Homework 3 of Numerical Analysis

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Theoretical questions

Problem I

Solution. Since $s \in \mathbb{S}_3^2$ and $s(0) = 0$, the constraints of $p(x)$ are

$$p(0) = 0, \quad p(1) = 1, \quad p'(1) = -3(2-1)^2 = -3, \quad p^{''}(1) = 6(2-1) = 6. \tag{1}$$

Suppose $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, then we have

$$\begin{aligned} 0 &= a_0, \\ 1 &= a_0 + a_1 + a_2 + a_3, \\ -3 &= a_1 + 2a_2 + 3a_3, \\ 6 &= 2a_2 + 6a_3. \end{aligned} \tag{2}$$

from which we know $p(x) = 12x - 18x^2 + 7x^3$, $s^{''}(0) = p^{''}(0) = -36 \neq 0$. Therefore, $s(x)$ is not a natural cubic spline. □

Problem II

(a)

Solution. From **Theorem 3.14**, $\mathbb{S}_2^1(x_1, \ldots, x_n)$ is a linear space with dimension $2 + n - 1 = n + 1$. As $f_i = f(x_i)$ only give n conditions, another one condition is needed for a unique element in the space. □

(b)

Solution. Since each p_i has constraints $p_i(x_i) = f_i$, $p_i(x_{i+1}) = f_{i+1}$ and $p'(x_i) = m_i$, the divided difference table for Hermite interpolation is

$$\begin{array}{c|ccc} x_i & f_i & & \\ x_i & f_i & m_i & \\ x_{i+1} & f_{i+1} & K_i & \frac{K_i - m_i}{x_{i+1} - x_i} \end{array}$$

where $K_i = f[x_i, x_{i+1}]$ is a constant under the condition. Thus $p_i(i = 1, 2 \ldots, n-1)$ can be determined as

$$p_i = f_i + m_i(x - x_i) + \frac{K_i - m_i}{x_{i+1} - x_i}(x - x_i)^2. \tag{3}$$

□

(c)

Solution. We know that $p_i'(x_{i+1}) = p_{i+1}'(x_{i+1}) = m_{i+1}$ must be held for $i = 1, 2, \ldots, n-2$. Then we have $n-2$ equations using (b)

$$m_i + 2(K_i - m_i) = m_{i+1} \quad \Longrightarrow \quad m_i + m_{i+1} = 2K_i = 2\frac{f_{i+1} - f_i}{x_{i+1} - x_i}. \tag{4}$$

Now $m_1 = f'(a)$ is given, then $m_2, m_3, \ldots, m_{n-1}$ can be computed by recursion $m_{i+1} = 2K_i - m_i$ ($m_1 = f'(a)$) or solving the linear equations. The uniqueness of m_i can also be attained from Cramer's rule since the determinant of the equations' coefficient matrix is 1. □

Problem III

Solution. Since $s \in \mathbb{S}_3^2$ and is natural cubic spline, the constraints of $s_2(x)$ are

$$s_2(0) = s_1(0) = 1 + c, \quad s_2'(0) = s_1'(0) = 3c, \quad s_2^{''}(0) = s_2^{''}(0) = 6c, \quad s_2^{''}(1) = 0. \tag{5}$$

Suppose $s_2(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, then we have

$$\begin{aligned} 1 + c &= a_0, \\ 3c &= a_1, \\ 6c &= 2a_2, \\ 0 &= 2a_2 + 6a_3. \end{aligned} \tag{6}$$

from which we know $s_2(x) = 1 + c + 3cx + 3cx^2 - cx^3$.

Furthermore, if $s(1) = -1$, then

$$s_2(1) = 6c + 1 = -1 \quad \Longrightarrow \quad c = -\frac{1}{3}. \tag{7}$$

□

Problem IV

(a)

Solution. Suppose the natural cubic spline interpolant satisfies

$$s(x) = \begin{cases} s_1(x), & x \in [-1, 0) \\ s_2(x), & x \in [0, 1] \end{cases}$$

Using the same denotation in **Lemma 3.4**, we have

$$s_1(x) = f_1 + s_1'(-1)(x+1) + \frac{M_1}{2}(x+1)^2 + \frac{s_1'''(-1)}{6}(x+1)^3. \tag{8}$$

For f_1, obviously $f_1 = f(-1) = 0$. For M_1, it's zero since s is a natural cubic spline. We also know that $f(0) = 1$, $f(1) = 0$, $f[-1, 0] = 1$, $f[0, 1] = -1$, $f[-1, 0, 1] = -1$. Similar to **Problem III**,

$$2M_2 = \frac{1}{2}M_1 + 2M_2 + \frac{1}{2}M_3 = 6f[-1, 0, 1] = -6. \tag{9}$$

thus $M_2 = -3$. Therefore, for $s_1'(-1)$, according to (3.10) in textbook, we have

$$s_1'(-1) = f[-1, 0] - \frac{1}{6}(M_2 + 2M_1)(0 + 1) = \frac{3}{2}. \tag{10}$$

For $s_1'''(-1)$, according to (3.9) in textbook, we have

$$s_1'''(-1) = \frac{M_2 - M_1}{0 + 1} = -3. \tag{11}$$

Hence, $s_1(x) = \frac{3}{2}(x+1) - \frac{1}{2}(x+1)^3$

The method to calculate $s_2(x)$ is almost the same,

$$s_2(x) = f_2 + s_2'(0)x + \frac{M_2}{2}x^2 + \frac{s_2'''(0)}{6}x^3. \tag{12}$$

where $f_2 = f(0) = 1$, $M_2 = -3$. For $s_2'(0)$ and $s_2'''(0)$,

$$s_2'(0) = f[0, 1] - \frac{1}{6}(M_3 + 2M_2)(1 - 0) = 0, \quad s_2'''(0) = \frac{M_3 - M_2}{1 - 0} = 3. \tag{13}$$

Hence, $s_2(x) = 1 - \frac{3}{2}x^2 + \frac{1}{2}x^3$. For a conclusion,

$$s(x) = \begin{cases} \frac{3}{2}(x+1) - \frac{1}{2}(x+1)^3, & x \in [-1, 0) \\ 1 - \frac{3}{2}x^2 + \frac{1}{2}x^3. & x \in [0, 1] \end{cases}$$

□

(b)

Solution. It's obvious that in case (i), $g(x) = -x^2 + 1$, so

$$\begin{aligned} \int_{-1}^{1} [s''(x)]^2 \, dx &= \int_{-1}^{0} [-3(x+1)]^2 \, dx + \int_{0}^{1} (-3+3x)^2 \, dx = 6, \\ \int_{-1}^{1} [g''(x)]^2 \, dx &= \int_{-1}^{1} 4 \, dx = 8 > 6, \\ \int_{-1}^{1} [f''(x)]^2 \, dx &= \int_{-1}^{1} [-\frac{\pi^2}{4} \cos(\frac{\pi}{2}x)]^2 \, dx = \frac{\pi^4}{16} \approx 6.088 > 6. \end{aligned} \tag{14}$$

The results verify Minimum bending energy of natural cubic spline. □

Problem V

(a)

Solution. From **Definition 3.23** and **Example 3.24**, the quadratic B-spline satisfies

$$B_i^2(x) = \frac{x - t_{i-1}}{t_{i+1} - t_{i-1}} \hat{B}_i(x) + \frac{t_{i+2} - x}{t_{i+2} - t_i} \hat{B}_{i+1}(x). \tag{15}$$

After applying **Definition 3.21**, we finally attain

$$B_i^2(x) = \begin{cases} \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} \hat{B}_i(x) = \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})}, & x \in (t_{i-1}, t_i] \\ \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} \hat{B}_i(x) + \frac{t_{i+2}-x}{t_{i+2}-t_i} \hat{B}_{i+1}(x) = \frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(t_{i+2}-x)(x-t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)}, & x \in (t_i, t_{i+1}] \\ \frac{t_{i+2}-x}{t_{i+2}-t_i} \hat{B}_{i+1}(x) = \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}, & x \in (t_{i+1}, t_{i+2}] \\ 0 & otherwise \end{cases}$$

□

(b)

Solution. According to what have been derived in (a), the left and right derivatives of $B_i^2(x)$ at t_i and t_{i+1} are

$$\begin{aligned}
\lim_{x\to t_i^-}\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) &= \frac{2(t_i-t_{i-1})}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} = \frac{2}{t_{i+1}-t_{i-1}},\\
\lim_{x\to t_i^+}\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) &= \frac{1}{t_{i+1}-t_{i-1}} + \frac{t_{i-1}-t_i}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{1}{t_{i+1}-t_i} = \frac{2}{t_{i+1}-t_{i-1}},\\
\lim_{x\to t_{i+1}^-}\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) &= \frac{1}{t_i-t_{i+1}} + \frac{t_{i+2}-t_{i+1}}{(t_{i+2}-t_i)(t_{i+1}-t_i)} + \frac{1}{t_i-t_{i+2}} = \frac{2}{t_i-t_{i+2}},\\
\lim_{x\to t_{i+1}^+}\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) &= \frac{-2(t_{i+2}-t_{i+1})}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} = \frac{2}{t_i-t_{i+2}}.
\end{aligned} \tag{16}$$

Therefore, $\lim_{x\to t_i^-}\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) = \lim_{x\to t_i^+}\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x)$, $\lim_{x\to t_{i+1}^-}\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) = \lim_{x\to t_{i+1}^+}\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x)$, thus verifying the continuity of $\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x)$ at t_i and t_{i+1}. □

(c)

Solution. From (a) we know

$$\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) = \begin{cases} \frac{2(x-t_{i-1})}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})}, & x\in(t_{i-1},t_i]\\ \frac{t_{i-1}+t_{i+1}-2x}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{t_i+t_{i+2}-2x}{(t_{i+2}-t_i)(t_{i+1}-t_i)}, & x\in(t_i,t_{i+1}]\\ \frac{2(x-t_{i+2})}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}, & x\in(t_{i+1},t_{i+2}]\\ 0. & otherwise \end{cases}$$

Therefore, if $x^*\in(t_{i-1},t_i]$, then $x^*-t_{i-1}=0$, and the solution is $x^*=t_{i-1}$, which is out of the interval's range.

If $x^*\in(t_i,t_{i+1}]$, then solving the equation we have

$$x^* = \frac{t_{i+1}t_{i+2}-t_{i-1}t_i}{(t_{i+1}+t_{i+2})-(t_{i-1}+t_i)}. \tag{17}$$

The following proof is to confirm $x^*\in(t_i,t_{i+1}]$. For the inequality $t_i<x^*$, it suffices to prove $t_{i+1}t_{i+2}-t_{i-1}t_i > t_i[(t_{i+1}+t_{i+2})-(t_{i-1}+t_i)]$. After simplifying the inequality, we get $t_i(t_{i+1}-t_i) < t_{i+2}(t_{i+1}-t_i)$, which is obviously correct since $t_i<t_{i+2}$. The other inequality $t_{i+1}>x^*$ can be attained similarly. Hence we complete the proof. □

(d)

Solution. Explicit expression in (a) shows that obviously $B_i^2(x) \geq 0$ because all terms of the expression in the support of $B_i^2(x)$ is greater than zero, and 0 can be obtained at every point outside the support. Therefore, we now only need to show that $B_i^2(x) < 1$.

Consider the interval of support, in which the maximum of $B_i^2(x)$ can be obtained. Since we have shown that the derivative of $B_i^2(x)$ is continuous and the function has value of 0 at the support's endpoints, $B_i^2(x)$ must reach its maximum at x^*, where its derivative vanishes.

That is to say,

$$\begin{aligned} B_i^2(x) \leq B_i^2(x^*) &= \frac{(t_{i+2}-t_{i-1})(t_{i+1}-t_{i-1})}{[(t_{i+1}+t_{i+2})-(t_{i-1}+t_i)]^2} + \frac{(t_{i+2}-t_{i-1})(t_{i+2}-t_i)}{[(t_{i+1}+t_{i+2})-(t_{i-1}+t_i)]^2} \\ &= \frac{t_{i+2}-t_{i-1}}{(t_{i+2}-t_{i-1})+(t_{i+1}-t_i)} \\ &< 1. \end{aligned} \tag{18}$$

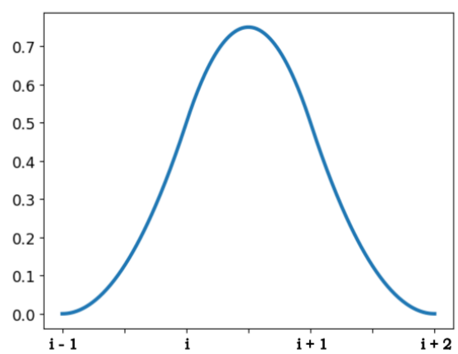
Hence we have finished the proof. $\square$

(e)

Solution. If $t_i = i$, then

$$B_i^2(x) = \begin{cases} = \frac{(x-i+1)^2}{2}, & x \in (i-1, i] \\ = \frac{(x-i+1)(i+1-x)}{2} + \frac{(i+2-x)(x-i)}{2}, & x \in (i, i+1] \\ = \frac{(i+2-x)^2}{2}, & x \in (i+1, i+2] \\ 0 & otherwise \end{cases}$$

The function image plotted by PYTHON is shown below.



□

Problem VI

Solution. By setting $f = t - x$, $g = (t-x)_+$ in Leibniz formula , we have

$$\begin{aligned}
&[t_{i-1}, t_i, t_{i+1}](t-x)_+^2 = (t_{i-1} - x)[t_{i-1}, t_i, t_{i+1}](t-x)_+ + [t_i, t_{i+1}](t-x)_+, \\
&[t_i, t_{i+1}, t_{i+2}](t-x)_+^2 = (t_i - x)[t_i, t_{i+1}, t_{i+2}](t-x)_+ + [t_{i+1}, t_{i+2}](t-x)_+.
\end{aligned} \tag{19}$$

Therefore, together with **Theorem 2.17** and **Example 3.31**,

$$\begin{aligned}
&(t_{i+2} - t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 \\
&= [t_i, t_{i+1}, t_{i+2}](t-x)_+^2 - [t_{i-1}, t_i, t_{i+1}](t-x)_+^2 \\
&= (t_i - x)[t_i, t_{i+1}, t_{i+2}](t-x)_+ + [t_{i+1}, t_{i+2}](t-x)_+ \\
&- (t_{i-1} - x)[t_{i-1}, t_i, t_{i+1}](t-x)_+ - [t_i, t_{i+1}](t-x)_+ \\
&= (t_i - x + t_{i+2} - t_i)[t_i, t_{i+1}, t_{i+2}](t-x)_+ - (t_{i-1} - x)[t_{i-1}, t_i, t_{i+1}](t-x)_+ \\
&= (t_{i+2} - x)[t_i, t_{i+1}, t_{i+2}](t-x)_+ + (x - t_{i-1})[t_{i-1}, t_i, t_{i+1}](t-x)_+ \\
&= (t_{i+2} - x)[\frac{(t_{i+2} - x)_+ - (t_{i+1} - x)_+}{(t_{i+2} - t_{i+1})(t_{i+2} - t_i)} - \frac{(t_{i+1} - x)_+ - (t_i - x)_+}{(t_{i+1} - t_i)(t_{i+2} - t_i)}] \\
&+ (x - t_{i-1})[\frac{(t_{i+1} - x)_+ - (t_i - x)_+}{(t_{i+1} - t_i)(t_{i+1} - t_{i-1})} - \frac{(t_i - x)_+ - (t_{i-1} - x)_+}{(t_i - t_{i-1})(t_{i+1} - t_{i-1})}].
\end{aligned} \tag{20}$$

From the "simplified" formula above, it's not difficult to use the definition of truncated power function and know that

$$(t_{i+2} - t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2$$
$$= \begin{cases}
(t_{i+2} - x) \cdot 0 + (x - t_{i-1})[\frac{1}{t_{i+1} - t_{i-1}} - \frac{(t_i - x)}{(t_i - t_{i-1})(t_{i+1} - t_{i-1})}] = \frac{(x - t_{i-1})^2}{(t_{i+1} - t_{i-1})(t_i - t_{i-1})}, & x \in (t_{i-1}, t_i] \\
\frac{(t_{i+2} - x)}{t_{i+2} - t_i} - \frac{(t_{i+1} - x)(t_{i+2} - x)}{(t_{i+1} - t_i)(t_{i+2} - t_i)} + \frac{(x - t_{i-1})(t_{i+1} - x)}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} = \frac{(x - t_{i-1})(t_{i+1} - x)}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} + \frac{(t_{i+2} - x)(x - t_i)}{(t_{i+2} - t_i)(t_{i+1} - t_i)}, & x \in (t_i, t_{i+1}] \\
(t_{i+2} - x)\frac{t_{i+2} - x}{(t_{i+2} - t_i)(t_{i+2} - t_{i+1})}, & x \in (t_{i+1}, t_{i+2}] \\
0, & otherwise
\end{cases}$$

which is algebraically identical to B_i^2. □

Problem VII

Solution. For any fixed n, from **Theorem 3.34**, $\frac{\mathrm{d}}{\mathrm{d}x}B_i^{n+1}(x)$ can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}x}B_i^{n+1}(x) = \frac{n}{t_{i+n} - t_{i-1}}B_i^n(x) - \frac{n}{t_{i+n+1} - t_i}B_{i+1}^n(x). \tag{21}$$

Integrate the left and right hand sides of the equation from t_{i-1} to t_{i+n+1} and we have

$$\frac{1}{n}\int_{t_{i-1}}^{t_{i+n+1}} \frac{\mathrm{d}}{\mathrm{d}x}B_i^{n+1}(x)\,dx = \frac{1}{t_{i+n} - t_{i-1}}\int_{t_{i-1}}^{t_{i+n+1}} B_i^n(x)\,dx - \frac{1}{t_{i+n+1} - t_i}\int_{t_{i-1}}^{t_{i+n+1}} B_{i+1}^n(x)\,dx. \tag{22}$$

Since $\frac{\mathrm{d}}{\mathrm{d}x}B_i^{n+1}(x)$ has primitive function $B_i^{n+1}(x)+C$ where C is an arbitrary constant, and the support of $B_i^{n+1}(x)+C$ is $[t_{i-1}, t_{i+n+1}]$, the LHS of the equation is equal to $\frac{1}{n}[B_i^{n+1}(t_{i+n+1}) - B_i^{n+1}(t_{i-1})] = 0$.

As for the RHS of the equation, **Lemma 3.27** told that the support of $B_i^n(x)$ is $[t_{i-1}, t_{i+n}]$, and the support of $B_{i+1}^n(x)$ is $[t_i, t_{i+n+1}]$. Therefore the equation is finally translated to

$$\frac{1}{t_{i+n} - t_{i-1}}\int_{t_{i-1}}^{t_{i+n}} B_i^n(x)\,dx = \frac{1}{t_{i+n+1} - t_i}\int_{t_i}^{t_{i+n+1}} B_{i+1}^n(x)\,dx, \tag{23}$$

and that shows the scaled integral of $B_i^n(x)$ is the same for any adjacent i, which implies the independence of index i for any knots' form by a simple recursion. □

Problem VIII

(a)

Solution. We now need to prove

$$\tau_2(x_i, x_{i+1}, x_{i+2}) = [x_i, x_{i+1}, x_{i+2}]x^4 \tag{24}$$

The relative table of divided difference of $[x_i, x_{i+1}, x_{i+2}]x^4$ is shown below.

$$\begin{array}{c|ccc}
x_i & x_i^4 & & \\
x_{i+1} & x_{i+1}^4 & K_1 := (x_{i+1}^2 + x_i^2)(x_{i+1} + x_i) & \\
x_{i+2} & x_{i+2}^4 & K_2 := (x_{i+2}^2 + x_{i+1}^2)(x_{i+2} + x_{i+1}) & \frac{K_2 - K_1}{x_{i+2} - x_i}
\end{array}$$

Therefore,

$$
\begin{aligned}
[x_i, x_{i+1}, x_{i+2}]x^4 &= \frac{(x_{i+2}^2 + x_{i+1}^2)(x_{i+2} + x_{i+1}) - (x_{i+1}^2 + x_i^2)(x_{i+1} + x_i)}{x_{i+2} - x_i} \\
&= \frac{(x_{i+2} - x_i)(x_{i+2}^2 + x_i x_{i+2} + x_i^2) + x_{i+1}^2(x_{i+2} - x_i) + x_{i+1}(x_{i+2} + x_i)(x_{i+2} - x_i)}{x_{i+2} - x_i} \\
&= x_{i+2}^2 + x_i x_{i+2} + x_i^2 + x_{i+1}^2 + x_{i+1}(x_{i+2} + x_i) \\
&= \tau_2(x_i, x_{i+1}, x_{i+2}).
\end{aligned} \tag{25}
$$

□

(b)

Solution. Exactly the same as the one in **Theorem 3.46**.

By **Lemma 3.45**, we have

$$
\begin{aligned}
&(x_{n+1} - x_1)\tau_k(x_1, \ldots, x_n, x_{n+1}) \\
&= \tau_{k+1}(x_1, \ldots, x_n, x_{n+1}) - \tau_{k+1}(x_1, \ldots, x_n) - x_1\tau_k(x_1, \ldots, x_n, x_{n+1}) \\
&= \tau_{k+1}(x_2, \ldots, x_n, x_{n+1}) + x_1\tau_k(x_1, \ldots, x_n, x_{n+1}) - \tau_{k+1}(x_1, \ldots, x_n) - x_1\tau_k(x_1, \ldots, x_n, x_{n+1}) \\
&= \tau_{k+1}(x_2, \ldots, x_n, x_{n+1}) - \tau_{k+1}(x_1, \ldots, x_n).
\end{aligned} \tag{26}
$$

The rest of the proof is an induction on n. For $n = 0$, it reduces to

$$
\tau_m(x_i) = [x_i]x^m, \tag{27}
$$

which is trivially true. Now suppose the condition holds for a non-negative integer $n < m$. Then (26) and the induction hypothesis yield

$$
\begin{aligned}
&\tau_{m-n-1}(x_i, \ldots, x_{i+n+1}) \\
&= \frac{\tau_{m-n}(x_{i+1}, \ldots, x_{i+n+1}) - \tau_{m-n}(x_i, \ldots, x_{i+n})}{x_{i+n+1} - x_i} \\
&= \frac{[x_{i+1}, \ldots, x_{i+n+1}]x^m - [x_i, \ldots, x_{i+n}]x^m}{x_{i+n+1} - x_i} \\
&= [x_i, \ldots, x_{i+n+1}]x^m,
\end{aligned} \tag{28}
$$

which completes the proof. □