Example 4: Prove Property 6 of two WSS processes *X(t)* and *Y(t)*, i.e.

$$|R_{XY}(\tau)| \le \left[R_{XX}(0)R_{YY}(0)\right]^{1/2}$$

Solution:

Start with the linear combination $Y(t+\tau)+\beta X(t)$, where β is any number, and find its MS value (positive)

$$E[\{Y(t+\tau)+\beta X(t)\}^2] = E[Y^2(t+\tau)] + \beta^2 E[X^2(t)] + 2\beta E[X(t)Y(t+\tau)]$$

$$= R_{YY}(0) + \beta^2 R_{XX}(0) + 2\beta R_{XY}(\tau)$$

To assure that this quadratic is non-negative $a\beta^2 + 2b\beta + c \ge 0$

$b^2 - ac \le 0$ or

$$|R_{XY}(\tau)| \le \left[R_{XX}(0)R_{YY}(0)\right]^{1/2}$$

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Example 5: Define two r.p.'s by $X(t) = p_1(t + \varepsilon)$ and $Y(t) = p_2(t + \varepsilon)$, when $p_1(t)$ and $p_2(t)$ are both periodic waveforms with period T and ε is a r.v. uniformly distributed on the interval (0,T). Find an expression for the cross-correlation function $E[X(t)Y(t + \tau)]$. Are these r.p.'s jointly WSS?

Solution:

$$R_{XY}(t,t+\tau) = E[X(t)Y(t+\tau)]$$

$$= E[p_1(t+\varepsilon)p_2(t+\varepsilon+\tau)]$$

$$Use \quad E[g(\varepsilon)] = \int_{-\infty}^{\infty} g(\varepsilon) f_E(\varepsilon) d\varepsilon \quad then$$

$$E[p_1(t+\varepsilon)p_2(t+\varepsilon+\tau)] = \int_{-\infty}^{\infty} p_1(t+\varepsilon)p_2(t+\varepsilon+\tau) f_{\mathrm{E}}(\varepsilon) d\varepsilon$$

$$= \frac{1}{T}\int_0^T p_1(t+\varepsilon)p_2(t+\varepsilon+\tau) d\varepsilon$$

Let $\eta = t + \varepsilon \quad \Rightarrow \quad d\eta = d\varepsilon$ and

$$R_{XY}(t,t+\tau)=\frac{1}{T}\int_{t}^{t+T}p_1(\eta)p_2(\eta+\tau)d\eta$$

Since $p_1(\eta)$ and $p_2(\eta)$ are periodic with period T, we can write,

$$R_{XY}(t,t+\tau)=\frac{1}{T}\int_{0}^{T}p_1(\eta)p_2(\eta+\tau)d\tau$$

$$=R_{XY}(0,\tau)=R_{XY}(\tau)$$ i.e. jointly WSS.

Example 6: An ensemble member of a stationary r.p. $X(t)$ is sampled at N times t_i, i = 1, 2, … N. By treating the samples as r.v.'s $X_i = X(t_i)$, an estimate or measurement $\hat{\bar{X}}$ of mean value $\bar{X}$ = E[X(t)] of the process is normally formed by *time averaging* the samples:

$$\hat{\bar{X}}=\frac{1}{N}\sum_{i=1}^{N}X_i$$

(a) Show that $E[\hat{\bar{X}}] = \bar{X}$

(b) If the samples are separated far enough in time so that the r.v.'s X_i can be considered statistically independent, show that the variance of the estimate of the process mean is

$$(\sigma_{\hat{\bar{X}}})^2 = \frac{\sigma_X^2}{N}$$

Solution:

(a)

$$E[\hat{\bar{X}}] = \frac{1}{N} E\left[\sum_{i=1}^{N} X_i\right] = \frac{1}{N}\sum_{i=1}^{N} E[X_i]$$

$$= \frac{1}{N}\sum_{i=1}^{N} E[X(t_i)] = \frac{1}{N}\sum_{i=1}^{N} \bar{X}$$

$$= \frac{1}{N}.N\bar{X} = \bar{X} \longleftarrow$$ i.e. unbiased in mean

(b)

$$\sigma^2_{\hat{\bar{X}}} = E\left[\left[\hat{\bar{X}} - E\left[\hat{\bar{X}}\right]\right]^2\right]$$

$$= E\left[\left[\frac{1}{N}\sum_{i=1}^{N} X_i - \frac{1}{N}\sum_{i=1}^{N} \bar{X}\right]^2\right]$$

$$= E\left[\left[\frac{1}{N}\sum_{i=1}^{N} \left(X_i - \bar{X}\right)\right]^2\right]$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E\left[\left(X_i - \bar{X}\right)\left(X_j - \bar{X}\right)\right]$$

But, $E\left[\left(X_i - \bar{X}\right)\left(X_j - \bar{X}\right)\right] = 0 \qquad \text{for } i \neq j$

$$= \sigma^2_{X_i} \qquad \text{for } i = j$$

Thus, $$\sigma^2_{\hat{X}} = \frac{1}{N^2}\sum_{i=1}^{N}\sigma^2_{X_i}$$

$$= \frac{1}{N^2}.N\sigma^2_{X_i} = \frac{\sigma^2_X}{N} \longleftarrow$$

Example 7: If X(t) is WSS with mean $\overline{X}$ and ACF $R_{XX}(\tau)$. Find the expression for mean, ACF and $R_{XY}(\tau)$ of

$$Y(t) = a\,X(t) + b\,X(t-1)$$

Solution:

(a) $$E[Y(t)] = \overline{Y} = aE[X(t)] + bE[X(t-1)]$$

$$= (a+b)\overline{X} \longleftarrow$$

$$\text{(b)}\quad R_{YY}(\tau) = E[Y(t)Y(t+\tau)]$$

$$= E[(aX(t) + bX(t-1))(aX(t+\tau) + X(t-1+\tau))]$$

$$= a^2 E[(X(t)X(t+\tau))] + abE[(X(t)X(t+\tau-1))] +$$

$$baE[(X(t-1)X(t+\tau))] + b^2 E[(X(t-1)X(t+\tau-1))]$$

$$= \left(a^2 + b^2\right) R_{XX}(\tau) + abR_{XX}(\tau-1) + baR_{XX}(\tau+1) \longleftarrow$$

$$\text{(c)}\quad R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

$$= E[X(t)(aX(t+\tau) + X(t-1+\tau))]$$

$$= aE[(X(t)X(t+\tau))] + bE[(X(t)X(t+\tau-1))]$$

$$= aR_{XX}(\tau) + bR_{XX}(\tau-1) \longleftarrow$$

Example 8: Consider $Z = XY$, where X and Y are Independent Gaussian r.v.'s with $X \sim N(\overline{X}, \sigma_X^2)$ and $Y \sim N(\overline{Y}, \sigma_Y^2)$. Find $\overline{Z}$ and σ_Z^2 in terms of statistics of X and Y.

Solution:

(a) $E[Z] = E[XY] = E[X]E[Y]$

$$\overline{Z} = \overline{X}.\overline{Y} \longleftarrow$$

(b) $E[Z^2] = E[X^2Y^2] = E[X^2]E[Y^2]$

$$= \left(\sigma_X^2 + \overline{X}^2\right)\left(\sigma_Y^2 + \overline{Y}^2\right)$$

Now, $\sigma_Z^2 = E[Z^2] - \overline{Z}^2$

$$= \left(\sigma_X^2 + \overline{X}^2\right)\left(\sigma_Y^2 + \overline{Y}^2\right) - \overline{X}^2.\overline{Y}^2$$

$$= \sigma_X^2\sigma_Y^2 + \overline{X}^2\sigma_Y^2 + \overline{Y}^2\sigma_X^2 \longleftarrow$$

Example 9: Given $X(t) = A\cos(\omega_0 t + \theta)$

$$Y(t) = B\cos(\omega_1 t + \phi)$$

A, B, ω_0, ω_1: Constants

θ, ϕ : Independent r.v.'s uniformly distributed over $(0, 2\pi)$.

(a) Show that $X(t)$, $Y(t)$ are not jointly WSS.

(b) If $\theta = \phi$, show that $X(t)$, $Y(t)$ are not jointly WSS unless $\omega_0 = \omega_1$.

Solution:

Use:

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$= A^2 E[\cos(\omega_0 t + \theta)\cos(\omega_0 t + \omega_0 \tau + \theta)]$$

$$= \frac{A^2}{2} E[\cos(\omega_0 \tau) + \cos(2\omega_0 t + \omega_0 \tau + 2\theta)]$$

$$R_{XX}(t,t+\tau)=\frac{A^2}{2}\cos(\omega_0\tau)+\frac{A^2}{2}E[\cos(2\omega_0 t+\omega_0\tau+2\theta)]$$

Now,

$$E[\cos(2\omega_0 t+\omega_0\tau+2\theta)]=\int_0^{2\pi}\cos(2\omega_0 t+\omega_0\tau+2\theta)\frac{1}{2\pi}d\theta=0$$

Thus, $$R_{XX}(t,t+\tau)=\frac{A^2}{2}\cos(\omega_0\tau)=R_{XX}(\tau)$$

and $$R_{YY}(t,t+\tau)=\frac{B^2}{2}\cos(\omega_1\tau)=R_{YY}(\tau)$$

Also E[X(t)] = E[Y(t)] = 0, since

$$E[X(t)]=A\int_0^{2\pi}\cos(2\omega_0 t+\theta)\frac{1}{2\pi}d\theta=0$$

i.e., X(t) and Y(t) are individually WSS. But now,

$$R_{XY}(t,t+\tau) = E[X(t)Y(t+\tau)]$$
$$= AB\,E[\cos(\omega_0 t+\theta)\cos(\omega_1 t+\omega_1\tau+\phi)]$$
$$= \frac{AB}{2}E[\cos\{(\omega_1-\omega_0)t+\omega_1\tau+\phi-\theta)\}$$
$$+\cos\{(\omega_1+\omega_0)t+\omega_1\tau+\phi+\theta)\}]$$
$$= 0$$

which is not a function of τ, so *X(t)*, *Y(t)* are not jointly WSS.

(b) If θ = ϕ ⇒ $R_{XY}(t,t+\tau) = \frac{AB}{2}\cos\{(\omega_1-\omega_0)t+\omega_1\tau\}$

which is again function of *t*. When $\omega_0 = \omega_1$,

$$R_{XY}(t,t+\tau) = \frac{AB}{2}\cos(\omega_1\tau) = R_{XY}(\tau) \longleftarrow$$

Time Averages and Ergodicity:

In practice, we would like to deal with only a single sample function rather than the ensemble of functions. For example, we may wish to infer the probability law or certain averages of the r.p. from the measurements on a single member of the ensemble set.

The time average for the ensemble set is defined as,

$$\bar{x} = \langle X(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt$$

→ Continuous time case

$$\bar{x} = \langle X(n) \rangle = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)$$

→ Random sequence case

Then, the time average autocorrelation function will be,

$$r_{XX}(\tau) = \langle X(t)X(t+\tau)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

$$r_{XX}(m) = \langle X(n)X(n+m)\rangle = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} x(n)x(n+m)$$

If these time averages are computed for each sample function within the ensemble set, these values, i.e., $\bar{x}$, $r_{XX}(\tau)$ or $r_{XX}(m)$ form r.v.'s themselves. Now, it is obvious (due to WSS) that taking *E[.]* (ensemble average) of these r.v.'s yields,

$$E[\bar{x}] = \bar{X}$$

and $$E[r_{XX}(\tau)] = R_{XX}(\tau)$$

Now if we assume that the deviations of these averages are zero from one sample to another, then

$$\bar{x} = \bar{X}$$

$$r_{XX}(\tau) = R_{XX}(\tau)$$

In other words, *the time averages obtained from one signal (time series) are equal to the ensemble averages*.

Ergodic r.p.'s:

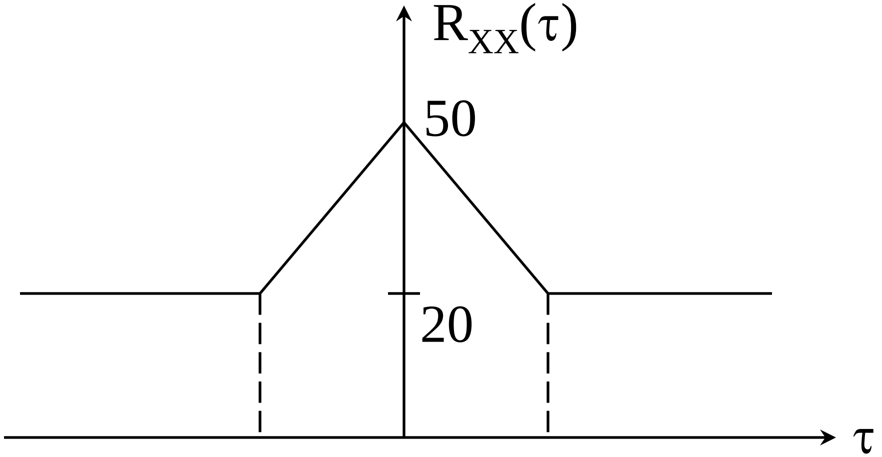
A r.p. is said to be "ergodic" if the *time averages computed for a sample function can be used as an approximation to the corresponding ensemble averages of the r.p*. This is a very restrictive form of stationarity (presupposes stationarity).

$$\bar{x} = \overline{X}$$

$$r_{XX}(\tau) = R_{XX}(\tau)$$

This assumption (even though not very practical) simplifies the inference of the statistics greatly as it allows all the statistics to be measured from only one sample function.

Example 10: Given an ergodic r.p. with ACF as shown



Find (a) $E[X(t)]$, (b) $E[X^2(t)]$, (c) σ_X^2.

Solution:

(a) From the properties of ergodic r.p.,

$$\lim_{\tau \to \infty} R_{XX}(\tau) = \mu_X^2$$

$$\text{Thus,} \quad 20 = \mu_X^2 \quad \Rightarrow \mu_X = E[X(t)] = \sqrt{20} \quad \longleftarrow$$

(b) $$E[X^2(t)] = R_{XX}(0) = 50 \quad \longleftarrow$$

(c) $$\begin{aligned} \sigma_X^2 &= E[X^2(t)] - \mu_X^2 \\ &= 50 - 20 \\ &= 30 \quad \longleftarrow \end{aligned}$$

Reading Assignment For Week 12: Sections 6.3-6.4 (Peebles)

A Brief Review on Digital Filters (Computer Assign 3):

Why Digital vs. Analog?

Advantages:

(1) Less (no) sensitivity to temperature variations / aging.
(2) Better reliability.
(3) Can be made adaptive.
(4) More compact and light weight.
(5) Less cost, etc.

Disadvantage:

Quantization effects due to rounding /truncation.

Two Types:

(1) Recursive or Infinite Impulse Response (IIR):

Input/output difference equation is

$$\underbrace{y(n)}_{\substack{\text{present}\\\text{output}\\\text{sample}}} = \underbrace{\sum_{i=0}^{M-1} b_i x(n-i)}_{\substack{\text{past + present}\\\text{input samples}\\\text{weighted by } b_i\text{'s}}} - \underbrace{\sum_{j=1}^{N-1} a_j y(n-j)}_{\substack{\text{past output samples}\\\text{weighted by } a_j\text{'s}}}$$

where x(n) : Input

y(n) : Output

a_i, b_i's : Filter coefficients

N : Order

Above equation is called an *ARMA (Auto-Regressive Moving Average)* process.

Special Case:

If $b_i = 0, \ \forall i \neq 0,$ then

$$y(n) = b_0 x(n) - \sum_{j=1}^{N-1} a_j y(n-j)$$

This system is referred to as *AR (Auto Regressive)* process.

(2) Non-Recursive or Finite Impulse Response (FIR):

$$\underbrace{y(n)}_{\substack{\text{present}\\ \text{output}\\ \text{sample}}} = \underbrace{\sum_{i=0}^{M-1} b_i x(n-i)}_{\substack{\text{past + present}\\ \text{input samples}\\ \text{weighted by } b_i\text{'s}}}$$

This is *Moving Average (MA)* filter.

Thus, this filter can be viewed as a special case of IIR when $a_j = 0, \ \forall \ j$.

MATLAB Filter Command:

$$\boxed{Y = \text{filter}\,(b, a, X)}$$

where X = Input signal (vector)
Y = Output signal (vector)
$a = [a_0, a_1, \ldots, a_{N-1}]$
$b = [b_0, b_1, \ldots, b_{M-1}]$

Thus, for an FIR filter,

$a = [a_0, 0, \ldots, 0]$ and $a_0 = 1$ usually.
$b = [b_{opt}]$ or $b = [1/6, 1/6, \ldots, 1/6]$ for a simple averaging filter of the same order M=6.

Magnitude and Phase Responses of the Optimal Filter: