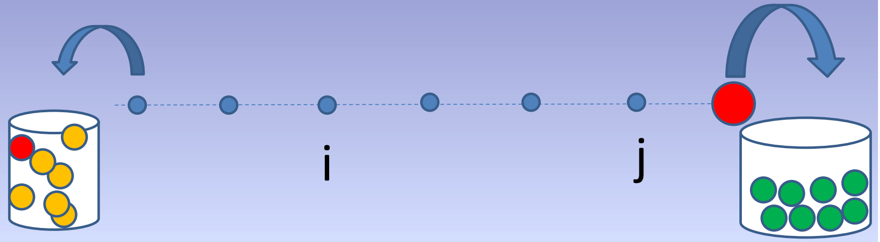
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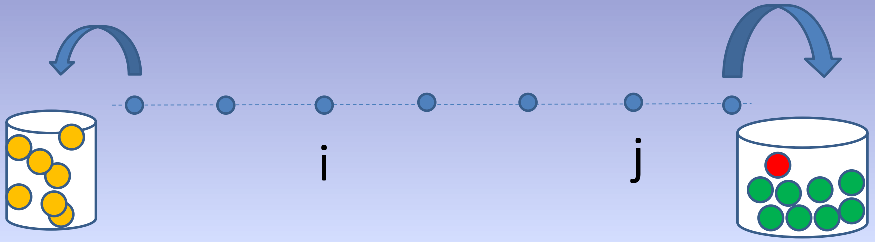
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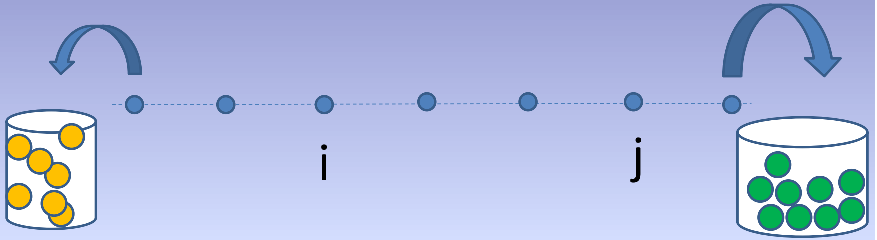

$$\mathbf{E}\left(x_i^2 x_j^2\right) = T_L^2\,\mathrm{P}\left(\genfrac{}{}{0pt}{}{\color{orange}\bullet}{\color{orange}\bullet}\right) + T_R^2\,\mathrm{P}\left(\genfrac{}{}{0pt}{}{\color{green}\bullet}{\color{green}\bullet}\right) + T_L\,T_R\left(\mathrm{P}\left(\genfrac{}{}{0pt}{}{\color{orange}\bullet}{\color{green}\bullet}\right) + \mathrm{P}\left(\genfrac{}{}{0pt}{}{\color{green}\bullet}{\color{orange}\bullet}\right)\right)$$

Energy covariance 1d linear chain

$$\mathbb{E}\left(x_i^2 x_j^2\right) - \mathbb{E}\left(x_i^2\right)\mathbb{E}\left(x_j^2\right) = \frac{2i(N+1-j)}{(N+3)(N+1)^2}(T_R - T_L)^2 \geq 0$$

Remark: up to a sign, covariance is the same in the boundary driven Exclusion Process with at most one particle per site.

Energy covariance 1d linear chain

$$\mathbb{E}\left(x_i^2 x_j^2\right) - \mathbb{E}\left(x_i^2\right)\mathbb{E}\left(x_j^2\right) = \frac{2i(N+1-j)}{(N+3)(N+1)^2}(T_R - T_L)^2 \geq 0$$

Remark: up to a sign, covariance is the same in the boundary driven Exclusion Process with at most one particle per site.

Remark: Long range correlations:

$$N\, Cov(x^2_{z_1 N}, x^2_{z_2 N}) \sim 2z_1(1-z_2)(T_R - T_L)^2$$

SIP Correlation Inequalities

Proposition

Let $\xi(t)$ be the SIP process and let ν_λ be its stationary measure. Then

$$\int \mathbb{E}_\xi \left(D\left(\xi_t, \sum_{i=1}^{n} \delta_{y_i}\right)\right) \nu_\lambda(d\xi) \geq \prod_{i=1}^{n} \int \mathbb{E}_\xi \left(D\left(\xi_t, \delta_{y_i}\right)\right) \nu_\lambda(d\xi)$$

In particular, the random variables $\{\xi_i(t)\}$ are **positively** correlated in the stationary state.

4.6 – Brownian energy process

Brownian Energy Process: BEP(1)

The energies of the Brownian Momentum Process

$$z_i(t) = x_i^2(t)$$

evolve as a diffusion process with generator

$$\mathscr{L}^{BEP(1)} = \sum_{(i,j)\in E} z_i z_j \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j}\right)^2 - \frac{1}{2}(z_i - z_j)\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j}\right)$$

Exercise: Show that stationary reversible measures are given by product measures with marginals Gamma distributions with shape $\frac{1}{2}$ and rate λ, i.e.

$$\mu_{rev}(dz) = \prod_{i=1}^{|V|} \frac{\lambda^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} z_i^{\frac{1}{2}-1} e^{-\lambda z_i} dz_i$$

Brownian Energy Process: BEP(m)

Start from the BMP process on a ladder graph with m layers

$$\mathscr{L}^{BMP}_{ladder} = \sum_{(i,j)\in E} \sum_{\alpha,\beta=1}^{m} \left(x_{i,\alpha} \frac{\partial}{\partial x_{j,\beta}} - x_{j,\beta} \frac{\partial}{\partial x_{i,\alpha}} \right)^2$$

The energies $z_i(t) = \sum_{\alpha=1}^{m} x_{i,\alpha}^2(t)$ evolve with the generator

$$\mathscr{L}^{BEP(m)} = \sum_{(i,j)\in E} z_i z_j \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)^2 - \frac{m}{2}(z_i - z_j) \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)$$

Exercise: Show that stationary reversible measures are given by product measures with marginals Gamma distributions with shape $\frac{m}{2}$ and rate λ, i.e.

$$\mu_{rev}(dz) = \prod_{i=1}^{|V|} \frac{\lambda^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} z_i^{\frac{m}{2}-1} e^{-\lambda z_i} dz_i$$

Adding-up $\mathfrak{su}(1,1)$ spins

$$\mathrm{L} = \sum_{(i,j)\in E} \left(\mathrm{K}_i^+ \mathrm{K}_j^- + \mathrm{K}_i^- \mathrm{K}_j^+ - 2\mathrm{K}_i^o \mathrm{K}_j^o + \frac{m^2}{8} \right)$$

$\left\{\mathrm{K}_i^+, \mathrm{K}_i^-, \mathrm{K}_i^o\right\}_{i\in V}$ satisfy $\mathfrak{su}(1,1)$ algebra

Adding-up $\mathfrak{su}(1,1)$ spins

$$\mathtt{L} = \sum_{(i,j)\in E} \left(\mathtt{K}_i^+ \mathtt{K}_j^- + \mathtt{K}_i^- \mathtt{K}_j^+ - 2\mathtt{K}_i^o \mathtt{K}_j^o + \frac{m^2}{8} \right)$$

$\left\{ \mathtt{K}_i^+, \mathtt{K}_i^-, \mathtt{K}_i^o \right\}_{i\in V}$ satisfy $\mathfrak{su}(1,1)$ algebra

$$\begin{cases} k_i^+ = z_i \\ k_i^- = z_i\, \partial_{z_i}^2 + \frac{m}{2} \partial_{z_i} \\ k_i^0 = z_i\, \partial_{z_i} + \frac{m}{4} \end{cases}$$

$$\begin{cases} K_i^+ |\xi_i\rangle = \left(\xi_i + \frac{m}{2}\right) |\xi_i + 1\rangle \\ K_i^- |\xi_i\rangle = \xi_i |\xi_i - 1\rangle \\ K_i^o |\xi_i\rangle = (\xi_i + m)\, |\xi_i\rangle \end{cases}$$

Duality between BEP(m) and SIP(m)

Theorem

The process $\{z(t)\}_{t\geq 0}$ with generator $\mathscr{L}^{BEP(m)}$ and the process $\{\xi(t)\}_{t\geq 0}$ with generator $\mathscr{L}^{SIP(m)}$ are dual on

$$D(z,\xi) = \prod_{i\in V} z_i^{\xi_i} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}+\xi_i)}$$

Duality between BEP(m) and SIP(m)

Theorem

The process $\{z(t)\}_{t\geq 0}$ with generator $\mathscr{L}^{BEP(m)}$ and the process $\{\xi(t)\}_{t\geq 0}$ with generator $\mathscr{L}^{SIP(m)}$ are dual on

$$D(z,\xi) = \prod_{i\in V} z_i^{\xi_i} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}+\xi_i)}$$

Proof: in the first representation $\mathbb{L} = \mathscr{L}^{BEP(m)}$, in the second representation $\mathbb{L} = \mathscr{L}^{SIP(m)}$, $D(z,\xi)$ is the intertwiner.

4.7 – Redistribution models

KMP model

Select a bond (i, j) at random and uniformly redistribute the energies (z_i, z_j) under the constraint of conserving the total energy $z_i + z_j$.

Generator

$$\mathscr{L}^{KMP} f(z) =$$

$$\sum_i \int_0^1 dp[f(z_1, \dots, p(z_i + z_{i+1}), (1-p)(z_i + z_{i+1}), \dots, z_N) - f(z)]$$

KMP model is an instantaneous thermalization limit of BEP(2).

Instantaneous thermalization limit

Start from a process $(z(t))_{t\geq 0}$ on a graph $G = (V, E)$ with generator

$$\mathscr{L} = \sum_{(i,j)\in E} \mathscr{L}_{i,j}$$

Assume the process $(z_i(t), z_j(t))_{t\geq 0}$ on two sites with generator $\mathscr{L}_{ij}$ has stationary measure $\mu_{stat}(\cdot \mid z_i, z_j)$ when $z_i(0) = z_i, z_j(0) = z_j$.

The instantaneously thermalized process is defined by the generator

$$\mathscr{L}^{IT} = \sum_{(i,j)\in E} \mathscr{L}^{IT}_{i,j}$$

where

$$\begin{aligned}
\mathscr{L}^{IT}_{i,j} f(z_i, z_j) &:= \lim_{t\to\infty} \left(e^{t\mathscr{L}_{i,j}} - 1\right) f(z_i, z_j) \\
&= \lim_{t\to\infty} \mathbb{E}_{z_i,z_j}[f(z_i(t), z_j(t))] - f(z_i, z_j) \\
&= \int d\mu_{stat}(z_i', z_j' \mid z_i, z_j)[f(z_i', z_j') - f(z_i, z_j)]
\end{aligned}$$

Instantaneous thermalization limit

From the definition

$$\mathscr{L}^{IT} = \sum_{(i,j)\in E} \mathscr{L}_{i,j}^{IT}$$

with

$$\mathscr{L}_{i,j}^{IT} f(z_i, z_j) := \lim_{t\to\infty} \left(e^{t\mathscr{L}_{i,j}} - 1 \right) f(z_i, z_j)$$

it follows that all dualities/self-dualities properties of the initial processes are inherited by the thermalized processes!

Energy redistribution models

KMP(m) $\equiv$ Instantaneous thermalization limit of BEP(m)

$$\mathscr{L}_{i,j}^{KMP(m)} f(z_i, z_j) = \lim_{t\to\infty} \left(e^{t\mathscr{L}_{i,j}^{BEP(m)}} - 1 \right) f(z_i, z_j)$$

$$= \int \int dz_i' dz_j' \; \rho_{stat}^{(m)}(z_i', z_j' \mid z_i' + z_j' = z_i + z_j)[f(z_i', z_j') - f(z_i, z_j)]$$

$$= \int_0^1 dp \; \nu^{(m)}(p) \; [f(p(z_i + z_j), (1-p)(z_i + z_j)) - f(z_i, z_j)]$$

where

$$\nu^{(m)}(p) = \frac{\Gamma(m)}{\Gamma(\frac{m}{2})\Gamma(\frac{m}{2})} p^{\frac{m}{2}-1}(1-p)^{\frac{m}{2}-1}$$

Indeed

$$X, Y \sim \mathrm{Gamma}\left(\frac{m}{2}, \lambda\right) \quad \text{i.i.d.} \quad \Longrightarrow \quad P = \frac{X}{X+Y} \sim \mathrm{Beta}\left(\frac{m}{2}, \frac{m}{2}\right)$$

For $m = 2$: uniform redistribution, KMP model

Particle redistribution models

Consider an interacting particle system on two sites $(\xi_i(t), \xi_j(t))$ with stationary measure $\mu(\cdot|\xi_i, \xi_j)$. Define

$$
\begin{aligned}
\mathscr{L}^{IT}_{i,j} f(\xi_i, \xi_j) &:= \lim_{t\to\infty} \left(e^{t\mathscr{L}_{i,j}} - 1\right) f(\xi_i, \xi_j) \\
&= \sum_{\xi_i'} \sum_{\xi_j'} \mu_{stat}(\xi_i', \xi_j' \mid \xi_i, \xi_j)[f(\xi_i', \xi_j') - f(\xi_i, \xi_j)]
\end{aligned}
$$

If $\xi_i(t) + \xi_j(t) = \xi_i + \xi_j$ then

$$
\mathscr{L}^{IT}_{i,j} f(\xi_i, \xi_j) = \sum_{r=0}^{\xi_i+\xi_j} \nu(r|\xi_i + \xi_j)\Big(f(r, \xi_i + \xi_j - r) - f(\xi)\Big)
$$

where

$$
\nu(r|\xi_i + \xi_j) = \frac{\mu_{stat}(r, \xi_i + \xi_j - r \mid \xi_i, \xi_j)}{\sum_{s=0}^{\xi_i+\xi_j} \mu_{stat}(r, \xi_i + \xi_j - r \mid \xi_i, \xi_j)}
$$

Particle redistribution models

Th-SIP(m) $\equiv$ Instantaneous thermalization limit of SIP(m)

$$\mathscr{L} f(\xi) = \sum_{(i,j)\in E} \sum_{s=0}^{\xi_i+\xi_j} \nu(r|\xi_i+\xi_j)\Big(f(\underbrace{\xi_1, \ldots, r}_{\text{site } i \nearrow}, \ldots, \underbrace{\xi_i+\xi_j-r}_{\text{site } j \nearrow}, \ldots, \xi_{|V|}) - f(\xi)\Big)$$

with

$$\nu(r|\xi_i+\xi_j) = \mathbb{E}\left[\binom{\xi_i+\xi_j}{r} P^x (1-P)^{\xi_i+\xi_j-x}\right] \qquad P \sim Beta\left(\frac{m}{2}, \frac{m}{2}\right)$$

Indeed

$$X, Y \sim \mathrm{NegBin}\left(\frac{m}{2}, p\right) \quad \text{i.i.d.}$$

$$\Longrightarrow \text{ conditioned on } X+Y=N, \quad X \sim \mathrm{Beta\,Binomial}\left(N, \frac{m}{2}, \frac{m}{2}\right)$$

For $m = 2$: uniform redistribution, dual-KMP model

Particle redistribution models

Th-IRW $\equiv$ Instantaneous thermalization limit of IND

$$\mathscr{L}f(\xi) = \sum_{(i,j)\in E} \sum_{s=0}^{\xi_i+\xi_j} \nu(r|\xi_i+\xi_j)\Big(f(\xi_1,\ldots,\underset{\text{site } i\ \nearrow}{r},\ldots,\underset{\text{site } j\ \nearrow}{\xi_i+\xi_j-r},\ldots,\xi_{|V|})-f(\xi)\Big)$$

with

$$\nu(r|\xi_i+\xi_j) = \binom{\xi_i+\xi_j}{r}\left(\frac{1}{2}\right)^{\xi_i+\xi_j}$$

Indeed

$$X, Y \sim \mathrm{Poi}(\lambda) \quad \text{i.i.d.}$$

$$\Longrightarrow \ \text{conditioned on } X+Y=N, \quad X \sim \mathrm{Binomial}\Big(N, \frac{1}{2}\Big)$$

Particle redistribution models

Th-SEP(n) $\equiv$ Instantaneous thermalization limit of SEP(n)

$$\mathscr{L}f(\xi) = \sum_{(i,j)\in E} \sum_{s=0}^{\xi_i+\xi_j} \nu(r|\xi_i+\xi_j)\Big(f(\xi_1,\ldots,\underset{\text{site } i \nearrow}{r},\ldots,\underset{\text{site } j \nearrow}{\xi_i+\xi_j-r},\ldots,\xi_{|V|})-f(\xi)\Big)$$

with

$$\nu(r|\xi_i+\xi_j) = \frac{\binom{n}{r}\binom{n}{\xi_i+\xi_j-r}}{\binom{2n}{\xi_i+\xi_j}}$$

Indeed

$$X, Y \sim \text{Binomial}(n,p) \quad \text{i.i.d.}$$

$$\Longrightarrow \text{ conditioned on } X+Y=N, \quad X \sim \text{Hypergeometric}(2n,n,N)$$

5. Further extensions

5.1 – Asymmetric systems: $\mathfrak{su}_q(2)$

5.2 – Asymmetric systems: $\mathfrak{su}_q(1,1)$

5.3 – Multi-type population dynamics

5.4 – Orthogonal polynomials

Lie algebraic approach to duality

- ⋆ Key idea: Markov generator in abstract form, i.e. as an element of a universal enveloping algebra of a Lie algebra.

I. Duality is related to a change of representation.
Duality functions are the intertwiners.

II. Dualities are associated to symmetries.
Acting with a symmetry on a duality fct. yields another duality fct.

Conversely, the approach can be turned into a constructive method.

Construction of Markov generators with algebraic structure

i) (*Lie Algebra*): Start from a Lie algebra $\mathfrak{g}$.

ii) (*Casimir*): Pick an element in the center of $\mathfrak{g}$, e.g. the Casimir C.

iii) (*Co-product*): Consider a co-product $\Delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ making the algebra a bialgebra and conserving the commutation relations.

iv) (*Quantum Hamiltonian*): Compute $H = \Delta(C)$.

v) (*Symmetries*): $S = \Delta(X)$ with $X \in \mathfrak{g}$ is a symmetry of H:

$$[H, S] = [\Delta(C), \Delta(X)] = \Delta([C, X]) = \Delta(0) = 0.$$

vi) (*Markov generator*): Apply a “ground state transformation” to turn H into a Markov generator L.

5.1 – Asymmetric systems and $\mathfrak{su}_q(2)$ deformed algebra

q-numbers

For $q \in (0,1)$ and $n \in \mathbb{N}_0$ introduce the q-number

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Remark: $\lim_{q \to 1} [n]_q = n$.

The first q-number's are:

$$[0]_q = 0, \qquad [1]_q = 1, \qquad [2]_q = q + q^{-1}, \qquad [3]_q = q^2 + 1 + q^{-2}, \quad \ldots$$

The quantum Lie algebra $\mathfrak{su}_q(2) \equiv U_q(\mathfrak{sl}_2)$

For $q \in (0,1)$ consider the algebra with generators J^+, J^-, J^0

$$[J^+, J^-] = [2J^0]_q, \qquad [J^0, J^\pm] = \pm J^\pm$$

where

$$[2J^0]_q := \frac{q^{2J^0} - q^{-2J^0}}{q - q^{-1}}$$

Irreducible representations are $(2j+1)-$dimensional, with $j \in \mathbb{N}/2$.
Casimir element:

$$C = J^- J^+ + [J^0]_q [J^0 + 1]_q$$

A standard representation ($n = 0, 1, \ldots 2j$)

$$\begin{cases} J^+ e^{(n)} &= \sqrt{[2j-n]_q[n+1]_q}\; e^{(n+1)} \\ J^- e^{(n)} &= \sqrt{[n]_q[2j-n+1]_q}\; e^{(n-1)} \\ J^0 e^{(n)} &= (n-j)\; e^{(n)} \end{cases}$$

In this representation $C\, e^{(n)} = [j]_q[j+1]_q\, e^{(n)}$

Co-product

A co-product $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)^{\otimes 2}$ is defined as

$$\begin{aligned} \Delta(J^\pm) &= J^\pm \otimes q^{-J^0} + q^{J^0} \otimes J^\pm \\ \Delta(J^0) &= J^0 \otimes 1 + 1 \otimes J^0 \end{aligned}$$

The co-product is an isomorphism s.t.

$$[\Delta(J^+), \Delta(J^-)] = [2\Delta(J^0)]_q \qquad [\Delta(J^0), \Delta(J^\pm)] = \pm\Delta(J^\pm)$$

Iteratively $\Delta^n : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)^{\otimes(n+1)}$, i.e. for $n \geq 2$

$$\begin{aligned} \Delta^n(J^\pm) &= \Delta^{n-1}(J^\pm) \otimes q^{-J^0_{n+1}} + q^{\Delta^{n-1}(J^0_i)} \otimes J^\pm_{n+1} \\ \Delta^n(J^0) &= \Delta^{n-1}(J^0) \otimes 1 + 1^{\otimes n} \otimes J^0_{n+1} \end{aligned}$$

Quantum Hamiltonian

$$\Delta(C_i) = -q^{J_i^0}\left\{J_i^+ \otimes J_{i+1}^- + J_i^- \otimes J_{i+1}^+ + B_i^- \otimes B_{i+1}^+\right\}q^{-J_{i+1}^0}$$

$$\begin{aligned} B_i^- \otimes B_{i+1}^+ \quad &= \quad \frac{(q^j + q^{-j})(q^{j+1} + q^{-(j+1)})}{2(q-q^{-1})^2}\left(q^{J_i^0} - q^{-J_i^0}\right) \otimes \left(q^{J_{i+1}^0} - q^{-J_{i+1}^0}\right) \\ &+ \quad \frac{(q^j - q^{-j})(q^{j+1} - q^{-(j+1)})}{2(q-q^{-1})^2}\left(q^{J_i^0} + q^{-J_i^0}\right) \otimes \left(q^{J_{i+1}^0} + q^{-J_{i+1}^0}\right) \end{aligned}$$

$$H \;:=\; \sum_{i=1}^{L-1}\left(1^{\otimes(i-1)} \otimes \Delta(C_i) \otimes 1^{\otimes(L-i-1)} + c_{q,j} 1^{\otimes L}\right)$$

$$c_{q,j} = \frac{(q^{2j} - q^{-2j})(q^{2j+1} - q^{-(2j+1)})}{(q-q^{-1})^2} \qquad s.t. \qquad H \cdot \left(\otimes_{i=1}^{L} e_i^{(0)}\right) = 0$$

Symmetries of H

Lemma

$$J^{\pm} := \sum_{i=1}^{L} q^{J_1^0} \otimes \cdots \otimes q^{J_{i-1}^0} \otimes J_i^{\pm} \otimes q^{-J_{i+1}^0} \otimes \ldots \otimes q^{-J_L^0}$$

$$J^0 := \sum_{i=1}^{L} \underbrace{1 \otimes \cdots \otimes 1}_{(i-1)\text{ times}} \otimes J_i^0 \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(L-i)\text{ times}} .$$

are symmetries of H.

Proof. Let $a \in \{+,-,0\}$, then $J^a = \Delta^{L-1}(J_1^a)$.

For $n = 2$: $[H, J^a] = [\Delta(C_1), \Delta(J_1^a)] = \Delta([C_1, J_1^a]) = \Delta(0) = 0.$

For $n > 2$: induction.

Ground state transformation

Lemma

Let H be a matrix with $H(\eta,\eta') \geq 0$ for $\eta \neq \eta'$.
Suppose g is a positive ground state. i.e. $H\,g = 0$ and $g(\eta) > 0$.
Let G be the matrix $G(\eta,\eta') = g(\eta)\delta(\eta,\eta')$. Then

$$L = G^{-1} H\, G$$

is a Markov generator.

Proof.

$$L(\eta,\eta') = \frac{H(\eta,\eta')g(\eta')}{g(\eta)}$$

Therefore

$$L(\eta,\eta') \geq 0 \quad \text{if} \quad \eta \neq \eta' \qquad\qquad \sum_{\eta'} L(\eta,\eta') = 0$$

Exponential symmetries

- $g^{(0)} = \otimes_{i=1}^{L} e_i^{(0)}$ is a ground state, i.e. $Hg^{(0)} = 0$.
- For every symmetry $[H, S] = 0$ another ground state is $g = Sg^{(0)}$.
- The exponential symmetry

$$S^{+} = \exp_{q^2}(E) = \sum_{n \geq 0} \frac{(E)^n}{[n]_q!} q^{-n(n-1)/2}$$

where

$$E = \Delta^{(L-1)}(q^{J_1^0}) \cdot \Delta^{(L-1)}(J_1^{+})$$

gives a positive ground state

$$g = S^{+} g^{(0)} = \sum_{\ell_1, \ldots, \ell_L} \otimes_{i=1}^{L} \left(\sqrt{\binom{2j}{\ell_i}_q} \cdot q^{\ell_i (1+j-2ji)} \right) e^{(\ell_i)}$$

ASEP(q,j) process

Definition

The Markov process ASEP(q,j) on $[1,L]\cap\mathbb{Z}$, denoted by $(\eta(t))_{t\geq 0}$, with state space $\{0,1,\ldots,2j\}^L$ is defined by

$$(\mathscr{L}^{ASEP(q,j)}f)(\eta)=\sum_{i=1}^{L-1}(\mathscr{L}_{i,i+1}f)(\eta)$$

with

$$\begin{aligned}(\mathscr{L}_{i,i+1}f)(\eta) &= q^{\eta_i-\eta_{i+1}-(2j+1)}[\eta_i]_q[2j-\eta_{i+1}]_q(f(\eta^{i,i+1})-f(\eta)) \\ &+ q^{\eta_i-\eta_{i+1}+(2j+1)}[2j-\eta_i]_q[\eta_{i+1}]_q(f(\eta^{i+1,i})-f(\eta))\end{aligned}$$

Remark: it follows from $L=G^{-1}HG$.

ASEP(q,j) process: special cases

$$
\begin{aligned}
(\mathscr{L}_{i,i+1}f)(\eta) &= q^{\eta_i-\eta_{i+1}-(2j+1)}[\eta_i]_q[2j-\eta_{i+1}]_q(f(\eta^{i,i+1})-f(\eta)) \\
&+ q^{\eta_i-\eta_{i+1}+(2j+1)}[2j-\eta_i]_q[\eta_{i+1}]_q(f(\eta^{i+1,i})-f(\eta))
\end{aligned}
$$

- $q=1 \to SEP(j)$: symmetric partial exclusion
  jump right at rate $\eta_i(2j-\eta_{i+1})$, jump left at rate $(2j-\eta_i)\eta_{i+1}$
- $j=1/2 \to ASEP(q)$: asymmetric exclusion
  jump right at rate q^{-1} , jump left at rate q
- $j=\infty \to TAZRP(q)$: totally asymmetric zero range
  after rescaling time $t \to q^{4j-1}t$, jump right at rate $\frac{1-q^{2\eta_i}}{1-q^2}$

Properties of ASEP(q,j)

Theorem

a) The $ASEP(q,j)$ is well-defined on $\mathbb{Z}$ and is a monotone process.

b) The $ASEP(q,j)$ on $\mathbb{Z}$ has a family (labeled by $\alpha > 0$) of reversible product measures with marginals

$$\mathbb{P}_\alpha(\eta_i = x) = \frac{\alpha^x}{Z_{i,\alpha}} \binom{2j}{x}_q \cdot q^{2x(1+j-2ji)}$$

c) The $ASEP(q,j)$ has translation invariant stationary product measures only for $j = 1/2$ and for $j \to \infty$.

Proof of b) for $\alpha = 1$: using $H = H^T$

$$LG^{-2} = (G^{-1}HG)G^{-2} = G^{-2}(GHG^{-1}) = G^{-2}L^T$$

Self-duality of ASEP(q,j)

Theorem

The ASEP(q,j) on $\mathbb{Z}$ is self-dual on

$$D(\eta,\xi)=\prod_{i\in\mathbb{Z}}\frac{[\eta_i]_q!}{[\eta_i-\xi_i]_q!}\frac{\Gamma_q(2j+1-\xi_i)}{\Gamma_q(2j+1)}\cdot q^{(\eta_i-\xi_i)[2\sum_{k=1}^{i-1}\xi_k+\xi_i]+4ji\xi_i}\cdot 1_{\eta_i\geq\xi_i}$$

Proof: it follows from the general method

- $d=G^{-2}$ is a trivial duality function
- $[H,S^+]$ and $L=G^{-1}HG$, thus $[L,G^{-1}S^+G]=0$.
- $D=(G^{-1}S^+G)G^{-2}=G^{-1}S^+G^{-1}$ is a duality fct.

Current of ASEP(q,j)

Definition

Let

$$N_i(t) := \sum_{k \geq i} \eta_k(t)$$

The current $J_i(t)$ during the time interval $[0,t]$ across the bond $(i-1,i)$ is defined as the net number of particles traversing the bond in the right direction:

$$J_i(t) = N_i(t) - N_i(0)$$

Remark: let $\xi^{(i)}$ be the configuration with 1 dual particle:

$$\xi_m^{(i)} = \begin{cases} 1 & \text{if } m = i \\ 0 & \text{otherwise} \end{cases}$$

then

$$D(\eta, \xi^{(i)}) = \frac{q^{4ji-1}}{q^{2j} - q^{-2j}} \cdot (q^{2N_i} - q^{2N_{i+1}})$$

First q^2-moment of the current

Theorem

$$\mathbb{E}_\eta\left[q^{2J_i(t)}\right] = q^{\sum_{k<i}\eta_k} - \sum_{k<i} q^{-4jk}\,\mathbb{E}\left[q^{4jX(t)}\left(1-q^{-2\eta_{X(t)}}\right)q^{2(N_{X(t)}(0)-N_i(0))} \mid X(0)=k\right]$$

with $X(t)$ a random walker on $\mathbb{Z}$ jumping left at rate $q^{2j}[2j]_q$ and jumping right at rate $q^{-2j}[2j]_q$

$$\mathbb{P}(X(t)=z \mid X(0)=k) = e^{-[4j]_q t} q^{-2j(z-k)} I_{z-i}(2[2j]_q t)$$

$I_n(t)$ modified Bessel fct.

First q^2-moment of the current

Proof: Duaility gives

$$\mathbb{E}_\eta(D(\eta(t),\xi^{(i)})) = \mathbb{E}_{\xi^{(i)}}(D(\eta,\xi^{(X(t))}))$$

$$\mathbb{E}_\eta\left[q^{4ji}\cdot(q^{2N_i(t)}-q^{2N_{i+1}(t)})\right] = \mathbb{E}_{\xi^{(i)}}\left[q^{4jX(t)}\cdot(q^{2N_{X(t)}}-q^{2N_{X(t)+1}})\right]$$

Therefore

$$\mathbb{E}_\eta[q^{2N_i(t)}] = \mathbb{E}_\eta[q^{2N_{i+1}(t)}] + q^{-4ji}\mathbb{E}_{\xi^{(i)}}\left[q^{4jX(t)}\cdot(q^{2N_{X(t)}}-q^{2N_{X(t)+1}})\right]$$

Multiply by $q^{-2N_i(0)}$ to get a recursion relation for the current and iterate.

Step initial condition

Proposition

For the step initial conditions $\eta^{\pm} \in \{0, \ldots, 2j\}^{\mathbb{Z}}$ defined as

$$\eta_i^{+} := \begin{cases} 0 & \text{for } i < 0 \\ 2j & \text{for } i \geq 0 \end{cases} \qquad \eta_i^{-} := \begin{cases} 2j & \text{for } i < 0 \\ 0 & \text{for } i \geq 0 \end{cases}$$

one has

$$\mathbb{E}_{\eta^+}\left[q^{2J_i(t)}\right] = q^{4j\max\{0,i\}}\left\{1 + q^{-4ji}\,\mathbf{E}_i\left[\left(1 - q^{4jX(t)}\right)\mathbf{1}_{X(t)\geq 1}\right]\right\}$$

$$\mathbb{E}_{\eta^-}\left[q^{2J_i(t)}\right] = q^{-4j\max\{0,i\}}\left\{1 - \mathbf{E}_i\left[\left(1 - q^{4jX(t)}\right)\mathbf{1}_{X(t)\geq 1}\right]\right\}$$

Step initial condition

Remark 1: asymptotics

$$\lim_{t\to\infty} \mathbb{E}_{\eta^+}\left[q^{2J_i(t)}\right] = q^{4j\max\{0,i\}}\left(1+q^{-4ji}\right) \qquad \text{shock}$$

$$\lim_{t\to\infty} \mathbb{E}_{\eta^-}\left[q^{2J_i(t)}\right] = 0 \qquad \text{rarefaction fan}$$

Remark 2: contour integral

$$\mathbb{E}_{\eta^+}\left[q^{2J_k(t)}\right] = \frac{q^{4j\max\{0,k\}}}{2\pi i}\int e^{-\frac{q^{2j}[2j]_q^3(q^{-1}-q)^2 z}{(1+q^{4j}z)(1+z)}t}\left(\frac{1+z}{1+q^{4j}z}\right)^k \frac{dz}{z}$$

where the integration contour includes 0 and $-q^{-4j}$ but does not include -1.

5.2 – Asymmetric systems and $\mathfrak{su}_q(1,1)$ deformed algebra

Quantum Lie algebra $\mathfrak{su}_q(1,1)$

For $q \in (0,1)$ consider the algebra with generators K^+, K^-, K^0

$$[K^0, K^\pm] = \pm K^\pm, \qquad [K^+, K^-] = -[2K^0]_q$$

where

$$[2K^0]_q := \frac{q^{2K^0} - q^{-2K^0}}{q - q^{-1}}$$

Irreducible representations are infinite dimensional. E.g., for $n \in \mathbb{N}$

$$\begin{cases} K^+ e^{(n)} & = & \sqrt{[n+2k]_q [n+1]_q}\; e^{(n+1)} \\ K^- e^{(n)} & = & \sqrt{[n]_q [n+2k-1]_q}\; e^{(n-1)} \\ K^o e^{(n)} & = & (n+k)\; e^{(n)} \end{cases}$$

Casimir element

$$C = [K^o]_q [K^o - 1]_q - K^+ K^-$$

In this representation

$$C\, e^{(n)} = [k]_q [k-1]_q\, e^{(n)} \qquad\qquad k \in \mathbb{R}_+$$

Co-product

Co-product $\Delta : U_q(\mathfrak{su}(1,1)) \to U_q(\mathfrak{su}(1,1))^{\otimes 2}$

$$\begin{aligned}\Delta(K^{\pm}) &= K^{\pm} \otimes q^{-K^o} + q^{K^o} \otimes K^{\pm} \\ \Delta(K^o) &= K^o \otimes 1 + 1 \otimes K^o\end{aligned}$$

The co-product is an isomorphism s.t.

$$[\Delta(K^o), \Delta(K^{\pm})] = \pm\Delta(K^{\pm}) \qquad [\Delta(K^+), \Delta(K^-)] = -[2\Delta(K^o)]_q$$

From co-associativity $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$

$\Delta^n : U_q(\mathfrak{su}(1,1)) \to U_q(\mathfrak{su}(1,1))^{\otimes(n+1)}$, i.e. for $n \geq 2$

$$\begin{aligned}\Delta^n(K^{\pm}) &= \Delta^{n-1}(K^{\pm}) \otimes q^{-K^o_{n+1}} + q^{\Delta^{n-1}(K^0_i)} \otimes K^{\pm}_{n+1} \\ \Delta^n(K^o) &= \Delta^{n-1}(K^o) \otimes 1 + 1^{\otimes n} \otimes K^0_{n+1}\end{aligned}$$

Quantum Hamiltonian

$$\Delta(C_i) = q^{K_i^0} \left\{ K_i^+ \otimes K_{i+1}^- + K_i^- \otimes K_{i+1}^+ - B_i \otimes B_{i+1} \right\} q^{-K_{i+1}^0}$$

Quantum Hamiltonian

$$\Delta(C_i) = q^{K_i^0} \left\{ K_i^+ \otimes K_{i+1}^- + K_i^- \otimes K_{i+1}^+ - B_i \otimes B_{i+1} \right\} q^{-K_{i+1}^0}$$

$$\begin{aligned} B_i \otimes B_{i+1} \quad &= \quad \frac{(q^k + q^{-k})(q^{k-1} + q^{-(k-1)})}{2(q - q^{-1})^2} \left(q^{K_i^0} - q^{-K_i^0} \right) \otimes \left(q^{K_{i+1}^0} - q^{-K_{i+1}^0} \right) \\ &+ \quad \frac{(q^k - q^{-k})(q^{k-1} - q^{-(k-1)})}{2(q - q^{-1})^2} \left(q^{K_i^0} + q^{-K_i^0} \right) \otimes \left(q^{K_{i+1}^0} + q^{-K_{i+1}^0} \right) \end{aligned}$$

Quantum Hamiltonian

$$\Delta(C_i) = q^{K_i^0}\left\{K_i^+ \otimes K_{i+1}^- + K_i^- \otimes K_{i+1}^+ - B_i \otimes B_{i+1}\right\}q^{-K_{i+1}^0}$$

$$H \;:=\; \sum_{i=1}^{L-1}\left(1^{\otimes(i-1)} \otimes \Delta(C_i) \otimes 1^{\otimes(L-i-1)} + c_{q,k}1^{\otimes L}\right)$$

$$c_{q,k} = \frac{(q^{2k} - q^{-2k})(q^{2k-1} - q^{-(2k-1)})}{(q - q^{-1})^2} \qquad s.t. \qquad H \cdot \left(\otimes_{i=1}^{L} e_i^{(0)}\right) = 0$$

Symmetries of H

Lemma

$$K^{\pm} := \sum_{i=1}^{L} q^{K_1^0} \otimes \cdots \otimes q^{K_{i-1}^0} \otimes K_i^{\pm} \otimes q^{-K_{i+1}^0} \otimes \ldots \otimes q^{-K_L^0}$$

$$K^{0} := \sum_{i=1}^{L} \underbrace{1 \otimes \cdots \otimes 1}_{(i-1) \text{ times}} \otimes K_i^0 \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(L-i) \text{ times}} .$$

are symmetries of H.

Proof. Let $a \in \{+,-,0\}$, then $K^a = \Delta^{L-1}(K_1^a)$

For $L = 2$: $[H, K^a] = [\Delta(C_1), \Delta(K_1^a)] = \Delta([C_1, K_1^a]) = \Delta(0) = 0$

For $L > 2$: induction.

Ground state transformation

Lemma

Let H be a matrix with $H(\eta,\eta') \geq 0$ if $\eta \neq \eta'$.
Suppose g is a positive ground state, i.e. $H\,g = 0$ and $g(\eta) > 0$.
Let G be the matrix $G(\eta,\eta') = g(\eta)\delta_{\eta,\eta'}$. Then

$$L = G^{-1} H\, G$$

is a Markov generator.

Indeed

$$L(\eta,\eta') = \frac{H(\eta,\eta')g(\eta')}{g(\eta)}$$

Therefore

$$L(\eta,\eta') \geq 0 \quad \text{if} \quad \eta \neq \eta' \qquad\qquad \sum_{\eta'} L(\eta,\eta') = 0$$

Exponential symmetries

- $g^{(0)} = \otimes_{i=1}^{L} e_i^{(0)}$ is a ground state, i.e. $Hg^{(0)} = 0$.
- For every symmetry $[H, S] = 0$ another ground state is $g = Sg^{(0)}$.
- The exponential symmetry

$$S^{+} = \exp_{q^2}(E) = \sum_{n \geq 0} \frac{(E)^n}{[n]_q!} q^{-n(n-1)/2}$$

with

$$E = \Delta^{(L-1)}(q^{K_1^0}) \cdot \Delta^{(L-1)}(K_1^{+})$$

gives a positive ground state

$$g = S^{+} g^{(0)} = \sum_{\ell_1, \ldots, \ell_L} \otimes_{i=1}^{L} \left(\sqrt{\binom{\ell_i + 2k - 1}{\ell_i}_q} \cdot q^{\ell_i(1-k+2ki)} \right) e^{(\ell_i)}$$

- Remark

$$\lim_{q \to 1} S^{+} = e^{\sum_i K_i^{+}} = \prod_i e^{K_i^{+}}$$

Asymmetric Inclusion Process: ASIP(q,4k)

For $k \in \mathbb{R}_+$ the interacting particle system ASIP$(q, 4k)$ on $[1, L] \cap \mathbb{Z}$ with state space $\mathbb{N}^L$ is defined by

$$(\mathscr{L}^{ASIP(q,4k)} f)(\eta) = \sum_{i=1}^{L-1} (\mathscr{L}_{i,i+1} f)(\eta)$$

with

$$\begin{aligned}(\mathscr{L}_{i,i+1} f)(\eta) &= q^{\eta_i - \eta_{i+1} + (2k-1)} [\eta_i]_q [2k + \eta_{i+1}]_q (f(\eta^{i,i+1}) - f(\eta)) \\ &+ q^{\eta_i - \eta_{i+1} - (2k-1)} [2k + \eta_i]_q [\eta_{i+1}]_q (f(\eta^{i+1,i}) - f(\eta))\end{aligned}$$

- $q \to 1 \Rightarrow$ *SIP*$(4k)$: symmetric inclusion
  jump right at rate $\eta_i(2k + \eta_{i+1})$, jump left at rate $(2k + \eta_i)\eta_{i+1}$

Properties of ASIP(q,k)

- The $ASIP(q, 4k)$ on $[1, L] \cap \mathbb{Z}$ has a family (labeled by $\alpha > 0$) of inhomogeneous reversible product measures with marginals

$$\mathbb{P}_\alpha(\eta_i = x) = \frac{\alpha^x}{Z_{i,\alpha}} \binom{x + 2k - 1}{x}_q \cdot q^{4kix}$$

- $q \to 1$: the reversible measure is homogeneous and product of Negative Binomials $(2k, \alpha)$

Asymmetric Brownian Energy Process: ABEP($\sigma, 4k$)

For $\sigma > 0$, let $(\eta^{(\epsilon)}(t))_{t\geq 0}$ be the $ASIP(1-\epsilon\sigma, 4k)$ process initialized with ϵ^{-1} particles. The scaling limit (weak asymmetry)

$$z_i(t) := \lim_{\epsilon \to 0} \epsilon\, \eta_i^{(\epsilon)}(t)$$

is the diffusion ABEP($\sigma, 4k$) with generator $\mathscr{L}^{ABEP(\sigma,4k)} = \sum_{i=1}^{L-1} \mathscr{L}_{i,i+1}$

$$\begin{aligned}
&\mathscr{L}_{i,i+1} = \\
&-\frac{1}{2\sigma}\left\{(1-e^{-2\sigma z_i})(e^{2\sigma z_{i+1}}-1) + 2k\left(2-e^{-2\sigma z_i}-e^{2\sigma z_{i+1}}\right)\right\}\left(\frac{\partial}{\partial z_i}-\frac{\partial}{\partial z_{i+1}}\right)\\
&+\frac{1}{4\sigma^2}(1-e^{-2\sigma z_i})(e^{2\sigma z_{i+1}}-1)\left(\frac{\partial}{\partial z_i}-\frac{\partial}{\partial z_{i+1}}\right)^2
\end{aligned}$$

Remark: $\mathscr{L}_{i,i+1}$ conserves $z_i(t) + z_{i+1}(t)$

Properties of ABEP($\sigma, 4k$)

- $\sigma \neq 0$

  the process is truly asymmetric, i.e. on the 1-d torus it carries a non-zero current.

  on the half-line it has inhomogeneous reversible product measures (labeld by $\gamma > -4\sigma k$) with marginal density

  $$\mu(dz_i) = \frac{1}{Z_{i,\alpha}}(1 - e^{-2\sigma z_i})^{(2k-1)} e^{-(4\sigma k i+\gamma) z_i} dz_i$$

- $\sigma \to 0^+$

  $$\mathscr{L}_{i,i+1}^{BEP(4k)} = -2k\,(z_i - z_{i+1})\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}}\right) + z_i z_{i+1}\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}}\right)^2$$

  The reversible measures are given by product of i.i.d. *Gamma*$(2k;\gamma)$

  $$\mu(dz_i) = \frac{1}{\gamma^{2k}\Gamma(2k)} z_i^{(2k-1)} e^{-\gamma z_i} dz_i$$

Asymmetric KMP-like processes

- *AKMP*$(\sigma, 4k)$

$$\mathscr{L}_{i,j}^{AKMP(\sigma,4k)} f(z_i, z_j) := \lim_{t\to\infty} \left(e^{t\mathscr{L}_{i,j}^{ABEP(\sigma,4k)}} - 1 \right) f(z_i, z_j)$$

$$= \int_0^1 dp\, \nu_\sigma^{(4k)}(p|z_i + z_j)\, [f(p(z_i + z_j), (1-p)(z_i + z_j)) - f(z_i, z_j)]$$

with

$$\nu_\sigma^{(4k)}(p|E) = \frac{1}{\mathcal{N}_{\sigma,k}} e^{2\sigma pE} \left\{ \left(e^{2\sigma pE} - 1 \right) \left(1 - e^{-2\sigma(1-p)E} \right) \right\}^{2k-1}$$

- Th-ASIP$(4k)$

$$(\eta_i, \eta) \to (R_q, \eta_i + \eta_j - R_q)$$

with R_q a q-deformed Beta-Binomial $(n+m, 2k, 2k)$

Duality between $ABEP(\sigma, k)$ and $SIP(k)$

Theorem [Carinci,G., Redig, Sasamoto (2016)]

- For every σ (including 0^+), the process $\{z(t)\}_{t\geq 0}$ with generator $\mathscr{L}^{ABEP(\sigma,4k)}$ and the process $\{\eta(t)\}_{t\geq 0}$ with generator $\mathscr{L}^{SIP(4k)}$ are dual on

$$D(z,\xi) = \prod_{i=1}^{L} \frac{\Gamma(2k)}{\Gamma(2k+\xi_i)} \left(\frac{e^{-2\sigma E_{i+1}(z)} - e^{-2\sigma E_i(z)}}{2\sigma} \right)^{\xi_i}$$

with

$$E_i(z) = \sum_{l=i}^{L} z_l \qquad E_{L+1}(z) = 0$$

- Same duality holds between $AKMP(\sigma, 4k)$ and Th-$SIP(4k)$

the symmetric case $\sigma = 0^+$

$$\mathtt{L} = \sum_{i=1}^{L-1} \left(\mathtt{K}_i^+ \mathtt{K}_{i+1}^- + \mathtt{K}_i^- \mathtt{K}_{i+1}^+ - 2\mathtt{K}_i^o \mathtt{K}_{i+1}^o + 2k^2 \right)$$

Two representations of the $\mathfrak{su}(1,1)$ algebra:

$$\begin{cases} K_i^+ e^{(\eta_i)} = (\eta_i + 2k)\, e^{(\eta_i+1)} \\ K_i^- e^{(\eta_i)} = \eta_i e^{(\eta_i-1)} \\ K_i^o e^{(\eta_i)} = (\eta_i + 4k)\, e^{(\eta_i)} \end{cases} \qquad \begin{cases} k_i^+ = z_i \\ k_i^- = z_i\, \partial_{z_i}^2 + 2k\partial_{z_i} \\ k_i^o = z_i\, \partial_{z_i} + k \end{cases}$$

$$\mathtt{L} = (L^{SIP(4k)})^* \qquad \mathtt{L} = \mathscr{L}^{BEP(4k)}$$

$$\frac{\Gamma(2k)}{\Gamma(2k+\xi_i)} z_i^{\xi_i}$$

Duality fct $\equiv$ intertwiner

the asymmetric case $\sigma \neq 0$

- The $ABEP(\sigma, 4k)$ can be mapped to $BEP(4k)$ via the non-local transformation

$$z \mapsto g(z) \qquad\qquad g_i(z) := \frac{e^{-2\sigma E_{i+1}(z)} - e^{-2\sigma E_i(z)}}{2\sigma}$$

Equivalently

$$L^{ABEP(\sigma,4k)} = C_g \circ L^{BEP(4k)} \circ C_{g^{-1}}$$

with

$$(C_g f)(z) = (f \circ g)(z)$$

- Therefore, despite the asymmetry, the symmetry group of $ABEP(\sigma, 4k)$ is the same as for $BEP(4k)$, namely $\mathfrak{su}(1,1)$. The representation is a non-local conjugation of the differential operator representation.

Self-duality of ASIP$(q, 4k)$

Theorem

The ASIP$(q, 4k)$ is self-dual on

$$D(\eta, \xi^{(\ell_1,\dots,\ell_n)}) = \frac{q^{-4k\sum_{m=1}^{n}\ell_m - n^2}}{(q^{2k} - q^{-2k})^n} \cdot \prod_{m=1}^{n} (q^{2N_{\ell_m}(\eta)} - q^{2N_{\ell_m+1}(\eta)})$$

where $\xi^{(\ell_1,\dots,\ell_n)}$ is the configuration with n particles at sites $\ell_1, \dots, \ell_n$ and

$$N_i(\eta) := \sum_{k=i}^{L} \eta_k$$

- It follows from the explicit knowledge of the reversible measure and from an exponential symmetry.

Application : bulk-driven ABEP$(\sigma, 4k)$

Definition

The current $J_i(t)$ during the time interval $[0, t]$ across the bond $(i-1, i)$ is defined as:

$$J_i(t) = E_i(z(t)) - E_i(z(0))$$

where

$$E_i(z) := \sum_{k \geq i} z_k$$

Remark: let $\xi^{(i)}$ be the configuration with 1 dual particle:

$$\xi_m^{(i)} = \begin{cases} 1 & \text{if } m = i \\ 0 & \text{otherwise} \end{cases}$$

then

$$D(z, \xi^{(i)}) = \left(\frac{e^{-2\sigma E_{i+1}(z)} - e^{-2\sigma E_i(z)}}{4k\sigma} \right)$$

5.3 – Multi-type population dynamics

Population dynamics models with $\mathfrak{su}(1,1)$ symmetry

It turns out that the diffusions of Wright-Fisher type have more structure than only the Heisenberg algebra. In the multi-type setting with parent independent mutations their generator can be written using $\mathfrak{su}(1,1)$ generators satisfying the commutation relations

$$[K^o, K^{\pm}] = \pm K^{\pm} \qquad\qquad [K^-, K^+] = 2K^o$$

$\mathfrak{su}(1,1)$ Heisenberg ferromagnet as a population model

$$\mathbb{L}_m = \frac{1}{2}\sum_{1\le i<j\le d}\left(\mathbb{K}_i^+\mathbb{K}_j^- + \mathbb{K}_i^-\mathbb{K}_j^+ - 2\mathbb{K}_i^o\mathbb{K}_j^o + \frac{m^2}{8}\right)$$

1. Written in terms of the continuous representation, $\mathbb{L}_m$ is the generator of the d-type Wright-Fisher diffusion with mutation rate $m/2$.
2. Written in terms of the discrete representation, $\mathbb{L}_m$ is the generator of the d-type Moran model with mutation rate $m/2$.
3. It commutes with

$$\sum_i \mathbb{K}_i^{\pm}, \sum_i \mathbb{K}_i^o$$

Multi-type Wright-Fisher diffusion with symmetric mutations

The d-types Wright-Fisher diffusion model with parent-independent mutation at rate $\theta \in \mathbb{R}$ is a diffusion process on the simplex $\sum_{i=1}^{d} x_i = 1$ with

$$\begin{aligned}
\mathscr{L}^{WF}_{d,\theta} g(x) &= \sum_{i=1}^{d-1} \frac{1}{2} x_i(1-x_i) \frac{\partial^2 g(x)}{\partial x_i^2} - \sum_{1\le i<j\le d-1} x_i x_j \frac{\partial^2 g(x)}{\partial x_i \partial x_j} \\
&+ \frac{\theta}{d-1} \sum_{i=1}^{d-1} (1-dx_i) \frac{\partial g(x)}{\partial x_i} .
\end{aligned}$$

$$\mathscr{L}^{BEP(m)}_{d} f(x_1,\ldots,x_{d-1},x_d) = \mathscr{L}^{WF}_{d,\frac{m}{4}(d-1)} g(x_1,\ldots,x_{d-1})$$

$$g(x_1,\ldots,x_{d-1}) = f(x_1,\ldots,x_{d-1},1-\sum_{j=1}^{d-1} x_j)$$

$$\begin{aligned}
\mathscr{L}^{BEP(m)}_{d} f(z) &= \frac{1}{2} \sum_{1\le i<j\le d} z_i z_j \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j}\right)^2 f(z) \\
&- \frac{m}{4} \sum_{1\le i<j\le d} (z_i - z_j) \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j}\right) f(z)
\end{aligned}$$

Multi-type Moran with symmetric mutations

The d-types Moran model with N individuals and parent-independent mutation at rate $\theta \in \mathbb{R}$ is a particle process on the simplex $\sum_{i=1}^{d} k_i = N$ where pair of individuals of types i and j are sampled uniformly at random, one dies with probability 1=2 and the other reproduces. In between reproduction events each individual accumulates mutations at a constant rate θ and his type mutates to any of the others with the same probability.

$$\mathscr{L}_d^{SIP(m)} f(k_1, \ldots, k_{d-1}, k_d) = \mathscr{L}^{Moran}_{N,d,\frac{m}{4}(d-1)} g(k_1, \ldots, k_{d-1})$$

$$g(k_1, \ldots, k_{d-1}) = f(k_1, \ldots, k_{d-1}, N - \sum_{j=1}^{d-1} k_j)$$

$$\begin{aligned} \mathscr{L}_d^{SIP(m)} f(k) &= \frac{1}{2} \sum_{1 \le i < j \le d} k_i \left(k_j + \frac{m}{2}\right) \left(f(k + e_i - e_j) - f(k)\right) \\ & \frac{1}{2} \sum_{1 \le i < j \le d} k_j \left(k_i + \frac{m}{2}\right) \left(f(k - e_i + e_j) - f(k)\right) \end{aligned}$$

Dualities for multi-type Wright-Fisher / Moral models

1. The d-type Wright Fisher diffusion with mutation rate $m/2$ and the d-type discrete Moran model with mutation rate $m/2$ are dual to each other with duality function

$$D(z_1, \ldots, z_d; k_1, \ldots, k_d) = \prod_i D_i(z_i, k_i)$$

with

$$D_i(z_i, k_i) = \frac{z_i^{k_i} \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2} + k_i\right)}$$

2. The d-type discrete Moran model with mutation rate $m/2$ is self-dual with self-duality function

$$D(n_1, \ldots, n_d; k_1, \ldots, k_d) = \prod_i D_i(n_i, k_i)$$

with

$$D_i(n_i, k_i) = \frac{n_i!}{(n_i - k_i)!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2} + k_i\right)}$$

5.4 – Orthogonal polynomials

Orthogonal polynomials

- Question: what is the relation between duality and stationary measure? How about orthogonal polynomials?

Orthogonal polynomials

- Question: what is the relation between duality and stationary measure? How about orthogonal polynomials?
- Answer: The SIP(k) is self-dual process with self-duality function

$$D(\eta, \xi) = \prod_x \frac{\Gamma(2k)}{\Gamma(2k + \xi_x)} M_{\xi_x}(\eta_x)$$

where $M_{\xi_x}(\eta_x)$ is the Meixner polynomials of degree ξ_x

$$\sum_{\eta=0}^{\infty} M_\xi(\eta) M_{\xi'}(\eta) \mu(\eta) = \delta_{\xi,\xi'} \frac{\xi! \Gamma(2k + \xi)}{p^\xi \Gamma(2k)}$$

with

$$\mu(\eta) = \frac{\Gamma(2k + \eta)}{\Gamma(2k)} \frac{p^\eta}{\eta!} (1 - p)^{2k}$$

Orthogonal polynomials (cont'd)

- Hypergeometric difference equation

$$\sigma(\eta)\Delta\nabla M_\xi(\eta) + \tau(\eta)\Delta M_\xi(\eta) + \lambda_\xi M_\xi(\eta) = 0$$

with

$$\Delta f(\eta) = f(\eta+1) - f(\eta) \qquad \nabla f(\eta) = f(\eta) - f(\eta-1)$$

$$\sigma(\eta) = \eta \qquad \tau(\eta) = 2kp - \eta(1-p) \qquad \lambda_\xi = \xi(1-p)$$

- 3-point recurrence relation

$$\eta M_\xi(\eta) = \alpha_\xi M_{\xi+1}(\eta) + \beta_\xi M_\xi(\eta) + \gamma_\xi M_{\xi-1}(\eta)$$

with

$$\alpha_\xi = \frac{p}{p-1} \qquad \beta_\xi = \frac{\xi + p\xi + 2kp}{1-p} \qquad \gamma_\xi = \frac{\xi(\xi - 1 + 2k)}{p-1}$$

- Raising operator

$$[p(\xi + 2k) + \eta p]M_\xi(\eta) - \eta M_\xi(\eta - 1) = pM_{\xi+1}(\eta)$$

Orthogonal polynomials (cont'd)

Other dualities with orthogonal polynomials

- Exclusion Process $\longrightarrow$ Krawtchouk polynomials
- Independent walkers $\longrightarrow$ Charlier polynomials
- Brownian momentum process $\longrightarrow$ Hermite polynomials

$$Lf(\eta) = \sum_{(x,y)\in E} \left(\eta_x \frac{\partial}{\partial \eta_y} - \eta_y \frac{\partial}{\partial \eta_x} \right)^2 f(\eta)$$

- Brownian energy process $\longrightarrow$ Laguerre polynomials

$$Lf(\eta) = \sum_{(i,l)\in E} \left[\eta_x \eta_y \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right)^2 + 2k(\eta_x - \eta_y) \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right) \right] f(\eta)$$