# EE 512

# Digital Signal Processing

## Session 5

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Start here



# Methods of Solution

## 1- Method of Undetermined Coefficients

The recursive method for finding the solution of difference equation does not provide a "closed form" solution.

Three steps are needed for finding the general or closed form Solution for a Constant Coefficient linear difference equation.

(zero-input response)

1- obtain the homogeneous or Complementary solution, $y_h(n)$.

2- obtain the particular solution, $y_p(n)$.
(zero-state response)

3- obtain the Constants in $y_h(n)$ by applying the initial Conditions.

The Complete solution is then

initial conditions

$$y(n) = y_h(n) + y_p(n)$$ then apply IC's to find the Constants in $y_h(n)$.

**Step 1** (Homogeneous Solution)

Arrange the Characteristic equation

$$a_0\lambda^N + a_1\lambda^{N-1} + \cdots + a_N = 0$$

and find its roots. Use table 1 to obtain the homogeneous solution $y_h(n)$ depending on the roots.

**Step 2**

As in the case of differential equations there are a set of rules that one must follow to form appropriate particular solutions while solving difference equation. These are summarized in Table 2.

Step 3

Solve for $C_i$'s using the initial conditions.

Example 1

Determine the Complete solution for

$$y(n) - \frac{3}{2} y(n-1) + \frac{1}{2} y(n-2) = 1 + 3^{-n} \qquad n \geq 0$$

with $y(-2) = 0$ and $y(-1) = 2$.

Step 1

$$y_h(n) = c_1 \lambda_1^n + c_2 \lambda_2^n$$

where $\lambda_1$ and $\lambda_2$ are the roots of

$$\lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = 0 \Rightarrow \lambda_1 = 1/2, \ \lambda_2 = 1$$

Thus

$$y_h(n) = c_1 2^{-n} + c_2 1^n = c_1 2^{-n} + c_2$$

Step 2

From the table 2 the choice for $y_p(n)$ is

$$d_1 + d_2 3^{-n}$$

but since this solution and the homogeneous solution have a common term that is the constant, the modified choice is

$$y_p(n) = d_1 n + d_2 3^{-n}$$

Substituting $y_p(n)$ into the Diff. Eq. gives

$$(d_1 n + d_2 3^{-n}) - \frac{3}{2}\left[d_1(n-1) + d_2\, 3^{-(n-1)}\right] + \frac{1}{2}\left[d_1(n-2) + d_2\, 3^{-(n-2)}\right]$$

$$= 1 + 3^{-n}$$

This gives

$$\frac{1}{2} d_1 + d_2 3^{-n} = 1 + 3^{-n} \implies d_1 = 2 \;,\; d_2 = 1$$

Thus

$$y(n) = y_h(n) + y_p(n)$$

$$= c_1 2^{-n} + c_2 + 2n + 3^{-n}$$

Step 3

Apply the initial conditions $y(-1) = 2 \;,\; y(-2) = 0$

$$\begin{cases} 2c_1 + c_2 = 1 \\ 4c_1 + c_2 = -5 \end{cases} \implies c_1 = -3 \;,\; c_2 = 7$$

or

$$y(n) = (-3)\, 2^{-n} + 7 + 2n + 3^{-n} \;,\quad n \geq 0$$

Example 2

Find the particular solution for the 1st order difference equation

$$y(n) - \frac{1}{2} y(n-1) = \sin\left(\frac{n\pi}{2}\right) \;,\quad n \geq 0$$

$$y_p(n) = d_1 \sin\left(\frac{n\pi}{2}\right) + d_2 \cos\left(\frac{n\pi}{2}\right)$$

Substitute in the equation

$$d_1 \sin\left(\frac{n\pi}{2}\right) + d_2 \cos\left(\frac{n\pi}{2}\right) - \frac{1}{2}\left[d_1 \sin\left((n-1)\frac{\pi}{2}\right) + d_2 \cos\left((n-1)\frac{\pi}{2}\right)\right]$$

$$= \sin\left(n\frac{\pi}{2}\right)$$

Recall the following trigonometric identities

$$\begin{cases} \sin\left((n-1)\frac{\pi}{2}\right) = \sin\left(\frac{n\pi}{2} - \frac{\pi}{2}\right) = -\cos\frac{n\pi}{2} \\ \cos\left((n-1)\frac{\pi}{2}\right) = \cos\left(\frac{n\pi}{2} - \frac{\pi}{2}\right) = \sin\frac{n\pi}{2} \end{cases}$$

Thus we get

$$\left(d_1 - \frac{1}{2} d_2\right) \sin\left(n\frac{\pi}{2}\right) + \left(\frac{1}{2} d_1 + d_2\right) \cos\left(\frac{n\pi}{2}\right) = \sin\left(\frac{n\pi}{2}\right)$$

$$\begin{cases} d_1 - \frac{1}{2} d_2 = 1 \\ \frac{1}{2} d_1 + d_2 = 0 \end{cases} \Longrightarrow \begin{array}{l} d_1 = 4/5 \\ d_2 = -2/5 \end{array}$$

Thus $y_p(n) = 4/5 \sin\left(\frac{n\pi}{2}\right) - \frac{2}{5} \cos\left(\frac{n\pi}{2}\right)$, $n \geq 0$

Table 2 : Rules for Choosing Particular Solutions

| Terms in the forcing function | Choice of particular solution † |
|---|---|
| 1. A Constant | $d$; $d$ is a Constant |
| 2. $b_1 n^K$ ; $b_1$ is a Constant | $d_0 + d_1 n + d_2 n^2 + \cdots + d_K n^K$; $d_i$'s are Constants |
| 3. $b_2 \alpha^{\pm n}$ ; $b_2$ and $\alpha$ are Constants | Proportional to $\alpha^{\pm n}$ |
| 4. $b_3 \cos n\omega$ } $b_3$ and $b_4$ are Constants | $d_1 \sin n\omega + d_2 \cos n\omega$ |
| 5. $b_4 \sin n\omega$ } $b_3$ and $b_4$ are Constants | |

† : If a term in any of the particular solutions in this column is a part of the Complementary Solution, it is necessary to modify the corresponding choice by multiplying it by "n" before using it. If such a term appears "r" times in the Complementary solution, the corresponding choice must be multiplied by $n^r$.

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| 1. A Constant | $d$; $d$ is a constant |
| 2. $b_1 n^K$ ; $b_1$ is a constant | $d_0 + d_1 n + d_2 n^2 + \cdots + d_K n^K$; $d_i$'s are constants |
| 3. $b_2 \alpha^{\pm n}$ ; $b_2$ and $\alpha$ are constants | proportional to $\alpha^{\pm n}$ |
| 4. $b_3 \cos n\omega$ } $b_3$ and $b_4$ are constants | $d_1 \sin n\omega + d_2 \cos n\omega$ |
| 5. $b_4 \sin n\omega$ } $b_3$ and $b_4$ are constants | |

† : If a term in any of the particular solutions in this column is a part of the complementary solution, it is necessary to modify the corresponding choice by multiplying it by "n" before using it. If such a term appears "r" times in the complementary solution, the corresponding choice must be multiplied by $n^r$.

Table 1

# Solution to Homogeneous Difference equation

| Difference Equation | Characteristic Equation |
|---|---|
| $a_0 y(n) + a_1 y(n-1) + \cdots a_N y(n-N) = 0$ | $a_0 \lambda^N + a_1 \lambda^{N-1} + \cdots a_N = 0$ |
| Case 1<br>$\lambda_1 \neq \lambda_2 \neq \lambda_3 = \cdots \neq \lambda_N$ | $y_h(n) = \sum_{i=1}^{N} c_i \lambda_i^n$ |
| Case 2<br>$\lambda_j \equiv$ root of multiplicity $m$ | $y_h(n) = \sum_{i=1}^{N-m} c_i \lambda_i^n + c_j \lambda_j^n + c_{j+1} n \lambda_j^n + \cdots c_{j+m} n^{m-1} \lambda_j^n$ |
| Case 3<br>Pair of roots $a \pm jb$ of multiplicity $m$ | $y_h(n) = \sum_{i=1}^{N-2m} c_i \lambda_i^n + d_1 r^n \cos n\theta + d_2 n r^n \cos n\theta + \cdots d_m n^{m-1} r^n \cos n\theta + d_{m+1} r^n \sin n\theta + d_{m+2} n r^n \sin n\theta + \cdots d_{2m} n^{m-1} r^n \sin n\theta$<br>$r = \sqrt{a^2+b^2}$<br>$\cos\theta = \frac{a}{r}$, $\sin\theta = b/r$<br>$-\pi < \theta < \pi$ |

## 2- The z- Transform

The role of z-transform to discrete-time systems is similar to that of the Laplace transform to continuous-time systems. In recent years, the time domain approach using state-space method has gained great significance in studies of discrete-time systems due to its great versatility and unified trentment to analysis and design problems. However, the importance of z-transform and its role in classical analysis and design has its own place in the real world.

The z-transform of sequence $\{x(n)\}$ is defined as

$$X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \qquad \text{Two-sided } z\text{-transform}$$

If $x(n) = 0$ for $\forall n<0$ then

$$X(z) = Z\{x(n)\} = \sum_{n=0}^{\infty} x(n)\, z^{-n} \qquad \text{One-sided } z\text{-transform}$$

Example 1

Find the z-transforms of

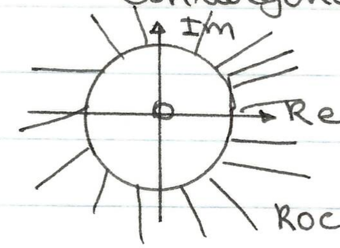
$$x(n) = a^n u(n) \text{ and } x(n) = a^n u(-n)$$

We have

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

Use $\displaystyle\sum_{r=0}^{\infty} r^n = \frac{1}{1-r} \quad |r|<1$ $\Longrightarrow$ $X(z) = \dfrac{1}{1-a/z}$ if $\left|\dfrac{a}{z}\right|<1$

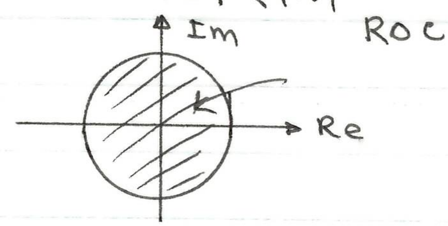
or $X(z) = \dfrac{z}{z-a}$ $\quad |z| > |a|$ $\quad$ ROC : Region of Convergence




For $x(n) = a^n u(-n)$ we have

$$X(z) = \sum_{n=-\infty}^{0} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n$$

$$= \frac{1}{1-\frac{z}{a}} = \frac{-a}{z-a} \qquad \text{for } \left|\frac{z}{a}\right| < 1 \text{ or } |z| < |a|$$




## Example 2

Find the z-Transform of

$$x(n) = A \sin(\omega_o n T) \quad , \quad \forall\, n \geq 0$$

Then

$$X(z) = \sum_{n=0}^{\infty} A \sin(\omega_o n T)\, z^{-n}$$

$$= A \sum_{n=0}^{\infty} \left( \frac{e^{j\omega_o n T} - e^{-j n \omega_o T}}{2j} \right) z^{-n}$$

$$= \frac{A}{2j} \left[ \sum_{n=0}^{\infty} \left(e^{j\omega_o T} z^{-1}\right)^n - \sum_{n=0}^{\infty} \left(e^{-j\omega_o T} z^{-1}\right)^n \right]$$

$$= \frac{A}{2j} \left[ \frac{1}{1-e^{j\omega_o T} z^{-1}} - \frac{1}{1-e^{-j\omega_o T} z^{-1}} \right]$$

$$= 2A \frac{z^{-1} \sin \omega_o T}{1 - 2z^{-1} \cos \omega_o T + z^{-2}}$$

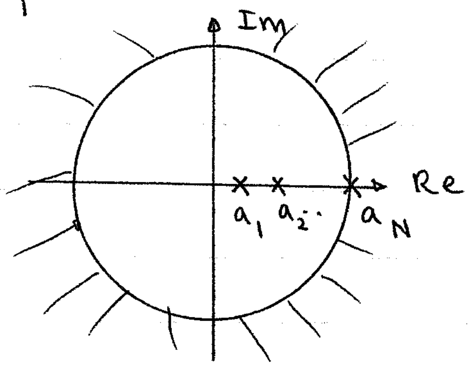
This exists when $\left| \dfrac{e^{\pm j\omega_o T}}{z} \right| < 1$ or $|z| > 1$ $\quad$ ROC

Remarks

1- For all right sided sequences the ROC is outside a circle; and for all the left-sided sequences the ROC is inside a circle.

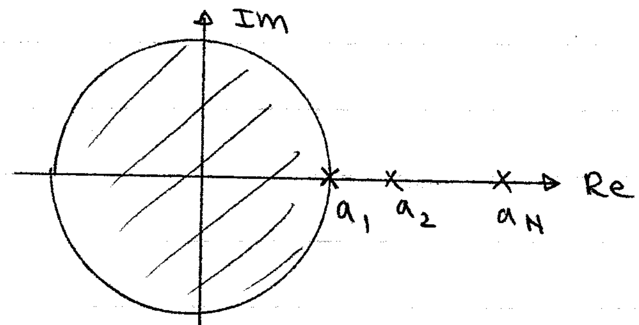
2- For right sided sequences the ROC is bounded on the inside by the pole with largest magnitude and on the outside by infinity ($\infty$).

For left-sided sequences the ROC is bounded on the outside by the pole with smallest magnitude and on the inside by zero.




R-Sided ROC

$a_N > a_{N-1} > \cdots > a_1$

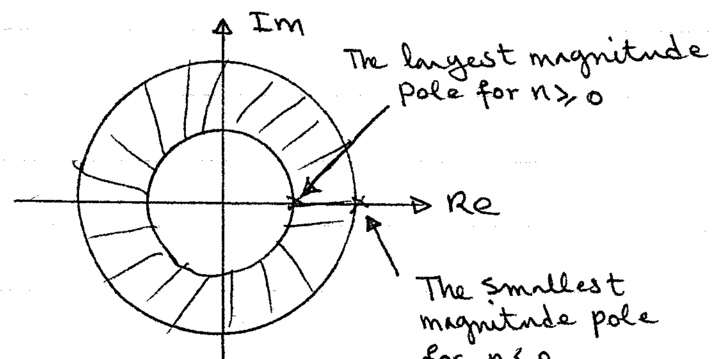



L-Sided ROC

$a_1 < a_2 < \cdots < a_N$

For a two-sided sequence ROC is bounded on the inside by the pole with the largest magnitude that contributes for $n \geq 0$ and on the outside by the pole with the smallest magnitude that contributes for $n \leq 0$.

Note: An ROC should not enclose a pole.



## Example

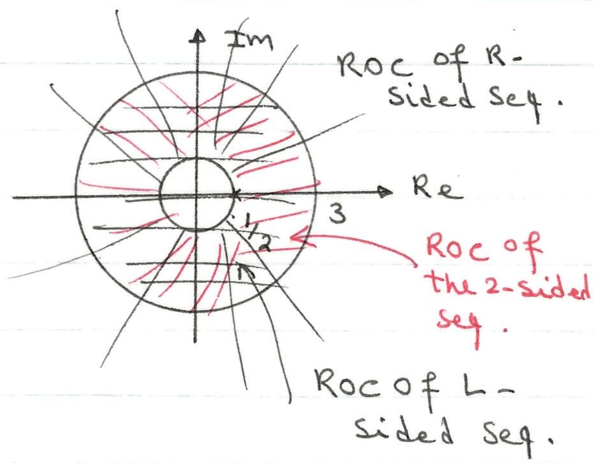
Let $x(n) = \begin{cases} (1/2)^n & n \geq 0 \\ 3^n & n \leq -1 \end{cases}$

Find $X(z)$ and the region of convergence.

$$X(z) = \sum_{n=-\infty}^{-1} x(n) z^{-n} + \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} 3^n z^{-n} + \sum_{n=0}^{\infty} (1/2)^n z^{-n}$$

$$= \sum_{n=-\infty}^{0} \left(\frac{3}{z}\right)^n - 1 + \sum_{n=0}^{\infty} \left(1/2z\right)^n$$




$$= \frac{1}{1-\frac{z}{3}} - 1 + \frac{1}{1-1/2z} = \frac{3}{3-z} - 1 + \frac{z}{z-1/2}$$

$$= \frac{-5/2\, z}{(z-3)(z-1/2)}$$

## Properties of Z-Transform

### 1) Linearity

Let $Z\{x_1(n)\} = X_1(z)$ and $\qquad R_1 < |z| < R_2$

$Z\{x_2(n)\} = X_2(z) \qquad R_3 < |z| < R_4$

Then $Z\{a x_1(n) + b x_2(n)\} = a X_1(z) + b X_2(z) \qquad R_5 < |z| < R_6$

If poles of $a X_1(z) + b X_2(z)$ = poles of $X_1(z) \cup$ poles of $X_2(z)$

then

$R_5 = \text{Max}[R_1, R_3]$ and $R_6 = \text{M}\overset{in}{\text{ax}}[R_2, R_4]$

## Remark

If the linear combination is such that some zeros are introduced which cancel poles, then the ROC may be larger. For example both $a^n u(n)$ and $a^n u(n-1)$ have ROC defined by $|z| > |a|$ but sequence $[a^n u(n) - a^n u(n-1)] = \delta(n)$ has a region of Convergence that is the entire $z$-plane.

$$Z\{a^n u(n)\} = \frac{z}{z-a} \qquad \text{ROC } |z| > |a|$$

$$Z\{a^n u(n-1)\} = \frac{z}{z-a} - 1 = \frac{a}{z-a} \qquad \text{ROC } |z| > |a|$$

and $Z\{a^n u(n) - a^n u(n-1)\} = Z\{\delta(n)\} = \frac{z}{z-a} - \frac{a}{z-a} = 1$

ROC is entire $z$-plane

## 2- Shifting in time-domain

Let $\quad Z\{x(n)\} = X(z) \qquad R_1 < |z| < R_2$

then $\quad Z\{x(n-n_0)\} = z^{-n_0} X(z) \qquad R_1 < |z| < R_2$ (when IC's are zero)

or $\quad Z\{x(n-n_0)\} = z^{-n_0}\left[X(z) + \sum_{k=-n_0}^{-1} x(k) z^{-k}\right]$, $x(-1), \dots x(-n_0)$ are IC's

ROC's of both are the same except may be at $z=0$ and $z=\infty$.

### Example

$Z\{\delta(n)\} = 1$ is Convergent anywhere in $z$-plane but $Z\{\delta(n-1)\}$ $= z^{-1}$ does not Converge at $z=0$ and the $z$-transform of $Z\{\delta(n+1)\} = z$ does not Converge at $z = \infty$.

Additionally we have

$$Z\{x(n+n_0)\} = z^{n_0}\left[X(z) - \sum_{k=0}^{n_0-1} x(k) z^{-k}\right]$$