Homework 2 of Numerical Analysis

刘陈若 3200104872
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Theoretical questions

Problem I

Solution. It's not difficult to have

$$p_1(f;x) = -\frac{1}{2}x + \frac{3}{2} \tag{1}$$

which passes through points $(1,1)$ and $(2,0.5)$. then we calculate

$$f(x) - p_1(f;x) = \frac{x^2-3x+2}{2x} = \frac{f^{''}(\xi(x))}{2}(x-1)(x-2) \quad \Longrightarrow \quad f^{''}(\xi(x)) = \frac{1}{x} \tag{2}$$

According to **Theorem 2.7**, noted that this is not a derivation of composite function. Instead we can get

$$\frac{2}{\xi^3(x)} = \frac{1}{x} \quad \Longrightarrow \quad \xi(x) = \sqrt[3]{2x} \tag{3}$$

Furthermore, obviously function $\xi(x) = \sqrt[3]{2x}$ is monotonic increasing, and $f^{''}(\xi(x)) = 1/x$ is monotonic decreasing in the interval $[1,2]$. Thus we have, in $[1,2]$,

$$max\ \xi(x) = \sqrt[3]{4} \qquad min\ \xi(x) = \sqrt[3]{2} \qquad max\ f^{''}(\xi(x)) = 1 \tag{4}$$

□

Problem II

Solution. As $f_i \geq 0$, by **Theorem 2.5**, there exists a unique polynomial $g \in \mathbb{P}_n$ such that $g(x_i) = f_i^{\frac{1}{2}}$ for $i = 0,1,\ldots,n$.

Set $p(x) = g^2(x)$, then it's easy to show that $p \in \mathbb{P}_{2n}^+$ as $g^2(x) \geq 0$. Additionally, $p(x_i) = g^2(x_i) = f_i$ for $i = 0,1,\ldots,n$. Therefore we successfully find the proper p. □

Problem III

Solution. For $n = 1$, the equation reduces to $f[t, t+1] = (e-1)e^t$, which can be easily concluded as

$$f[t, t+1] = \frac{f[t+1] - f[t]}{t+1-t} = e^{t+1} - e^t = (e-1)e^t \tag{5}$$

Suppose the equation holds. For the inductive step, we have

$$\begin{aligned} f[t, t+1, \ldots, t+n+1] &= \frac{f[t+1, \ldots, t+1+n] - f[t, \ldots, t+n]}{n+1} \\ &= \frac{\frac{(e-1)^n}{n!}e^{t+1} - \frac{(e-1)^n}{n!}e^t}{n+1} \\ &= \frac{(e-1)^n}{(n+1)!}e^t(e-1) \\ &= \frac{(e-1)^{n+1}}{(n+1)!}e^t \end{aligned} \tag{6}$$

thus finishing the proof.

Furthermore, if $t = 0$, we have

$$f[0, 1, \ldots, n] = \frac{(e-1)^n}{n!} = \frac{1}{n!}f^{(n)}(\xi) \tag{7}$$

Similar to Problem I, that implies

$$e^{\xi(n)} = (e-1)^n \quad \Longrightarrow \quad \xi(n) = nln(e-1) \tag{8}$$

As $ln(e-1) > \frac{1}{2}$, $\xi > \frac{n}{2}$ respectively. □

Problem IV

Solution. By $Definition 2.18$, we can construct the following table divided difference,

0	5			
1	3	-2		
3	5	1	1	
4	12	7	2	0.25

By $Definition 2.14$, the interpolating polynomial is generated from the main diagonal and the first column of the above table as follows.

$$p_3(f;x) = 5 - 2x + x(x-1) + 0.25x(x-1)(x-3) \tag{9}$$

Furthermore, calculating the first-order derivative of p_3 we have

$$p_3^{'}(f;x) = 0.75x^2 - 2.25 \tag{10}$$

Then we can find its root in the interval $(1, 3)$, which is $\sqrt{3}$. Therefore we can estimate the location of the minimum as $x_{min} = \sqrt{3}$. □

Problem V

Solution. The Hermite interpolation problem can be expressed as

$$f_0 = 0, \quad f_1 = 1, \quad f_1' = 7, \quad f_1^{''} = 42, \quad f_2 = 128, \quad f_2' = 448 \tag{11}$$

And the table of divided difference has the form

0	0					
1	1	1				
1	1	7	6			
1	1	7	21	15		
2	128	127	120	99	42	
2	128	448	321	201	102	30

The table shows that $f[0, 1, 1, 1, 2, 2] = 30$.

Furthermore, we know that the divided difference is expressible in terms of $\frac{1}{5!}f^{(5)}(\xi)$. Therefore, by solving the equation

$$\frac{1}{5!}f^{(5)}(\xi) = 30 \tag{12}$$

we have $\xi = \sqrt{\frac{10}{7}}$. □

Problem VI

Solution. Similar to the question above, the table of the divided differences has the form

$$\begin{array}{c|ccccc}
0 & 1 & & & & \\
1 & 2 & 1 & & & \\
1 & 2 & -1 & -2 & & \\
3 & 0 & -1 & 0 & \frac{2}{3} & \\
3 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & -\frac{5}{36}
\end{array}$$

so we know from the table that

$$p_4(f;x) = 1 + x - 2x(x-1) + \frac{2}{3}x(x-1)^2 - \frac{5}{36}x(x-1)^2(x-3) \tag{13}$$

Set $x = 2$ in p_4, we estimate $f(2)$ as $p_4(f;2) = \frac{11}{18}$.

Furthermore, according to **Theorem 2.35**, we have

$$R_4(f;x) = \frac{f^{(5)}(\xi)}{5!}x(x-1)^2(x-3)^2 \tag{14}$$

So, using the condition $|f^{(5)}(x)| \leq M$ on $[0,3]$, we have

$$|R_4(f;2)| = |\frac{1}{60}f^{(5)}(\xi)| \leq \frac{M}{60} \tag{15}$$

Hence we estimate the maximum error of the above answer as $\frac{M}{60}$. □

Problem VII

Solution. We prove these by inductions.

For the first equation, when $k = 0$, obviously $LHS = RHS = f(x) = f(x_0)$ as $0! = 1$. Suppose the equation holds. For the inductive step, we have

$$\begin{aligned}
\Delta^{k+1}f(x) &= \Delta^k f(x+h) - \Delta^k f(x) \\
&= k!h^k(f[x_1, x_2, \ldots, x_{k+1}] - f[x_0, x_1, \ldots, x_k]) \\
&= k!h^k(x_{k+1} - x_0)f[x_0, x_1, \ldots, x_{k+1}] \\
&= (k+1)!h^{k+1}f[x_0, x_1, \ldots, x_{k+1}]
\end{aligned} \tag{16}$$

thus finishing the proof.

For the second equation, when $k = 0$, obviously $LHS = RHS = f(x) = f(x_0)$ as $0! = 1$. Suppose the equation holds. For the inductive step, we have

$$
\begin{aligned}
\nabla^{k+1} f(x) &= \nabla^k f(x) - \nabla^k f(x-h) \\
&= k!h^k (f[x_0, x_{-1}, \ldots, x_{-k}] - f[x_{-1}, x_{-2}, \ldots, x_{-k-1}]) \\
&= k!h^k (x_0 - x_{-k-1}) f[x_0, x_{-1}, \ldots, x_{-k-1}] \\
&= (k+1)!h^{k+1} f[x_0, x_{-1}, \ldots, x_{-k-1}]
\end{aligned}
\tag{17}
$$

thus finishing the proof. □

Problem VIII

Solution. Using the definition of partial derivative, the continuity of divided difference, **Corollary 2.15** and **Theorem 2.17**, we have

$$
\begin{aligned}
\frac{\partial}{\partial x_0} f[x_0, x_1, \ldots, x_n] &= \lim_{h \to 0} \frac{f[x_0 + h, x_1, \ldots, x_n] - f[x_0, x_1, \ldots, x_n]}{h} \\
&= \lim_{h \to 0} \frac{f[x_1, \ldots, x_n, x_0 + h] - f[x_0, x_1, \ldots, x_n]}{h} \\
&= \lim_{h \to 0} f[x_0, x_1, \ldots, x_n, x_0 + h] \\
&= \lim_{h \to 0} f[x_0, x_0 + h, x_1, \ldots, x_n] \\
&= f[x_0, x_0, x_1, \ldots, x_n]
\end{aligned}
\tag{18}
$$

thus finishing the proof. Similarly, for any x_i $(i = 1, 2, \ldots, n)$, we have

$$
\begin{aligned}
\frac{\partial}{\partial x_i} f[x_0, x_1, \ldots, x_n] &= \lim_{h \to 0} \frac{f[x_0, x_1, \ldots, x_i + h, \ldots, x_n] - f[x_0, x_1, \ldots, x_i, \ldots, x_n]}{h} \\
&= \lim_{h \to 0} \frac{f[x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_i + h] - f[x_i, x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]}{h} \\
&= \lim_{h \to 0} f[x_i, x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_i + h] \\
&= \lim_{h \to 0} f[x_0, x_1, \ldots, x_{i-1}, x_i, x_i + h, x_{i+1}, \ldots, x_n] \\
&= f[x_0, x_1, \ldots, x_i, x_i, \ldots, x_n]
\end{aligned}
\tag{19}
$$

□

Problem IX

Solution. Without loss of generality, we translate interval $[a, b]$ to $[-m, m]$, where m satisfies $m = (b-a)/2$.

From Chebyshev Theorem we know

$$min \max_{x\in[-1,1]} |a_0x^n + a_1x^{n-1} + \cdots + a_n| = \frac{|a_0|}{2^{n-1}} \tag{20}$$

Therefore, by a linear transformation $x \to mx$, we have

$$min \max_{x\in[-m,m]} |a_0x^n + a_1x^{n-1} + \cdots + a_n| = \frac{|a_0|m^n}{2^{n-1}} \tag{21}$$

And from this we finally determine

$$min \max_{x\in[a,b]} |a_0x^n + a_1x^{n-1} + \cdots + a_n| = \frac{|a_0|(\frac{b-a}{2})^n}{2^{n-1}} = \frac{|a_0|(b-a)^n}{2^{2n-1}} \tag{22}$$

□

Problem X

Solution. By **Theorem 2.42**, $T_n(x)$ assumes its extrema $n+1$ times at points x'_k defined in (2.44) in textbook. Suppose $||\hat{p_n}||_\infty \leq ||p||_\infty$ does not hold. Then it implies that

$$\exists p \in \mathbb{P}_n^a \quad s.t. \quad ||p||_\infty \leq ||\hat{p_n}||_\infty \tag{23}$$

Consider the polynomial $Q(x) = \hat{p_n}(x) - p(x)$, then

$$Q(x'_k) = \frac{(-1)^k}{T_n(a)} - p(x'_k) \quad k = 0, 1, \ldots, n \tag{24}$$

By the condition $||p||_\infty \leq ||\hat{p_n}||_\infty$, $Q(x)$ has alternating signs at these $n+1$ points. Hence $Q(x)$ must have n zeros. Additionally, $Q(a) = 1 - 1 = 0$, so Q has at least $n+1$ zeros.

However, by the construction of $Q(x)$, the degree of $Q(x)$ is at most n. Therefore, $Q(x) \equiv 0$, which means $p(x) = \hat{p_n}(x)$. This is a contradiction to the assumption, thus finishing the proof. □

Problem XI

Solution. Firstly, based on Binomial expansion we have

$$\sum_{k=0}^{n} b_{n,k}(t) = (t + 1 - t)^n = 1 \tag{25}$$

Secondly, based on the equation above, we have

$$\begin{aligned}
\sum_{k=0}^{n} \frac{k}{nt} b_{n,k}(t) &= \sum_{k=1}^{n} \frac{k}{nt} b_{n,k}(t) \\
&= \sum_{k=1}^{n} \frac{kn!}{nt(n-k)!k!} t^k (1-t)^{n-k} \\
&= \sum_{k=1}^{n} C_{n-1}^{k-1} t^{k-1} (1-t)^{(n-1)-(k-1)} \\
&= \sum_{k=1}^{n} b_{n-1,k-1}(t) = \sum_{k=0}^{n} b_{n-1,k}(t) \\
&= 1
\end{aligned} \tag{26}$$

Therefore, $\sum_{k=0}^{n} k b_{n,k}(t) = nt$.

Thirdly, similar with the proof of (2.50c), we first have

$$\begin{aligned}
\sum_{k=0}^{n} \frac{k(k-1)}{n(n-1)t^2} b_{n,k}(t) &= \sum_{k=2}^{n} \frac{k(k-1)}{n(n-1)t^2} b_{n,k}(t) \\
&= \sum_{k=2}^{n} \frac{k(k-1)n!}{n(n-1)t^2(n-k)!k!} t^k (1-t)^{n-k} \\
&= \sum_{k=2}^{n} C_{n-2}^{k-2} t^{k-2} (1-t)^{(n-2)-(k-2)} \\
&= \sum_{k=2}^{n} b_{n-2,k-2}(t) = \sum_{k=0}^{n} b_{n-2,k}(t) \\
&= 1
\end{aligned} \tag{27}$$

Therefore, $\sum_{k=0}^{n} k(k-1)b_{n,k}(t) = n(n-1)t^2$. Then, according to all the euqations above, we have

$$\begin{aligned}\sum_{k=0}^{n}(k-nt)^2 b_{n,k}(t) &= \sum_{k=0}^{n}[k(k-1)+k-2nkt+n^2t^2]b_{n,k}(t) \\ &= n(n-1)t^2+nt-2n^2t^2+n^2t^2 \\ &= nt-nt^2 = nt(1-t)\end{aligned} \tag{28}$$

$\square$