Active neutrino self-interaction with NSI(August 10, 2021)

Contents

1 Particle model

We consider an effective scalar and vector NSI between active neutrinos in addition to SM processes.
If the self-interaction is strong, it could keep active neutrinos in thermal equilibrium with themselves longer than the weak interaction, and enhance DW production rate of sterile neutrinos.
General NSI Lagrangian for active neutrino self-interaction,

$$\mathcal{L}_j = -\frac{G_F}{\sqrt{2}} \left(\epsilon_{j,\nu}\right)^{\alpha\beta\gamma\delta} \left(\bar{\nu}_\alpha \mathcal{O}_j \nu_\beta\right) \left(\bar{\nu}_\gamma \mathcal{O}_j' \nu_\delta\right) \tag{1.1}$$

Since we want to study testability in KATRIN, we would use electron flavor as the one and only flavor in our model.
Hence, $\alpha = \beta = \gamma = \delta = e$
Then (1.1),

$$\mathcal{L}_j = -\frac{G_F}{\sqrt{2}} \left(\epsilon_{j,\nu_e}\right)^{eeee} \left(\bar{\nu}_e \mathcal{O}_j \nu_e\right) \left(\bar{\nu}_e \mathcal{O}_j' \nu_e\right) \tag{1.2}$$

1.1 Properties of ν_e

- The "neutrino" ν_e in the model is a **Dirac fermion**.
  what about the point that if ν is Majorana, they can't be sterile?
- This neutrino has two chiral components, $\nu_e = \nu_L + \nu_R$
- ν_L represent LH component of neutrino field and **active neutrino** ν_a.
- ν_R represent RH component of neutrino field and **sterile neutrino** ν_s.
- neutrino oscillation : $\nu_a \leftrightarrow \nu_s (\nu_L \leftrightarrow \nu_R)$

1.2 Scalar NSI

We would introduce a scalar interaction so that we can compare our results with [1](Not sure whether they make use of same operators)

$$\mathcal{O}_S = (1-\gamma^5) \qquad \mathcal{O}_S' = (1-\gamma^5)$$

Here $(1-\gamma^5)$ implies that we are dealing with left-handed active neutrinos.

$$\mathcal{L}_S = -\frac{G_F}{\sqrt{2}} \left(\epsilon_{S,\nu_e}\right)^{eeee} \left(\bar{\nu}_e (1-\gamma^5) \nu_e\right) \left(\bar{\nu}_e (1-\gamma^5) \nu_e\right) \tag{1.3}$$

We could see this as a low energy EFT for a scalar heavy mediator ϕ ($m_\phi \gg T$) with interaction,

$$\mathcal{L}_s \supset \frac{\lambda_\phi}{2} \nu_e \nu_e \phi + \text{h.c} \tag{1.4}$$

then, coefficients could be,

$$\frac{G_F}{\sqrt{2}} \left(\epsilon_{j,\nu_e}\right)^{eeee} = \frac{\lambda_\phi^2}{m_\phi^2}$$

and this would correspond to Case `A` of Fig.2 of [1]

1.3 Vector NSI

We would introduce a vector interaction so that we can compare our results with [2]

$$\mathcal{O}_V = \gamma^\mu(1-\gamma^5) \qquad \mathcal{O}'_V = \gamma_\mu(1-\gamma^5)$$

$$\mathcal{L}_V = -\frac{G_F}{\sqrt{2}} \left(\epsilon_{V,\nu_e}\right)^{eeee} \left(\bar{\nu}_e\gamma^\mu(1-\gamma^5)\nu_e\right) \left(\bar{\nu}_e\gamma_\mu(1-\gamma^5)\nu_e\right) \tag{1.5}$$

2 Interaction rate

In the interaction rate calculations, we consider processes involving active neutrinos.
We have SM processes with interaction rate[3],

$$\Gamma_a = C_a(p,T)G_F^2 pT^4 \tag{2.1}$$

where $C_a(p,T)$ is a momentum- and temperature dependent coefficient.
In our extension, following processes will get extra contribution from active neutrino self-interaction.

$$\nu_e + \nu_e \to \nu_e + \nu_e$$
$$\nu_e + \bar{\nu}_e \to \nu_e + \bar{\nu}_e$$

Check Appendix A for detailed calculation

2.1 Scalar NSI

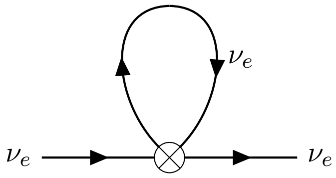
$$\Gamma_S(E,T) = \frac{7\pi G_F^2 \epsilon_S^2}{450} ET^4 \tag{2.2}$$

2.2 Vector NSI

$$\Gamma_V(E,T) = \frac{56\pi G_F^2 \epsilon_V^2}{225} ET^4 \tag{2.3}$$

3 Thermal potential

In addition to SM self-energy diagrams, self-interaction can contribute a new diagram for thermal potential calculation.



Check Appendix B for detailed calculation

3.1 Scalar NSI(B.3.1)

$$\mathcal{V}_{\text{scalar}} = \pm\sqrt{2}G_F\left(\epsilon_{S,\nu_e}\right)^{eeee}\left(n_\nu - n_{\bar{\nu}}\right) \tag{3.1}$$

$(+)$ for ν and $(-)$ for $\bar{\nu}$

3.2 Vector NSI(B.3.2)

$$\mathcal{V}_{\text{vector}} = \mp 2\sqrt{2}G_F\left(\epsilon_{S,\nu_e}\right)^{eeee}\left(n_\nu - n_{\bar{\nu}}\right) \tag{3.2}$$

$(-)$ for ν and $(+)$ for $\bar{\nu}$

3.3 Scalar Mediator(B.3.3)

$$\mathcal{V}_\phi = \mp\frac{\lambda_\phi^2}{m_\phi^2}\left(n_\nu - n_{\bar{\nu}}\right) - \frac{7\pi^2\lambda_\phi^2}{90m_\phi^4}ET^4 \tag{3.3}$$

- We have to go one order higher in $\mathcal{L}_{\text{NSI}}$ to get relevant thermal potential terms.

A Calculation of Interaction rate

Possible mistakes:

- co-efficients

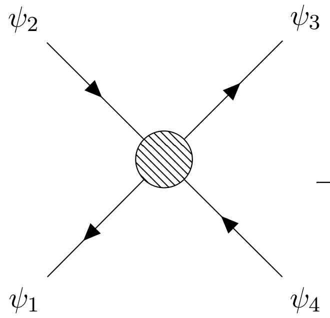
A.1 Matrix elements

EFT vertex

From [4], 4-Fermion EFT vertex for a general Lagrangian

$$\mathcal{L}_{\text{int}} \equiv \sum C^{f_1 f_2 f_3 f_4} \Gamma_{s_1 s_2} \Gamma_{s_3 s_4} \bar{\psi}^{f_1}_{s_1} \psi^{f_2}_{s_2} \bar{\psi}^{f_3}_{s_3} \psi^{f_4}_{s_4} \tag{A.1}$$

is given by,

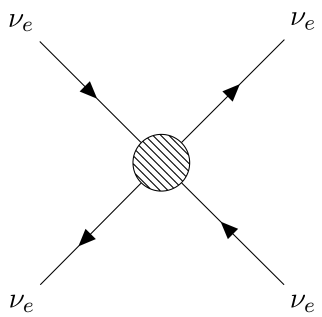


$$\longrightarrow i\Gamma_{1234} =\equiv 2iC^{f_1 f_2 f_3 f_4} \Gamma_{s_1 s_2} \Gamma_{s_3 s_4} - 2iC^{f_1 f_4 f_3 f_2} \Gamma_{s_1 s_4} \Gamma_{s_3 s_2} \tag{A.2}$$

For NSI Lagrangian,

$$\mathcal{L}_j = -\frac{G_F}{\sqrt{2}} \left(\epsilon_{j,\nu_e}\right)^{eeee} \left(\bar{\nu}_e \mathcal{O}_j \nu_e\right) \left(\bar{\nu}_e \mathcal{O}_j \nu_e\right) \tag{A.3}$$

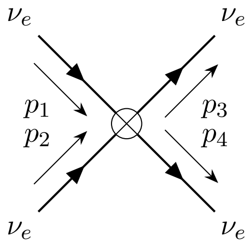
Removed a factor of 2 as we mentioned in the meeting



$$\longrightarrow i\Gamma_{1234} =\equiv -i\frac{G_F}{\sqrt{2}} \left(\epsilon_{j,\nu_e}\right)^{eeee} \left[\mathcal{O}_{s_1 s_2} \mathcal{O}_{s_3 s_4} - \mathcal{O}_{s_1 s_4} \mathcal{O}_{s_3 s_2}\right] \tag{A.4}$$

$$\begin{aligned} u(p_i) \text{ or } \bar{u}(p_i) &\equiv \text{spinors for neutrinos} \\ v(p_i) \text{ or } \bar{v}(p_i) &\equiv \text{spinors for anti-neutrinos} \end{aligned}$$

A.1.1 $\nu_e + \nu_e \to \nu_e + \nu_e$



$$-\,i\mathcal{M} = [\bar{u}(p_4)\bar{u}(p_3)]\,[i\Gamma_{4231}]\,[u(p_1)u(p_2)] \tag{A.5}$$

$$-i\mathcal{M} = [\bar{u}(p_4)\bar{u}(p_3)]\left(-i\frac{G_F}{\sqrt{2}}\left(\epsilon_{j,\nu_e}\right)^{eeee}[\mathcal{O}_{42}\mathcal{O}_{31} - \mathcal{O}_{41}\mathcal{O}_{32}]\right)[u(p_1)u(p_2)] \tag{A.6}$$

$$\mathcal{M} = \frac{G_F}{\sqrt{2}}\left(\epsilon_{j,\nu_e}\right)^{eeee}\left(\,[\bar{u}(p_4)\mathcal{O}u(p_2)\bar{u}(p_3)\mathcal{O}u(p_1)] - [\bar{u}(p_4)\mathcal{O}u(p_1)\bar{u}(p_3)\mathcal{O}u(p_2)]\,\right) \tag{A.7}$$

$$\mathcal{M}^\dagger = \frac{G_F}{\sqrt{2}}\left(\epsilon_{j,\nu_e}\right)^{eeee}\left(\,\left[\bar{u}(p_1)\bar{\mathcal{O}}u(p_3)\bar{u}(p_2)\bar{\mathcal{O}}u(p_4)\right] - \left[\bar{u}(p_2)\bar{\mathcal{O}}u(p_3)\bar{u}(p_1)\bar{\mathcal{O}}u(p_4)\right]\,\right) \tag{A.8}$$

where $\bar{\mathcal{O}} = \gamma^0\mathcal{O}^\dagger\gamma^0$

Take average over initial spins and sum over final spins:

$$\overline{|\mathcal{M}_j|}^2 = \frac{1}{4}\sum \mathcal{M}_j^\dagger\mathcal{M}_j \tag{A.9}$$

For Scalar NSI:

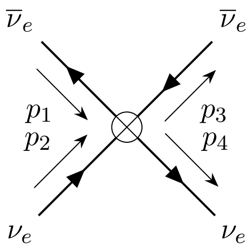
$$\overline{|\mathcal{M}_S|}^2 = 8\left(\epsilon_{S,\nu_e}^{eeee}\right)^2 G_F^2\,(p_1\cdot p_2)\,(p_3\cdot p_4) \tag{A.10}$$

For Vector NSI:

$$\overline{|\mathcal{M}_V|}^2 = 128\left(\epsilon_{V,\nu_e}^{eeee}\right)^2 G_F^2\,(p_1\cdot p_2)\,(p_3\cdot p_4) \tag{A.11}$$

Comparing with https://journals.aps.org/prd/abstract/10.1103/PhysRevD.52.1764 which has a factor 32 including $\mathcal{S} = \frac{1}{2}$, we seem to miss another $\frac{1}{2}$.

A.1.2 $\nu_e + \bar{\nu}_e \to \nu_e + \bar{\nu}_e$



$$-\,i\mathcal{M} = [\bar{u}(p_3)v(p_4)]\,[i\Gamma_{1234}]\,[\bar{v}(p_1)u(p_2)] \tag{A.12}$$

$$i\mathcal{M} = [\bar{u}(p_3)v(p_4)]\left(-i\frac{G_F}{\sqrt{2}}\left(\epsilon_{j,\nu_e}\right)^{eeee}[\mathcal{O}_{12}\mathcal{O}_{34} - \mathcal{O}_{14}\mathcal{O}_{32}]\right)[\bar{v}(p_1)u(p_2)] \tag{A.13}$$

$$\mathcal{M} = \frac{G_F}{\sqrt{2}}\left(\epsilon_{j,\nu_e}\right)^{eeee}\left(\,[\bar{v}(p_1)\mathcal{O}u(p_2)\bar{u}(p_3)\mathcal{O}v(p_4)] - [\bar{v}(p_1)\mathcal{O}v(p_4)\bar{u}(p_3)\mathcal{O}u(p_2)]\,\right) \tag{A.14}$$

$$\mathcal{M}^\dagger = \frac{G_F}{\sqrt{2}} \left(\epsilon_{j,\nu_e}\right)^{eeee} \left(\left[\bar{v}(p_4)\bar{\mathcal{O}}u(p_3)\bar{u}(p_2)\bar{\mathcal{O}}v(p_1)\right] - \left[\bar{u}(p_2)\bar{\mathcal{O}}u(p_3)\bar{v}(p_4)\bar{\mathcal{O}}v(p_1)\right] \right) \tag{A.15}$$

where $\bar{\mathcal{O}} = \gamma^0 \mathcal{O}^\dagger \gamma^0$

Take average over initial spins and sum over final spins:

$$\overline{|\mathcal{M}_j|}^2 = \frac{1}{4} \sum \mathcal{M}_j^\dagger \mathcal{M}_j \tag{A.16}$$

For Scalar NSI:

$$\overline{|\mathcal{M}_S|}^2 = 8 \left(\epsilon_{S,\nu_e}^{eeee}\right)^2 G_F^2 \left(p_1 \cdot p_3\right) \left(p_2 \cdot p_4\right) \tag{A.17}$$

For Vector NSI:

$$\overline{|\mathcal{M}_V|}^2 = 128 \left(\epsilon_{V,\nu_e}^{eeee}\right)^2 G_F^2 \left(p_1 \cdot p_3\right) \left(p_2 \cdot p_4\right) \tag{A.18}$$

Comparing with https://journals.aps.org/prd/abstract/10.1103/PhysRevD.52.1764, factor 128 seems correct.
$(p_1 \cdot p_4)(p_2 \cdot p_3)$ comes from different assignment of momenta. But if we acknowledge that there are s-,t-channel diagrams and do calculations accordingly, it won't be a problem?

A.2 Total cross-section

We will work in the Center of Mass frame.

$$\begin{aligned}
p_1 &= (E_1, \mathbf{p}) \\
p_2 &= (E_2, -\mathbf{p}) \\
p_3 &= (E_3, \mathbf{p}^*) \\
p_4 &= (E_4, -\mathbf{p}^*) \\
s &= (p_1 + p_2)^2 = 2p_1 \cdot p_2 = (E_1 + E_2)^2 \\
t &= (p_1 - p_3)^2 = -2p_1 \cdot p_3 = 2(\mathbf{p} \cdot \mathbf{p}^* - E_1 E_3)
\end{aligned}$$

angle between $\mathbf{p}$ and $\mathbf{p}^*$ is θ.
For Matrix elements are of the form,

$$\begin{aligned}
\overline{|\mathcal{M}|}^2_{\nu\nu-\nu\nu} &= K_1 \left(p_1 \cdot p_2\right) \left(p_3 \cdot p_4\right) \\
&= K_1 \Big(E_1 E_2 E_3 E_4 - E_1 E_2 \left(\vec{p}_3 \cdot \vec{p}_4\right) - E_3 E_4 \left(\vec{p}_1 \cdot \vec{p}_2\right) + \left(\vec{p}_1 \cdot \vec{p}_2\right) \left(\vec{p}_3 \cdot \vec{p}_4\right) \Big) \\
\overline{|\mathcal{M}|}^2_{\nu\bar{\nu}-\nu\bar{\nu}} &= K_1 \left(p_1 \cdot p_3\right) \left(p_2 \cdot p_4\right) \\
&= K_1 \Big(E_1 E_2 E_3 E_4 - E_1 E_3 \left(\vec{p}_2 \cdot \vec{p}_4\right) - E_2 E_4 \left(\vec{p}_1 \cdot \vec{p}_3\right) + \left(\vec{p}_1 \cdot \vec{p}_3\right) \left(\vec{p}_2 \cdot \vec{p}_4\right) \Big)
\end{aligned} \tag{A.19}$$

In CoM frame,

$$\begin{aligned}
\overline{|\mathcal{M}|}^2_{\nu\nu-\nu\nu} &= \frac{K_1}{4} s^2 \\
\overline{|\mathcal{M}|}^2_{\nu\bar{\nu}-\nu\bar{\nu}} &= \frac{K_1}{4} t^2
\end{aligned} \tag{A.20}$$

Starting from,

$$d\sigma = \frac{S\hbar^2}{4\sqrt{\left(p_1 \cdot p_2\right)^2 - \left(m_1 m_2 c^2\right)^2}} |\mathcal{M}|^2 d\Pi_{\text{LIPS}} \tag{A.21}$$

For a $2 \to j$ process, Lorentz Invariant Phase Space,

$$d\Pi_{\text{LIPS}} \equiv \prod_{\text{final states } j} \frac{d^3 p_j}{(2\pi)^3} \frac{1}{2E_{p_j}} (2\pi)^4 \delta^4(\Sigma p) \tag{A.22}$$

Now for a $1 + 2 \to 3 + 4$ process with massless particles,

$$d\sigma = \frac{S}{2s} |\mathcal{M}|^2 \frac{d^3 p_3}{2E_3 (2\pi)^3} \frac{d^3 p_4}{2E_4 (2\pi)^3} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \tag{A.23}$$

In CoM frame,

$$\begin{aligned} d\sigma &= \frac{S}{2s} |\mathcal{M}|^2 \frac{d^3 p_3}{2E_3 (2\pi)^3} \frac{d^3 p_4}{2E_4 (2\pi)^3} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \\ &= \frac{S}{2s} \mathcal{M}|^2 \frac{d^3 p_3}{2E_3 (2\pi)^3} \frac{d^3 p_4}{2E_4 (2\pi)^3} (2\pi)^4 \delta(\sqrt{s} - E_3 - E_4) \delta^3(\vec{p}_3 + \vec{p}_4) \\ &= \frac{S}{2 \cdot (2\pi)^2 \cdot s} |\mathcal{M}|^2 \frac{d^3 p_3}{2E_3} \frac{d^3 p_4}{2E_4} \delta(\sqrt{s} - E_3 - E_4) \delta^3(\vec{p}_3 + \vec{p}_4) \end{aligned} \tag{A.24}$$

$$\sigma_{\nu\nu-\nu\nu} = \frac{\frac{1}{2}}{2 \cdot (2\pi)^2 \cdot s} \int \frac{d^3 p_3}{2E_3} \frac{d^3 p_4}{2E_4} \delta(\sqrt{s} - E_3 - E_4) \delta^3(\vec{p}_3 + \vec{p}_4) \cdot \frac{K_1}{4} s^2 \tag{A.25}$$

integrating out $\mathbf{p}_4$ with $\delta^3(\vec{p}_3 + \vec{p}_4)$, and we use,

$$\int dx\, g\left(x\right) \delta\left(f\left(x\right)\right) = g\left(x_0\right) \frac{1}{f'\left(x_0\right)}$$

where x_0 solves $f\left(x_0\right) = 0$.

$$\sigma_{\nu\nu-\nu\nu} = \frac{1}{4 \cdot (2\pi)^2 \cdot s} \cdot \frac{K_1}{4} s^2 \int \frac{d^3 p_3}{(2E_3)^2} \delta(\sqrt{s} - 2E_3) \tag{A.26}$$

$$d^3 p_3 = p_3^2 dp_3 d\Omega_3$$
$$E_3 dE_3 = |\mathbf{p}_3| dp_3$$

$$\begin{aligned} \sigma_{\nu\nu-\nu\nu} &= \frac{K_1 s^2}{16 \cdot (2\pi)^2 \cdot s} \int \frac{p_3^2 dp_3 d\Omega_3}{(2E_3)^2} \delta(2E_3 - \sqrt{s}) \\ &= \frac{K_1 s}{16 \cdot (2\pi)^2} \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot 4\pi \\ &= \frac{K_1 s}{128\pi} \end{aligned} \tag{A.27}$$

For $\nu_e + \bar{\nu}_e \to \nu_e + \bar{\nu}_e$,

$$\sigma_{\nu\bar{\nu}-\nu\bar{\nu}} = \frac{1}{2 \cdot (2\pi)^2 \cdot s} \int \frac{d^3 p_3}{2E_3} \frac{d^3 p_4}{2E_4} \delta(\sqrt{s} - E_3 - E_4) \delta^3(\vec{p}_3 + \vec{p}_4) \cdot \frac{K_1}{4} t^2 \tag{A.28}$$

$t = (p_1 - p_3)^2 = -2p_1 \cdot p_3 = 2\left(\vec{p}_1 \cdot \vec{p}_3 - E_1 E_3\right) = 2|p_1||p_3|(\cos\theta - 1) t^2 = 4|p_1|^2|p_3|^2(1 - \cos\theta)^2$

$$\sigma_{\nu\bar{\nu}-\nu\bar{\nu}} = \frac{1}{2 \cdot (2\pi)^2 \cdot s} \int \frac{d^3p_3}{2E_3}\frac{d^3p_4}{2E_4}\delta(\sqrt{s} - E_3 - E_4)\delta^3(\vec{p}_3 + \vec{p}_4) \cdot \frac{K_1}{4} 4|p_1|^2|p_3|^2(1 - \cos\theta)^2 \tag{A.29}$$

integrating out $\mathbf{p}_4$ with $\delta^3(\vec{p}_3 + \vec{p}_4)$,

$$\sigma_{\nu\bar{\nu}-\nu\bar{\nu}} = \frac{K_1|p_1|^2}{8\pi^2 s} \int \frac{d^3p_3}{4}\delta(\sqrt{s} - 2E_3)(1 - \cos\theta)^2 \tag{A.30}$$

$$d^3p_3 = p_3^2 dp_3 d\Omega_3 = p_3^2 dp_3 d(\cos\theta) d\phi$$
$$E_3 dE_3 = |\mathbf{p}_3| dp_3$$

$$\begin{aligned}
\sigma_{\nu\bar{\nu}-\nu\bar{\nu}} &= \frac{K_1|p_1|^2}{32\pi^2 s} \int p_3^2 dp_3 d(\cos\theta) d\phi \delta(2p_3 - \sqrt{s})(1 - \cos\theta)^2 \\
&= \frac{K_1 \frac{s}{4}}{32\pi^2 s} \int p_3^2 \delta(2p_3 - \sqrt{s}) dp_3 \int_{-1}^{1} d(\cos\theta)(1 - \cos\theta)^2 \int_0^{2\pi} d\phi \\
&= \frac{K_1}{128\pi^2} \cdot \frac{1}{2}\left(\frac{\sqrt{s}}{2}\right)^2 \frac{8}{3} \cdot 2\pi = \frac{K_1 s}{3 \cdot 128\pi}
\end{aligned} \tag{A.31}$$

$$\sigma_{tot} = \sigma_{\nu\nu-\nu\nu} + \sigma_{\nu\bar{\nu}-\nu\bar{\nu}} = \frac{K_1 s}{128\pi}\left(1 + \frac{1}{3}\right) = \frac{K_1 s}{96\pi} \tag{A.32}$$

A.3 Interaction rate

Now Interaction rate from Eq. 3.6 of [2] for a neutrino with fixed energy E, by averaging over the phase space of the other neutrino (or anti-neutrino) it scatters with,

$$\Gamma_V(E, T) = 2 \int \frac{d^3\vec{p}_2}{(2\pi)^3} f_\nu\left(E_2, T\right) \sigma_{\text{tot}}\left(\vec{p}_1, \vec{p}_2\right) v_{\text{rel}} \tag{A.33}$$

where $f_\nu\left(E_2, T\right) = 1/\left[1 + \exp\left(E_2/T\right)\right]$ is the Fermi-Dirac distribution for active neutrino or antineutrino, σ_{tot} is the sum of the two cross-sections. The relative velocity is equal to $\sqrt{2(1 - \cos\theta)}$ for ultra-relativistic neutrinos [2]. The prefactor of 2 captures both helicity states of the active neutrinos and antineutrinos. Hence, Γ_V also includes the reaction rate of antineutrinos

Now for $p_1 = (E, \vec{p}), p_2 = (E', \vec{p'})$

$$s = 2p_1 \cdot p_2 = 2(EE' - \vec{p} \cdot \vec{p'}) = 2|p||p'|(1 - \cos\theta)$$

$$
\begin{aligned}
\Gamma(E,T) &= 2\int \frac{d^3\vec{p}_2}{(2\pi)^3} f_\nu\left(E_2,T\right)\sigma_{\text{tot}}\left(\vec{p}_1,\vec{p}_2\right)v_{\text{rel}} \\
&= 2\int \frac{|p'|^2 dp' d\Omega}{(2\pi)^3}\frac{1}{1+\exp\left(E'/T\right)}\frac{K_1}{96\pi}\cdot 2|p||p'|(1-\cos\theta)\cdot\sqrt{2(1-\cos\theta)} \\
&= \frac{4K_1E}{96\pi}\int \frac{E'^2 dE' d(\cos\theta) d\phi}{(2\pi)^3}\frac{1}{1+\exp\left(E'/T\right)}E'\sqrt{2}(1-\cos\theta)^{3/2} \\
&= \frac{K_1E}{24\pi}\int \frac{E'^3 dE'}{(2\pi)^3}\frac{1}{1+\exp\left(E'/T\right)}\int_{-1}^{1} d(\cos\theta)\sqrt{2}(1-\cos\theta)^{3/2}\int_0^{2\pi} d\phi \qquad \text{(A.34)} \\
&= \frac{K_1E}{24\pi\cdot(2\pi)^2}\int \frac{E'^3 dE'}{1+\exp\left(E'/T\right)}\int_{-1}^{1} d(\cos\theta)\sqrt{2}(1-\cos\theta)^{3/2} \\
&= \frac{K_1E}{24\pi\cdot(2\pi)^2}\frac{7\pi^4T^4}{120}\cdot\frac{16}{5} \\
\Gamma(E,T) &= \frac{7K_1\pi}{3600}ET^4
\end{aligned}
$$

(E' integration in Mathematica.
If we use MB distribution, we get $6T^4$ instead of $\frac{7\pi^4T^4}{120} \approx 5.68T^4$)

$K_S = 8G_F^2\epsilon_S^2$ for scalar NSI and $K_V = 128G_F^2\epsilon_V^2$ for Vector NSI
ie,

$$
\begin{aligned}
\Gamma_S(E,T) &= \frac{7\pi G_F^2\epsilon_S^2}{450}ET^4 \\
\Gamma_V(E,T) &= \frac{56\pi G_F^2\epsilon_V^2}{225}ET^4
\end{aligned}
\qquad \text{(A.35)}
$$

B Calculation of Thermal potential

We follow the method presented in *"Finite temperature corrections to the effective potential of neutrinos in a medium"*, (J. C. D'Olivo, J. F. Nieves, M. Torres)[5] for SM thermal potential

B.1 Neutrino Dispersion Relation

The properties of a neutrino that propagates through a medium are determined from the Dirac equation, which in momentum space is

$$(\not{k} - \Sigma_{\text{eff}})\,\psi = 0 \tag{B.1}$$

Here k_μ is the neutrino momentum and Σ_{eff} is the neutrino self-energy, which includes the effects of the background. The chiral nature of the neutrino interactions implies that the self-energy of a (left-handed) neutrino is of the form

Chiral nature of neutrino interactions imply [6],

$$\Sigma_{\text{eff}} = P_R \Sigma P_L \tag{B.2}$$

Where $P_{R/L} = \frac{1\pm\gamma_5}{2}$
In vacuum, the only term self energy can depend on is the neutrino four-momentum k_μ. i.e.,

$$\Sigma = a\not{k} \tag{B.3}$$

But in a medium with velocity u_μ(we would assume rest frame where $u_\mu = (1, 0, 0, 0)$),

$$\Sigma = a\not{k} + b\not{u} + c\,[\not{k}, \not{u}] \tag{B.4}$$

From [7], we can see that at one-loop level, $c = 0$. Then,

$$\Sigma = a\not{k} + b\not{u} \tag{B.5}$$

a,b depend on invariant quantities $\omega = k \cdot u$ and $\kappa = \sqrt{\omega^2 - k^2}$.
Now Dirac Equation,

$$(\not{k} - \Sigma_{\text{eff}})\,\psi = 0 \tag{B.6}$$

$$(\not{k} - P_R(a\not{k} + b\not{u})P_L)\,\psi = 0 \tag{B.7}$$

for active neutrinos, $\psi = \psi_L = P_L\psi$ also $P_L^2 = P_L$

$$\begin{aligned}
(\not{k} - P_R(a\not{k} + b\not{u})P_L)\,P_L\psi &= 0\\
(\not{k}P_L - P_R(a\not{k} + b\not{u})P_L)\,\psi &= 0\\
\left(\not{k}P_L^2 - P_R(a\not{k} + b\not{u})P_L\right)\psi &= 0\\
(P_R\not{k}P_L - P_R(a\not{k} + b\not{u})P_L)\,\psi &= 0
\end{aligned} \tag{B.8}$$

$$P_R\left[(1-a)\,\not{k} - b\not{u}\right]P_L\psi = 0 \Rightarrow \not{V}\psi = 0 \tag{B.9}$$

where $V_\mu = (1-a)\,k_\mu - bu_\mu$ For non-trivial solutions of $\not{V}\psi$=0, $V^2 = 0$

$$\begin{aligned}
[(1-a)\,k_\mu - bu_\mu)]\,[(1-a)\,k^\mu - b\mu^\mu] &= 0 \\
(1-a)^2\,k_\mu k^\mu + b^2 u_\mu u^\mu - 2\,(1-a)\,b\,(k\cdot u) &= 0 \\
(1-a)^2\,k^2 + b^2 u^2 - 2b\,(1-a)\,\omega &= 0 \\
(1-a)^2\left(\omega^2 - \kappa^2\right) + b^2 - 2b\,(1-a)\,\omega &= 0 \\
[(1-a)\,(\omega-\kappa) - b]\,[(1-a)\,(\omega+\kappa) - b] &= 0 \\
f\,(\omega)\times\overline{f}\,(\omega) &= 0
\end{aligned} \tag{B.10}$$

So, $V^2 = 0 \Rightarrow f\,(\omega)\,\overline{f}\,(\omega) = 0$
This equation has the solution:

$$f\,(\omega_\kappa) = 0 \tag{B.11}$$

or,

$$\overline{f}\,(-\overline{\omega}_\kappa) = 0 \tag{B.12}$$

Since we can separate self-energy into gauge dependent and independent parts, we can separate f as,

$$f = f_0 + f_\xi = [(1-a_0)\,(\omega-\kappa) - b_0] + [(1-a_\xi)\,(\omega-\kappa) - b_\xi] \tag{B.13}$$

Dispersion relation is independent of gauge parameter. f_0 and f_ξ must vanish separately at $\omega = \omega_\kappa$

$$\begin{aligned}
f_0\,(\omega_\kappa) \equiv (1-a_0)\,(\omega_\kappa-\kappa) - b_0 &= 0 \\
\Rightarrow b_0 = (1-a_0)\,(\omega_\kappa-\kappa) \Rightarrow \omega_\kappa - \kappa &= b_0 + a_0\,(\omega_\kappa-\kappa) \\
\omega_\kappa &= \kappa + b_0 + a_0\,(\omega_\kappa-\kappa)
\end{aligned} \tag{B.14}$$

In zeroth order, $\omega_\kappa = \kappa$. Substituting this in (B.14)

$$\omega_\kappa = \kappa + b_0\,(\omega_\kappa = \kappa)$$

So for neutrinos,

$$\omega_\kappa = \kappa + b_0\,(\kappa) + \text{h.o} \tag{B.15}$$

Similarly, for anti-neutrinos,

$$\overline{f}\,(-\overline{\omega}_\kappa) = 0 \tag{B.16}$$

$$\begin{aligned}
\overline{f}_0\,(-\overline{\omega}_\kappa) \equiv (1-a_0)\,(-\overline{\omega}_\kappa+\kappa) - b_0 &= 0 \\
\Rightarrow b_0 = (1-a_0)\,(-\overline{\omega}_\kappa+\kappa) \Rightarrow \omega_\kappa &= \kappa - b_0 - a_0\,(-\overline{\omega}_\kappa+\kappa)
\end{aligned} \tag{B.17}$$

Keeping terms in the first order,

$$\omega_\kappa = \kappa - b_0\,(\omega_\kappa = -\kappa)$$

So for anti-neutrinos,

$$\overline{\omega}_\kappa = \kappa - b_0 \left(-\kappa\right) + \text{h.o} \tag{B.18}$$

B.2 Effective potential

We can see the effective potential as a contribution to the energy of the particle by background medium.ie,
Energy of Neutrino propagating through a medium =
. Energy of neutrino in vacuum + Effective potential in the medium

$$\omega_\kappa = \kappa + \mathcal{V}_l \tag{B.19}$$

We are only considering the lowest order, hence $\mathcal{V}_l$ coincide with b_0, but in general it depends on a_0 too.
Now our problem has reduced to finding b_0 in $\Sigma = (a_0 + a_\xi)\not{k} + (b_0 + b_\xi)\not{u}$

B.3 Self energy calculation

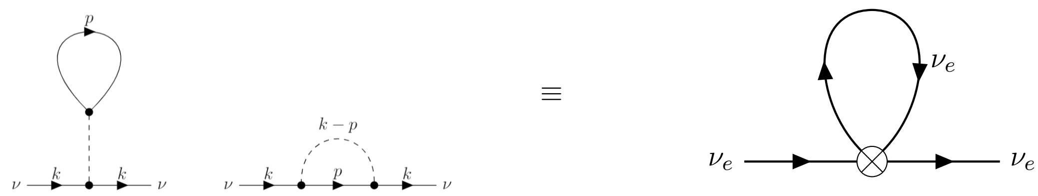
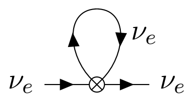


Figure 1: Self energy diagrams in NSI

From [4] (or we can infer from Fig.1) we can write Feynman rules for 4-fermion vertex Matrix elements for an NSI Lagrangian,

$$\mathcal{L}_j = -\frac{G_F}{\sqrt{2}} \left(\epsilon_{j,\nu_e}\right)^{eeee} \left(\bar{\nu}_e \mathcal{O}_j \nu_e\right) \left(\bar{\nu}_e \mathcal{O}'_j \nu_e\right) \tag{B.20}$$



$$\equiv \quad i\mathcal{M} = \frac{i^2 G_F}{\sqrt{2}} \left(\epsilon_{j,\nu_e}\right)^{eeee} \int \frac{d^4p}{(2\pi)^4} \left(\bar{u}\mathcal{O}_j S_F(p) \mathcal{O}'_j u - \bar{u}\mathcal{O}_j u \text{Tr}\left[\mathcal{O}'_j S_F(p)\right]\right) \tag{B.21}$$

where $S_F(p)$ is the fermion propagator at finite temperature,

$$S_F\left(p\right) = \left(\not{p} + m_l\right) \left[\frac{1}{p^2 - m_l^2} + 2\pi i \delta\left(p^2 - m_l^2\right) \eta\left(p \cdot u\right)\right] \tag{B.22}$$

with,

$$\eta\left(p \cdot u\right) = \frac{\theta\left(p \cdot u\right)}{e^x + 1} + \frac{\theta\left(-p \cdot u\right)}{e^{-x} + 1}, x = \frac{\left(p \cdot u - \mu\right)}{T} \tag{B.23}$$

for neutrinos,

$$S_F(p) = \not{p}\left[\frac{1}{p^2+i\eta} + 2\pi i\delta\left(p^2-m_\nu^2\right)\eta\left(p\cdot u\right)\right] \tag{B.24}$$

Background-independent part of the fermion propagator only renormalizes the wave function and does not contribute to the dispersion relation in the lowest order[5]. To simplify the calculations, we will only consider background-dependent terms.

$$S_F^T(p) = 2\pi i\delta\left(p^2-m_l^2\right)\eta\left(p\cdot u\right)\left(\not{p}+m_l\right) \tag{B.25}$$

Now,

$$i\mathcal{M} = \bar{u}\left[i\Sigma_{\text{eff}}\right]u = \bar{u}\left[\frac{i^2 G_F}{\sqrt{2}}\left(\epsilon_{j,\nu_e}\right)^{eeee}\int\frac{d^4p}{(2\pi)^4}\left(\mathcal{O}_j S_F(p)\mathcal{O}_j' - \mathcal{O}_j\text{Tr}\left[\mathcal{O}_j' S_F(p)\right]\right)\right]u \tag{B.26}$$

$$i\Sigma_{\text{eff}} = \frac{-G_F}{\sqrt{2}}\left(\epsilon_{j,\nu_e}\right)^{eeee}\int\frac{d^4p}{(2\pi)^4}\left(\mathcal{O}_j S_F^T(p)\mathcal{O}_j' - \mathcal{O}_j\text{Tr}\left[\mathcal{O}_j' S_F^T(p)\right]\right) \tag{B.27}$$

Correct order is, $\mathcal{O}_j S_F^T(p)\mathcal{O}_j'$

B.3.1 Scalar NSI(this will change once we introduce h.o terms in Lagrangian)

For scalar NSI, $\mathcal{O}_S = (1-\gamma^5)$ and $\mathcal{O}_S' = (1-\gamma^5)$

$$\begin{aligned} i\Sigma_{\text{eff}} &= \frac{-G_F}{\sqrt{2}}\left(\epsilon_{S,\nu_e}\right)^{eeee}\int\frac{d^4p}{(2\pi)^4}\left(S_F^T(p)\mathcal{O}_S\mathcal{O}_S' - \mathcal{O}_S\text{Tr}\left[\mathcal{O}_S' S_F^T(p)\right]\right) \\ &= -\frac{G_F}{\sqrt{2}}\left(\epsilon_{S,\nu_e}\right)^{eeee}\int\frac{d^4p}{(2\pi)^4}\left(S_F^T(p)(1-\gamma^5)(1-\gamma^5) - (1-\gamma^5)\text{Tr}\left[(1-\gamma^5)S_F^T(p)\right]\right) \end{aligned} \tag{B.28}$$

$$i\Sigma_{\text{eff}} = -\frac{G_F}{\sqrt{2}}\left(\epsilon_{S,\nu_e}\right)^{eeee}\int\frac{d^4p}{(2\pi)^4}\left(2\cdot S_F^T(p) - \text{Tr}\left[(1-\gamma^5)S_F^T(p)\right]\right)(1-\gamma^5) \tag{B.29}$$

$\text{Tr}\left[(1-\gamma^5)S_F^T(p)\right] = 2\pi i\delta\left(p^2-m_\nu^2\right)\eta\left(p\cdot u\right)\text{Tr}\left[(1-\gamma^5)\not{p}\right] = 0$ ($m_\nu\approx 0$)

$$\begin{aligned} i\Sigma_{\text{eff}} &= -\frac{G_F}{\sqrt{2}}\left(\epsilon_{S,\nu_e}\right)^{eeee}\int\frac{d^4p}{(2\pi)^4}\left(2\cdot 2\pi i\delta\left(p^2-m_\nu^2\right)\eta\left(p\cdot u\right)\gamma_\mu p^\mu\right)(1-\gamma^5) \\ &= -\frac{2iG_F}{\sqrt{2}}\left(\epsilon_{S,\nu_e}\right)^{eeee}\int\frac{d^4p}{(2\pi)^3}\delta\left(p^2-m_\nu^2\right)\eta\left(p\cdot u\right)\gamma_\mu p^\mu\underbrace{u_\mu u^\mu}_{=1(B.1)}(1-\gamma^5) \\ &= -\sqrt{2}iG_F\left(\epsilon_{S,\nu_e}\right)^{eeee}\int\frac{d^4p}{(2\pi)^3}\delta\left(p^2-m_\nu^2\right)\eta\left(p\cdot u\right)\left(p\cdot u\right)\not{u}(1-\gamma^5) \\ &= -\sqrt{2}iG_F\left(\epsilon_{S,\nu_e}\right)^{eeee}J_1^{(\nu)}\not{u}(1-\gamma^5) \end{aligned} \tag{B.30}$$

$$\Sigma_{\text{eff}} = -2\sqrt{2}G_F\left(\epsilon_{S,\nu_e}\right)^{eeee}J_1^{(\nu)}\not{u}P_L \tag{B.31}$$

where(check B.3.3),

$$J_n^{(f)} = \int\frac{d^4p}{(2\pi)^3}\delta\left(p^2-m_f^2\right)\eta(p\cdot u)(p\cdot u)^n \tag{B.32}$$

Now comparing (B.31) with $\Sigma_{\text{eff}} = P_R \Sigma P_L = P_R(b_0 \not{u})P_L = -b_0 \not{u} P_L$,

$$b_0 = 2\sqrt{2}G_F \left(\epsilon_{S,\nu_e}\right)^{eeee} J_1^{(\nu)} = \sqrt{2}G_F \left(\epsilon_{S,\nu_e}\right)^{eeee} (n_\nu - n_{\bar{\nu}}) \tag{B.33}$$

From (B.2) and (B.1), Effective potential $\mathcal{V}_l = b_0(\omega_k = \kappa)$ for ν and $\mathcal{V}_l = -b_0(\omega_k = -\kappa)$ for $\bar{\nu}$

i.e. in the lowest order,

$$\mathcal{V}_{\text{scalar}} = \pm\sqrt{2}G_F \left(\epsilon_{S,\nu_e}\right)^{eeee} (n_\nu - n_{\bar{\nu}}) \tag{B.34}$$

$(+)$ for ν and $(-)$ for $\bar{\nu}$

We will add $\sim (\bar{\nu}_e \mathcal{O}_j \nu_e) \Box (\bar{\nu}_e \mathcal{O}'_j \nu_e)$ to the NSI Lagrangian to get more accurate thermal potential.

B.3.2 Vector NSI(this will change once we introduce h.o terms in Lagrangian)

For Vector NSI, $\mathcal{O}_V = \gamma_\mu(1-\gamma^5)$ and $\mathcal{O}'_V = \gamma^\mu(1-\gamma^5)$

$$\begin{aligned} i\Sigma_{\text{eff}} &= \frac{-G_F}{\sqrt{2}} \left(\epsilon_{V,\nu_e}\right)^{eeee} \int \frac{d^4p}{(2\pi)^4} \left(\mathcal{O}_V S_F^T(p) \mathcal{O}'_V - \mathcal{O}_V \text{Tr}\left[\mathcal{O}'_V S_F^T(p)\right]\right) \\ &= -\frac{G_F}{\sqrt{2}} \left(\epsilon_{V,\nu_e}\right)^{eeee} \int \frac{d^4p}{(2\pi)^4} \left(\gamma_\mu(1-\gamma^5) S_F^T(p) \gamma^\mu (1-\gamma^5) - \gamma_\mu(1-\gamma^5)\text{Tr}\left[\gamma^\mu(1-\gamma^5) S_F^T(p)\right]\right) \end{aligned} \tag{B.35}$$

$$i\Sigma_{\text{eff}} = -\frac{G_F}{\sqrt{2}} \left(\epsilon_{V,\nu_e}\right)^{eeee} \int \frac{d^4p}{(2\pi)^4} \left(\gamma_\mu(1-\gamma^5) S_F^T(p) - \text{Tr}\left[\gamma_\mu(1-\gamma^5) S_F^T(p)\right]\right) \gamma^\mu(1-\gamma^5) \tag{B.36}$$

$$\begin{aligned} \text{Tr}\left[\gamma^\mu(1-\gamma^5) S_F^T(p)\right] &= 2\pi i \delta\left(p^2 - m_\nu^2\right) \eta\left(p\cdot u\right) \text{Tr}\left[\gamma^\mu(1-\gamma^5)\not{p}\right] \\ &= 2\pi i \delta\left(p^2 - m_\nu^2\right) \eta\left(p\cdot u\right) \text{Tr}\left[\gamma^\mu \not{p}\right] = 2\pi i \delta\left(p^2 - m_\nu^2\right) \eta\left(p\cdot u\right) \cdot 4p^\mu \end{aligned}$$

$$\gamma_\mu(1-\gamma^5)\not{p}\gamma^\mu(1-\gamma^5) = -\not{p}\gamma_\mu(1+\gamma^5)\gamma^\mu(1-\gamma^5) = -\not{p}\gamma_\mu\gamma^\mu(1-\gamma^5)^2 = -\not{p}(-4I_4)\cdot 2\cdot(1-\gamma^5) = 8\not{p}(1-\gamma^5)$$

$$\begin{aligned} \Sigma_{\text{eff}} &= -\frac{G_F}{\sqrt{2}} \left(\epsilon_{V,\nu_e}\right)^{eeee} \int \frac{d^4p}{(2\pi)^3} \delta\left(p^2 - m_\nu^2\right) \eta\left(p\cdot u\right) \left(\gamma_\mu(1-\gamma^5)\not{p} - 4p_\mu\right) \gamma^\mu(1-\gamma^5) \\ &= -\frac{G_F}{\sqrt{2}} \left(\epsilon_{V,\nu_e}\right)^{eeee} \int \frac{d^4p}{(2\pi)^3} \delta\left(p^2 - m_\nu^2\right) \eta\left(p\cdot u\right) \left(8\not{p} - 4\not{p}\right) (1-\gamma^5) \\ &= \frac{4G_F}{\sqrt{2}} \left(\epsilon_{V,\nu_e}\right)^{eeee} \int \frac{d^4p}{(2\pi)^3} \delta\left(p^2 - m_\nu^2\right) \eta\left(p\cdot u\right) \not{p}(1-\gamma^5) \\ &= \frac{4G_F}{\sqrt{2}} \left(\epsilon_{V,\nu_e}\right)^{eeee} \int \frac{d^4p}{(2\pi)^3} \delta\left(p^2 - m_\nu^2\right) \eta\left(p\cdot u\right) \gamma_\mu p^\mu \underbrace{u_\mu u^\mu}_{=1(B.1)} (1-\gamma^5) \\ &= \frac{4G_F}{\sqrt{2}} \left(\epsilon_{V,\nu_e}\right)^{eeee} J_1^{(\nu)} \not{u}(1-\gamma^5) \end{aligned} \tag{B.37}$$

where(check B.3.3),

$$J_n^{(f)} = \int \frac{d^4p}{(2\pi)^3} \delta\left(p^2 - m_f^2\right) \eta(p\cdot u)(p\cdot u)^n \tag{B.38}$$

$$\Sigma_{\text{eff}} = 4\sqrt{2}G_F \left(\epsilon_{V,\nu_e}\right)^{eeee} J_1^{(\nu)} \not{u} P_L \tag{B.39}$$

Now comparing (B.39) with $\Sigma_{\text{eff}} = P_R \Sigma P_L = P_R(b_0\not{u})P_L = -b_0\not{u}P_L$,

$$b_0 = -4\sqrt{2}G_F \left(\epsilon_{V,\nu_e}\right)^{eeee} J_1^{(\nu)} = -2\sqrt{2}G_F \left(\epsilon_{S,\nu_e}\right)^{eeee} (n_\nu - n_{\bar{\nu}}) \tag{B.40}$$

From (B.2) and (B.1), Effective potential $\mathcal{V}_l = b_0(\omega_k = \kappa)$ for ν and $\mathcal{V}_l = -b_0(\omega_k = -\kappa)$ for $\bar{\nu}$
ie,

$$\mathcal{V}_{\text{vector}} = \mp 2\sqrt{2}G_F \left(\epsilon_{S,\nu_e}\right)^{eeee} (n_\nu - n_{\bar{\nu}}) \tag{B.41}$$

$(-)$ for ν and $(+)$ for $\bar{\nu}$

B.3.3 Scalar mediator

Let's assume we have a scalar mediator with mass m_ϕ as in [1].

$$\mathcal{L}_s \supset \frac{\lambda_\phi}{2}\nu_e\nu_e\phi + \text{h.c} \tag{B.42}$$

There will be two diagrams(Fig 1),

$$\begin{aligned} i\Sigma_{\text{eff}} &= 2\lambda_\phi^2 \int \frac{d^4p}{(2\pi)^4} S_F^T(p) \frac{(1-\gamma^5)}{2} \frac{1}{(k-p)^2 - m_\phi^2} \frac{(1-\gamma^5)}{2} - \\ &\qquad \frac{(1-\gamma^5)}{2} \frac{1}{(k-p)^2 - m_\phi^2} \text{Tr}\left[\frac{(1-\gamma^5)}{2} S_F^T(p)\right] \\ &= 2\lambda_\phi^2 \int \frac{d^4p}{(2\pi)^4} S_F^T \left(\frac{1}{(k-p)^2 - m_\phi^2}\right) \frac{(1-\gamma^5)}{2} \\ &= 2\lambda_\phi^2 \int \frac{d^4p}{(2\pi)^4} \frac{2\pi i \delta\left(p^2 - m_\nu^2\right) \eta\left(p\cdot u\right) \gamma_\mu p^\mu}{(k-p)^2 - m_\phi^2} P_L \end{aligned} \tag{B.43}$$

ie,

$$\Sigma_{\text{eff}} = 2\lambda_\phi^2 \gamma_\mu \mathcal{P}^\mu(1) P_L \tag{B.44}$$

where,

$$\mathcal{P}_\mu(1) = \int \frac{d^4p}{(2\pi)^3} \frac{\delta\left(p^2 - m_l^2\right)\eta(p\cdot u)}{(p-k)^2 - m_\phi^2} p_\mu \tag{B.45}$$

Now, with $\mathcal{M}^2 = m_l^2 + k^2 - m_\phi^2$

$$\mathcal{P}_\mu(1) = \int \frac{d^4p}{(2\pi)^3} \frac{\delta\left(p^2 - m_l^2\right)\eta(p\cdot u)}{p^2 + k^2 - 2\cdot k - m_\phi^2} p_\mu \tag{B.46}$$

$$\mathcal{P}_\mu(1) = \int \frac{d^4p}{(2\pi)^3} \frac{\delta\left(p^2 - m_l^2\right)\eta(p\cdot u)}{\mathcal{M}^2 - 2p\cdot k} p_\mu \tag{B.47}$$

$$\mathcal{P}_\mu(1) = \int \frac{d^4p}{(2\pi)^3}\delta\left(p^2 - m_l^2\right)\eta(p\cdot u)\frac{p_\mu}{\mathcal{M}^2}\left(1 - \frac{2p\cdot k}{\mathcal{M}^2}\right)^{-1} \tag{B.48}$$

Only keeping terms up to $\mathcal{M}^{-4}$,

$$\mathcal{P}_\mu(1) = \frac{1}{\mathcal{M}^2}\int \frac{d^4p}{(2\pi)^3}\delta\left(p^2 - m_l^2\right)\eta(p\cdot u)p_\mu\left(1 + \frac{2p\cdot k}{\mathcal{M}^2} + O\left(\mathcal{M}^{-4}\right)\right) \tag{B.49}$$

$$\mathcal{P}_\mu(1) = \frac{1}{\mathcal{M}^2}\int \frac{d^4p}{(2\pi)^3}\delta\left(p^2 - m_l^2\right)\eta(p\cdot u)p_\mu\left(1 + \frac{2p_\nu k^\nu}{\mathcal{M}^2} + O\left(\mathcal{M}^{-4}\right)\right) \tag{B.50}$$

$$\mathcal{P}_\mu(1) = \frac{1}{\mathcal{M}^2}\left[I_\mu + \frac{2k^\nu}{\mathcal{M}^2}I_{\mu\nu}\right] \tag{B.51}$$

Where

$$\begin{aligned} I_\mu &= \int \frac{d^4p}{(2\pi)^3}\delta\left(p^2 - m_l^2\right)\eta(p\cdot u)p_\mu \\ I_{\mu\nu} &= \int \frac{d^4p}{(2\pi)^3}\delta\left(p^2 - m_l^2\right)\eta(p\cdot u)p_\mu p_\nu \end{aligned} \tag{B.52}$$

I_μ is manifestly covariant and depends only on the vector u_μ.

$$I_\mu = Au_\mu \tag{B.53}$$

Therefore, contracting I_μ with u_μ,

$$I_\mu u^\mu = Au_\mu u^\mu = A \tag{B.54}$$

$$I_\mu u^\mu = \int \frac{d^4p}{(2\pi)^3}\delta\left(p^2 - m_l^2\right)\eta(p\cdot u)p_\mu u^\mu = J_1^{(l)} \tag{B.55}$$

ie,

$$I_\mu = J_1^{(l)}u_\mu \tag{B.56}$$

Similarly, $I_{\mu\nu} = \int \frac{d^4p}{(2\pi)^3}\delta\left(p^2 - m_l^2\right)\eta(p\cdot u)p_\mu p_\nu$ depends only on u

$$I_{\mu\nu} = Ag_{\mu\nu} + Bu_\mu u_\nu \tag{B.57}$$

By contracting this expression with $u_\mu u_v$ and $g_{\mu v}$, we obtain two equations for A and B ,

$$g^{\mu\nu}I_{\mu\nu} = \int \frac{d^4p}{(2\pi)^3}\delta\left(p^2 - m_l^2\right)\eta(p\cdot u)p_\mu g^{\mu\nu}p_\nu = m_l^2 J_0^{(l)} = 4A + B \tag{B.58}$$

$$u^\mu u^\nu I_{\mu\nu} = \int \frac{d^4p}{(2\pi)^3}\delta\left(p^2 - m_l^2\right)\eta(p\cdot u)p_\mu u^\mu u^\nu p_\nu = J_2^{(l)} = A + B \tag{B.59}$$

$$I^\mu_\mu = m_l^2 J_0^{(l)} \tag{B.60}$$

Solving for A and B, these are then determined as

$$\begin{aligned} A &= \tfrac{1}{3}\left(m_l^2 J_0^{(l)} - J_2^{(l)}\right) \\ B &= \tfrac{1}{3}\left(4J_2^{(l)} - m_l^2 J_0^{(l)}\right) \end{aligned}$$

Substituting the expressions for I_μ and $I_{\mu v}$ into (B.51),

$$\mathcal{P}_\mu(1) = \frac{1}{\mathcal{M}^2}\left\{\left[J_1^{(l)} + \frac{2\omega}{3\mathcal{M}^2}\left(4J_2^{(l)} - m_l^2 J_0^{(l)}\right)\right] u_\mu + \frac{2k_\mu}{3\mathcal{M}^2}\left(m_l^2 J_0^{(l)} - J_2^{(l)}\right)\right\} \tag{B.61}$$

Now,

$$\Sigma_{\text{eff}} = 2\lambda_\phi^2 \gamma_\mu \mathcal{P}^\mu(1) P_L \tag{B.62}$$

$m_l = m_\nu \approx 0$

$$\Sigma_{\text{eff}} = \frac{2\lambda_\phi^2}{\mathcal{M}^2}\gamma_\mu \left\{\left[J_1^{(\nu)} + \frac{8\omega}{3\mathcal{M}^2} J_2^{(\nu)}\right] u^\mu - \frac{2J_2^{(\nu)}}{3\mathcal{M}^2} k^\mu\right\} P_L \tag{B.63}$$

$$\Sigma_{\text{eff}} = \frac{2\lambda_\phi^2}{\mathcal{M}^2}\left\{\left[J_1^{(\nu)} + \frac{8\omega}{3\mathcal{M}^2} J_2^{(\nu)}\right] \not{u} - \frac{2J_2^{(\nu)}}{3\mathcal{M}^2} \not{k}\right\} P_L \tag{B.64}$$

Comparing with $\Sigma_{\text{eff}} = P_R \Sigma P_L = P_R(a_0 \not{k} + b_0 \not{u}) P_L = -(a_0 \not{k} + b_0 \not{u}) P_L$,

$$b_0 = \frac{4\lambda_\phi^2}{\mathcal{M}^2}\left[J_1^{(\nu)} + \frac{8\omega}{3\mathcal{M}^2} J_2^{(\nu)}\right] \tag{B.65}$$

$\mathcal{M}^2 \approx -m_\phi^2$

$$b_0 = -\frac{2\lambda_\phi^2}{(-m_\phi^2)}\left[\frac{1}{2}\left(n_\nu - n_{\bar\nu}\right) - \frac{8\omega}{3m_\phi^2} \cdot \frac{1}{2}\left(n_\nu \langle E_\nu \rangle + n_{\bar\nu}\langle E_{\bar\nu}\rangle\right)\right] \tag{B.66}$$

Effective potential $\mathcal{V}_l = b_0(\omega_k = \kappa)$ for ν and $\mathcal{V}_l = -b_0(\omega_k = -\kappa)$ for $\bar\nu$

$$\mathcal{V}_\phi = \pm\frac{\lambda_\phi^2}{m_\phi^2}\left(n_\nu - n_{\bar\nu}\right) - \frac{8\lambda_\phi^2}{3m_\phi^4} \cdot \omega \cdot \frac{7\pi^2 T^4}{240} \tag{B.67}$$

$$\mathcal{V}_\phi = \pm\frac{\lambda_\phi^2}{m_\phi^2}\left(n_\nu - n_{\bar\nu}\right) - \frac{7\pi^2\lambda_\phi^2}{90m_\phi^4} E T^4 \tag{B.68}$$

Evaluating $J_n^{(f)}$

$$J_n^{(f)} = \int \frac{d^4p}{(2\pi)^3}\delta\left(p^2 - m_f^2\right)\eta(p\cdot u)(p\cdot u)^n \tag{B.69}$$

δ-function,

$$\begin{aligned} \delta\left(p^2 - m_f^2\right) &= \delta\left(\left(p - m_f\right)\left(p + m_f\right)\right) \\ &= \delta\left(p_0^2 - \vec{p}^2 - m_f^2\right) \\ &= \delta\left(p_0^2 - \omega_p^2\right) \\ &= \delta\left(\left(p_0 - \omega_p\right)\left(p_0 + w_p\right)\right) \\ &= \frac{1}{2\omega_p}\left[\delta\left(p_0 - w_p\right) + \delta\left(p_0 + w_p\right)\right] \end{aligned} \tag{B.70}$$

where $\omega_p = \sqrt{p^2 + m_f^2} = E_p$.

$$\eta\left(p\cdot u\right) = \frac{\theta\left(p\cdot u\right)}{e^{\frac{(p\cdot u-\mu)}{T}}+1} + \frac{\theta\left(-p\cdot u\right)}{e^{-\frac{(p\cdot u-\mu)}{T}}+1} \tag{B.71}$$

In the rest frame of the medium, $u_\mu = (1,0,0,0)$,

$$\eta\left(p\cdot u\right) = \frac{\theta\left(p_0\right)}{e^{\frac{(p_0-\mu)}{T}}+1} + \frac{\theta\left(-p_0\right)}{e^{-\frac{(p_0-\mu)}{T}}+1} = \theta\left(p_0\right) f_f(p_0) + \theta\left(-p_0\right) f_{\bar{f}}(-p_0) \tag{B.72}$$

where, we have introduced the particle and antiparticle momentum distributions,

$$f_{f,\bar{f}}(E) = \frac{1}{e^{\beta(E\mp\mu)}+1}$$

number densities are given by

$$n_{f,\bar{f}} = g_f \int \frac{d^3\mathbf{p}}{(2\pi)^3} f_{f,\bar{f}}$$

and the thermal average of $\mathcal{E}^n$,

$$\left\langle \mathcal{E}^n_{f,\bar{f}} \right\rangle \equiv \frac{g_f}{n_{f,\bar{f}}} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathcal{E}^n f_{f,\bar{f}}$$

Now,

$$\begin{aligned}
J_n^{(f)} &= \int \frac{d^4p}{(2\pi)^3} \delta\left(p^2 - m_f^2\right) \eta(p\cdot u)(p\cdot u)^n \\
&= \int \frac{d^3\mathbf{p}dp_0}{(2\pi)^3} \frac{1}{2\omega_p} \left[\delta\left(p_0-\omega_p\right) + \delta\left(p_0+\omega_p\right)\right] \left[\theta\left(p_0\right) f_f(p_0) + \theta\left(-p_0\right) f_{\bar{f}}(-p_0)\right] (p_0)^n \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left[\frac{\omega_p^n}{2w_p} f_f\left(\omega_p\right) + \frac{\left(-\omega_p\right)^n}{2\omega_p} f_{\bar{f}}\left(\omega_p\right)\right] \\
&= \frac{1}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left[E_f^{n-1} f_f(E_p) + (-1)^n E_f^{n-1} f_{\bar{f}}(E_p)\right] \\
&= \frac{1}{2} \left[\frac{n_f}{g_f} \left\langle E_f^{n-1} \right\rangle + (-1)^n \frac{n_{\bar{f}}}{g_{\bar{f}}} \left\langle E_{\bar{f}}^{n-1} \right\rangle\right]
\end{aligned} \tag{B.73}$$

For neutrinos,
$g_\nu = g_{\bar{\nu}} = 1$

$$J_0^{(\nu)} = \frac{1}{2}\left[n_\nu \left\langle \frac{1}{E_\nu} \right\rangle + n_{\bar{\nu}} \left\langle \frac{1}{E_{\bar{\nu}}} \right\rangle\right] \tag{B.74}$$

$$J_1^{(\nu)} = \frac{1}{2}\left[\frac{n_\nu}{g_\nu} - \frac{n_{\bar{\nu}}}{g_{\bar{\nu}}}\right] = \frac{1}{2}\left(n_\nu - n_{\bar{\nu}}\right) \tag{B.75}$$

$$J_2^{(\nu)} = \frac{1}{2}\left[n_\nu \left\langle E_\nu \right\rangle + n_{\bar{\nu}} \left\langle E_{\bar{\nu}} \right\rangle\right] = \frac{7\pi^2 T^4}{240} \tag{B.76}$$

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