Problem Chosen	2025	Team Control Number
A	**MCM/ICM**	**2522824**
	Summary Sheet	

Our task was to create a mathematical model to help archaeologists use analysis of the wear patterns on stairs to understand more about their history. This would provide insight into their usage, traffic patterns, and creation.

For our model we incorporated concepts from material science physics and archaeology to quantitatively analyze stair wear. We started our model by reviewing the pieces of a staircase, and how forces from human motion affect a staircase. When people ascend stairs, they exert force on the tread to elevate themselves. On the other hand, when people descend stairs, they strike the front edge of the stair with force. These forces leave wear patterns on the staircase, which allow us to qualitatively and quantitatively analyze wear. If indentations are along the treads, we determine that portion of the step was used for ascension, while if indentations are along the edge of the stair, we determine that portion of the step was used for descension. Similarly, if there are multiple columns of wear, then a stairwell was used simultaneously, potentially in opposite directions. Quantitatively analyzing these wear patterns by measuring indentation depth via height measurement creates heat maps that can be used to determine the volume of erosion from human footsteps.

Our next step was to translate this heat map into steps taken. Using the Archard Wear Equation, we determined the frictional shear required to remove a specific amount of material from a staircase. Then, we translated this to the average person to determine the number of steps taken in order to create that shear. Finally, we refined our model using the Abbott-Firestone curve to better approximate the original height of the staircase, and applied our model to a real-world staircase as an example.

Our model utilizes measurements such as the depth of wear and height variations, which can be collected even with limited resources. It's also quite versatile since the parameters can be adjusted respectively to the specific staircase in question. Overall, our model provides a reliable framework for understanding the usage of stairs over time using physical signs of wearing.

Contents

1. Introduction

1.1 Problem Background

Over time, stairs in historic structures have undergone wear from repeated use and in doing so, they leave behind physical evidence of how they were used. These wear patterns such as uneven erosion or bowed surfaces are shaped by factors such as traffic volume, movement direction, environmental conditions, and the materials used to construct the stairs. In addition to the physical evidence, historical and environmental contexts are needed to more accurately interpret the way stairs abrade. Stair erosion is a complex process that is the result of many coinciding factors, but rough approximations of stair wear can be made given details on the stairs' end condition.

1.2 Restatement of the Problem

Stairs in ancient structures often show signs of wear that can provide insight into their usage throughout history. Archaeologists are particularly interested in the way stairs were used and historical traffic patterns that may have occurred. Our task is to create a model that can give guidance on the information that archaeologists can figure out from worn stairs, using data available from the stairs' current condition. This information includes things like when the stairs were constructed, if people were going up or down, how many people were going up at once, and how often the stairs were used. We should be able to minimize the cost and simplify the process needed to collect the measurements necessary in our model as well.

1.3 Problem Analysis

To address this problem, we need to develop a model that utilizes wear patterns on stairs to provide archaeologists with insights into their history and usage. First, we need to identify key factors that we need to consider in our model. The traffic volume on a stairwell is the number of people who were using the stairs over time. The more frequently a set of stairs were used and the more people used it, the more indentations we would expect it to have, and we would need to rely on those physical dents to figure out the traffic. Directionality depends on if people predominantly went up or down the staircase. This could likely be figured out by considering where the most pressure is put and comparing the indentation patterns on the stairs to simulated or predicted indentation patterns based on this pressure analysis. When ascending a staircase, people tend to put more pressure on their heel to push off the steps. When descending, there's more pressure on the back of the foot, which may create a different indentation pattern. We also need to consider how many people were traveling at once. Single file lines would create more concentrated dents in the stairs while larger groups would leave more spread out marks. Most of these factors can be figured out from the physical marks left on the stairs. However, we also need to consider some other things. Environmental factors play a big role in this as well. For instance, if the stairs are in an area that experiences frequent rainfall, there will be more damage outside of just people using the stairs because of weathering. Considering historical context is also useful. With the location of the stairs, such as if they're at a temple or marketplace, and the population size of the place the stairs are, we can make more accurate interpretations of the traffic intensity and usage patterns.

We also need to figure out what data archaeologists must collect in order to figure out all the key factors mentioned previously. Pertaining to the physical wearing of the stairs, things like the depth and shape of the wears, the distribution, micro-scratches, and erosion patterns on the risers and edges of steps should be measured in order to more accurately identify the history behind them.

2. Assumptions & Notations

2.1 Assumptions

Assumption 1: Indentations within stairs are formed by erosion of the staircase. This erosion is largely from foot traffic on the stairwell removing microscopic pieces of the stairwell, similar to how raindrops over time can carve rock.
Justification: We rule out compression of the stairwell as a possibility, as the compressive force on the stairs from human footsteps is not enough to cause plastic deformation in the vertical direction. In order for a staircase to form indentation, mass must be removed from the stairwell.

Assumption 2: Damage to staircases is entirely from shear force (force parallel to the horizontal surface of the stair), not normal force (force perpendicular to the horizontal surface of the stair).
Justification: The normal forces of the elements involved in the erosion of the staircase (footsteps and rain) are not strong enough to cause fracture or significant damage to the rock. The main form of erosion on the stair should be due to frictional stress caused by the movement of feet and water parallel to the stair surface.

Assumption 3: All steps taken on stairs were done with skin, leather, or another hide-like material that all have similar material properties for the sake of the friction of the material on a stone stair, and so the stair's hardness is the only important material factor in determining the wear.
Justification: Compared to granite, materials like skin, leather, and other common shoe soles are all very similar in terms of hardness and material resilience, making this a decent approximation that removes a lot of complexity from having to consider shoe material when determining wear. Stairs are also likely already damaged and somewhat impure when they are walking on, meaning that the leather will be able to damage the asperities in the stairs and wear them down in a frictional manner.

Assumption 4: The force imparted by people walking on a stair is always constant.
Justification: While the forces in a real world dynamics problem are never constant, because the relationships that we use to model them are linear, the exact dynamics of the forces are not important as long as the average is known. Furthermore, the deviations from this average should be small enough as to not significantly change the scale of our approximations, and over time the forces should average out.

3. Developing the Model

3.1 Defining Parts of a Staircase

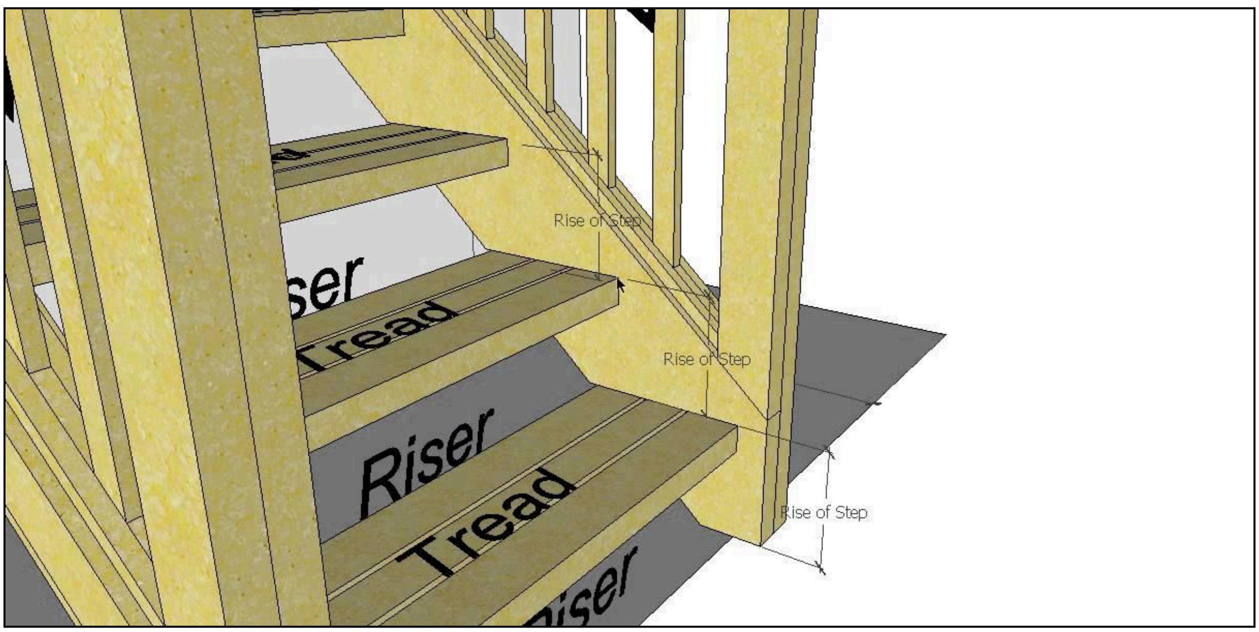

Figure 1: Diagram of a staircase

Step: One pair of riser and tread.

Tread: A step's horizontal surface.

Riser: A stair's vertical surface.

Edge: the outer perpendicular connection between a tread and a riser

3.2 Wear Patterns

Different distributions of wear on steps can give insights into how a stair was used. For determining which parts of a stairwell were used (whether the stairwell was used on the right side, the left side, both simultaneously, and/or how many columns of people walked on the stairs), we give the examples below:

	
When people ascend stairs, they exert force on the ball of their foot to elevate themselves, and do so on the top surface (the tread). This step shows damage to the tread (farther from the riser), signifying that it was more often used to travel upwards.	When people descend stairs, they often strike the front edge of the stair as a result of/to gain forward momentum. This stairwell shows damage to the right edges of the stairs (not the tread), signifying that people walked down the stairs on the right side.

Figure 2: Example Staircases.

Similarly, we can determine the locations of usage on a stair based on its indentations.

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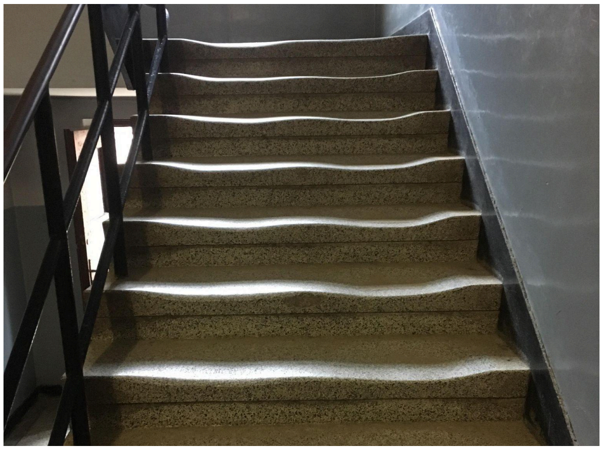
<tr>
<td>
Here, we can see two columns of indentations on the stairwell, signifying that people walked on this staircase in two simultaneous columns. Within each column there are also two minima visible, which are the two step locations of each person.</td>
<td>
Here, we can see one column of indentations down the middle, signifying that typically people walked on this stairwell one at a time, in the center of the stairwell.</td>
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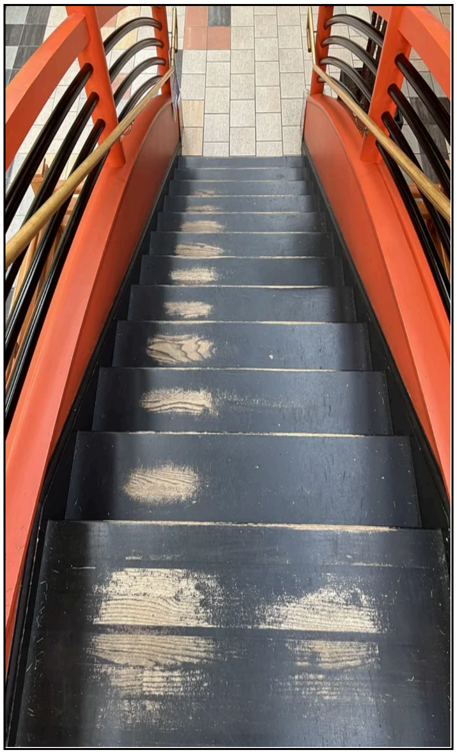
Figure 3: Example Staircases.



For this staircase, we can see two patterns of wear: on the left side of the staircase paint loss is concentrated on the treads, and on the right side of the staircase paint loss is concentrated on the edges. Thus, for the left side of the staircase wear is concentrated on the treads, while on the right side of the staircase wear is concentrated on the edges. Thus, using our knowledge from the above scenarios, we can determine that these stairs were likely used simultaneously as there are two columns of wear (left and right). We can also determine that people traveled upwards on the left because wear is concentrated on the treads there, and people traveled downwards on the right because wear is concentrated on the edges there.

Figure 4: Example Staircases.

3.3 Quantitatively Determining the Wear on Stairs

Now that we can qualitatively determine how a staircase was used, we will quantify the wear on a stairwell. To start, we will make a few assumptions to simplify our process:

Assumption 2.1: Each tread on a stair at construction was originally perfectly horizontal.

Assumption 2.2: The maximum height of the tread is the tread's original height (that location we will assume to have zero wear.) Thus, all heights below the maximum will have some measurable wear.

Assumption 2.3: For all locations where the height of the tread is below the original height, wear must have occurred and removed a volume of material composing the stairwell. This wear must have occurred from human traffic.

Justification: Stairs are typically used as a method of transport by humans to ascent/descend terrain.

With these assumptions, we can create a heatmap of how much volume was removed, and where that volume was removed from via the equation below:

$$\Delta H = H_{max} - H_{l,w}$$

Where ΔH is the height difference, H_{max} is the maximum height of the stair, and $H_{l,w}$ is the height at a certain length l and width w.

Example:

Step: 1m x 0.4m	0m --- 0.1m	0.1m --- 0.2m	0.2m --- 0.3m	0.3m --- 0.4m	0.4m --- 0.5m	0.5m --- 0.6m	0.6m --- 0.7m	0.7m --- 0.8m	0.8m --- 0.9m	0.9m --- 1m
0m --- 0.1m		0.002	0.003	0.002			0.006	0.006	0.006	
0.1m --- 0.2m		0.002	0.003	0.002						
0.2m --- 0.3m		0.001	0.002	0.001						
0.3m --- 0.4m			0.001							

Figure 4: An example heatmap for the height difference (ΔH) due to wear for a staircase. These values would be measured by researchers, and are the average difference between the maximum height of the step and the height of the step in that section of the stair. This stair is sectioned into 0.1m x 0.1m areas, but for more precise measurements a tread could be separated into more areas.

We can convert the above heatmap into volume removed via:

$$\Delta V = \Delta H \times A_{x,y}$$

Where V represents the volume removed and $A_{x,y}$ represents a section's area.

Step: 1m x 0.4m	0m --- 0.1m	0.1m --- 0.2m	0.2m --- 0.3m	0.3m --- 0.4m	0.4m --- 0.5m	0.5m --- 0.6m	0.6m --- 0.7m	0.7m --- 0.8m	0.8m --- 0.9m	0.9m --- 1m
0m --- 0.1m		0.00002	0.00003	0.00002			0.00006	0.00006	0.00006	
0.1m --- 0.2m		0.00002	0.00003	0.00002						
0.2m --- 0.3m		0.00001	0.00002	0.00001						
0.3m --- 0.4m			0.00001							Note: Units are in m^2

Figure 5: Heatmap from Figure 4 for the height difference (ΔV) due to wear for a staircase.

Quantitatively, we can see patterns in this tread: between 0.1m and 0.4m from the left there is indentation in the stair along the horizontal surface, while between 0.6m and 0.9m there is indentation towards the front of the stair, towards the edge. Based on Section 3.2, the wear on this tread shows that it was simultaneously used for two columns of people: ascending on the left and descending on the right.

In the next section, we will determine how often the stairs were used/the age of the stairs using these quantitative measurements of material removed.

3.4 The Archard Wear Equation

A common method of measuring the wear on a surface due to frictional shear is to use the Archard wear equation. This equation models the lost volume of particulate when two surfaces rub against each other, and can be used to calculate the particulate loss due to shear stress between two solid surfaces. This equation is integral to our understanding of the foot-stair interaction, as we can use it to provide a quantitative estimate for how many steps it takes to produce a given indentation. The equation is an empirical estimate that condenses a large number of complex factors into a simple formula, meaning that it serves as a very rough approximation that still can provide a useful estimation. The equation is as follows:

$$Q = \frac{KWL}{H}$$

Where Q is the volume of material removed, K is a dimensionless constant of wear, W is the normal load, L is the sliding distance, and H is the hardness of the stair. The work of applying this equation appears in trying to approximate these factors. Since the hardness of a stone stair is around 10^9Pa, and leather is generally much softer than stone, K will be taken to be around 10^{-3} to represent medium wear, where more impure and gravelly stairs will have a higher K value. The load from each step varies depending on whether the person is going up or down. When going up, the load is the force needed to keep the person in dynamic equilibrium in the vertical direction plus the force needed to actually ascend, whereas when going down, most people move at a close to constant speed, and the force should just be equal to the person's weight so that they remain in dynamic equilibrium. If we take the weight of the average person to be around 70kg, their weight is 686N, so in the case of going up, we have:

$$686N + F_{up}$$

where F_{up} can be found using the work that must be done by the person to ascend the stairs. While this will vary wildly depending on the stair type, the general amount of work done against gravity can be found using the formula for gravitational potential energy, W=mgh. Since we are assuming that the force is applied roughly evenly as people move up the staircase, the height of the staircase cancels with the height in the expression for gravitational potential energy, and we are left with the additional force up being an average of 686N, meaning that the force doubles when people are going up, but is unchanged when going down. In order for there to be any wear from walking, there has to be a sliding distance associated with the foot, which we will take to be quite small (a few millimeters, around 0.005m), as this effect is small but present, especially on gravelly or wet stairs. Finally, we need to divide the total volume of particulate removed by the area it is removed over to get the height indentation. The area of the balls of the feet (which would be where most of the force is concentrated) is around 500 cm^2, and plugging all of the numbers in yields 7.2 million steps per millimeter of indentation for going upwards, and 14.4 million steps for going downwards. It is worth noting that the damage going down is applied to the edge of the stair tread, and damage going up is applied to the back of the stair tread.

4. Refining the Model

4.1 Preface

Recall from Section 3.3:

Assumption 2.1: Each tread on a stair at construction was originally perfectly horizontal.

Assumption 2.2: The maximum height of the tread is the tread's original height (that location we will assume to have zero wear.) Thus, all heights below the maximum will have some measurable wear.

In reality, these assumptions are not always true. For example, some staircases may start with rough surfaces and thus the maximum height may not be representative of the tread. Other staircases may have some irregularities creating slanted surfaces.

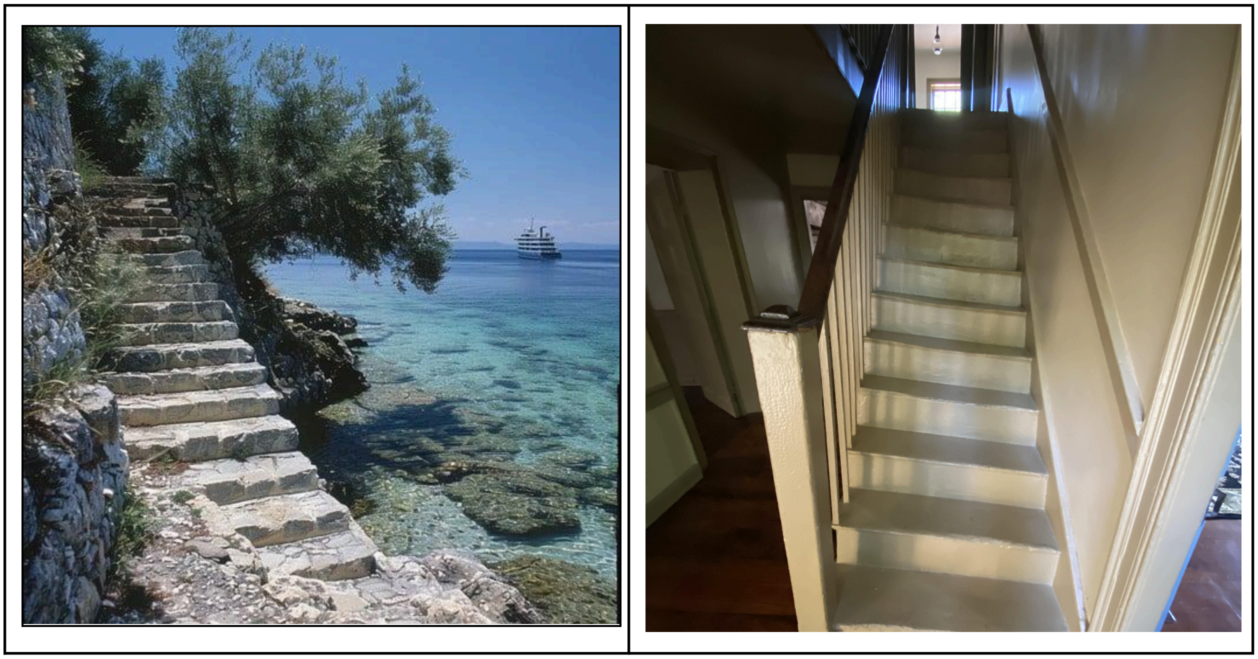


Figure 6: Some example irregular staircases

4.2 The Abbott-Firestone Curve

The Abbott-Firestone curve describes the distribution of heights on a surface. Using the Abbott-Firestone curve, we will identify the mode of a tread, which is likely representative of the least worn areas.

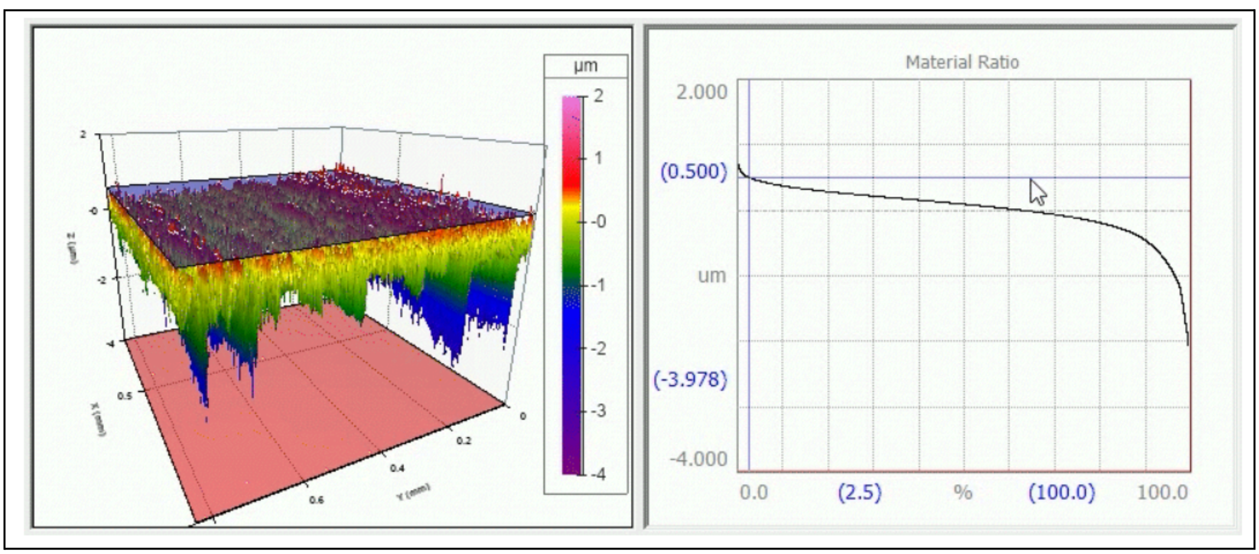


Figure 7. This is a visual representation of an Abbott-Firestone Curve. On the left is an example surface which has a large quantity of indentations (the blue portions). On the

right is our Abbott-Firestone curve that represents this surface. As we travel down the y-axis of the curve, the probability that a randomly selected height from the surface is below the selected height (the y-axis) decreases.

We will derive our Abbott-Firestone curve from the cumulative distribution function (CDF) of the surface height measurements.

Original CDF: $F(x) = P(X \leq x)$

Where *X* represents the random variable and *x* is a specific value for the variable, such that $P(X \leq x)$ is the probability that a randomly selected individual will have a height less than or equal to *x*. Applying this to our scenario, we can write:

Abbott-Firestone Curve: $F(x) = \frac{1}{A}\int\int G(x)\, dl\, dw$

Where *A* represents the total surface area of the tread, and

$$G(x) = 1 \text{ if } h_{l,w} \leq x$$

$$G(x) = 0 \text{ if } h_{l,w} > x$$

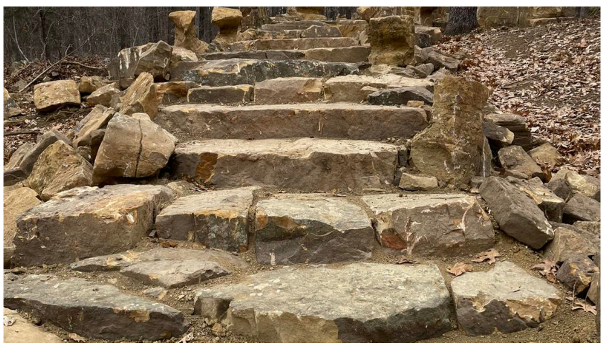
This basically becomes the probability that for a specified height *x*, a randomly selected height on the tread is less than or equal to that height. Now, in order to find the mode (*m*) of the Abbott-Firestone curve, we will maximize the derivative of $F(x)$.

$$m = \max(\frac{dF}{dx})$$

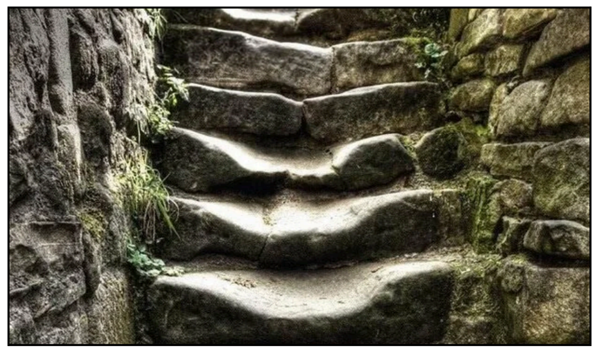
Here, *x* is constrained to lie between the minimum and maximum surface heights of the tread. The mode *m* is a relative maximum of the curve and often represents the least worn areas. In many scenarios, *m* roughly represents the original average height of the stair. Using this value for H_{max} in Section 3.3 in cases where assumptions for H_{max} do not hold will yield a more precise result.

4.3 Limitations of the Abbott-Firestone Curve

The Abbott-Firestone Curve has limitations. For example, if the majority of a stair were to be worn down, the Abbott-Firestone curve may show a maximum at a worn-down height, making it difficult to estimate the original height. Thus, the Abbott-Firestone curve should be used by researchers in conjunction with H_{max} as an alternative method when stairs aren't as worn.



Good for Abbott-Firestone Curve provided the rocks aren't majority worn-down (irregular treads mean that the absolute maximum height will vastly differ from the actual original average height).



Not good for the Abbott-Firestone Curve (the stone steps are very worn-down, so the absolute maximum height of a step will be more accurate than the mode, which may give a value from an eroded portion).

5. Applying the Model/Sensitivity Analysis

In order to evaluate the accuracy of our model, we need to apply it to some staircases with well-documented histories. This allows us to determine if the model produces results that align with historical usage and also validates its reliability in interpreting wear patterns.

For instance, recalling *Figure 3*, a staircase from Auschwitz (*left*) exhibits two distinct columns of indentions which is consistent with its history, as people probably marched up and down the stairs in organized lines. In contrast, a staircase from a house (*right*) displaying more concentrated indentations indicates single-person use. An example of a sample heatmap from the Auschwitz staircase is provided below:

Step: 1m x 0.4m	0m --- 0.075m	0.075m --- 0.15m	0.15m --- 0.225m	0.225m --- 0.3m	0.3m --- 0.375m	0.375m --- 0.45m	0.45m --- 0.525m	0.525m --- 0.6m	0.6m --- 0.675m	0.675m --- 0.75m	0.75m --- 0.825m	0.825m --- 0.9m	0.9m --- 9.75m	0.9.75m --- 1.05m	1.05m --- 1.125m	1.125m --- 1.2m	1.2m --- 1.275m	1.275m --- 1.35m	1.35m --- 1.425m	1.425m --- 1.5m
0m --- 0.1m					0.001	0.002	0.001							0.001	0.002	0.001				
0.1m --- 0.2m			0.001	0.003	0.006	0.005	0.003	0.002	0.001		0.001	0.003	0.004	0.007	0.006	0.007	0.004	0.001		
0.2m --- 0.3m		0.001	0.004	0.007	0.014	0.013	0.015	0.008	0.004	0.002	0.002	0.007	0.009	0.018	0.016	0.018	0.011	0.003	0.001	
0.3m --- 0.4m		0.003	0.008	0.010	0.018	0.016	0.018	0.009	0.007	0.003	0.003	0.009	0.012	0.019	0.019	0.019	0.013	0.006	0.002	

Figure 8. This is an example heatmap of a single stair taken from a staircase at Auschwitz. There is significant indentation and there appear to be a total of roughly four peaks. The x and y axis represent the position on the stairs measured from the back left corner, and the numbers on the heatmap are the indentation in millimeters. The original stair height is probably around 190 millimeters, the average stair height for most staircases.

Since we know the indentation at any one point, we can use the figures calculated earlier to determine how many steps were taken. The majority of the indentation occurs near the edge of the stair tread – for this reason, we can assume that the stairs were mostly used to travel downwards. There would be no impact from rain or weather, as these are indoor stairs, so the primary and only source of indentation would be from footsteps on the stair. Using the calculated figure of 14.4 million steps per millimeter of indentation, we find that for each hotspot of around 20 millimeters of indentation, it would have taken roughly 280 million steps to cause it. Due to the scale of each square being a width of 75 millimeters, the large hotspots cannot be the result of one foot, as a foot is only around 100 millimeters in width. Thus, the two individual hotspots must have been caused by two separate sets of people walking on the stairs, indicating that this was a dual lane staircase, with very heavy foot traffic. A quick check reveals that there must have been around 7000 steps per day for around 100 years to account for this level of wear, which seems like a reasonable estimate for a place that is presumed to have had high traffic.

6. Analysis of the Model

6.1 Strengths

Our model incorporates quantitative tools such as the Archard wear equation and the Abbott-Firestone curve to analyze wear patterns. This allows for more precise estimates for usage intensity and enables archaeologists to infer specific details such as the number of steps, traffic direction, and usage frequency. Relying on a more quantitative foundation ensures that the model has measurable and repeatable results, so it is reliable and can be applied to different situations.

Our model is also adaptable. By adjusting parameters such as hardness of materials, the coefficient of wear, and the effects of water erosion, the model can be applied to a wider range of scenarios. Versatility is a crucial strength in our model. Also, the data needed for our model can be collected using relatively simple tools, so the model is accessible even for archaeologists with limited resources.

6.2 Weaknesses

Our model most accurately applies to stone staircases because of this material's durability and resistance to erosion and weathering over time. Other materials like wood would be more prone to degradation from environmental factors, making it difficult to reliably assess the wear patterns on wooden staircases using our model. Still, the parameters can be adjusted to account for different characteristics of different materials, although environmental factors would cause inaccuracy. The model also can't accurately differentiate between wear caused by humans and those caused by environmental factors, such as rain and wind. This could lead to some overestimations or underestimations of human usage patterns especially if the stairs are outdoors, meaning our model has much more accuracy for stairs located indoors away from the weather.

Our model relies on assumptions that simplify concepts much more than they'd be in reality. This includes things like constant force per step and linear wear rates which might not always be the case in real life. Varied walking styles or uneven step usage can't be accounted for.

7. Conclusion

This model serves as a rough approximation of foot traffic on a set of stone stairs using measurements of the final indentation on the stairs. It is relatively lightweight to use, requiring only basic measurements taken with something as simple as a ruler, and does not require any complex calculations as the majority of the values that we use are approximations. The model is also relatively insensitive to minor changes in hardness or other conditions due to the linearity of all of the relationships, and the model's nature as a ballpark estimate device. Even if the hardness of a stair is two or three times as hard as another set of stairs, it will not significantly change the estimate of the end traffic on the stairs, as the order of magnitude of the differences in traffic will be much higher than this. The model functions best in indoor conditions, where factors like weather, humidity, temperature, and others tend to be more constant. However, when used on these indoor stairs, it provides a decent indication of the level of traffic that was present on the stairs. If the amount of traffic on the stairs was roughly known, then the model could also be used in reverse to find the approximate age of the stairs, with this once again only truly being reasonably accurate on a set of stone stairs found indoors.

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