CSC236 Jan 11, 2018

Ex. Prove that $\forall n \in \mathbb{N}$ $n>4$, $2^n > n^2$.

① Let $P(n): 2^n > n^2$. ② Base case $n=5$: $2^5 = 32 > 5^2 = 25 \therefore P(5)$ ✓

Note: Base case here is not 0 because $\exists$ 1-to-1 mapping between the infinite countable sets $\mathbb{N}$ and $\mathbb{N} - \{0,1,2,3,4\}$

③ Assume $P(k)$ holds for arbitrary $k \in \mathbb{N}$, $k>4$, $\therefore 2^k > k^2$

We have $2^k > k^2 \therefore 2 \cdot 2^k > 2k^2 \therefore 2^{k+1} > 2k^2$

Must show $2k^2 \geq (k+1)^2 = k^2 + 2k + 1$, $\therefore$ must show $k^2 > 2k+1$, $k>4$

$[k^2 \geq 2k+1 \rightarrow k^2 - 2k - 1 \geq 0 \rightarrow k^2 - 2k + 1 \geq 2 \rightarrow (k-1)^2 \geq 2]$ TRUE

$\therefore 2^{k+1} > 2k^2 \therefore 2^{k+1} > k^2 + k^2 > k^2 + 2k + 1 = (k+1)^2 \therefore 2^{k+1} > (k+1)^2$

So we have shown that $P(k) \rightarrow P(k+1)$

④ We have $P(5) \wedge [\forall k \in \mathbb{N}, k>4, P(k) \rightarrow P(k+1)] \therefore$ by PSI we can conclude that $\forall n \in \mathbb{N}$, $n>4$, $P(n): 2^n > n^2$.

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Ex: Suppose $h_0, h_1, h_2 \ldots$ is a sequence defined as follows:

$h_0 = 1$, $h_1 = 2$, $h_2 = 3$, $h_k = h_{k-1} + h_{n-2} + h_{k-3}$ for all $k \in \mathbb{Z}$, $k \geq 3$

Prove that $h_n \leq 3^n$ for all integers $n \geq 0$

① Let $P(n): h_n \leq 3^n$.

② Base $n=0$: $h_n = 1 \leq 3^0 = 1 \therefore P(0)$ true.

Base $n=1$: $h_n = 2 \leq 3^1 = 3 \therefore P(2)$ true

Base $n=2$: $h_n = 3 \leq 3^2 = 9 \therefore P(2)$ true.

③ Let k be arbitrary natural number. Assume $\forall j \in \mathbb{N}$, $0 \leq j \leq k$, $P(j)$.

From definition: $h_{k+1} = h_k + h_{k-1} + h_{k-2} \leq 3^k + 3^{k-1} + 3^{k-2}$

$3^k + 3^{k-1} + 3^{k-2} \leq 3^k + 3^k + 3^k = 3(3^k) = 3^{k+1}$

$\therefore$ we have $h_{k+1} \leq 3^{k+1}$ $\therefore \forall j \in \mathbb{N}$ $0 \leq j \leq k$ $P(j) \rightarrow P(k+1)$

④ We have $P(0) \wedge P(1) \wedge P(2) \wedge$ [arbitrary $k \in \mathbb{N} \geq 3$, $\forall j \in \mathbb{N}$ $0 \leq j \leq k$ $P(k) \rightarrow P(k+1)$]

so by principle of complete induction can conclude $\forall n \in \mathbb{N}$ $P(n)$.

Slide:

9 • Use induction to prove that the number of nodes in a full binary tree is odd.

<u>Recall</u>: full binary tree has 2 children for every parent.

Let n = height of binary tree, such than $n=1$ is root.

① Let $P(n)$: fbt of height n has odd number of nodes, $n>0$

② Base case $n=1$: tree has one node; $1=2(0)+1 \therefore P(1)$ true!

③ Assume $P(k)$ true for some arbitrary $k \in \mathbb{N}$, $k>0$. So a tree of height k has z nodes st. $z=2f+1$ for some $f \in \mathbb{Z}$. (definition of odd numbers)

An fbt of height k has 2^{k-1} nodes on the lowest level, and by definition every parent in fbt must have 2 children. $\therefore$ adding another level adds $2(2^{k-1}) = 2^k$ children.

So fbt of height $k+1$ has $2f+1+2^k = 2(f+2^{k-1})+1 = 2s+1$ nodes where $s=f+2^{k-1}$, so fbt of height $k+1$ has odd # nodes!

So we have shown that $P(k) \longrightarrow P(k+1)$.

④ $P(1) \wedge P(k) \rightarrow P(k+1) \therefore$ via P.I. we know $\forall n \in \mathbb{N}$ $n>0$, $P(n)$.