CSC236 - Week 2

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Ex. Prove that $\forall n \in \{\mathbb{N} \setminus \{0, 1, 2, 3, 4\}\},\ n > 4,\ 2^n > n^2$

- Notation: Let $\mathbb{N}_{n>4} \equiv \{\mathbb{N} \setminus \{0, 1, 2, 3, 4\}\}$
- Let $P(n) : 2^n > n^2$.
- Base case $n = 5$: $2^5 = 32 > 25 = 5^2 \ \therefore P(5) = \mathit{true}$.

Note: $\mathbb{N}_{n>4}$ is an infinite countable set. We can use a non-zero base case (e.g. 5) and use induction to prove that our predicate is true over an infinite countable subset of $\mathbb{N}$.

- Assume $p(k)$ holds for some arbitrary $k \in \mathbb{N}_{n>4} \ \therefore\ 2^k > k^2$.
  * We have $2^k > k^2 \ \therefore\ 2 \cdot 2^k > 2k^2 \ \equiv\ \boxed{2^{k+1} > 2k^2}$
  * Note that $[[2^{k+1} > 2k^2] \wedge [2k^2 > (k+1)^2] \rightarrow [P(k+1) : 2^{k+1} > (k+1)^2]]$. So if we can show that $2k^2 > (k+1)^2$, we can conclude $P(k) \rightarrow P(k+1)$.
  * $2k^2 > (k+1)^2 \ \equiv\ 2k^2 > k^2 + 2k + 1 \ \equiv\ k^2 > 2k + 1$
    $k^2 > 2k + 1 \ \equiv\ k^2 - 2k - 1 > 0 \ \equiv\ k^2 - 2k + 1 > 2 \ \equiv\ (k-1)^2 > 2$
  * $k \in \mathbb{N}_{n>4} \rightarrow k > 4 \ \therefore\ (k-1)^2 > (4-1)^2 = 3^2 = 9 > 2 \ \therefore\ (k-1)^2 > 2 = \mathit{true}$
  * $\boxed{(k-1)^2 > 2 = \mathit{true}} \rightarrow \boxed{2k^2 > (k+1)^2 = \mathit{true}} \rightarrow \boxed{[P(k) \rightarrow P(k+1)] = \mathit{true}}$

  So we have shown that $P(k) \rightarrow P(k+1)$.
- $P(5) \wedge [\,\forall k \in \mathbb{N}_{n>4},\ P(k) \rightarrow P(k+1)],\ \therefore$ by principle of simple induction, we can conclude that $\forall n \in \mathbb{N}_{n>4},\ P(n)$.

Ex. Suppose that $h_0, h_1, h_2...$ is a sequence defined as follows:
$h_0 = 1;\qquad h_1 = 2;\qquad h_2 = 3;\qquad h_k = h_{k-1} + h_{k-2} + h_{k-3}\ \ \forall k \in \mathbb{Z},\ k \geq 3.$
Prove that $\forall n \in \mathbb{N}, h_n \leq 3^n$.

- Let $P(n) : h_n \leq 3^n$.
- Base case $n = 0$: $h_n = 1 \leq 1 = 3^0 = 3^n$.
  Base case $n = 1$: $h_n = 2 \leq 3 = 3^1 = 3^n$.
  Base case $n = 2$: $h_n = 3 \leq 9 = 3^3 = 3^n$.
- Let k be some arbitrary $k \in \mathbb{N}$. Assume