Portfolio optimisation: bridging the gap between theory and practice

Appendices

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This document contains references to sections and equations from the main paper.

A Extended first stage formulation

In Section 4 of the main paper, we introduce the first stage formulation with a selection of real-world constraints. In this appendix, we present an extended model with other classes of exogenous constraints that are appropriate for the first stage. Here we gather several constraints that are scattered through many papers into a single, standardised framework combining equities and futures. For this appendix to be a self-contained document, we repeat here the text from Section 4 of the main paper.

Some of the criteria for selecting optimal portfolios are defined by ratios of two affine functions, which make use of the Charnes and Cooper [1962] linearisation procedure. This procedure requires a transformation of the underlying decision variables, which in turn also requires updating most of the exogenous constraints. In Appendix B, we rewrite all constraints in this appendix for ratio-based models.

Let n be the number of assets in which we may invest and let N be a set containing all n assets ($|N| = n$). We split N into two sets, namely N_1 as the set containing all equities and N_2 as the set containing all futures contracts (assets that require margin accounts). We have that $N_1, N_2 \subseteq N$ and that $N_1 \cup N_2 = N$. We informally refer to assets in N_1 as "equities" and assets in N_2 as "futures".

Let C be the current mark-to-market valuation of the portfolio, and let there also be two optional risk-free assets $\ell, b \in N_1$ in which we may invest. Risk-free asset ℓ represents the interest rate obtained when we hold a long position in cash, that is, when we lend money to the bank. Asset b represents the interest rate applied when we hold a short position in cash, that is, when we borrow money from the bank.

Let:

$w_i^+ \geq 0, w_i^- \geq 0$	the proportion of the portfolio held in (respectively) long/short positions in asset $i \in N$.
z_i^+, z_i^-	$\begin{cases} 1 & \text{if any of asset } i \text{ is held in long/short positions} \\ 0 & \text{otherwise} \end{cases}$

Depending on which exogenous constraints are applicable to the problem at hand, binary variables z_i^+ and z_i^- are not strictly necessary, although many commercial and open-source solvers automatically detect whether a binary variable can be removed during their presolve phase.

In portfolio selection literature, the proportion invested in asset i is more often represented with a single variable w_i, where $w_i > 0$ means a long position and $w_i < 0$ means a short position. A single variable, however, prevents us from modelling certain classes of exogenous constraints, such shorting limits [Jacobs et al., 2005, Kumar et al., 2008] or even short positions in futures.

Let $\rho(\boldsymbol{w})$, where $\boldsymbol{w} = (w_1^+, w_1^-, \ldots, w_n^+, w_n^-)$, be the objective function to be minimised (we assume a minimisation problem without loss of generality). The basic portfolio selection model is given by:

$$\min \quad \rho(\boldsymbol{w}) \tag{A.1}$$

$$\text{subject to} \quad \sum_{i \in N_1} (w_i^+ - w_i^-) + \sum_{i \in N_2} (w_i^+ + w_i^-) = 1 \tag{A.2}$$

$$w_i^+ \leq U z_i^+ \quad \forall i \in N \tag{A.3}$$

$$w_i^- \leq U z_i^- \quad \forall i \in N \tag{A.4}$$

$$z_i^+ + z_i^- \leq 1 \quad \forall i \in N \tag{A.5}$$

$$z_i^+, z_i^- \in \mathbb{B} \tag{A.6}$$

$$w_i^+, w_i^- \geq 0 \quad \forall i \in N \tag{A.7}$$

Constraint (A.2) is the budget constraint. For equities, the contribution of $w_i^-, i \in N_1$ towards the budget is negative, while for futures the contribution of $w_i^-, i \in N_2$ is positive. Constraints (A.3)-(A.4) link variables $\boldsymbol{w}$ and $\boldsymbol{z}$. We use the U symbol loosely throughout this text to indicate any constant that is large enough in its given context. Constraints (A.5) ensure that an asset, if picked, cannot be both long and short at the same time. Constraints (A.6)-(A.7) define variable bounds. Depending on the choice of $\rho(\boldsymbol{w})$, the specific formulation may require additional variables and constraints (i.e. minimising CVaR).

A.1 Shorting limits in equities

Let τ^-, τ^+ be the minimum and maximum proportions of C allowed in a collection of short positions in equities. In order to enforce such limits, we add the following constraint:

$$\tau^- \leq \sum_{i \in N_1} w_i^- \leq \tau^+ \tag{A.8}$$

With this definition, we replace Constraints (A.3)-(A.4) by:

$$w_i^+ \leq (1 + \tau^+) z_i^+ \quad \forall i \in N \tag{A.9}$$

$$w_i^- \leq (1 + \tau^+) z_i^- \quad \forall i \in N_2 \tag{A.10}$$

$$w_i^- \leq \tau^+ z_i^- \quad \forall i \in N_1 \tag{A.11}$$

If $\tau^+ > 0$, then it is possible to invest up to $1 + \tau^+$ in long positions.

Several constraints are expressed as limits applicable to individual assets or collections of assets. When shorting is allowed, it is possible to express these limits as either absolute or relative (with regards to the total long/short exposure). Absolute limits make it possible to include buy-in thresholds and work as a link between binary and continuous variables. Relative limits, on the other hand, make it easier to ensure model feasibility when the long/short exposure is variable ($\tau^- < \tau^+$). In this case, let $\tau = \sum_{i \in N_1} w_i^-$ represent the actual short exposure in equities (as decided by the model itself).

A.2 Shorting limits in futures

Let υ^-, υ^+ be the minimum and maximum proportions of $1 + \tau$ allowed in a collection of short positions in futures. To enforce these limits, we add the following constraint:

$$\Big(1 + \sum_{i \in N_1} w_i^-\Big)\upsilon^- \leq \sum_{i \in N_2} w_i^- \leq \Big(1 + \sum_{i \in N_1} w_i^-\Big)\upsilon^+ \tag{A.12}$$

By defining these limits as relative to the total long exposure, we have that $0 \leq \upsilon^- \leq \upsilon^+ \leq 1$. As an illustration, let $\tau^- = \tau^+ = 0.5$ and $\upsilon^- = \upsilon^+ = 0.5$. In this case, the portfolio must hold 50% of C in short positions in equities, 75% of C must be held in short positions in futures and another 75% must be held in long positions in any of the assets.

A.3 Exposure limits in the risk-free assets

Risk-free assets generally behave as equities, with a slight difference: the asset representing a lend rate cannot be held in short positions, while the asset representing a borrow rate cannot be held in long positions. Let c^-, c^+ be the minimum and maximum proportions of $1+\tau$ allowed in the risk-free assets, such that $-1 \leq c^- \leq c^+ \leq 1$. We add the following constraints:

$$w_\ell^- = 0 \tag{A.13}$$

$$w_b^+ = 0 \tag{A.14}$$

$$\Big(1 + \sum_{i \in N_1} w_i^-\Big)\Big(\max\{0, c^-\}\Big) \leq w_\ell^+ \leq \Big(1 + \sum_{i \in N_1} w_i^-\Big)\Big(\max\{0, c^+\}\Big) \tag{A.15}$$

$$\Big(\sum_{i \in N_1} w_i^-\Big)\Big(\max\{0, -c^+\}\Big) \leq w_b^- \leq \Big(\sum_{i \in N_1} w_i^-\Big)\Big(\max\{0, -c^-\}\Big) \tag{A.16}$$

A.4 Minimum expected return

Let μ_i be the expected return of asset $i \in N$. In scenario-based formulations, $\mu_i = \sum_{s \in S} p_s r_{is}$, where S is the number of discrete scenarios, p_s is the probability of scenario $s \in S$ and r_{is} is the return of asset i in scenario s. We also remind the reader that μ_i takes leverage into account. If a given futures i has expected return 1% and leverage level equal to $5\times$, then effectively $\mu_i = 5\%$.

The following constraint enforces minimum expected portfolio return given a target return $\bar{\mu}$:

$$\sum_{i \in N} \mu_i (w_i^+ - w_i^-) \geq \bar{\mu} \tag{A.17}$$

Note that short positions in futures contribute negatively to the portfolio expected return.

A.5 General lower and upper bounds

Let l^+, l^- be the lower limits ($0 \leq l^+, l^-$) on the proportion invested in long/short positions in any asset **if** we choose to invest in that asset. Likewise, let u^+, u^- be the upper limits ($0 \leq u^+, u^-$) on the proportion invested in long/short positions in any asset. The constraints that enforce these limits are given by:

$$l^+ z_i^+ \leq w_i^+ \leq u^+ z_i^+ \qquad \forall i \in N \setminus \{\ell, b\} \tag{A.18}$$

$$l^- z_i^- \leq w_i^- \leq u^- z_i^- \qquad \forall i \in N \setminus \{\ell, b\} \tag{A.19}$$

The limits above are applicable (by our choice) to all but the risk-free assets. It is also possible to enforce different limits per asset, or even enforce them selectively. Lower bounds allow the enforcement of a minimum holding level, or as it is commonly referred to in literature, a buy-in threshold. This is especially important in cardinality constraints, since we can only enforce a minimum number of nonzero positions if l^+ and l^- are both non-zero. Upper bounds, if applicable, usually define stronger (tighter) bounds than Constraints (A.9)-(A.11).

A.6 Asset specific lower and upper bounds

Let ε_i be a mandatory lower limit ($-1 \leq \varepsilon_i \leq 1$) on the proportion invested in asset i. Likewise, let δ_i be a mandatory upper limit ($-1 \leq \delta_i \leq 1$) on the proportion invested in asset i. We define these as relative bounds: negative limits indicate a proportion of τ and positive limits indicate a proportion $1 + \tau$. The constraints for a given asset i are defined as:

$$w_i^- \leq -\Big(\sum_{j\in N_1} w_j^-\Big)\varepsilon_i \qquad \text{if } \varepsilon_i < 0 \text{ and } i \in N_1 \tag{A.20}$$

$$w_i^- \leq -\Big(1+\sum_{j\in N_1} w_j^-\Big)\varepsilon_i \qquad \text{if } \varepsilon_i < 0 \text{ and } i \in N_2 \tag{A.21}$$

$$w_i^+ \geq \Big(1+\sum_{j\in N_1} w_j^-\Big)\varepsilon_i \qquad \text{if } \varepsilon_i > 0 \tag{A.22}$$

$$w_i^- \geq -\Big(\sum_{j\in N_1} w_j^-\Big)\delta_i \qquad \text{if } \delta_i < 0 \text{ and } i \in N_1 \tag{A.23}$$

$$w_i^- \geq -\Big(1+\sum_{j\in N_1} w_j^-\Big)\delta_i \qquad \text{if } \delta_i < 0 \text{ and } i \in N_2 \tag{A.24}$$

$$w_i^+ \leq \Big(1+\sum_{j\in N_1} w_j^-\Big)\delta_i \qquad \text{if } \delta_i > 0 \tag{A.25}$$

A.7 Class specific lower and upper bounds

Asset classes are a general term for any subset of assets, such as types of financial instruments (equities, derivatives, etc.) or sectors of the economy. Let $Q \subseteq N \setminus \{\ell, b\}$, E_Q be a mandatory lower limit ($-1 \leq E_Q \leq 1$) on the proportion invested jointly in all assets in Q, and Δ_Q be a mandatory upper limit ($-1 \leq \Delta_Q \leq 1$) on the proportion invested jointly in all assets in Q. The joint limit constraints are:

$$\sum_{i\in Q} w_i^- \leq -\Big(\sum_{i\in N_1} w_i^-\Big)E_Q \qquad \text{if } E_Q < 0 \text{ and } Q \cap N_1 = Q \tag{A.26}$$

$$\sum_{i\in Q} w_i^- \leq -\Big(1+\sum_{i\in N_1} w_i^-\Big)E_Q \qquad \text{if } E_Q < 0 \text{ and } Q \cap N_2 = Q \tag{A.27}$$

$$\sum_{i\in Q} w_i^+ \geq \Big(1+\sum_{i\in N_1} w_i^-\Big)E_Q \qquad \text{if } E_Q > 0 \tag{A.28}$$

$$\sum_{i\in Q} w_i^- \geq -\Big(\sum_{i\in N_1} w_i^-\Big)\Delta_Q \qquad \text{if } \Delta_Q < 0 \text{ and } Q \cap N_1 = Q \tag{A.29}$$

$$\sum_{i\in Q} w_i^- \geq -\Big(1+\sum_{i\in N_1} w_i^-\Big)\Delta_Q \qquad \text{if } \Delta_Q < 0 \text{ and } Q \cap N_2 = Q \tag{A.30}$$

$$\sum_{i\in Q} w_i^+ \leq \Big(1+\sum_{i\in N_1} w_i^-\Big)\Delta_Q \qquad \text{if } \Delta_Q > 0 \tag{A.31}$$

Due to the operational differences in shorting equities and futures, it is not straightforward to establish short limits when Q contains assets in both N_1 and N_2.

A.8 Turnover constraints

Turnover constraints are especially important in portfolio rebalancing, as a way to prevent liquidity risk by avoiding major changes to the portfolio composition. Schreiner [1980] defines portfolio turnover as the rate of the total amount of new securities exchanged over a particular period, divided by the total mark-to-market value of the fund. The usual understanding is that we should calculate it as either the amount of securities bought or the amount of securities sold. Here, however, we adopt a more broad definition as the total amount of securities purchased AND sold as a proportion of C. We assume that cash is liquid enough so that transactions in the risk-free assets are not included in the turnover calculation. This causes an asymmetry in purchases and sales that, in our view, justify the broad definition of turnover.

As an illustration, let C = \$10,000. Suppose we sell \$3,000 in assets and then buy \$2,000 in other assets (ignoring any transaction costs), leaving the remaining \$1,000 in cash. In this case, we consider the turnover to be $(3 + 2)/10 = 50\%$ of C. Notice also that shorting increases the total exposure: if we completely replace a 150/50 portfolio by another 150/50 portfolio, all in risky assets, we have a turnover of 400% of C.

Let W_i be the current proportion invested in asset $i \in N \setminus \{\ell, b\}$, which can be positive (long) or negative (short). Also, let Γ be the maximum turnover allowed as a fraction of C. We need the additional decision variables $t_i = |W_i - (w_i^+ - w_i^-)|$ as the difference in proportion of asset $i \in N \setminus \{\ell, b\}$ between the current and the new portfolio. The following equations limit the total turnover:

$$t_i \geq W_i - (w_i^+ - w_i^-) \qquad \forall i \in N \setminus \{\ell, b\} \tag{A.32}$$

$$t_i \geq (w_i^+ - w_i^-) - W_i \qquad \forall i \in N \setminus \{\ell, b\} \tag{A.33}$$

$$\sum_{i \in N} t_i \leq \Gamma \tag{A.34}$$

Constraints (A.32)-(A.33) linearise the absolute value. It is theoretically possible that $t_i > |W_i - (w_i^+ - w_i^-)|$. This is not critical however since Constraint (A.34) ensures that the sum of t_i lies within the maximum turnover allowed.

We may also define individual turnover limits Γ_i for asset i, and individual turnover bounds accordingly as:

$$t_i \leq \Gamma_i \tag{A.35}$$

An application of asset-specific turnover limits is to choose an optimal "liquidity-feasible" portfolio. Suppose that the average trading volume of i is \$10 million. An experienced investor knows that purchasing over 5% of this value in a day might result in liquidity problems. If $C =$ \$2 million, we have that $\Gamma_i = 25\%$.

A.9 Cardinality constraints

Cardinality refers to the number of assets for which we hold non-zero positions. Let $K_{\min}, K_{\max}$ be the minimum and maximum number of assets that must/can be held in the portfolio. Likewise, let $K^+_{\min}, K^+_{\max}$, $K^-_{\min}, K^-_{\max}$ be the minimum and maximum number of assets held in long/short positions in the portfolio. We enforce cardinality restrictions as:

$$K_{\min} \leq \sum_{i \in N \setminus \{\ell, b\}} (z_i^+ + z_i^-) \leq K_{\max} \tag{A.36}$$

$$K^+_{\min} \leq \sum_{i \in N \setminus \{\ell, b\}} z_i^+ \leq K^+_{\max} \tag{A.37}$$

$$K^-_{\min} \leq \sum_{i \in N \setminus \{\ell, b\}} z_i^- \leq K^-_{\max} \tag{A.38}$$

where the risk-free assets are excluded from the cardinality count. For enforcing non-zero minimum cardinality, it is necessary that l^+ and l^- are both non-zero to ensure that $z_i^+ = 1 \iff w_i^+ > 0$ and $z_i^- = 1 \iff w_i^- > 0$.

Let $Q \subseteq N \setminus \{\ell, b\}$, $K^Q_{\min}, K^Q_{\max}$ be the minimum and maximum number of assets in Q that must/can be held in the portfolio. Cardinality constraints for a subset of assets are enforced as:

$$K^Q_{\min} \leq \sum_{i \in Q} (z_i^+ + z_i^-) \leq K^Q_{\max} \tag{A.39}$$

A.10 Enforcement of the UCITS 5/10/40 rule

The Undertakings for the Collective Investment in Transferable Securities (UCITS) is the European Commission regulatory framework for managing and selling mutual funds. UCITS funds are often perceived as safe and well-regulated investments. One of its most recognised restrictions is the 5-10-40 rule, which states that (i) no asset can take more than $\iota = 10\%$ of the portfolio and (ii) the sum of assets over $\zeta = 5\%$ must be less or equal to $\eta = 40\%$. UCITS requirements also forbid short-selling, but here we assume that both long and short positions together must adhere to this rule. There are exceptions to this rule, e.g. for funds replicating a stock market index, the maximum limit per issuer is $\iota = 20\%$ of net assets. Scozzari et al. [2013] proposed an initial set of constraints and decision variables to enforce the 5-10-40 rule, which was later improved by Strub and Trautmann [2019]. Let:

b_i^+, b_i^- $\quad \begin{cases} 1 & \text{if } w_i^+ > \zeta, w_i^- > \zeta \text{ respectively} \\ 0 & \text{otherwise} \end{cases}$

$v_i^+ \geq 0, v_i^- \geq 0$ $\quad$ be the maximum between 0 and $w_i^+ - \zeta, w_i^- - \zeta$ respectively.

Then in order to enforce the 5-10-40 rule, we add:

$$v_i^+ \geq w_i^+ - \zeta \qquad \forall i \in N \setminus \ell \tag{A.40}$$
$$v_i^- \geq w_i^- - \zeta \qquad \forall i \in N \setminus b \tag{A.41}$$
$$v_i^+ \leq (\iota - \zeta) b_i^+ \qquad \forall i \in N \setminus \ell \tag{A.42}$$
$$v_i^- \leq (\iota - \zeta) b_i^- \qquad \forall i \in N \setminus b \tag{A.43}$$
$$\sum_{i \in N \setminus \ell, b} (v_i^+ + v_i^-) + (b_i^+ + b_i^-)\zeta \leq \eta \tag{A.44}$$

B First stage formulation for ratio-based models

Several portfolio optimisation models require optimising a ratio of two affine functions, which can be linearised with the standard transformation introduced by Charnes and Cooper [1962]. If $\rho(\boldsymbol{w}) = \frac{\boldsymbol{c}^T \boldsymbol{w} + \alpha}{\boldsymbol{d}^T \boldsymbol{w} + \beta}$, define $t = \frac{1}{\boldsymbol{d}^T \boldsymbol{w} + \beta}$ such that $\rho(\boldsymbol{w}) = \boldsymbol{c}^T \boldsymbol{w} t + \alpha t$. Let $\hat{\boldsymbol{w}} = \boldsymbol{w} t$. Add:

$$t = \frac{1}{\boldsymbol{d}^T \boldsymbol{w} + \alpha} \implies (\boldsymbol{d}^T \boldsymbol{w} + \beta) t = 1 \implies \boldsymbol{d}^T \hat{\boldsymbol{w}} + \beta t = 1$$

If $(\hat{\boldsymbol{w}}^*, t^*)$ solves the linearised problem, $\boldsymbol{w}^* = \hat{\boldsymbol{w}}^*/t^*$ solves the original problem. Since this transformation requires a change of the variables that appear in the numerator, all constraints where these same variables appear must be updated by multiplying them by t. Therefore, most of the exogenous constraints described in Appendix A need to be adapted accordingly, with the notable exception being cardinality Constraints (A.36)-(A.39). In this appendix, we summarise the updated constraints.

The generic ratio-based portfolio optimisation model is defined as:

$$\max \quad \boldsymbol{c}^T \hat{\boldsymbol{w}} + t\alpha \tag{B.1}$$
$$\text{subject to} \quad \boldsymbol{d}^T \hat{\boldsymbol{w}} + \beta t = 1 \tag{B.2}$$
$$\sum_{i \in N_1} (\hat{w}_i^+ - \hat{w}_i^-) + \sum_{i \in N_2} (\hat{w}_i^+ + \hat{w}_i^-) = t \tag{B.3}$$
$$\hat{w}_i^+ \leq U z_i^+ \quad \forall i \in N \tag{B.4}$$
$$\hat{w}_i^- \leq U z_i^- \quad \forall i \in N \tag{B.5}$$
$$(A.5) - (A.6)$$
$$\hat{w}_i^+, \hat{w}_i^- \geq 0 \qquad \forall i \in N \tag{B.6}$$
$$t \geq 0 \tag{B.7}$$

Some of the constraints below become nonlinear when applying the transformation. In order to linearise them, we also add continuous auxiliary variables $\hat{z}_i^+ = z_i^+ t$, $\hat{z}_i^- = z_i^- t$ and the following additional constraints:

$$0 \leq \hat{z}_i^+ \leq U z_i^+ \tag{B.8}$$
$$\hat{z}_i^+ \leq t \tag{B.9}$$
$$t - \hat{z}_i^+ + U z_i^+ \leq U \tag{B.10}$$

$$0 \leq \hat{z}_i^- \leq U z_i^- \tag{B.11}$$
$$\hat{z}_i^- \leq t \tag{B.12}$$
$$t - \hat{z}_i^- + U z_i^- \leq U \tag{B.13}$$

This linearisation was introduced by Guastaroba et al. [2016], and they ensure that if $z_i^+ = 0$, $\hat{z}_i^+ = 0$ and that if $z_i^+ = 1$, $\hat{z}_i^+ = t$. The same is valid for the corresponding shorting variables.

Constraints (A.8)-(A.12), required for shorting limits and linking the $\boldsymbol{w}$ and $\boldsymbol{z}$ variables, are replaced by:

$$t\tau^- \;\leq\; \sum_{i\in N_1} \hat{w}_i^- \;\leq\; t\tau^+ \tag{B.14}$$

$$\hat{w}_i^+ \;\leq\; (1+\tau^+)\hat{z}_i^+ \qquad \forall i \in N \tag{B.15}$$

$$\hat{w}_i^- \;\leq\; (1+\tau^+)\hat{z}_i^- \qquad \forall i \in N_2 \tag{B.16}$$

$$\hat{w}_i^- \;\leq\; \tau^+\hat{z}_i^- \qquad \forall i \in N_1 \tag{B.17}$$

$$\Big(t + \sum_{i\in N_1} \hat{w}_i^-\Big)\upsilon^- \;\leq\; \sum_{i\in N_2} \hat{w}_i^- \;\leq\; \Big(t + \sum_{i\in N_1} \hat{w}_i^-\Big)\upsilon^+ \tag{B.18}$$

Constraints (A.13)-(A.16), which add limits to the risk-free assets, are replaced by:

$$\hat{w}_\ell^- \;=\; 0 \tag{B.19}$$

$$\hat{w}_b^+ \;=\; 0 \tag{B.20}$$

$$\Big(t + \sum_{i\in N_1} \hat{w}_i^-\Big)\Big(\max\{0, c_-\}\Big) \leq \hat{w}_\ell^+ \leq \Big(t + \sum_{i\in N_1} \hat{w}_i^-\Big)\Big(\max\{0, c_+\}\Big) \tag{B.21}$$

$$\Big(\sum_{i\in N_1} \hat{w}_i^-\Big)\Big(\max\{0, -c_+\}\Big) \leq \hat{w}_b^- \leq \Big(\sum_{i\in N_1} \hat{w}_i^-\Big)\Big(\max\{0, -c_-\}\Big) \tag{B.22}$$

The minimum expected return constraints (A.17) is replaced by:

$$\sum_{i\in N} \mu_i(\hat{w}_i^+ - \hat{w}_i^-) \geq t\bar{\mu} \tag{B.23}$$

With the help of the continuous variables defined above, the absolute bounds and buy-in threshold constraints, which replace Constraints (A.18)-(A.19), can be enforced as:

$$l^+\hat{z}_i^+ \leq \hat{w}_i^+ \leq u^+\hat{z}_i^+ \qquad \forall i \in N \setminus \{\ell, b\} \tag{B.24}$$

$$l^-\hat{z}_i^- \leq \hat{w}_i^- \leq u^-\hat{z}_i^- \qquad \forall i \in N \setminus \{\ell, b\} \tag{B.25}$$

Asset specific bounds, enforced by Constraints (A.20)-(A.25), are replaced by:

$$\hat{w}_i^- \leq -\Big(\sum_{j\in N_1} \hat{w}_j^-\Big)\varepsilon_i \qquad \text{if } \varepsilon_i < 0 \text{ and } i \in N_1 \tag{B.26}$$

$$\hat{w}_i^- \leq -\Big(t + \sum_{j\in N_1} \hat{w}_j^-\Big)\varepsilon_i \qquad \text{if } \varepsilon_i < 0 \text{ and } i \in N_2 \tag{B.27}$$

$$\hat{w}_i^+ \geq \Big(t + \sum_{j\in N_1} \hat{w}_j^-\Big)\varepsilon_i \qquad \text{if } \varepsilon_i > 0 \tag{B.28}$$

$$\hat{w}_i^- \geq -\Big(\sum_{j\in N_1} \hat{w}_j^-\Big)\delta_i \qquad \text{if } \delta_i < 0 \text{ and } i \in N_1 \tag{B.29}$$

$$\hat{w}_i^- \geq -\Big(t + \sum_{j\in N_1} \hat{w}_j^-\Big)\delta_i \qquad \text{if } \delta_i < 0 \text{ and } i \in N_2 \tag{B.30}$$

$$\hat{w}_i^+ \leq \Big(t + \sum_{j\in N_1} \hat{w}_j^-\Big)\delta_i \qquad \text{if } \delta_i > 0 \tag{B.31}$$

Class constraints (A.26)-A.31) are replaced by:

$$\sum_{i\in Q} \hat{w}_i^- \leq -\Big(\sum_{i\in N_1} \hat{w}_i^-\Big)E_Q \qquad \text{if } E_Q < 0 \text{ and } Q\cap N_1 = Q \tag{B.32}$$

$$\sum_{i\in Q} \hat{w}_i^- \leq -\Big(t + \sum_{i\in N_1} \hat{w}_i^-\Big)E_Q \qquad \text{if } E_Q < 0 \text{ and } Q\cap N_2 = Q \tag{B.33}$$

$$\sum_{i\in Q} \hat{w}_i^+ \geq \Big(t + \sum_{i\in N_1} \hat{w}_i^-\Big)E_Q \qquad \text{if } E_Q > 0 \tag{B.34}$$

$$\sum_{i\in Q} \hat{w}_i^- \geq -\Big(\sum_{i\in N_1} \hat{w}_i^-\Big)\Delta_Q \qquad \text{if } \Delta_Q < 0 \text{ and } Q\cap N_1 = Q \tag{B.35}$$

$$\sum_{i\in Q} \hat{w}_i^- \geq -\Big(t + \sum_{i\in N_1} \hat{w}_i^-\Big)\Delta_Q \qquad \text{if } \Delta_Q < 0 \text{ and } Q\cap N_2 = Q \tag{B.36}$$

$$\sum_{i\in Q} \hat{w}_i^+ \leq \Big(t + \sum_{i\in N_1} \hat{w}_i^-\Big)\Delta_Q \qquad \text{if } \Delta_Q > 0 \tag{B.37}$$

Turnover constraints (A.32)-(A.34) are replaced by:

$$t_i \geq W_i t - (\hat{w}_i^+ - \hat{w}_i^-) \qquad \forall i \in N \setminus \{\ell, b\} \tag{B.38}$$

$$t_i \geq (\hat{w}_i^+ - \hat{w}_i^-) - W_i t \qquad \forall i \in N \setminus \{\ell, b\} \tag{B.39}$$

$$\sum_{i\in N} t_i \leq \Gamma t \tag{B.40}$$

$$t_i \leq \Gamma_i t \qquad \text{for any } i \text{ necessary} \tag{B.41}$$

Finally, for enforcing the 5/10/40 UCITS rule in ratio-based models, we make use of following additional sets of variables:

b_i^+, b_i^-	$\begin{cases} 1 & \text{if } \hat{w}_i^+/t > \zeta, \hat{w}_i^-/t > \zeta \text{ respectively} \\ 0 & \text{otherwise} \end{cases}$
$\hat{b}_i^+, \hat{b}_i^-$	continuous variables such that $\hat{b}_i^+ = b_i^+ t$ and $\hat{b}_i^- = b_i^- t$
$\hat{v}_i^+ \geq 0, \hat{v}_i^- \geq 0$	the maximum between 0 and $\hat{w}_i^+ - \zeta t, \hat{w}_i^- - \zeta t$ respectively.

as well as the following constraints:

$$0 \leq \hat{b}_i^+ \leq U b_i^+ \tag{B.42}$$

$$\hat{b}_i^+ \leq t \tag{B.43}$$

$$t - \hat{b}_i^+ + U b_i^+ \leq U \tag{B.44}$$

$$\hat{v}_i^+ \geq \hat{w}_i^+ - \zeta t \qquad \forall i \in N \setminus \ell \tag{B.45}$$

$$\hat{v}_i^+ \leq (\iota - \zeta)\hat{b}_i^+ \qquad \forall i \in N \setminus \ell \tag{B.46}$$

$$0 \leq \hat{b}_i^- \leq U b_i^- \tag{B.47}$$

$$\hat{b}_i^- \leq t \tag{B.48}$$

$$t - \hat{b}_i^- - U b_i^- \leq U \tag{B.49}$$

$$\hat{v}_i^- \geq \hat{w}_i^- - \zeta t \qquad \forall i \in N \setminus b \tag{B.50}$$

$$\hat{v}_i^- \leq (\iota - \zeta)\hat{b}_i^- \qquad \forall i \in N \setminus b \tag{B.51}$$

$$\sum_{i\in N\setminus \ell, b} (\hat{v}_i^+ + \hat{v}_i^-) + (\hat{b}_i^+ + \hat{b}_i^-)\zeta \leq \eta t \tag{B.52}$$

C Statistics

In this appendix, we detail the calculations for the statistics presented in tables in Section 6. Let:

$\boldsymbol{P}$	be a series of $0..T$ daily portfolio values, where P_t is the value on day t.
$\boldsymbol{r}$	be a series of $1..T$ daily returns, where r_t is the return on day t calculated as $\frac{P_t - P_{t-1}}{P_{t-1}}$.
$\boldsymbol{r_f}$	be a series of $1..T$ daily risk free returns, where r_{ft} is the risk free rate on day t.
$\boldsymbol{b}$	be a series of $1..T$ daily benchmark returns, where b_t is the benchmark return on day t.

In our experiments, $\boldsymbol{r_f}$ varies over time. To calculate some measures we convert it into a constant daily rate r_f^d and an annual rate r_f^y. To do that we compute:

$$\begin{aligned}
R_{f0} &= 1 \\
R_{ft} &= R_{t-1}(1 + r_{ft}) && \forall t = 1, \ldots, T \\
r_f^d &= R_T^{1/T} - 1 \\
r_f^y &= (1 + r_f^d)^{252} - 1
\end{aligned}$$

The following metrics are shown in performance tables:

1. **FV** (Final value):

$$\frac{P_T}{P_0}$$

2. **CAGR** (Compound annual growth rate, in %):

$$\begin{aligned}
Y &= \frac{T}{252} \\
\text{CAGR} &= 100 \times \left(\left(\frac{P_T}{P_0} \right)^{\frac{1}{Y}} - 1 \right)
\end{aligned}$$

3. **Vol** (annualised volatility):

$$\begin{aligned}
\mu(\boldsymbol{r}) &= \frac{1}{T} \sum_{t=1}^{T} r_t \\
\sigma(\boldsymbol{r}) &= \sqrt{\frac{1}{T-1} \sum_{t=1}^{T} \big(r_t - \mu(\boldsymbol{r})\big)^2} \\
\text{Vol} &= \sigma(\boldsymbol{r}) \times \sqrt{252}
\end{aligned}$$

4. **Sharpe** (annualised Sharpe ratio):

$$\text{Sharpe} = \sqrt{252}\, \frac{\mu(\boldsymbol{r}) - r_f^d}{\sigma(\boldsymbol{r})}$$

5. **Sortino** (annualised Sortino ratio):

$$\begin{aligned}
\sigma^-(\boldsymbol{r}) &= \sqrt{\frac{1}{T} \sum_{t=1}^{T} (\min[0, r_t - r_f^d])^2} \\
\text{Sortino} &= \sqrt{252}\, \frac{\mu(\boldsymbol{r}) - r_f^d}{\sigma^-(\boldsymbol{r})}
\end{aligned}$$

6. **MDD** (maximum drawdown, in %):

$$\text{MDD} = \max\left(0, 100 \times \max_{0 \leq t < u \leq T} \frac{P_t - P_u}{P_t}\right)$$

D Long only charts, evaluation of the second stage objective function

In Section 6.4 of the main paper, we reported two figures that showed the average total deviation and the average surplus costs for varying values of θ. Those two figures included only the long/short experiments - those were the only ones where we observed an increase in surpluses as θ decreased. In this appendix, we display the long only results for both odd and round lots in Figures 1 and 2.

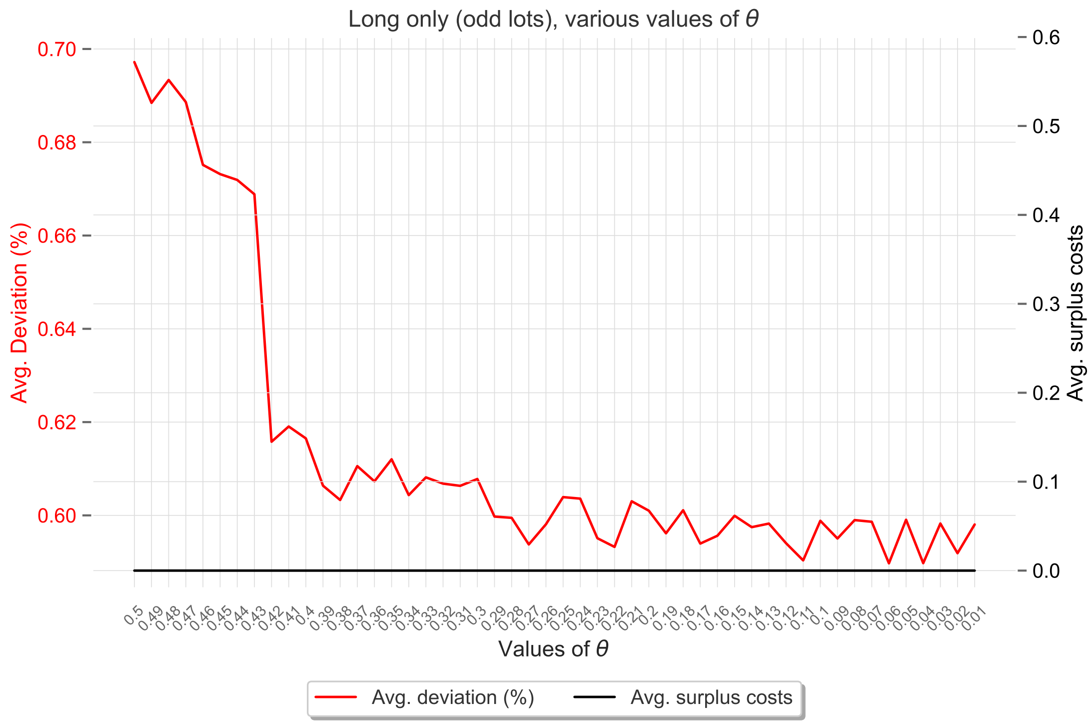

Figure 1: Long only with odd lots, various values of θ

In these charts, we observed no surplus costs for all reported values of θ. However, we still see a decay in the total average deviation from the desired first stage portfolio proportions. By visual inspection, with odd lots the average total deviation tends to "stabilise" for $\theta \leq 0.3$, while with round lots that is observed for $\theta \leq 0.13$. We believe this empirical findings further justify our default choice of $\theta = 0.05$.

E Woodside-Oriakhi et al. [2013] formulation

In this appendix, we adapt the formulation from Woodside-Oriakhi et al. [2013] to the first stage model of our framework. The formulation is both relative and absolute, that is, it includes both portfolio proportions and units/shares held/purchased/sold. Let D be the limit on the total amount of transaction costs that can be spent and let U_i^b and U_i^s be respectively the upper bounds on the amount of shares/units of asset i to be bought/sold. The updated model requires the following additional decision variables:

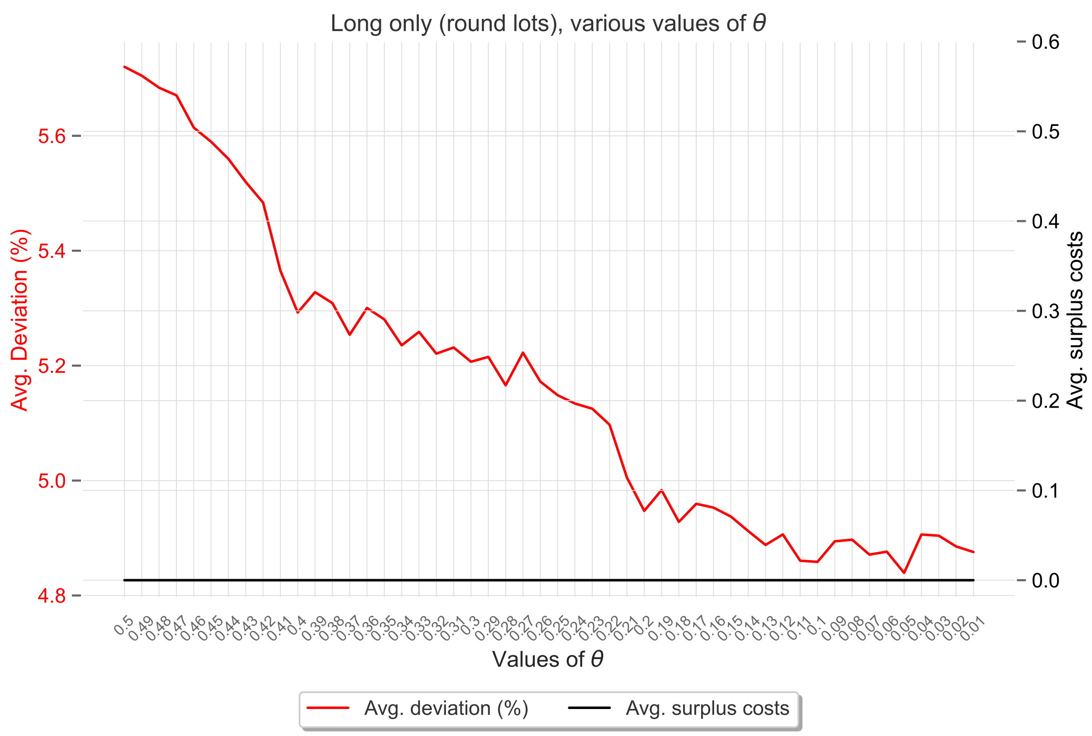

Figure 2: Long only with round lots, various values of θ

x_i	number of lots (if $i \in N^L$) or units (if $i \in \overline{N}^{\not{L}}$) of asset i in the rebalanced portfolio,
x_c	amount of cash to be held in the rebalanced portfolio,
$y_i^b \geq 0$	the number of lots (if $i \in N^L$) or units (if $i \in \overline{N}^{\not{L}}$) of asset i bought,
$y_i^s \geq 0$	the number of lots (if $i \in N^L$) or units (if $i \in \overline{N}^{\not{L}}$) of asset i sold,
α_i^b	1 if we buy any of asset i, 0 otherwise
α_i^s	1 if we sell any of asset i, 0 otherwise
$0 \leq w^{\text{tc}} \leq 1$	the proportion of P consumed by transaction costs.

The original formulation considers fixed transaction costs. Here for simplicity we do not add them.

$$\sum_{i=1}^{N} f_i V_i \left(y_i^b + y_i^s\right) \leq D \tag{E.1}$$

$$x_i = X_i + y_i^b - y_i^s \qquad i \in N \setminus \ell, b \tag{E.2}$$

$$x_c + \sum_{i=1}^{N} \frac{V_i x_i}{L_i} = P - \sum_{i=1}^{N} f_i V_i \left(y_i^b + y_i^s\right) \tag{E.3}$$

$$w_\ell^+ - w_b^- = \frac{x_c}{P} \tag{E.4}$$

$$w_i^+ - w_i^- = \frac{V_i x_i}{L_i P} \qquad i \in N \setminus \ell, b \tag{E.5}$$

$$w^{\text{tc}} = \frac{\sum_{i=1}^{N} f_i V_i \left(y_i^b + y_i^s\right)}{P} \tag{E.6}$$

$$y_i^b \leq U_i^b \alpha_i^b \tag{E.7}$$

$$y_i^s \leq U_i^b \alpha_i^s \tag{E.8}$$

$$\alpha_i^b + \alpha_i^s \leq 1 \tag{E.9}$$

Woodside-Oriakhi et al. [2013] consider only nonnegative positions, imposing that $x_i \geq 0$. Here we let x_i be a free variable, and the shorting exposure is controlled by the w_i^- variables. Constraint (E.1) define the upper limit on transaction costs. Constraints (E.2) define the amounts to be held as a function of the lots/units currently held. Constraint (E.3) define the discounted portfolio value. Constraints (E.4)-(E.5) link the holdings to proportions, and Constraint (E.6) defines the proportion of P spent on transaction costs. Constraints (E.7)-(E.9) forbid both buying and selling the same asset.

Since the weights are defined according to the non-discounted portfolio value, it is also necessary to replace Eq. (A.2) (budget constraint) with:

$$\sum_{i \in N_1} (w_i^+ - w_i^-) + \sum_{i \in N_2} (w_i^+ + w_i^-) + w^{\text{tc}} = 1 \tag{E.10}$$

As the budget constraint includes the proportion spent in transaction costs, it is unclear if every portfolio selection model can be solved. Some examples include those which explicitly compares portfolio returns to benchmark returns (e.g. enhanced indexation models) or ratio-based models, in which a variable transformation on the weights is required. We leave the exploration of these possibilities for future work.

Due to combining financial values with proportions, we observed that the Woodside model is prone to numerical inaccuracies. Consider the example used in Section 6.6. In the main experiments reported, we assumed an initial value of \$1 million, and transaction costs of 5 basis points. With this configuration, which is not unusual in financial markets, we observe in the model coefficients and values ranging from 1×10^9, in Constraint (E.3), to $V_i \times 10^{-13}$, in Constraint (E.6). Upon solving several instances, we noticed that minor changes to the CPLEX configurations resulted in very different optimal solutions, well outside the default tolerance levels.

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