Example 4: Prove Property 6 of two WSS processes $X(t)$ and $Y(t)$, i.e.

$$|R_{XY}(\tau)| \le [R_{XX}(0)R_{YY}(0)]^{1/2}$$

Solution:

Start with the linear combination $Y(t+\tau)+\beta X(t)$, where β is any number, and find its MS value (positive)

$$E[\{Y(t+\tau)+\beta X(t)\}^2] = E[Y^2(t+\tau)] + \beta^2 E[X^2(t)] + 2\beta E[X(t)Y(t+\tau)]$$

$$= R_{YY}(0) + \beta^2 R_{XX}(0) + 2\beta R_{XY}(\tau)$$

To assure that this quadratic is non-negative $a\beta^2 + 2b\beta + c \ge 0$

$b^2 - ac \le 0$ or

$$|R_{XY}(\tau)| \le [R_{XX}(0)R_{YY}(0)]^{1/2}$$

Example 5: Define two r.p.'s by $X(t) = p_1(t+\varepsilon)$ and $Y(t) = p_2(t+\varepsilon)$, when $p_1(t)$ and $p_2(t)$ are both periodic waveforms with period T and ε is a r.v. uniformly distributed on the interval (0,T). Find an expression for the cross-correlation function $E[X(t)Y(t+\tau)]$. Are these r.p.'s jointly WSS?

Solution:

$$R_{XY}(t,t+\tau) = E[X(t)Y(t+\tau)]$$

$$= E[p_1(t+\varepsilon)p_2(t+\varepsilon+\tau)]$$

$$Use \quad E[g(\varepsilon)] = \int_{-\infty}^{\infty} g(\varepsilon) f_E(\varepsilon)d\varepsilon \quad then$$

$$E[p_1(t+\varepsilon)p_2(t+\varepsilon+\tau)] = \int_{-\infty}^{\infty} p_1(t+\varepsilon)p_2(t+\varepsilon+\tau)f_E(\varepsilon)d\varepsilon$$

$$= \frac{1}{T}\int_0^T p_1(t+\varepsilon)p_2(t+\varepsilon+\tau)d\varepsilon$$

Let $\eta = t + \varepsilon \Rightarrow d\eta = d\varepsilon$ and

$$R_{XY}(t,t+\tau) = \frac{1}{T}\int_t^{t+T} p_1(\eta)p_2(\eta+\tau)d\eta$$

Since $p_1(\eta)$ and $p_2(\eta)$ are periodic with period T, we can write,

$$R_{XY}(t,t+\tau) = \frac{1}{T}\int_0^T p_1(\eta)p_2(\eta+\tau)d\tau$$

$$= R_{XY}(0,\tau) = R_{XY}(\tau) \quad \text{i.e. jointly WSS.}$$

Example 6: An ensemble member of a stationary r.p. $X(t)$ is sampled at N times t_i. i = 1, 2, … N. By treating the samples as r.v.'s $X_i = X(t_i)$, an estimate or measurement $\hat{X}$ of mean value $\overline{X}$ = E[X(t)] of the process is normally formed by *time averaging* the samples:

$$\hat{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$$

(a) Show that $E[\hat{X}] = \overline{X}$

(b) If the samples are separated far enough in time so that the r.v.'s X_i can be considered statistically independent, show that the variance of the estimate of the process mean is

$$(\sigma_{\hat{X}})^2 = \frac{\sigma_X^2}{N}$$

Solution:

(a)

$$E[\hat{X}] = \frac{1}{N}E\left[\sum_{i=1}^{N} X_i\right] = \frac{1}{N}\sum_{i=1}^{N} E[X_i]$$

$$= \frac{1}{N}\sum_{i=1}^{N} E[X(t_i)] = \frac{1}{N}\sum_{i=1}^{N} \overline{X}$$

$$= \frac{1}{N}.N\overline{X} = \overline{X} \longleftarrow \quad \text{i.e. unbiased in mean}$$

(b) $$\sigma^2_{\hat{\bar{X}}} = E\left[\left[\hat{\bar{X}} - E\left[\hat{\bar{X}}\right]\right]^2\right]$$

$$= E\left[\left[\frac{1}{N}\sum_{i=1}^{N} X_i - \frac{1}{N}\sum_{i=1}^{N}\bar{X}\right]^2\right]$$

$$= E\left[\left[\frac{1}{N}\sum_{i=1}^{N}\left(X_i - X\right)\right]^2\right]$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E\left[\left(X_i - \bar{X}\right)\left(X_j - \bar{X}\right)\right]$$

But, $E\left[\left(X_i - \bar{X}\right)\left(X_j - \bar{X}\right)\right] = 0$ for $i \neq j$

$= \sigma_X^2$ for $i = j$

Thus, $$\sigma^2_{\hat{\bar{X}}} = \frac{1}{N^2}\sum_{i=1}^{N}\sigma^2_{X_i}$$

$$= \frac{1}{N^2}.N\sigma^2_{X_i} = \frac{\sigma_X^2}{N} \longleftarrow$$

Example 7: If X(t) is WSS with mean $\bar{X}$ and ACF $R_{XX}(\tau)$. Find the expression for mean, ACF and $R_{XY}(\tau)$ of

$$Y(t) = a\,X(t) + b\,X(t-1)$$

Solution:

(a) $$E[Y(t)] = \bar{Y} = aE[X(t)] + bE[X(t-1)]$$
$$= (a+b)\bar{X} \longleftarrow$$

(b) $$R_{YY}(\tau) = E[Y(t)Y(t+\tau)]$$
$$= E[(aX(t) + bX(t-1))(aX(t+\tau) + X(t-1+\tau))]$$
$$= a^2E[(X(t)X(t+\tau))] + abE[(X(t)X(t+\tau-1))] +$$
$$baE[(X(t-1)X(t+\tau))] + b^2E[(X(t-1)X(t+\tau-1))]$$
$$= \left(a^2 + b^2\right)R_{XX}(\tau) + abR_{XX}(\tau-1) + baR_{XX}(\tau+1) \longleftarrow$$

(c) $$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$
$$= E[X(t)(aX(t+\tau) + X(t-1+\tau))]$$
$$= aE[(X(t)X(t+\tau))] + bE[(X(t)X(t+\tau-1))]$$
$$= aR_{XX}(\tau) + bR_{XX}(\tau-1) \longleftarrow$$

Example 8: Consider $Z = XY$, where X and Y are Independent Gaussian r.v.'s with $X \sim N(\bar{X}, \sigma_X^2)$ and $Y \sim N(\bar{Y}, \sigma_Y^2)$. Find $\bar{Z}$ and σ_Z^2 in terms of statistics of X and Y.

Solution:

(a) $$E[Z] = E[XY] = E[X]E[Y]$$
$$\bar{Z} = \bar{X}.\bar{Y} \longleftarrow$$

(b) $$E[Z^2] = E[X^2Y^2] = E[X^2]E[Y^2]$$
$$= \left(\sigma_X^2 + \bar{X}^2\right)\left(\sigma_Y^2 + \bar{Y}^2\right)$$

Now, $$\sigma_Z^2 = E[Z^2] - \bar{Z}^2$$
$$= \left(\sigma_X^2 + \bar{X}^2\right)\left(\sigma_Y^2 + \bar{Y}^2\right) - \bar{X}^2.\bar{Y}^2$$
$$= \sigma_X^2\sigma_Y^2 + \bar{X}^2\sigma_Y^2 + \bar{Y}^2\sigma_X^2 \longleftarrow$$

Example 9: Given $X(t) = A\cos(\omega_0 t + \theta)$

$Y(t) = B\cos(\omega_1 t + \phi)$

A, B, ω_0, ω_1: Constants

θ, ϕ : Independent r.v.'s uniformly distributed over $(0, 2\pi)$.

(a) Show that $X(t)$, $Y(t)$ are not jointly WSS.

(b) If $\theta = \phi$, show that $X(t)$, $Y(t)$ are not jointly WSS unless $\omega_0 = \omega_1$.

Solution:

Use: $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$= A^2 E[\cos(\omega_0 t + \theta)\cos(\omega_0 t + \omega_0\tau + \theta)]$$

$$= \frac{A^2}{2} E[\cos(\omega_0\tau) + \cos(2\omega_0 t + \omega_0\tau + 2\theta)]$$

$$R_{XX}(t, t+\tau) = \frac{A^2}{2}\cos(\omega_0\tau) + \frac{A^2}{2} E[\cos(2\omega_0 t + \omega_0\tau + 2\theta)]$$

Now,

$$E[\cos(2\omega_0 t + \omega_0\tau + 2\theta)] = \int_0^{2\pi} \cos(2\omega_0 t + \omega_0\tau + 2\theta)\frac{1}{2\pi}d\theta = 0$$

Thus, $R_{XX}(t, t+\tau) = \frac{A^2}{2}\cos(\omega_0\tau) = R_{XX}(\tau)$

and $R_{YY}(t, t+\tau) = \frac{B^2}{2}\cos(\omega_1\tau) = R_{YY}(\tau)$

Also E[X(t)] = E[Y(t)] = 0, since

$$E[X(t)] = A\int_0^{2\pi} \cos(2\omega_0 t + \theta)\frac{1}{2\pi}d\theta = 0$$

i.e., X(t) and Y(t) are individually WSS. But now,

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)]$$

$$= AB\, E[\cos(\omega_0 t + \theta)\cos(\omega_1 t + \omega_1\tau + \phi)]$$

$$= \frac{AB}{2} E[\cos\{(\omega_1 - \omega_0)t + \omega_1\tau + \phi - \theta\}$$

$$+ \cos\{(\omega_1 + \omega_0)t + \omega_1\tau + \phi + \theta\}]$$

$$= 0$$

which is not a function of τ, so $X(t)$, $Y(t)$ are not jointly WSS.

(b) If $\theta = \phi \Rightarrow R_{XY}(t, t+\tau) = \frac{AB}{2}\cos\{(\omega_1 - \omega_0)t + \omega_1\tau\}$

which is again function of t. When $\omega_0 = \omega_1$,

$$R_{XY}(t, t+\tau) = \frac{AB}{2}\cos(\omega_1\tau) = R_{XY}(\tau) \longleftarrow$$

Time Averages and Ergodicity:

In practice, we would like to deal with only a single sample function rather than the ensemble of functions. For example, we may wish to infer the probability law or certain averages of the r.p. from the measurements on a single member of the ensemble set.

The time average for the ensemble set is defined as,

$$\bar{x} = \langle X(t)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)dt \quad \rightarrow \text{Continuous time case}$$

$$\bar{x} = \langle X(n)\rangle = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} x(n) \quad \rightarrow \text{Random sequence case}$$

Then, the time average autocorrelation function will be,

$$r_{XX}(\tau) = \langle X(t)X(t+\tau)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

$$r_{XX}(m) = \langle X(n)X(n+m)\rangle = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} x(n)x(n+m)$$

If these time averages are computed for each sample function within the ensemble set, these values, i.e., $\bar{x}$, $r_{XX}(\tau)$ or $r_{XX}(m)$ form r.v.'s themselves. Now, it is obvious (due to WSS) that taking $E[.]$ (ensemble average) of these r.v.'s yields,

$$E[\bar{x}] = \overline{X}$$

and $$E[r_{XX}(\tau)] = R_{XX}(\tau)$$

Now if we assume that the deviations of these averages are zero from one sample to another, then

$$\bar{x} = \overline{X}$$
$$r_{XX}(\tau) = R_{XX}(\tau)$$

In other words, *the time averages obtained from one signal (time series) are equal to the ensemble averages.*

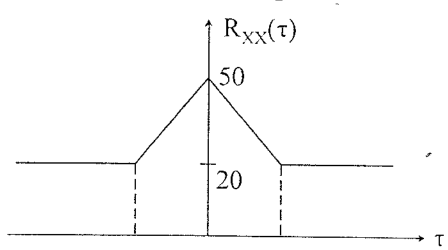
Ergodic r.p.'s:

A r.p. is said to be "ergodic" if the *time averages computed for a sample function can be used as an approximation to the corresponding ensemble averages of the r.p.* This is a very restrictive form of stationarity (presupposes stationarity).

$$\bar{x} = \overline{X}$$
$$r_{XX}(\tau) = R_{XX}(\tau)$$

This assumption (even though not very practical) simplifies the inference of the statistics greatly as it allows all the statistics to be measured from only one sample function.

Example 10: Given an ergodic r.p. with ACF as shown



Find (a) $E[X(t)]$, (b) $E[X^2(t)]$, (c) σ_X^2.

Solution:

(a) From the properties of ergodic r.p.,

$$\lim_{\tau\to\infty} R_{XX}(\tau) = \mu_X^2$$

Thus, $20 = \mu_X^2 \quad \Rightarrow \mu_X = E[X(t)] = \sqrt{20}$

(b) $E[X^2(t)] = R_{XX}(0) = 50$

(c) $\sigma_X^2 = E[X^2(t)] - \mu_X^2$
$= 50 - 20$
$= 30$

Reading Assignment For Week 12: Sections 6.3-6.4 (Peebles)

Power Spectrum or Spectral Density Function (SDF):

Let X(t) be a WSS r.p. with ACF $R_{XX}(\tau)$, then SDF is

n k

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau = \sum_{k=-\infty}^{\infty} R_{XX}(k)\, e^{-j\Omega k}$$

$e^{j\Omega}$ π $e^{j\Omega k}$

and $$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega \quad \Omega$$

k π

change all to discrete case

Thus, $R_{XX}(\tau) \xleftrightarrow{F} S_{XX}(\omega)$ ***Wiener-Khintchine Relations***

k $e^{j\Omega}$

Properties:

(1) $S_{XX}(\omega)$ is real, and $S_{XX}(0) \geq 0$

(2) Since $R_{XX}(\tau)$ is real, $S_{XX}(-\omega) = S_{XX}(\omega)$, i.e., symmetrical.

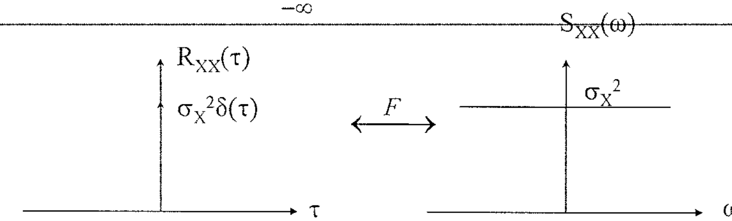
(3) $S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau)d\tau$

(4) $\sigma_X^2 = R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega$

Remark:

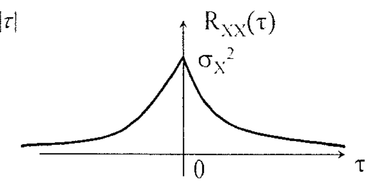
For a "white" noise process, $R_{XX}(\tau) = \sigma_X^2\delta(\tau)$

Thus, $S_{XX}(\omega) = \int_{-\infty}^{\infty} \sigma_X^2\delta(\tau)e^{-j\omega\tau}d\tau = \sigma_X^2$



i.e. contains all the frequencies with equal contribution.

Example 3: Random process X(t), which is WSS has an ACF given by, $R_{XX}(\tau) = \sigma_X^2 e^{-|\tau|}$



Find SDF.

Solution:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau = \sigma_X^2\int_{-\infty}^{\infty} e^{-|\tau|}e^{-j\omega\tau}d\tau$$

$$= \sigma_X^2\int_{-\infty}^{0} e^{+\tau}e^{-j\omega\tau}d\tau + \sigma_X^2\int_{0}^{\infty} e^{-\tau}e^{-j\omega\tau}d\tau$$

$$= \sigma_X^2\left[\int_{-\infty}^{0} e^{+\tau}e^{-j\omega\tau}d\tau + \int_{0}^{\infty} e^{-\tau}e^{-j\omega\tau}d\tau\right]$$

$$= \sigma_X^2\left[\frac{1}{1-j\omega} + \frac{1}{1+j\omega}\right] = \frac{2\sigma_X^2}{1+\omega^2} \longleftarrow$$

Example 4: A WSS r.p. X(t) has an SDF given by,

$$S_{XY}(\omega) = \frac{8}{(\omega^2+9)^2}$$

Find its ACF.

Solution: Let us rewrite $S_{XY}(\omega)$ as

$$S_{XY}(\omega) = G(\omega).G(\omega) = \left[\frac{\sqrt{8}}{(\omega^2+9)}\right]^2$$

Now, using the convolution property, we have

$$R_{XX}(\tau) = g(\tau) * g(\tau) = \int g(\xi)g(\tau-\xi)d\xi$$

But, from Table E.1. (page 434), we have

$$G(\omega) = \left[\frac{\sqrt{8}}{(\omega^2+9)}\right] \xleftrightarrow{F^{-1}} g(\tau) = \frac{\sqrt{8}}{6}e^{-3|\tau|}$$

Thus,

$$R_{XX}(\tau) = \frac{8}{36}\int_{-\infty}^{\infty} e^{-3|\xi|}e^{-3|\tau-\xi|}d\xi$$

$$= \frac{8}{36}\int_{-\infty}^{0} e^{3\xi}e^{-3(\tau-\xi)}d\xi + \frac{8}{36}\int_{0}^{\tau} e^{-3\xi}e^{-3(\tau-\xi)}d\xi + \frac{8}{36}\int_{\tau}^{\infty} e^{-3\xi}e^{3(\tau-\xi)}d\xi$$

$$= \frac{2}{9}e^{-3\tau}(\tau+1/3), \quad \tau \ge 0$$

Since $R_{XX}(-\tau) = R_{XX}(\tau)$, then

$$R_{XX}(\tau) = \frac{2}{9}e^{-3|\tau|}(|\tau|+1/3)$$

Cross Power Spectrum:

Consider two r.p.'s X(t) and Y(t) with cross-correlation function $R_{XY}(\tau)$. We assume that X(t), Y(t) are individually and jointly WSS. Then, *cross power spectrum* is defined by,

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}d\tau$$

and
$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}d\omega$$

Properties:

(1) $S_{XY}(\omega) = S_{YX}(-\omega) = S^*_{YX}(\omega)$.

(2) The real part of $S_{XY}(\omega)$ or $S_{YX}(\omega)$ has even symmetry while the imaginary part has odd symmetry.

$$\mathrm{Re}\{S_{XY}(\omega)\} = \mathrm{Re}\{S_{XY}(-\omega)\} = \mathrm{Re}\{S_{YX}(\omega)\} = \mathrm{Re}\{S_{YX}(-\omega)\}$$

and

$$\mathrm{Im}\{S_{XY}(\omega)\} = -\mathrm{Im}\{S_{XY}(-\omega)\} = \mathrm{Im}\{S_{YX}(-\omega)\} = -\mathrm{Im}\{S_{YX}(\omega)\}$$

(3) If X(t) and Y(t) are orthogonal, then $S_{XY}(\omega) = S_{YX}(\omega) = 0$.

(4) $$R_{XY}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)d\omega$$

Example 5: If X(t) is stationary and $Y(t) = A + B\,X(t)$

with A, B real constants. Find R_{YY}, R_{XY}, S_{YY} and S_{XY}.

Solution:

$$R_{YY}(\tau) = E[Y(t)Y(t+\tau)]$$
$$= E[(A+BX(t))(A+BX(t+\tau))]$$
$$= E\left[A^2 + ABX(t) + ABX(t+\tau) + B^2X(t)X(t+\tau)\right]$$

$$R_{YY}(\tau) = A^2 + 2AB\mu_X + B^2R_{XX}(\tau)$$

$$S_{YY}(\omega) = F\{R_{YY}(\tau)\}$$
$$= \left(A^2 + 2AB\mu_X\right)2\pi\delta(\omega) + B^2S_{XX}(\omega)$$

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$
$$= E[X(t)(A+BX(t+\tau))] = E[AX(t) + AB\,X(t)X(t+\tau)]$$
$$= A\mu_X + AB\,R_{XX}(\tau)$$

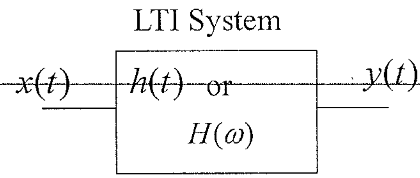
$$S_{XY}(\omega) = F\{R_{XY}(\tau)\}$$
$$= A\mu_X 2\pi\delta(\omega) + AB\,S_{XX}(\omega)$$

Convert to discrete-time case

Consider a causal stable LTI system with impulse response h(t) and let x(t) be a real input signal i.e. a sample function of a WSS random process X(t). Then, we have

$$y(t) = \int_{-\infty}^{\infty} x(\eta)h(t-\eta)d\eta$$

$$= \int_{-\infty}^{\infty} h(\eta)x(t-\eta)d\eta$$



The same convolution equation holds for the entire ensemble sets X(t) and Y(t) i.e. $Y(t) = \int_{-\infty}^{\infty} X(\eta)h(t-\eta)d\eta = \int_{-\infty}^{\infty} h(\eta)X(t-\eta)d\eta$

Now, given the mean and autocorrelation function, μ_X and $R_{XX}(\tau)$ of X(t), the statistics of Y(t) can be determined as follows.

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Mean

$$\mu_Y = E[Y(t)] = \int_{-\infty}^{\infty} h(\eta)E[X(t-\eta)]d\eta = \mu_X \int_{-\infty}^{\infty} h(\eta)d\eta$$

$= \mu_X H(0)$ i.e. mean of Y(t) is constant

where $H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$ is the Frequency Response of the LTI System

Cross-Correlation and Cross Power Spectrum

$$R_{XY}(t,t+\tau) = E[X(t)Y(t+\tau)] = E\left[X(t)\int_{-\infty}^{\infty} h(\eta)X(t+\tau-\eta)d\eta\right]$$

$$= \int_{-\infty}^{\infty} h(\eta)E[X(t)X(t+\tau-\eta)]d\eta$$

or $R_{XY}(t,t+\tau) = \int_{-\infty}^{\infty} h(\eta)R_{XX}(\tau-\eta)d\eta$

If X(t) is WSS, then $\boxed{R_{XY}(\tau) = h(\tau) * R_{XX}(\tau)}$

The cross-power spectrum is then given by

$$\boxed{S_{XY}(\omega) = H(\omega)S_{XX}(\omega)}$$

Remark

For a white process $R_{XX}(\tau) = \sigma_X^2\delta(\tau)$ and $S_{XX}(\omega) = \sigma_X^2$

Thus,

$$\boxed{\begin{aligned} S_{XY}(\omega) &= \sigma_X^2 H(\omega) \\ R_{XY}(\tau) &= \sigma_X^2 h(\tau) \end{aligned}}$$

i.e. for a white-noise input, the cross-correlation between input and output of a linear system is proportional to the impulse response of the system (or the cross-power spectrum is proportional to the frequency response).

Auto-Correlation and Power Spectrum

$$R_{YY}(t,t+\tau) = E[Y(t)Y(t+\tau)] = E\left[Y(t)\int_{-\infty}^{\infty} h(\eta)X(t+\tau-\eta)d\eta\right]$$

$$= \int_{-\infty}^{\infty} h(\eta)E[Y(t)X(t+\tau-\eta)]d\eta$$

$$= \int_{-\infty}^{\infty} h(\eta)R_{XY}(\eta-\tau)d\eta$$

If X(t) is WSS, then $R_{YY}(\tau) = h(\tau) * R_{XY}(-\tau)$

On the other hand,

$$R_{XY}(\tau) = h(\tau) * R_{XX}(\tau) \Longrightarrow R_{XY}(-\tau) = h(-\tau) * R_{XX}(-\tau)$$

Thus,

$$\boxed{\begin{aligned} R_{YY}(\tau) &= h(\tau) * h(-\tau) * R_{XX}(-\tau) \\ &= h(\tau) * h(-\tau) * R_{XX}(\tau) \end{aligned}}$$

The power spectrum is

$$\boxed{\begin{aligned} S_{YY}(\omega) &= H(\omega)H^*(\omega)S_{XX}(\omega) \\ &= |H(\omega)|^2 S_{XX}(\omega) \end{aligned}}$$ Spectral Factorization

Important Remarks

1. For a white process $R_{XX}(\tau) = \sigma_X^2 \delta(\tau)$ and $S_{XX}(\omega) = \sigma_X^2$

Thus, $$S_{YY}(\omega) = \sigma_X^2 |H(\omega)|^2$$

i.e. the output signal is ***NOT*** white and is correlated (colored noise) since the power spectrum is dependent on frequency.

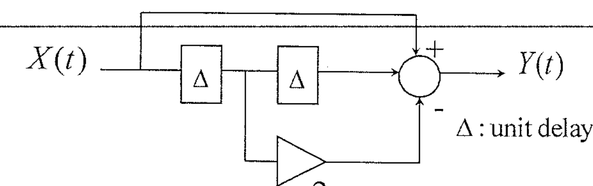
2. Since mean of Y(t) is constant and its auto-correlation and cross-correlation functions are only dependent on τ, the output r.p. is also individually and jointly WSS with the input X(t).

Example 1:

A white WSS random process X(t) with uniform PDF

$$f_{X(t)}(x(t)) = \begin{cases} 1 & -1/2 \le x(t) \le 1/2 \\ 0 & elsewhere \end{cases}$$

is applied to the input of a system with structure shown below. Find the output *power spectral density.*



Solution

$$S_{YY}(\omega) = \sigma_X^2 H(\omega)H(-\omega) = \sigma_X^2 |H(\omega)|^2$$

We first find σ_X^2 and then $H(\omega)$.

$$\mu_X = E[X(t)] = \int_{-1/2}^{1/2} x f_X(x)dx = 0 \qquad \text{Since WSS}$$

and

$$\sigma_X^2 = E[X^2(t)] = \int_{-1/2}^{1/2} x^2 f_X(x)dx = 1/12$$

Since the input is a stationary white process

$$R_{XX}(\tau) = E[X(t)X(t+\tau)] = \sigma_X^2 \delta(\tau) = 1/12\ \delta(\tau)$$

Thus, the power spectral density of the input is $S_{XX}(\omega) = \sigma_X^2$

From the system structure, $y(t) = x(t-2) - 2x(t-1)\cos\omega_o + x(t)$

Thus, $H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = e^{-j2\omega} - 2e^{-j\omega}\cos\omega_o + 1$

$$\begin{aligned} S_{YY}(\omega) &= \sigma_X^2 H(\omega)H(-\omega) \\ &= \frac{1}{12}(e^{2j\omega} + e^{-2j\omega} + 2 + 4\cos^2\omega_o - 4e^{j\omega}\cos\omega_o - 4e^{-j\omega}\cos\omega_o) \\ &= \frac{1}{12}(4\cos^2\omega + 4\cos^2\omega_o - 8\cos\omega\cos\omega_o) = \frac{1}{3}(\cos\omega - \cos\omega_o)^2 \end{aligned}$$

Example 2:

Consider a r.p. Y(t) defined as

$$Y(t) = \frac{1}{2T}\int_{t-T}^{t+T} X(\eta)\,d\eta$$

where X(t) is a WSS r.p. Show that

$$S_{YY}(\omega) = S_{XX}(\omega)\,\text{sinc}^2(\omega T) = S_{XX}(\omega)\left[\frac{\sin(\omega T)}{\omega T}\right]^2$$

Solution

Rewrite the expression for Y(t) as a convolution integral i.e.

$$Y(t) = \frac{1}{2T}\int_{t-T}^{t+T} X(\eta)\,d\eta = \int_{-\infty}^{\infty} X(\eta)h(t-\eta)\,d\eta$$

where

$$h(t-\eta) = \begin{cases} \frac{1}{2T} & t-T \le \eta \le t+T \\ 0 & elsewhere \end{cases}$$

That is, Y(t) is the output of a LTI system with symmetric impulse response h(t) given by

$$h(t) = \begin{cases} \frac{1}{2T} & -T \le t \le T \\ 0 & elsewhere \end{cases} \quad \text{and hence} \quad H(\omega) = \operatorname{sinc}(\omega T)$$

Thus, using the power spectrum equation for Y(t), we get

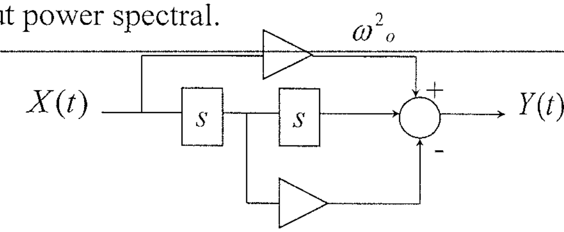
$$S_{YY}(\omega) = S_{XX}(\omega)\,|H(\omega)|^2 = S_{XX}(\omega)\operatorname{sinc}^2(\omega T)$$

Example 3-Another Version of Example 1:

A white WSS random process X(t) with uniform PDF

$$f_{X(t)}(x(t)) = \begin{cases} 1 & -1/2 \le x(t) \le 1/2 \\ 0 & elsewhere \end{cases}$$

is applied to the input of a filter with realization shown below. Find the output power spectral.



Solution

Use $\quad S_{YY}(\omega) = \sigma_X^2 H(\omega)H(-\omega) = \sigma_X^2\,|H(\omega)|^2$

We first find σ_X^2 and then $H(\omega)$.

$$\mu_X = E[X(t)] = \int_{-1/2}^{1/2} x f_X(x)\,dx = 0$$

$$\sigma_X^2 = E[X^2(t)] = \int_{-1/2}^{1/2} x^2 f_X(x)\,dx = 1/12$$

Since the process is white

$R_{XX}(\tau) = E[X(t)X(t+\tau)] = \sigma_X^2\delta(\tau) = 1/12\ \delta(\tau)$

Thus, the power spectral density of the input is $S_{XX}(\omega) = \sigma_X^2$

From the realization $\quad H(s) = s^2 - 2s\omega_o + \omega^2{}_o$

$$H(\omega) = (j\omega)^2 - 2j\omega\omega_o + \omega^2{}_o = (-\omega^2 + \omega^2{}_o) + j(-2\omega\omega_o)$$

$$|H(\omega)|^2 = (-\omega^2 + \omega^2{}_o)^2 + (-2\omega\omega_o)^2 = (\omega^2 + \omega^2{}_o)^2$$

Thus, $\quad S_{YY}(\omega) = \sigma_X^2\,|H(\omega)|^2 = \sigma_X^2(\omega^2 + \omega^2{}_o)^2$