CSC236 Tutorial #3

Sample Solutions

1. Let $\Sigma = \{a, b\}$ be a set of characters. Let A be a set of strings of characters in Σ. Assume A is defined as follows:

   - $a \in A$;
   - if $s \in A$, then $s \cdot a \in A$ and $s \cdot b \in A$, where $\cdot$ denotes string concatenation;
   - nothing else belongs to A.

   Use structural induction to prove that every string in A begins with an a.

   Solution: $P(s)$: the string s begins with an a.

   The goal is to prove for all $s \in A$, $P(s)$.

   Base Case: Let $s = a$.
   Then s begins with an a, and so $P(s)$.

   Induction Step: Let $s \in A$. By definition, $s \cdot a \in A$ and $s \cdot b \in A$.

   Suppose $P(s)$, i.e., s begins with an a. **[IH]**

   WTP: $P(s \cdot a)$ and $P(s \cdot b)$.

   By IH, s begins with an a. Therefore, both $s \cdot a$ and $s \cdot b$ begin with an a. So, $P(s \cdot a)$ and $P(s \cdot b)$.

2. Let $S \subset \mathbb{Z}^2$ be a set defined as follows:

   - $(0, 0) \in S$;
   - if $(x, y) \in S$, then $(x, y + 1) \in S$, $(x + 1, y + 1) \in S$, and $(x + 2, y + 1) \in S$;
   - nothing else belongs to S.

   (a) List seven elements in S.

   (b) Use structural induction to prove that for all $(x, y) \in S, x \leq 2y$.

   Solution:

   (a) $(0, 0), (0, 1), (1, 1), (2, 1), (0, 2), (1, 2), (2, 2)$.

   (b) $P((x, y)) : x \leq 2y$.
   The goal is to prove for all $(x, y) \in S$, $P((x, y))$.
   Base Case: Let $(x, y) = (0, 0)$.
   Then $P((x, y))$ since 0 is equal to 2×0.
   Induction Step: Let $(x, y) \in S$. By definition, S includes $(x, y+1)$, $(x+1, y+1)$, and $(x+2, y+1)$.
   Suppose $P((x, y))$, i.e., $x \leq 2y$. **[IH]**

WTP:
(A) $P((x, y+1))$, i.e., $x \leq 2(y+1)$;
(B) $P((x+1, y+1))$, i.e., $x+1 \leq 2(y+1)$;
(C) $P((x+2, y+1))$ i.e., $x+2 \leq 2(y+1)$;

Case (A):
By IH, $x \leq 2y$.
Then $x \leq 2y+2$.
Then $x \leq 2(y+1)$.

Case (B):
By IH, $x \leq 2y$.
Then $x+1 \leq 2y+2$.
Then $x+1 \leq 2(y+1)$.

Case (C):
By IH, $x \leq 2y$.
Then $x+2 \leq 2y+2$.
Then $x+2 \leq 2(y+1)$.

3. Let E be a set of expressions defined as follows:

- $\texttt{two} \in E$;
- if e_1 and e_2 are in E, then so are $(e_1 \texttt{ plus } e_2)$ and $(e_1 \texttt{ times } e_2)$;
- nothing else belongs to E.

(a) Show six expressions that are in E.

(b) For $e \in E$, let $L(e)$ be the total number of `two`'s, `plus`'s, and `times`'s in e, and $T(e)$ be the number of `two`'s in e. For example

$$L((\texttt{two plus two})) = 3, \qquad T((\texttt{two plus two})) = 2,$$
$$L((\texttt{two plus (two plus two)})) = 5, \qquad T((\texttt{two plus (two plus two)})) = 3.$$

Prove by Structural Induction that for all $e \in E$, $T(e) = \frac{L(e)+1}{2}$.

(c) For $e \in E$, let $V(e)$ be the value of expression e, i.e.,

- $V(\texttt{two}) = 2$;
- if $e_1, e_2 \in E$, then
  $V((e_1 \texttt{ plus } e_2)) = V(e_1) + V(e_2)$ and
  $V((e_1 \texttt{ times } e_2)) = V(e_1) \times V(e_2)$.

Prove by Structural Induction that for all $e \in E$, $V(e) \geq 2T(e)$.

Solution:

(a) `two`, `(two plus two)`, `(two times two)`, `(two plus (two plus two))`,
`(two plus (two times two))`, `(two times (two times two))`.

(b) $P(e) : T(e) = \frac{L(e)+1}{2}$.
The goal is to prove for all $e \in S$, $P(e)$.

Base Case: Let $e = \texttt{two}$.
Then $L(e) = 1$ and $T(e) = 1$.
Then $\frac{L(e)+1}{2} = \frac{1+1}{2} = 1 = T(e)$.
So $P(\texttt{two})$ holds.

Induction Step: Let $e_1, e_2 \in E$. By definition, E includes $(e_1 \texttt{ plus } e_2)$ and $(e_1 \texttt{ times } e_2)$.
Suppose $P(e_1)$ and $P(e_2)$, i.e., $T(e_1) = \frac{L(e_1)+1}{2}$ and $T(e_2) = \frac{L(e_2)+1}{2}$. **[IH]**
WTP:
(A) $P((e_1 \texttt{ plus } e_2))$, i.e., $T((e_1 \texttt{ plus } e_2)) = \frac{L((e_1 \texttt{ plus } e_2))+1}{2}$;
(B) $P((e_1 \texttt{ times } e_2))$, i.e.,$T((e_1 \texttt{ times } e_2)) = \frac{L((e_1 \texttt{ times } e_2))+1}{2}$;

Case (A): Note that $L((e_1 \texttt{ plus } e_2)) = L(e_1) + L(e_2) + 1$.

$$\begin{aligned}
T((e_1 \texttt{ plus } e_2)) &= T(e_1) + T(e_2) \\
&= \frac{L(e_1)+1}{2} + \frac{L(e_2)+1}{2} && \text{\# By IH} \\
&= \frac{L(e_1) + L(e_2) + 1 + 1}{2} \\
&= \frac{L((e_1 \texttt{ plus } e_2)) + 1}{2} && \text{\# Sine } L((e_1 \texttt{ plus } e_2)) = L(e_1) + L(e_2) + 1.
\end{aligned}$$

Case (B): Note that $L((e_1 \texttt{ times } e_2)) = L(e_1) + L(e_2) + 1$.

$$\begin{aligned}
T((e_1 \texttt{ times } e_2)) &= T(e_1) + T(e_2) \\
&= \frac{L(e_1)+1}{2} + \frac{L(e_2)+1}{2} && \text{\# By IH} \\
&= \frac{L(e_1)+L(e_2)+1+1}{2} \\
&= \frac{L((e_1 \texttt{ times } e_2))+1}{2} && \text{\# Since } L((e_1 \texttt{ times } e_2)) = L(e_1)+L(e_2)+1.
\end{aligned}$$

(c) $P(e) : V(e) \geq 2 \times T(e)$.
The goal is to prove for all $e \in S$, $P(e)$.

Base Case: Let $e = \texttt{two}$.
Then $V(e) = 2$ and $T(e) = 1$.
Then $V(e) \geq 2 \times T(e)$, and so $P(\texttt{two})$.

Induction Step: Let $e_1, e_2 \in E$. By definition, E includes $(e_1 \texttt{ plus } e_2)$ and $(e_1 \texttt{ times } e_2)$.
Suppose $P(e_1)$ and $P(e_2)$, i.e., $V(e_1) \geq 2T(e_1)$ and $V(e_2) \geq 2T(e_2)$. **[IH]**
WTP:
(A) $P((e_1 \texttt{ plus } e_2))$, i.e., $V((e_1 \texttt{ plus } e_2)) \geq 2T((e_1 \texttt{ plus } e_2))$;
(B) $P((e_1 \texttt{ times } e_2))$, i.e., $V((e_1 \texttt{ times } e_2)) \geq 2T((e_1 \texttt{ times } e_2))$;

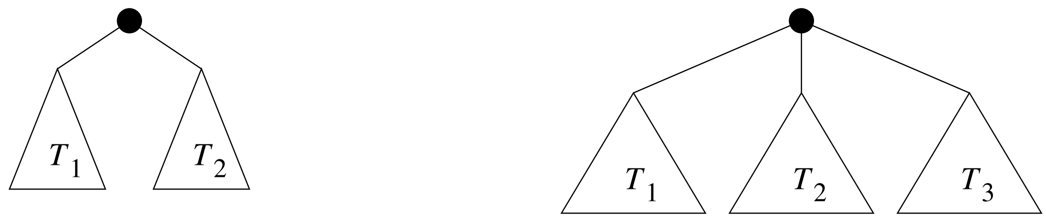
Case (A): Note that $T((e_1 \texttt{ plus } e_2)) = T(e_1) + T(e_2)$.
By definition, $V((e_1 \texttt{ plus } e_2)) = V(e_1) + V(e_2)$.
Then $V((e_1 \texttt{ plus } e_2)) \geq 2T(e_1) + 2T(e_2)$, since by IH, $V(e_1) \geq 2T(e_1)$ and $V(e_2) \geq 2T(e_2)$.
Then $V((e_1 \texttt{ plus } e_2)) \geq 2(T(e_1) + T(e_2))$.
Then $V((e_1 \texttt{ plus } e_2)) \geq 2T((e_1 \texttt{ plus } e_2))$, since $T((e_1 \texttt{ plus } e_2)) = T(e_1) + T(e_2)$.

Case (B): Note that $T((e_1 \texttt{ times } e_2)) = T(e_1) + T(e_2)$.
By definition, $V((e_1 \texttt{ times } e_2)) = V(e_1) \times V(e_2)$.
Then $V((e_1 \texttt{ times } e_2)) \geq 2T(e_1) \times 2T(e_2)$, since by IH, $V(e_1) \geq 2T(e_1)$ and $V(e_2) \geq 2T(e_2)$.
Then $V((e_1 \texttt{ times } e_2)) \geq 4(T(e_1) \times T(e_2)) \geq 2(T(e_1) + T(e_2))$, since $T(e_1), T(e_2) \geq 1$.[1]
Then $V((e_1 \texttt{ times } e_2)) \geq 2T((e_1 \texttt{ times } e_2))$, since $T((e_1 \texttt{ times } e_2)) = T(e_1) + T(e_2)$.

[1] On an assignment we would expect you to prove that $a, b \geq 1$ implies $4ab \geq 2(a+b)$.

4. Consider the recursive definition of *complete binary trees*:

   - a single node is a complete binary tree;
   - if T_1 and T_2 are complete binary trees *of the same height*, then the tree constructed by placing T_1 and T_2 under a new root node (as illustrated below on the left) is also a complete binary tree;
   - nothing else is a complete binary tree.



complete binary tree   complete ternary tree

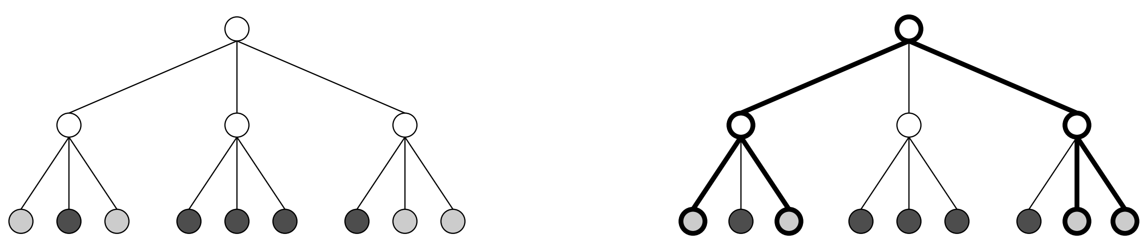
   We define *complete* **ternary** *trees* as:

   - a single node is a complete ternary tree;
   - if t_1, t_2, and t_3 are complete ternary trees *of the same height*, then the tree constructed by placing t_1, t_2, and t_3 under a new root node (as illustrated above on the right) is also a complete ternary tree;
   - nothing else is a complete ternary tree.

   A *coloured* complete ternary tree is a complete ternary tree in which every leaf node has been assigned one of two colours (red or blue). Only the leaves are coloured — internal nodes are colourless.

   A *subtree* of a tree t is a subset of t's leaves along with all of the leaves' ancestors (and the edges connecting them). Every complete ternary tree contains many complete binary subtrees — simply pick two children to keep and one to remove for every internal node.

   A *monochromatic* complete binary subtree of a coloured complete ternary tree is a complete binary subtree all of whose leaves are the same colour. For example, the coloured complete ternary tree on the left below contains a monochromatic complete binary subtree, indicated on the right by thick edges and node outlines.



a coloured complete ternary tree   a monochromatic complete binary subtree

   Use structural induction to prove that every coloured complete ternary tree contains some monochromatic complete binary subtree.

Hint: It suffices that you show there is always one such binary tree. Note that you cannot choose how the ternary tree is coloured. Rather, your proof must establish that the above statement is valid no matter how colours are assigned to the leaves of the ternary tree.

Solution: Let T denote the set of all colored complete ternary trees.
$P(t)$: arbitrarily coloured tree t has a complete monochromatic binary subtree.

The goal is to prove for all $t \in T$, $P(t)$. Proof by structural induction on the set of complete ternary trees.

Base Case: Let t be a single node.
A single node is a complete binary tree, and has a single colour assigned to it. Therefore, t is a complete monochromatic binary tree. Moreover, it is a subtree of itself.
Therefore, t contains a monochromatic complete binary subtree, and so $P(t)$.

Induction Step: Let $t_1, t_2, t_3 \in T$, and have the same height. By definition, T includes a complete ternary tree t consisting of a root with t_1, t_2 and t_3 as subtrees.
Suppose $P(t_1), P(t_2)$, and $P(t_3)$, i.e., each of t_1, t_2 and t_3 has a complete monochromatic binary subtree. **[IH]**
WTP: $P(t)$, i.e., t has a complete monochromatic binary subtree.

Since t_1, t_2 and t_3 have the same height, they contain complete binary subtrees of the same height. By IH, each of t_1, t_2 and t_3 has a complete monochromatic binary subtree. That is, each of these trees has a complete binary subtree such that all leaves of each of the subtrees are coloured either red or blue. Since there are 2 colours and 3 trees, by the pigeonhole principle, at least two of these complete monochromatic subtrees have to be of the same colour. Without loss of generality, let's assume that both t_1 and t_2 have complete binary subtrees (of the same height) that are red. In this case, t has a complete binary subtree that is red. Namely, the one obtained by placing t_1 and t_2 under the root of t. Since the root of t is not contained in t_1 or t_2, the result is a complete binary tree, by the recursive definition of a complete binary tree. By construction, it is a subtree of t and monochromatically red. Thus, $P(t)$.