Day 4: MTH 144 : Precalculus

Combining functions and Inverses

1. Adding, subtracting, multiplying and dividing functions

Two functions f and g can be combined to form new functions $f+g$, $f-g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers.

ALGEBRA OF FUNCTIONS

Let f and g be functions with domains A and B. Then the functions $f+g$, $f-g$, fg, and f/g are defined as follows.

$(f+g)(x) = f(x) + g(x)$	Domain $A \cap B$
$(f-g)(x) = f(x) - g(x)$	Domain $A \cap B$
$(fg)(x) = f(x)g(x)$	Domain $A \cap B$
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	Domain $\{x \in A \cap B \mid g(x) \neq 0\}$

Practice: Perform the following function operations for f and g and find the domains of the resulting functions.

$f(x) = \dfrac{1}{x-2}$ and $g(x) = \sqrt{x}$.

(a) $f+g$

(b) $f-g$

(c) fg

(d) $\dfrac{f}{g}$

2. Composition of Functions

Another way to combine two functions f and g is to compose them. Composing the function $f(x)$ with the function $g(x)$, denoted $(f \circ g)(x)$, means that the output of g becomes the input of f. In other words $(f \circ g)(x) = f(g(x))$ or the function f is evaluated at $g(x)$.

For example if $f(x) = 5x - 3$ and $g(x) = 2 - x^2$ then $(f \circ g)(x) =$

What is the domain of $f \circ g$?

Example: Let $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{4}{x-1}$. Find the following:

(a) $f \circ g$ and its domain

(b) $g \circ f$ and its domain

NOTE The domain of $f \circ g$ is the intersection of the domain of inner function g and the resulting function $f \circ g$.

3. Applications with composition of functions

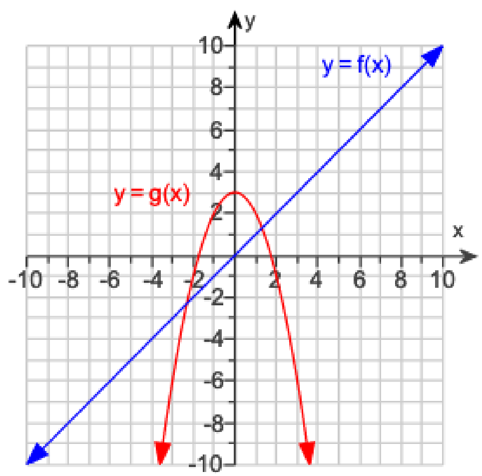
The weekly cost C of producing x units is given by $C(x) = 60x + 750$. The number x of units produced in t hours is given by $x(t) = 50t$.

(a) Find and interpret $(C \circ x)(t)$

(b) Find the time that must elapse in order for the cost to increase to $\$15,000$.

Practice: Exercise 1

Use a graphical approach to answer the following questions about the two functions graphed. Find $(f+g)(0), (f-g)(-1), (fg)(1), (f/g)(2), (f \circ g)(1), (g \circ f)(1)$.



Practice: Exercise 2

Let $f(x) = \dfrac{x}{x+3}$ and $g(x) = 8x - 3$. Find the following

1. $(f \circ g)(x)$ and its domain

2. $(g \circ f)(x)$ and its domain

Practice: Exercise 3

Let $f(x) = x^3 + 8$ and $g(x) = x - 5$ and $h(x) = \sqrt{x}$. Find $f \circ g \circ h$.

Practice: Exercise 5

A spherical balloon is being inflated. The radius of the balloon is increasing at a rate of 3cm/s.

1. Find a function f that models the radius as a function of time t, in seconds.

2. Find a function g that models the volume as a function of the radius r, in cm.

3. Find and interpret $g \circ f$.

8. One-to-one Functions

A function f is **one-to-one** if there is a one-to-one correspondence between the inputs and the outputs. In other words, f never takes on the same values twice.

Examples: Is f one-to-one?

$f(x) = x^2$

$f(x) = x^3$

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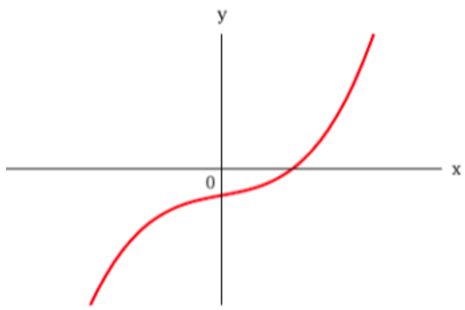
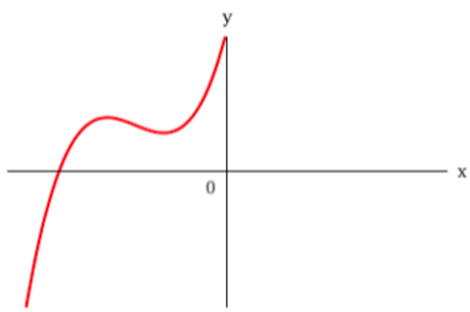
How to determine if a function f is one-to-one - Graphically
Determine if the functions f and g are one-to-one.



(a)



(b)

Practice - Restricted Domain
Graph the function $f(x) = (x+2)^2 - 1$ and decide if it is one-to-one on its entire domain.

9. Inverse Functions

A one-to-one function with domain A and range B has an **inverse function** which takes on as domain the range of f and gives as output the domain of f. In other words the inverse of a function f undoes what f does. The inverse, denoted as f^{-1} takes as input the output of f and gives as output the input of f, i.e. if $f(x) = y$ then $f^{-1}(y) = x$.

For example if f is the function that takes the input x multiplies it by 5, adds 2 then takes the 5th power of the result, what would the inverse f^{-1} have to be to undo what f did?

The inverse function f^{-1} reverses the effect of f!

INVERSE FUNCTION PROPERTY

Let f be a one-to-one function with domain A and range B. The inverse function f^{-1} satisfies the following cancellation properties:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Conversely, any function f^{-1} satisfying these equations is the inverse of f.

Practice: Are f and g inverses of each other?

$f(x) = 8 - 4x, \quad g(x) = \dfrac{8-x}{4}$

NOTE: The function and its inverse switch domain and range:
Domain of f =
Range of f =

10. How to find the inverse of a one-to-one function - Algebraically

Let f be a one-to one function. Then $f(x) = y$ for any x in the domain of f and f has an inverse denoted f^{-1}.

Example: Let $f(x) = \dfrac{x-9}{x+9}$.

(a) Is f one-to-one?

(b) Find the domain of f.

(c) Can you find the range of f without graphing it?

(d) Find the inverse of f.

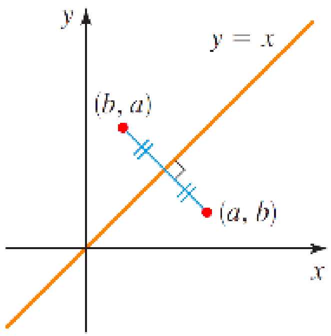
(e) Can you find the range of f now using the information you have about f^{-1}?

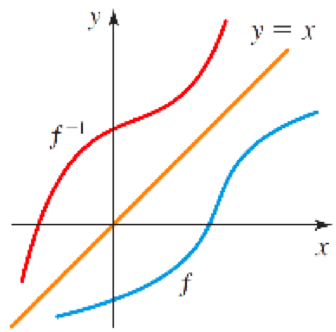
Find the inverse $g(x) = \sqrt[11]{7x}$.

11. How to find the inverse of a one-to-one function - Graphically

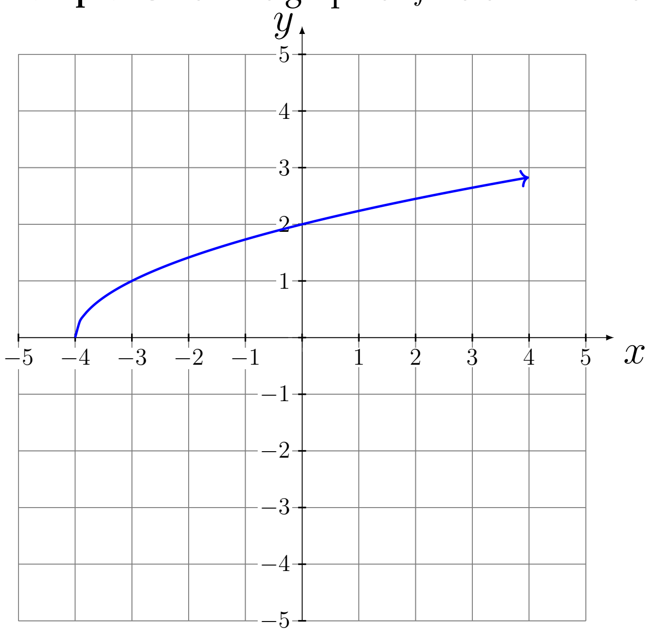
Let f be a one-to one function. Then $f(x) = y$ for any x in the domain of f and f has an inverse f^{-1}. A point (a, b) on the graph of f is going to become (b, a) on the graph of f^{-1} since the input is switched with the output.

The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.





Example: Given the graph of f below find the graph of f^{-1}.



Practice: Exercise 1

Find $(g \circ f^{-1})(2)$

x	0	1	2	3	4
f(x)	1	4	5	2	3

x	−1	1	2	3	4
g(x)	5	1	4	2	3

Practice: Exercise 2

Find $f(1), f^{-1}(-3), f^{-1}(0), (f^{-1} \circ f)(2)$