Jan 18, 2016

- ~~Note~~: missed first part of lecture due to forgotten bug in last lecture hall. Class started at 6:34

Week 2:

- Well-ordering principle definition

- Structure of well-ordering proofs.

(16) EX: Use the principle of well-ordering to prove that $\forall n \in \mathbb{N}, \sum_{i=0}^{n} i = n(n+1)/2$.

(i) $P(n): \sum_{i=0}^{n} i = n(n+1)/2$.

(ii) Assume that $P(n)$ does not hold $\forall n \in \mathbb{N}$. $\therefore \exists m \in \mathbb{N}$ st. $P(m) = \text{false}$.

(iii) Define $S: \{k \in \mathbb{N}, P(k) \text{ is false}\}$

(iv) By definition $S \subset \mathbb{N}$, and $m \in S$ and $\therefore S \neq \emptyset$. (S is non-empty)

(v) By principle of well-ordering, S contains a smallest element. Let a be the smallest element in S.

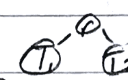
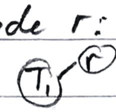
(vi) $P(0): \sum_{i=0}^{0} i = \frac{0(0+1)}{2} : 0 = 0 \therefore P(0)$ is true $\therefore 0 \notin S, m \neq 0, a > 0$.
Since a is smallest element in S, $a-1 \notin S$. $(a > 0) \rightarrow (a-1) \geq 0$.
Since $(a-1) \notin S$, $P(a-1)$ holds $\therefore \sum_{i=0}^{a-1} = (a-1)(a)/2 \therefore \sum_{i=0}^{a} = a(a+1)/2$.
So $P(a-1) \rightarrow P(a)$, $\therefore$ Contradiction! $\therefore$ our assumption that $\exists$ non-empty set S has smallest element a is false.

(vii) Thus we can conclude $\forall n \in \mathbb{N}$, $P(n)$ holds.

Week 3:

(3) - Defining recursive sets:
1) Indicate smallest, simplest objects
2) Indicate how larger, more complex objects built from smaller ones.
3) Close definition

(5) - Example: $0 \in \mathbb{N}$, $k \in \mathbb{N} \rightarrow k+1 \in \mathbb{N}$, nothing else in $\mathbb{N}$

(4) - Example: non empty binary trees
1) single node
2) given 2 bt's T_1, T_2, node r:  is a bt.
2i) given bt T_1, node r:  is a bt
3) nothing else is a non-empty BT

- Structural Induction:
  1) P holds for simplest elements
  2) assume P holds for smaller elements, show P holds for elements constructed out of those smaller elements.

(8) Ex: Prove that every non-empty binary tree has one more node than edge.

(i) Base case: T = bin tree w/ single node; nodes $= 1 =$ edges $+ 1$ $\therefore$ true

Missing explicit definitions of $P(T)$ and t_x.

(ii) Let T_1, T_2 be non-empty bin trees. Assume property holds for T_1, T_2.
Bin tree constructed following definition $(T_1)\text{-}(r)\text{-}(T_2) = T_3$
t_3 nodes $= t_1 + t_2 + 1$; t_3 edges $= t_1 - 1 + (t_2 - 1) + 2 = t_3$ nodes $- 1$
Property holds for T_3 constructed from T_1, T_2, r.
Bin tree constructed following definition $(T_1)\text{-}(r) = T_3$:
t_3 nodes $= t_1 + 1$; t_3 edges $= t_1 - 1 + 1 = t_1 = t_3$ nodes $- 1$
Property holds for T_3 constructed from T_1, r.

(iii) Conclude that property holds for all non-empty bin trees.

(10) - Consider the recursively defined set $S \subseteq \mathbb{N}^2$:
(a) $(0,0) \in S$; (b) if $(a,b) \in S$, so are $(a+1, b+1)$ and $(a+3, b)$
Show that $[\forall (x,y) \in S,\ x \geq y$ and $x - y = 3f$ for some $f \in \mathbb{Z}]$: $P(x,y)$
(i) Base: $(0,0) = (x,y)$; $0 \geq 0$, $0 - 0 = 3 \cdot 0$ $\therefore P(0,0)$
(ii) Let (x,y) be an element in S for which $P(x,y)$ is true.
Let $(t,v) = (x+1, y+1)$; $x \geq y \rightarrow x+1 \geq y+1$; $t - v = x+1-y-1 = x-y = 3f$
So we can conclude that $P(t,v)$ is true.
Let $(q,r) = (x+3, y)$; $x+3 > x \geq y$; $x+3-y = x-y+3 = 3f+3 = 3(f+1)$
So we can conclude that $P(q,r)$ is true.

(11) - Will not rewrite problem for want of space on the page.
(i) Base let $x \in E$. $vr(x) = 1 = 0 + 1 = op(x) + 1$ $\therefore P(x)$ true.
(ii) For $e_1, e_2 \in E$, assume $P(e_1)$, $P(e_2)$. Let $e_3 = (e_1 \circledcirc e_2)$.
$vr(e_3) = vr(e_2) + vr(e_1)$ $op(e_3) = op(e_1) + op(e_2) + 1$
$vr(e_3) = op(e_2) + 1 + op(e_1) + 1 = op(e_1) + op(e_2) + 2 = op(e_3) + 1$ $\therefore P(e_3)$ true.
(iii) We can conclude $P(x)$ $\forall x \in E$.