Chapter 1

Randomized BA

Problem 1.1 Let $A \in \mathbb{R}^{d\times p}$ and $b \in \mathbb{R}^p$. For every $i \in I := \{1, \dots, N\}$, let $\{B_k\}_{i=1}^N \in \mathbb{R}^{p\times d}$. $b \in \mathbb{R}^p$. The problem is to find $\bar{x} \in \mathbb{R}^p$ such that

$$(\forall i \in I) \quad \bar{x} \in S_i := \{x \in \mathbb{R}^p \mid B_i A x = B_i b\}, \tag{1.1}$$

under the assumption that $S := \bigcap_{i\in I} S_i \neq \emptyset$

We refer to the modified algorithm as the random iteration of BA, which guarantees the convergence of the iterations for a appropriate choice of the parameter α:

$$x^{k+1} = x^k + \alpha_{\xi_k} B_{\xi_k} \left(b - Ax^k\right) \tag{1.2}$$

where $(\xi_k)_{k=1}^{\infty}$ is a famililly of i.i.d. S-valued random variables.

For every $i \in I$, define $C_i = B_i A$ and $T_i = \mathrm{Id} - \alpha_i C_i$. Then the iteration matrix (1.2) is equal to:

$$x^{k+1} = T_{\xi_k} x^k + \alpha_{\xi_k} B_{\xi_k} b. \tag{1.3}$$

Moreover, assume that

$$\begin{aligned} T\colon \quad & \mathbb{R}^p \times I && \mapsto \mathbb{R}^p \\ & (x, i) && \mapsto x + \alpha_i B_i \left(b - Ax\right) \end{aligned}$$

is measurable.

This scheme is motivated by the following problem,

$$\bar{x} := \mathrm{argmin}_{x\in\mathbb{R}^p}\{\|Ax - b\|^2\} = (A^T A)^{-1} A^T b \tag{1.4}$$

Theorem 1.2 *Let $x_0 \in \mathbb{R}^p$. For every $i \in I$, let α_i the stepsize, such that*

$$(\forall i \in I) \quad 2\alpha_i C_i - \alpha_i^2 C_i^T C_i \quad \textit{is positive semidefinite.} \tag{1.5}$$

Then the sequence generated by Random BA iteration (1.3) converges almost surely to a S-valued random variable.

Proof. Let $\bar{x} \in S_i$ and $x \notin S_i$

$$\begin{aligned}
\|Tx - \bar{x}\|^2 =& \|x - \bar{x}\|^2 - 2\alpha_i \Big\langle B_i \left(b - Ax^k\right) \mid x - \bar{x} \Big\rangle + \alpha_i^2 \|B_i \left(b - Ax^k\right)\|^2 \\
=& \|x - \bar{x}\|^2 - 2\alpha_i \Big\langle B_i A \left(\bar{x} - x^k\right) \mid x - \bar{x} \Big\rangle + \alpha_i^2 \|B_i A \left(x^k - \bar{x}\right)\|^2 \\
=& \|x - \bar{x}\|^2 - \Big\langle 2\alpha_i C_i \left(\bar{x} - x^k\right) \mid x - \bar{x} \Big\rangle + \Big\langle \alpha_i^2 C_i^T C_i \left(x^k - \bar{x}\right) \mid x^k - \bar{x} \Big\rangle \\
=& \|x - \bar{x}\|^2 - \Big\langle (2\alpha_i C_i - \alpha_i^2 C_i^T C_i) \left(\bar{x} - x^k\right) \mid x - \bar{x} \Big\rangle \\
<& \|x - \bar{x}\|^2,
\end{aligned} \tag{1.6}$$

and for every $i \in I$ T_i is paracontractive and the result follows from [1, Theorem 3.6] □

Theorem 1.3 *Let $x_0 \in \mathbb{R}^p$. For every $i \in I$, let α_i the stepsize and let $p_i = \mathbb{P}(\xi^{-1}(i)) > 0$, such that*

$$\rho(\mathrm{Id} - \sum_{i \in I} (p_i w_i B_i A)) < 1 \tag{1.7}$$

Then, for every $\bar{x} \in S$

$$\mathbb{E}(x^k - \bar{x}) \to 0 \quad \textit{and} \quad k \to 0 \tag{1.8}$$

Proof. Let $\bar{x} \in S$ and $k \in \mathbb{N}$. It follows from (1.2) that

$$\begin{aligned}
\mathbb{E}(x^{k+1} - \bar{x}) &= x^k - \bar{x} - \sum_{i \in I} p_i w_i B_i (A_i x^k - b) \\
&= x^k - \bar{x} - \sum_{i \in I} p_i w_i B_i A (x^k - \bar{x}) \\
&= \left(\mathrm{Id} - \sum_{i \in I} (p_i w_i B_i A)\right) (x^k - \bar{x}) \\
&= \left(\mathrm{Id} - \sum_{i \in I} (p_i w_i B_i A)\right)^{k+1} (x^0 - \bar{x})
\end{aligned} \tag{1.9}$$

□

Remark 1.4 We explore different instances of this method

- If $N = 1$ and $B_1 = A^T$ we have that $C = A^T A$ and

  $$2\alpha x^T A^T A x - \alpha^2 x^T C^T C x = 2\alpha \|Ax\|^2 - \alpha^2 \|A^T A x\|^2 \geq (2\alpha - \alpha^2 \|A\|^2) \|Ax\|^2, \tag{1.10}$$

  which is positive if $\alpha \in]0, \frac{2}{\|A\|^2}[$. Then (1.2) is reduced to the method proposed in [2] and its convergence is deduced from Theorem 1.2.

Bibliography

[1] N. Hermer, L.D. Russell, and A. Sturm, *Random Function Iterations for Consistent Stochastic Feasibility* 2019.

[2] H. W. Engl, M. Hanke, and A. Neubauer. Regularization of Inverse Problems (Mathematics and its Applications). Dordrecht, The Netherlands: Springer, vol. 375, 1996.

[3] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces.*2nd edn. Springer, New York (2017).