Homework 1 of Numerical Analysis

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Theoretical questions

Problem I

Solution. The width of the interval at nth step (Because the definition of 'at' is not clear enough, here we consider the interval as the one BEFORE nth bisection *e.g.* When $n = 1$, the width is 2), denoted by I_n, is

$$I_n = \frac{3.5 - 1.5}{2^{n-1}} = 2^{2-n} \tag{1}$$

Furthermore, the maximum possible distance between root r and the midpoint of the interval (also refers to the interval BEFORE nth step) is $\frac{1}{2}I_n = 2^{1-n}$. □

Problem II

Solution. We first define *relative error* as the DISTANCE between root r and one of the end point of the intervals where it is located DIVIDED BY r.

By definition, suppose c_n is one of the interval end point of r after n steps, then we should have

$$\frac{|r - c_n|}{r} \leq \epsilon \quad \Longleftarrow \quad \frac{b_0 - a_0}{2^n a_0} \leq \epsilon \tag{2}$$

In the transformation, we make use of $|r - c_n| \leq I_{n+1} = (b_0 - a_0)/2^n$ and $a_0 \leq r$. Separate variable n to one side of the inequality, then we have

$$n \geq \log_2 \frac{b_0 - a_0}{\epsilon a_0} \tag{3}$$

thus finishing the proof. □

Problem III

Solution. we use $p(x) = 4x^3 - 2x^2 + 3$, $p'(x) = 12x^2 - 4x$ and $Newton's\ Method$

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)}, \quad x_0 = -1 \tag{4}$$

for iteration. The results are shown in the table. Noted that infinite decimals is accurate to fifth decimal places.

n	x_n	$p(x_n)$	$p'(x_n)$
0	-1	-3	16
1	-0.8125	-0.46582	11.17188
2	-0.77080	-0.02014	10.21289
3	-0.76883	-0.00004	10.16859
4	-0.76883	-0.00000	10.16847

□

Problem IV

Solution. It should be noted that C must depend on the root r. A strict proof has given to assistant Hu Shuang through Dingding, so here we only give a simple explanation.

Suppose C only depend on x_n, f and its finite order (otherwise the problem is meaningless) derivatives. Then considering $f(x) = e^x - 1$ with root $r = 0$, for the part of the function to the left of x_n ($x_n > 0$), we can at least construct countable functions (using Taylor expansion of f at $x = x_n$) which have the same value (C depends) as $e^x - 1$. Therefore, no matter what these functions on the left of x_n look like, they have the same C and s. However, obviously their roots are different, thus producing contradictions.

In fact, under the condition that C also depends on the root r, s can be any nonzero constant. Suppose $s = s_0$ ($s_0 \neq 0$), by iteration formula we have

$$x_{n+1} - r = x_n - r - \frac{f(x_n)}{f'(x_0)} \tag{5}$$

$$\Longrightarrow \quad e_{n+1} = e_n^{s_0}\left[(x_n - r)^{1-s_0} - (x_n - r)^{-s_0}\frac{f(x_n)}{f'(x_0)}\right] \tag{6}$$

Therefore, s can be ANY nonzero constant s_0, and the corresponding C is

$$C = (x_n - r)^{1-s_0} - (x_n - r)^{-s_0}\frac{f(x_n)}{f'(x_0)} \tag{7}$$

□

Problem V

Solution. The iteration converges. For the initial value x_0, we discuss it in three cases.

If $x_0 = 0$, it's obvious that $x_n = 0$ for any n and the iteration is convergent.

If $0 < x_0 < \frac{\pi}{2}$, firstly we can conclude $x_n > 0$ for any n by an easy induction. Secondly, according to a common inequality $x < tanx$ $(x > 0)$, we have

$$x_n = tanx_{n+1} > x_{n+1} \quad (n = 1, 2, \dots) \tag{8}$$

Therefore, the sequence x_n is monotonically decreasing with a lower bound 0. Then it converges by **Theorem 1.12**.

If $0 > x_0 > -\frac{\pi}{2}$, similarly we know the sequence $\{x_n\}$ is monotonically increasing with a upper bound 0. Then it converges by **Theorem 1.12**. $\square$

Problem VI

Solution. set $x_1 = \frac{1}{p}$, $x_2 = \frac{1}{p+\frac{1}{p}}, \dots$,and so forth. We first prove that the sequence $\{x_n\}$ converges.

As $p > 1$, it's not difficult to have $x_n > 0$ for any positive integer n. Then by the iteration $x_{n+1} = 1/(x_n + p)$, we set $f(x) = 1/(x + p)$ which satisfies

$$|f(x) - f(y)| = \frac{|x - y|}{(p + x)(p + y)} < \frac{1}{p^2}|x - y|, \quad \forall x, y > 0 \tag{9}$$

By **Definition 1.36** and **Theorem 1.38**, the sequence $\{x_n\}$ is convergent.

Now we can denote $x = lim_{x\to\infty} x_n$. Then by definition, x satisfies the equation

$$x = \frac{1}{p + x}, \quad x \geq 0 \tag{10}$$

Solve this equation we have the value of $x = \frac{-p+\sqrt{p^2+4}}{2}$. $\square$

Problem VII

Solution. If $a_0 < 0 < b_0$, then the inequality in Problem II is meaningless because $loga_0$ is undefined.

Additionally, using the previous symbols in Problem I and II, we redefine *relative error* as the DISTANCE between root r and one of the end point of the intervals where it is located DIVIDED BY

$|r|$. Then, after making use of $|r - c_n| \le I_{n+1} = (b_0 - a_0)/2^n$, similarly we have

$$\frac{|r - c_n|}{|r|} \le \epsilon \quad \Longleftarrow \quad n \ge \log_2 \frac{b_0 - a_0}{\epsilon|r|} \tag{11}$$

However, this inequality has to depend on r, because $|r|$ has no absolute relationship with a_0 and b_0. Besides, as $r \to 0$, according to the inequality, $n \to \infty$, which is meaningless. Hence we conclude that the relative error is NOT a appropriate measure. $\square$