Singular Perturbation Analysis for Parabolic Partial Differential Equations

Cristopher Arenas
cristopher.arenas@alumnos.usm.cl

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Abstract

Commonly, exists some phenomena that can be represented as mathematical models with differential equations. However, this representations are often a simplifications of the situation that pretend to model, because some factors are not taken into acount. The present work attemps to analize the efect of having small terms in the mathematical model of the parabolic partial differential equiations, where the assumption of discating them can cause huge differences with the real behavior of the context that is represented. For this, some perturbations in parabolic partial differential equations will be shown, and next some approximations will be used, considering the boundary function method.

1 Introduction

Exist some phenomena that can be represented as mathematical models using differential equations. In some cases simplifications in the moment of making the model can reduce the difficulty of the representation, but in change the model could not consider the effect of small factors that are occurring. For example, consider the following parabolic partial differential equation, valid for a domain $\Omega = \{(x,t) : 0 < x < 1, \quad 0 < t < T\}$, where T is some time value greater that 0:

$$\varepsilon \left(\frac{\partial u}{\partial t} - a\,\Delta\,u \right) = f(u,x,t,\varepsilon) \tag{1}$$

$$u(x,0,\varepsilon) = \phi(x) \tag{2}$$

$$\frac{\partial u}{\partial x}(0,t,\varepsilon) = 0 \tag{3}$$

$$\frac{\partial u}{\partial x}(1,t,\varepsilon) = 0 \tag{4}$$

As can be seen, the term ε, that assume a tiny value, is involving with the derivatives in the mathematical representation of the equations. The usual approach [1] consist in the approximation of the unknown function u into the series:

$$u(x,t,\varepsilon) = u_0(x,t) + \varepsilon\,u_1(x,t) + \varepsilon^2\,u_2(x,t) + \dots \tag{5}$$

where tipically only a few terms of the series are considered to the approximation. If x is a one-dimensional variable in space, there is necesary to realize a boundary analysis in order to determinate the effect of ε. On the other hand, if x is a m-dimensional variable in space, some others considerations appears. Specificaly if x is a 2-dimentional variable in space, the corners of the domain are affected for the approximations realized in the boundaries, so a new function, called corner boundary function must be obtained [2].

2 Objectives

- Describe the singular perturbation analysis for parabolic partial differential equations.
- Show aproximations of the perturbed heat equation with newmann conditions (1D).
- Show the singular perturbation analysis in the heat equation in thin bodies (2D).

3 Examples

1. Singularly perturbed parabolic equation

$$\varepsilon^2\left(\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}\right)=-(1+x^2)u+1 \tag{6}$$
$$u(x,0,\varepsilon)=0 \tag{7}$$
$$\frac{\partial u}{\partial x}(0,t,\varepsilon)=0 \tag{8}$$
$$\frac{\partial u}{\partial x}(1,t,\varepsilon)=0 \tag{9}$$

2. Heat conduction in thin bodies

$$\varepsilon^2\frac{\partial u}{\partial t}-\varepsilon^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=-\varepsilon^2\,u \tag{10}$$
$$u(x,0)=x \tag{11}$$
$$u(0,t)=0 \tag{12}$$
$$u(1,t)=1 \tag{13}$$
$$\frac{\partial u}{\partial y}(x,0)-\varepsilon^2\,u(x,0)=0 \tag{14}$$
$$\frac{\partial u}{\partial y}(x,1)+\varepsilon^2\,u(x,1)=0 \tag{15}$$

References

[1] Logan J David. *Applied Mathematics.* Wiley, 2006.

[2] Adelaida B Vasil'eva, Valentin F Butuzov, and Leonid V Kalachev. *The boundary function method for singular perturbation problems.* SIAM, 1995.