PCA, SVD and Other Related Acronyms

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UdS & DFKI, Saarbrücken, Germany

Summer Semester Seminar

15th May 2019

The Problem

High Dimensional Data

How do we deal with high dimensional data?

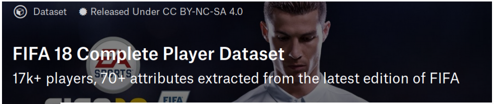
The Problem

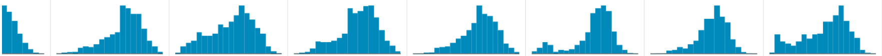
High Dimensional Data

time = (1.844012, 0.590383, 1.003636, -0.577031, 1.515419, 1.097797, 1.812856, 0.933615, -2.396581, -0.931116, -0.719396, -0.376134, -1.204231, 0.045771, -0.287482, 1.084627, 4.399265, 1.516829, -0.838133, -1.881685, 0.108117, 2.345857, -1.292667, -2.286168, 3.419926, 4.260052, -1.016988, 3.140229, -3.161504, -0.800707, -1.433775, 2.290546, 1.932333, 0.714649, -3.033084, -0.958289, -1.704687, -1.597345, 1.525060, 3.337017, -2.787743, 1.479353, 3.452092, -3.242210, 0.532302, -0.551804, 2.344314, -0.919049, -1.872516, 0.080137, 1.208913, -2.136555, -2.218254, 0.206410, 0.133225, -1.521032, 1.735609, 2.885288, -2.048691, 2.375038, 0.316599, -0.254595, 2.159168, 1.118603, -0.775468, 0.933521, -0.351797, 2.193516, 2.499064, 2.818742, -0.213898, 0.446962, 1.767461, 1.342941, 1.117215, -0.042004, 4.199081, 3.041796, -1.770649, -0.528354, -2.067354, 0.283046, -0.099049, -0.105402, 2.823484, -2.583724, -2.906962, 0.592174, -3.029664, -0.170582, 0.406366, 1.963008, -3.229250, -3.499467, -0.136623, -1.551140, 0.348241, -1.597526, 0.703598, 3.122618, 0.466473, -0.113320, -2.119155, 1.092863, -0.908410, 0.253259, -1.082862, 4.408773, 2.419691, 2.343239, 0.703793, 1.270707, 0.410221, -1.293057, -0.799147, 2.214563, -0.212623, 1.206766, -0.731273, 2.308388, -1.029362, -2.080709, 0.749148, -1.412619, 1.073051, -2.498955, -0.520858, 1.391912, -1.181121, 1.523457, -1.245448, -0.290742, -2.589719, -0.366162, 3.586508, 0.908829, -1.125176, -0.937035, -1.163619, 1.759209, 3.678231, 0.019263, -0.395732, 1.142848, -0.500150, -3.005232, 2.287069, -0.524648, -0.944902, 0.038368, -1.093538, -0.697787, 0.767664, 2.399855, 2.425945, 1.563581, -1.086811, 0.372100, 1.400303, -2.278863, 0.643208, -0.459837, 1.756295, 2.057359, 3.140241, -1.740582, 1.386243, -1.822378, 1.528883, -1.984250, 1.214508, -1.336822, -0.321478, -0.162113, 0.272326, -2.673072, 0.612675, -0.657483, -0.557969, -3.358420, -2.559981, -1.683046, -1.314229, -2.425110, -2.506184, -1.606668, 1.332781, -2.760878, -2.400824, -1.830618, -2.406664,-1.169146, -1.838281, 0.588559, 2.285466, -0.401462, 1.632473, -0.510084, -2.072332, -2.627897, 2.531830, -2.524195, 2.035469, 1.906113, -1.257332, -4.039220, -0.467614, -2.275054, -3.409202, -0.014383, 0.445576, 1.461529, -1.318478, 0.061049, 0.280523, 2.173227, -0.027133, 2.791830, -0.728346, -1.804815, 1.245291, 0.970318, 2.646388, 0.246842, -1.823608, 1.888760, 0.265116, -2.027269, -0.089802, 0.389976, -0.654499, 2.565478, -2.647825, 2.658914, 1.385568, 2.306623, 0.476923, -0.869644, -0.170338, 0.495097, -2.604649, 0.610231, 0.739677, 0.322778, -2.042915, -1.353154, 0.177016, 1.840185, -0.271689, -0.401560, -0.421108, -0.185526, 1.041765, -4.599578, -0.829409, 0.076258, -0.503421, 1.891007, -0.931777, 0.434825, -0.467926, -1.417658, -0.320597, -4.084039, -3.899607, 0.977403, 0.774670, 3.269479, -1.031264, -0.433907, -2.30576 0, 0.811788, 2.347483, -1.254061, -0.861366, 0.080974, -3.666142, -0.363376, -2.384475, -4.290071, -0.924723, 1.257435, 1.223927, 0.276726, 1.541471, 1.274240, 1.883040, -1.987514, -0.809325, 1.252716, 1.812783, -0.511801, -1.657522, 1.196169, 0.804855, -1.861488, -2.113367, 0.429888, -0.920844, 0.377247)

The Problem

High Dimensional Data





	# Acceleration	# Aggression	# Agility	# Balance	# Ball control	# Composure	# Crossing
2.29k	11 – 96	11 – 96	14 – 96	11 – 96	8 – 95	5 – 96	5 – 91
2228	89	63	89	63	93	95	85
2154	92	48	90	95	95	96	77
2100	94	56	96	82	95	92	75
2291	88	78	86	60	91	83	77
1493	58	29	52	35	48	70	15
2143	79	80	78	80	89	87	62
1458	57	38	60	43	42	64	17
2096	93	54	93	91	92	87	80

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High Dimensional Data

How do we deal with high dimensional data?

- Are all dimensions relevant?
- Are they correlated?
- All of them describe the data?

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These questions are related to **feature selection**, **data compression**, **visualisation...**

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Let's **project** into a lower dimensional space, but **wisely**...

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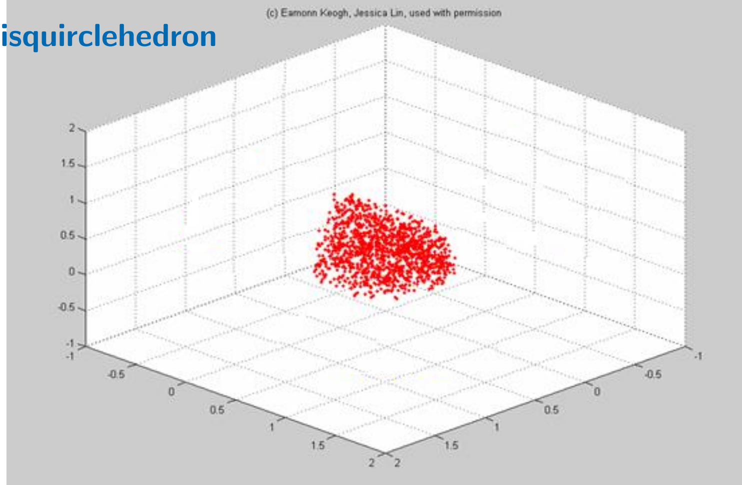
Is any kind of projection valid?

Trisquirclehedron



https://cs.gmu.edu/∼jessica/DimReducDanger.htm

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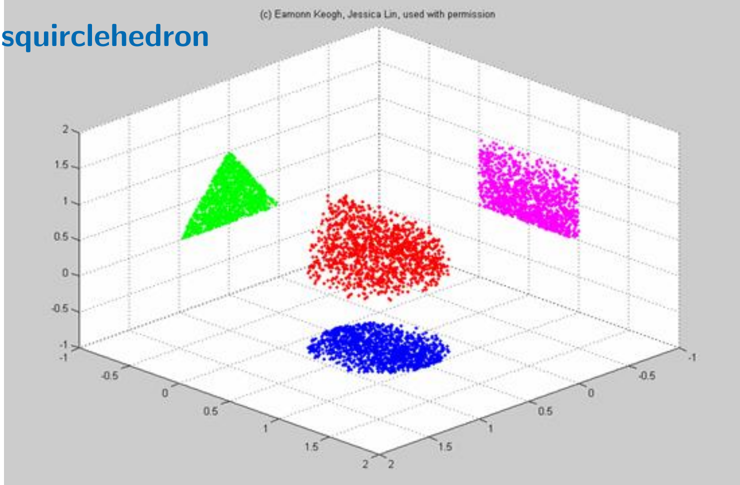

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Principal Component Analysis

Outline

Description:

Linear projection of a set of data points from a space defined by many correlated variables into a space defined by fewer uncorrelated coordinates called principal components while still retaining most of the variability present in the data.

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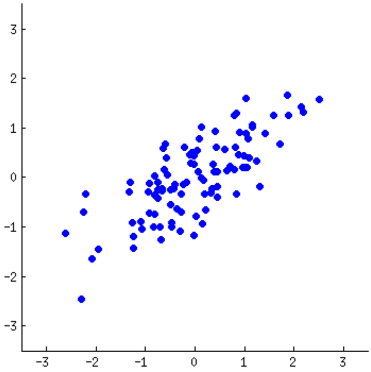
PCA: Visualisation in 2D⇒1D



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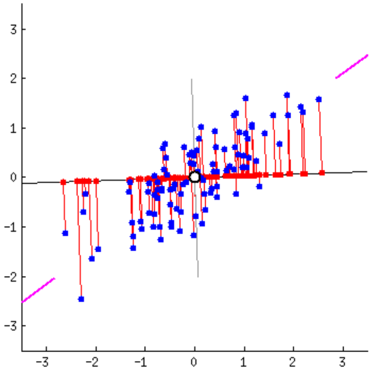

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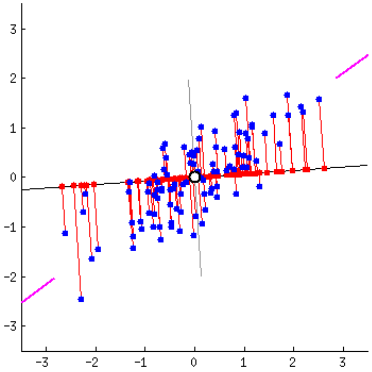

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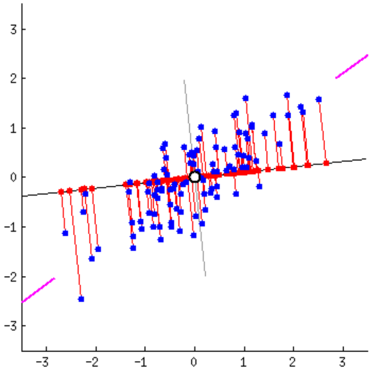

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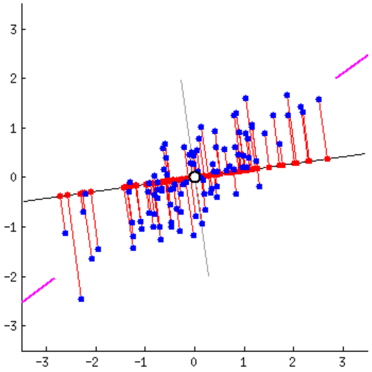

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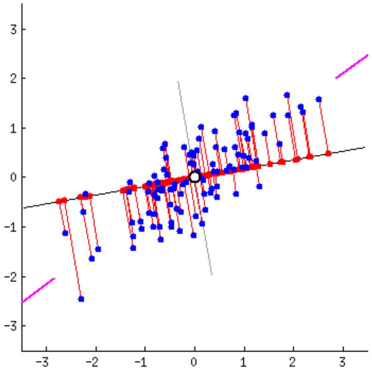

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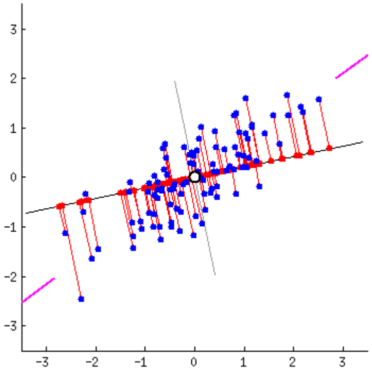

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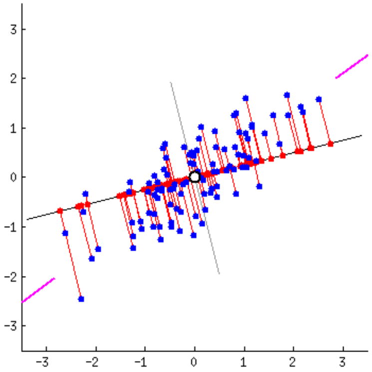

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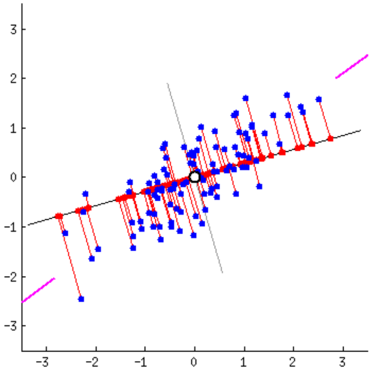

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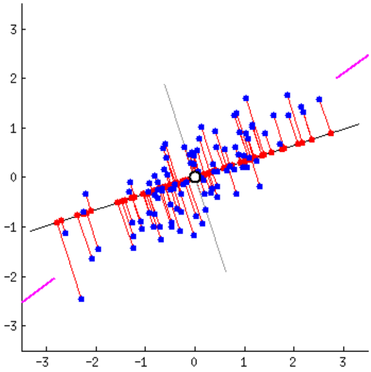

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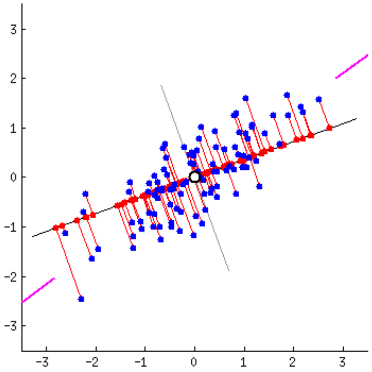

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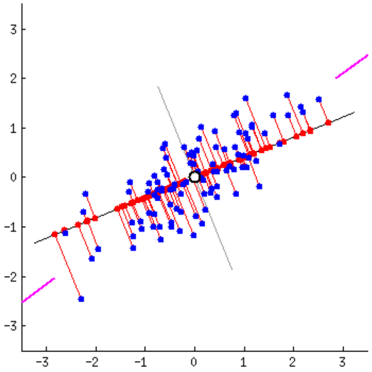

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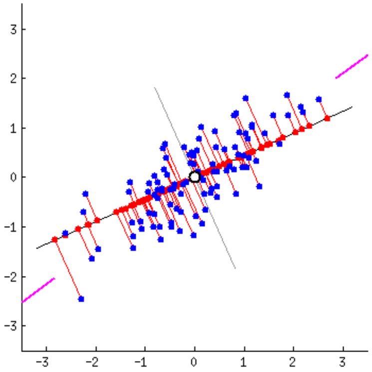

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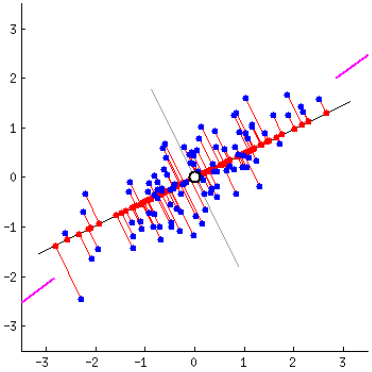

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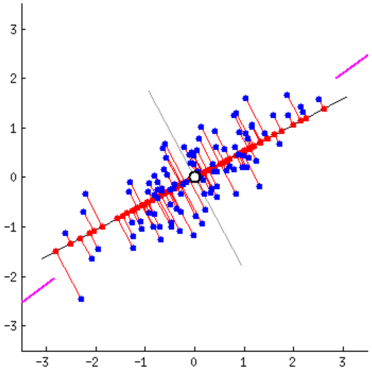

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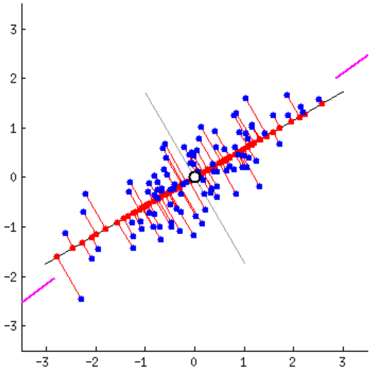

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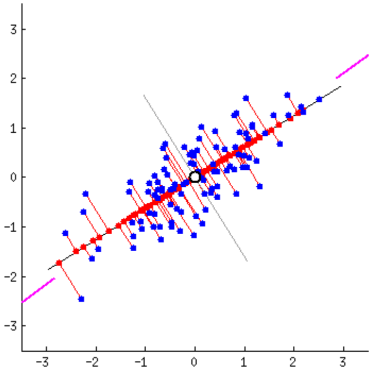

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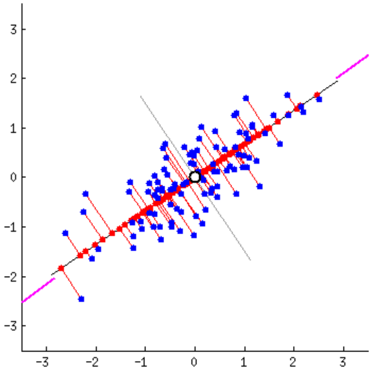

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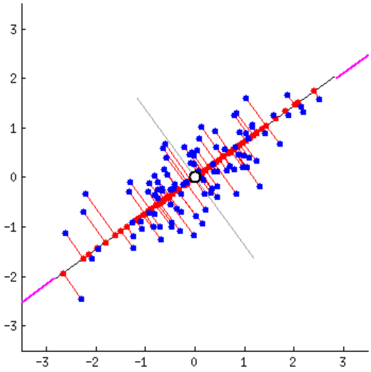

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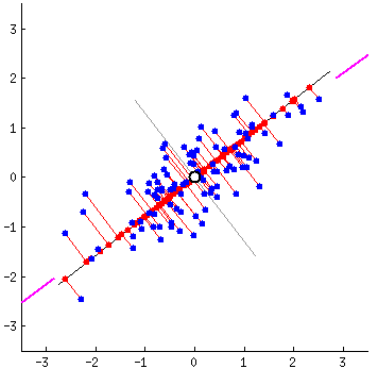

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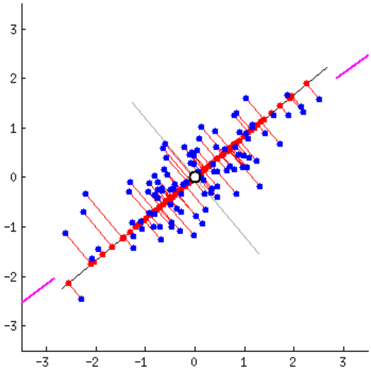

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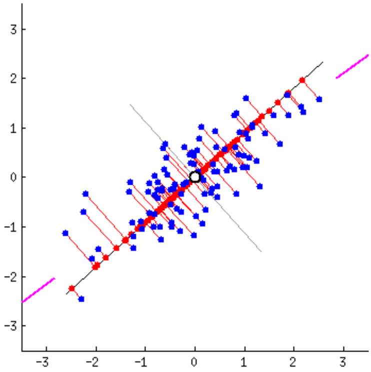

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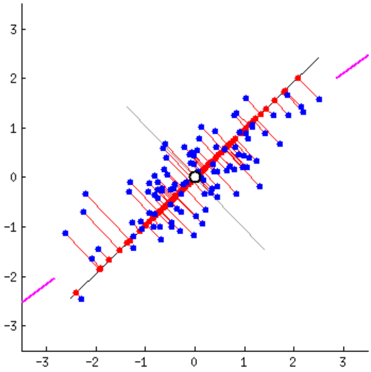

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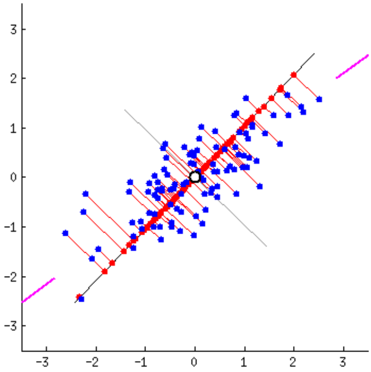

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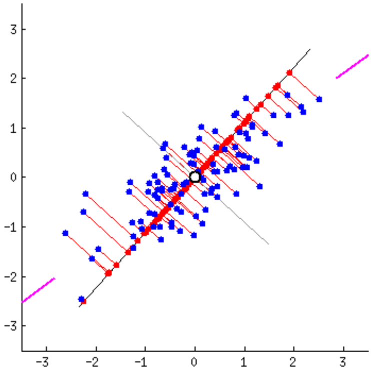

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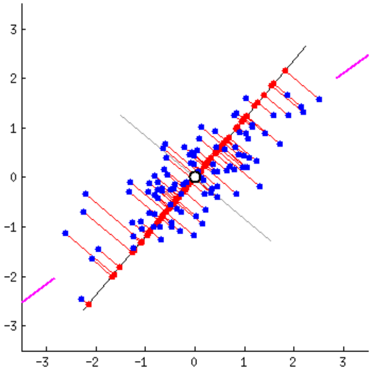

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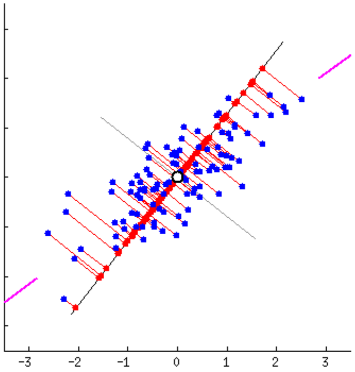

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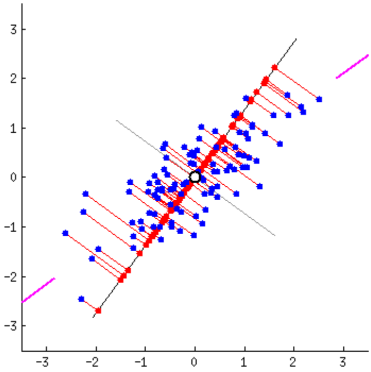

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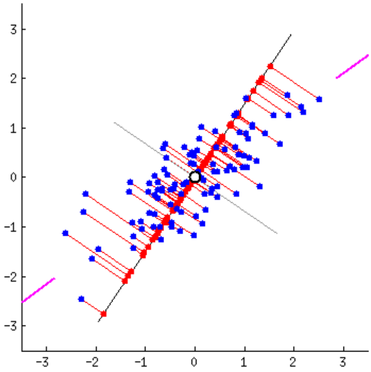

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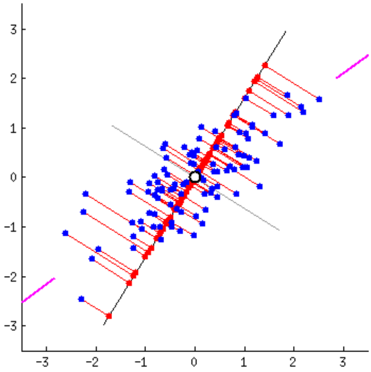

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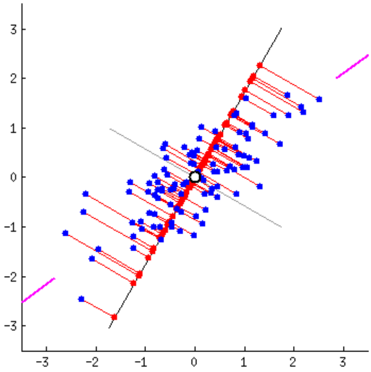

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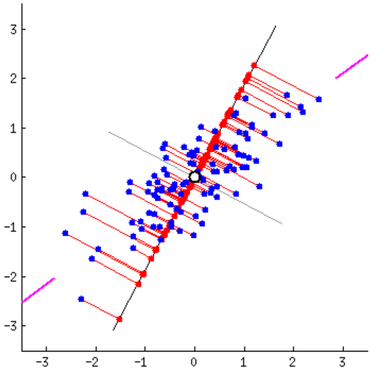

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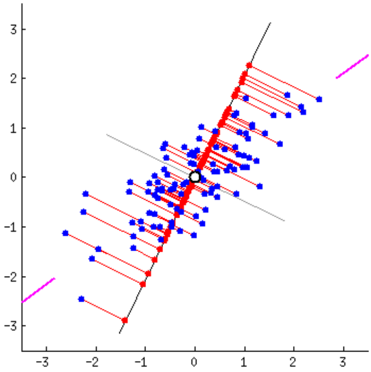

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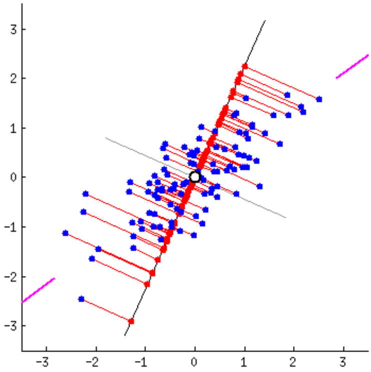

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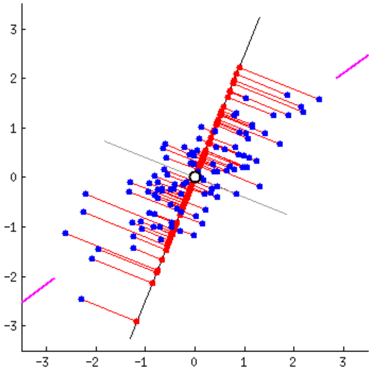

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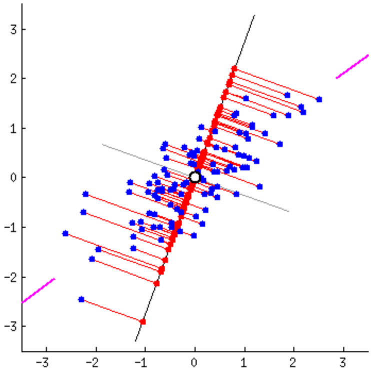

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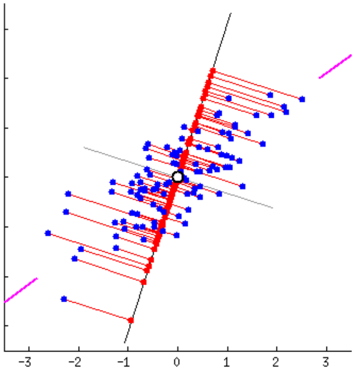

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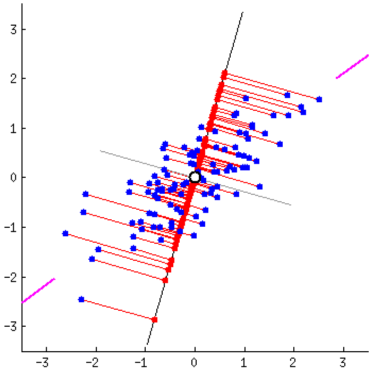

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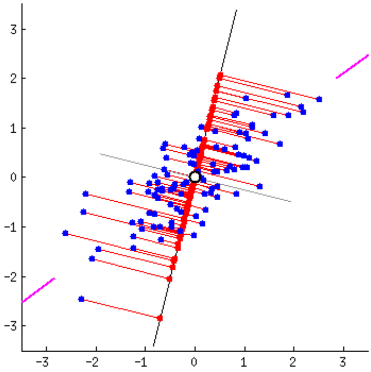

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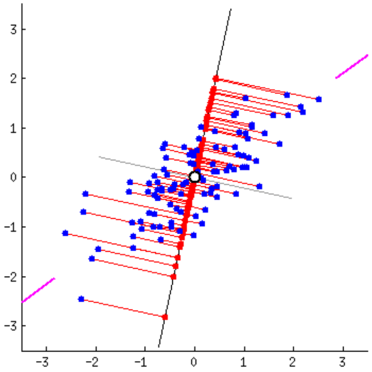

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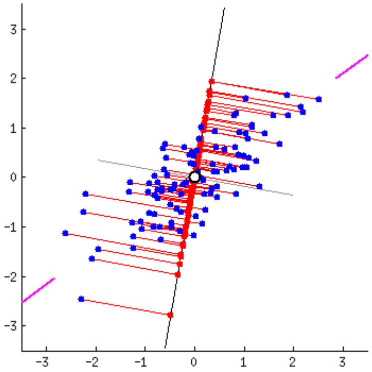

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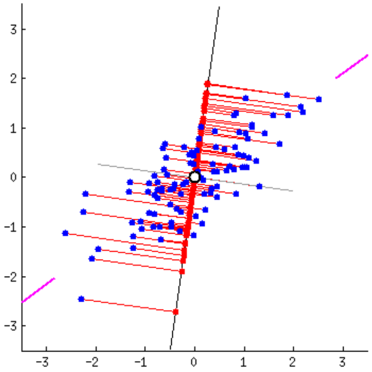

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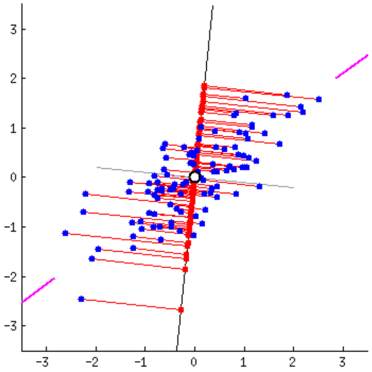

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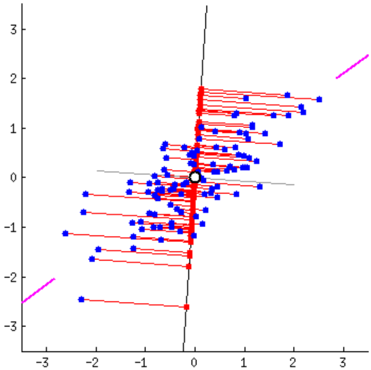

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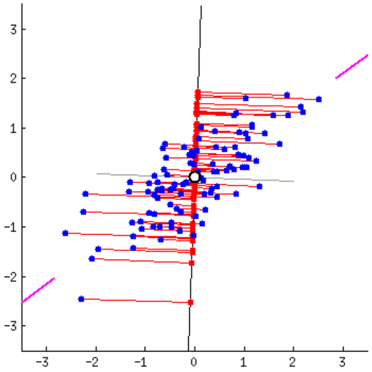

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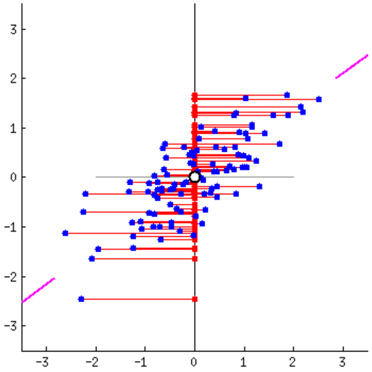

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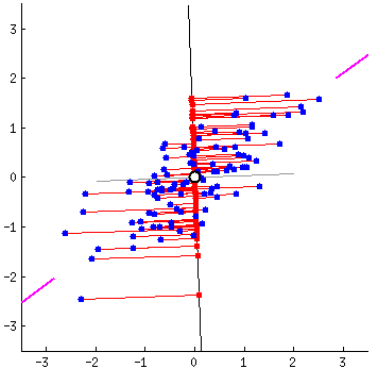

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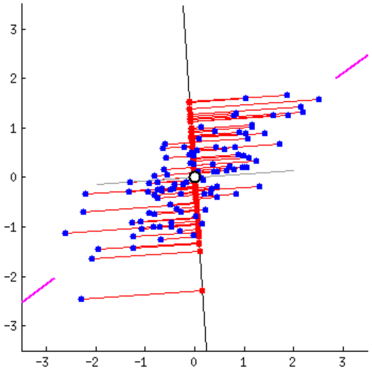

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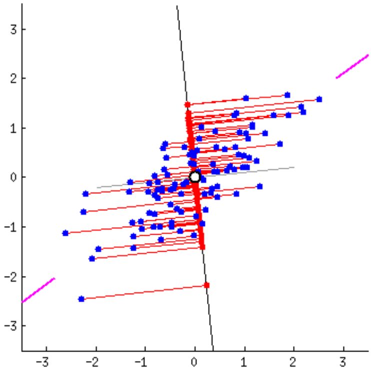

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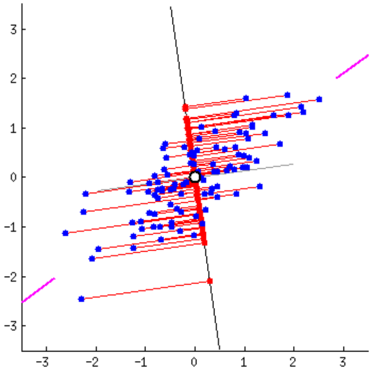

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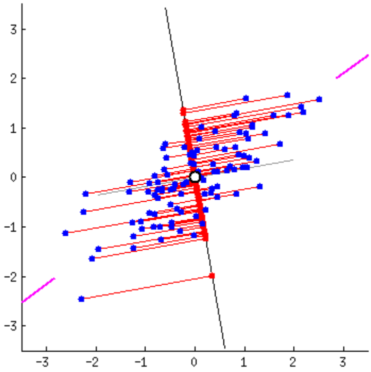

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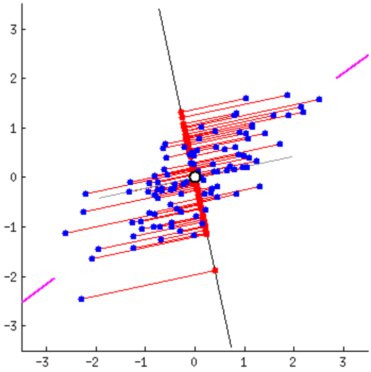

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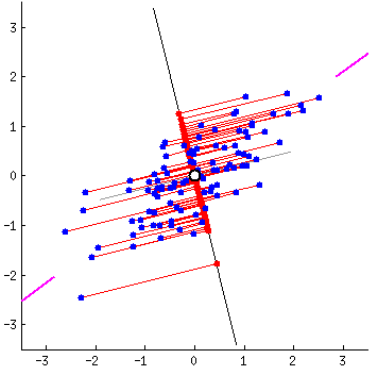

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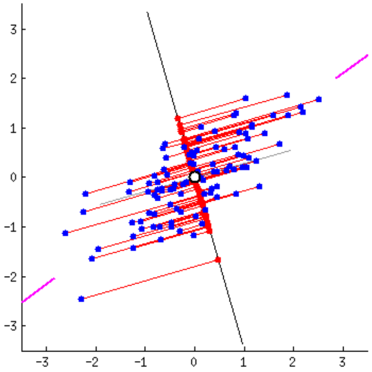

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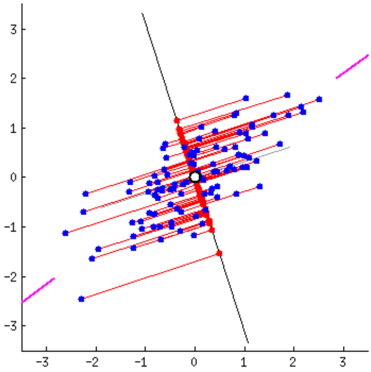

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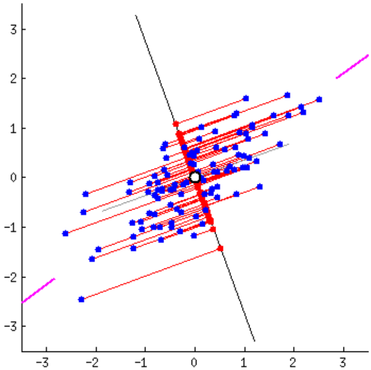

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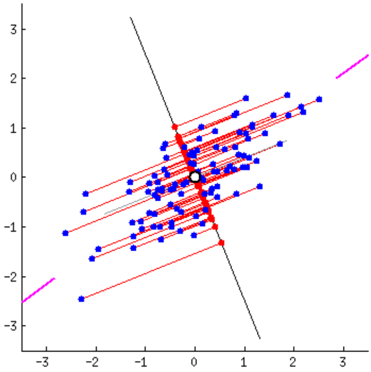

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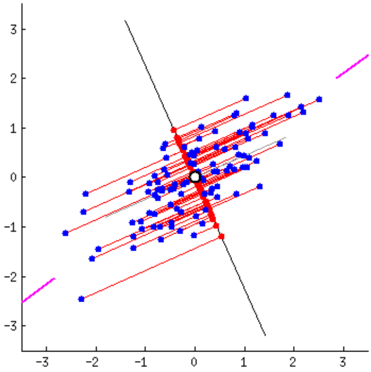

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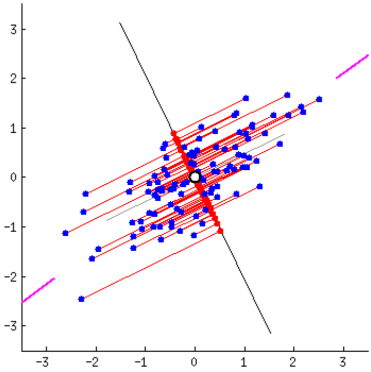

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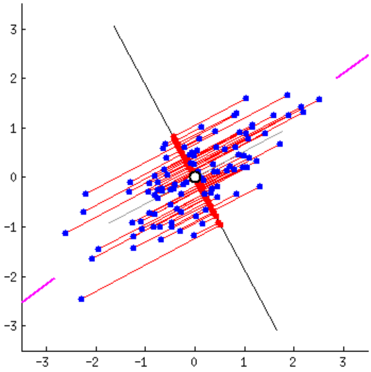

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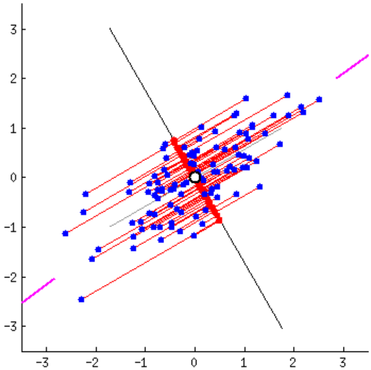

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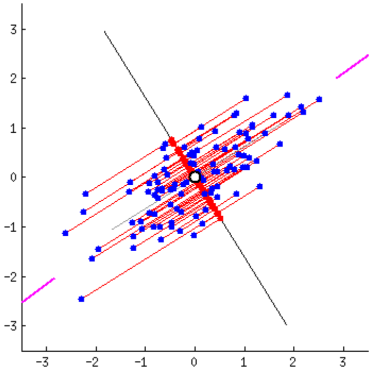

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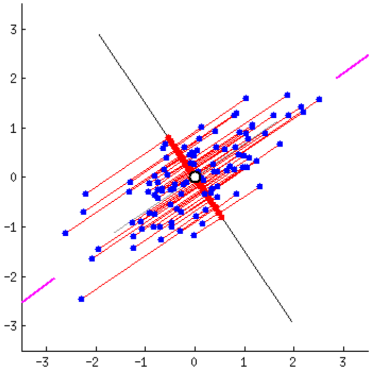

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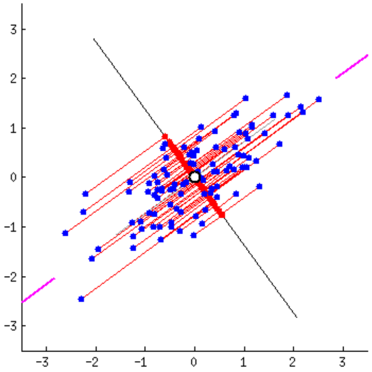

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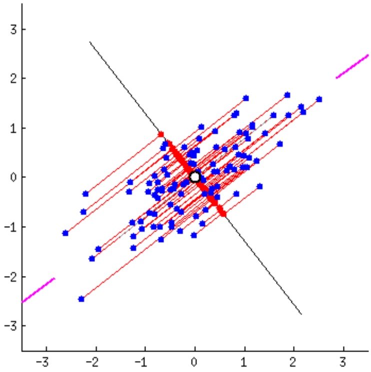

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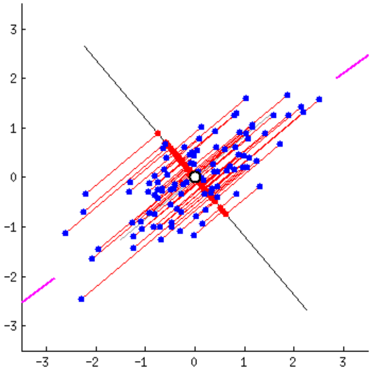

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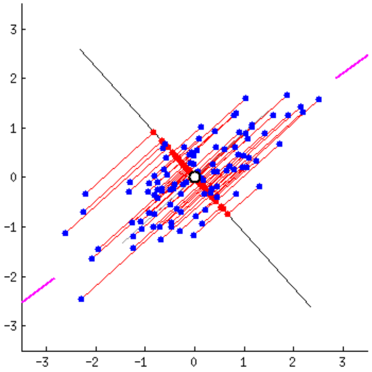

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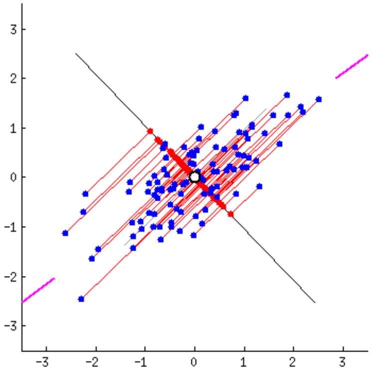

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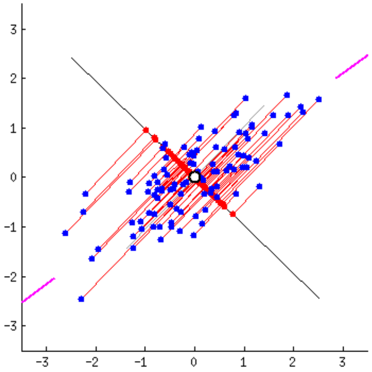

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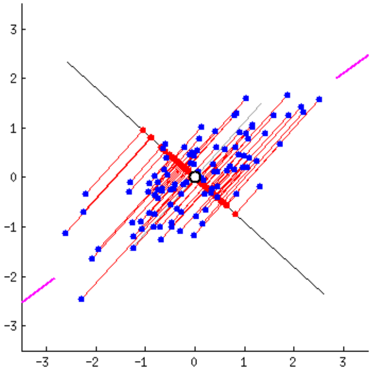

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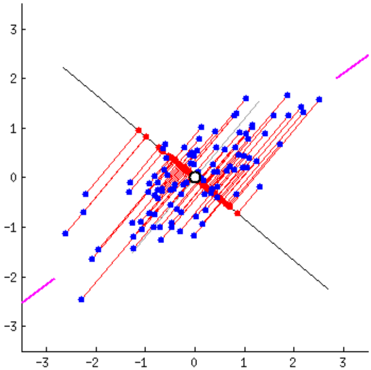

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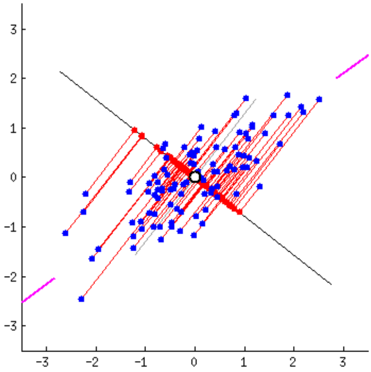

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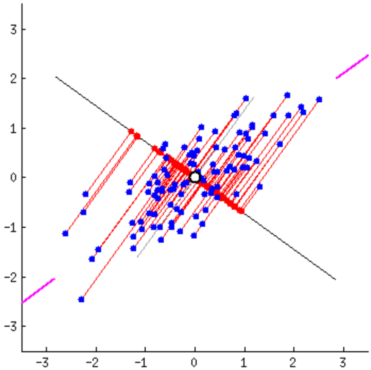

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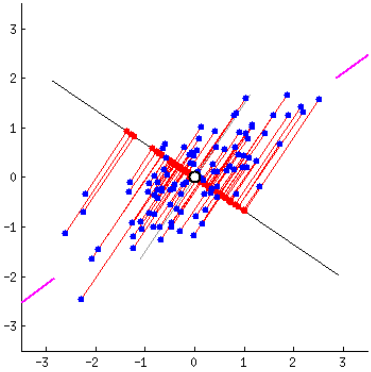

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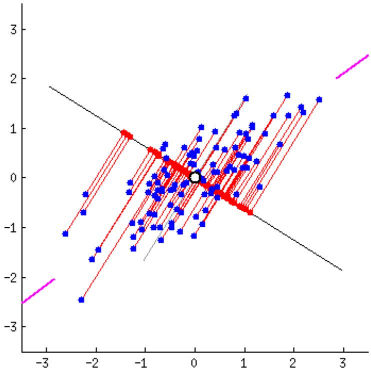

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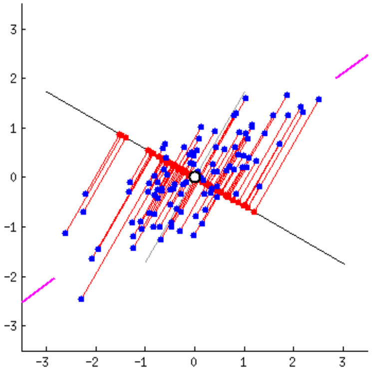

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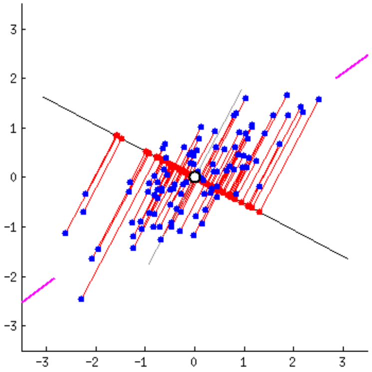

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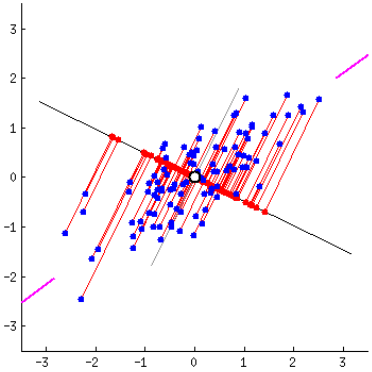

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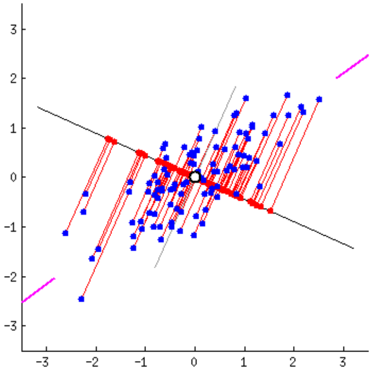

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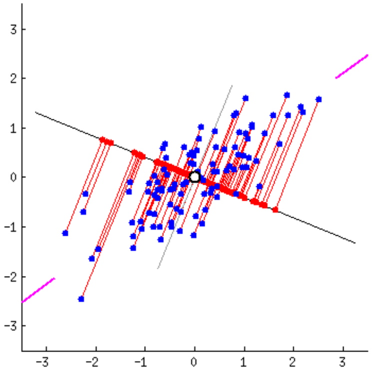

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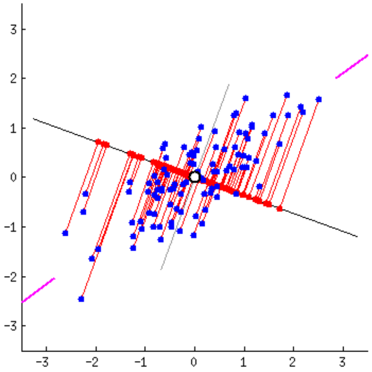

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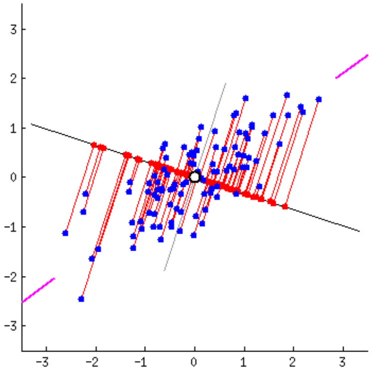

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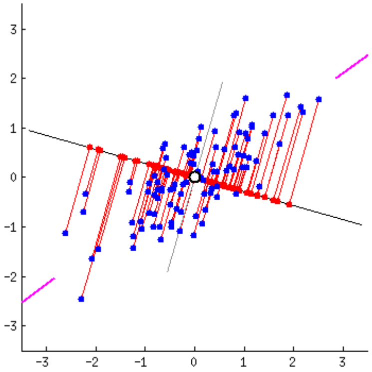

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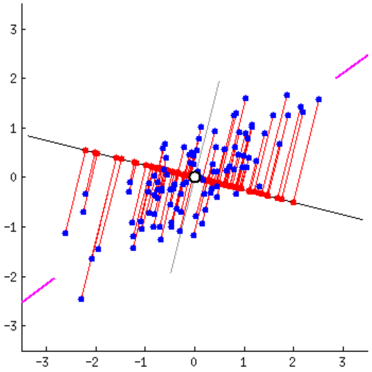

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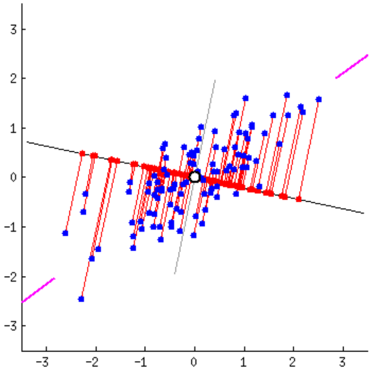

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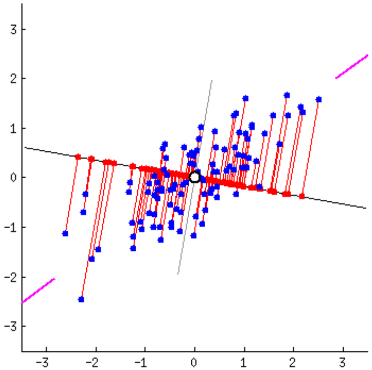

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/making-sense-of-principal-component-analysis-eigenvectors-eigenvalues

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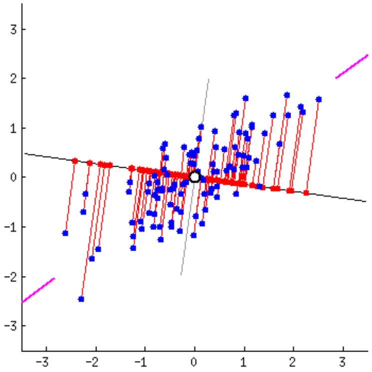

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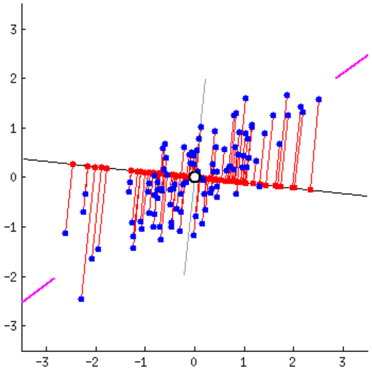

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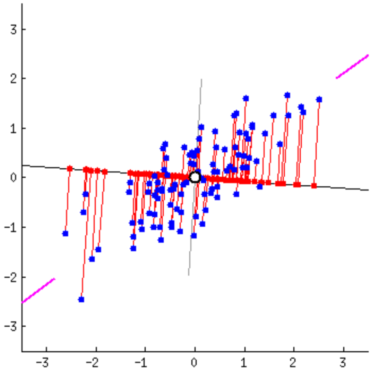

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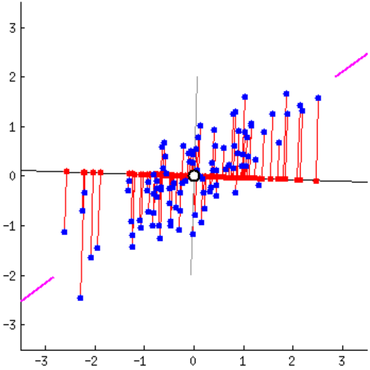

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PCA: Usages

- Dimensionality reduction/projection
  - Compression, clustering, classification...
- Orthogonalisation of the dimensions
  - Input for other algorithms
- Visualisation
- Data interpretation
  - Feature selection
- ...

PCA: How?

1. Eigenvalue decomposition of a data covariance matrix
2. Singular value decomposition of a data matrix

$$\mathbf{Av} = \lambda \mathbf{v}$$

Eigenvectors $\mathbf{v}$ and eigenvalues λ of a matrix $\mathbf{A}$

Principal Component Analysis, PCA

Eigenvectors and Eigenvalues

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

Eigenvectors $\mathbf{v}$ and eigenvalues λ of a matrix $\mathbf{A}$

For a given $\mathbf{A}$, how do we obtain them?

$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = 0, \quad$ where $\mathbf{I}$ is the identity matrix

Principal Component Analysis, PCA

Eigenvectors and Eigenvalues

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For a given $\mathbf{A}$, how do we obtain them?

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1. Eigenvalues $\mid \mathbf{A} - \lambda\mathbf{I} \mid = 0 \quad \Rightarrow \lambda_i$
2. Eigenvectors $(\mathbf{A} - \lambda_i\mathbf{I})\,\mathbf{v}_{\lambda_i} = 0 \quad \Rightarrow \mathbf{v}_{\lambda_i}$

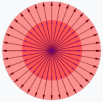
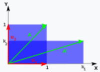
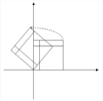
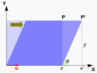
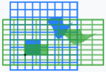
Principal Component Analysis, PCA

Eigenvectors and Eigenvalues

Matrix	$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$	$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$	$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ $c = \cos\theta$ $s = \sin\theta$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} c & s \\ s & c \end{bmatrix}$ $c = \cosh\varphi$ $s = \sinh\varphi$

Principal Component Analysis, PCA

Eigenvectors and Eigenvalues

	Scaling	Unequal scaling	Rotation	Horizontal shear	Hyperbolic rotation
Illustration					
Matrix	$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$	$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$	$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ $c = \cos\theta$ $s = \sin\theta$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} c & s \\ s & c \end{bmatrix}$ $c = \cosh\varphi$ $s = \sinh\varphi$
Characteristic polynomial	$(\lambda - k)^2$	$(\lambda - k_1)(\lambda - k_2)$	$\lambda^2 - 2c\lambda + 1$	$(\lambda - 1)^2$	$\lambda^2 - 2c\lambda + 1$
Eigenvalues, λ_i	$\lambda_1 = \lambda_2 = k$	$\lambda_1 = k_1$ $\lambda_2 = k_2$	$\lambda_1 = e^{\mathbf{i}\theta} = c + s\mathbf{i}$ $\lambda_2 = e^{-\mathbf{i}\theta} = c - s\mathbf{i}$	$\lambda_1 = \lambda_2 = 1$	$\lambda_1 = e^{\varphi}$ $\lambda_2 = e^{-\varphi}$,
Eigenvectors	All non-zero vectors	$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$u_1 = \begin{bmatrix} 1 \\ -\mathbf{i} \end{bmatrix}$ $u_2 = \begin{bmatrix} 1 \\ +\mathbf{i} \end{bmatrix}$	$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors

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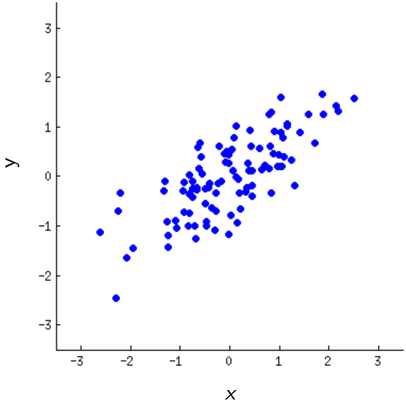
2. Singular value decomposition of a data matrix

Principal Component Analysis, PCA

Covariance Matrix



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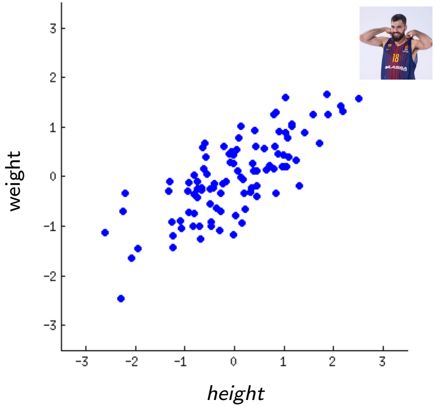

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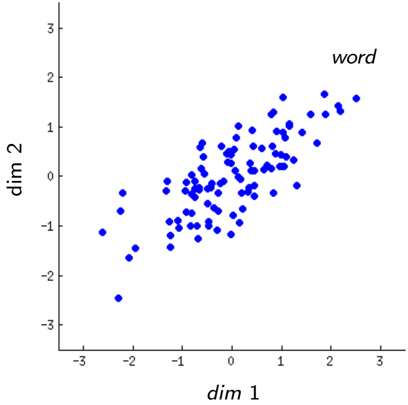

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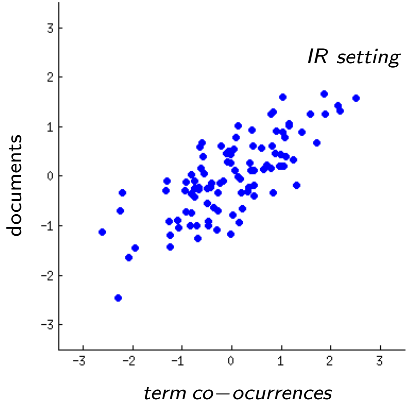

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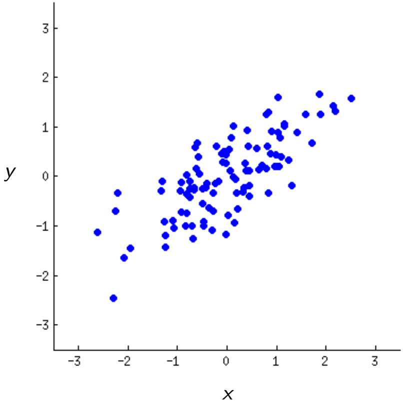
Principal Component Analysis, PCA

Covariance Matrix

$$\mu_x = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad \mu_y = \frac{1}{n}\sum_{i=1}^{n} y_i$$



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Principal Component Analysis, PCA

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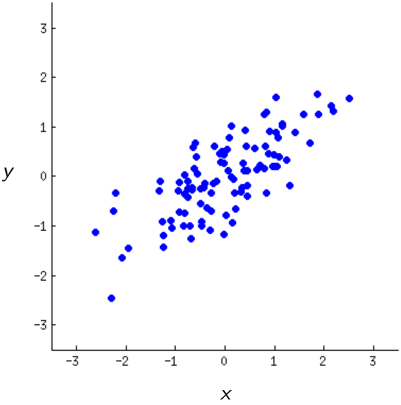
$$\mu_x = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad \mu_y = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$\sigma_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu_x)^2 = var(x)$$

$$\sigma_{xy}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu_x)(y_i - \mu_y)$$

$$= covar(x, y)$$



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Principal Component Analysis, PCA

Covariance Matrix, Σ

$$\text{In 2D,} \quad \Sigma = \begin{pmatrix} \sigma_{xx}^2 & \sigma_{yx}^2 \\ \sigma_{xy}^2 & \sigma_{yy}^2 \end{pmatrix} \qquad \text{with } \sigma_{xy}^2 = \sigma_{yx}^2,$$

Principal Component Analysis, PCA

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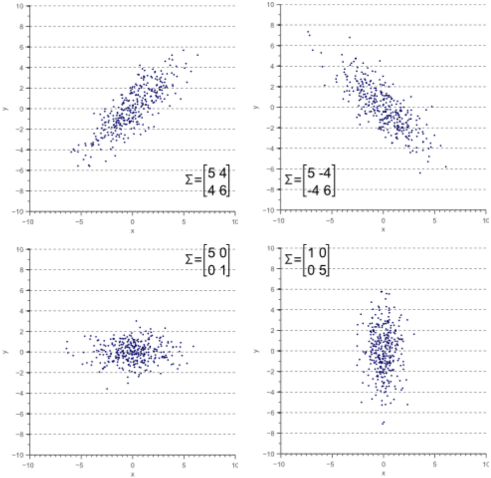
$$\text{In general,} \quad \Sigma = \begin{pmatrix} \sigma^2_{11} & \sigma^2_{12} & \sigma^2_{13} & \dots & \sigma^2_{1n} \\ \sigma^2_{21} & \sigma^2_{22} & \sigma^2_{23} & \dots & \sigma^2_{2n} \\ \sigma^2_{31} & \sigma^2_{32} & \sigma^2_{33} & \dots & \sigma^2_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \sigma^2_{n1} & \sigma^2_{n2} & \sigma^2_{n3} & \dots & \sigma^2_{nn} \end{pmatrix}$$

Principal Component Analysis, PCA

Covariance Matrix, Σ



http://www.visiondummy.com/2014/04/geometric-interpretation-covariance-matrix/

Principal Component Analysis, PCA

Principal Components and Maximum Variance

$$\sigma_x^2 = \frac{1}{n-1}\sum_{i=1}^{n} x_i^2 = var(x)$$

What do we want?

Principal Component Analysis, PCA

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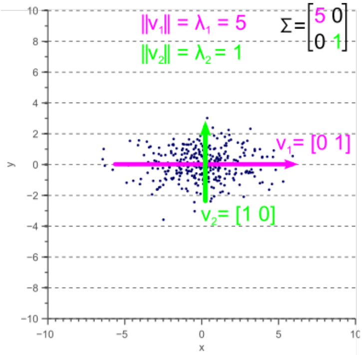
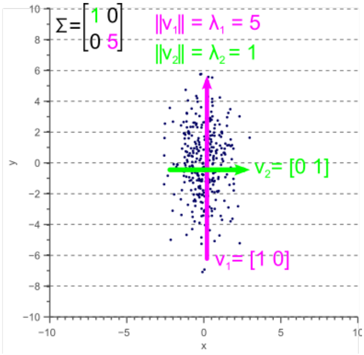
Some linear algebra and Lagrange multipliers later...

$$\boldsymbol{\Sigma}\mathbf{v} = \lambda\mathbf{v}$$





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Principal Component Analysis, PCA

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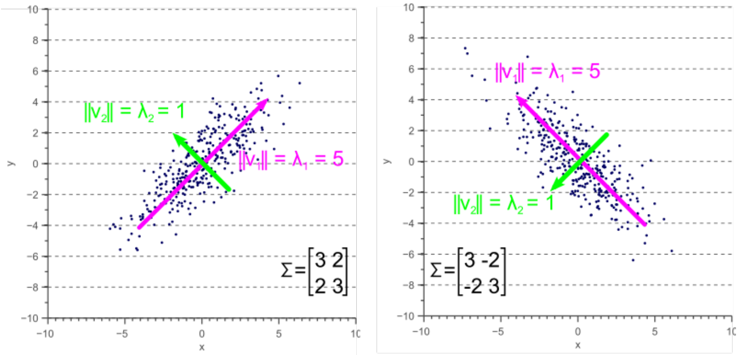

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Principal Component Analysis, PCA

PCA: How?

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Principal Component Analysis, PCA

Eigenvalue Decomposition Method, Wrap Up

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   - Calculate its eigenvalues
   - Rank the eigenvalues
   - Take the top-k to get the new k uncorrelated dimesions

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Is this *pca.fit_transform(x)*?

- Numerically unstable

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2. Singular value decomposition of a data matrix

Outline

- Linear algebra

Singular-Value Decomposition, SVD

Basics

- Linear algebra
- **Factorisation** of a matrix $\mathbf{A}$ as $\mathbf{A} = \mathbf{USV^T}$

Singular-Value Decomposition, SVD

Basics

- Linear algebra
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  - ✓ $\mathbf{U}$ is an $m \times m$ orthogonal matrix,

Singular-Value Decomposition, SVD

Basics

- Linear algebra
- **Factorisation** of a matrix $\mathbf{A}$ as $\mathbf{A} = \mathbf{USV^T}$
  - ✓ $\mathbf{U}$ is an $m \times m$ **orthogonal matrix**,
    - $\mathbf{U^T U} = \mathbf{UU^T} = \mathbf{I}$
    - or, equivalently, $\mathbf{U^T} = \mathbf{U}^{-1}$

Singular-Value Decomposition, SVD

Basics

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- **Factorisation** of a matrix $\mathbf{A}$ as $\mathbf{A} = \mathbf{USV^T}$
  - ✓ $\mathbf{U}$ is an $m \times m$ orthogonal matrix,
  - ✓ $\mathbf{S}$ is a diagonal $m \times n$ matrix with non-negative real numbers,

Singular-Value Decomposition, SVD

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  - ✓ $\mathbf{U}$ is an $m \times m$ orthogonal matrix,
  - ✓ $\mathbf{S}$ is a diagonal $m \times n$ matrix with non-negative real numbers,
  - ✓ $\mathbf{V^T}$ is the conjugate transpose of an $n \times n$ orthogonal matrix

Singular-Value Decomposition, SVD

Basics

$$
\begin{pmatrix} \\ m \times n \\ \\ \end{pmatrix} = \begin{pmatrix} \\ \quad m \times m \quad \\ \\ \end{pmatrix} \begin{pmatrix} \\ m \times n \\ \\ \end{pmatrix} \begin{pmatrix} n \times n \end{pmatrix}
$$

$$
\mathbf{A} \quad = \quad \mathbf{U} \quad \mathbf{S} \quad \mathbf{V}^{\mathsf{T}}
$$

Singular-Value Decomposition, SVD

SVD: Singular Values

$$\mathbf{S} = \begin{pmatrix} s_1 & & & \\ & \ddots & \mathbf{0} & \\ & \mathbf{0} & s_r & \\ & & & 0 \end{pmatrix};$$

s_1 ... s_r, singular values of $\mathbf{A}$ (in decreasing order)

r, rank of $\mathbf{A}$

Singular-Value Decomposition, SVD

SVD: Singular Values

$$\mathbf{S} = \begin{pmatrix} s_1 & & & \\ & \ddots & \text{\Large 0} & \\ \text{\Large 0} & & s_r & \\ & & & 0 \end{pmatrix} ; \qquad \mathbf{A_r} = \sum_{i=1}^{r} s_i \overrightarrow{u}_i \overrightarrow{v}_i^T$$

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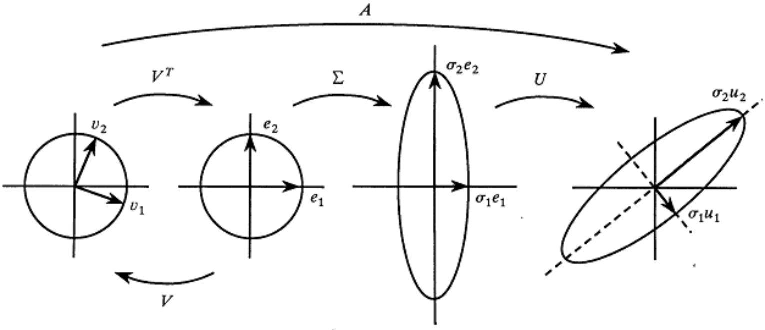
Singular-Value Decomposition, SVD

SVD: 2×2 Geometric Interpretation



$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^{\mathbf{T}}$: a linear transformation is a rotation or reflection, followed by a scaling, followed by another rotation or reflection

`https://blogs.sas.com`

Data $\mathbf{X}$ (is now our $\mathbf{A}$), Data Covariance $\mathbf{\Sigma} = \frac{1}{n-1}\mathbf{X}^{\mathsf{T}}\mathbf{X}$

$$\mathbf{X} = \mathbf{U}\mathbf{S}\mathbf{V}^{\mathsf{T}}$$

Singular-Value Decomposition, SVD

Relation with Eigenvalue Decomposition

Data $\mathbf{X}$ (is now our $\mathbf{A}$), Data Covariance $\mathbf{\Sigma} = \frac{1}{n-1}\mathbf{X}^\mathsf{T}\mathbf{X}$

$$\mathbf{X} = \mathbf{U}\mathbf{S}\mathbf{V}^\mathsf{T}$$

Let's SVD-decompose the data and estimate the covariance:

$$\mathbf{\Sigma} = \frac{1}{n-1}(\mathbf{U}\mathbf{S}\mathbf{V}^\mathsf{T})^\mathsf{T}(\mathbf{U}\mathbf{S}\mathbf{V}^\mathsf{T})$$

Singular-Value Decomposition, SVD

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Singular-Value Decomposition, SVD

Relation with Eigenvalue Decomposition

Data $\mathbf{X}$ (is now our $\mathbf{A}$), Data Covariance $\boldsymbol{\Sigma} = \frac{1}{n-1}\mathbf{X}^\mathsf{T}\mathbf{X}$

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Singular-Value Decomposition, SVD

Relation with Eigenvalue Decomposition

Data $\mathbf{X}$ (is now our $\mathbf{A}$), $\quad$ Data Covariance $\boldsymbol{\Sigma} = \frac{1}{n-1}\mathbf{X}^\mathsf{T}\mathbf{X}$

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On the other hand, $\boldsymbol{\Sigma}$ is symmetric, so can be diagonalised:

$$\boldsymbol{\Sigma} = \mathbf{V}\mathbf{L}\mathbf{V}^\mathsf{T}$$

Singular-Value Decomposition, SVD

Relation with Eigenvalue Decomposition

Let's SVD-decompose the data and estimate the covariance:

$$\mathbf{\Sigma} = \frac{1}{n-1}(\mathbf{USV^T})^\mathbf{T}(\mathbf{USV^T}) = \frac{1}{n-1}\mathbf{VSU^TUSV^T} = \mathbf{V}\frac{\mathbf{S}^2}{n-1}\mathbf{V^T}$$

On the other hand, $\mathbf{\Sigma}$ is symmetric, so can be diagonalised:

$$\mathbf{\Sigma} = \mathbf{VLV^T}$$

Singular-Value Decomposition, SVD

Relation with Eigenvalue Decomposition

Let's SVD-decompose the data and estimate the covariance:

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On the other hand, $\mathbf{\Sigma}$ is symmetric, so can be diagonalised:

$$\mathbf{\Sigma} = \mathbf{V}\mathbf{L}\mathbf{V}^\mathsf{T}$$

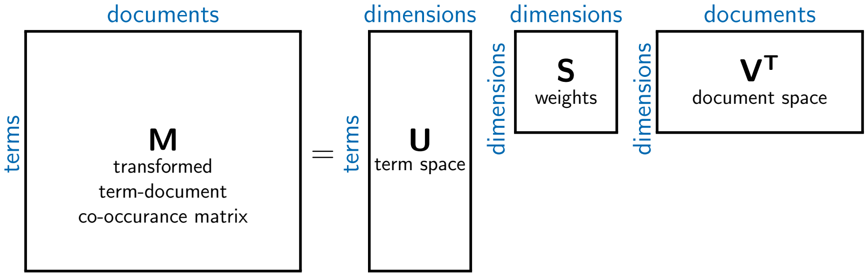
- $\mathbf{L} = \frac{\mathbf{S}^2}{n-1}$, eigenvalues $\propto$ singular values
- $\mathbf{XV} = \mathbf{US}$, principal components ($\mathbf{XV}$) $\propto$ left singular vectors

`https://nlp.stanford.edu/IR-book/pdf/18lsi.pdf`

Christopher D. Manning, Prabhakar Raghavan, and Hinrich Schütze. 2008. **Introduction to Information Retrieval**. Cambridge University Press, New York, NY, USA.

Comments & Final Remarks

SVD: Application, Latent Semantic Analysis (LSA/LSI)

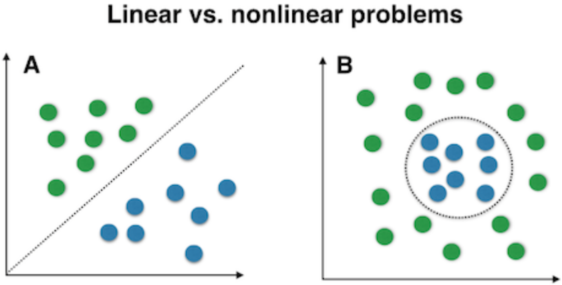


$$S_r \Longrightarrow \mathbf{M}_r$$

Comments & Final Remarks

Limitations of PCA

The useful projections **might not** be **linear**

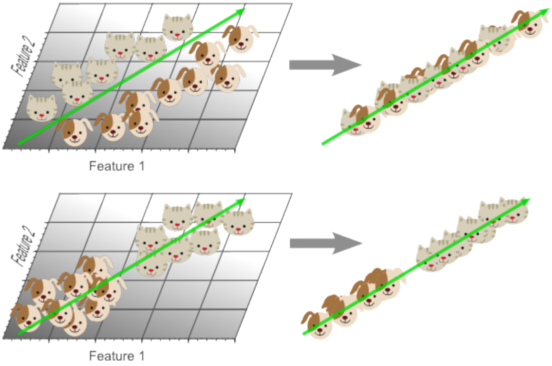


https://sebastianraschka.com/Articles/2014_kernel_pca.html

Comments & Final Remarks

Limitations of PCA

The most discriminative information **might not** be captured by the largest **variance**

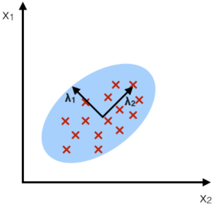


http://www.visiondummy.com/2014/05/feature-extraction-using-pca/

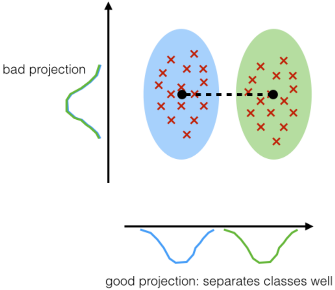
Comments & Final Remarks

General PCA vs. Linear Discriminant Analysis (LDA)

PCA:
component axes that maximize the variance



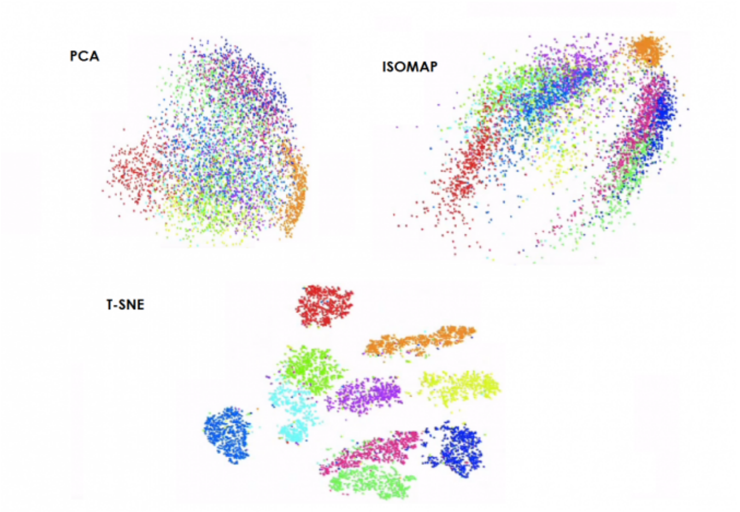
LDA:
maximizing the component axes for class-separation



`https://sebastianraschka.com/Articles/2014_python_lda.html`

Comments & Final Remarks

Visualisation: t-SNE



https://quantdare.com/tsne/

t-Distributed Stochastic Neighbor Embedding

- Completely different approach to PCA
- Non-linear technique

t-Distributed Stochastic Neighbor Embedding

- Completely different approach to PCA
- Non-linear technique

Description

- Probability of similarity of points in high-D space
- Probability of similarity of points in low-D space
- Minimise the difference between these conditional probabilities in high-D and low-D spaces

Comments & Final Remarks

Comments on Visualisation

Thanks!

Questions?

PCA, SVD and Other Related Acronyms

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Summer Semester Seminar

15th May 2019