

**Cauchy-Schwarz:**  $|\mathbf{u}^\top \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$ .

**Spectral norm:**  $\|A\| := \max_{\|\mathbf{v}\|=1} \|A\mathbf{v}\|$ .

**Mean-value theorem:** If  $a < b$  and  $h : [a, b] \rightarrow \mathbb{R}$  continuous and differentiable in  $(a, b)$ , then there exists  $c \in (a, b)$  such that

$$h'(c) = \frac{h(b) - h(a)}{b - a}.$$

**Fundamental theorem of calculus:** If  $a < b$  and  $h$  differentiable on an open domain  $(a, b)$  and  $h'$  continuous on  $[a, b]$ , then

$$h(b) - h(a) = \int_a^b h'(t) dt.$$

**Differentiable:**  $f : \text{dom}(f) \rightarrow \mathbb{R}^m$ , where  $\text{dom}(f) \subseteq \mathbb{R}^d$  is differentiable at  $\mathbf{x}$  if there exists  $A \in \mathbb{R}^{m \times d}$  and an error function  $r : \mathbb{R}^d \rightarrow \mathbb{R}^m$  defined in some neighborhood of  $\mathbf{0} \in \mathbb{R}^d$  such that for all  $\mathbf{y}$  in the neighborhood of  $\mathbf{x}$ ,

$$f(\mathbf{y}) = f(\mathbf{x}) + A(\mathbf{y} - \mathbf{x}) + r(\mathbf{y} - \mathbf{x}),$$

where

$$\lim_{\mathbf{v} \rightarrow \mathbf{0}} \frac{\|r(\mathbf{v})\|}{\|\mathbf{v}\|} = \mathbf{0}.$$

$A$  is then the Jacobian of  $f$  at  $\mathbf{x}$ .

**B-Lipschitz:**  $f$  is  $B$ -Lipschitz if

$$\|f(\mathbf{x}) - f(\mathbf{y})\| \leq B\|\mathbf{x} - \mathbf{y}\|,$$

which is equivalent to bounded gradients on open domains (in closed domains, only  $\Leftarrow$  holds)

$$\|\nabla f(\mathbf{x})\| \leq B.$$

**Cosine theorem:**  $2\mathbf{v}^\top \mathbf{w} = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2$ .

## 2 Convexity

Domain must be convex. Strict convexity if inequalities become strict inequalities. Equivalent definitions  $\forall \mathbf{x}, \mathbf{y} \in \text{dom}(f)$ :

- $f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y})$ .
- First-order exists:  $f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x})$ .
- First-order exists:  $(\nabla f(\mathbf{y}) - \nabla f(\mathbf{x}))^\top (\mathbf{y} - \mathbf{x}) \geq 0$ .
- Second-order exists:  $\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}$ .

Intuition:  $f$  is above its tangential hyperplane at  $(\mathbf{x}, f(\mathbf{x}))$ .

**Jensen's inequality:** If  $f$  convex, and  $\sum_{i=1}^m \lambda_i = 1$ , then

$$f\left(\sum_{i=1}^m \lambda_i \mathbf{x}_i\right) \leq \sum_{i=1}^m \lambda_i f(\mathbf{x}_i).$$

The other direction holds for concave functions ( $-f$  is convex).

**Preserving convexity:** Max, sum, and multiplication with positive scalars preserve convexity.  $f \circ g$  is convex on  $\text{dom}(f \circ g) := \{\mathbf{x} \in \mathbb{R}^m \mid g(\mathbf{x}) \in \text{dom}(f)\}$  if  $g$  is affine.

**Local minimum:** A point  $\mathbf{x}$ , such that there exists  $\epsilon > 0$  with

$$f(\mathbf{x}) \leq f(\mathbf{y}), \quad \forall \mathbf{y} \in \text{dom}(f) \text{ satisfying } \|\mathbf{y} - \mathbf{x}\| < \epsilon.$$

**Global minimum:** A point  $\mathbf{x}$  such that

$$f(\mathbf{x}) \leq f(\mathbf{y}), \quad \forall \mathbf{y} \in \text{dom}(f).$$

If  $f$  is convex and differentiable over an open domain, then  $\nabla f(\mathbf{x}) = \mathbf{0}$  if and only if  $\mathbf{x}$  is a global minimum.

**Sublevel set:** Let  $f$  be continuous (not convex). If there exists a nonempty and bounded sublevel set  $f^{\leq \alpha}$ , then  $f$  has a global minimum.

TODO: Convex programs.

## 3 Gradient descent

$f$  must be differentiable, then we use the update rule:

$$\mathbf{x}_{t+1} := \mathbf{x}_t - \gamma_t \nabla f(\mathbf{x}_t).$$

**Vanilla analysis:** Assuming only convexity, we get a bound on the summed error

$$\sum_{t=0}^{T-1} (f(\mathbf{x}_t) - f^*) \leq \frac{\gamma_t}{2} \sum_{t=0}^{T-1} \|\nabla f(\mathbf{x}_t)\|^2 + \frac{1}{2\gamma_t} \|\mathbf{x}_0 - \mathbf{x}^*\|^2.$$

Proof by using first-order convexity on  $\mathbf{x}_t$  and  $\mathbf{x}^*$ , and rewrite the gradient descent update rule.

**Lipschitz functions** ( $\mathcal{O}(1/\epsilon^2)$ ): Setting  $\gamma := R/B\sqrt{T}$ , we get

$$\frac{1}{T} \sum_{t=0}^{T-1} (f(\mathbf{x}_t) - f^*) \leq \frac{RB}{\sqrt{T}}.$$

Using bound  $\|\mathbf{x}_0 - \mathbf{x}^*\| \leq R$ .

## 3 Smooth functions

$L$ -smooth with equivalent definitions  $\forall \mathbf{x}, \mathbf{y} \in X$ :

- $f(\mathbf{y}) \leq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) + \frac{L}{2} \|\mathbf{x} - \mathbf{y}\|^2$ .
- Lemma 3.3:  $\frac{L}{2} \mathbf{x}^\top \mathbf{x} - f(\mathbf{x})$  is convex.
- Lemma 3.5:  $\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|$ .
- Lemma 6.1:  $\|\nabla^2 f(\mathbf{x})\| \leq L$  ( $\Leftarrow$  only if  $X$  is open).

Intuition:  $f$  is below a not-too-steep tangential paraboloid at  $(\mathbf{x}, f(\mathbf{x}))$ .

**Affine functions** (Lemma 3.4):  $f(\mathbf{x}) = \mathbf{x}^\top Q\mathbf{x} + \mathbf{b}^\top \mathbf{x} + c$  is smooth with parameter  $2\|Q\|$  if  $Q$  is symmetric.

**Sufficient decrease** (Lemma 3.7): Choosing  $\gamma := 1/L$ , gradient descent satisfies

$$f(\mathbf{x}_{t+1}) \leq f(\mathbf{x}_t) - \frac{1}{2L} \|\nabla f(\mathbf{x}_t)\|^2, \quad t \geq 0.$$

(Already holds if  $f$  is  $L$ -smooth over line segment connecting  $\mathbf{x}_t$  and  $\mathbf{x}_{t+1}$ .) Proof by first definition of smoothness, cosine theorem, and gradient descent update rule.

**Convergence** ( $\mathcal{O}(1/\epsilon)$ ) (Theorem 3.8): Choosing  $\gamma := 1/L$ , gradient descent yields

$$f(\mathbf{x}_T) - f^* \leq \frac{L}{2T} \|\mathbf{x}_0 - \mathbf{x}^*\|^2, \quad T > 0.$$

Proof by starting from vanilla analysis and bounding gradient sum with sufficient decrease.

Accelerated gradient descent achieves  $\mathcal{O}(1/\sqrt{\epsilon})$  by using an intermediate variable.

## 3 Strongly convex functions

$\mu$ -strongly convex with equivalent definitions  $\forall \mathbf{x}, \mathbf{y} \in X$ :

- $f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) + \frac{\mu}{2} \|\mathbf{x} - \mathbf{y}\|^2$ .
- Lemma 3.11:  $f(\mathbf{x}) - \frac{\mu}{2} \mathbf{x}^\top \mathbf{x}$  is convex.

Intuition:  $f$  is above a not-too-flat tangential paraboloid at  $(\mathbf{x}, f(\mathbf{x}))$ .

**Strict convexity** (Lemma 3.12): If  $f$  is  $\mu$ -strongly convex, then  $f$  is strictly convex.

**Geometrically decreasing distances** (Theorem 3.14): Choosing  $\gamma := 1/L$ , gradient descent satisfies

$$\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2 \leq \left(1 - \frac{\mu}{L}\right) \|\mathbf{x}_t - \mathbf{x}^*\|^2, \quad t \geq 0.$$

Proof by rewriting vanilla analysis with first definition of strong convexity and sufficient decrease.

**Convergence**  $\mathcal{O}(\log 1/\epsilon)$  (Theorem 3.14): Choosing  $\gamma := 1/L$ , gradient descent yields

$$f(\mathbf{x}_T) - f^* \leq \frac{L}{2} \left(1 - \frac{\mu}{L}\right)^T \|\mathbf{x}_0 - \mathbf{x}^*\|^2, \quad T > 0.$$

Proof by using geometrically decreasing distances and smoothness with  $\nabla f(\mathbf{x}^*) = \mathbf{0}$ .

#### 4 Projected gradient descent

Optimization within closed convex subset  $X \subseteq \mathbb{R}^d$ .

$$\begin{aligned} \mathbf{y}_{t+1} &:= \mathbf{x}_t - \gamma \nabla f(\mathbf{x}) \\ \mathbf{x}_{t+1} &:= \Pi_X(\mathbf{y}_{t+1}) := \operatorname{argmin}_{\mathbf{x} \in X} \|\mathbf{x} - \mathbf{y}_{t+1}\|^2. \end{aligned}$$

After every step, project back onto  $X$ .

**Projection properties** (Fact 4.1):  $\mathbf{x} - \Pi_X(\mathbf{y})$  and  $\mathbf{y} - \Pi_X(\mathbf{y})$  form an obtuse angle,

- $(\mathbf{x} - \Pi_X(\mathbf{y}))^\top (\mathbf{y} - \Pi_X(\mathbf{y})) \leq 0$ .
- $\|\mathbf{x} - \Pi_X(\mathbf{y})\|^2 + \|\mathbf{y} - \Pi_X(\mathbf{y})\|^2 \leq \|\mathbf{x} - \mathbf{y}\|^2$ .

**Lipschitz functions** ( $\mathcal{O}(1/\epsilon^2)$ ) (Theorem 4.2): Same bound as gradient descent. Proof by replacing  $\mathbf{x}_{t+1}$  by  $\mathbf{y}_{t+1}$  in the vanilla analysis and using the second projection property with  $\mathbf{x} = \mathbf{x}^*$  and  $\mathbf{y} = \mathbf{y}_{t+1}$ .

**Sufficient decrease** (Lemma 4.3): If  $f$  is  $L$ -smooth, choosing stepsize  $\gamma := 1/L$ , we get

$$f(\mathbf{x}_{t+1}) \leq f(\mathbf{x}_t) - \frac{1}{2L} \|\nabla f(\mathbf{x}_t)\|^2 + \frac{L}{2} \|\mathbf{y}_{t+1} - \mathbf{x}_{t+1}\|^2.$$

Proof by the same as gradient descent, but then with projection step.

**Smooth functions** ( $\mathcal{O}(1/\epsilon)$ ) (Theorem 4.4): Same result as in gradient descent. Proof by compensating for the extra term in sufficient decrease by the vanilla analysis.

**Strongly convex** ( $\mathcal{O}(\log 1/\epsilon)$ ) (Theorem 4.5): Decreasing distances still holds, but extra term in convergence bound when choosing  $\gamma := 1/L$ ,

$$\begin{aligned} f(\mathbf{x}_T) - f^* &\leq \|\nabla f(\mathbf{x}^*)\| \left(1 - \frac{\mu}{L}\right)^{T/2} \|\mathbf{x}_0 - \mathbf{x}^*\| \\ &\quad + \frac{L}{2} \left(1 - \frac{\mu}{L}\right)^T \|\mathbf{x}_0 - \mathbf{x}^*\|^2. \end{aligned}$$

This is due to the fact that we cannot use  $\nabla f(\mathbf{x}^*) = \mathbf{0}$  in the constrained case.

#### 5 Coordinate descent

Update only one coordinate of  $\mathbf{x}_t$  at a time, meaning that we only need to compute the gradient of one coordinate of  $\nabla f(\mathbf{x}_t)$ .

**PL inequality:**  $f$  has a global minimum  $\mathbf{x}^*$ . Definition  $\forall \mathbf{x} \in X$ :

$$\frac{1}{2} \|\nabla f(\mathbf{x})\|^2 \geq \mu(f(\mathbf{x}) - f(\mathbf{x}^*)).$$

**Strong convexity**  $\Rightarrow$  **PL inequality** (Lemma 5.2).

**Coordinate-wise smoothness:**  $f$  is coordinate-wise smooth with  $\mathcal{L} = [L_1, \dots, L_d] \in \mathbb{R}_+^d$  if  $\forall \mathbf{x}, \mathbf{y} \in X, i \in [d]$ :

$$f(\mathbf{x} + \lambda \mathbf{e}_i) \leq f(\mathbf{x}) + \lambda \nabla_i f(\mathbf{x}) + \frac{L_i}{2} \lambda^2.$$

This gives a more fine-grained picture of  $f$  than smoothness. It might be the case that all  $L_i$  are significantly smaller than the best possible  $L$ -smoothness.

**Update rule:**

choose an active coordinate  $i \in [d]$

$$\mathbf{x}_{t+1} := \mathbf{x}_t - \gamma_i \nabla_i f(\mathbf{x}_t) \mathbf{e}_i.$$

**Coordinate-wise sufficient decrease** (Lemma 5.5): With stepsize  $\gamma_i = 1/L_i$ , coordinate descent satisfies

$$f(\mathbf{x}_{t+1}) \leq f(\mathbf{x}_t) - \frac{1}{2L_i} |\nabla_i f(\mathbf{x}_t)|^2.$$

**Randomized coordinate descent convergence** (Theorem 5.6):  $f$  is coordinate-wise smooth with  $L$  and satisfies PL-inequality with  $\mu$ . Choosing  $\gamma_i = \frac{1}{L}$ , we get

$$\mathbb{E}[f(\mathbf{x}_T) - f^*] \leq \left(1 - \frac{\mu}{dL}\right)^T (f(\mathbf{x}_0) - f^*).$$

Proof by using coordinate-wise sufficient decrease and taking expectation with respect to  $i$  on both sides. Then, expectation over  $\mathbf{x}_t$  to remove condition.

**Importance sampling convergence** (Theorem 5.7): Sample  $i$  with probability  $L_i / \sum_{j=1}^d L_j$ . Let  $\bar{L} = 1/d \sum_{i=1}^d L_i$ . Choosing  $\gamma_i = 1/L_i$ , we get

$$\mathbb{E}[f(\mathbf{x}_T) - f^*] \leq \left(1 - \frac{\mu}{d\bar{L}}\right)^T (f(\mathbf{x}_0) - f^*).$$

Proof by the same method as randomized coordinate descent.

**Steepest coordinate descent convergence** (Corollary 5.8): Choose index with largest absolute gradient. Same conditions as randomized coordinate descent. Then, we get

$$f(\mathbf{x}_T) - f^* \leq \left(1 - \frac{\mu}{dL}\right)^T (f(\mathbf{x}_0) - f^*).$$

TODO: Strong convexity with respect to  $\ell_1$ -norm.

**Greedy coordinate descent:** Choose the index by one of the above methods, but then perform a line search over that coordinate and minimize by solving a 1-dimensional optimization problem (easy). This does not require  $f$  to be differentiable. But, this does not always return the global minimum, since there are functions with points where it can make no progress.

Theorem 5.11: Let  $f$  be of the form  $f(\mathbf{x}) := g(\mathbf{x}) + h(\mathbf{x})$  with  $h(\mathbf{x}) = \sum_i h_i(x_i)$ ,  $h_i$  convex, and  $g$  convex and differentiable. If  $\mathbf{x}$  is a point that greedy coordinate descent cannot make progress in any coordinate, then  $\mathbf{x}$  is a global minimum of  $f$ .

#### 6 Nonconvex functions

For nonconvex functions, gradient descent may get stuck in a local minimum, stuck in a saddle point (flat region), or infinitely decrease, but never reach a critical point (e.g.  $1/e^x$ ).

**Gradient convergence** (Theorem 6.2):  $f$  is  $L$ -smooth. Choosing  $\gamma := 1/L$ , then

$$\frac{1}{T} \sum_{t=0}^{T-1} \|\nabla f(\mathbf{x}_t)\|^2 \leq \frac{2L}{T} (f(\mathbf{x}_0) - f^*).$$

In particular,  $\|\nabla f(\mathbf{x}_t)\|^2 \leq \frac{2L}{T} (f(\mathbf{x}_0) - f^*)$  for some  $t \in [T-1]$ , and  $\lim_{t \rightarrow \infty} \|\nabla f(\mathbf{x}_t)\|^2 = 0$ . This does not mean that it converges to a critical point, since it may never reach a point with 0 gradient, but only move toward it asymptotically. Proof by sufficient decrease, which does not require convexity.

$\gamma := 1/L$  **does not overshoot critical points** (Lemma 6.3).

TODO: Trajectory analysis.

#### 7 The Frank-Wolfe algorithm

Constrained optimization algorithm without projection (which can be very complex) by making use of linear minimization oracle:

$$\text{LMO}_X(\mathbf{g}) := \operatorname{argmin}_{\mathbf{z} \in X} \mathbf{g}^\top \mathbf{z}.$$

The algorithm is then

$$\begin{aligned}\mathbf{s}_t &:= \text{LMO}_X(\nabla f(\mathbf{x}_t)) \\ \mathbf{x}_{t+1} &:= (1 - \gamma_t)\mathbf{x}_t + \gamma_t \mathbf{s}_t.\end{aligned}$$

Reduces non-linear constrained optimization to linear optimization over the same set. Rationale is that the gradient defines the best linear approximation of  $f$  at  $\mathbf{x}_t$ .

**Properties:** (1) iterates are always feasible, i.e., in  $X$ , (2) projection-free, which can be very complex, and (3) iterates have a simple sparse representation, i.e.,  $\mathbf{x}_t$  is always a convex combination of  $\mathbf{x}_0$  and the minimizers  $\mathbf{s}_{1:t-1}$ .

Let  $X = \text{conv}(\mathcal{A})$ , then every  $\mathbf{s} := \text{LMO}_X(\mathbf{g}) \in \text{conv}(X)$  is a convex combination of atoms,  $\mathbf{s} = \sum_{i=1}^n \lambda_i \mathbf{a}_i$  with  $\sum_{i=1}^n \lambda_i = 1$ . Furthermore, there is always an atom in  $\mathcal{A}$  that minimizes the LMO.

**$\ell_1$ -ball:** The LMO for  $X = \{\mathbf{x} \in \mathbb{R}^d \mid \|\mathbf{x}\|_1 \leq 1\}$  is given by

$$\text{LMO}_X(\mathbf{g}) = -\text{sign}(g_i)\mathbf{e}_i \text{ with } i := \underset{i \in [d]}{\text{argmax}} |g_i|.$$

TODO: Spectahedron.

**Duality gap** (Lemma 7.2): We can easily compute an upper bound of the optimality gap,

$$g(\mathbf{x}) := \nabla f(\mathbf{x})^\top (\mathbf{x} - \mathbf{s}) \geq f(\mathbf{x}) - f^*,$$

with  $\mathbf{s} := \text{LMO}_X(\nabla f(\mathbf{x}))$ . At any optimal point  $\mathbf{x}^*$ ,  $g(\mathbf{x}^*) = 0$ .

Proof by using  $\nabla f(\mathbf{x})^\top \mathbf{s} \leq \nabla f(\mathbf{x})^\top \mathbf{x}^*$ , and the first-order characterization of convexity.

**Descent** (Lemma 7.4): For a step  $\mathbf{x}_{t+1} := \mathbf{x}_t - \gamma_t(\mathbf{s} - \mathbf{x}_t)$  with stepsize  $\gamma_t \in [0, 1]$ , it holds that

$$f(\mathbf{x}_{t+1}) \leq f(\mathbf{x}_t) - \gamma_t f'(\mathbf{x}_t) + \gamma_t^2 \frac{L}{2} \|\mathbf{s} - \mathbf{x}_t\|^2,$$

with  $\mathbf{s} := \text{LMO}_X(\nabla f(\mathbf{x}))$ . Proof by first definition of smoothness and duality gap.

**Convergence analysis** ( $\mathcal{O}(1/\epsilon)$ ) (Theorem 7.3):  $f$  is  $L$ -smooth and convex. With  $\gamma_t = 2/(t+2)$ , Frank-Wolfe yields

$$f(\mathbf{x}_T) - f^* \leq \frac{2L \text{diam}(X)^2}{T+1}.$$

Proof by duality gap and descent lemma, and then induction.

**Linear search stepsize:** Choose  $\gamma_t \in [0, 1]$  such that the progress is maximized,

$$\gamma_t := \underset{\gamma \in [0,1]}{\text{argmin}} f((1 - \gamma)\mathbf{x}_t + \gamma \mathbf{s}).$$

The descent lemma still holds for this stepsize, since this stepsize can only be better than a predetermined stepsize. And, thus the convergence also holds.

TODO: Gap-based stepsize.

TODO: Affine invariance.

TODO: Curvature constant.