

Computational Intelligence Lab: Linear Algebra Recap

Cristian Perez Jensen

April 14, 2024

Contents

1	<i>Inner product and norms</i>	1
2	<i>Vector spaces</i>	2
3	<i>Matrices</i>	2
3.1	<i>Invertible matrices</i>	3
3.2	<i>Trace</i>	3
3.3	<i>Orthogonal projection</i>	3
3.4	<i>Special matrices</i>	4

1 Inner product and norms

Definition 1 (Inner product). An inner product $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ is an operation defined on a vector space \mathcal{V} that satisfies the following properties $\forall x, y, z, a, b \in \mathbb{R}$,

- Commutativity,

$$\langle x, y \rangle = \langle y, x \rangle;$$

- Linearity,

$$\langle x, ay + bz \rangle = a\langle x, y \rangle + b\langle x, z \rangle;$$

- Positive definiteness,

$$x \neq 0 \implies \langle x, x \rangle > 0$$

$$x = 0 \iff \langle x, x \rangle = 0;$$

- Bilinearity (follows from commutativity and linearity),

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$$

$$\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle.$$

Corollary. $\langle x + y, x + y \rangle = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle.$

The vector space \mathcal{V} , along with an inner product, defines an inner vector space. During this course, we will assume that we always work with real vectors in \mathbb{R}^n . An example of an inner product is the dot product,¹

$$x \cdot y = x^\top y \in \mathbb{R}.$$

¹ Usually, this operation is what is meant by the inner product.

Definition 2 (Norm). A norm $\| \cdot \| : \mathbb{R}^n \rightarrow \mathbb{R}$ is a function that can be thought of as a way of measuring the distance from the origin. Norms satisfy the following properties,

- Positive definiteness, $x \neq 0 \implies \|x\| > 0$;
- Triangle inequality, $\|x + y\| \leq \|x\| + \|y\|$;
- Cauchy-Schwarz inequality, $|\langle x, y \rangle| \leq \|x\| \|y\|$.

Corollary. For the Euclidean norm, the following holds,

$$\cos \theta = \frac{\langle x, y \rangle}{\|x\| \|y\|},$$

where θ is the angle between x and y .

Each inner product defines a canonical norm $\|x\| \doteq \sqrt{\langle x, x \rangle}$. For ex-

ample, the Euclidean norm is defined by the dot product,

$$\|x\|_2 = \sqrt{x^\top x} = \sqrt{\sum_{i=1}^n x_i^2}.$$

The p -norm is a generalization of the Euclidean norm,

$$\|x\|_p = \sqrt[p]{\sum_{i=1}^n |x_i|^p}.$$

2 Vector spaces

The *vector space* \mathbb{R}^m consists of all column vectors with m elements. For a set of vectors $\mathcal{C} = \{c_1, \dots, c_n \mid c_i \in \mathbb{R}^m\}$, we can define a subspace spanned by this set, denoted by $\text{span}(\mathcal{C})$. It is the set of all possible linear combinations of elements of \mathcal{C} . If a set of vectors that span a subspace are independent, they are called a *basis*, $b_1, \dots, b_k \in \mathbb{R}^m$. The number of basis vectors defines the *dimensionality* of the subspace.²

Observation. The following facts hold about subspaces,

- Every subspace contains the zero vector 0 ;
- If x and y are in the subspace, then $x + y$ is also in the subspace;
- If x is in the subspace and $a \in \mathbb{R}$, then ax is also in the subspace.

Definition 3 (Orthogonal subspaces). Subspaces \mathcal{V} and \mathcal{W} are orthogonal when $v^\top w = 0$ for all $v \in \mathcal{V}, w \in \mathcal{W}$.

3 Matrices

The rank r of a matrix $A \in \mathbb{R}^{m \times n}$ is the dimensionality of its *column space*. It is bounded by

$$r \leq \min\{m, n\}.$$

A matrix $A \in \mathbb{R}^{m \times n}$ defines 4 fundamental subspaces,

- Column space $\subseteq \mathbb{R}^m$ (r dimensional), $\{b \mid Ax = b\}$;
- Null space $\subseteq \mathbb{R}^n$ ($n - r$ dimensional), $\{x \mid Ax = 0\}$;
- Row space $\subseteq \mathbb{R}^n$ (r dimensional), $\{b \mid A^\top x = b\}$;
- Left null space $\subseteq \mathbb{R}^m$ ($m - r$ dimensional), $\{x \mid A^\top x = 0\}$.

The row space $\text{row}(A)$ is the *orthogonal complement* of the null space $\text{null}(A)$, thus $\text{row}(A) + \text{null}(A) = \mathbb{R}^n$. Similarly, $\text{col}(A) + \text{null}(A^\top) = \mathbb{R}^m$.

² We know that the amount of basis vectors must be smaller than the amount of vectors that span the subspace, which must be smaller than the dimensionality of the space,

$$k \leq m \leq n.$$

The matrix is full-rank if $r = \min\{m, n\}$.

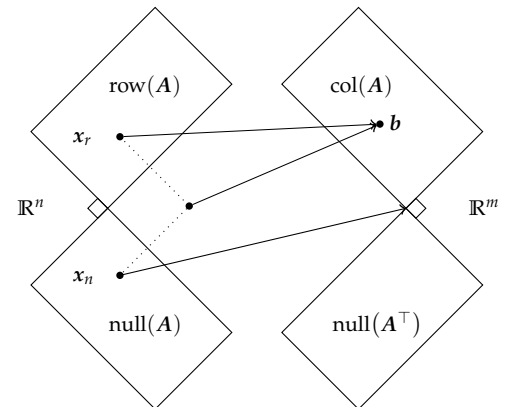


Figure 1. Illustration of the 4 spaces defined by a matrix A . It shows the perpendicular spaces. Furthermore, it shows that $Ax_r = b$ for some $x_r \in \text{col}(A)$. Also, if you add a vector from the null space to the row vector, it still maps to the same b , $A(x_r + x_n) = Ax_r + Ax_n = Ax_r = b$.

3.1 Invertible matrices

A matrix is only invertible if it is a square full-rank matrix, i.e., $r = m = n$.

Properties. Let $A \in \mathbb{R}^{n \times n}$ be a full-rank matrix and $k \in \mathbb{R}$, then

$$\begin{aligned} A^{-1}A &= AA^{-1} = I \\ (kA)^{-1} &= \frac{1}{k}A^{-1} \\ \det(A^{-1}) &= \frac{1}{\det(A)} \\ (AB)^{-1} &= B^{-1}A^{-1}. \end{aligned}$$

The Moore-Penrose inverse (pseudo-inverse) is a generalization of the inverse. It is the solution to the general least squares problem $\min_x \|Ax - b\|_2$. For full-rank matrices, the left pseudo-inverse can be computed by

$$A^+ = (A^\top A)^{-1}A^\top.$$

The right pseudo-inverse can be computed as

$$A^+ = A^\top(AA^\top)^{-1}.$$

The left pseudo-inverse can be computed if $r = n$ and has the property $A^+A = I_{n \times n}$.

The right pseudo-inverse can be computed if $r = m$ and has the property $AA^+ = I_{m \times m}$.

3.2 Trace

Definition 4 (Trace). The trace of a square matrix $A \in \mathbb{R}^{n \times n}$ is the sum of its diagonal,

$$\text{tr}(A) = \sum_{i=1}^n a_{ii}.$$

Properties. Let $A, B, C \in \mathbb{R}^{n \times n}$, $x, y \in \mathbb{R}^n$, $c, d \in \mathbb{R}$, then

$$\begin{aligned} \text{tr}(cA + dB) &= c \cdot \text{tr}(A) + d \cdot \text{tr}(B) \\ \text{tr}(A) &= \text{tr}(A^\top) \\ \text{tr}(AB) &= \text{tr}(BA) \\ \text{tr}(ABC) &= \text{tr}(CAB) = \text{tr}(BCA) \\ \text{tr}(A^\top B) &= \text{tr}(B^\top A) = \text{tr}(AB^\top) = \text{tr}(BA^\top) \\ x^\top y &= \text{tr}(x^\top y) = \text{tr}(xy^\top). \end{aligned}$$

Linearity.

Cyclic property.

xy^\top is a rank-1 matrix.

Furthermore, the trace of a matrix is equal to the sum of the eigenvalues of the matrix.

3.3 Orthogonal projection

The projection a_1 of a vector a on another vector b can be computed as

$$a_1 = \frac{b^\top a}{b^\top b} b.$$

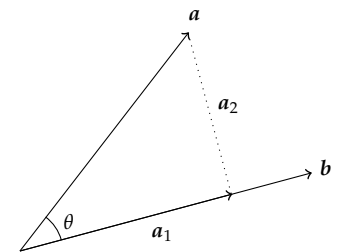


Figure 2. Projection of a on b , denoted a_1 , and the rejection of a from b , denoted a_2 .

The rejection can then be computed as

$$a_2 = a - a_1.$$

We can also project a vector a onto the column space of a matrix $B \in \mathbb{R}^{m \times n}$, denoted a_B . This can be computed by a matrix multiplication,

$$P = B(B^\top B)^{-1}B^\top,$$

which is equivalent to the left Moore-Penrose inverse.

Definition 5 (Projection matrix). A square matrix $P \in \mathbb{R}^{n \times n}$ is called a projection matrix if it is idempotent, *i.e.*, $PP = P$.

Properties. Let P be an orthogonal projection matrix, then $P = P^\top$, *i.e.*, P is symmetric. Furthermore, the eigenvalues of a projection matrix are all ones and zeros, because of idempotency.

3.4 Special matrices

Definition 6 (Orthogonal matrix). An orthogonal matrix is an invertible matrix whose columns q_1, \dots, q_n are all orthogonal to each other and of unit length, *i.e.*,

$$\begin{aligned} q_i^\top q_j &= 0 \quad \forall i \neq j \in [n] \\ q_i^\top q_i &= 1 \quad \forall i \in [n]. \end{aligned}$$

Properties. Let $Q \in \mathbb{R}^{n \times n}$ be an orthogonal matrix,

$$\begin{aligned} Q^\top &= Q^{-1} \\ \langle x, y \rangle &= \langle Qx, Qy \rangle. \end{aligned}$$