Test

1. $v_1 = (1,1,3)$, $v_2 = (2,1,0)$, $v_3 = (4,3,1)$, $v_4 = (3,2,1)$

b) Fie $B = \{v_1, v_2, v_3\}$

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 3 \\ 3 & 0 & 1 \end{pmatrix} \quad rg\,A = rg\begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 3 \\ 3 & 0 & 1 \end{pmatrix} = 3 \Rightarrow B \text{ e SLI}$$

$$\dim_{\mathbb{R}} \mathbb{R}^3 = 3 = |B| \quad \Rightarrow$$

$\Rightarrow$ $\left.\begin{array}{l} B \text{ e SG} \\ B \text{ e SLI} \end{array}\right\} \Rightarrow \underline{B = \{v_1, v_2, v_3\}}$ bază a lui $\mathbb{R}^3$

a) $B = \{v_1, v_2, v_3\}$ bază $\Rightarrow$ B e SG.

$\left.\text{Orice supramulțime a unei SG e SG}\right\} \Rightarrow B' = \{v_1, v_2, v_3, v_4\}$ e SG în $\mathbb{R}^3$

c) $x = (1,1,1)$

Fie $a, b, c \in \mathbb{R}$

$x = (1,1,1) = av_1 + bv_2 + cv_3 = a(1,1,3) + b(2,1,0) + c(4,3,1)$

$= (a+2b+4c, a+b+3c, 3a+c)$

$$\begin{cases} a+2b+4c=1 \\ a+b+3c=1 \\ 3a+c=1 \end{cases} \Leftrightarrow \begin{cases} a+b+3c=1 \\ a+2b+4c=1 \\ 3a+c=1 \end{cases} \Leftrightarrow \begin{cases} a+b+3c=1 \\ b+c=0 \\ -3b-8c=-2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a+b+3c=1 \\ b+c=0 \\ -5c=-2 \end{cases} \Rightarrow \begin{array}{l} a = \frac{1}{5} \\ b = -\frac{2}{5} \\ c = \frac{2}{5} \end{array}$$

$\Rightarrow$ coordonatele lui x în raport cu B sunt $\left(\frac{1}{5}, -\frac{2}{5}, \frac{2}{5}\right)$

② $f: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x,y,z) = (2x+2y+2z, 2x+3y+z, 2x+y+3z)$

a) Fie $A = [f]_{R_0,R_0} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}$

(R_0 baza canonică)

Fie C matricea de trecere de la R_0 la $R = \{(-2,1,1), (0,-1,1), (1,1,1)\}$

$$C = \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$C^{-1} = \frac{1}{\det C} \cdot C^* = \begin{pmatrix} -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Se știe că $A' = [f]_{R,R} = C^{-1}AC = \begin{pmatrix} -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

a) Val. proprii sunt rădăcinile din $\mathbb{K}$ ale pol. caract.

$P(\lambda) = \det(A - \lambda I_3) = \lambda^3 - \sigma_1 \lambda^2 + \sigma_2 \lambda - \sigma_3 = 0$

$\sigma_1 = Tr(A) = 8$

$\sigma_2 = \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 2 + 2 + 8 = 12$

$\sigma_3 = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 0$

$\lambda^3 - 8\lambda + 12\lambda = 0 \Rightarrow \lambda(\lambda^2 - 8\lambda + 12) = 0 \Rightarrow \lambda(\lambda-2)(\lambda-6) = 0$

$\Rightarrow$ Val proprii sunt: $\lambda_1 = 0, m_1 = 1$

$\lambda_2 = 2, m_2 = 1$

$\lambda_3 = 6, m_2 = 1$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_1 x\} = \{x \in \mathbb{R}^3 \mid f(x) = 0\}$

$$\begin{cases} 2x_1 + 2x_2 + 2x_3 = 0 \\ 2x_1 + 3x_2 + x_3 = 0 \\ 2x_1 + x_2 + 3x_3 = 0 \end{cases} \Rightarrow \begin{cases} 3x_2 + x_3 = -2x_1 \\ x_2 + 3x_3 = -2x_1 \end{cases} \Rightarrow$$

$$\Rightarrow x_2 = -\frac{1}{2}x_1, \quad x_3 = -\frac{1}{2}x_1$$

$V_{\lambda_1} = \{(x_1, -\frac{1}{2}x_1, -\frac{1}{2}x_1) \mid x_1 \in \mathbb{R}\}$

$= \langle\{(1, -\frac{1}{2}, -\frac{1}{2})\}\rangle$

$V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid f(x) = 2x\}$

$$\begin{cases} 2x_1 + 2x_2 + 2x_3 = 2x_1 \\ 2x_1 + 3x_2 + x_3 = 2x_2 \\ 2x_1 + x_2 + 3x_3 = 2x_3 \end{cases} \Leftrightarrow \begin{cases} 2x_2 + 2x_3 = 0 \\ 2x_1 + x_2 + x_3 = 0 \\ 2x_1 + x_2 + x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 2x_2 = -2x_3 \\ 2x_1 + x_2 = -x_3 \end{cases} \Leftrightarrow \begin{cases} x_2 = -x_3 \\ 2x_1 + x_2 = -x_3 \end{cases} \Rightarrow \begin{matrix} x_2 = -x_3 \\ x_1 = 0 \end{matrix}$$

$V_{\lambda_2} = \{(0, -x_3, x_3) \mid x_3 \in \mathbb{R}\} = \langle\{(0, -1, 1)\}\rangle$

$V_{\lambda_3} = \{x \in \mathbb{R}^3 \mid f(x) = 6x\}$

$$\begin{cases} 2x_1 + 2x_2 + 2x_3 = 6x_1 \\ 2x_1 + 3x_2 + x_3 = 6x_1 \\ 2x_1 + x_2 + 3x_2 = 6x_2 \end{cases} \Leftrightarrow \begin{cases} -4x_1 + 2x_2 + 2x_3 = 0 \\ 2x_1 - 3x_2 + x_3 = 0 \\ 2x_1 + x_2 - 3x_3 = 0 \end{cases} \Leftrightarrow$$

$$\Rightarrow \begin{cases} -3x_2 + x_3 = -2x_1 \\ x_2 - 3x_3 = -2x_1 \end{cases} \Rightarrow x_1 = x_2 = x_3$$

$V_{\lambda_3} = \{(x_1, x_1, x_1) \mid x_1 \in \mathbb{R}\} = \langle\{(1, 1, 1)\}\rangle$

(3) $Q(x) = (\alpha - 2)x_1^2 + (\alpha - 2)x_2^2 + (\alpha + 1)x_3^2 - 2x_1x_2 + 4x_1x_3 - 4x_2x_3$

a) $Q(x) = x_1^2 + x_2^2 + 4x_3^2 - 2x_1x_2 + 4x_1x_3 - 4x_2x_3 + (\alpha - 3)x_1^2 + (\alpha - 3)x_2^2$
$+ (\alpha - 3)x_3^2$

$Q(x) = (x_1 - x_2 + 2x_3)^2 - (\alpha - 3)(x_1^2 + x_2^2 + x_3^2)$

~~Q(x)~~ . Q e pozitiv definită $\Leftrightarrow \forall x \in \mathbb{R}^3 \setminus \{0\}, Q(x) = 0$

2) $Q(x) = \ldots$

(3) $Q: \mathbb{R}^3 \to \mathbb{R}$

$$Q(x_1, x_2, x_3) = (\alpha-2)x_1^2 + (\alpha-2)x_2^2 + (\alpha+1)x_3^2 - 2x_1x_2 + 4x_1x_3 - 4$$

b) $$Q(x_1,x_2,x_3) = x_1^2 + x_2^2 + 4x_3^2 - 2x_1x_2 + 4x_1x_3 - 4x_2x_3$$

$$= \underbrace{(x_1 - x_2 + 2x_3)}_{y_1}{}^2 =$$

$$Q(x) = y_1^2$$

a) $$Q(x_1,x_2,x_3) = (\alpha-2)x_1^2 + (\alpha-2)x_2^2 + (\alpha+1)x_3^2 - 2x_1x_2 + 4x_1x_3 -$$

$$= (x_1 - x_2 + 2x_3)^2 + (\alpha-3)(x_1^2 + x_2^2 + x_3^2)$$

$Q(x)$ poz def $\Leftrightarrow$ $Q(x) > 0, \forall x \in \mathbb{R}^3 \setminus \{0\}$

$Q(x) = 0 \Leftrightarrow x = 0$

Pentru $\alpha > 3$: $\underbrace{(x_1 - x_2 + 2x_3)^2}_{\geq 0} + \underbrace{(\alpha-3)}_{\geq 0}\underbrace{(x_1^2 + x_2^2 + x_3^2)}_{\geq 0}$

$\Rightarrow Q(x) > 0, \forall x \in \mathbb{R}^3 \setminus \{0\}$

$Q(x) = 0 \Leftrightarrow x = 0$

$\to$ Pt $\alpha \in (3, \infty)$, Q e poz. def