Day 4, Monday Feb 3rd

- Written HW1 posted and due next mondy
- Exam 1 next Wednesday

1. Adding, subtracting, multiplying and dividing functions

Two functions f and g can be combined to form new functions $f+g$, $f-g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers.

ALGEBRA OF FUNCTIONS

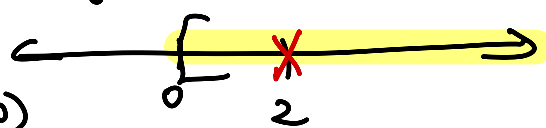
Let f and g be functions with domains A and B. Then the functions $f+g$, $f-g$, fg, and f/g are defined as follows.

$(f+g)(x) = f(x) + g(x)$	Domain $A \cap B$
$(f-g)(x) = f(x) - g(x)$	Domain $A \cap B$
$(fg)(x) = f(x)g(x)$	Domain $A \cap B$
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	Domain $\{x \in A \cap B \mid g(x) \neq 0\}$

Practice: Perform the following function operations for f and g and find the domains of the resulting functions.

$f(x) = \frac{1}{x-2}$ and $g(x) = \sqrt{x}$.

D for f: $(-\infty, 2) \cup (2, \infty)$
g: $[0, \infty)$



(a) $(f+g)(x) = \frac{1}{x-2} + \sqrt{x}$

Domain $f+g$: $[0,2) \cup (2,\infty)$

(b) $f - g$

$(f-g)(x) = \frac{1}{x-2} - \sqrt{x}$ same

(c) $(fg) = \frac{1}{x-2} \cdot \sqrt{x} = \frac{\sqrt{x}}{x-2}$ same

(d) $\left(\frac{f}{g}\right)(x) = \frac{\frac{1}{x-2}}{\sqrt{x}} = \frac{1}{x-2} \cdot \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{x}(x-2)}$

Domain for $\frac{f}{g}$ $\boxed{(0,2) \cup (2,\infty)}$

$f(x) = \frac{x+1}{x+2}$

$g(x) = x+1$

$\frac{f}{g} = \frac{\frac{x+1}{x+2}}{x+1} = \frac{\cancel{x+1}}{x+2} \cdot \frac{1}{\cancel{x+1}}$

$= \frac{1}{x+2}$

2. Composition of Functions

Another way to combine two functions f and g is to compose them. Composing the function $f(x)$ with the function $g(x)$, denoted $(f \circ g)(x)$, means that the output of g becomes the input of f. In other words $(f \circ g)(x) = f(g(x))$ or the function f is evaluated at $g(x)$.

For example if $f(x) = 5x - 3$ and $g(x) = 2 - x^2$ then $(f \circ g)(x) =$ $f(g(x))$

$= f(2-x^2) = 5(2-x^2)-3$

$= 10-5x^2-3 = 7-5x^2$

What is the domain of $f \circ g$?

$(-\infty, \infty)$

$(g \circ f)(x) = g(f(x)) = 2-(5x-3)^2$

$= 2-(25x^2-30x+9)$

$= -25x^2+30x-7$

D: $(-\infty,-2) \cup (-2,\infty)$

Example: Let $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{4}{x-1}$. Find the following:

D: $(-\infty,1) \cup (1,\infty)$

g

(a) $f \circ g$ and its domain

$$f(g(x)) = \frac{1}{g(x)+2} = \frac{1}{\frac{4}{x-1}+\frac{2}{1}\frac{(x-1)}{(x-1)}} = \frac{1}{\frac{4+2x-2}{x-1}}$$

$$= \frac{1}{\frac{2x+2}{x-1}} = 1 \cdot \frac{x-1}{2x+2} = \frac{x-1}{2x+2}$$

Domain: $(-\infty,-1) \cup (-1,\infty)$ & $x \neq 1$

(b) $g \circ f$ and its domain

$$g(f(x)) = \frac{4}{\frac{1}{x+2}-\frac{1}{1}\frac{(x+2)}{(x+2)}} = \frac{4}{\frac{1-x-2}{x+2}} = \frac{4}{\frac{-x-1}{x+2}} = 4 \cdot \frac{x+2}{-x-1}$$

$$= \frac{4x+8}{-x-1} = \frac{4x+8}{-(x+1)} = \frac{-(4x+8)}{x+1}$$

$x \neq -1$
$x \neq -2$ Domain

NOTE The domain of $f \circ g$ is the intersection of the domain of inner function g and the resulting function $f \circ g$.

3. Applications with composition of functions

The weekly cost C of producing x units is given by $C(x) = 60x + 750$. The number x of units produced in t hours is given by $x(t) = 50t$.

(a) Find and interpret $(C \circ x)(t)$ $= C(x(t)) = C(50t) = 60(50t)+750$

$= 3000t + 750$

$C(x(t)) = 3000t + 750$

it gives cost in terms of hours.

(b) Find the time that must elapse in order for the cost to increase to $\$15,000$.

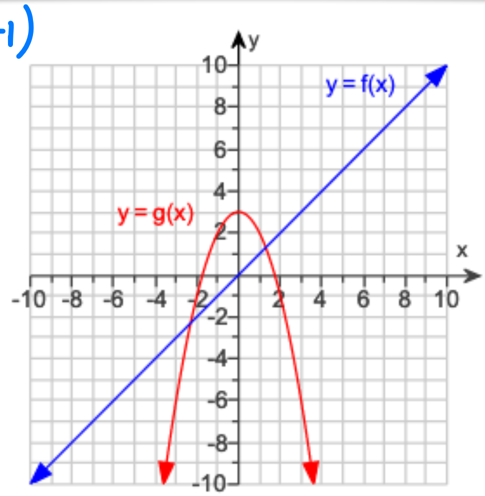
$15,000 = 3000t + 750 \Rightarrow t = 4.75$ hours

Practice: Exercise 1

Use a graphical approach to answer the following questions about the two functions graphed. Find $(f+g)(0)$, $(f-g)(-1)$, $(fg)(1)$, $(f/g)(2)$, $(f \circ g)(1)$, $(g \circ f)(1)$.

$f(0)+g(0)$
$0+3$
3

$f(-1)-g(-1)$
$-1-2$
-3



$f(1)\,g(1)$
$1 \cdot 2$
2

$\dfrac{f(2)}{g(2)} = \dfrac{2}{(-1)} = -2$

$f(g(1)) = f(2) = 2$

$g(f(1)) = g(1) = 2$

Practice: Exercise 2

Let $f(x) = \dfrac{x}{x+3}$ and $g(x) = 8x - 3$. Find the following

1. $(f \circ g)(x)$ and its domain

$$f(g(x)) = f(8x-3) = \frac{8x-3}{8x-3+3} = \frac{8x-3}{8x}$$

Dom: $(-\infty, 0) \cup (0, \infty)$

2. $(g \circ f)(x)$ and its domain

$g(f(x)) = g\left(\frac{x}{x+3}\right) = 8\,\frac{x}{x+3} - 3$

$x \neq -3$

Dom: $(-\infty, -3) \cup (-3, \infty)$

Practice: Exercise 3

Let $f(x) = x^3 + 8$ and $g(x) = x - 5$ and $h(x) = \sqrt{x}$. Find $f \circ g \circ h$.

$(f \circ g \circ h)(x) = f(g(h(x))) = f(\sqrt{x} - 5)$

$= (\sqrt{x} - 5)^3 + 8$

$V_{sphere} = \frac{4\pi r^3}{3}$

Practice: Exercise 5

A spherical balloon is being inflated. The radius of the balloon is increasing at a rate of 3cm/s.

1. Find a function f that models the radius as a function of time t, in seconds.

   $f(t) = 3t$

Related rates
Implicit diff

2. Find a function g that models the volume as a function of the radius r, in cm.

   $g(r) = \boxed{\frac{4\pi r^3}{3}}$

3. Find and interpret $g \circ f$.

   $g(f(t)) = g(3t) = \frac{4\pi(3t)^3}{3} = \frac{108\pi t^3}{3}$

   $\boxed{V(t) = 36\pi t^3}$

8. One-to-one Functions

A function f is **one-to-one** if there is a one-to-one correspondence between the inputs and the outputs. In other words, f never takes on the same values twice.

Examples: Is f one-to-one?

$f(x) = x^2$

NO

$-1 \to 1$
$1 \to 1$

$f(x) = x^3$

yes

$-1 \to -1$
$0 \to 0$
$1 \to 1$

$1 \to 5$
$2 \to 3$
$3 \to 5$

NOT one-to-one

$1 \to 5$
$2 \to 3$
$3 \to 4$

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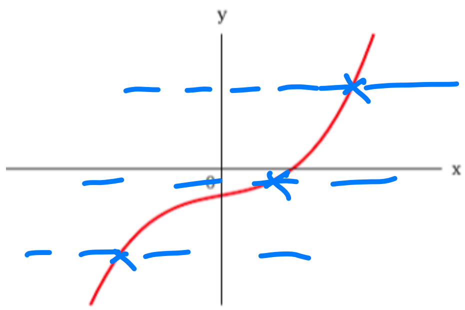
How to determine if a function f is one-to-one - Graphically
Determine if the functions f and g are one-to-one.

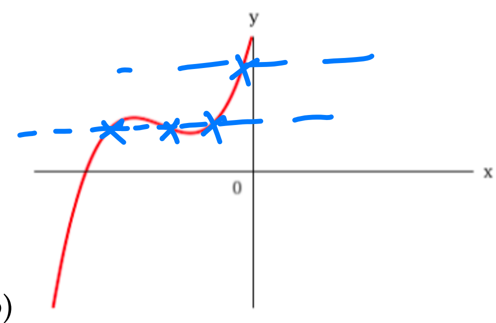
we do the Horizontal line test !

(a)



Yes

(b)



NOT one-to-one

Practice - Restricted Domain
Graph the function $f(x) = (x+2)^2 - 1$ and decide if it is one-to-one on its entire domain.

* next class

f^{-1} means the inverse of f

9. Inverse Functions

A one-to-one function with domain A and range B has an **inverse function** which takes on as domain the range of f and gives as output the domain of f. In other words the inverse of a function f undoes what f does. The inverse, denoted as f^{-1} takes as input the output of f and gives as output the input of f, i.e. if $f(x) = y$ then $f^{-1}(y) = x$.

For example if f is the function that takes the input x multiplies it by 5, adds 2 then takes the 5th power of the result, what would the inverse f^{-1} have to be to undo what f did?

$$f(x) = (5x+2)^5$$

$$f^{-1}(x) = \frac{\sqrt[5]{x} - 2}{5}$$

$$f(x) = \frac{x+3}{x+2}$$

The inverse function f^{-1} reverses the effect of f!

INVERSE FUNCTION PROPERTY

Let f be a one-to-one function with domain A and range B. The inverse function f^{-1} satisfies the following cancellation properties:

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$ for every x in A

$(f \circ f^{-1})(x)$ $f(f^{-1}(x)) = x$ for every x in B

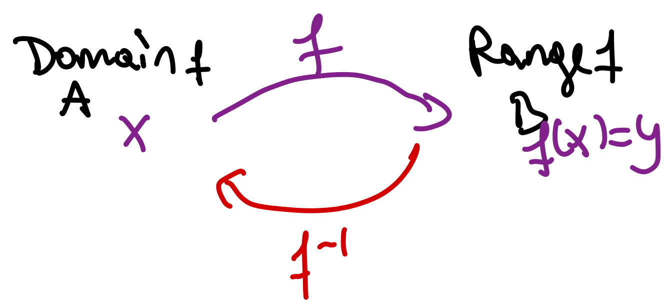
Conversely, any function f^{-1} satisfying these equations is the inverse of f.

Practice: Are f and g inverses of each other?

$f(x) = 8 - 4x,$ | $g(x) = \dfrac{8-x}{4}$

$f^{-1}(x) = \,?$

$f^{-1}(x) = \dfrac{x}{-4} - 8$



$$f^{-1}(x) = \frac{-x}{4} - \frac{8}{1}\cdot\frac{4}{4} = \frac{-x-32}{4}$$

NOTE: The function and its inverse switch domain and range:
Domain of f = Range of f^{-1}
Range of f = Domain of f^{-1}

$$(f \circ g)(x) = f\left(\frac{8-x}{4}\right) = \cancel{8} - \cancel{4}\cdot\frac{\cancel{8}-x}{\cancel{4}} = x$$

$$(g \circ f)(x) = g(8-4x) = \frac{8-(8-4x)}{4} = \frac{\cancel{8}-\cancel{8}+4x}{4} = \frac{4x}{4} = x$$

10. How to find the inverse of a one-to-one function - Algebraically

Let f be a one-to one function. Then $f(x) = y$ for any x in the domain of f and f has an inverse denoted f^{-1}.

Example: Let $f(x) = \dfrac{x-9}{x+9}$.

(a) Is f one-to-one?

yes graph HTL

(b) Find the domain of f.

$(-\infty,-9)\cup(-9,\infty)$

(c) Can you find the range of f without graphing it?

range of f = domain of f^{-1}

(d) Find the inverse of f.

1) $y = \dfrac{x-9}{x+9}$

2) $\dfrac{x}{1} = \dfrac{y-9}{y+9}$

3) $x(y+9) = y-9$

$xy + 9x = y - 9$

$xy - y = -9 - 9x$

$y(x-1) = -9-9x \Rightarrow y = \dfrac{-9-9x}{x-1} = f^{-1}(x)$

$f(x) = 8-4x$

1) Step 1 write it as
$y = 8-4x$

2) switch the x and y

$\boxed{x = 8-4y}$

3) Solve for y

$8-4y = x$
$-8 \quad -8$

$-4y = x-8 \quad \div -4$

$y = \dfrac{x-8}{-4}$

$y = \dfrac{8-x}{4}$

4) Step 4
write f^{-1} instead of y

$f^{(-1)} = \dfrac{8-x}{4}$

(e) Can you find the range of f now using the information you have about f^{-1}?

yes range of f = domain of f^{-1} : $(-\infty,1)\cup(1,\infty)$

Find the inverse $g(x) = \sqrt[11]{7x}$.

$y = \sqrt[11]{7x}$

$x = \sqrt[11]{7y}$

$x = (7y)^{\frac{1}{11}}$

$x^{11} = (7y)^{\frac{1}{11}\cdot 11}$

$x^{11} = 7y$

$\dfrac{x^{11}}{7} = y \Rightarrow f^{-1}(x) = \dfrac{x^{11}}{7}$