Day 2 Notes (we finished the notes from Day 1)

- Make sure you register for AleKS
- Homework 1 is open and due by Thursday Jan 30th

Definition. An **alternating sequence** is a sequence with terms that alternate in sign. $-,+,-,+,\dots$ $n=1$ $(-1)^n$

$+,-,+,-,\dots$ $n=1$ $(-1)^{n+1}$

2. Recursively defined sequences

Some sequences do not have simple defining formulas like those of the preceding example.

Definition. When the nth term of a sequence may depend on some or all of the terms preceding it we call the sequence recursive sequence.

Example 4. Example of recursive sequence:

$a_1 = 4$ and $\underbrace{a_n}_{\text{next term}} = 2\underbrace{a_{n-1}}_{\text{previous term}} + 1$ for $n > 1$.

ex: $5, 10, 15, 20, \dots, a_{n-1}, a_n, a_{n+1}, \dots$

$a_n = 5n$

$\begin{cases} a_1 = 5 \\ a_n = a_{n-1} + 5 \end{cases}$ same sequence defined recursively

$\{a_n\} = \{4, 9, 19, 39, \dots\}$

$a_1 = 4$

$n=2$: $a_2 = 2a_{2-1} + 1 = 2a_1 + 1 = 2(4)+1 = 9$

$n=3$: $a_3 = 2a_2 + 1 = 2\cdot 9 + 1 = 19$

$n=4$: $a_4 = 2a_3 + 1 = 2\cdot 19 + 1 = 39$

Example 5. The Fibonacci Sequence:

Find the first 7 terms of the sequence defined recursively by

$a_1 = 1$, $a_2 = 1$ and $\boxed{a_n = a_{n-1} + a_{n-2}}$. next is sum of previous two terms

$\{1, 1, 2, 3, 5, 8, 13, \dots$

$a_1 = 1$

$a_2 = 1$

$a_3 = a_2 + a_1 = 1+1 = 2$

$a_4 = a_3 + a_2 = 2+1 = 3$

$a_5 = a_4 + a_3 = 3+2 = 5 \dots$

If we add the terms of a sequence we obtain a **series**.

Sequence: $\{a_n\} = \{a_1, a_2, a_3, \dots\}$

ex: $\{a_n\} = \{5, 10, 15, \dots\}$

Series: $a_1 + a_2 + a_3 + \dots = \sum_{i=1}^{\infty} a_i$ infinite sum

ex: $5 + 10 + 15 + 20 + 25 + \dots$

Partial sum: is a finite series or a finite sum of numbers or a sum of numbers which can simply be added to a total

ex: $a_1 + a_2 + a_3 = S_3 \rightarrow$ the third partial sum.

nth partial sum $S_n = a_1 + a_2 + \dots + a_n$

ex: $S_3 = 5 + 10 + 15 = 30$

$S_5 = 5 + 10 + 15 + 20 + 25 = 75$

Sigma notation: [1]

is a short way (compact) of writing sums.

We use the greek letter $\sum$ to denot sum.

sigma

$S_{10} = a_1 + a_2 + a_3 + \dots + a_{10} = \sum_{k=1}^{10} a_k$

index

$S_5 = \sum_{i=1}^{5} a_i \qquad S_n = \sum_{j=1}^{n} a_j$

general term

Example 6. Write the sum using summation notation.

1. $5 + 10 + 15 + 20 + 25 + 30 = \sum_{i=1}^{6} 5i = S_6$ (a_1, a_2, a_6)
2. $\dfrac{\sqrt{3}}{1} + \dfrac{\sqrt{4}}{2} + \dfrac{\sqrt{5}}{3} + \dots + \dfrac{\sqrt{n+2}}{n} = S_n = \sum_{t=1}^{n} \dfrac{\sqrt{t+2}}{t}$
3. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \boxed{\sum_{c=1}^{\infty} \frac{1}{2^c}}$ or $\boxed{\sum_{n=1}^{\infty} \frac{1}{2^n}}$

[1] We use **Sigma notation** for both infinite series and finite series.

Example 7. Write the terms for ~~each series~~ sum and evaluate the sum.
$\sum_{n=1}^{5} 2n+3$ $= (2\cdot1+3) + (2\cdot2+3) + (2\cdot3+3) + (2\cdot4+3) + (2\cdot5+3)$

$n=1 \quad n=2 \quad n=3 \quad n=4 \quad n=5$

$= 5 + 7 + 9 + 11 + 13 = 45$

Note: In Calculus you will learn how to find what an infinite series adds up to for certain types of series.

Example 8. Consider the series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

Lets investigate the first few partial sums:

$S_1 =$ $1/2$
$S_2 =$ $1/2 + 1/4 = 0.75$
$S_3 =$ $1/2 + 1/4 + 1/8 = 0.875$
$S_4 =$ $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 0.9375$

$S_5 = S_4 + \frac{1}{32} = 0.9375 + \frac{1}{32}$
$S_5 = 0.96875$

As we continue adding terms, the sum gets closer and closer to 1.

Example 9. Consider the series:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$$

Lets investigate the first few partial sums:

$S_1 =$ $1/1 = 1$
$S_2 =$ 1.5
$S_3 =$ 1.833
$S_4 =$ 2.0833 . .. ..

As we continue adding terms, the sum grows larger and larger without bound. This series does not add up to an actual value.

4. Factorial notation

Factorials form the basis for important series like Taylor series expansions.

Definition. The factorial of a positive integer n, written as $n!$, is the product of all positive integers less than or equal to n.

ex: $4! = 4\cdot3\cdot2\cdot1 = 24$ $\quad 4! = 1\cdot2\cdot3\cdot4$

For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3628800$

$n! = n \cdot (n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 = 1 \cdot 2 \cdot 3 \cdots (n-1)\, n$

By definition, $0! = 1$

$\frac{4}{\cancel{2} \cdot 3} = \frac{2}{3}$

Example 10. Evaluate the following expressions.

(a) $\frac{8!}{3! \cdot 5!} = \frac{8 \cdot 7 \cdot \cancel{6} \cdot \cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{(\cancel{3} \cdot \cancel{2} \cdot 1)(\cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1})} = 56$

$\frac{8!}{3! \cdot 5!} = \frac{8 \cdot 7 \cdot \cancel{6} \cdot \cancel{5!}}{\cancel{3!} \cdot \cancel{5!}} = 56$

(b) $\frac{n!}{(n+2)!} = \frac{\cancel{n!}}{\underbrace{\cancel{1 \cdot 2 \cdot 3 \cdots n}}(n+1)(n+2)}$

$= \frac{1}{(n+1)(n+2)}$

Example 11. Given the sequence defined by $\boxed{b_n = \frac{n^2}{(n+1)!}}$ find b_1 and b_6.

$b_1 = \frac{1^2}{(1+1)!} = \frac{1}{2!} = \frac{1}{2}$

$b_6 = \frac{6^2}{(6+1)!} = \frac{6^2}{7!} = \frac{\cancel{36}}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot 4 \cdot 5 \cdot \cancel{6} \cdot 7} = \frac{1}{140}$

5. Self Assessment

Take a moment to reflect on your understanding of each learning objective. For each topic, write the appropriate number from 1 to 5, where 1 star means you don't understand the concept at all, and 5 means you feel completely confident about it. Be honest with your assessment—this will help you identify areas where you might need additional practice or support.

1. **Writing terms of a sequence from the nth term:**
2. **Writing terms of a sequence from a recursive definition:**
3. **Using factorial notation:**
4. **Using summation notation:**