CSC236H
Introduction to the Theory of Computation

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A **simple** induction proof consists of two parts:

- **Base Case**: $P(0)$.
- **Induction Step**: Let $k \in \mathbb{N}$. Assume $P(k)$. **[IH]**
  WTP: $P(k+1)$.

A **complete** induction proof consists of two parts:

- **Base Case**: $P(0)$.
- **Induction Step**: Assume for all $0 \leq j < k$, $j \in \mathbb{N}$, $P(j)$. **[IH]**
  WTP: $P(k)$.

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Another way of stating **complete induction**

- **Base Case**: $P(0)$.
- **Induction Step**: Assume for all $0 \leq j \leq k$, $j \in \mathbb{N}$, $P(j)$. **[IH]**
  WTP: $P(k+1)$.

Simple Induction:

- **Induction Step**: Let $k \in \mathbb{N}$. Assume $P(k)$. **[IH]**
  WTP: $P(k+1)$.

Complete Induction

- **Induction Step**: Assume for all $0 \leq j \leq k$, $j \in \mathbb{N}$, $P(j)$. **[IH]**
  WTP: $P(k+1)$.

Example:

1. Suppose that $h_0, h_1, h_2, \ldots$ is a sequence defined as follows:

$$\begin{aligned} h_0 &= 1, h_1 = 2, h_2 = 3, \\ h_k &= h_{k-1} + h_{k-2} + h_{k-3} \qquad \text{for all integers } k \geq 3. \end{aligned}$$

Prove that $h_n \leq 3^n$ for all integers $n \geq 0$.

IS: Suppose ... . **[IH]**
WTP: $P(k+1)$.

$$\text{Possible IH} : \begin{cases} \text{Simple Induction:} & P(k) \\ \text{Complete Induction:} & P(0), P(1), P(2), ..., P(k) \end{cases}$$

Summary of steps in proof by induction:

- Step 1: Define the predicate.
- Step 2: Prove the predicate holds for the **Base Case**.
- Step 3: Set up the **Induction Step (IS)**, indicate **Induction Hypothesis (IH)**, indicate **What to Prove (WTP)**.
- Step 4: Prove that the predicate holds for all natural numbers using **IH** (make sure to explicitly indicate where you use IH).

- The predicate must denote the statement that we are asked to prove.

Examples:

1. Use induction to prove that for all natural numbers $n \geq 3$,

$$(1 + \frac{1}{n})^n \leq n.$$

2. Let $a_0, a_1, \ldots$ be a sequence of natural numbers such that:
   $a_0 = 1$, and for all $n \geq 1$, $a_n = 2a_{n-1} + 1$.
   Prove that for all $n \in \mathbb{N}$, $n \geq 1$, $a_n = 2^{n+1} - 1$.

Examples:

1. (Exercise 7 in Chapter 1 of the course notes) Use induction to prove that, for any integers $m \geq 2$ and $n \geq 1$,

$$\sum_{t=0}^{n} m^t = \frac{m^{n+1} - 1}{m - 1}.$$

Use induction to prove that the number of nodes in a full binary tree is odd.

- Be careful about multiple base cases.

Example:

$$\begin{aligned} h_0 &= 1, h_1 = 2, h_2 = 3, \\ h_k &= h_{k-1} + h_{k-2} + h_{k-3} \qquad \text{for all integers } k \geq 3. \end{aligned}$$

Prove that $h_n \leq 3^n$ for all integers $n \geq 0$.

- The importance of proving the base case(s).

Example: Prove that For any $n \in \mathbb{N}$, $\sum_{t=0}^{n} 2^t = 2^{n+1}$.

- Is it possible to use induction to prove statements about members of the following sets? If yes, how? If no, why?

  - The set of even natural numbers.

  - The set of integer numbers $\mathbb{Z}$.

  - The set of rational numbers $\mathbb{Q}$.

PWO: Any nonempty subset A of $\mathbb{N}$ contains a minimum element.
That is, for any $A \subset \mathbb{N}$ such that $A \neq \varnothing$, there is some $a \in A$ such that for all $b \in A$, $a \leq b$.

Theorem: The principles of well-ordering, simple induction, and complete induction are equivalent.

Proof: Page 19 of course notes (optional).

Summary of Steps in Proof by PWO

- Step 1: Define the predicate P.
- Step 2: Assume for **contradiction** that $\neg(\forall n \in \mathbb{N}, P(n))$. Note that this is equivalent to $\exists n \in \mathbb{N}, \neg P(n)$.
- Step 3: Define the set S such that $\underline{k \in S \text{ if and only if } \neg P(k)}$. In other words:

$$S = \{k \in \mathbb{N} | P(k) \text{ is false}\}.$$

- Step 4: indicate that by assumption S is nonempty.
- Step 5: Use the **Principle of Well Ordering**, there will be a smallest element $a \in S$.
- Step 6: Reach a **contradiction** – often by using a to show that there is another member of S that is smaller than a (the open-ended part of the proof).
- Step 7: Conclude that our original assumption (in Step 2) is false, and so $\forall n \in \mathbb{N}, P(n)$.

Use the *Principle of Well-Ordering* to prove that for all $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

Example of Proof by PWO

Use the *Principle of Well-Ordering* to prove that any natural number $n \geq 2$ has a prime factorization.