# CSC236 Tutorial #1

1. Aaron and Bianca play the following game: they place on a table two piles containing an equal number of matches. They take turns removing some (non-zero) number of matches from one of the piles; the player whose turn it is can choose which pile. The player removing the last match from the table wins. Use induction to show that if Aaron goes first, Bianca has a winning strategy. That is, she can win no matter how Aaron plays.

   **Solution:** As the first step, we define the predicate which we want to prove:
   $P(n)$ : starting from two piles of $n$ matches each, and given the rules of the game, if Aaron moves first, Bianca can win.

   The goal is to prove for all $n \in \mathbb{N}, n \geq 1$, $P(n)$.

   **Base Case:** Let $k = 1$.
   Then $P(k)$ is true since the only move possible for Aaron is to take the only match from one of the two piles. Bianca removes the 1 match in the other pile and wins.

   **Induction Step:** Let $k \in \mathbb{N}$. Suppose for all $j \in \mathbb{N}$, $1 \leq j < k$, $P(j)$ is true. **[IH]**
   **WTP:** $P(k)$ holds.

   Starting from two piles of $k$ matches, Aaron will have to choose a pile and remove $i$ matches from it where $1 \leq i \leq k$. There are two possibilities:

   *Case 1:* $i = k$. After Aaron's move, there is only one pile left with $k$ matches. Bianca can remove all those $n$ matches on her turn and win.

   *Case 2:* $1 \leq i < k$. After Aaron's move, Bianca will remove exactly $i$ matches from the other pile. This leaves two piles of $k - i$ matches, and it is again Aarons turn to play. We can view this as the beginning of a fresh game with $k - i$ matches in each pile. Since $1 \leq k - i < k$, by IH, Bianca can win the game.
   In conclusion, by the Principle of Simple Induction, $\forall n \in \mathbb{N}, P(n)$

2. Use induction to prove that for all natural numbers $n \geq 3$, $(1 + \frac{1}{n})^n \leq n$.

   **Solution:** Let $P(n)$ denote the assertion that the inequality $(1 + \frac{1}{n})^n \leq n$ holds.

   **Base Case:** Let $k = 3$. (Note that since we're asked to prove the statement for $n \geq 3$, the base case is to prove $P(3)$.)

   Then we have
   $$(1 + \frac{1}{3})^3 = 2.\overline{370} < 3.$$

   Therefore, $P(3)$ is true.

**Induction Step:** Let $k \in \mathbb{N}$, and $k \geq 3$. Suppose $P(k)$ is true, i.e., $(1+\frac{1}{k})^k \leq k$. **[IH]**
**WTP:** $P(k+1)$ holds, i.e., $(1+\frac{1}{k+1})^{k+1} \leq k+1$.

$(1+\frac{1}{k})^k \leq k$ # By IH

Then $(1+\frac{1}{k})^k \times (1+\frac{1}{k}) \leq k(1+\frac{1}{k})$ # Multiply both sides by the positive value $(1+\frac{1}{k})$

Then $(1+\frac{1}{k})^{k+1} \leq k+1$ # Algebra

On the other hand, $(1+\frac{1}{k+1})^{k+1} \leq (1+\frac{1}{k})^{k+1}$ # Since $\frac{1}{k+1} \leq \frac{1}{k}$

Therefore, $(1+\frac{1}{k+1})^{k+1} \leq k+1$.

In conclusion, by the Principle of Simple Induction, $\forall n \in \mathbb{N}, P(n)$

3. (Exercise 7 in Chapter 1 of the course notes) Use induction to prove that, for any integers $m \geq 2$ and $n \geq 1$,

$$\sum_{t=0}^{n} m^t = \frac{m^{n+1}-1}{m-1}.$$

**Solution:** Let $m \in \mathbb{N}$, and $m \geq 2$. Let $P(n)$ denote the assertion that the equality $\sum_{t=0}^{n} m^t = \frac{m^{n+1}-1}{m-1}$ holds.

**Base Case:** Since we're asked to prove the statement for $n \geq 1$, we can ignore the case where $n = 0$ (although $P(0)$ is also true). So, the base case is to prove $P(1)$ holds.

Let $k = 1$.

$$\begin{aligned}
\sum_{t=0}^{k} m^t &= \sum_{t=0}^{1} m^t \\
&= m^0 + m^1 \\
&= m + 1 \\
&= \frac{(m+1)(m-1)}{m-1} \\
&= \frac{m^2-1}{m-1} \\
&= \frac{m^{k+1}-1}{m-1}
\end{aligned}$$

Therefore, $P(1)$ is true.

**Induction Step:** Let $k \in \mathbb{N}$, and $k \geq 1$. Suppose $P(k)$ is true, i.e., $\sum_{t=0}^{k} m^t = \frac{m^{k+1}-1}{m-1}$. **[IH]**
**WTP:** $P(k+1)$ holds, i.e., $\sum_{t=0}^{k+1} m^t = \frac{m^{k+2}-1}{m-1}$.

$$
\begin{aligned}
\sum_{t=0}^{k+1} m^t &= \sum_{t=0}^{k} m^t + m^{k+1} && \text{\# By definition of summation} \\
&= \frac{m^{k+1}-1}{m-1} + m^{k+1} && \text{\# By IH} \\
&= \frac{m^{k+1}-1+m^{k+1}(m-1)}{m-1} \\
&= \frac{m^{k+1}(1+m-1)-1}{m-1} && \text{\# Factoring out } m^{k+1} \\
&= \frac{m^{k+2}-1}{m-1}
\end{aligned}
$$

Therefore, $P(k+1)$ is true.
In conclusion, by the Principle of Simple Induction, $\forall n \in \mathbb{N}, P(n)$