Șiruri de numere reale

Ex1: Să se calculeze următoarele limite de șiruri:

a) $\lim\limits_{n\to\infty} \frac{n^p}{a^n}; a,p>0$

$$\lim_{n\to\infty} a^n = \begin{cases} +\infty, & a>1 \\ 1, & a=1 \\ 0, & a\in(-1,1) \\ \nexists, & a\le -1 \end{cases}$$

$\lim\limits_{n\to\infty} \frac{n^p}{a^n}; a,p>0$

I. $a\in(0,1) \Rightarrow \lim\limits_{n\to\infty} \frac{n^p}{a^n} = \lim\limits_{n\to\infty} \frac{n^p}{0_+} = +\infty$

II. $a=1 \Rightarrow \lim\limits_{n\to\infty} \frac{n^p}{a^n} = \lim\limits_{n\to\infty} \frac{n^p}{1} = \lim\limits_{n\to\infty} n^p = +\infty$

III $a\in(1,\infty) \Rightarrow \lim\limits_{n\to\infty} \frac{n^p}{a^n} = \left[\frac{\infty}{\infty}\right]$

Not $x_n = \frac{n^p}{a^n}$

$x_n \ge 0, (\forall) n\in \mathbb{N}^*$

$\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim\limits_{n\to\infty} \frac{(n+1)^p}{\cancel{a^{n+1}}} \cdot \frac{\cancel{a^n}}{n^p} = \lim\limits_{n\to\infty} \frac{1}{a} \cdot \frac{(n+1)^p}{n^p} =$

$= \frac{1}{a} \cdot \lim\limits_{n\to\infty} \left(\underbrace{\frac{n+1}{n}}_{1}\right)^p = \frac{1}{a} \in \overline{\mathbb{R}}$

I. Dacă $l<1 \Rightarrow \exists \lim x_n, \lim x_n = 0$

II. Dacă $l>1 \Rightarrow \exists \lim x_n, \lim x_n = \infty$

$\left.\begin{array}{l} l=\frac{1}{a} \\ a\in(1,\infty)\end{array}\right| \Rightarrow l<1 \Rightarrow \lim\limits_{n\to\infty} x_n = 0$

b) $\lim\limits_{n\to\infty} \frac{n}{a^{\sqrt{n}}}, a>0$

I. $a\in(0,1) \Rightarrow \lim\limits_{n\to\infty} \frac{n}{a^{\sqrt{n}}} = +\infty$

II $a=1 \Rightarrow \lim\limits_{n\to\infty} \frac{n}{a^{\sqrt{n}}} = \lim\limits_{n\to\infty} \frac{n}{1} = \lim\limits_{n\to\infty} n = +\infty$

II. $a\in(1,\infty) \Rightarrow \lim\limits_{n\to\infty} \frac{n}{a^{\sqrt{n}}} = \left[\frac{\infty}{\infty}\right]$

Not $a_n = n$

$b_n = a^{\sqrt{n}}$

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} a^{\sqrt{n}} = +\infty$

$b_{n+1} - b_n = a^{\sqrt{n+1}} - a^{\sqrt{n}} > 0 \Rightarrow b_s \nearrow$

$\lim\limits_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{n+1-n}{a^{\sqrt{n+1}}-a^{\sqrt{n}}} = \lim\limits_{n\to\infty} \frac{1}{a^{\sqrt{n+1}}-a^{\sqrt{n}}} = \lim\limits_{n\to\infty} \frac{1}{\underset{\infty}{a^{\sqrt{n}}}\,(\underset{0}{a^{\sqrt{n+1}-\sqrt{n}}-1})}$

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$\lim\limits_{n\to\infty} \frac{n}{a^{\sqrt{n}}} \overset{n=x^2}{=} \lim\limits_{x\to\infty} \frac{x^2}{a^x} \overset{L'H}{=} \lim\limits_{x\to\infty} \frac{2x}{a^x \ln a} = 2 \lim\limits_{x\to\infty} \frac{x}{a^x \ln a} \overset{L'H}{=}$

$= 2 \lim\limits_{x\to\infty} \frac{1}{\underset{\infty}{a^x}(\ln a)^2} = 0 \Rightarrow \lim\limits_{n\to\infty} a_n = 0$

c) $\lim\limits_{n\to\infty} \sqrt[n]{1^p+2^p+\dots+n^p}, p>0$

$\lim\limits_{n\to\infty} (1^p+2^p+\dots+n^p)^{\frac{1}{n}} = [\infty^0]$

<u>Notăm</u> $x_n = 1^p+2^p+\dots+n^p$

$x_n > 0, (\forall)\, n\in\mathbb{N}^*$

Dacă $\exists \lim \frac{x_{n+1}}{x_n} = \ell \in \overline{\mathbb{R}} \Rightarrow \exists \lim \sqrt[n]{x_n} = \ell$

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\frac{1^p+2^p+\dots+n^p+(n+1)^p}{1^p+2^p+\dots+n^p}=\lim_{n\to\infty}\left[1+\frac{(n+1)^p}{1^p+2^p+\dots+n^p}\right]$$

<u>Notăm</u> $a_n=1^p+2^p+\dots+(n+1)^p$

$b_n=1^p+2^p+\dots+n^p$

$$\lim_{n\to\infty} b_n=\lim_{n\to\infty} 1^p+2^p+\dots+n^p=\infty$$

$b_{n+1}-b_n=(n+1)^p>0 \Rightarrow b_n$ s $\nearrow$

$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=\lim_{n\to\infty}\frac{(n+2)^p}{(n+1)^p}=\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^p=1 \Longrightarrow$$

$$\overset{L-S-C}{\Longrightarrow}\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=1\Rightarrow\lim_{n\to\infty}\sqrt[n]{x_n}=1$$

d) $\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n$; $a,b>0$

$$\lim_{n\to\infty}\left(\frac{a^{\frac1n}+b^{\frac1n}}{2}\right)^n=[1^\infty]$$

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n=\lim_{n\to\infty}\left(1+\frac{a^{\frac1n}+b^{\frac1n}-2}{2}\right)^n=$$

$$=\lim_{n\to\infty}\left[\left(1+\frac{a^{\frac1n}+b^{\frac1n}-2}{2}\right)^{\frac{2}{a^{\frac1n}+b^{\frac1n}-2}}\right]^{\frac{a^{\frac1n}+b^{\frac1n}-2}{2}\cdot n}=$$

$$=e^{\lim_{n\to\infty}\cancel{\frac{1}{n}}\cdot\frac{a^{\frac1n}+b^{\frac1n}-2}{2}\cdot n}$$

$$\lim_{n\to\infty}\frac{1}{\frac1n}\cdot\frac{a^{\frac1n}+b^{\frac12}-2}{2}\overset{x=\frac1n}{=}\lim_{x\to0}\frac1x\cdot\frac{a^x+b^x-2}{2}=\lim_{x\to0}\frac{a^x+b^x-2}{2x}=$$

$$\overset{L'H}{=}\lim_{x\to0}\frac{a^x\ln a+b^x\ln b}{2}=\frac{\ln a+\ln b}{2}=\ln ab$$

$\Rightarrow \ell = e^{\ln\sqrt{ab}} = \sqrt{ab}$

Ex2: Se consideră șirurile $(x_n)_{n\geq 1}$ = def. prin recurența:

$x_{n+1} = x_n(1-x)$, $(\forall)\ n \in \mathbb{N}$, $x_1 \in (0,1)$

a) Șirul e convergent și calculați limita

b) $\lim\limits_{n\to\infty} n x_n$

a) Dem prin inducție că $x_n \in (0,1)$, $(\forall)\ n \in \mathbb{N}^*$

I. Etapa verificării: $x_1 \in (0,1)$ (ip) Adev

II. Etapa 2, $P(k) = x_k \in (0,1)$

$P(k)$ adev $\Rightarrow$ $P(k+1)$ adev

$P(k)$ adev $\Rightarrow x_k \in (0,1)$

$x_{k+1} = x_k(1-x_k)$

$$\left.\begin{array}{l}\left.\begin{array}{l} x_k > 0 \\ x_k < 1 \Rightarrow 1 - x_k > 0 \end{array}\right\} \Rightarrow x_{k+1} > 0 \\ x_{k+1} = x_k(1-x_k) \\ \left.\begin{array}{l} x_k < 1 \\ x_k > 0 \Rightarrow 1 - x_k < 1 \end{array}\right\} \Rightarrow x_{k+1} < 1 \end{array}\right\} \Rightarrow x_{k+1} \in (0,1) \Rightarrow$$

$\Rightarrow P(k+1)$ adev

$x_n \in (0,1)$, $(\forall)\ n \in \mathbb{N} \Rightarrow x_n$ mărginit (1)

$$\left.\begin{array}{l}\dfrac{x_{n+1}}{x_n} = \dfrac{\cancel{x_n}(1-x_n)}{\cancel{x_n}} = 1 - x_n \\ 0 < x_n < 1 \end{array}\right\} \Rightarrow \frac{x_{n+1}}{x_n} < 1 \Rightarrow (x_n)\ s \searrow \Rightarrow$$

$\Rightarrow x_n$ monoton (2)

(1) (2) $\xRightarrow{W.}$ (x_n) convergent

(x_n) converget $\Rightarrow \exists\, \ell$ a.î. $\lim x_{n+1} = \lim x_n$, $\ell \in \overline{\mathbb{R}}$

trecem la limită în relația de recurență

$\ell = \ell(1-\ell)$

$\ell = \ell - \ell^2 \Rightarrow \ell^2 = 0 \Rightarrow \ell = 0 \Rightarrow \lim\limits_{n\to\infty} x_n = 0$

b) $\lim\limits_{n\to\infty} n x_n = [\infty \cdot 0] = \lim\limits_{n\to\infty} \dfrac{n}{\frac{1}{x_n}}$

$a_n = n$

$b_n = \dfrac{1}{x_n}$

$\lim\limits_{n\to\infty} b_n = \infty$

$\dfrac{b_{n+1}}{b_n} = \dfrac{\frac{1}{x_{n+1}}}{\frac{1}{x_n}} = \dfrac{x_n}{x_{n+1}}$

(x_n) S $\searrow$

$\Rightarrow \dfrac{b_{n+1}}{b_n} > 1 \Rightarrow (b_n)_S \nearrow$

$\lim\limits_{n\to\infty} \dfrac{a_{n+1}-a_n}{b_{n+1}-b_n} = \dfrac{1}{\frac{1}{x_{n+1}} - \frac{1}{x_n}} = \lim\limits_{n\to\infty} \dfrac{1}{\frac{1}{x_n(1-x_n)} - \frac{1}{x_n}} = \lim\limits_{n\to\infty} \dfrac{1}{\underbrace{\frac{1}{x_n}}_{\infty}\Big(\underbrace{\frac{1}{1-x_n}-1}_{0}\Big)} =$

$= \lim\limits_{n\to\infty} \dfrac{1}{\frac{1-(1-x_n)}{x_n(1-x_n)}} = \lim\limits_{n\to\infty} \dfrac{x_n(1-x_n)}{\cancel{1}-(\cancel{1}-x_n)} = \lim\limits_{n\to\infty} 1 - x_n = 0 \Rightarrow$

$\Rightarrow \exists \lim\limits_{n\to\infty} a_n, \lim\limits_{n\to\infty} a^n = 0$

<u>Ex 3</u> Fie $(x_n)_{n\geq 1}$, un şir def. prin relaţia de recurentă $(\mathbb{R}_+^*)$

$x_n(1-x_{n+1}) > \frac{1}{4}$, $(\forall)\ n \in \mathbb{N}^*$

a) (x_n) convergent + limita

$x_n \geq 0, (\forall)\ x_n$

$x_n(1-x_{n+1}) > \frac{1}{4} \Rightarrow 1 - x_{n+1} > \dfrac{1}{4x_n} > 0 \Rightarrow$

$\Rightarrow 1 - x_{n+1} > 0 \Rightarrow x_{n+1} < 1 \Rightarrow x$

$(\forall), a,b > 0 \Rightarrow M_a > M_g \Rightarrow \frac{a+b}{2} \geq \sqrt{ab} \Rightarrow$

$\Rightarrow \sqrt{x_n(1-x_{n+1})} \leq \cancel{x_n+x_{n+1}} \frac{x_n + (1-x_{n+1})}{2} \quad /()^2$

$x_n(1-x_{n+1}) \leq \frac{x_n^2 + x_{n+1}^2 + 1 - \cancel{2x_nx_n} \, 2x_{n+1}}{4}$

$x_n(1-x_n) > \frac{1}{4} \Rightarrow$

$\Rightarrow \frac{x_n+(1-x_{n+1})}{2} > \sqrt{\frac{1}{4}} \Rightarrow \frac{x_n + (1-x_{n+1})}{2} > \frac{1}{2} \Rightarrow$

$\Rightarrow x_n + 1 - x_{n+1} > 1$

$x_n - x_{n+1} > 0 \Rightarrow x_n$ s $\searrow \Rightarrow (x_n)$ monoton

(x_n) mărginit $\Big\} \xRightarrow{W.} x_n$ convergent $\Rightarrow$

$\Rightarrow \exists\, l \in \overline{\mathbb{R}}$, $\lim x_n = \lim x_{n+1}$

Trecem la limită în relația de recurență:

$l(1-l) \geq \frac{1}{4}$

$l - l^2 \geq \frac{1}{4}$

$0 \geq \frac{1}{4} + l^2 - l$

$l^2 - l + \frac{1}{4} \leq 0$

$(l-\frac{1}{2})^2 \leq 0 \Rightarrow \boxed{l = \frac{1}{2}}$