An optimisation framework for building realistic portfolios with equities and futures contracts

Appendices

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This document contains references to sections and equations from the main paper.

A First stage formulation for ratio-based models

Several portfolio optimisation models require optimising a ratio of two affine functions, which can be linearised with the standard transformation introduced by Charnes and Cooper [1962]. If $\rho(\boldsymbol{w}) = \frac{\boldsymbol{c}^T\boldsymbol{w}+\alpha}{\boldsymbol{d}^T\boldsymbol{w}+\beta}$, define $t = \frac{1}{\boldsymbol{d}^T\boldsymbol{w}+\beta}$ such that $\rho(\boldsymbol{w}) = \boldsymbol{c}^T\boldsymbol{w}t + \alpha t$. Let $\hat{\boldsymbol{w}} = \boldsymbol{w}t$. Add:

$$t = \frac{1}{\boldsymbol{d}^T\boldsymbol{w}+\alpha} \implies (\boldsymbol{d}^T\boldsymbol{w}+\beta)t = 1 \implies \boldsymbol{d}^T\hat{\boldsymbol{w}} + \beta t = 1$$

If $(\hat{\boldsymbol{w}}^*, t^*)$ solves the linearised problem, $\boldsymbol{w}^* = \hat{\boldsymbol{w}}^*/t^*$ solves the original problem. Since this transformation requires a change of the variables that appear in the numerator, all constraints where these same variables appear must be updated by multiplying them by t. Therefore, most of the exogenous constraints described in Section 4 need to be adapted accordingly, with the notable exception being cardinality Constraints (48)-(51). In this appendix, we summarise the updated constraints.

The generic ratio-based portfolio optimisation model is defined as:

$$\max \quad \boldsymbol{c}^T\hat{\boldsymbol{w}} + t\alpha \tag{A.1}$$

$$\text{subject to} \quad \boldsymbol{d}^T\hat{\boldsymbol{w}} + \beta t = 1 \tag{A.2}$$

$$\sum_{i\in N_1}(\hat{w}_i^+ - \hat{w}_i^-) + \sum_{i\in N_2}(\hat{w}_i^+ + \hat{w}_i^-) = t \tag{A.3}$$

$$\hat{w}_i^+ \le Uz_i^+ \quad \forall i \in N \tag{A.4}$$

$$\hat{w}_i^- \le Uz_i^- \quad \forall i \in N \tag{A.5}$$

$$(17) - (18)$$

$$\hat{w}_i^+, \hat{w}_i^- \ge 0 \quad \forall i \in N \tag{A.6}$$

$$t \ge 0 \tag{A.7}$$

Some of the constraints below become nonlinear when applying the transformation. In order to linearise them, we also add continuous auxiliary variables $\hat{z}_i^+ = z_i^+ t$, $\hat{z}_i^- = z_i^- t$ and the following additional constraints:

$$0 \le \hat{z}_i^+ \le U z_i^+ \tag{A.8}$$
$$\hat{z}_i^+ \le t \tag{A.9}$$
$$t - \hat{z}_i^+ + U z_i^+ \le U \tag{A.10}$$

$$0 \le \hat{z}_i^- \le U z_i^- \tag{A.11}$$
$$\hat{z}_i^- \le t \tag{A.12}$$
$$t - \hat{z}_i^- + U z_i^- \le U \tag{A.13}$$

This linearisation was introduced by Guastaroba et al. [2016], and they ensure that if $z_i^+ = 0$, $\hat{z}_i^+ = 0$ and that if $z_i^+ = 1$, $\hat{z}_i^+ = t$. The same is valid for the corresponding shorting variables.

Constraints (20)-(24), required for shorting limits and linking the $\boldsymbol{w}$ and $\boldsymbol{z}$ variables, are replaced by:

$$t\tau^- \ \le \ \sum_{i \in N_1} \hat{w}_i^- \ \le \ t\tau^+ \tag{A.14}$$
$$\hat{w}_i^+ \ \le \ (1+\tau^+)\hat{z}_i^+ \qquad \forall i \in N \tag{A.15}$$
$$\hat{w}_i^- \ \le \ (1+\tau^+)\hat{z}_i^- \qquad \forall i \in N_2 \tag{A.16}$$
$$\hat{w}_i^- \ \le \ \tau^+ \hat{z}_i^- \qquad \forall i \in N_1 \tag{A.17}$$
$$\Big(t + \sum_{i \in N_1} \hat{w}_i^-\Big)\upsilon^- \ \le \ \sum_{i \in N_2} \hat{w}_i^- \ \le \ \Big(t + \sum_{i \in N_1} \hat{w}_i^-\Big)\upsilon^+ \tag{A.18}$$

Constraints (25)-(28), which add limits to the risk-free assets, are replaced by:

$$\hat{w}_\ell^- \ = \ 0 \tag{A.19}$$
$$\hat{w}_b^+ \ = \ 0 \tag{A.20}$$
$$\Big(t + \sum_{i \in N_1} \hat{w}_i^-\Big)\Big(\max\{0, c_-\}\Big) \le \hat{w}_\ell^+ \ \le \ \Big(t + \sum_{i \in N_1} \hat{w}_i^-\Big)\Big(\max\{0, c_+\}\Big) \tag{A.21}$$
$$\Big(\sum_{i \in N_1} \hat{w}_i^-\Big)\Big(\max\{0, -c_+\}\Big) \le \hat{w}_b^- \ \le \ \Big(\sum_{i \in N_1} \hat{w}_i^-\Big)\Big(\max\{0, -c_-\}\Big) \tag{A.22}$$

The minimum expected return constraints (29) is replaced by:

$$\sum_{i \in N} \mu_i(\hat{w}_i^+ - \hat{w}_i^-) \ge t\bar{\mu} \tag{A.23}$$

With the help of the continuous variables defined above, the absolute bounds and buy-in threshold constraints, which replace Constraints (30)-(31), can be enforced as:

$$l^+ \hat{z}_i^+ \le \hat{w}_i^+ \le u^+ \hat{z}_i^+ \qquad \forall i \in N \setminus \{\ell, b\} \tag{A.24}$$
$$l^- \hat{z}_i^- \le \hat{w}_i^- \le u^- \hat{z}_i^- \qquad \forall i \in N \setminus \{\ell, b\} \tag{A.25}$$

Asset specific bounds, enforced by Constraints (32)-(37), are replaced by:

$$\hat{w}_i^- \le -\Big(\sum_{j\in N_1} \hat{w}_j^-\Big)\varepsilon_i \qquad \text{if } \varepsilon_i < 0 \text{ and } i \in N_1 \tag{A.26}$$

$$\hat{w}_i^- \le -\Big(t + \sum_{j\in N_1} \hat{w}_j^-\Big)\varepsilon_i \qquad \text{if } \varepsilon_i < 0 \text{ and } i \in N_2 \tag{A.27}$$

$$\hat{w}_i^+ \ge \Big(t + \sum_{j\in N_1} \hat{w}_j^-\Big)\varepsilon_i \qquad \text{if } \varepsilon_i > 0 \tag{A.28}$$

$$\hat{w}_i^- \ge -\Big(\sum_{j\in N_1} \hat{w}_j^-\Big)\delta_i \qquad \text{if } \delta_i < 0 \text{ and } i \in N_1 \tag{A.29}$$

$$\hat{w}_i^- \ge -\Big(t + \sum_{j\in N_1} \hat{w}_j^-\Big)\delta_i \qquad \text{if } \delta_i < 0 \text{ and } i \in N_2 \tag{A.30}$$

$$\hat{w}_i^+ \le \Big(t + \sum_{j\in N_1} \hat{w}_j^-\Big)\delta_i \qquad \text{if } \delta_i > 0 \tag{A.31}$$

Class constraints (38)-43) are replaced by:

$$\sum_{i\in Q} \hat{w}_i^- \le -\Big(\sum_{i\in N_1} \hat{w}_i^-\Big)E_Q \qquad \text{if } E_Q < 0 \text{ and } Q \cap N_1 = Q \tag{A.32}$$

$$\sum_{i\in Q} \hat{w}_i^- \le -\Big(t + \sum_{i\in N_1} \hat{w}_i^-\Big)E_Q \qquad \text{if } E_Q < 0 \text{ and } Q \cap N_2 = Q \tag{A.33}$$

$$\sum_{i\in Q} \hat{w}_i^+ \ge \Big(t + \sum_{i\in N_1} \hat{w}_i^-\Big)E_Q \qquad \text{if } E_Q > 0 \tag{A.34}$$

$$\sum_{i\in Q} \hat{w}_i^- \ge -\Big(\sum_{i\in N_1} \hat{w}_i^-\Big)\Delta_Q \qquad \text{if } \Delta_Q < 0 \text{ and } Q \cap N_1 = Q \tag{A.35}$$

$$\sum_{i\in Q} \hat{w}_i^- \ge -\Big(t + \sum_{i\in N_1} \hat{w}_i^-\Big)\Delta_Q \qquad \text{if } \Delta_Q < 0 \text{ and } Q \cap N_2 = Q \tag{A.36}$$

$$\sum_{i\in Q} \hat{w}_i^+ \le \Big(t + \sum_{i\in N_1} \hat{w}_i^-\Big)\Delta_Q \qquad \text{if } \Delta_Q > 0 \tag{A.37}$$

Turnover constraints (44)-(46) are replaced by:

$$t_i \ge W_i t - (\hat{w}_i^+ - \hat{w}_i^-) \qquad \forall i \in N \setminus \{\ell, b\} \tag{A.38}$$

$$t_i \ge (\hat{w}_i^+ - \hat{w}_i^-) - W_i t \qquad \forall i \in N \setminus \{\ell, b\} \tag{A.39}$$

$$\sum_{i\in N} t_i \le \Gamma t \tag{A.40}$$

$$t_i \le \Gamma_i t \qquad \text{for any } i \text{ necessary} \tag{A.41}$$

Finally, for enforcing the 5/10/40 UCITS rule in ratio-based models, we make use of following additional sets of variables:

b_i^+, b_i^-	$\begin{cases} 1 & \text{if } \hat{w}_i^+/t > \zeta, \hat{w}_i^-/t > \zeta \text{ respectively} \\ 0 & \text{otherwise} \end{cases}$
$\hat{b}_i^+, \hat{b}_i^-$	continuous variables such that $\hat{b}_i^+ = b_i^+ t$ and $\hat{b}_i^- = b_i^- t$
$\hat{v}_i^+ \ge 0, \hat{v}_i^- \ge 0$	the maximum between 0 and $\hat{w}_i^+ - \zeta t, \hat{w}_i^- - \zeta t$ respectively.

as well as the following constraints:

$$0 \le \hat{b}_i^+ \le U b_i^+ \tag{A.42}$$
$$\hat{b}_i^+ \le t \tag{A.43}$$
$$t - \hat{b}_i^+ + U b_i^+ \le U \tag{A.44}$$
$$\hat{v}_i^+ \ge \hat{w}_i^+ - \zeta t \qquad \forall i \in N \setminus \ell \tag{A.45}$$
$$\hat{v}_i^+ \le (\iota - \zeta)\hat{b}_i^+ \qquad \forall i \in N \setminus \ell \tag{A.46}$$

$$0 \le \hat{b}_i^- \le U b_i^- \tag{A.47}$$
$$\hat{b}_i^- \le t \tag{A.48}$$
$$t - \hat{b}_i^- - U b_i^- \le U \tag{A.49}$$
$$\hat{v}_i^- \ge \hat{w}_i^- - \zeta t \qquad \forall i \in N \setminus b \tag{A.50}$$
$$\hat{v}_i^- \le (\iota - \zeta)\hat{b}_i^- \qquad \forall i \in N \setminus b \tag{A.51}$$

$$\sum_{i \in N \setminus \ell, b} (\hat{v}_i^+ + \hat{v}_i^-) + (\hat{b}_i^+ + \hat{b}_i^-)\zeta \le \eta t \tag{A.52}$$

B Statistics

In this appendix, we detail the calculations for the statistics presented in tables in Section 6. Let:

$\boldsymbol{P}$	be a series of $0..T$ daily portfolio values, where P_t is the value on day t.
$\boldsymbol{r}$	be a series of $1..T$ daily returns, where r_t is the return on day t calculated as $\frac{P_t - P_{t-1}}{P_{t-1}}$.
$\boldsymbol{r_f}$	be a series of $1..T$ daily risk free returns, where r_{ft} is the risk free rate on day t.
$\boldsymbol{b}$	be a series of $1..T$ daily benchmark returns, where b_t is the benchmark return on day t.

In our experiments, $\boldsymbol{r_f}$ varies over time. To calculate some measures we convert it into a constant daily rate r_f^d and an annual rate r_f^y. To do that we compute:

$$
\begin{aligned}
R_{f0} &= 1 \\
R_{ft} &= R_{t-1}(1 + r_{ft}) && \forall t = 1, \ldots, T \\
r_f^d &= R_T^{1/T} - 1 \\
r_f^y &= (1 + r_f^d)^{252} - 1
\end{aligned}
$$

The following metrics are shown in performance tables:

1. **FV** (Final value):
$$\frac{P_T}{P_0}$$
2. **CAGR** (Compound annual growth rate, in %):
$$Y = \frac{T}{252}$$
$$\text{CAGR} = 100 \times \left(\left(\frac{P_T}{P_0} \right)^{\frac{1}{Y}} - 1 \right)$$

3. **Vol** (annualised volatility):

$$\mu(\boldsymbol{r}) = \frac{1}{T}\sum_{t=1}^{T} r_t$$

$$\sigma(\boldsymbol{r}) = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\big(r_t - \mu(\boldsymbol{r})\big)^2}$$

$$\text{Vol} = \sigma(\boldsymbol{r}) \times \sqrt{252}$$

4. **Sharpe** (annualised Sharpe ratio):

$$\text{Sharpe} = \sqrt{252}\,\frac{\mu(\boldsymbol{r}) - r_f^d}{\sigma(\boldsymbol{r})}$$

5. **Sortino** (annualised Sortino ratio):

$$\sigma^-(\boldsymbol{r}) = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(\min[0, r_t - r_f^d])^2}$$

$$\text{Sortino} = \sqrt{252}\,\frac{\mu(\boldsymbol{r}) - r_f^d}{\sigma^-(\boldsymbol{r})}$$

6. **Omega** (Omega ratio):

$$\text{Omega}_\tau = \frac{\sum_{t=1}^{T}\max\{r_t - \tau, 0\}}{\sum_{t=1}^{T}\max\{\tau - r_t, 0\}}$$

We employ $\tau = 0$.

7. **MDD** (maximum drawdown, in %):

$$\text{MDD} = \max\left(0, 100 \times \max_{0 \le t < u \le T}\frac{P_t - P_u}{P_t}\right)$$

8. **MDR** (maximum days to recovery):

$$\text{MDR} = \max_{0 \le t < u \le T}\{u - t : P_u < P_t \,\wedge\, \nexists v > t, v < u \mid P_v \ge P_t\}$$

If the portfolio values form a monotonic increasing sequence, then $\text{MDR} = 0$.

9. **Beta** (portfolio β in relation to a benchmark):

$$\sigma(\boldsymbol{r}, \boldsymbol{b}) = \frac{1}{T-1}\sum_{t=1}^{T}\big(r_t - \mu(\boldsymbol{r})\big)\big(b_t - \mu(\boldsymbol{b})\big)$$

$$\sigma^2(\boldsymbol{b}) = \frac{1}{T-1}\sum_{t=1}^{T}\big(b_t - \mu(\boldsymbol{b})\big)^2$$

$$\text{Beta} = \frac{\sigma(\boldsymbol{r}, \boldsymbol{b})}{\sigma^2(\boldsymbol{b})}$$

C Woodside-Oriakhi et al. [2013] formulation

In this appendix, we adapt the formulation from Woodside-Oriakhi et al. [2013] to the first stage model of our framework. The formulation is both relative and absolute, that is, it includes both portfolio proportions and units/shares held/purchased/sold. Let D be the limit on the total amount of transaction costs that can be spent and let U_i^b and U_i^s be respectively the upper bounds on the amount of shares/units of asset i to be bought/sold. The updated model requires the following additional decision variables:

x_i	number of lots (if $i \in N^L$) or units (if $i \in \overline{N^{L}}$) of asset i in the rebalanced portfolio,
x_c	amount of cash to be held in the rebalanced portfolio,
$y_i^b \geq 0$	the number of lots (if $i \in N^L$) or units (if $i \in \overline{N^{L}}$) of asset i bought,
$y_i^s \geq 0$	the number of lots (if $i \in N^L$) or units (if $i \in \overline{N^{L}}$) of asset i sold,
α_i^b	1 if we buy any of asset i, 0 otherwise
α_i^s	1 if we sell any of asset i, 0 otherwise
$0 \leq w^{\mathrm{tc}} \leq 1$	the proportion of P consumed by transaction costs.

The original formulation considers fixed transaction costs. Here for simplicity we do not add them.

$$\sum_{i=1}^{N} f_i V_i \left(y_i^b + y_i^s\right) \leq D \tag{C.53}$$

$$x_i = X_i + y_i^b - y_i^s \qquad i \in N \setminus \ell, b \tag{C.54}$$

$$x_c + \sum_{i=1}^{N} \frac{V_i x_i}{L_i} = P - \sum_{i=1}^{N} f_i V_i \left(y_i^b + y_i^s\right) \tag{C.55}$$

$$w_\ell^+ - w_b^- = \frac{x_c}{P} \tag{C.56}$$

$$w_i^+ - w_i^- = \frac{V_i x_i}{L_i P} \qquad i \in N \setminus \ell, b \tag{C.57}$$

$$w^{\mathrm{tc}} = \frac{\sum_{i=1}^{N} f_i V_i \left(y_i^b + y_i^s\right)}{P} \tag{C.58}$$

$$y_i^b \leq U_i^b \alpha_i^b \tag{C.59}$$

$$y_i^s \leq U_i^b \alpha_i^s \tag{C.60}$$

$$\alpha_i^b + \alpha_i^s \leq 1 \tag{C.61}$$

Woodside-Oriakhi et al. [2013] consider only nonnegative positions, imposing that $x_i \geq 0$. Here we let x_i be a free variable, and the shorting exposure is controlled by the w_i^- variables. Constraint (C.53) define the upper limit on transaction costs. Constraints (C.54) define the amounts to be held as a function of the lots/units currently held. Constraint (C.55) define the discounted portfolio value. Constraints (C.56)-(C.57) link the holdings to proportions, and Constraint (C.58) defines the proportion of P spent on transaction costs. Constraints (C.59)-(C.61) forbid both buying and selling the same asset.

Since the weights are defined according to the non-discounted portfolio value, it is also necessary to replace Eq. (14) (budget constraint) with:

$$\sum_{i \in N_1} (w_i^+ - w_i^-) + \sum_{i \in N_2} (w_i^+ + w_i^-) + w^{\mathrm{tc}} = 1 \tag{C.62}$$

As the budget constraint includes the proportion spent in transaction costs, it is unclear if every portfolio selection model can be solved. Some examples include those which explicitly compares portfolio returns to benchmark returns (e.g. enhanced indexation models) or ratio-based models, in which a variable transformation on the weights is required. We leave the exploration of these possibilities for future work.

Due to combining financial values with proportions, we observed that the Woodside model is prone to numerical inaccuracies. Consider the example used in Section 6.7. In the main experiments reported, we assumed an initial value of \$1 million, and transaction costs of 5 basis points. With this configuration, which is not unusual in financial markets, we observe in the model coefficients and values ranging from 1×10^9, in Constraint (C.55), to $V_i \times 10^{-13}$, in Constraint (C.58). Upon solving several instances, we noticed that minor changes to the CPLEX configurations resulted in very different optimal solutions, well outside the default tolerance levels.

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