Herbrand-satisfiability of a quantified set-theoretic fragment*

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Abstract. In the last decades, several fragments of set theory have been studied in the context of *Computable Set Theory*. In general, the semantics of set-theoretic languages differs from the canonical first-order semantics in that the interpretation domain of set-theoretic terms is fixed to a given universe of sets, as for instance the von Neumann standard cumulative hierarchy of sets, i.e., the class consisting of all the pure sets. Because of this, theoretical results and various machinery developed in the context of first-order logic cannot be easily adapted to work also in the set-theoretic realm. Recently, quantified fragments of set theory which allow one to explicitly handle ordered pairs have been studied for decidability purposes, in view of applications in the field of knowledge representation. Among other results, a NexpTime decision procedure for satisfiability of formulae in one of these fragments, $\forall_{\mathbf{0}}^{\pi}$, has been provided. In this paper we exploit the main features of such a decision procedure to reduce the satisfiability problem for the fragment $\forall_{\mathbf{0}}^{\pi}$ to the problem of Herbrand satisfiability for a first-order language extending it. In addition, it turns out that such a reduction maps formulae of the *Disjunctive Datalog* subset of $\forall_{\mathbf{0}}^{\pi}$ into Disjunctive Datalog formulae.[1]

1 Introduction

The quantified fragment of set theory $\forall_{\mathbf{0}}^{\pi}$ (see [5]) allows the explicit manipulation of ordered pairs. It is expressive enough to include a relevant amount of set-theoretic constructs, in particular map-related ones: in fact, it is characterized by the presence of terms of the form $[\cdot,\cdot]$ (ordered pair) and $\bar{\pi}(\cdot)$ (collection of the non-pair members of its argument). This language has applications in the field of *knowledge representation*. In fact, a large amount of *description logic* constructs are expressible in it. In particular, the very expressive description logic $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$ can be expressed in a fragment of $\forall_{\mathbf{0}}^{\pi}$ that has an NP-complete decision problem, in contrast with the description logic $\mathcal{SROIQ}$ (cf. [13]), which

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[1] $\mathsf{DATALOG}^{\vee,\neg}$-formulae are often referred to also as *programs*, so that the expressions "$\mathsf{DATALOG}^{\vee,\neg}$-formulae" and "$\mathsf{DATALOG}^{\vee,\neg}$-programs" must be regarded as synonyms.

underpins the current standard language for Semantic Web OWL2[2] and whose decision problem is N2ExpTime-complete (see [15]). Despite of these desirable properties of the language $\forall_{\mathbf{0}}^{\pi}$, no decision procedure for it has been implemented yet.

The semantics of set-theoretic languages (see [16] for an introduction to set theory, and [3, 7, 18] for an overview on decidable fragments of set theory) differs from the *canonical* first-order logic semantics (see [10]) in that, in general, the interpretation domain of set-theoretic terms is based on the *von Neumann standard cumulative hierarchy of (pure) sets* $\mathcal{V} = \bigcup_{\gamma \in On} \mathcal{V}_\gamma$, where

$$\begin{aligned} \mathcal{V}_{\gamma+1} &= \mathcal{P}(\mathcal{V}_\gamma)\,, \quad \text{for each ordinal } \gamma \\ \mathcal{V}_\lambda &= \textstyle\bigcup_{\mu<\lambda} \mathcal{V}_\mu\,, \text{ for each limit ordinal } \lambda \ \ (\text{thus, } \mathcal{V}_0 = \emptyset) \end{aligned}$$

(with $\mathcal{P}(\cdot)$ the powerset operator and *On* the class of all ordinals). Because of this, results and techniques developed in the context of first-order logic are not easily reusable in a set-theoretic domain.

In this paper we show that these difficulties can be circumvented for the fragment $\forall_{\mathbf{0}}^{\pi}$ by *encoding* some axioms of set theory (namely regularity and a weak form of extensionality) as sentences in a first-order language extending $\forall_{\mathbf{0}}^{\pi}$. More specifically, we will prove that, for every $\forall_{\mathbf{0}}^{\pi}$-sentence φ, one can construct in polynomial time a corresponding formula χ_φ such that φ admits a set-theoretic model if and only if $\chi_\varphi \wedge \varphi$ admits a Herbrand model (cf. [12]), when set-theoretic predicates in $\chi_\varphi \wedge \varphi$ are regarded as uninterpreted predicates. It turns out that these formulae χ_φ can be considered as $\mathsf{DATALOG}^{\vee,\neg}$-formulae. As a consequence, the reduction we are going to discuss can be seen as a reduction from $\forall_{\mathbf{0},\mathbf{D}\vee}^{\pi}$ to $\mathsf{DATALOG}^{\vee,\neg}$, where $\forall_{\mathbf{0},\mathbf{D}\vee}^{\pi}$ consists of the formulae in $\forall_{\mathbf{0}}^{\pi}$ which satisfy the syntactic constraints required to be regarded also as $\mathsf{DATALOG}^{\vee,\neg}$-formuale.

We recall that $\mathsf{DATALOG}^{\vee,\neg}$ (read *Disjunctive Datalog*) extends Datalog by allowing disjunctions in the head of rules. Such additional feature considerably increments the expressive power of the basic language and makes $\mathsf{DATALOG}^{\vee,\neg}$ suitable to be used in several applications such as reasoning for ontologies of the semantic web [14] and planning problems [8]. $\mathsf{DATALOG}^{\vee,\neg}$ has been introduced in [9], where its formal semantics, its expressive power, and the complexity of its reasoning problems have been treated in depth. Decidability and complexity of Datalog extensions with various combinations of disjunction and negation have been studied in [1, 11]. Also, optimization strategies have been provided for algorithms devised in this context, and a considerable amount of academic and commercial software implementing these algorithms is available (see for example [2]). Thus, the reduction we present in this paper allows one to reuse the machinery available for $\mathsf{DATALOG}^{\vee,\neg}$ in the implementation of an optimized reasoning engine for the language $\forall_{\mathbf{0},\mathbf{D}\vee}^{\pi}$ and, consequently, for the description logic $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$ mentioned above, as all the constructs in $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$ are expressible in $\forall_{\mathbf{0},\mathbf{D}\vee}^{\pi}$.

The link between set-theoretic languages and logic programming we start to investigate here is of a certain interest to projects related with the production

[2] http://www.w3.org/TR/owl2-primer/

and usage of *open data* such as PRISMA*: on one hand, it provides a way to implement with small efforts a solid and efficient reasoning and query engine for the very expressive representation language $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$; on the other hand, it allows one to implement typical logic-programming tasks for $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$ knowledge bases such as, for example, answer-set programming and non-monotonic reasoning, which may be of some interest for public utility applications.

The rest of the paper is organized as follows. In Section 2 we review some notions and definitions from first-order logic, including some results by Herbrand and the definition of the Disjunctive Datalog fragment of first-order logic. Then, in Section 3, after recalling the syntax and semantics of the fragment of set theory $\forall_{\mathbf{0}}^{\pi}$, we briefly review a decision procedure for $\forall_{\mathbf{0}}^{\pi}$-formulae, together with some useful properties. A polynomial-time reduction of the satisfiability problem for $\forall_{\mathbf{0}}^{\pi}$-formulae to the Herbrand satisfiability problem for first-order formulae is described in Section 4. Finally, we draw our conclusions and provide some hints for future research in Section 5.

2 First-order logic

We briefly review some notations and definitions from first-order logic which will be used throughout the paper.

We shall denote with $\mathit{Vars} = \{x, y, z, \ldots\}$, $\mathit{Consts} = \{a, b, c, \ldots\}$, $\mathit{Functs} = \{f, g, h, \ldots\}$, and $\mathit{Preds} = \{P, Q, R, \ldots\}$ four denumerably infinite and pairwise disjoint collections of *variable*, *constant*, *function*, and *predicate* symbols, respectively. In addition, a positive *arity* $ar(\nu) \in \mathbb{N}^+$ is associated to each function and predicate symbol $\nu \in \mathit{Functs} \cup \mathit{Preds}$.

A *first-order language* $\mathcal{L}$ is characterized by a triple $(\mathit{Preds}_{\mathcal{L}}, \mathit{Functs}_{\mathcal{L}}, \mathit{Consts}_{\mathcal{L}})$, where $\mathit{Preds}_{\mathcal{L}} \subseteq \mathit{Preds}$, $\mathit{Functs}_{\mathcal{L}} \subseteq \mathit{Functs}$ and $\mathit{Consts}_{\mathcal{L}} \subseteq \mathit{Consts}$.

The collection $\mathit{Terms}_{\mathcal{L}}$ of the *terms* of $\mathcal{L}$ is defined as the smallest set of expressions such that:

- $\mathit{Vars} \cup \mathit{Consts}_{\mathcal{L}} \subseteq \mathit{Terms}_{\mathcal{L}}$;
- $f(t_1, \ldots, t_n) \in \mathit{Terms}_{\mathcal{L}}$, for all $f \in \mathit{Functs}_{\mathcal{L}}$ and $t_1, \ldots, t_n \in \mathit{Terms}_{\mathcal{L}}$ (where $n = ar(f)$).

A term of $\mathcal{L}$ is *closed* if it involves no variable. We denote by $\mathit{ClTerms}_{\mathcal{L}}$ the set of the closed terms of $\mathcal{L}$.

The collection $\mathit{Forms}_{\mathcal{L}}$ of the *formulae* of $\mathcal{L}$ is the smallest set of expressions of the following forms:

- *atomic formulae*: $P(t_1, \ldots, t_n) \in \mathit{Forms}_{\mathcal{L}}$, for all $P \in \mathit{Preds}_{\mathcal{L}}$ and $t_1, \ldots, t_n \in \mathit{Terms}_{\mathcal{L}}$ (where $n = ar(P)$);
- *universally and existentially quantified formulae*: $(\forall x)\varphi, (\exists x)\varphi \in \mathit{Forms}_{\mathcal{L}}$, for every $\varphi \in \mathit{Forms}_{\mathcal{L}}$ (where $x \in \mathit{Vars}$);
- *propositional formulae*: $\neg\varphi, \varphi \wedge \psi, \varphi \vee \psi \in \mathit{Forms}_{\mathcal{L}}$, for all $\varphi, \psi \in \mathit{Forms}_{\mathcal{L}}$.[3]

[3] To simplify the notation, we shall also make use as abbreviations of the propositional connectives $\rightarrow$ and $\leftrightarrow$. In particular, $\varphi \rightarrow \psi$ will stand for $\neg\varphi \vee \psi$, whereas $\varphi \leftrightarrow \psi$

Next we recall some further definitions which are particularly useful to our purposes:

- a *literal* of $\mathcal{L}$ is an atomic formula $P(t_1, \ldots, t_n)$ or its negation $\neg P(t_1, \ldots, t_n)$;
- a *quantifier-free formula* of $\mathcal{L}$ is a propositional combination of atomic formulae of $\mathcal{L}$;
- a *prenex formula* of $\mathcal{L}$ is a formula of the form $(\mathcal{Q}_1 x_1) \ldots (\mathcal{Q}_n x_n)\psi$, where ψ is a quantifier-free formula of $\mathcal{L}$ and $\mathcal{Q}_i \in \{\forall, \exists\}$, for $i = 1, \ldots, n$;
- a *universally quantified prenex formula* of $\mathcal{L}$ is a prenex formula whose quantifiers are all universal (i.e., a formula of the form $(\forall x_1) \ldots (\forall x_n)\psi$, with ψ quantifier-free);
- a quantifier occurrence $(\mathcal{Q}x)$ in a formula φ (involving only primitive propositional connectives) is *essentially existential* if either it has the form $(\exists x)$ and it occurs within the scope of an even number of the negation connective $\neg$, or it has the form $(\forall x)$ and it occurs in the scope of an odd number of the negation connective $\neg$. Otherwise it is *essentially universal*. For instance, all four quantifier occurrences in $\neg(\exists x)(\exists y)P(x, y) \vee (\forall x)(\forall y)P(x, y)$ are essentially universal, whereas all four quantifier occurrences in $\neg(\forall x)(\forall y)P(x, y) \vee (\exists x)(\exists y)P(x, y)$ are essentially existential;
- an occurrence of a variable x in a formula φ is *free* if it is not contained in any quantified subformula of φ of the form $(\forall x)\psi$ or $(\exists x)\psi$;
- we denote with $\varphi[x \to y]$ the formula obtained from φ by replacing every free occurrence of x with y;
- a formula is *closed* (*sentence*) if it contains no free variable;
- a formula is *ground* if it contains no variable.[4]

In the rest of the paper we shall sometimes abbreviate quantifier prefixes as $(\forall x_1) \ldots (\forall x_n)$ by $(\forall x_1, \ldots, x_n)$. Notice that quantifier-free formulae can be considered as universally quantified prenex formulae with an empty quantifier prefix. Given a formula φ, we denote with $\mathit{Preds}(\varphi)$, $\mathit{Functs}(\varphi)$, $\mathit{Consts}(\varphi)$, and $\mathit{Vars}(\varphi)$ the sets of the predicate, function, constant, and variable symbols occurring in φ, respectively. Similar notations will also be used with sets of formulae.

First-order *semantics* is given in terms of *interpretations*. Let $\mathcal{L} = (\mathit{Preds}_{\mathcal{L}}, \mathit{Functs}_{\mathcal{L}}, \mathit{Consts}_{\mathcal{L}})$ be a first-order language $\mathcal{L}$. An *interpretation* $\mathbf{I}$ for $\mathcal{L}$ is a pair $(D^{\mathbf{I}}, \cdot^{\mathbf{I}})$ where $D^{\mathbf{I}}$, the *interpretation domain*, is a generic nonempty set (i.e., a set not necessarily in the von Neumann hierarchy), and $\cdot^{\mathbf{I}}$ is a map which associates

- an element $a^{\mathbf{I}}$ in $D^{\mathbf{I}}$ to each constant a of $\mathit{Consts}_{\mathcal{L}}$,
- an n-ary *relation* $P^{\mathbf{I}}$ over $D^{\mathbf{I}}$ (i.e., a set of n-tuples of elements of $D^{\mathbf{I}}$) to each n-ary predicate symbol P in $\mathit{Preds}_{\mathcal{L}}$, with $n \in \mathbb{N}$, and
- an n-ary *function* $f^{\mathbf{I}}$ over $D^{\mathbf{I}}$ (i.e., a map associating an n-tuple of elements of $D^{\mathbf{I}}$ to each element in $D^{\mathbf{I}}$) to each n-ary function symbol f in $\mathit{Functs}_{\mathcal{L}}$, with $n \in \mathbb{N}$.

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will stand for $(\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$. Therefore we shall refer to $\neg$, $\wedge$, and $\vee$ as *primitive propositional connectives*.

[4] Plainly, any ground formula is closed.

In addition, given an interpretation $\mathbf{I} = (D^{\mathbf{I}}, \cdot^{\mathbf{I}})$ for $\mathcal{L}$, an *assignment* A in $\mathbf{I}$ is a map associating an element x^A in $D^{\mathbf{I}}$ to each variable $x \in \mathit{Vars}$. Given two assignments A, A' in $\mathbf{I}$ and a variable $x \in \mathit{Vars}$, A' is said to be an *x-variant* of A iff A and A' coincide on all variables but x, i.e., iff $y^A = y^{A'}$ for all $y \in \mathit{Vars}$ such that $y \neq x$.

A pair $(\mathbf{I}, A)$, where $\mathbf{I}$ is an interpretation for $\mathcal{L}$ and A is an assignment in $\mathbf{I}$, associates to each term of $\mathcal{L}$ a *value* in $D^{\mathbf{I}}$ as follows:

– $x^{\mathbf{I},A} =_{\mathrm{Def}} x^A$, for all $x \in \mathit{Vars}$;
– $a^{\mathbf{I},A} =_{\mathrm{Def}} a^{\mathbf{I}}$, for all $a \in \mathit{Consts}_{\mathcal{L}}$;
– $f(t_1, \ldots, t_n)^{\mathbf{I},A} =_{\mathrm{Def}} f^{\mathbf{I}}(t_1^{\mathbf{I},A}, \ldots, t_n^{\mathbf{I},A})$, for all $f \in \mathit{Functs}_{\mathcal{L}}$ and $t_1, \ldots, t_n \in \mathit{Terms}_{\mathcal{L}}$, where $n = ar(f)$.

Moreover, the pair $(\mathbf{I}, A)$ *evaluates* each formula φ of $\mathcal{L}$ to a truth value **true** or **false** according to the following rules:

– $P(t_1, \ldots, t_n)^{\mathbf{I},A} = \mathbf{true}$ iff $\left[t_1^{\mathbf{I},A}, \ldots, t_n^{\mathbf{I},A}\right] \in P^{\mathbf{I}}$, for every atomic formula $P(t_1, \ldots, t_n)$ of $\mathcal{L}$;
– $((\forall x)\varphi)^{\mathbf{I},A} = \mathbf{true}$ iff $\varphi^{\mathbf{I},A'} = \mathbf{true}$, for *every* x-variant A' of A, for every formula φ of $\mathcal{L}$ and variable $x \in \mathit{Vars}$;
– $((\exists x)\varphi)^{\mathbf{I},A} = \mathbf{true}$ iff $\varphi^{\mathbf{I},A'} = \mathbf{true}$, for *some* x-variant A' of A, for every formula φ of $\mathcal{L}$ and variable $x \in \mathit{Vars}$;
– propositional combinations of formulae of $\mathcal{L}$ are recursively evaluated according to the usual semantics of propositional logic.

We shall use the notation $(\mathbf{I}, A) \models_{\mathsf{FO}} \varphi$ to indicate that the pair $(\mathbf{I}, A)$ evaluates the formula φ to **true**, whereas we shall write $(\mathbf{I}, A) \not\models_{\mathsf{FO}} \varphi$ to indicate that $(\mathbf{I}, A)$ evaluates φ to **false**.

Notice that, when evaluating a closed formula φ, the assignment plays no role and may be omitted, since $\varphi^{\mathbf{I},A} = \varphi^{\mathbf{I},B}$, for every interpretation $\mathbf{I}$ and assignments A, B in $\mathbf{I}$. In such a case, we shall simply write $\varphi^{\mathbf{I}}$ and $\mathbf{I} \models_{\mathsf{FO}} \varphi$ in place of $\varphi^{\mathbf{I},A}$ and $(\mathbf{I}, A) \models_{\mathsf{FO}} \varphi$, respectively, for any assignment A.

A formula φ of $\mathcal{L}$ is said to be *satisfiable* if and only if $(\mathbf{I}, A) \models_{\mathsf{FO}} \varphi$ for *some* interpretation $\mathbf{I}$ for $\mathcal{L}$ and assignment A in $\mathbf{I}$. In addition, φ is said to be *valid* if and only if $(\mathbf{I}, A) \models_{\mathsf{FO}} \varphi$ for *all* the interpretations $\mathbf{I}$ for $\mathcal{L}$ and assignments A in $\mathbf{I}$. Two formulae φ and ψ of $\mathcal{L}$ are said to be *equivalent* (and we write $\varphi \equiv \psi$) iff $\varphi^{\mathbf{I},A} = \psi^{\mathbf{I},A}$, for all interpretations $\mathbf{I}$ for $\mathcal{L}$ and assignments A in $\mathbf{I}$.

We conclude this section by recalling the notion of *restricted universal quantifiers*. These are quantifiers of the form

$$(\forall x_1, \ldots, x_n | P(y_1, \ldots, y_m)) \qquad \text{and} \qquad (\exists x_1, \ldots, x_n | P(y_1, \ldots, y_m)),$$

with $x_1, \ldots, x_n, y_1, \ldots, y_m \in \mathit{Vars}$ such that $\{y_1, \ldots, y_m\} \subseteq \{x_1, \ldots, x_n\}$, and where P is a predicate symbol in $\mathit{Preds}_{\mathcal{L}}$ of arity m. The intended meaning of such quantifiers is that quantification is restricted to all the $x_1, \ldots, x_n$ such that the predicate $P(y_1, \ldots, y_m)$ holds. Plainly, restricted universal quantifiers can be expressed by standard quantifiers since we have:

$$\begin{aligned}
(\forall x_1, \ldots, x_n | P(y_1, \ldots, y_m))\psi &\equiv (\forall x_1, \ldots, x_n)(P(y_1, \ldots, y_m) \rightarrow \psi) \\
(\exists x_1, \ldots, x_n | P(y_1, \ldots, y_m))\psi &\equiv (\exists x_1, \ldots, x_n)(P(y_1, \ldots, y_m) \wedge \psi)
\end{aligned}$$

for any formula ψ of $\mathcal{L}$.

2.1 Herbrand logic

A first-order interpretation $\mathcal{H} = (D^{\mathcal{H}}, \cdot^{\mathcal{H}})$ for a language $\mathcal{L}$ is said to be a *Herbrand interpretation* for $\mathcal{L}$ if

- $D^{\mathcal{H}} = \mathit{ClTerms}_{\mathcal{L}}$, and
- $t^{\mathcal{H}} = t$, for all $t \in \mathit{ClTerms}_{\mathcal{L}}$.

A formula φ of $\mathcal{L}$ is said to be *Herbrand-satisfiable* iff $(\mathcal{H}, A) \models_{\mathsf{FO}} \varphi$, for some Herbrand interpretation $\mathcal{H}$ for $\mathcal{L}$ and assignment A in $\mathcal{H}$.

The following theorem is a fundamental result of first-order logic, reported here from [10, Theorem 5.9.4] in a simplified form:

Theorem 1. *Let $\mathcal{L}$ be a first-order language such that $\mathit{Consts}_{\mathcal{L}} = \mathit{Consts}$, and let φ be a sentence of $\mathcal{L}$. Then φ is satisfiable if and only if it is Herbrand-satisfiable.* □

This result can be further refined for specific classes of formulae. We shall focus our attention on function-free universally quantified prenex formulae.

We begin by recalling some useful definitions. Let c be a fixed constant symbol. Given a sentence φ in a first-order language $\mathcal{L}$, the *extended set of constant symbols* occurring in φ is defined as:

$$\mathit{Consts}^*(\varphi) =_{\mathrm{Def}} \begin{cases} \mathit{Consts}(\varphi) & \text{if } \mathit{Consts}(\varphi) \neq \emptyset \\ \{c\} & \text{otherwise.} \end{cases}$$

Then the *Herbrand universe* of φ is the set of terms which can be constructed from the extended set of constant symbols and the function symbols occurring in φ. More precisely, it is the minimal set $\mathtt{U}$ such that

- $\mathit{Consts}^*(\varphi) \subseteq \mathtt{U}$;
- $f(t_1, \ldots, t_n) \in \mathtt{U}$, for all $f \in \mathit{Functs}(\varphi)$ and $t_1, \ldots, t_n \in \mathtt{U}$, where $ar(f) = n$.

Observe that if no function symbol occurs in φ, then $\mathtt{U}$ is finite and coincides with the extended set of constant symbols of φ.

Every nonempty finite subset of $\mathtt{U}$ is said to be a *Herbrand domain* of φ.

Definition 1. *Let $\mathcal{L}$ be a first-order language, and let $D = \{t_1, \ldots, t_n\}$ be a nonempty finite set of closed terms of $\mathcal{L}$. For any formula χ of $\mathcal{L}$, the* Herbrand expansion $\mathit{Exp}(\chi, D)$ *of χ over D is the formula of $\mathcal{L}$ recursively defined as follows:*

$$\begin{aligned}
\mathit{Exp}(\mathcal{A}, D) &=_{\mathit{Def}} \mathcal{A} && \textit{(if } \mathcal{A} \textit{ is atomic)} \\
\mathit{Exp}(\neg\varphi, D) &=_{\mathit{Def}} \neg \mathit{Exp}(\varphi, D) \\
\mathit{Exp}(\varphi \oplus \psi, D) &=_{\mathit{Def}} \mathit{Exp}(\varphi, D) \oplus \mathit{Exp}(\psi, D) \\
\mathit{Exp}((\forall x)\varphi, D) &=_{\mathit{Def}} \mathit{Exp}(\varphi[x \to t_1], D) \wedge \ldots \wedge \mathit{Exp}(\varphi[x \to t_n], D) \\
\mathit{Exp}((\exists x)\varphi, D) &=_{\mathit{Def}} \mathit{Exp}(\varphi[x \to t_1], D) \vee \ldots \vee \mathit{Exp}(\varphi[x \to t_n], D),
\end{aligned}$$

where $\oplus \in \{\wedge, \vee\}$. □

Herbrand expansions enjoy some interesting properties. Indeed, if D is a Herbrand domain for a sentence φ, then all the terms occurring in the expansion $Exp(\varphi, D)$ of φ over D are in the Herbrand universe of φ and, more important, $Exp(\varphi, D)$ is a quantifier-free sentence. Thus, in such a case, $Exp(\varphi, D)$ can be regarded as a propositional formula, whose propositional variables are the atomic formulae occurring in it.

The following useful technical result on Herbrand expansions can be proved by structural induction.

Lemma 1. *Let φ be a sentence of $\mathcal{L}$ and let D, D' be Herbrand domains of φ such that $D \subseteq D'$. Then*

(a) if all quantifiers in φ are essentially existential, then $Exp(\varphi, D) \rightarrow Exp(\varphi, D')$ is valid;
(b) if all quantifiers in φ are essentially universal, then $Exp(\varphi, D') \rightarrow Exp(\varphi, D)$ is valid. $\square$

Herbrand expansions are a useful tool to reduce the validity problem of first-order sentences to a (potentially infinite) set of propositional problems, as shown in the *Herbrand's Theorem* (Theorem 8.6.5 in [10]) reported below. In the theorem statement, the notion of *Skolemization* is used. The interested readers can refer to [17] for a complete discussion about such technique. For our purposes, it is enough to recall that Skolemization is a process which allows one to eliminate existential quantifiers from a prenex sentence by means of terms involving *fresh* function symbols, in such a way that the resulting sentence is satisfiable if and only if so is the original one. When no existential quantifier occurs in a prenex sentence φ, the only *Skolemized version* of φ is φ itself. Skolemization can be easily generalized also to sentences φ which are not in prenex form. In such a case, it allows one to eliminate from φ all quantifiers that are *essentially* existential, by means of suitable terms involving newly introduced function symbols. As in the case of prenex sentences, it turns out that when no essentially existential quantifier occurs in a sentence φ, the only Skolemized version of φ is φ itself. In addition, every Skolemized version of a sentence of the form $\neg\varphi$ has the form $\neg\psi$, for some sentence ψ.

Theorem 2 (Herbrand's Theorem). *A sentence φ is valid if and only if for each Skolemized version $\neg\psi$ of $\neg\varphi$ there exists a Herbrand domain D of ψ such that $Exp(\psi, D)$ is a tautology.* $\square$

The decidability of the satisfiability problem for finite conjunctions of function-free universally quantified prenex sentences, and more in general for function-free sentences involving no essentially existential quantifier, follows easily from Herbrand's Theorem, as clarified in the following lemma.

Lemma 2. *Let φ be a sentence of $\mathcal{L}$ involving no essentially existential quantifier and no function symbol, and let $\mathtt{U}$ be the (finite) Herbrand universe of φ. Then φ is satisfiable if and only if $Exp(\varphi, \mathtt{U})$ is (propositionally) satisfiable. Hence, the satisfiability problem for function-free sentences of $\mathcal{L}$ involving no essentially existential quantifier is decidable.*

Proof. Let φ be a sentence of $\mathcal{L}$ as in the hypotheses. Plainly, φ if satisfiable if and only if $\neg\varphi$ is not valid. Thus, from Theorem 2, φ is satisfiable if and only if for every Skolemized version $\neg\psi$ of $\neg\neg\varphi$ and every Herbrand domain D of ψ, the sentence $Exp(\psi, D)$ is not a tautology. But $Exp(\psi, D)$ is not a tautology if and only if $\neg Exp(\psi, D)$ is satisfiable, hence if and only if $Exp(\neg\psi, D)$ is satisfiable. In addition, since, just like φ, the sentence $\neg\neg\varphi$ involves no essentially existential quantifier, the sole Skolemized version of $\neg\neg\varphi$ is $\neg\neg\varphi$ itself. Thus, we can conclude that φ is satisfiable if and only if $Exp(\neg\neg\varphi, D)$ is satisfiable for every Herbrand domain D of $\neg\varphi$, and therefore if and only if $Exp(\varphi, D)$ is satisfiable for every Herbrand domain D of φ. Finally, since by Lemma 1(b) $Exp(\varphi, \mathtt{U}) \rightarrow Exp(\varphi, D)$ is valid for every Herbrand domain D of φ (where $\mathtt{U} = Consts^*(\varphi)$ is the finite Herbrand universe of φ), we have that φ is satisfiable if and only if $Exp(\varphi, \mathtt{U})$ is satisfiable.

From the first part of the proof, it follows that the satisfiability problem for function-free sentences of $\mathcal{L}$ involving no essentially existential quantifier can be effectively reduced to the satisfiability problem for propositional formulae, and therefore it is decidable. □

2.2 Disjunctive Datalog

We close the section by briefly reviewing the $\mathsf{DATALOG}^{\vee,\neg}$ first-order fragment of $\mathcal{L}$. A $\mathsf{DATALOG}^{\vee,\neg}$*-formula* is a finite conjunction of *rules*, i.e., closed formulae of the following form

$$(\forall x_1) \dots (\forall x_n)(\varphi \rightarrow \psi) ,$$

where φ (the rule *body*) is a conjunction of literals, ψ (the rule *head*) is a disjunction of literals, and $Vars(\psi) \subseteq Vars(\varphi)$ (*safety condition*). *Facts* are special ground rules whose body is valid. For this kind of rules, one may omit to indicate the rule body. Thus facts can just be regarded as disjunctions of ground literals. They are used to express facts about real world items, such as for example $childOf(Alice, Bob)$, $isMale(Bob)$, etc.

Finally, we observe that restricted universal quantifiers can easily be embedded in $\mathsf{DATALOG}^{\vee,\neg}$-rules and -formulae. Indeed, let us consider a sentence of the form

$$(\forall x_1, \dots, x_n | P(y_1, \dots, y_m))(\varphi \rightarrow \psi) , \tag{1}$$

where $\{y_1, \dots, y_m\} \subseteq \{x_1, \dots, x_n\}$, φ is a conjunction of literals, ψ is a disjunction of literals, $Vars(\psi) \subseteq Vars(\varphi) \cup \{y_1, \dots, y_m\}$, and P is a predicate symbol of arity m. Then, as remarked earlier, (1) is equivalent to the sentence

$$(\forall x_1, \dots, x_n)(P(y_1, \dots, y_m) \rightarrow (\varphi \rightarrow \psi)),$$

which, in its turn, is equivalent to the sentence

$$(\forall x_1, \dots, x_n)((P(y_1, \dots, y_m) \wedge \varphi) \rightarrow \psi). \tag{2}$$

Plainly, (2) is a (standard) $\mathsf{DATALOG}^{\vee,\neg}$-rule, with body $P(y_1, \dots, y_m) \wedge \varphi$.

3 The language $\forall_{\mathbf{0}}^{\pi}$

We recall the syntax and semantics of the set-theoretic language $\forall_{\mathbf{0}}^{\pi}$, whose decision problem has been studied in [5]. *Atomic* $\forall_{\mathbf{0}}^{\pi}$*-formulae* are of the following types

$$\nu \in \bar{\pi}(\mu), \quad [\nu, \nu'] \in \mu, \quad \nu = \mu \tag{3}$$

with $\nu, \nu', \mu \in \mathit{Vars} \cup \mathit{Consts}$. Intuitively, a clause of type $\nu \in \bar{\pi}(\mu)$ expresses that ν is a *non-pair* member of μ, whereas a clause of type $[\nu, \nu'] \in \mu$ expresses that the pair $[\nu, \nu']$ belongs to μ. Atomic $\forall_{\mathbf{0}}^{\pi}$-formulae and their negations are called $\forall_{\mathbf{0}}^{\pi}$*-literals*. *Quantifier-free* $\forall_{\mathbf{0}}^{\pi}$*-formulae* are propositional combinations of atomic $\forall_{\mathbf{0}}^{\pi}$-formulae. *Simple-prenex* $\forall_{\mathbf{0}}^{\pi}$*-formulae* have the following form:

$$(\forall x_1 \in \bar{\pi}(a_1)) \dots (\forall x_n \in \bar{\pi}(a_n))(\forall\, [y_1, z_1] \in b_1) \dots (\forall\, [y_m, z_m] \in b_m)\psi, \tag{4}$$

where $n, m \geq 0$, $x_i \in \mathit{Vars}$ and $a_i \in \mathit{Consts}$, for $1 \leq i \leq n$, $y_j, z_j \in \mathit{Vars}$ and $b_j \in \mathit{Consts}$, for $1 \leq j \leq m$, and ψ is a quantifier-free $\forall_{\mathbf{0}}^{\pi}$-formula. The constants $a_1, \dots, a_n, b_1, \dots b_m$ are the *domain constants* of (4). Finally, $\forall_{\mathbf{0}}^{\pi}$*-formulae* are finite conjunctions of simple-prenex $\forall_{\mathbf{0}}^{\pi}$-formulae. In this paper we are mainly interested in $\forall_{\mathbf{0}}^{\pi}$*-sentences*, namely finite conjunctions of *closed* simple-prenex $\forall_{\mathbf{0}}^{\pi}$-formulae. The collection of the domain constants of a $\forall_{\mathbf{0}}^{\pi}$-sentence φ, which we denote with $\mathit{Consts}_D(\varphi)$, consists of the domain constants of all of its conjuncts.

$\forall_{\mathbf{0}}^{\pi}$-sentences can be regarded as first-order formulae in the language $\mathcal{L} = (\mathit{Preds}_{\mathcal{L}}, \mathit{Functs}_{\mathcal{L}}, \mathit{Consts}_{\mathcal{L}})$, with $\mathit{Preds}_{\mathcal{L}} = \{=, \in\}$, $\mathit{Functs}_{\mathcal{L}} = \{[\cdot,\cdot], \bar{\pi}(\cdot)\}$, and $\mathit{Consts}_{\mathcal{L}} = \mathit{Consts}$. However, $\forall_{\mathbf{0}}^{\pi}$-sentences have some syntactic restrictions which do not apply to general first-order languages: for instance, *nested terms* such as $[[a, b], c]$ are not allowed in them. Likewise, the semantics of the fragment $\forall_{\mathbf{0}}^{\pi}$ is restricted only to *set-theoretic* interpretations $\mathbf{I} = (D^{\mathbf{I}}, \cdot^{\mathbf{I}})$, where $D^{\mathbf{I}}$ is the von Neumann hierarchy of pure sets $\mathcal{V}$ and $=^{\mathbf{I}}$ and $\in^{\mathbf{I}}$ are the standard equality and membership relations among sets in $\mathcal{V}$, so that they comply with the Zermelo-Fraenkel axioms of set theory, as reported for instance in [16] (we will see below that also the interpretations $[\cdot,\cdot]^{\mathbf{I}}$ and $\bar{\pi}^{\mathbf{I}}$ are restricted). Here we list only the *extensionality* axiom and a consequence of the *regularity* axiom, namely the acyclicity of membership relation:

(C1) *Extensionality*: two sets are *equal* if and only if they have the same members, i.e.,

$$u =^{\mathbf{I}} v \iff (\forall x)(x \in^{\mathbf{I}} u \leftrightarrow x \in^{\mathbf{I}} v);$$

(C2) *Acyclicity*: membership is acyclic, i.e.,

$$(\forall x)(\neg(x\ (\in^{\mathbf{I}})^{+}\ x)),$$

where $(\in^{\mathbf{I}})^{+}$ is the *transitive closure* of $\in^{\mathbf{I}}$.

Concerning the interpretation of pair terms, $[\cdot,\cdot]^{\mathbf{I}}$ has to be a *pairing function*, in the sense described below. Given a binary operation $\boldsymbol{\pi} : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$, the *Cartesian product* of $u, v \in \mathcal{V}$ with respect to $\boldsymbol{\pi}$, denoted by $u \times^{\boldsymbol{\pi}} v$, is the class defined by

$$u \times^{\boldsymbol{\pi}} v =_{\mathrm{Def}} \{\boldsymbol{\pi}(u', v') \,|\, u' \in u \wedge v' \in v\}.$$

Then a binary operation $\boldsymbol{\pi} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is said to be a *pairing function* if and only if the following two conditions hold for all $u, v, u', v' \in \mathcal{V}$:

(C3) $\boldsymbol{\pi}(u, v) = \boldsymbol{\pi}(u', v') \iff u = u' \wedge v = v'$,
(C4) $u \times^{\boldsymbol{\pi}} v$ is a set in $\mathcal{V}$.

Finally, the interpretation of the *non-pairs* operator is constrained by the chosen pairing function $[\cdot, \cdot]^{\mathbf{I}}$, as expressed by

(C5) $u \in^{\mathbf{I}} \bar{\pi}^{\mathbf{I}}(v) \iff u \in^{\mathbf{I}} v \wedge (\forall u', v')([u', v']^{\mathbf{I}} \in^{\mathbf{I}} v \to [u', v']^{\mathbf{I}} \neq u)$

which must hold for all $u, v \in \mathcal{V}$.

We shall refer to interpretations $\mathbf{I} = (D^{\mathbf{I}}, \cdot^{\mathbf{I}})$ of the type we have just described as *pair-aware set-theoretic interpretations*. Proceeding along the same lines as Section 2, we write $(\mathbf{I}, A) \models_{\mathcal{V}} \varphi$ to indicate that the pair-aware set-theoretic interpretation $\mathbf{I}$ and the assignment A in $\mathbf{I}$ evaluate the $\forall_{\mathbf{0}}^{\pi}$-formula φ to **true**. A $\forall_{\mathbf{0}}^{\pi}$-formula φ is said to be *(set-theoretically) satisfiable* if and only there exist a pair-aware set-theoretic interpretation $\mathbf{I}$ and an assignment A in $\mathbf{I}$ satisfying it, i.e., such that $(\mathbf{I}, A) \models_{\mathcal{V}} \varphi$. In case φ is a $\forall_{\mathbf{0}}^{\pi}$-sentence, we can simply write $\mathbf{I} \models_{\mathcal{V}} \varphi$ to indicate that $\mathbf{I}$ evaluates φ to **true**, for a given pair-aware set-theoretic interpretation $\mathbf{I}$. A $\forall_{\mathbf{0}}^{\pi}$-sentence φ is said to be *(set-theoretically) satisfiable* if and only if $\mathbf{I} \models_{\mathcal{V}} \varphi$, for some pair-aware set-theoretic interpretation $\mathbf{I}$.

3.1 A decision procedure for $\forall_{\mathbf{0}}^{\pi}$-sentences

The satisfiability test for $\forall_{\mathbf{0}}^{\pi}$-sentences reported in [5] relies on the existence of finite structures of bounded size, called *skeletal representations*, which witness the existence of certain particular interpretations, called *realizations* (see Definitions 2 and 3 below).

Skeletal representations are finite collections of ground atomic $\forall_{\mathbf{0}}^{\pi}$-formulae of the form (3), subject to certain conditions (see Definition 2 below). As such, a skeletal representation induces a (first-order logic) Herbrand interpretation of $\forall_{\mathbf{0}}^{\pi}$-sentences, which we continue to denote also by $\mathcal{S}$, such that

$$\mathcal{S} \models_{\mathsf{FO}} \gamma \iff \gamma \in \mathcal{S},$$

holds, for every ground atomic $\forall_{\mathbf{0}}^{\pi}$-formula γ. The definition of skeletal representation adopted in this paper differs slightly from the one presented in [5]. In particular, here we extend the notion of skeletal representation so as to also encapsulate the notion of *V-extensionality* and the technical condition (i) of Theorem 3 in [5]. It turns out that all the results in [5] can be adapted to cope with this extended definition of skeletal representation. A reformulation of Theorem 3 in [5], which is central for the set-theoretic satisfiability problem for $\forall_{\mathbf{0}}^{\pi}$-sentences, is reported in Theorem 3 below.

In the following definition, we shall make use of the *membership closure* relation $\in_{\mathcal{S}}^{+}$ relative to a finite set $\mathcal{S}$ of ground atomic $\forall_{\mathbf{0}}^{\pi}$-formulae. This is the *minimal transitive relation* such that, for all constants a, b, c occurring in $\mathcal{S}$, we have

$$\begin{aligned}\mathcal{S} \models_{\mathsf{FO}} a \in \bar{\pi}(b) &\Longrightarrow a \in^{+}_{\mathcal{S}} b,\\ \mathcal{S} \models_{\mathsf{FO}} [a,b] \in c &\Longrightarrow a \in^{+}_{\mathcal{S}} c \wedge b \in^{+}_{\mathcal{S}} c.\end{aligned}$$

(Notice that equality clauses of type $a = b$ in $\mathcal{S}$ play no role in the definition of the relation $\in^{+}_{\mathcal{S}}$.)

Definition 2 (Skeletal representation). *Let V, T be two disjoint sets of constants. A* skeletal representation *$\mathcal{S}$ relative to (V,T) is a finite set of ground atomic $\forall^{\pi}_{0}$-formulae (of type (3)) such that the following conditions hold:*

- *(S1) all the constants occurring in $\mathcal{S}$ are in $V \cup T$;*
- *(S2) the membership closure relation induced by $\mathcal{S}$ is irreflexive, i.e., $a \notin^{+}_{\mathcal{S}} a$, for every $a \in V \cup T$;*
- *(S3) if $\mathcal{S} \models_{\mathsf{FO}} a = b$, for distinct constants a and b, then $a, b \in V$;*
- *(S4) if $\mathcal{S} \models_{\mathsf{FO}} a = b$, then $\mathcal{S} \models_{\mathsf{FO}} \gamma[a \to b]$ and $\mathcal{S} \models_{\mathsf{FO}} \gamma[b \to a]$, for every $\gamma \in \mathcal{S}$;*
- *(S5) if $\mathcal{S} \not\models_{\mathsf{FO}} a = b$, for some $a, b \in V$, then a and b must be* distinguished *in $\mathcal{S}$ by some constant c, in the sense that $\mathcal{S} \models_{\mathsf{FO}} c \in \bar{\pi}(a)$ iff $\mathcal{S} \not\models_{\mathsf{FO}} c \in \bar{\pi}(b)$, or by some pair $[c,d]$, in the sense that $\mathcal{S} \models_{\mathsf{FO}} [c,d] \in a$ iff $\mathcal{S} \not\models_{\mathsf{FO}} [c,d] \in b$.* □

Condition (S2), which is closely related to the acyclicity of the membership relation in set theory, guarantees that a skeletal representation $\mathcal{S}$ can be turned into a corresponding pair-aware set-theoretic interpretation (i.e., its realization; see below). Conditions (S4) and (S5) are related to the extensionality axiom of set theory. Finally, (S1) and (S3) are technical conditions which, together with (S2), (S4), and (S5) in Definition 2, guarantee some desirable properties of realizations, as shown in Lemma 3 below.

In the following definition of *realization of a skeletal representation*, taken from [5, Definition 1], we shall make use of the family $\{\pi_n\}_{n\in\mathbb{N}}$ of pairing functions recursively defined by

$$\begin{aligned}\pi_0(u,v) &=_{\mathrm{Def}} \{u, \{u,v\}\}\\ \pi_{n+1}(u,v) &=_{\mathrm{Def}} \{\pi_n(u,v)\}\,,\end{aligned}$$

for $u, v \in \mathcal{V}$.

Definition 3 (Realization [5]). *Let V, T be two finite, nonempty, and disjoint sets of constants, where $T = \{t_1, t_2, \ldots, t_n\}$, and let $\mathcal{S}$ be a skeletal representation relative to (V,T). Then the* realization *of $\mathcal{S}$ relative to (V,T) is the pair-aware set-theoretic interpretation $\mathcal{R}_{\mathcal{S}}$ defined as follows:*

$$\begin{aligned}[\cdot,\cdot]^{\mathcal{R}_{\mathcal{S}}} &=_{\mathit{Def}} \pi_h\\ a^{\mathcal{R}_{\mathcal{S}}} &=_{\mathit{Def}} \{b^{\mathcal{R}_{\mathcal{S}}} \mid \mathcal{S} \models_{\mathsf{FO}} b \in \bar{\pi}(a)\} \cup \{[b,c]^{\mathcal{R}_{\mathcal{S}}} \mid \mathcal{S} \models_{\mathsf{FO}} [b,c] \in a\}, \qquad \textit{for } a \in V,\\ t_i^{\mathcal{R}_{\mathcal{S}}} &=_{\mathit{Def}} \{b^{\mathcal{R}_{\mathcal{S}}} \mid \mathcal{S} \models_{\mathsf{FO}} b \in \bar{\pi}(t_i)\}\\ &\qquad \cup \{[b,c]^{\mathcal{R}_{\mathcal{S}}} \mid \mathcal{S} \models_{\mathsf{FO}} [b,c] \in t_i\} \cup \{\{k+1, k, i\}\}\,, \qquad \textit{for } t_i \in T,\end{aligned}$$

where $h = |V| + |T|$ *and* $k = |V| \cdot (|V| + |T| + 3)$.[5] $\square$

The following lemma is a direct consequence of Definition 3 above and of Lemma 2 in [5].

Lemma 3. *Let* V, T *be two finite, nonempty, and disjoint sets of constants,* $\mathcal{S}$ *a skeletal representation relative to* (V, T)*, and* $\mathcal{R}_{\mathcal{S}}$ *the realization of* $\mathcal{S}$ *relative to* (V, T)*. Then, for all* $a \in V$,

$$a^{\mathcal{R}_{\mathcal{S}}} \subseteq \left\{ b^{\mathcal{R}_{\mathcal{S}}} \mid b \in V \cup T \right\} \cup \left\{ [b, c]^{\mathcal{R}_{\mathcal{S}}} \mid b, c \in V \cup T \right\}.$$

In addition, the following conditions hold, for all $a, b, c \in V \cup T$:

$$\begin{aligned} \mathcal{R}_{\mathcal{S}} \models_{\mathcal{V}} a = b &\iff \mathcal{S} \models_{\mathsf{FO}} a = b \text{ or } a, b \text{ are the same symbol} \\ \mathcal{R}_{\mathcal{S}} \models_{\mathcal{V}} a \in \bar{\pi}(b) &\iff \mathcal{S} \models_{\mathsf{FO}} a \in \bar{\pi}(b) \\ \mathcal{R}_{\mathcal{S}} \models_{\mathcal{V}} [a, b] \in c &\iff \mathcal{S} \models_{\mathsf{FO}} [a, b] \in c. \end{aligned}$$

$\square$

Let V, T, and T' be finite nonempty sets of constants such that $T \subseteq T'$ and $V \cap T' = \emptyset$ (so that we have also that $V \cap T = \emptyset$), and let $\mathcal{S}$ be a skeletal representation relative to (V, T). Then, plainly, $\mathcal{S}$ is also a skeletal representation relative to (V, T'). Let $\mathcal{R}_{\mathcal{S}}$ be the realization of $\mathcal{S}$ relative to (V, T), and $\mathcal{R}'_{\mathcal{S}}$ the realization of $\mathcal{S}$ relative to (V, T'). Then, relative to $\forall_{\mathbf{0}}^{\pi}$-sentences φ such that $\mathit{Consts}(\varphi) \subseteq V$, it is not hard to check that the realizations $\mathcal{R}_{\mathcal{S}}$ and $\mathcal{R}'_{\mathcal{S}}$ are equivalent, in the sense that $\varphi^{\mathcal{R}_{\mathcal{S}}} = \varphi^{\mathcal{R}'_{\mathcal{S}}}$.

In view of such considerations and of Definition 2, Theorem 3 in [5] can then be restated as follows.

Theorem 3. *Let* φ *be a* $\forall_{\mathbf{0}}^{\pi}$*-sentence, let* $V = \mathit{Consts}(\varphi)$*, and let* T *be a set of constants disjoint from* V *and such that* $|T| = 2 \cdot |V|$*. Then* φ *is (set-theoretically) satisfiable if and only if there exists a skeletal representation* $\mathcal{S}$ *relative to* (V, T) *such that the realization* $\mathcal{R}_{\mathcal{S}}$ *of* $\mathcal{S}$ *relative to* (V, T) *satisfies* φ. $\square$

The decidability of the set-theoretic satisfiability problem for $\forall_{\mathbf{0}}^{\pi}$-sentences easily follows from Theorem 3, as the number of possible skeletal representations for any given $\forall_{\mathbf{0}}^{\pi}$-sentence is finitely bounded, and it is effectively verifiable whether the realization of a skeletal representation is a pair-aware set-theoretic model for a $\forall_{\mathbf{0}}^{\pi}$-sentence. Moreover, in analogy with Herbrand logic, skeletal representations and Lemma 3 provide means to reduce the satisfiability of $\forall_{\mathbf{0}}^{\pi}$-sentences to the satisfiability problem for propositional logic. To this purpose, we prove the following preliminary lemma.

Lemma 4. *Let* V, T *be two finite, nonempty, and disjoint sets of constants, and let* $\mathcal{S}$ *and* $\mathcal{R}_{\mathcal{S}}$ *be, respectively, a skeletal representation and its realization, both relative to* (V, T)*. Then we have*

$$[\mathit{Exp}(\varphi, V \cup T)]^{\mathcal{S}} = \varphi^{\mathcal{R}_{\mathcal{S}}},$$

for every $\forall_{\mathbf{0}}^{\pi}$*-sentence* φ *such that* $\mathit{Consts}(\varphi) \subseteq V \cup T$ *and* $\mathit{Consts}_D(\varphi) \subseteq V$.

[5] We are assuming that integers are represented *à la* von Neumann, namely $0 =_{\text{Def}} \emptyset$ and, recursively, $n + 1 =_{\text{Def}} n \cup \{n\}$.

Proof. It will be enough to prove the lemma in the case of closed simple-prenex $\forall_{\mathbf{0}}^{\pi}$-formulae, as $\forall_{\mathbf{0}}^{\pi}$-sentences are just finite conjunctions of closed simple-prenex $\forall_{\mathbf{0}}^{\pi}$-formulae. Accordingly, in the rest of this proof we will assume that φ is a closed simple-prenex $\forall_{\mathbf{0}}^{\pi}$-formula.

We proceed by induction on the length n of the quantifier prefix of φ. If $n = 0$, i.e. φ is quantifier-free, then $Exp(\varphi, V \cup T) = \varphi$, so that $[Exp(\varphi, V \cup T)]^{\mathcal{S}} = \varphi^{\mathcal{S}}$. But then $\varphi^{\mathcal{S}} = \varphi^{\mathcal{R}_{\mathcal{S}}}$, since, by Lemma 3, the interpretations $\mathcal{S}$ and $\mathcal{R}_{\mathcal{S}}$ coincide on all atomic $\forall_{\mathbf{0}}^{\pi}$-formulae occurring in φ.

For the inductive case, let us first assume that φ has the form $(\forall x \in \bar{\pi}(a))\psi$, where $a \in Consts_D(\varphi) \subseteq V$ and ψ is a simple-prenex $\forall_{\mathbf{0}}^{\pi}$-formula with one less quantifier than φ. Since $\mathcal{R}_{\mathcal{S}}$ is the realization of $\mathcal{S}$ and $a \in V$, we plainly have

$$(\bar{\pi}(a))^{\mathcal{R}_{\mathcal{S}}} = \{b^{\mathcal{R}_{\mathcal{S}}} \mid \mathcal{S} \models_{\mathsf{FO}} b \in \bar{\pi}(a)\}, \tag{5}$$

so that

$$((\forall x \in \bar{\pi}(a))\psi)^{\mathcal{R}_{\mathcal{S}}} = \bigwedge_{\mathcal{S} \models_{\mathsf{FO}} b \in \bar{\pi}(a)} (\psi[x \to b])^{\mathcal{R}_{\mathcal{S}}}. \tag{6}$$

Observe that, for every constant b such that $\mathcal{S} \models_{\mathsf{FO}} b \in \bar{\pi}(a)$, we have $Consts_D(\psi[x \to b]) = Consts_D(\psi) \subseteq Consts_D(\varphi) \subseteq V$, and $Consts(\psi[x \to b]) \subseteq Consts(\psi) \cup \{b\} \subseteq Consts(\varphi) \cup \{b\} \subseteq V \cup T$, as we have assumed that $Consts(\varphi) \cup Consts(\mathcal{S}) \subseteq V \cup T$. Thus, by inductive hypothesis, we have

$$[Exp(\psi[x \to b], V \cup T)]^{\mathcal{S}} = (\psi[x \to b])^{\mathcal{R}_{\mathcal{S}}},$$

for every b such that $\mathcal{S} \models_{\mathsf{FO}} b \in \bar{\pi}(a)$, and, therefore, from (5) and (6), and from the definition of Herbrand expansion, we have:

$$\begin{aligned}
[Exp((\forall x \in \bar{\pi}(a))\psi, V \cup T)]^{\mathcal{S}} &= \bigwedge_{\mathcal{S} \models_{\mathsf{FO}} b \in \bar{\pi}(a)} [Exp(\psi[x \to b], V \cup T)]^{\mathcal{S}} \\
&= \bigwedge_{\mathcal{S} \models_{\mathsf{FO}} b \in \bar{\pi}(a)} (\psi[x \to b])^{\mathcal{R}_{\mathcal{S}}} = ((\forall x \in \bar{\pi}(a))\psi)^{\mathcal{R}_{\mathcal{S}}}.
\end{aligned}$$

Concerning the case in which φ has the form $(\forall [x, y] \in a)\psi$, where $a \in Consts_D(\varphi) \subseteq V$ and ψ is a simple-prenex $\forall_{\mathbf{0}}^{\pi}$-formula with one less quantifier than φ, by Lemma 3, we have $a^{\mathcal{R}_{\mathcal{S}}} \setminus (\bar{\pi}(a))^{\mathcal{R}_{\mathcal{S}}} = \{[b, c]^{\mathcal{R}_{\mathcal{S}}} \mid \mathcal{S} \models_{\mathsf{FO}} [b, c] \in a\}$, so that

$$((\forall [x, y] \in a)\psi)^{\mathcal{R}_{\mathcal{S}}} = \bigwedge_{\mathcal{S} \models_{\mathsf{FO}} [b,c] \in a} (\psi[x \to b][y \to c])^{\mathcal{R}_{\mathcal{S}}}.$$

Much as before, by inductive hypothesis we have

$$[Exp(\psi[x \to b][y \to c], V \cup T)]^{\mathcal{S}} = (\psi[x \to b][y \to c])^{\mathcal{R}_{\mathcal{S}}},$$

for all b, c such that $\mathcal{S} \models_{\mathsf{FO}} [b, c] \in a$, and therefore

$$\begin{aligned}
[Exp((\forall[x,y] \in a)\psi, V \cup T)]^{\mathcal{S}} &= \bigwedge_{\mathcal{S} \models_{\mathsf{FO}} [b,c] \in a} [Exp(\psi[x \to b][y \to c], V \cup T)]^{\mathcal{S}} \\
&= \bigwedge_{\mathcal{S} \models_{\mathsf{FO}} [b,c] \in a} (\psi[x \to b][y \to c])^{\mathcal{R}_{\mathcal{S}}} \\
&= ((\forall[x,y] \in a)\psi)^{\mathcal{R}_{\mathcal{S}}},
\end{aligned}$$

concluding the proof of the lemma. $\square$

Now, Theorem 3 can be restated in a form which resembles more closely Lemma 2.

Corollary 1. *Let φ be a $\forall_{\mathbf{0}}^{\pi}$-sentence, $V = Consts(\varphi)$, and T a set of constants disjoint from V and such that $|T| = 2 \cdot |V|$. Then φ is (set-theoretically) satisfiable if and only if $Exp(\varphi, V \cup T)$ is satisfied by a Herbrand interpretation $\mathcal{S}$ which is a skeletal representation relative to (V, T), i.e., $[Exp(\varphi, V \cup T)]^{\mathcal{S}} = \mathbf{true}$.*

Proof. From Theorem 3, φ is satisfiable if and only if there exists a skeletal representation $\mathcal{S}$ relative to (V, T) such that the realization $\mathcal{R}_{\mathcal{S}}$ of $\mathcal{S}$ relative to (V, T) satisfies φ. But since, by Lemma 4, we have

$$\varphi^{\mathcal{R}_{\mathcal{S}}} = [Exp(\varphi, V \cup T)]^{\mathcal{S}},$$

the thesis follows. $\square$

3.2 The Disjunctive Datalog fragment of $\forall_{\mathbf{0}}^{\pi}$

We close the section by defining the *Disjunctive Datalog subset* $\forall_{\mathbf{0},\mathbf{D}^{\vee}}^{\pi}$ *of* $\forall_{\mathbf{0}}^{\pi}$ as the collection of the $\forall_{\mathbf{0}}^{\pi}$-sentences whose conjuncts are closed simple-prenex $\forall_{\mathbf{0}}^{\pi}$-formulae of the form

$$(\forall x_1 \in \bar{\pi}(a_1)) \ldots (\forall x_n \in \bar{\pi}(a_n))(\forall\,[y_1, z_1] \in b_1) \ldots (\forall\,[y_m, z_m] \in b_m)\psi, \tag{7}$$

such that ψ has the form

$$(\gamma_1 \wedge \ldots \wedge \gamma_l) \to (\sigma_1 \vee \ldots \vee \sigma_h)\,,$$

where $l, h \geq 0$ and $\gamma_1, \ldots, \gamma_l, \sigma_1, \ldots \sigma_h$ are $\forall_{\mathbf{0}}^{\pi}$-literals. Since (7) is closed, each variable x which may occur in the head of the rule ψ must be bound, so that x occurs in at least one atom of the rule body, as required for Disjunctive Datalog rules, when restricted quantifiers in (7) are expanded as indicated in (2).

In the next section we present a reduction of the set-theoretic satisfiability problem for $\forall_{\mathbf{0}}^{\pi}$-sentences to the satisfiability problem of first-order formulae, and thus to Herbrand satisfiability. In addition, we shall see that the same reduction maps $\forall_{\mathbf{0},\mathbf{D}^{\vee}}^{\pi}$-sentences to $\mathsf{DATALOG}^{\vee,\neg}$-sentences, so that reducibility of the set-theoretic satisfiability problem for $\forall_{\mathbf{0},\mathbf{D}^{\vee}}^{\pi}$-sentences to the satisfiability problem for $\mathsf{DATALOG}^{\vee,\neg}$-sentences will follow readily.

4 Herbrand-satisfiability of $\forall_{\mathbf{0}}^{\pi}$-sentences

In this section we show how to reduce the set-theoretic satisfiability problem for $\forall_{\mathbf{0}}^{\pi}$-sentences to the satisfiability problem in first-order logic. To this purpose, we introduce the function-free first-order language $\mathcal{L}_0^{\pi}$ which involves, besides constants, also the following predicate symbols:

binary	ternary	4-ary
$\widehat{\in}$, $P^{=}$, $P^{\bar{\pi}}$	$P^{[\cdot,\cdot]}$, *distBy*	$distBy_{\pi}$

We shall provide a polynomial-time reduction $\varphi \mapsto \Theta_{\varphi}$ from $\forall_{\mathbf{0}}^{\pi}$-sentences into $\mathcal{L}_0^{\pi}$-sentences such that the $\forall_{\mathbf{0}}^{\pi}$-sentence φ is set-theoretically satisfiable if and only if the corresponding $\mathcal{L}_0^{\pi}$-sentence Θ_{φ} is Herbrand-satisfiable. In our reduction, the predicate $\widehat{\in}$ will represent the transitive closure of the membership relation among sets, whereas the predicates *distBy* and $distBy_{\pi}$ will model the fact that two sets are distinct. In particular, the predicate *distBy* will take care of the case in which two sets are distinguished by a set that is not a pair, whereas the predicate $distBy_{\pi}$ will take care of the case in which two sets are distinguished by a pair. Finally, the predicate $P^{=}(\nu_1, \nu_2)$ will model equality between ν_1 and ν_2, the predicate $P^{\bar{\pi}}(\nu_1, \nu_2)$ will represent the set-theoretic atomic formula $\nu_1 \in \bar{\pi}(\nu_2)$, and $P^{[\cdot,\cdot]}(\nu_1, \nu_2, \nu_3)$ will represent the atomic formula $[\nu_1, \nu_2] \in \nu_3$, where $\nu_1, \nu_2, \nu_3 \in \mathit{Vars} \cup \mathit{Consts}$.

For the sake of clarity, with a slight abuse of notation, we shall write $\nu_1 = \nu_2$, $\nu_1 \in \bar{\pi}(\nu_2)$, and $[\nu_1, \nu_2] \in \nu_3$ in place of $P^{=}(\nu_1, \nu_2)$, $P^{\bar{\pi}}(\nu_1, \nu_2)$ and $P^{[\cdot,\cdot]}(\nu_1, \nu_2, \nu_3)$, respectively, for all $\nu_1, \nu_2, \nu_3 \in \mathit{Vars} \cup \mathit{Consts}$. With such an understanding, $\forall_{\mathbf{0}}^{\pi}$ can be regarded as a sublanguage of $\mathcal{L}_0^{\pi}$, and skeletal representations as particular Herbrand interpretations (over $\mathcal{L}_0^{\pi}$), subject to the conditions reported in Definition 2. For example, the skeletal representation $\mathcal{S} = \{a \in \bar{\pi}(b)\}$ can be regarded as a Herbrand interpretation such that $\mathcal{S} \models_{\mathsf{FO}} \gamma \iff \gamma = a \in \bar{\pi}(b)$. Conversely, let us consider the two set of constants $V = \{a, b\}$ and $T = \{t\}$ and let $\mathcal{H} = \{a \in \bar{\pi}(b), a = t\}$, $\mathcal{H}' = \mathcal{H} \cup \{t \in \bar{\pi}(b)\}$ be two Herbrand interpretations.[6] Then $\mathcal{H}'$ is a skeletal representation relative to (V, T) whereas $\mathcal{H}$ is not, as $\mathcal{H}$ does not comply with condition (S3). aggiunto

$\forall_{\mathbf{0}}^{\pi}$-formulae have a set-theoretic semantics. However, Lemma 2 and Corollary 1 suggest how to circumvent this issue by mapping skeletal representations into Herbrand interpretations. Indeed, skeletal representations are Herbrand interpretations of a very particular type, but, of course, there are Herbrand interpretations which do not comply with the conditions of Definition 2.

Our reduction $\varphi \mapsto \Theta_{\varphi}$ from $\forall_{\mathbf{0}}^{\pi}$-sentences into $\mathcal{L}_0^{\pi}$-sentences will be defined in terms of an $\mathcal{L}_0^{\pi}$-formula $\chi^{(V,T)}$, depending on two nonempty and disjoint sets of constants V and T, where $V = \mathit{Consts}(\varphi)$ and $|T| = 2 \cdot |V|$, intended to enforce the conditions of Definition 2, in such a way that

[6] With a slight abuse of notation and in analogy with skeletal representations, here we are representing Herbrand interpretations as sets with posing $\gamma \in \mathcal{H} \iff \mathcal{H} \models_{\mathsf{FO}} \gamma$, for every Herbrand interpretation $\mathcal{H}$ and for every ground atomic formula γ.

– every Herbrand interpretation $\mathcal{H}$ satisfying $\chi^{(V,T)}$ is a skeletal representation relative to (V,T), and
– every skeletal representation relative to (V,T) satisfies $\chi^{(V,T)}$.

Then we shall put $\Theta_\varphi \equiv_{\text{Def}} \chi^{(V,T)} \wedge \varphi$ and prove that φ is set-theoretically satisfiable if and only if Θ_φ is satisfiable in first-order logic.

Thus, let V, T be two nonempty and disjoint sets of constants (such that $|T| = 2 \cdot |V|$). The formula $\chi^{(V,T)}$ is defined as follows:

$$\chi^{(V,T)} \equiv_{\text{Def}} \chi_1 \wedge \chi_2 \wedge \chi_3^{(V,T)} \wedge \chi_4^{(V,T)},$$

where

$$\begin{aligned}\chi_1 \equiv_{\text{Def}}\; & (\forall x,y)(x \in \bar{\pi}(y) \rightarrow x \widehat{\in} y) \wedge (\forall x,y,z)([x,y] \in z \rightarrow x \widehat{\in} z \wedge y \widehat{\in} z) \\ & \wedge (\forall x,y,z)(x \widehat{\in} y \wedge y \widehat{\in} z \rightarrow x \widehat{\in} z) \wedge (\forall x)\neg(x \widehat{\in} x)\end{aligned}$$

$$\begin{aligned}\chi_2 \equiv_{\text{Def}}\; & (\forall x,y)(x = y \rightarrow y = x) \wedge (\forall x,y,z)(x = y \wedge y = z \rightarrow x = z) \\ & \wedge (\forall x,y,z)(x \in \bar{\pi}(y) \wedge x = z \rightarrow z \in \bar{\pi}(y)) \\ & \wedge (\forall x,y,z)(x \in \bar{\pi}(y) \wedge y = z \rightarrow x \in \bar{\pi}(z)) \\ & \wedge (\forall x,y,z,v)([x,y] \in z \wedge x = v \rightarrow [v,y] \in z) \\ & \wedge (\forall x,y,z,v)([x,y] \in z \wedge y = v \rightarrow [x,v] \in z) \\ & \wedge (\forall x,y,z,v)([x,y] \in z \wedge z = v \rightarrow [x,y] \in v)\end{aligned}$$

$$\begin{aligned}\chi_3^{(V,T)} \equiv_{\text{Def}}\; & \bigwedge_{x,y \in V} \big(\neg(x = y) \rightarrow \mathsf{dist}^{(V,T)}(x,y) \vee \mathsf{dist}_\pi^{(V,T)}(x,y)\big) \\ & \wedge (\forall x,y,z)\big(\mathit{distBy}(x,y,z) \rightarrow z \in \bar{\pi}(x) \wedge \neg(z \in \bar{\pi}(y))\big) \\ & \wedge (\forall x,y,z,v)\big(\mathit{distBy}_\pi(x,y,z,v) \rightarrow [z,v] \in x \wedge \neg([z,v] \in y)\big)\end{aligned}$$

$$\chi_4^{(V,T)} \equiv_{\text{Def}} \bigwedge_{x \in V \cup T, t \in T, x \neq t} \neg(x = t),$$

and where we are using the following abbreviations

$$\mathsf{dist}^{(V,T)}(x,y) \equiv_{\text{Def}} \bigvee_{z \in V \cup T} \big(\mathit{distBy}(x,y,z) \vee \mathit{distBy}(y,x,z)\big)$$

$$\mathsf{dist}_\pi^{(V,T)}(x,y) \equiv_{\text{Def}} \bigvee_{z,v \in V \cup T} \big(\mathit{distBy}_\pi(x,y,z,v) \vee \mathit{distBy}_\pi(y,x,z,v)\big).$$

It can be easily verified that $\chi^{(V,T)}$ is a $\mathsf{DATALOG}^{\vee,\neg}$-formula. In addition, $V \cup T$ is the Herbrand universe of $\chi^{(V,T)}$. Indeed, no constant occurs in χ_1 and χ_2, whereas $\chi_3^{(V,T)}$ and $\chi_4^{(V,T)}$ involve all the constants in $V \cup T$ and no other.

The formulae χ_1, χ_2, $\chi_3^{(V,T)}$, and $\chi_4^{(V,T)}$ are intended to formalize the conditions (S2), (S4), (S5), and (S3) of Definition 2, as clarified in the following lemma.

Lemma 5. *Let V, T be two disjoint and nonempty sets of constants. Let $\mathcal{H}$ be a Herbrand interpretation such that:*

- *$Consts(\mathcal{H}) \subseteq V \cup T$, i.e., all the atomic formulae of $\mathcal{L}_0^\pi$ involving some constant not in $V \cup T$ are evaluated to* **false** *by $\mathcal{H}$, and*
- *$\mathcal{H} \models_{\mathsf{FO}} Exp(\chi^{(V,T)}, V \cup T)$.*

Then $\mathcal{H}$ is a skeletal representation relative to (V, T).

Proof. We must prove that $\mathcal{H}$ satisfies all the conditions reported in Definition 2. In view of Lemma 2, $\mathcal{H}$ satisfies $\chi^{(V,T)}$, as we are assuming that $\mathcal{H}$ satisfies $Exp(\chi^{(V,T)}, V \cup T)$ and $V \cup T$ is the Herbrand universe of $\chi^{(V,T)}$.

Concerning (S1), there is nothing to prove, since it follows directly from the hypotheses.

Next, we observe that the relation $\widehat{\in}^{\mathcal{H}}$ is acyclic, as $\mathcal{H}$ satisfies the conjunct χ_1 of $\chi^{(V,T)}$. Hence, $\in_{\mathcal{H}}^+$ must be acyclic as well, as required by condition (S2), since $\in_{\mathcal{H}}^+ \subseteq \widehat{\in}^{\mathcal{H}}$.[7]

Concerning (S3), let $\mathcal{H} \models_{\mathsf{FO}} a = b$, where a and b are distinct constants, and assume by way of contradiction that $b \notin V$. Then we must have $b \in T$, as $b \in Consts(\mathcal{H}) \subseteq V \cup T$. But since $\mathcal{H} \models_{\mathsf{FO}} \chi_4^{(V,T)}$, then, *a fortiori*, $\mathcal{H} \models_{\mathsf{FO}} \neg(a = b)$, a contradiction, proving that $b \in V$. Since $\mathcal{H} \models_{\mathsf{FO}} a = b$ and $\mathcal{H} \models_{\mathsf{FO}} Exp(\chi_2, V \cup T)$, then we have also $\mathcal{H} \models_{\mathsf{FO}} Exp((\forall x)(x = y \rightarrow y = x), V \cup T)$, so that $\mathcal{H} \models_{\mathsf{FO}} b = a$ holds. Thus, the same argument as before shows that we have also $a \in V$, proving (S3).

Next, let $\mathcal{H} \models_{\mathsf{FO}} a = b$, for some constants $a, b \in V \cup T$, and let γ be a ground atomic $\forall_0^\pi$-formula such that $\mathcal{H} \models_{\mathsf{FO}} \gamma$. We have to prove that $\mathcal{H} \models_{\mathsf{FO}} \gamma[a \rightarrow b]$ and $\mathcal{H} \models_{\mathsf{FO}} \gamma[b \rightarrow a]$. Such a formula γ can be of one of the types listed in (3). Since, as observed earlier, we have $\mathcal{H} \models_{\mathsf{FO}} Exp(\chi_2, V \cup T)$, then (S4) easily follows. Suppose that, for example, $\gamma = a \in \bar{\pi}(c)$, for some $c \in V \cup T$. It can be easily verified that $a \in \bar{\pi}(c) \wedge a = b \rightarrow b \in \bar{\pi}(c)$ is a conjunct of $Exp(\chi_2, V \cup T)$ as $(\forall x, y, z)(x \in \bar{\pi}(y) \wedge x = z \rightarrow z \in \bar{\pi}(y))$ is a conjunct of χ_2, and thus $\mathcal{H} \models_{\mathsf{FO}} \gamma[b \rightarrow a]$ follows from $\mathcal{H} \models_{\mathsf{FO}} Exp(\chi_2, V \cup T)$ in the case under consideration. The remaining possible cases for γ can be proved analogously. modificato

Finally, let $a, b \in V$ be such that $\mathcal{H} \not\models_{\mathsf{FO}} a = b$. Hence, $\mathcal{H} \models_{\mathsf{FO}} \mathsf{dist}^{(V,T)}(a, b) \vee \mathsf{dist}_\pi^{(V,T)}(a, b)$, as, from the hypotheses, we have $\mathcal{H} \models_{\mathsf{FO}} Exp(\chi_3^{(V,T)}, V \cup T)$. It can easily be verified that if $\mathcal{H} \models_{\mathsf{FO}} \mathsf{dist}^{(V,T)}(a, b)$, then a and b are distinguished in $\mathcal{H}$ by a constant (in the sense of Definition 2), whereas if $\mathcal{H} \models_{\mathsf{FO}} \mathsf{dist}_\pi^{(V,T)}(a, b)$, then a and b are distinguished in $\mathcal{H}$ by a pair term. Indeed, if $\mathcal{H} \models_{\mathsf{FO}} \mathsf{dist}^{(V,T)}(a, b)$, then $\mathcal{H} \models_{\mathsf{FO}} distBy(a, b, c) \vee distBy(b, a, c)$, for some constant $c \in V \cup T$, so that

[7] The relation $\in_{\mathcal{H}}^+$ does not necessarily coincide with $\widehat{\in}_{\mathcal{H}}$, as the minimality of $\widehat{\in}_{\mathcal{H}}$ is not enforced by χ_1.

either $\mathcal{H} \models_{\mathsf{FO}} c \in \bar{\pi}(a) \wedge \neg(c \in \bar{\pi}(b))$ or $\mathcal{H} \models_{\mathsf{FO}} c \in \bar{\pi}(b) \wedge \neg(c \in \bar{\pi}(a))$, and in either case a and b are distinguished in $\mathcal{H}$ by some constant $c \in V \cup T$. By reasoning much as above, it can also be checked that if $\mathcal{H} \models_{\mathsf{FO}} \mathsf{dist}_{\pi}^{(V,T)}(a,b)$, then a and b are distinguished in $\mathcal{H}$ by some pair $[c,d]$, with $c,d \in V \cup T$. Thus, $\mathcal{H}$ satisfies condition (S5). $\square$

In addition, every skeletal interpretation can be extended to comply with the constraints imposed by $\chi^{(V,T)}$.

Lemma 6. *Given two nonempty and disjoint sets of constants V, T (such that $|T| = 2 \cdot |V|$), every skeletal representation $\mathcal{S}$ relative to (V,T) can be extended/redefined in such a way that $\mathcal{S} \models_{\mathsf{FO}} Exp(\chi^{(V,T)}, V \cup T)$.*

Proof. Let $\mathcal{S}$ be a skeletal representation relative to (V,T), where V and T are sets of constants as in the hypotheses. We extend $\mathcal{S}$ as follows. To begin with, $\widehat{\in}^{\mathcal{S}}$ is defined as the minimal transitive relation over $V \cup T$ such that

$$\begin{aligned}(a \in \bar{\pi}(b))^{\mathcal{S}} &\Longrightarrow (a \widehat{\in} b)^{\mathcal{S}}\\ ([a,b] \in c)^{\mathcal{S}} &\Longrightarrow (a \widehat{\in} c)^{\mathcal{S}} \wedge (b \widehat{\in} c)^{\mathcal{S}}.\end{aligned}$$

Next, the relations $\mathsf{dist}^{\mathcal{S}}$, $\mathsf{dist}_{\pi}^{\mathcal{S}}$, $distBy^{\mathcal{S}}$ and $distBy_{\pi}^{\mathcal{S}}$ are defined as indicated below:

$$\begin{aligned}\mathsf{dist}(a,b)^{\mathcal{S}} &\Longleftrightarrow (\exists c)\, distBy(a,b,c)^{\mathcal{S}}\\ distBy(a,b,c)^{\mathcal{S}} &\Longleftrightarrow ((c \in \bar{\pi}(a))^{\mathcal{S}} \leftrightarrow \neg(c \in \bar{\pi}(b))^{\mathcal{S}})\\ \mathsf{dist}_{\pi}(a,b)^{\mathcal{S}} &\Longleftrightarrow (\exists c,d)\, distBy_{\pi}(a,b,c,d)^{\mathcal{S}}\\ distBy_{\pi}(a,b,c,d)^{\mathcal{S}} &\Longleftrightarrow (([c,d] \in a)^{\mathcal{S}} \leftrightarrow \neg([c,d] \in b)^{\mathcal{S}}).\end{aligned}$$

In order to show that the Herbrand interpretation $\mathcal{S}$, extended in this way, satisfies $\chi^{(V,T)}$, we have to prove that $\mathcal{S}$ satisfies the formulae χ_1, χ_2, $\chi_3^{(V,T)}$, and $\chi_4^{(V,T)}$.

By the very construction process, $\mathcal{S}$ satisfies the first four conjuncts of χ_1. In addition, $\mathcal{S}$ satisfies also the last conjunct $(\forall x)\neg(x \widehat{\in} x)$ of χ_1. Indeed, if this were not the case, $\mathcal{S}$ would evaluate to **true** an atom of the form $c \widehat{\in} c$, for some $c \in V \cup T$. But this would be possible only if one of the following situations occurs:

(a) $\mathcal{S}$ evaluates to **true** an atom of one of the following three forms

$$c \in \bar{\pi}(c), \;\; [c,b] \in c, \;\; [a,c] \in c\,,$$

for some $a, b \in V \cup T$;

(b) $\mathcal{S}$ evaluates to **true** all atoms in a sequence of the form

$$c \widehat{\in} a_1, \; a_1 \widehat{\in} a_2, \; \ldots, \; a_{n-1} \widehat{\in} a_n, \; a_n \widehat{\in} c\,,$$

with $a_i \in V \cup T$ for $1 \leq i \leq n$.

However, case (a) cannot occur, since, by condition (S2), the membership relation induced by $\mathcal{S}$ is acyclic. We show that also case (b) cannot occur, thereby proving that $\mathcal{S}$ cannot contain any atom of the form $c\widehat{\in}c$. Indeed, if (b) were true, then $\mathcal{S}$ would evaluate to **true** a maximal sequence of atoms of the form

$$a_0\widehat{\in}a_1,\ a_1\widehat{\in}a_2,\ \ldots,\ a_{n-1}\widehat{\in}a_n,\ a_n\widehat{\in}a_{n+1}\,,$$

where a_0 and a_{n+1} coincide. But then, for each $i = 0, 1, \ldots, n$, $\mathcal{S}$ would evaluate to **true** at least an atom of one of the following types

$$a_i \in \bar{\pi}(a_{i+1}),\ \ [a_i, b] \in a_{i+1},\ \ [b, a_i] \in a_{i+1}\,,$$

with $b \in V \cup T$, and therefore the membership relation $\in^+_{\mathcal{S}}$ induced by $\mathcal{S}$ would contain a cycle, contradicting condition (S2). Summing up, $\mathcal{S}$ satisfies also the last conjunct of χ_1, and hence it satisfies the whole formula χ_1.

To show that $\mathcal{S}$ satisfies χ_2 and $\chi_3^{(V,T)}$, it is enough to observe that $Consts(\mathcal{S}) \subseteq V \cup T$ and that $\mathcal{S}$ satisfies conditions (S4) and (S5), respectively, as, in our assumptions, it is a skeletal representation.

Finally, since (again) $\mathcal{S}$ comply with condition (S3), it follows that $\mathcal{S}$ satisfies also the formula $\chi_4^{(V,T)}$, as no new atom of the form $a = b$ can possibly be introduced into $\mathcal{S}$ during the extension process indicated above.

Hence, in conclusion, $\mathcal{S}$ satisfies formula $\chi^{(V,T)}$ at whole. $\square$

Finally, the next theorem states that the satisfiability of every $\forall^{\pi}_{\mathbf{0}}$-sentence φ can be decided by checking the Herbrand satisfiability of the corresponding $\mathcal{L}^{\pi}_{0}$-formula $\chi^{(V,T)} \wedge \varphi$, thus concluding the verification of the correctness of our reduction.

Theorem 4. *Let φ be a $\forall^{\pi}_{\mathbf{0}}$-sentence, let $V = Consts(\varphi)$, and let T be any set of constants disjoint from V such that $|T| = 2 \cdot |V|$. Then φ is set-theoretically satisfiable if and only if $\chi^{(V,T)} \wedge \varphi$ is satisfiable in first-order logic.*

Proof. To begin with, let us assume that φ is set-theoretically satisfiable. Then, by Corollary 1 there exists a skeletal representation $\mathcal{S}$ relative to (V, T) such that $\mathcal{S} \models_{\mathsf{FO}} Exp(\varphi, V \cup T)$. Since $\mathcal{S}$ is a skeletal representation relative to (V, T), by Lemma 6 we can assume that $\mathcal{S} \models_{\mathsf{FO}} Exp(\chi^{(V,T)}, V \cup T)$ holds as well, so that $\mathcal{S} \models_{\mathsf{FO}} Exp(\chi^{(V,T)} \wedge \varphi, V \cup T)$. Hence, considering that the Herbrand universe of $\chi^{(V,T)}$ coincides with $V \cup T$, and the one of φ is a subset of $V \cup T$, by Lemma 2 we can conclude that $\chi^{(V,T)} \wedge \varphi$ is satisfiable in the sense of first-order logic.

Conversely, let $\chi^{(V,T)} \wedge \varphi$ be satisfiable, in the sense of first-order logic. Then, by Lemma 2 there is a Herbrand interpretation $\mathcal{H}$ such that $\mathcal{H} \models_{\mathsf{FO}} Exp(\chi^{(V,T)} \wedge \varphi, V \cup T)$, so that $\mathcal{H} \models_{\mathsf{FO}} Exp(\chi^{(V,T)}, V \cup T)$ and $\mathcal{H} \models_{\mathsf{FO}} Exp(\varphi, V \cup T)$ must hold as well. Since $Exp(\chi^{(V,T)} \wedge \varphi, V \cup T)$ involves constants in $V \cup T$ only, we can assume, without loss of generality, that $Consts(\mathcal{H}) \subseteq V \cup T$. In addition, by Lemma 5, $\mathcal{H}$ is a skeletal representation relative to (V, T), so that we can conclude that φ is set-theoretically satisfiable, as consequence of Corollary 1.

$\square$

We conclude this section by observing that, if φ is a $\forall^{\pi}_{\mathbf{0},\mathbf{D}^{\vee}}$-sentence, then $\chi^{(V,T)} \wedge \varphi$ is a $\mathsf{DATALOG}^{\vee,\neg}$-sentence, as $\chi^{(V,T)}$ is a $\mathsf{DATALOG}^{\vee,\neg}$-sentence. Thus the satisfiability problem for $\forall^{\pi}_{\mathbf{0},\mathbf{D}^{\vee}}$-sentences can be reduced in polynomial time to the Herbrand satisfiability problem for $\mathsf{DATALOG}^{\vee,\neg}$-sentences.

Corollary 2. *Let φ be a $\forall^{\pi}_{\mathbf{0},\mathbf{D}^{\vee}}$-sentence. Let $V = Consts(\varphi)$ and let T be any set of constants disjoint from V and such that $|T| = 2 \cdot |V|$. Then φ is set-theoretically satisfiable if and only if the corresponding $\mathsf{DATALOG}^{\vee,\neg}$-sentence $\chi^{(V,T)} \wedge \varphi$ is satisfiable, in the sense of Disjunctive Datalog.* □

5 Conclusions and future work

In this paper we have identified a correspondence between the fragment of set theory $\forall^{\pi}_{\mathbf{0}}$ and first-order logic (in particular Herbrand logic) by providing a polynomial-time reduction of $\forall^{\pi}_{\mathbf{0}}$-sentences to formulae in a first-order language, called $\mathcal{L}^{\pi}_{0}$, suitable for this purpose. In addition, we have shown that if we limit ourselves to the Disjunctive Datalog restriction of $\forall^{\pi}_{\mathbf{0}}$, called $\forall^{\pi}_{\mathbf{0},\mathbf{D}^{\vee}}$, then our reduction maps formulae in this subfragment to $\mathsf{DATALOG}^{\vee,\neg}$-formulae.

Such a correspondence, and its consequences, has to be further investigated. For instance, applications of techniques and results devised in the context of logic programming (such as, for example, answer-set programming and negation-as-failure) to the $\forall^{\pi}_{\mathbf{0},\mathbf{D}^{\vee}}$ subfragment need to be studied. In view of our reduction, a satisfiability checker for $\forall^{\pi}_{\mathbf{0},\mathbf{D}^{\vee}}$-formulae can be implemented by reusing some machinery from logic programming, for example the Disjunctive Datalog system *DLV* described in [2].[8]

We intend to develop analogous reductions for other decidable fragments of set theory, such as, for instance, the quantified fragment $\forall^{\pi}_{0,2}$, whose decision procedure is based on a reduction to $\forall^{\pi}_{\mathbf{0}}$ (see [4]), and the unquantified fragment $\mathsf{MLSS}^{\times}_{2,m}$, whose satisfiability problem can in turn be reduced to the satisfiability problem for $\forall^{\pi}_{0,2}$ (see [6]).

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[8] http://www.dlvsystem.com/

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