Interaction rate for a $2 \to 2$ process $1+2 \longrightarrow 3+4$,

$$\Gamma(E_1) = \frac{1}{2E_1} \int d^3\tilde{p}_2 \, d^3\tilde{p}_3 \, d^3\tilde{p}_4 \, (2\pi)^4 \delta^4 (p_{initial} - p_{final}) \, S|M|^2 \, \mathcal{F}[f]$$

where, $d^3\tilde{p}_i = \dfrac{d^3\vec{p}_i}{(2\pi)^3 2E(\vec{p})}$ $\quad S|M|^2 \equiv$ relevant matrix element.

$$\mathcal{F}[f] = f_2(E_2)(1 \pm f_3(E_3))(1 \pm f_4(E_4))$$

[from 1507.06655 (sterile-dm paper) Eq. B7]

$$\mathcal{F}[f] \equiv \Lambda[f_1, f_2, f_3, f_4] = (1-f_1)(1-f_2) f_3 f_4 - f_1 f_2 (1-f_3)(1-f_4)$$

[ astro-ph/9506015 ]

Useful expressions:

$$d^3\vec{p}_i = p_i^2 \, dp_i \, d\Omega_i = p_i^2 \, dp_i \, d(\cos\theta_i) \, d\phi_i$$

$$d^3\lambda = \lambda^2 d\lambda \, d\Omega_\lambda \qquad p_i \cdot \lambda = p_i \cos\theta_i \lambda$$

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \times \delta(E_1 + E_2 - E_3 - E_4)$$

$$\delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) = \int \frac{d^3\lambda}{(2\pi)^3} e^{i(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\cdot\lambda}$$

Now rewriting $\Gamma(E_1)$,

$$\Gamma(E_1) = \frac{1}{2E_1} \int \frac{p_2^2 p_3^2 p_4^2 \, dp_2 \, dp_3 \, dp_4 \, (2\pi)^4 \delta(E_1 + E_2 - E_3 - E_4) \times S|M|^2 \times \mathcal{F}[f]}{(2\pi)^9 \, 2^3 E_2 E_3 E_4} \int \frac{d^3\lambda}{(2\pi)^3} e^{i(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\cdot\lambda} \, d\Omega_2 \, d\Omega_3 \, d\Omega_4$$

$$p_i^2 = E_i^2 - m_i^2 \Rightarrow \frac{p_i^2}{E_i^2} = 1 - \left(\frac{m_i}{E_i}\right)^2 \quad , 2p_i dp_i = 2E_i dE_i$$

$$P(E_1) = \frac{1}{16 E_1 p_1} \frac{1}{(2\pi)^8} \int \frac{p_2 p_3 p_4 \, dp_2 dp_3 dp_4}{E_2 E_3 E_4} \delta(E_1+E_2-E_3-E_4) \, \cancel{S|M|^2} \, \mathcal{F}[f]$$

$$p_1 p_2 p_3 p_4 \int \lambda^2 d\lambda \int e^{i p_1 \cdot \lambda} d\Omega_\lambda \int e^{i p_2 \cdot \lambda} d\Omega_2 \int e^{-i p_3 \cdot \lambda} d\Omega_3 \int e^{-i p_4 \cdot \lambda} d\Omega_4$$

$$\times S|M|^2$$

$$= \frac{1}{64\pi^3 E_1 p_1} \int \frac{p_2 dp_2}{E_2} \frac{p_3 dp_3}{E_3} \frac{p_4 dp_4}{E_4} \delta(E_1+E_2-E_3-E_4) \mathcal{F}[f] \times D(p_1, p_2, p_3, p_4)$$

where $D(p_1, p_2, p_3, p_4) = \dfrac{p_1 p_2 p_3 p_4}{64\pi^5} \displaystyle\int \lambda^2 d\lambda \int e^{i p_1 \cdot \lambda} d\Omega_\lambda \int e^{i p_2 \cdot \lambda} d\Omega_2 \int e^{-i p_3 \cdot \lambda} d\Omega_3 \int d\Omega_4 e^{-i p_4 \cdot \lambda}$

$$\times S|M|^2$$

$$\int e^{\pm i p_i \cdot \lambda} d\Omega_i = \int e^{\pm i p_i \lambda \cos\theta_i} d\cos\theta_i \, d\phi_i = 2\pi \times \int_0^\pi d\cos\theta_i \, e^{\pm i p_i \lambda \cos\theta_i}$$

$$= 2\pi \int_1^{-1} du \, e^{\pm i p_i \lambda u} = 2\pi \times \left. \frac{e^{\pm i p_i \lambda u}}{\pm i p_i \lambda} \right|_1^{-1}$$

$$= 2\pi \times 2 \, \frac{e^{\mp i p_i \lambda} - e^{\pm i p_i \lambda}}{\pm 2 p_i \lambda i} = -4\pi \frac{\sin(p_i \lambda)}{p_i \lambda}$$

then, $D(p_1, p_2, p_3, p_4) = \dfrac{p_1 p_2 p_3 p_4}{64\pi^5} \displaystyle\int \lambda^2 d\lambda \, (4\pi)^4 \prod_{i=1}^{4} \frac{\sin p_i \lambda}{p_i \lambda} \times S|M|^2$

(if $S|M|^2$ is independent of $p_1, p_2, p_3, p_4$)

If $S|M|^2 = K(p_1 \cdot p_2)(p_3 \cdot p_4)$

## 0.1 D-Integrals

Following Appendix A of [1]https://arxiv.org/pdf/hep-ph/9703315v1.pdf and Appendix C of [2]https://arxiv.org/pdf/2005.01629v1.pdf

Assume matrix elements are of the form;

$$S\overline{|\mathcal{M}|}^2 = K_1(p_1 \cdot p_2)\,(p_3 \cdot p_4) + K_2 m^2\,(p_3 \cdot p_4)$$

Let's define angles $\varphi, \theta$,

$$\vec{p_i} = p_i\,(\sin\theta_i \cos\varphi_i, \sin\theta_i \sin\varphi_i, \cos\theta_i)$$

Then

$$\begin{aligned}
\vec{p_i} \cdot \vec{p_j} &= p_i p_j\,(\sin\theta_i \cos\varphi_i \sin\theta_j \cos\varphi_j + \sin\theta_i \sin\varphi_i \sin\theta_j \sin\varphi_j + \cos\theta_i \cos\theta_j) \\
\vec{p_i} \cdot \vec{p_j} &= p_i p_j\,(\sin\theta_i \sin\theta_j \cos(\varphi_i - \varphi_j) + \cos\theta_i \cos\theta_j) \\
\vec{p_i} \cdot \vec{p_j} &= p_i p_j \cos\phi_{ij}
\end{aligned} \tag{1}$$

$$\begin{aligned}
(p_1 \cdot p_2)\,(p_3 \cdot p_4) &= (E_1 E_2 - \vec{p_1} \cdot \vec{p_2})\,(E_3 E_4 - \vec{p_3} \cdot \vec{p_4}) \\
&= E_1 E_2 E_3 E_4 - E_1 E_2\,(\vec{p_3} \cdot \vec{p_4}) - E_3 E_4\,(\vec{p_1} \cdot \vec{p_2}) + (\vec{p_1} \cdot \vec{p_2})\,(\vec{p_3} \cdot \vec{p_4})
\end{aligned} \tag{2}$$

$$D(p_1, p_2, p_3, p_4) = \frac{p_1 p_2 p_3 p_4}{64\pi^5} \int_0^\infty \lambda^2 d\lambda d\Omega_\lambda \prod_{i=2}^{4} d\Omega_i e^{-i(\mathbf{p_1}+\mathbf{p_2}-\mathbf{p_3}-\mathbf{p_4})\cdot\lambda} S\overline{|\mathcal{M}|}^2$$

$$D(p_1, p_2, p_3, p_4) = \frac{p_1 p_2 p_3 p_4}{64\pi^5} \int_0^\infty \lambda^2 d\lambda \int e^{-i\vec{p_1}\cdot\lambda} d\Omega_\lambda \int e^{-i\vec{p_2}\cdot\lambda} d\Omega_2 \int e^{+i\vec{p_3}\cdot\lambda} d\Omega_3 \int e^{+i\vec{p_4}\cdot\lambda} d\Omega_4 \times S\overline{|\mathcal{M}|}^2 \tag{3}$$

$$\begin{aligned}
D(p_1, p_2, p_3, p_4) = \frac{p_1 p_2 p_3 p_4}{64\pi^5} \int_0^\infty \lambda^2 d\lambda \int e^{-i\vec{p_1}\cdot\lambda} d\Omega_\lambda \int e^{-i\vec{p_2}\cdot\lambda} d\Omega_2 \int e^{+i\vec{p_3}\cdot\lambda} d\Omega_3 \int e^{+i\vec{p_4}\cdot\lambda} d\Omega_4 \times \\
K_1\,[E_1 E_2 E_3 E_4 - E_1 E_2\,(\vec{p_3} \cdot \vec{p_4}) - E_3 E_4\,(\vec{p_1} \cdot \vec{p_2}) + (\vec{p_1} \cdot \vec{p_2})\,(\vec{p_3} \cdot \vec{p_4})] + \\
K_2 m^2\,[E_3 E_4 - \vec{p_3} \cdot \vec{p_4}]
\end{aligned} \tag{4}$$

There can be four types of integrals possible from the expression above(extra factors are neglected):

$$D_{SS} = \frac{p_1 p_2 p_3 p_4}{64\pi^5} \int_0^\infty \lambda^2 d\lambda \frac{4\pi \cdot \sin(p_1\lambda)}{p_1\lambda} \frac{4\pi \cdot \sin(p_2\lambda)}{p_2\lambda} \frac{4\pi \cdot \sin(p_3\lambda)}{p_3\lambda} \frac{4\pi \cdot \sin(p_4\lambda)}{p_4\lambda} \tag{5}$$

$$\begin{aligned}
D_{SC} &= \frac{p_1 p_2 p_3 p_4}{64\pi^5} \int d\Omega_\lambda e^{ip_1\cdot\lambda} \int d\Omega_2 e^{ip_2\cdot\lambda} \int d\Omega_3 d\Omega_4 e^{-i(p_3+p_4)\cdot\lambda} \cdot (\vec{p_3} \cdot \vec{p_4}) \\
&= \frac{p_1 p_2 p_3 p_4}{64\pi^5} \int \lambda^2 d\lambda \frac{4\pi \cdot \sin(p_1\lambda)}{p_1\lambda} \frac{4\pi \cdot \sin(p_2\lambda)}{p_2\lambda} \cdot \\
&\quad \times \int d\phi_3 d\phi_4 d(\cos\theta_3) d(\cos\theta_4) e^{-ip_3\lambda\cos\theta_3} e^{-ip_4\lambda\cos\theta_4} \\
&\quad p_3 p_4\,(\sin\theta_3 \sin\theta_4 \cos(\varphi_3 - \varphi_4) + \cos\theta_3 \cos\theta_4) \\
&= \frac{p_1 p_2 (p_3 p_4)^2}{64\pi^5} \int \lambda^2 d\lambda \frac{4\pi \cdot \sin(p_1\lambda)}{p_1\lambda} \frac{4\pi \cdot \sin(p_2\lambda)}{p_2\lambda} \cdot \\
&\quad \times (2\pi)^2 \int d(\cos\theta_3) \cos\theta_3 e^{-ip_3\lambda\cos\theta_3} \int d(\cos\theta_4) \cos\theta_4 e^{-ip_4\lambda\cos\theta_4}
\end{aligned} \tag{6}$$

$$\begin{aligned}
\int d(\cos\theta) \cos\theta e^{-ip\lambda\cos\theta} &= \int u \cdot e^{-ip\lambda u} du \\
&= \frac{2\mathrm{i}\,(\sin(p\lambda) - p\lambda\cos((p\lambda)))}{(p\lambda)^2}
\end{aligned} \tag{7}$$

$$D_{SC} = \frac{-4}{\pi} \int \frac{d\lambda}{\lambda^4} \cdot \sin(p_1\lambda) \cdot \sin(p_2\lambda) \\ \times (\sin(p_3\lambda) - p_3\lambda\cos((p_3\lambda)) \cdot (\sin(p_4\lambda) - p_4\lambda\cos((p_4\lambda)) \tag{8}$$

Similarly,

$$D_{CS} = \frac{-4}{\pi} \int \frac{d\lambda}{\lambda^4} \cdot (\sin(p_1\lambda) - p_1\lambda\cos((p_1\lambda)) \cdot (\sin(p_2\lambda) - p_2\lambda\cos((p_2\lambda)) \\ \times \sin(p_3\lambda) \cdot \sin(p_4\lambda) \tag{9}$$

and finally,

$$D_{CC} = \frac{p_1p_2p_3p_4}{64\pi^5} \int d\Omega_\lambda d\Omega_2 e^{i(p_1+p_2)\cdot\lambda} \cdot (\vec{p}_1 \cdot \vec{p}_2) \int d\Omega_3 d\Omega_4 e^{-i(p_3+p_4)\cdot\lambda} \cdot (\vec{p}_3 \cdot \vec{p}_4) \tag{10}$$

$$D_{CC} = \frac{p_1p_2p_3p_4}{64\pi^5} \int \lambda^2 d\lambda \int d\phi_1 d\phi_2 d(\cos\theta_1) d(\cos\theta_2) e^{+ip_1\lambda\cos\theta_1} e^{+ip_2\lambda\cos\theta_2} p_1p_2 \\ (\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2) + \cos\theta_1\cos\theta_2) \\ \times \int d\phi_3 d\phi_4 d(\cos\theta_3) d(\cos\theta_4) e^{-ip_3\lambda\cos\theta_3} e^{-ip_4\lambda\cos\theta_4} p_3p_4 \\ (\sin\theta_3\sin\theta_4\cos(\varphi_3-\varphi_4) + \cos\theta_3\cos\theta_4) \tag{11}$$

$$D_{CC} = \frac{(p_1p_2p_3p_4)^2}{64\pi^5} \int \lambda^2 d\lambda \cdot (2\pi)^2 \int d(\cos\theta_1)\cos\theta_1 e^{+ip_1\lambda\cos\theta_1} \int d(\cos\theta_2)\cos\theta_2 e^{+ip_2\lambda\cos\theta_2} \cdot \\ \times (2\pi)^2 \int d(\cos\theta_3)\cos\theta_3 e^{-ip_3\lambda\cos\theta_3} \int d(\cos\theta_4)\cos\theta_4 e^{-ip_4\lambda\cos\theta_4} \tag{12}$$

$$D_{CC} = \frac{4}{\pi} \int \frac{d\lambda}{\lambda^6} \cdot (\sin(p_1\lambda) - p_1\lambda\cos((p_1\lambda)) \cdot (\sin(p_2\lambda) - p_2\lambda\cos((p_2\lambda)) \\ \times (\sin(p_3\lambda) - p_3\lambda\cos((p_3\lambda)) \cdot (\sin(p_4\lambda) - p_4\lambda\cos((p_4\lambda)) \tag{13}$$

These can be evaluated with Mathematica and have 4 different possibilities for different conditions for momentum:

1. $p_1 + p_2 > p_3 + p_4$ and $p_1 + p_4 > p_2 + p_3$,
2. $p_1 + p_2 > p_3 + p_4$ and $p_1 + p_4 < p_2 + p_3$,
3. $p_1 + p_2 < p_3 + p_4$ and $p_1 + p_4 < p_2 + p_3$,
4. $p_1 + p_2 < p_3 + p_4$ and $p_1 + p_4 > p_2 + p_3$,

# References

[1] A.D. Dolgov, S.H. Hansen, and D.V. Semikoz. "Non-equilibrium corrections to the spectra of massless neutrinos in the early universe". In: *Nuclear Physics B* 503.1 (1997), pp. 426–444. ISSN: 0550-3213. DOI: https://doi.org/10.1016/S0550-3213(97)00479-3. URL: http://www.sciencedirect.com/science/article/pii/S0550321397004793.

[2] Xuheng Luo, Werner Rodejohann, and Xun-Jie Xu. "Dirac neutrinos and $N_{\mathrm{eff}}$". In: *JCAP* 06 (2020), p. 058. DOI: 10.1088/1475-7516/2020/06/058. arXiv: 2005.01629 [hep-ph].