Journal of Automated Reasoning manuscript No.
(will be inserted by the editor)

A Framework for the Verification of Certifying Computations

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the date of receipt and acceptance should be inserted later

Abstract Formal verification of complex algorithms is challenging. Verifying their implementations goes beyond the state of the art of current automatic verification tools and usually involves intricate mathematical theorems. Certifying algorithms compute in addition to each output a witness certifying that the output is correct. A checker for such a witness is usually much simpler than the original algorithm – yet it is all the user has to trust. The verification of checkers is feasible with current tools and leads to computations that can be completely trusted. We describe a framework to seamlessly verify certifying computations. We use the automatic verifier VCC for establishing the correctness of the checker and the interactive theorem prover Isabelle/HOL for high-level mathematical properties of algorithms. We demonstrate the effectiveness of our approach by presenting the verification of typical examples of the industrial-level and widespread algorithmic library LEDA.

1 Introduction

One of the most prominent and costly problems in software engineering is correctness of software. In this article, we are concerned with software for difficult algorithmic problems, e.g., in the domain of graphs. The algorithms for such problems are complex; formal verification of the resulting programs is not tractable, which explains why few graph algorithms have been verified. We show how to obtain *formal instance correctness*, i.e., formal proofs that outputs for particular inputs are correct. We do so by combining the concept of certifying algorithms with methods for code verification and theorem proving.

A *certifying algorithm* [6,36,23] produces with each output a *certificate* or *witness* that the *particular output* is correct. The accompanying *checker* for a certifying

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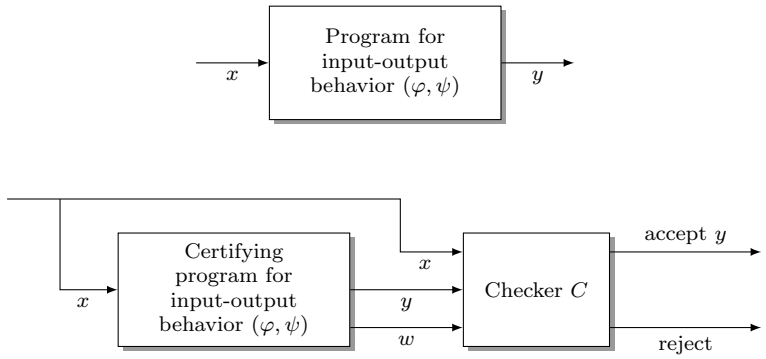


Fig. 1: The top figure shows the input-output behavior of a conventional program for input-output behavior (φ, ψ). The user feeds an input x satisfying $\varphi(x)$ to the program, and the program returns an output y satisfying $\psi(x, y)$. A certifying algorithm for input-output behavior (φ, ψ) computes y and a witness w. The checker C accepts the triple (x, y, w) if and only if w is a valid witness for the postcondition $\psi(x, y)$, i.e., it proves $\psi(x, y)$.

algorithm with input x, output y, and witness w takes as input the triple (x, y, w) and accepts the triple if w proves that y is a correct output for input x. Otherwise, the checker rejects the output or witness as buggy[1].

Figure 1 contrasts a standard algorithm with a certifying algorithm for inout-output behavior (φ, ψ). An algorithm for input-output behavior (φ, ψ) receives an input x satisfying a precondition $\varphi(x)$ and is supposed to deliver an output y satisfying the postcondition $\psi(x, y)$. We call such a y a *correct output*. If the input does not satisfy the precondition, the result of the computation is unspecified. A user of a standard algorithm has, in general, no means of knowing that y is a correct output and has not been compromised by a bug. In contrast, if the accompanying checker of a certifying algorithm accepts, the user may proceed with the complete confidence that output y has not been compromised by a bug. If the checker rejects, either y is incorrect or w is not a proof of the correctness of y.

We illustrate the concept of certifying algorithms with an example. The greatest common divisor of two nonnegative integers a and b, not both zero, is the largest integer g that divides a and b. We write $g = gcd(a, b)$. The extended Euclidean algorithm is a certifying algorithm for greatest common divisor. In addition to the output $g = gcd(a, b)$, it also computes as a witness[2] integers s and t such that $g = s \cdot a + t \cdot b$. The checker checks that g divides a and b and that $g = s \cdot a + t \cdot b$. Why does this prove that g is the greatest common divisor of a and b? Consider any integer d that divides a and b. Then $g = s \cdot a + t \cdot b = (s \cdot (a/d) + t \cdot (b/d)) \cdot d$, and hence, d divides g.

Certifying algorithms are a key design principle of the algorithmic library LEDA [24]: Checkers are an integral part of the library and may (optionally) be invoked after every execution of a LEDA algorithm. The adoption of this principle

[1] Throughout the paper, we say the checker *accepts* if the checker returns *True*; otherwise, we say it *rejects*.

[2] It is known that such integers s and t always exist.

greatly improved the reliability of the library. However, how can one be sure that the checker programs are correct? The third author used to answer: "Checkers are simple programs with little algorithmic complexity. Hence, one may assume that their implementations are correct." We give a better answer in this paper.

We take the certifying-algorithms approach a step further by developing a methodology to verify the checkers. We demonstrate it on three examples: the connectivity of graphs, single-source shortest paths in graphs with nonnegative edge weights, and maximum cardinality matching in graphs. The latter is one of the more complex algorithms in LEDA. The description of the algorithm and its implementation in [24] comprises 15 pages. In contrast, the checker fits on one page. Our formalization effort has revealed that the checker program in LEDA is incorrect in that it does not check that the matching of a graph is a subset of the edges of the graph (see Section 4.3 for a definition of matching).

We introduce our methodology in Section 2 and give detailed case studies in Section 4 before evaluating our approach and the obtained results in Section 5. In Section 3, we survey the verification tools VCC and Isabelle/HOL. Section 6 discusses related work, and Section 7 offers conclusions. The companion web page `http://www21.in.tum.de/~boehmes/certifying_computations.html` contains additional material, in particular, the program listings including VCC annotations and the Isabelle/HOL proofs.

This article is a revised and extended version of a paper published by the same authors at CAV 2011 [1]. We have added two case studies to underline the feasibility and elegance of our approach. Moreover, we have strengthened and simplified our approach. We now prove the total correctness of the checkers instead of only partial correctness. Furthermore, we establish that the checker accepts a triple (x, y, w) if and only if w is a valid witness for output y. Previously, we only proved the if direction. The simplification results from having only one kind of specification in VCC instead of two. Furthermore, a new version of VCC allows for shorter specifications and proofs.

2 Outline of Methodology

We consider algorithms that take an input from a set X and produce an output in a set Y and a witness in a set W. The input $x \in X$ is supposed to satisfy a precondition $\varphi(x)$, and the input together with the output $y \in Y$ is supposed to satisfy a postcondition $\psi(x, y)$. A *witness predicate* for a specification with precondition φ and postcondition ψ is a predicate $\mathcal{W} \subseteq X \times Y \times W$, where W is a set of witnesses with the following *witness property*:

$$\varphi(x) \wedge \mathcal{W}(x, y, w) \longrightarrow \psi(x, y). \tag{1}$$

In contrast to algorithms that work on abstract sets X, Y, and W, the implementing programs operate on concrete representations of abstract objects. We use $\overline{X}$, $\overline{Y}$, and $\overline{W}$ for the set of representations of objects in X, Y, and W, respectively, and assume the mappings $i_X : \overline{X} \to X$, $i_Y : \overline{Y} \to Y$, and $i_W : \overline{W} \to W$.

We illustrate these definitions through the example from the introduction. In the case of greatest common divisors, X and W are the set of pairs of integers, and Y is the set of integers. For input (a, b), output g and witness (s, t), the precondition

$\varphi((a,b))$ states that the inputs a and b are nonnegative integers and that at least one of them is not zero. The postcondition $\psi((a,b),g)$ states that $g = gcd(a,b)$. The witness predicate $\mathcal{W}((a,b),g,(s,t))$ states that $g = sa+tb$ and that g divides a and b. A typical representation of integers in implementations is via bitstrings. Hence, $\overline{X}$ and $\overline{W}$ are each the set of pairs of bit strings, and $\overline{Y}$ is the set of bit strings. The mappings i_X, i_Y, and i_W map (pairs of) bit strings to the corresponding (pairs of) integers.

The checker program C receives a triple $(\overline{x},\overline{y},\overline{w})$ and is supposed to check whether it fulfills the witness property. More precisely, let $x = i_X(\overline{x})$, $y = i_Y(\overline{y})$, and $w = i_W(\overline{w})$. If $\neg\varphi(x)$, C may do anything (run forever or halt with an arbitrary output). If $\varphi(x)$, C must halt and either accept or reject. It is required to accept if $\mathcal{W}(x,y,w)$ holds and is required to reject otherwise. In order to achieve formal instance correctness, we will always prove the following two proof obligations: Checker Correctness and Witness Property. Together they give Theorem 1.

Checker Correctness: We need to prove that C checks the witness predicate, assuming that the precondition[3] holds, i.e., on input $(\overline{x},\overline{y},\overline{w})$ and with $x = i_X(\overline{x})$, $y = i_Y(\overline{y})$, and $w = i_W(\overline{w})$:

1. If $\varphi(x)$, C halts.
2. If $\varphi(x)$ and $\mathcal{W}(x,y,w)$, C accepts $(\overline{x},\overline{y},\overline{w})$, and if $\varphi(x)$ and $\neg\mathcal{W}(x,y,w)$, C rejects the triple.

Witness Property: We need to prove implication (1).

In our running example, the witness property is

$$a \geq 0 \wedge b \geq 0 \wedge a+b > 0 \wedge g|a \wedge g|b \wedge g = a\cdot s + b\cdot t \longrightarrow g = gcd(a,b).$$

Here a, b, g, s and t are assumed to be nonnegative integers.

Theorem 1 *Assume that the proof obligations are discharged. Let* $(\overline{x},\overline{y},\overline{w}) \in \overline{X}\times\overline{Y}\times\overline{W}$ *and let* $x = i_X(\overline{x})$, $y = i_Y(\overline{y})$, *and* $w = i_W(\overline{w})$.

If C accepts a triple $(\overline{x},\overline{y},\overline{w})$, $\varphi(x) \longrightarrow \psi(x,y)$ *by a formal proof. If C rejects a triple* $(\overline{x},\overline{y},\overline{w})$, $\neg\varphi(x) \vee \neg\mathcal{W}(x,y,w)$ *by a formal proof.*

Proof If C accepts $(\overline{x},\overline{y},\overline{w})$, we have $\varphi(x) \longrightarrow \mathcal{W}(x,y,w)$ by the correctness proof of C. Then by (1) we have a formal proof for $\varphi(x) \longrightarrow \psi(x,y)$. Conversely, if C rejects the triple, the correctness proof of C establishes $\neg\varphi(x) \vee \neg\mathcal{W}(x,y,w)$. □

We discuss next how to fulfill the proof obligations in a *comprehensive* and *efficient* framework . Comprehensive means that the final proof formally combines (as much as possible at the syntactic level) the correctness arguments for all levels (implementation, abstraction, and mathematical theory). Efficient means using the right tool for the right task. For example, applying a general theorem prover to verify imperative code would involve a lot of language-specific overhead and lead to less automation; similarly, a specialized code verifier is often not powerful enough to cover nontrivial mathematical properties. The goals of comprehensiveness and efficiency seem to be conflicting because different tools usually come with different languages, axiomatization sets, etc. Our solution is to use second-order logic as a common interface language.

[3] We stress that the checker has the same precondition as the algorithm.

LEDA is written in C++ [24]. Our aim is to verify code which is as close as possible to the original implementation; by this, we demonstrate the feasibility of verifying already established libraries written in imperative languages such as C. Thus, we verify code with VCC [14], an automatic code verifier for full C. Verifying the C++ implementations remains open. Our choosing of VCC is motivated by the maturity of the tool and the provision of an assertion language that is rich enough for our requirements. In the Verisoft XT project [38], VCC was successfully used to verify tens of thousands of lines of C code. The assertion language offers ghost code and ghost types such as maps and unbounded integers. This gives enough expressiveness to quantify over graphs, labelings, etc., and simplifies the translation to other proof systems. For verifying the mathematical part, we use Isabelle/HOL, a higher-order-logic interactive theorem prover [26], because of the large amount of already formalized mathematics, its descriptive proof format, and its various automatic proof methods and tools. In Section 3, we review both systems.

Checker Verification: The starting point is the checker code written in C. Using VCC, we annotate the functions and data structures such that the witness predicate $\mathcal{W}$ can be established as the postcondition of the checker function. We define the witness predicate and the pre- and postcondition as well as the mappings i_X, i_Y, and i_W as pure mathematical objects using VCC ghost types and ghost functions. Note that as a precondition to all our programs we assume that the concrete input values $\overline{x}$ are valid (i.e., they are not NULL pointers and do not point outside array bounds nor to memory protected addresses nor outside the address space). This is ensured by using the VCC invariant \wrapped or other invariants that ensure that the objects are owned by the current thread.

Export to Isabelle/HOL: Establishing the witness property involves, in general, mathematical reasoning beyond what is conveniently done in VCC. We therefore translate the precondition, witness predicate, postcondition, and the abstract representations of the input, output, and witness from VCC to Isabelle/HOL. Since we formulated them as pure mathematical objects in VCC, this translation is purely syntactical and does not involve any VCC specifics. The translation could easily be automated.

Witness Property: We prove the witness property using Isabelle/HOL. It is convenient to formulate this theorem on yet a higher level of abstraction and provide linking proofs to connect the exported VCC predicates with their abstracted counterparts.

We stress that the overall correctness theorem, i.e., the witness property, can be formulated in VCC; this is important for usability. The user of a verified checker only has to look at its VCC specification; the fact that we outsource the proof of the witness property to Isabelle/HOL is of no concern to the user. We formulate the witness property as an axiom in VCC. This is sound since we restrict the language for describing the witness property to second-order logic, which guarantees that we can express it equivalently in Isabelle's higher-order logic (cf. Section 3). More precisely, since the VCC formulation of the witness property is valid if and only if its translation to Isabelle is valid, and since Isabelle is consistent, and hence, only valid statements can be proven, it is sound to add the witness property as an axiom to VCC.

The reader may wonder why we do not formally prove the existence of a witness:

$$\forall x\, y.\ \varphi(x) \wedge \psi(x, y) \longrightarrow \exists w.\ \mathcal{W}(x, y, w).$$

The existence of a witness is part of the correctness argument of the solution algorithm (e.g., the shortest-path algorithm, the maximum-matching algorithm). As previously mentioned, we do not verify the solution algorithms. Rather, the execution of the solution algorithm establishes the existence of a witness whenever it is called for a specific input $\overline{x}$. It returns $\overline{y}$ and $\overline{w}$, which we then hand to the checker C. In this way, we obtain formal instance correctness without having to verify the solution algorithm. Of course, this leaves the possibility that the solution algorithm is incorrect and does not always provide a $\overline{y}$ and $\overline{w}$ such that the checker accepts $(\overline{x}, \overline{y}, \overline{w})$.

For a user, the checker is what counts. The user can trust it because it has been formally verified. Moreover, if it accepts a triple $(\overline{x}, \overline{y}, \overline{w})$, the user can be sure that y is a correct output, provided that x satisfies the precondition of the algorithm. This is because the witness property has been formally verified. If the checker rejects a triple, the user knows that either x does not satisfy the precondition or (x, y, w) does not satisfy the witness predicate. The method by which $\overline{y}$ and $\overline{w}$ were produced is of no concern to the user.

The witness property is formulated with respect to a certain input-output behavior (φ, ψ) and not with respect to a particular algorithm that realizes the input-output behavior. Therefore, a checker can be used in connection with any certifying algorithm for input-output behavior (φ, ψ) that produces the appropriate witnesses.

3 Tool Overview: Isabelle/HOL and VCC

Isabelle/HOL. Isabelle/HOL [26] is an interactive theorem prover for classical higher-order logic based on Church's simply typed lambda calculus. The system is built on top of an inference kernel, which provides only a small number of rules to construct theorems; complex deductions (especially by automatic proof methods) ultimately rely on these rules only. This approach, called LCF due to its pioneering system [18], guarantees the correctness of theorems proven in the system as long as the inference kernel is correct. Isabelle/HOL comes with a rich set of already formalized theories, among which are natural numbers and integers as well as sets, finite sets, and—as a recent addition [30]—graphs. New types can, for example, be introduced by defining them as records (isomorphic to tuples with named update and selector functions). New constants can be introduced, for example, via definitions relative to already existing constants.

Proofs in Isabelle/HOL can be written in a style close to that of mathematical textbooks. The user structures the proof, and the system fills in the gaps with its automatic proof methods. Moreover, the user can use locales, which allow defining local scopes in which constants are defined and assumptions are made. Theorems proven in the context of a locale and can use the constants and depend on the assumptions of this locale. A locale can be instantiated to concrete entities if the user is able to show that the latter fulfill the locale assumptions.

VCC. VCC [14] is an assertional, automatic, deductive code verifier for full C. Specifications in the form of function contracts, data invariants, and loop invariants as well as further annotations to maintain inductively defined information or to guide VCC otherwise, are added directly into the C source code as comments. During builds with a C compiler, these annotations are ignored. From the annotated program, VCC generates verification conditions for partial or total correctness, which it then tries to discharge using the automatic theorem prover Z3 [25] through the Boogie verifier [3].

Verification in VCC makes heavy use of ghost data and code, enclosed by _(and), for reasoning about the program but omitted from the concrete implementation. VCC provides ghost objects, ghost fields of structured data types, local ghost variables, ghost function parameters, and ghost code. Ghost data and ghost code can use both C data types and additional mathematical data types, e.g., mathematical integers (**integer**) and natural numbers (**natural**), records (similar to C structures), and maps (with a syntax similar to C arrays). VCC ensures that information does not flow from a ghost state to a non-ghost state and that all ghost code terminates; these checks guarantee that program execution, when projected to the non-ghost code, is not affected by the ghost code.

Export from VCC to Isabelle/HOL. For the types and propositions that we pass from VCC to Isabelle/HOL, we restrict ourselves to a subset of VCC's specification language. Simple types are natural numbers, integers, algebraic datatypes over simple types, and ghost records whose fields are simple types. Rich types are simple types, ghost records whose fields are rich types, and maps from simple types to rich types. Propositions can be formed by usual logical connectives, quantifiers over variables of rich types, arithmetic expressions, equalities, and user-defined pure, stateless functions whose argument and result types are rich and whose definitions or contracts are again propositions, possibly using pattern matching over algebraic datatypes. Any type or function of this subset can be expressed equivalently in Isabelle/HOL, essentially by syntactic rewriting. More precisely, VCC algebraic datatypes can be translated into Isabelle datatypes, VCC ghost records can be translated into Isabelle records, and pure VCC ghost functions can be translated into Isabelle function definitions. The former two translations are sound and complete because the semantics of datatypes and records is the same in both systems; the latter is sound and complete because VCC's underlying logic is subsumed by the higher-order logic of Isabelle/HOL. The translation maps VCC specification types (\bool, **natural**, **integer**, and map types) to equivalent Isabelle types (*bool*, *nat*, *int*, and function types) and maps VCC expressions comprising logical connectives, quantifiers, arithmetic operations, equality, and specification functions to corresponding Isabelle terms.

4 Case Studies

We present three case studies from the domain of graphs. We start with the certifying algorithms and the corresponding checkers in LEDA. We give formal proofs for the correctness of the checkers, for the related witness properties, and for the connection between them. Except for the witness properties, which are proven in

Isabelle/HOL, all presented abstractions and functions have been formally verified using VCC.

All files related to our formalization can be obtained from the following URL: http://www21.in.tum.de/~boehmes/certifying_computations.html

4.1 Connected Components in Graphs

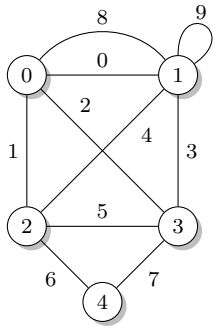
Our first case study considers the connected components problem. Given an undirected graph $G = (V, E)$, we consider an algorithm that decides whether G is connected, i.e., whether there is a path between any pair of vertices [24, Section 7.4]. In the negative case, i.e., when the graph is not connected, there is a simple witness. It consists of a cut S, i.e., a nonempty subset S of the vertices with $S \neq V$ such that every edge of the graph has either both or no endpoint in S. In other words, no edge crosses the cut. In the positive case, i.e., when the given graph is connected, the algorithm can produce a spanning tree of G as a witness. A spanning tree of G is a subgraph of G, which is a tree and contains all vertices of G. On a high level, we instantiate our general approach as follows:

input x	=	an undirected graph $G = (V, E)$
output y	=	either *True* or *False*, indicating whether G is connected
witness w	=	a cut or a spanning tree
$\varphi(x)$	=	V and E are finite sets and G is wellformed i.e., a pair of vertices in $V \times V$ is associated with every $e \in E$
$\mathcal{W}(x, y, w)$	=	y is *True* and w is a spanning tree of G, or y is *False* and w is a cut
$\psi(x, y)$	=	if y is *True*, G is connected, and if y is *False*, G is not connected.

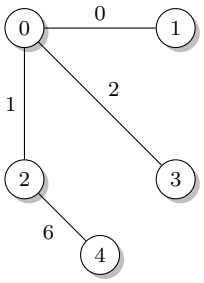
We restrict ourselves to the positive case $y = \mathit{True}$. We describe a checker for the spanning tree witness and the verification of this checker. Figure 2 shows a graph G and its spanning tree. We represent spanning trees by functions *parent-edge* and *num* and by a root vertex r, and we view the edges of the tree oriented towards r: for $v \neq r$, *parent-edge*(v) is the first edge on the path from v to r, *parent-edge*$(r) = \perp$, and *num*(v) is the length of the path from v to r for all v. The function *num* is needed in order to show that *parent-edge* encodes a forest. We present the implementation of a checker in Section 4.1.1, detail the formalization of the witness predicate and the verification of the checker in Section 4.1.2, and prove the witness property in Section 4.1.3. The relevant files in the companion website are check_connected.c (C-code and checker correctness), check_connected.thy (Isabelle/HOL representation of the VCC checker specification and witness property), ConnectedComponents.thy (abstract verification of the witness property in a locale), and ConnectedComponents_Link.thy (instantiation of the locale with the Isabelle/HOL representation of the VCC specification).

4.1.1 Connected-Components Checker

We begin by fixing a representation of graphs in the programming language C, see Listing 1. Vertices are numbered consecutively from 0 to `n` − 1. Edges are pairs where the first vertex is labeled `s` (for source), and the second vertex is labeled `t`



(a) A connected graph G



(b) A spanning tree of G

vertex	0	1	2	3	4
parent-edge	$\perp$	0	1	2	6
num	0	1	1	1	2

(c) Spanning tree representation

Fig. 2: An example of a connected graph G and a spanning tree of G witnessing its connectivity. The vertices belong to the set $\{0, \ldots, n-1\}$ and the edges are pairs of vertices indexed by an identifier ranging from 0 to $m-1$, where n and m are the number of vertices and edges in G. The spanning tree in (b) can be represented by a root vertex $r = 0$ and functions *parent-edge* and *num* as shown in the table in (c). Graphs may have self-loops and parallel edges.

```
typedef unsigned Nat;
typedef Nat Vertex;
typedef Nat Edge_Id;
typedef struct { Nat s; Nat t; } Edge;
typedef struct { Nat m; Nat n; Edge* es; } Graph;
```

Listing 1: A representation of graphs in C. The field m gives the number of edges (and hence the length of the array es), and n gives the number of vertices in the graph.

(for target). Edges are stored in an array es, which is indexed by edge identifiers ranging from 0 to m − 1. We require that the two vertices of each edge belong to the graph, i.e., that they are from the range $\{0, \ldots, \mathsf{n}-1\}$, and call graphs with this property *wellformed*. We use the same data structure for directed and undirected graphs. For directed graphs, an edge e with e.s = u and e.t = v is directed from u to v. For undirected graphs, it represents the unordered pair {u,v}.

We represent spanning trees as explained before. Instead of functions, we use two arrays, parent_edge and num, in addition to a root vertex r. The parent_edge array maps r to a negative value, i.e., to a value that does not identify any edge.

The connected-graph checker is a function that accepts if the two functions check_r and check_parent_num (Listing 2) accept. The first function checks that r is indeed the root of the spanning tree. The second function checks for every vertex v different from r that the edge parent_edge[v] is incident to v and that the other endpoint of the edge has a number one smaller than num[v].

```
int check_r(Graph* G, Vertex r, int* parent_edge, Nat* num)
{
  return r < G→n && num[r] == 0 && parent_edge[r] < 0;
}

int check_parent_num(Graph* G, Vertex r, int* parent_edge, Nat* num)
{
  Vertex v, a, b; Edge_Id e;

  for (v = 0; v < G→n; v++)
  {
    if (v == r) continue;

    if (parent_edge[v] < 0 || ((Edge_Id)parent_edge[v]) ≥ G→m) return FALSE;

    e = (Edge_Id)parent_edge[v];
    a = G→es[e].s;
    b = G→es[e].t;

    if (v == a && num[v] == num[b] + 1) continue;
    if (v == b && num[v] == num[a] + 1) continue;
    return FALSE;
  }
  return TRUE;
}
```

Listing 2: The connected-components checker.

```
_(typedef \natural \Vertex)
_(typedef \natural \Edge_Id)
_(record \Edge {
  \Vertex src;
  \Vertex trg;
})
_(record \Graph {
  \natural num_verts;
  \natural num_edges;
  \Edge edge[\Edge_Id];
})
```

```
_(def \bool \wellformed(\Graph G)
{
  return
    ∀ \Edge_Id i; i < G.num_edges ⟶
      G.edge[i].src < G.num_verts ∧
      G.edge[i].trg < G.num_verts;
})
```

Listing 3: Abstract graphs and a predicate to describe wellformed graphs.

4.1.2 Checker Correctness

To prove the two checker functions correct, we need to provide abstract representations for graphs and paths. We decided to keep them close to the concrete representation for two reasons. First, it makes detecting differences, and hence potential bugs, easier for the programmer. Second, it also makes reasoning for VCC simpler. The declaration of abstract graphs is given in Listing 3 together with the ghost predicate \wellformed for describing when an abstract graph is wellformed. This ghost predicate plays the role of the precondition φ in this case study. Our abstract version of the num array is a mapping from vertices to natural numbers. The abstract version of the parent_edge array is a mapping from vertices to the set

$\mathbb{N} \cup \{\bot\}$; we use $\bot$ to model an undefined value. To represent this set, we define an algebraic datatype Option:

```
_(datatype \Option
{
  case \none();
  case \some(\Edge_Id e);
})
```

with operations \is_some(o) for the test $o \neq \bot$ and \the(o) for extracting an edge identifier. The abstraction functions that map concrete data to pure mathematical data are straightforward to define. For example,

```
_(def \Graph \abs_graph(Graph* G)
{
  return (\Graph) {
    .num_verts = G→n,
    .num_edges = G→m,
    .edge =
      \lambda \Edge_Id i;
        (i < G→m) ?
          (\Edge) { .src = G→es[i].s, .trg = G→es[i].t } :
          (\Edge) { .src = 0, .trg = 0 }};
})
```

abstracts a concrete graph G into an abstract graph of type \Graph. Similarly abstraction functions \abs_parent_edge and \abs_num are defined to abstract tparent_edge and num respectively; we will refer to \abs_parent_edge(G, parent_edge) as P.

Using the abstract types, we define the witness predicate as a conjunction of two properties, one for each of the checker functions in Listing 2.

check_r: Vertex r is the root of the spanning tree:

$$r < G.num\_verts \land \neg\backslash is\_some(parent\_edge[r]) \land num[r] = 0$$

check_parent_num: Every vertex of the graph is connected to some other vertex closer to r:

$$\begin{aligned}&\forall\ \backslash Vertex\ v;\ v < G.num\_verts \land v \neq r \longrightarrow\\ &\quad \backslash is\_some(parent\_edge[v]) \land \backslash the(parent\_edge[v]) < G.num\_edges \land\\ &\quad (G.edge[\backslash the(parent\_edge[v])].trg == v \land\\ &\qquad num[v] == num[G.edge[\backslash the(parent\_edge[v])].src] + 1 \lor\\ &\quad G.edge[\backslash the(parent\_edge[v])].src == v \land\\ &\qquad num[v] == num[G.edge[\backslash the(parent\_edge[v])].trg] + 1)\end{aligned}$$

Thanks to the low level of abstraction in the above predicates, the two checker functions are easily verified. For the verification of check_parent_num, we need to annotate the loop with the check_parent_num property in which G.num_verts is replaced by the loop variable as a loop invariant. Moreover, for every **return** FALSE, we need to assert, or restate, on the abstract level the properties that are violated to guide VCC. Otherwise, it would fail to show completeness of the checker. For instance,

```
if (parent_edge[v] < 0 || ((Edge_Id)parent_edge[v]) ≥ G→m)
{
  _(assert ¬\is_some(P[v]) ∨ \the(P[v]) ≥ \abs_graph(G).num_edges)
  return FALSE;
}
```

is one of the two occurrences of such extra assertions in check_parent_num.

We express the postcondition of the checker, i.e., that any pair of vertices of the graph G is connected by a path as follows:

```
∀ \Vertex u, v; u < G.num_verts ∧ v < G.num_verts ⟶
   ∃ \Path p; \natural n; \is_path(G, p, n, u, v)
```

Here, the type \Path is a sequence of vertices, represented as a mapping from natural numbers to vertices, and the predicate \is_path(G, p, n, u, v) holds if the path p of length n starts at u, ends at v, and only contains pairwise distinct vertices that are connected by edges of the graph:

```
p[0] == u ∧
p[n] == v ∧
(∀ \natural i; i ≤ n ⟶ p[i] < G.num_verts) ∧
(∀ \natural i; i < n ⟶ \is_edge(G, p[i], p[i+1])) ∧
(∀ \natural i, j; i ≤ n ∧ j ≤ n ∧ i ≠ j ⟶ p[i] ≠ p[j])
```

The predicate \is_edge(G, u, v), for any two vertices u and v of G, is true if and only if u and v are the endpoints of an edge of G:

```
∃ \Edge_Id i; i < G.num_edges ∧
   (G.edge[i].src = u ∧ G.edge[i].trg = v ∨
    G.edge[i].src = v ∧ G.edge[i].trg = u)
```

We give a formal proof for the implication, from precondition and witness predicate to the postcondition, in the next section.

4.1.3 Proof of the Witness Property for the Connected-Components Checker

We prove in Isabelle that a spanning tree witnesses the connectivity of a graph. We do so in two steps. First, we perform a high-level proof in which we abstract from concrete representations of graphs and spanning trees. We then instantiate this proof with the data structures and corresponding properties exported from VCC.

Our formalization builds on the Isabelle graph library developed by Lars Noschinski [30]. Graphs in this library are directed. A *pseudo-digraph* is a wellformed directed graph with a finite set of vertices and a finite set of edges; the library reserves the word *digraph* for graphs without parallel edges and self-loops. Unlike in VCC, in Isabelle we represent undirected graphs as bidirected graphs[4], i.e., directed graphs containing for every edge (u, v) also the reversed edge (v, u). The function *mk-symmetric* maps a pseudo-digraph to a bidirected pseudo-digraph by appropriately extending the set of edges with missing reversed edges. A vertex v is *reachable* from a vertex u in a (bi)directed graph G if there exists a directed *walk* from u to v in G, i.e., a sequence $(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)$ of edges with $u_1 = u$, $v_k = v$, and $v_i = u_{i+1}$ for $1 \leq i < k$. An alternative and equivalent formalization of reachability between vertices u and v in G is via sequences of vertices $v_1, v_2, \ldots, v_k$, where $v_1 = u$, $v_k = v$ and (v_i, v_{i+1}) is an edge of G for $1 \leq i < k$. We say a vertex v is *reachable through a path* from a vertex u in G if v is reachable from u through a path in G. An undirected graph is *connected* if for any two vertices of the graph, one is reachable through a path from the other.

[4] We do so in order to directly use the Isabelle graph library.

```
locale connected-components-locale = pseudo-digraph +
  fixes num :: α ⇒ nat
  fixes parent-edge :: α ⇒ β option
  fixes r :: α
  assumes r_assms: r ∈ verts G ∧ parent-edge r = None ∧ num r = 0
  assumes parent-num-assms:
    ⋀v. v ∈ verts G ∧ v ≠ r ⟹
      ∃e ∈ edges G.
        parent-edge v = Some e ∧
        target G e = v ∧
        num v = num (start G e) + 1
```

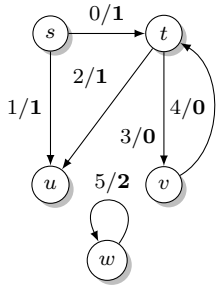
Listing 4: Locale for the connected-components proof in Isabelle. The symbol ⋀ stands for universal quantification.

Our high-level proof rests on the Isabelle locale *connected-components-locale* (Listing 4) that describes the assumptions of our theorem. We fix G to be a pseudo-digraph where α is an abstraction of the type of vertices and β is an abstraction of the type of edges. Furthermore, we fix a representation of spanning trees with functions *parent-edge* and *num* and vertex r as the root. Based on these assumptions we prove that G is connected. We first show that every vertex v in the graph is reachable from the root r by induction on *num* v, i.e., the length of the walk from r to v in the spanning tree. The base case follows directly from our assumptions. For the inductive step, we can assume a walk from r to the parent of a vertex v. Using the assumptions, this walk can be extended to a walk from r to v since there is an edge between v and its parent. Now, since G is bidirected, we can establish that there is a walk between any two vertices of G by combining the walks that connect them with the root r. If there is a walk between two vertices, there is also a path between them. Therefore, all vertices in G are reachable through a path from one another, and hence, G is connected.

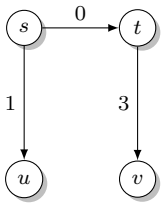
The final part of the formal proof—linking the high-level proofs with the properties exported from VCC to Isabelle—is fairly straightforward. Proving that the precondition and the witness predicate (cf. Section 4.1.2) match the assumptions specified in the locale *connected-components-locale* involves no reasoning beyond syntactical rewriting. To instantiate these assumptions, we provide lifting functions that abstract from the concrete representations of graphs and spanning trees stemming from our VCC specification to the high-level representation used by the Isabelle graph library. Thus, if the checker accepts, the lifted high-level graph is connected. Establishing the checker postcondition (the connectivity of unlifted graphs) requires showing that any high-level path witnessing reachability between two vertices corresponds to an unlifted path. This is straightforward because our representation of paths in the VCC formalization (cf. Section 4.1.2) is close to the path representation of the Isabelle graph library.

4.2 Shortest Paths in Graphs

Our second case study is about the single-source shortest-path problem in directed graphs with nonnegative edge weights. It can be solved, for instance, using Dijkstra's algorithm [24, Sections 6.6 and 7.5]. Instead of verifying this algorithm, we



(a) A directed graph G



(b) A shortest-path tree of G

vertex	s	t	u	v	w
parent-edge	$\perp$	0	1	3	$\perp$
num	0	1	1	2	∞
dist	0	1	1	1	∞

(c) Tree representation

Fig. 3: A directed graph $G = (V, E)$ with the edges labeled $i/\mathbf{k}$, where i is a unique edge index and where $\mathbf{k}$ is the cost of that edge, and a shortest-path tree of G rooted at start vertex $s \in V$. The tree is encoded by *parent-edge*, *num* and *dist* according to the table in (c). Observe that vertex w is not reachable from s and that the cycle $t \to v \to t$ has cost zero.

request that it returns, along with the computed shortest distance for each vertex of a graph, the corresponding shortest path as witness. That is, we instantiate our general approach as follows:

input x	=	a directed graph $G = (V, E)$, a function $cost : E \to \mathbb{N}$ for edge weights, a vertex s
output y	=	a mapping $dist : V \to (\mathbb{N} \cup \infty)$
witness w	=	a tree rooted at s
$\varphi(x)$	=	G is wellformed, V and E are finite sets, and $s \in V$
$\mathcal{W}(x, y, w)$	=	w is a shortest-path tree, i.e., for each v reachable from s, the tree path from s to v has length $dist(v)$
$\psi(x, y)$	=	for each $v \in V$, $dist(v)$ is the cost of a shortest path from s to v (or ∞ if there is no path from s to v).

Figure 3 shows a directed graph and a shortest-path tree rooted at s. We encode a shortest-path tree by functions *parent-edge*, *dist*, and *num*. For each v reachable from s, $dist(v)$ is the shortest-path distance from s to v and $num(v)$ is the depth of v in the shortest-path tree. For vertices v that are not reachable from s, $dist(v) = num(v) = \infty$. For reachable vertices v different from s, the edge $parent\text{-}edge(v)$ is the last edge on a shortest path from s to v. This witness is somewhat verbose. As we will see in Section 4.2.3, we could do without the *parent-edge* function. If all edge costs are positive, no witness is required beyond the *dist* function. If one also allows cost zero for edges as we do, the depth function *num* is indispensable [23, Section 2.4].

We present the implementation of a checker in Section 4.2.1, detail the formalization of the witness predicate and the verification of the checker in Section 4.2.2, and prove the witness property in Section 4.2.3. The proof of the witness property is also available at [33]. The proof in this paper uses an older version of the Isabelle graph library. The relevant files in the companion website are check_shortest_path.c (C-code and checker correctness), check_shortest_path.thy (Isabelle/HOL representation of the VCC checker specification, and witness property), ShortestPath.thy (abstract verification of the witness property in a locale),

and ShortestPath_Link.thy (instantiation of the locale with the Isabelle/HOL representation of the VCC specification).

4.2.1 Shortest-Paths Checker

We adopt the data structures of the previous case study (Section 4.1.1) with the exception that the num array stores elements of type int instead of Nat. This is because vertices may now also be unreachable from the source vertex, and we encode this by requiring that num takes a negative value for such vertices. We represent distances from the source vertex to any other vertex by an array dist with elements of type int. Any negative value encodes ∞. Finally, the edge weights are modeled by an array cost that gives for every edge a value of type ushort (an abbreviation for **unsigned short**).

Based on these types, we implement the shortest-path checker as a function that accepts when all of the four functions given in Listing 5 accept. That is, we check that the source vertex s is indeed the starting point (in check_start_val), that the dist and num arrays are consistent with respect to unreachable vertices, i.e., either both are finite or both are infinite (in check_no_path), that the triangle inequality property (Section 4.2.3) is fulfilled (in check_trian), and that the parent edge of every vertex v defines its distance value (in check_just).

There is a subtle point in the checker code. We want to establish the triangle inequality ($dist(u) + cost(u,v) \geq dist(v)$ for all edges (u,v)) and the distance justification ($dist(u) + cost(u,v) = dist(v)$ if (u,v) is the parent edge of v) over the extended natural numbers $\mathbb{N} \cup \{\infty\}$. However, C knows only finite precision arithmetic. We solve the case of infinite distances by appropriate case distinctions. We solve the case of potential overflow in finite precision arithmetic as follows: Distances are of type int, i.e., from the set $\{-2^{31}, \ldots, 2^{31}-1\}$ on a 32-bit platform, and edge costs are of type ushort, i.e., between 0 and $2^{16}-1$, and hence contained in the set of nonnegative values of type int. In arithmetic expressions, we cast all nonnegative values to **unsigned** with range $0 \ldots 2^{32}-1$. This guarantees that bounded integer arithmetic is exact and allows VCC to conclude equalities and inequalities between natural numbers.

Note that there is an alternative approach where parent_edge is not part of the witness. In that case check_just has to be rewritten. When considering a node v, it has to iterate over all edges into v to find the edge that defines dist[v]. An efficient implementation of this iteration requires providing each vertex with the list of edges into it.

4.2.2 Checker Correctness

We now define our abstract specification for the shortest-path checker. We use the same data structures as in the previous case study (Section 4.1.2) with the exception that the num mapping now takes vertices to extended naturals ($\mathbb{N} \cup \{\infty\}$), represented by the type Enat. Extended naturals provide an explicit value for infinity:

```
_(datatype \Enat
{
  case \enat_inf();
  case \enat_val(\natural n);
```

```
bool check_start_val(Vertex s, int* dist)
{
   return dist[s] == 0;
}

bool check_no_path(Graph* G, int* dist, int* num)
{
   Vertex v;

   for (v = 0; v < G→n; v++)
   {
      if (INF(dist[v]) != INF(num[v])) return FALSE;
   }
   return TRUE;
}

int check_trian(Graph* G, ushort* cost, int* dist)
{
   Edge_Id e; Vertex source, target;

   for (e = 0; e < G→m; e++)
   {
      source = G→es[e].s;
      target = G→es[e].t;

      if (INF(dist[source])) continue;
      if (INF(dist[target])) return FALSE;
      if (VAL(dist[target]) > VAL(dist[source]) + cost[e]) return FALSE;
   }
   return TRUE;
}

bool check_just(Graph* G, Vertex s, ushort* cost, int* dist, int* parent_edge, int* num)
{
   Vertex v, source; Edge_Id e;

   for (v = 0; v < G→n; v++)
   {
      if (v == s || INF(num[v])) continue;
      if (parent_edge[v] < 0 || ((Edge_Id)parent_edge[v]) ≥ G→m) return FALSE;

      e = (Edge_Id)parent_edge[v];
      source = G→es[e].s;

      if (G→es[e].t != v) return FALSE;
      if (INF(dist[source]) || VAL(dist[v]) != VAL(dist[source]) + cost[e]) return FALSE;
      if (INF(num[source]) || VAL(num[v]) != VAL(num[source]) + 1) return FALSE;
   }
   return TRUE;
}
```

Listing 5: Functions composing the shortest-path checker. The predicate INF(x) abbreviates x < 0, and VAL(x) stands for the type cast (Nat)x; Nat is the C type **unsigned** as defined in Listing 1.

```
})
```

We define functions \is_enat_inf to check whether an extended natural is infinity and \enat_val_of to convert an extended natural distinct from infinity into the corresponding natural number. For better readability, we will write $\mathsf{a} =_e \infty$ for \is_enat_inf(a). Moreover, we provide the predicates \enat_eq (abbreviated by $=_e$) and \enat_le ($\leq_e$) to decide equality and less-or-equal of two extended naturals as well as a function \enat_add ($+_e$) for the sum of an extended natural and a natural number:

```
_(def \bool \enat_eq(\Enat e1, \Enat e2)
{
  return
    (e1 =_e ∞ ∧ e2 =_e ∞) ∨
    (e1 ≠_e ∞ ∧ e2 ≠_e ∞ ∧ \enat_val_of(e1) = \enat_val_of(e2));
})

_(def \bool \enat_le(\Enat e1, \Enat e2)
{
  return e2 =_e ∞ ∨ (e1 ≠_e ∞ ∧ e2 ≠_e ∞ ∧ \enat_val_of(e1) ≤ \enat_val_of(e2));
})

_(def \Enat \enat_add(\Enat e, \natural n)
{
  return (e =_e ∞) ? \enat_inf() : \enat_val(\enat_val_of(e) + n);
})
```

The type of extended natural numbers is also used for the abstract representation of the dist array. Again, as in the previous case study, concrete types and abstract types are sufficiently similar such that abstraction functions relating one to the other are straightforward to define. We omit them here.

The preconditions of this case study are that G is a wellformed graph and that the source vertex s is a vertex of G, i.e., that $\mathsf{s} < \mathsf{G.num\_verts}$ holds. We formalize the witness predicate as a conjunction of four properties, one for each of the four checker functions in Listing 5.

check_start_val: Vertex s is indeed the starting point:

$$\mathsf{dist[s]} =_e \backslash\mathsf{enat\_val(0)}$$

check_no_path: The num mapping and the dist mapping are consistent with respect to unreachable vertices, i.e., both are either finite or infinite:

$$\forall\ \backslash\mathsf{Vertex\ v};\ \mathsf{v} < \mathsf{G.num\_verts} \longrightarrow (\mathsf{dist[v]} =_e \infty \longleftrightarrow \mathsf{num[v]} =_e \infty)$$

check_trian: The triangle inequality holds for all edges of the graph:

$$\forall\ \backslash\mathsf{Edge\_Id\ i};\ \mathsf{i} < \mathsf{G.num\_edges} \longrightarrow \mathsf{dist[G.edge[i].trg]} \leq_e \mathsf{dist[G.edge[i].src]} +_e \mathsf{cost[i]}$$

check_just: The parent edges encode a tree rooted at s and define the distance values of reachable vertices:

$$\begin{aligned}&\forall\ \backslash\mathsf{Vertex\ v};\\ &\quad \mathsf{v} < \mathsf{G.num\_verts} \wedge \mathsf{v} \neq \mathsf{s} \wedge \mathsf{num[v]} \neq_e \infty \longrightarrow\\ &\quad \backslash\mathsf{is\_some(parent\_edge[v])} \wedge \backslash\mathsf{the(parent\_edge[v])} < \mathsf{G.num\_edges} \wedge\\ &\quad \mathsf{v} = \mathsf{G.edge[}\backslash\mathsf{the(parent\_edge[v])]\text{.}trg} \wedge\\ &\quad \mathsf{dist[v]} =_e \mathsf{dist[G.edge[}\backslash\mathsf{the(parent\_edge[v])]\text{.}src]} +_e \mathsf{cost[}\backslash\mathsf{the(parent\_edge[v])]} \wedge\\ &\quad \mathsf{num[v]} =_e \mathsf{num[G.edge[}\backslash\mathsf{the(parent\_edge[v])]\text{.}src]} +_e 1\end{aligned}$$

```
_(def \bool \is_walk(\Graph G, \Path p, \Vertex u, \Vertex v)
{
  switch (p)
  {
    case none(): return u = v;
    case path(i, q):
      return i < G.num_edges ∧ u = G.edge[i].src ∧ \is_walk(G, q, G.edge[i].trg, v);
  }
})
```

Listing 6: A walk from vertex u to vertex v is a finite sequence of connected edges of graph G where the source vertex of the first edge is u and the target vertex of the last edge is v.

We have verified that each of these four properties holds if and only if the corresponding checker function accepts. The three functions check_no_path, check_trian, and check_just need additional annotations before VCC can verify their correctness. The loops in these functions have to be annotated with loop invariants that are, just as in the previous case study (Section 4.1.2), only simple variants of the postconditions above. Also, as for the connected-components checker, we need to explicitly state properties that are violated before every **return** FALSE statement. Such properties are reformulations of concrete properties on the abstract level. In addition, both check_trian and check_just require the graph under consideration to be wellformed, and check_just, furthermore, requires that num and dist are consistent (the postcondition of check_no_path). We add these requirements as preconditions to the checker functions.

In order to be able to express the postcondition of the shortest-path checker, we define sequences of edges as a recursive datatype:

```
_(datatype \Path
{
  case none();
  case path(\Edge_Id i, \Path p);
})
```

Only particular instances of this datatype are paths in the given graph G. To qualify valid paths, we proceed in two steps. We first define a predicate that expresses the conditions under which a sequence of edges constitutes a walk in graph G from vertex u to vertex v (Listing 6). Second, we define a predicate to describe when the set of vertices of an edge sequence is distinct (Listing 7). A path from vertex u to vertex v in G is a walk p from u to v with distinct vertices. We define this as a predicate \is_path(G, p, u, v).

With a recursive function \path_cost that computes for a given path its length using the cost mapping, we can finally state the postcondition of the shortest path checker:

```
(∀ \Vertex v; v < G.num_verts ⟶
  ¬\is_enat_inf(dist[v]) ⟷ (∃ \Path p; \is_path(G, p, s, v))) ∧
(∀ \Vertex v; v < G.num_verts ∧ ¬\is_enat_inf(dist[v]) ⟶
  (∀ \Path p; \is_path(G, p, s, v) ⟶ \enat_val_of(dist[v]) ≤ \path_cost(cost, p)) ∧
  (∃ \Path p; \is_path(G, p, s, v) ∧ \enat_val_of(dist[v]) = \path_cost(cost, p)))
```

```
_(def \bool \occurs(\Graph G, \Vertex u, \Vertex v, \Path p)
   _(decreases \size(p))
{
   switch (p)
   {
      case none(): return u = v;
      case path(i, q): return u = G.edge[i].src ∨ \occurs(G, u, G.edge[i].trg, q);
   }
})

_(def \bool \distinct_verts(\Graph G, \Path p)
{
   switch (p)
   {
      case none(): return \true;
      case path(i, q): return ¬\occurs(G, G.edge[i].src, G.edge[i].trg, q) ∧ \distinct_verts(G, q);
   }
})
```

Listing 7: Predicate \distinct_verts(G, p) holds if the set of vertices connected by path p is distinct. Predicate \occurs(G, u, v, p) is true if and only if u is either equal to v or equal to any vertex touched by path p.

We formally prove this property, under the assumption of the precondition and the witness predicate, in Isabelle below.

4.2.3 Proof of the Witness Property for the Shortest-Path Checker

We present here the outline of the Isabelle/HOL proof of the witness property, namely that if G is wellformed (the precondition of this case study) and if the witness predicate holds, then $dist(v)$ is indeed the shortest-path distance from s to v for all vertices $v \in V$. For the full formal proof, please check the companion website.

Listing 8 shows our Isabelle locales. We separate the assumptions into three locales to avoid the use of unneeded assumptions when proving intermediate lemmas. This makes the intermediate lemmas more general, and hence, usable in other contexts[5]. The first locale *basic-sp* subsumes the locale *pseudo-digraph* mentioned in Section 4.1.3. Moreover, it assumes it is given the function $dist : V \to (\mathbb{R} \cup \{\infty, -\infty\})$, an edge cost function $c : E \to \mathbb{R}$, and a start vertex s.

Let μ be a function that takes a cost function c on edges and two vertices s and v and returns the cost of a shortest path from s to v in G using the cost function c. We split the proof of the witness property into two parts. First, we prove a lemma *dist-le-μ* using the locale *basic-sp*. The lemma states that $dist\ v \leq \mu\ c\ s\ v$ for every vertex $v \in V$. Then, we prove the lemma *dist-ge-μ* using the locale *basic-just-sp*. The lemma states that $dist\ v \geq \mu\ c\ s\ v$ for every vertex $v \in V$ under some extra assumptions (Listing 9). Later, we show that these extra assumptions

[5] For example, we used some of those lemmas also for the verification of a checker for the shortest-path problem with general edge weights (not only nonnegative edge weights as in this case study) [33].

```
locale basic-sp = pseudo-digraph +
  fixes dist :: α ⇒ ereal
  fixes c :: β ⇒ real
  fixes s :: α
  assumes general_source_val: dist s ≤ 0
  assumes trian: ⋀e. e ∈ edges G ⟹ dist (target G e) ≤ dist (start G e) + c e

locale basic-just-sp = basic-sp +
  fixes num :: α ⇒ enat
  assumes just:
    ⋀v. v ∈ verts G ⟹ v ≠ s ⟹ num v ≠ ∞ ⟹
      ∃ e ∈ edges G.
        v = target G e ∧
        dist v = dist (start G e) + c e  ∧
        num v = num (start G e) + enat 1

locale shortest-path-pos-cost = basic-just-sp +
  assumes s-in-G: s ∈ verts G
  assumes start_val: dist s = 0
  assumes no-path: ⋀v. v ∈ verts G ⟹ (dist v = ∞ ⟷ num v = ∞)
  assumes pos-cost: ⋀e. e ∈ edges G ⟹ 0 ≤ c e
```

Listing 8: Locales for the shortest-paths proof in Isabelle.

```
lemma (in basic-just-sp) dist-ge-μ:
  fixes v :: α
  assumes v ∈ verts G
  assumes num v ≠ ∞
  assumes dist v ≠ −∞
  assumes μ c s s = ereal 0
  assumes dist s = 0
  assumes ⋀u. u ∈ verts G ⟹ u ≠ s ⟹ num u ≠ ∞ ⟹ num u ≠ enat 0
  shows dist v ≥ μ c s v
```

Listing 9: The central lemma of the shortest-paths proof in Isabelle

hold in the locale *shortest-path-pos-cost*. Hence, we obtain a theorem stating that $dist\ v = \mu\ c\ s\ v$ for every $v \in V$ using the locale *shortest-path-pos-cost*.

Linking this Isabelle proof with the specification exported from VCC is a matter of translating from one representation to another. We intentionally chose to define paths and their costs in VCC (Section 4.2.2) similar to the way they are defined in the Isabelle graph library to ease our translation proofs. Since there are several more concepts to relate than in the previous checker (Section 4.1.3), our proofs for the shortest-path checker are more tedious. Nevertheless, no complex reasoning is required. We establish that the assumptions of the *shortest-path-pos-cost* locale are implied by the checker precondition and witness predicate, and we prove that our final theorem proved in that locale implies the checker postcondition.

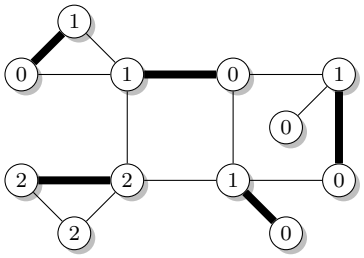


Fig. 4: The vertex labels certify that the indicated matching is of maximum cardinality: All edges of the graph have either both endpoints labeled as 2 or at least one endpoint labeled as 1. Any matching can hence use at most one edge with both endpoints labeled 2 and at most four edges that have an endpoint labeled 1. Therefore, no matching has more than five edges. The matching shown consists of five edges (in bold).

4.3 Maximum Cardinality Matching in Graphs

Our third case study is about maximum cardinality matching in general graphs. A *matching* in a graph G is a subset M of the edges of G such that no two share an endpoint. A matching has maximum cardinality if its cardinality is at least as large as that of any other matching. Figure 4 shows a graph, a maximum cardinality matching, and a witness of this fact. An *odd-set cover* L of a graph G is a labeling of the vertices of G with integers such that every edge of G is either incident to a vertex labeled 1 or connects two vertices labeled with the same number i and $i \geq 2$.

Theorem 2 (Edmonds [16]) *Let M be a matching in a graph G, and let L be an odd-set cover of G. For any $i \geq 0$, let n_i be the number of vertices labeled i. If*

$$|M| = n_1 + \sum_{i \geq 2} \lfloor n_i/2 \rfloor, \tag{2}$$

then M is a maximum cardinality matching.

Proof Let N be any matching in G. For $i \geq 2$, let N_i be the edges in N that connect two vertices labeled i, and let N_1 be the remaining edges in N. Then, by the definition of odd-set cover, every edge in N_1 is incident to a vertex labeled 1. Since edges in a matching do not share endpoints, we have

$$|N_1| \leq n_1 \quad \text{and} \quad |N_i| \leq \lfloor n_i/2 \rfloor \quad \text{for } i \geq 2.$$

Thus, $|N| \leq n_1 + \sum_{i \geq 2} \lfloor n_i/2 \rfloor = |M|$. □

For every maximum cardinality matching M there is an odd-set cover L satisfying equality (2); the proof of this is nontrivial and of no importance for the purpose of this paper. The cover uses nonnegative vertex labels in the range 0 to $|V| - 1$ and all n_i's with $i \geq 2$ are odd. The *certifying algorithm for maximum cardinality matching* in LEDA returns a matching M and an odd-set cover L such that (2) holds. The relationship to Section 2 is as follows:

input x	=	an undirected graph G
output y	=	a set of edges M
witness w	=	a vertex labeling L
$\varphi(x)$	=	G and M are wellformed and have no self-loops
$\mathcal{W}(x, y, w)$	=	M is a matching in G, L is an odd-set cover for G, and Equation (2) holds
$\psi(x, y)$	=	M is a maximum cardinality matching in G.

Theorem 2 is the witness property. We give a formal proof for it in Section 4.3.3. It is easy to write a correct program that checks whether a set of edges is a matching and a vertex labeling is an odd-set cover, which together satisfy Equation 2, see Section 4.3.1. In Section 4.3.2, we describe the verification of such a checker. The relevant files in the companion website are check_matching.c (C-code and checker correctness), check_matching.thy (Isabelle/HOL representation of the VCC checker specification and witness property), Matching.thy (abstract verification of the witness property in a locale), and Matching_Link.thy (instantiation of the locale with the Isabelle/HOL representation of the VCC specification).

This case study is a modified version of the one we present in [1]. The proof of the witness property is very similar to the one published in [32] except that it uses the Isabelle graph library, which has been developed meanwhile.

4.3.1 Maximum-Cardinality-Matching Checker

We build the checker using the graph data structure as in the previous case studies (Listing 1). We assume that graphs are wellformed and have neither self-loops nor duplicate edges. We treat the edges of a graph as undirected edges. Matchings are also represented by graphs. We require an additional witness in the form of an array f that maps edge identifiers of the matching to edge identifiers of the input graph. For instance, if a graph consists of three edges (identified as 0, 1 and 2) and the computed matching consists of the third edge (i.e., 2), then f would be an array with a single element 2 indicating how the only edge of the matching corresponds to the edges of the input graph. Finally, the vertex labeling is represented by an array osc, which is indexed by vertices and stores elements of type Nat. The checker function requires an auxiliary array check that can store as many elements of type Nat as there are vertices in the input graph, but at least two. We expect that this array is allocated elsewhere and given as input to the checker.

In addition to the checker function, there are four helper functions (Listing 10). The checker accepts if the first three of them accept and if the fourth function returns a value that is equal to the number of edges of the matching M. In short, the helper functions perform the following tasks. The function check_subset checks whether M is a subgraph of G with respect to the mapping f. The function check_matching checks that M is indeed a matching (contains no two edges that are incident). The function check_osc checks whether the vertex labeling is an odd-set cover and that vertex labels are in the range $\{0, \ldots, \mathsf{G}{\to}\mathsf{n} - 1\}$. Finally, the function weight computes the sum on the right-hand side of Equation (2). This computation is optimized by first searching for the greatest vertex label, which can be considerably smaller than the maximal $\mathsf{G}{\to}\mathsf{n} - 1$, and then summing up partial sums only until this greatest label. The main checker function passes the auxiliary array check to check_matching as the degree_in_M argument and to weight as the count argument.

```
bool check_subset(Graph* G, Graph* M, Nat* f)
{
   Edge_Id e;

   for (e = 0; e < M→m; e++)
   {
      if (f[e] ≥ G→m) return FALSE;
      if (M→es[e].s == G→es[f[e]].s && M→es[e].t == G→es[f[e]].t) continue;
      if (M→es[e].s == G→es[f[e]].t && M→es[e].t == G→es[f[e]].s) continue;
      return FALSE;
   }
   return TRUE;
}

bool check_matching(Graph* M, Nat* degree_in_M)
{
   Vertex v; Edge_Id e;

   for (v = 0; v < M→n; v++) degree_in_M[v] = 0;
   for (e = 0; e < M→m; e++)
   {
      if (degree_in_M[M→es[e].s] == 1 || degree_in_M[M→es[e].t] == 1) return FALSE;
      degree_in_M[M→es[e].s] = 1;
      degree_in_M[M→es[e].t] = 1;
   }
   return TRUE;
}

bool check_osc(Graph* G, Nat* osc)
{
   Edge_Id e; Vertex v, w;

   for (v = 0; v < G→n; v++) if (osc[v] ≥ G→n) return FALSE;
   for (e = 0; e < G→m; e++)
   {
      v = G→es[e].s;
      w = G→es[e].t;
      if (osc[v] == 1 || osc[w] == 1 || (osc[v] == osc[w] && osc[v] ≥ 2)) continue;
      return FALSE;
   }
   return TRUE;
}

Nat weight(Graph* G, Nat* osc, Nat* count)
{
   Vertex v; Nat c, s, max = 1, r = (G→n > 2) ? G→n : 2;

   for (c = 0; c < r; c++) count[c] = 0;
   for (v = 0; v < G→n; v++)
   {
      count[osc[v]] = count[osc[v]] + 1;
      if (osc[v] > max) max = osc[v];
   }
   s = count[1];
   for (c = 2; c < max + 1; c++) s += count[c] / 2;
   return s;
}
```

Listing 10: Helper functions of the maximum-cardinality-matching checker.

4.3.2 Checker Correctness

We build on the abstract graph data structure of Listing 3. We require that graphs are wellformed and contain no self-loops:

```
∀ \Edge_Id i; i < G.num_edges ⟶ G.edge[i].src ≠ G.edge[i].trg
```

nor duplicate edges:

```
∀ \Edge_Id i1, i2; i1 < G.num_edges ∧ i2 < G.num_edges ∧ i1 ≠ i2 ⟶
  G.edge[i1].src ≠ G.edge[i2].src ∨ G.edge[i1].trg ≠ G.edge[i2].trg
```

An abstract vertex labeling L is a mapping from vertices to natural numbers. The mapping f from edge identifiers to edge identifiers has a straightforward representation as an abstract mapping. We omit here, as in the previous case studies, the description of abstraction functions from concrete to abstract values.

The witness predicate is a conjunction of four predicates, each related to one of the helper functions in Listing 10.

check_subset: M must be a subgraph of G w.r.t. the edge mapping f, i.e., every edge of M must also be an edge of G modulo symmetry of edges:

```
∀ \Edge_Id i; i < M.num_edges ⟶
  f[i] < G.num_edges ∧
  (M.edge[i].src = G.edge[f[i]].src ∧ M.edge[i].trg = G.edge[f[i]].trg ∨
   M.edge[i].src = G.edge[f[i]].trg ∧ M.edge[i].trg = G.edge[f[i]].src)
```

check_matching: M must be a matching, i.e., no two edges of M have a vertex in common:

```
∀ \Edge_Id i1, i2;
  i1 < M.num_edges ∧ i2 < M.num_edges ∧ i1 ≠ i2 ⟶
  M.edge[i1].src ≠ M.edge[i2].src ∧ M.edge[i1].src ≠ M.edge[i2].trg ∧
  M.edge[i1].trg ≠ M.edge[i2].src ∧ M.edge[i1].trg ≠ M.edge[i2].trg
```

check_osc: L must be an odd-set cover of G, i.e., for every edge of G, one of the edge's vertices is labeled 1 or both vertices are labeled by the same number greater than or equal to 2:

```
∀ \Edge_Id i; i < G.num_edges ⟶
  L[G.edge[i].src] = 1 ∨
  L[G.edge[i].trg] = 1 ∨
  L[G.edge[i].src] = L[G.edge[i].trg] ∧ L[G.edge[i].src] ≥ 2
```

weight: Equation (2) must hold. We define it stepwise. The number of vertices labeled with c is defined recursively:

```
_(def \natural \label_count(\Label L, \natural c, \natural i)
{
  return (i = 0) ? 0 : ((L[i − 1] = c) ? 1 : 0) + \label_count(L, c, i − 1);
})
```

We have n_c = \label_count(L, c, G.num_verts) for a vertex label c. The sum of these numbers for labels greater than 1 is again defined recursively:

```
_(def \natural \rec_weight(\Label L, \natural n, \natural i)
{
  return (i < 2) ? 0 : \label_count(L, i, n) / 2 + \rec_weight(L, n, i − 1);
})
```

We have $\sum_{i\geq 2}\lfloor n_i/2 \rfloor$ = \rec_weight(L, G.num_verts, m) where m is the greatest label assigned to any vertex by L. The complete sum is then:

```
_(def \natural \full_weight(\Label L, \natural n, \natural i)
{
  return \label_count(L, 1, n) + \rec_weight(L, n, i);
})
```

That is, we have $n_1 + \sum_{i\geq 2}\lfloor n_i/2 \rfloor$ = \full_weight(L, G.num_verts, m) with the same m as before. Finally, the predicate capturing Equation (2) is as follows:

```
M.num_edges = \full_weight(L, G.num_verts, m) ∧
∀ \Vertex v; v < G.num_verts ⟶ L[v] ≤ m
```

Verifying the correctness of the checker (Section 4.3.1) is done in the same way as the earlier case studies for the first three predicates above. We only have to provide the right loop invariants, and simple variations of the predicates to be proved are sufficient. In check_matching, we need additional loop invariants. Along with the first loop, we accumulate the knowledge about the initialization of the degree_in_M array by specifying that the first positions of the array have already been set to 0:

```
∀ Nat u; u < v ⟶ degree_in_M[u] = 0
```

Moreover, on the second loop, we need three additional loop invariants. One invariant states that values stored in degree_in_M are in range:

```
∀ Nat v; v < M→n ⟶ degree_in_M[v] ≤ 1
```

Another invariant states that vertices, for which degree_in_M is still 0, cannot be part of any already checked edge:

```
∀ Nat v; v < M→n ∧ degree_in_M[v] = 0 ⟶
  ∀ Nat e1; e1 < e ⟶ M→es[e1].s ≠ v ∧ M→es[e1].t ≠ v
```

Finally, vertices for which degree_in_M has already been set to 1 are mapped by a ghost mapping E to their adjacent edge in the matching M:

```
∀ Vertex v; v < M→n ∧ degree_in_M[v] = 1 ⟶
  E[v] < e ∧ (M→es[E[v]].s = v ∨ M→es[E[v]].t = v)
```

This invariant is required to prove completeness. We maintain this invariant by updating the ghost mapping E in the loop body accordingly.

Proving the weight function correct is the most intricate part of the checker verification. There are two properties that need to be shown: functional correctness and the absence of overflows. Functional correctness requires that the function computes the n_i and the overall sum of Equation (2) correctly, as specified by the weight predicate above. Absence of overflows requires that the additions in both the second and third loop do not overflow. Surprisingly, the absence of overflows is much harder to establish than functional correctness.

We concentrate first on functional correctness. The second loop updates the count array in a way that maintains the following property:

```
∀ Nat j; j < r ⟶ count[j] = \label_count(L, j, v)
```

From this property follows this loop invariant on the third loop:

```
s = \full_weight(L, G→n, c − 1))
```

Together with a further loop invariant for the second loop to guarantee that max is the greatest label seen so far, we can conclude that the weight function is functionally correct.

The addition in the second loop can never overflow because in each loop iteration, the loop variable is an upper limit on the value count[i] for each label i. Concerning the addition in the third loop, we observe that in each loop iteration, the value of s is bounded by the number of vertices in G. To establish this property, we build up a ghost map sum in the second loop in such a way that in every iteration of that loop, this map fulfills the following invariant:

```
sum[1] = count[1] ∧
(∀ Nat j; 1 < j ∧ j < r ⟶ sum[j] = sum[j − 1] + count[j]) ∧
(∀ Nat j; 1 < j ∧ j < r ⟶ sum[j] ≤ v)
```

Maintaining this invariant requires updating the sum map during each iteration of the second loop. We do so in a nested ghost loop in which we propagate the increment that happened on the count array to every possibly affected element sum[j].

The postcondition of the checker expresses that the cardinality of any matching of G cannot be smaller than the cardinality of M:

```
∀ \Graph M2; \Edge_Map I2; \is_subset(G, M2, I2) ∧ \is_matching(M2) ⟶
  M2.num_edges ≤ M.num_edges
```

We give a formal proof that the checker preconditions and the witness predicate imply this property in the following section.

4.3.3 Proof of the Witness Property for the Maximum-Cardinality-Matching Checker

We explain the Isabelle proof for the witness property, i.e., for Theorem 2. See Listing 11 for an excerpt of our formal Isabelle proof development that can be found in file Matching.thy. The formal proof follows the scheme of the textbook proof and is split into two main parts.

For $i \geq 2$, let M_i be the edges in M that connect two vertices labeled i, and let M_1 be the remaining edges in M. The sets M_i, $i \geq 1$, are disjoint. We use the definition of an odd-set cover to prove $M \subseteq \bigcup_{i\geq 1} M_i$, and thus $|M| \leq \sum_{i\geq 1} |M_i|$ by disjointness of the sets M_i. Let V_i be the vertices labeled i, and let $n_i = |V_i|$. We formally prove: $|M_1| \leq n_1$ and $|M_i| \leq \lfloor n_i/2 \rfloor$.

In order to prove $|M_1| \leq n_1$, we exhibit an injective function from M_1 to V_1. We first prove, using the definition of an odd-set cover, that every edge $e \in M_1$ has at least one endpoint in V_1. This gives rise to a function $endpoint_{V_1}$ that maps from M_1 to V_1. We then use that edges in a matching do not share endpoints (i.e., edges in a matching are disjoint when interpreted as sets) to conclude that $endpoint_{V_1}$ is injective. This establishes $|M_1| \leq |V_i|$.

For $i \geq 2$, the proof of the inequality $|M_i| \leq \lfloor n_i/2 \rfloor$ is similar but more involved. M_i is a set of edges. If we represent edges as sets, each with cardinality two, then M_i is a collection of sets. We define the set of vertices V_i' to be $\bigcup_{i\geq 2} M_i$ and use the definition of an odd-set cover to prove $V_i' \subseteq V_i$. Since the edges in a matching are pairwise disjoint, we obtain $\left|V_i'\right| = 2 \cdot |M_i|$. Note also that $\left|V_i'\right|$ must be even since $|M_i|$ is a natural number. Thus, we can prove that $|M_i| \leq \left\lfloor \left|V_i'\right| / 2 \right\rfloor$, and hence, $|M_i| \leq \left\lfloor \left|V_i'\right| / 2 \right\rfloor \leq \lfloor |V_i| / 2 \rfloor = \lfloor n_i/2 \rfloor$.

```
type_synonym label = nat

definition disjoint-edges :: (α, β) pre-graph ⇒ β ⇒ β ⇒ bool where
  disjoint-edges G e1 e2 = (
    start G e1 ≠ start G e2 ∧ start G e1 ≠ target G e2 ∧
    target G e1 ≠ start G e2 ∧ target G e1 ≠ target G e2)

definition matching :: (α, β) pre-graph ⇒ β set ⇒ bool where
  matching G M = (
    M ⊆ edges G ∧
    (∀e1 ∈ M. ∀e2 ∈ M. e1 ≠ e2 ⟶ disjoint-edges G e1 e2))

definition OSC :: (α, β) pre-graph ⇒ (α ⇒ label) ⇒ bool where
  OSC G L = (
    ∀e ∈ edges G.
      L (start G e) = 1 ∨ L (target G e) = 1 ∨
      L (start G e) = L (target G e) ∧ L (start G e) ≥ 2)

definition weight :: label set ⇒ (label ⇒ nat) ⇒ nat where
  weight LV f = f 1 + ∑i ∈ LV. (f i) div 2

definition N :: α set ⇒ (α ⇒ label) ⇒ label ⇒ nat where
  N V L i = card {v ∈ V. L v = i}

locale matching-locale = digraph +
  fixes maxM :: β set
  fixes L :: α ⇒ label
  assumes matching: matching G maxM
  assumes OSC: OSC G L
  assumes weight: card maxM = weight {i ∈ L ‘ verts G. i > 1} (N (verts G) L)
```

Listing 11: Definitions and locale for the matching proof in Isabelle.

Instantiating this Isabelle proof for the data structures and properties exported from VCC is fairly straightforward since both formalizations have been chosen intentionally close to each other. We prove by induction that $N\ \{0\ ..<\ n\}\ L\ l$ equals \label_count(L, l, n) for every label l. Moreover, we prove by induction that *weight* $\{2\ ..\ k\}\ f$ equals \full_weight(L, n, k) if $f\ l$ and \label_count(L, l, n) coincide. After showing that $|M|$ equals M.num_edges, we can establish the witness property for the matching checker.

5 Evaluation

We have shown that our methodology allows us to lift the trustworthiness of certifying algorithms to a new level: one can obtain formal instance correctness of certifying algorithms. A certifying algorithm returns for any input $\overline{x}$ satisfying the precondition φ a pair $(\overline{y}, \overline{w})$. It is accompanied by a witness predicate $\mathcal{W}$ and an associated checker program C. If $\overline{x}$ satisfies the precondition, C is supposed to halt on input $(\overline{x}, \overline{y}, \overline{w})$ and decide on the witness predicate. We have shown in three cases that checker correctness can be established formally using VCC. Precondition plus witness predicate are supposed to imply the postcondition ψ. We have shown in the same three cases that the witness property can be established formally using

Isabelle/HOL. In all cases, we followed the same approach: (1) write the checker in C and annotate it so that VCC can establish that the checker decides the witness property, (2) translate precondition, witness predicate, and postcondition from VCC to Isabelle/HOL, and (3) prove the witness property in Isabelle/HOL and add the VCC-version of it as an axiom to VCC. We translated manually from VCC to Isabelle/HOL. Since the translation is purely syntactical, it can be automated.

We believe that our approach would work for all certifying algorithms discussed in the survey on certifying algorithms [23]. This might require formalizing the subject matter in Isabelle/HOL first. Our work profited from the recent formalization of graphs in Isabelle/HOL [30]. A similar effort would be necessary before geometric or randomized algorithms can be tackled.

It is always a gratifying experience to see that a formalization reveals gaps in paper and pen reasoning. In our case, we found that the checker for a maximum cardinality matching in LEDA does not verify that the computed matching M is a subgraph of G.

The matching algorithm for general graphs and its efficient implementation are advanced topics in graph algorithms. The algorithm is highly complex and is not covered in the standard textbooks on algorithms. The following page numbers illustrate the complexity gap between the original algorithm and the checker: In the LEDA book, the description of the algorithm for computing the maximum cardinality matching and the proof of its correctness fills about 15 pages, compared to a one-page description of the checker implementation.

All described theorems and lemmas have been formally verified using VCC and Isabelle/HOL. Table 1 summarizes our effort in lines of code, lines of annotations, and lines of Isabelle specifications and proofs. We observe an average ratio of 2.5 for the VCC annotation overhead, which includes our specifications for the witness predicates besides annotations for code verification. The overhead grows to an average ratio of 6.7 when also considering our Isabelle proofs. We believe that this overhead is a small price to pay for verifying full functional correctness of code with nontrivial mathematical properties. Observe that the Isabelle linking proofs are much shorter than the high-level proofs in the Isabelle locales for the two more complex checkers. Linking proofs, hence, cause only a small overhead and, as we found, are fairly straightforward. In our opinion, splitting the proofs into high-level proofs for mathematical concepts and lower-level linking proofs improved productivity and certainly helped to reduce the overall proof size and effort. The overall VCC proof time for all checkers is 10 seconds on a 2.9 GHz Intel Core i7 machine. The overall Isabelle proof time is also well below one minute on the same machine.

It took several months to develop the framework and complete the first example as described in [1]. For this paper, we have reworked the framework, thereby strengthening and simplifying it at the same time. We now prove total correctness of the checkers and not only partial correctness. Moreover, we now establish that the checker accepts a triple (x, y, w) if and only if w is a valid witness for output y, whereas previously, we only proved one direction. The simplification results from dropping concrete specifications in VCC. Moreover, a new version of VCC has allowed for shorter specifications and proofs.

Our methodology is not confined to verifying certifying computations only. Rather, any proposition that VCC fails to establish can be exported to Isabelle and proved there; see [7, Section 4.5] for an example. Hence, our methodology

		VCC		Isabelle	
	C code	Annotations	Exports	Locales	Link
Connected components	61	162	63	102	163
Shortest paths	107	318	109	279	198
Matching	124	263	78	318	164

Table 1: Lines of C code, annotations for VCC, and Isabelle declarations and proofs, excluding empty lines in all cases. For Isabelle, we give the numbers for the specifications exported from VCC, abstract proofs performed within locales, and proofs that link the concrete representation exported from VCC with the abstract proofs separately.

extends the applicability of VCC and allows, in conjunction with Isabelle, the verification of complicated problems that were infeasible up to now.

6 Related Work

The notion of a certifying algorithm is ancient. Al-Khawarizmi already described how to (partially) check the correctness of a multiplication in his book on algebra. The extended Euclidean algorithm for greatest common divisors is also certifying; it dates back to the 17th century. Yet, formal verification of checkers is recent.

In 1997, Bright et al. [11] verified a checker for a sorting algorithm that has been formalized in the Boyer-Moore theorem prover [10]. De Nivelle and Piskac formally verified the checker for priority queues implemented in LEDA [27]. Bulwahn et al. [12] describe a verified SAT checker, i.e., a checker for certificates of unsatisfiability produced by a SAT solver. They develop the checker and prove its correctness within Isabelle/HOL. Similar proof checkers have been formalized in the Coq [5] proof assistant [15,2]. CeTA [37], a tool for certified termination analysis, is also based on formally verified checkers. In contrast to our approach, all mentioned checkers are entirely developed and verified within the language of a theorem prover.

VCC has been applied to verify tens of thousands of lines of C code. So far, the majority of its verification targets have been restricted to system-level code from the domain of microkernels and hypervisors [4,21,35]. Our work extends the range of VCC applications to graph algorithms and, in general, to any code that requires nontrivial mathematical reasoning to establish full functional correctness.

In contrast to using an automatic program verifier such as VCC as we did, imperative code can also be verified in interactive theorem provers such as Coq, HOL [19], or Isabelle/HOL. This requires a formalization of the imperative language and its semantics within the theorem prover. Charguéraud describes a tool, CFML, that is based on the idea of extracting characteristic formulae from programs [13]. CFML is embedded in Coq and targets imperative Caml programs. It has been applied to verify imperative data structures such as mutable lists, sparse arrays and union-find. Norrish presented a formal semantics of C formalized in the HOL theorem prover [29]. Parallel to this work, a subset of C, called C0, was formalized in Isabelle/HOL [22]. Schirmer developed a verification environment

for sequential imperative programs within Isabelle and embedded C0 into this environment [34]; his verification environment is written in the generic imperative programming language Simpl. Schirmer's work has been applied, for instance, to verify a compiler for C0 [31]. More recently, the seL4 microkernel, written in C, has been verified in Isabelle/HOL [21] based on the work of Norrish and Schirmer. The reverse process of turning C code into a human readable abstraction can partly be automated [20]. The latter work opens up an alternative path for the verification of certifying computations. One writes the checker in C, uses the trusted translation to obtain an equivalent Simpl program, and then proves checker correctness and the witness property in Isabelle/HOL. We plan to also try this alternative route. There are advantages and disadvantages for both approaches, and it is too early to claim the superiority of one approach over the other with regards to verification effort. An advantage of our current approach is that we annotate the source program and that VCC is an efficient tool to prove that the checker decides on the witness property. An advantage of the alternative approach is that one has to trust less components in the verification process. The trusted components in this case are the parser that translates from C to Isabelle and Isabelle's inference kernel (which is based on the LCF principle). In our current approach one has to trust VCC's translation from VCC to logic, and more importantly one has to trust both the prover Z3 and the VCC engine neither of which are based on the LCF principle.

Since checkers are fairly simple programs that are not performance critical, code generation is a viable alternative to our approach. However, generating C programs from theorem provers is still beyond the state of the art.

Previous work that proposes, as we do, the use of interactive theorem provers as backends to code verification systems comprises, for instance, the link between Boogie and Isabelle/HOL [9] and the link between Why and Coq [17]. Both systems have a C verifier frontend. Such approaches for connecting code verifiers and proof assistants usually give proof assistants the same information that is made available to the first-order engine, overwhelming the users of the proof assistants with a mass of detail. Instead, we allow only clean chunks of mathematics to move between the verifier and the proof assistant. This hides details of the underlying programming languages from the proof assistant, thus requiring the user to discharge only interesting proof obligations.

Shortest-path algorithms, especially imperative implementations thereof, are popular as case studies for demonstrating code verification [13,28,8]. They target full functional correctness as opposed to instance correctness. Verifying instance correctness is orthogonal to verifying the implementation of a particular shortest path algorithm. Our work is directly applicable to any implementation of shortest-path that is instrumented to provide the necessary witness expected by our checker.

To our knowledge, there has been no other attempt to verify algorithms or checkers for connected components or maximum cardinality matchings.

7 Conclusion and Future Work

We have described a framework for the verification of certifying computations and applied it to three nontrivial combinatorial problems: connectivity of graphs,

shortest paths in graphs, and maximum cardinality matchings in graphs. Our work greatly increases the trustworthiness of certifying algorithms.

Specifically, for each instance of the considered three problems, we can now give a formal proof of the correctness of the result. Thus, the user has neither to trust the implementation of the original algorithm nor the checker, nor does he have to understand why the witness property holds. We stress that we did not prove the correctness of the original programs but only verified the results of their computations.

Our methodology can be applied to any other problem for which a certifying algorithm is known; see [23] for a survey. The fourth author is currently verifying a checker for shortest paths with arbitrary edge costs [33], and Lars Noschinski is verifying a checker for non-planarity testing of graphs.

Our methodology is not restricted to verifying certifying computations. The integration of VCC and Isabelle/HOL should be useful whenever verification of a program requires nontrivial mathematical reasoning.

Acknowledgements We thank Ernie Cohen for his advice on VCC idioms, Mark Hillebrand for insightful discussions on VCC, and Norbert Schirmer for his initial Isabelle support. Moreover, we thank Lars Noschinski for developing a powerful graph library in Isabelle/HOL. We are grateful to Jasmin Christian Blanchette for providing useful comments on an earlier version of this article. We also thank the reviewers for their insightful comments.

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