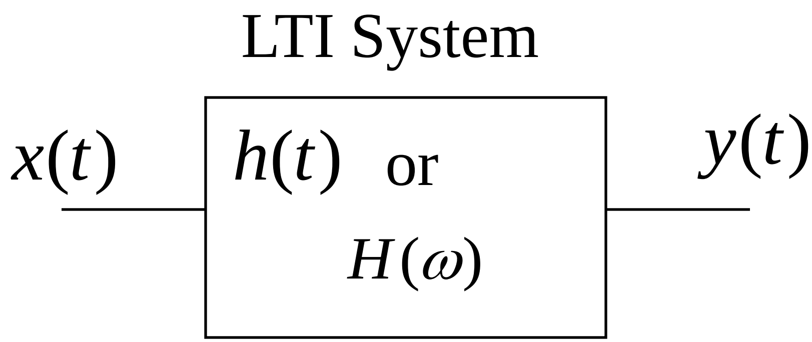
# Chapter 8: Linear Systems with Random Inputs

Consider a causal stable LTI system with impulse response h(t) and let x(t) be a real input signal i.e. a sample function of a WSS random process X(t). Then, we have

$$y(t) = \int_{-\infty}^{\infty} x(\eta)h(t-\eta)d\eta$$

$$= \int_{-\infty}^{\infty} h(\eta)x(t-\eta)d\eta$$




The same convolution equation holds for the entire ensemble sets X(t) and Y(t) i.e.

$$Y(t) = \int_{-\infty}^{\infty} X(\eta)h(t-\eta)d\eta = \int_{-\infty}^{\infty} h(\eta)X(t-\eta)d\eta$$

Now, given the mean and autocorrelation function, $\mu_X$ and $R_{XX}(\tau)$ of X(t), the statistics of Y(t) can be determined as follows.

## Mean

$$\mu_Y = E[Y(t)] = \int_{-\infty}^{\infty} h(\eta)E[X(t-\eta)]d\eta = \mu_X \int_{-\infty}^{\infty} h(\eta)d\eta$$

$$= \mu_X H(0)$$ i.e. mean of Y(t) is constant

where $H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$ is the Frequency Response of the LTI System

## Cross-Correlation and Cross Power Spectrum

$$R_{XY}(t,t+\tau) = E[X(t)Y(t+\tau)] = E\left[ X(t)\int_{-\infty}^{\infty} h(\eta)X(t+\tau-\eta)d\eta \right]$$

$$= \int_{-\infty}^{\infty} h(\eta)E[X(t)X(t+\tau-\eta)]d\eta$$

or $R_{XY}(t,t+\tau) = \int_{-\infty}^{\infty} h(\eta)R_{XX}(\tau-\eta)d\eta$

If X(t) is WSS, then $\boxed{R_{XY}(\tau) = h(\tau) * R_{XX}(\tau)}$

The cross-power spectrum is then given by

$$\boxed{S_{XY}(\omega) = H(\omega) S_{XX}(\omega)}$$

**Remark**

For a white process $R_{XX}(\tau) = \sigma_X^2 \delta(\tau)$ and $S_{XX}(\omega) = \sigma_X^2$

Thus,

$$\boxed{\begin{aligned} S_{XY}(\omega) &= \sigma_X^2 H(\omega) \\ R_{XY}(\tau) &= \sigma_X^2 h(\tau) \end{aligned}}$$

i.e. for a white-noise input, the cross-correlation between input and output of a linear system is proportional to the impulse response of the system (or the cross-power spectrum is proportional to the frequency response).

## Auto-Correlation and Power Spectrum

$$R_{YY}(t,t+\tau) = E[Y(t)Y(t+\tau)] = E\left[ Y(t)\int_{-\infty}^{\infty} h(\eta)X(t+\tau-\eta)d\eta \right]$$

$$= \int_{-\infty}^{\infty} h(\eta)E[Y(t)X(t+\tau-\eta)]d\eta$$

$$= \int_{-\infty}^{\infty} h(\eta)R_{XY}(\eta-\tau)d\eta$$

If X(t) is WSS, then $R_{YY}(\tau) = h(\tau) * R_{XY}(-\tau)$

On the other hand,

$$R_{XY}(\tau) = h(\tau) * R_{XX}(\tau) \Rightarrow R_{XY}(-\tau) = h(-\tau) * R_{XX}(-\tau)$$

Thus,

$$\boxed{\begin{aligned} R_{YY}(\tau) &= h(\tau) * h(-\tau) * R_{XX}(-\tau) \\ &= h(\tau) * h(-\tau) * R_{XX}(\tau) \end{aligned}}$$

The power spectrum is

$$\boxed{\begin{aligned} S_{YY}(\omega) &= H(\omega)H^*(\omega)S_{XX}(\omega) \\ &= |H(\omega)|^2 S_{XX}(\omega) \end{aligned}}$$

Spectral Factorization

## Important Remarks

1. For a white process $R_{XX}(\tau) = \sigma_X^2 \delta(\tau)$ and $S_{XX}(\omega) = \sigma_X^2$

Thus,

$$S_{YY}(\omega) = \sigma_X^2 |H(\omega)|^2$$

i.e. the output signal is ***NOT*** white and is correlated (colored noise) since the power spectrum is dependent on frequency.

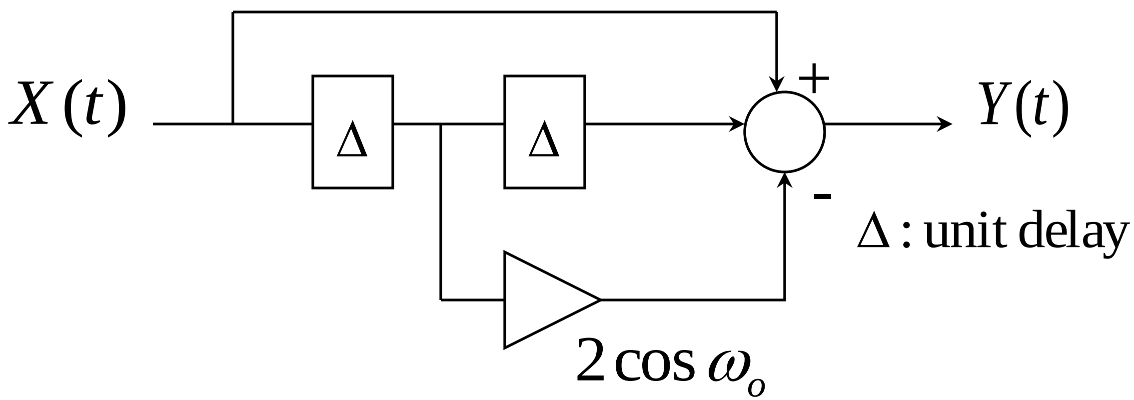
2. Since mean of Y(t) is constant and its auto-correlation and cross-correlation functions are only dependent on $\tau$, the output r.p. is also individually and jointly WSS with the input X(t).

## Example 1:

A white WSS random process X(t) with uniform PDF

$$f_{X(t)}(x(t)) = \begin{cases} 1 & -1/2 \le x(t) \le 1/2 \\ 0 & elsewhere \end{cases}$$

is applied to the input of a system with structure shown below. Find the output *power spectral density*.




Solution

$$S_{YY}(\omega) = \sigma_X^2 H(\omega) H(-\omega) = \sigma_X^2 \,|\, H(\omega) \,|^2$$

We first find $\sigma_X^2$ and then $H(\omega)$.

$$\mu_X = E[X(t)] = \int_{-1/2}^{1/2} x f_X(x)dx = 0 \qquad \text{Since WSS}$$

and

$$\sigma_X^2 = E[X^2(t)] = \int_{-1/2}^{1/2} x^2 f_X(x)dx = 1/12$$

Since the input is a stationary white process

$$R_{XX}(\tau) = E[X(t)X(t+\tau)] = \sigma_X^2 \delta(\tau) = 1/12\, \delta(\tau)$$

Thus, the power spectral density of the input is $S_{XX}(\omega) = \sigma_X^2$

From the system structure, $y(t) = x(t-2) - 2x(t-1)\cos\omega_o + x(t)$

Thus, $H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = e^{-j2\omega} - 2e^{-j\omega}\cos\omega_o + 1$

$$S_{YY}(\omega) = \sigma_X^2 H(\omega) H(-\omega)$$

$$= \frac{1}{12}(e^{2j\omega} + e^{-2j\omega} + 2 + 4\cos^2\omega_o - 4e^{j\omega}\cos\omega_o - 4e^{-j\omega}\cos\omega_o)$$

$$= \frac{1}{12}(4\cos^2\omega + 4\cos^2\omega_o - 8\cos\omega\cos\omega_o) = \frac{1}{3}(\cos\omega - \cos\omega_o)^2$$

Example 2:

Consider a r.p. Y(t) defined as

$$Y(t) = \frac{1}{2T}\int_{t-T}^{t+T} X(\eta)\, d\eta$$

where X(t) is a WSS r.p. Show that

$$S_{YY}(\omega) = S_{XX}(\omega)\,\mathrm{sinc}^2(\omega T) = S_{XX}(\omega)\left[\frac{\sin(\omega T)}{\omega T}\right]^2$$

## Solution

Rewrite the expression for Y(t) as a convolution integral i.e.

$$Y(t) = \frac{1}{2T}\int_{t-T}^{t+T} X(\eta)\,d\eta = \int_{-\infty}^{\infty} X(\eta)h(t-\eta)\,d\eta$$

where

$$h(t-\eta) = \begin{cases} \dfrac{1}{2T} & t-T \le \eta \le t+T \\ 0 & elsewhere \end{cases}$$

That is, Y(t) is the output of a LTI system with symmetric impulse response h(t) given by

$$h(t) = \begin{cases} \dfrac{1}{2T} & -T \le t \le T \\ 0 & elsewhere \end{cases}$$ and hence $H(\omega) = \operatorname{sinc}(\omega T)$

Thus, using the power spectrum equation for Y(t), we get

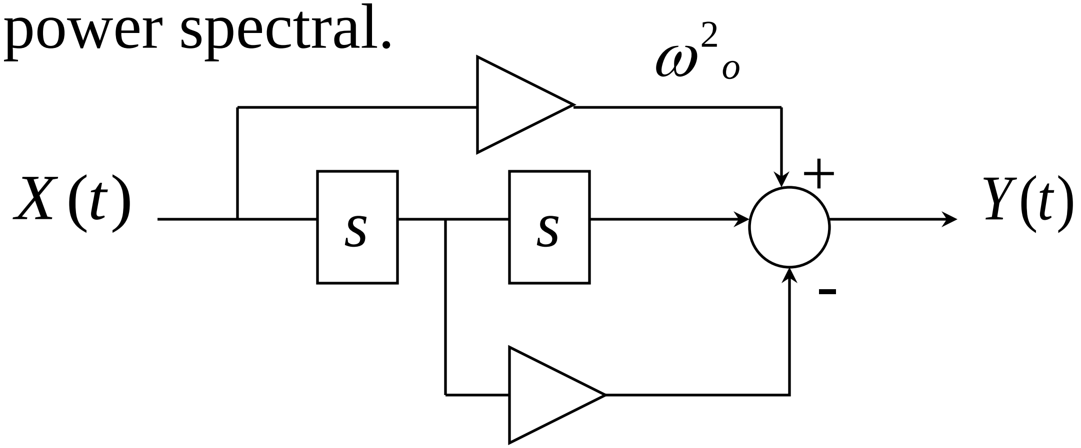
$$S_{YY}(\omega) = S_{XX}(\omega)\,|\,\mathrm{H}(\omega)\,|^2 = S_{XX}(\omega)\operatorname{sinc}^2(\omega T)$$

# Example 3-Another Version of Example 1:

A white WSS random process X(t) with uniform PDF

$$f_{X(t)}(x(t)) = \begin{cases} 1 & -1/2 \le x(t) \le 1/2 \\ 0 & elsewhere \end{cases}$$

is applied to the input of a filter with realization shown below. Find the output power spectral.




Solution

Use $S_{YY}(\omega) = \sigma_X^2 H(\omega)H(-\omega) = \sigma_X^2 \,|\,H(\omega)\,|^2$

We first find $\sigma_X^2$ and then $H(\omega)$.

$$\mu_X = E[X(t)] = \int_{-1/2}^{1/2} x\, f_X(x)dx = 0$$

$$\sigma_X^2 = E[X^2(t)] = \int_{-1/2}^{1/2} x^2 f_X(x)dx = 1/12$$

Since the process is white

$$R_{XX}(\tau) = E[X(t)X(t+\tau)] = \sigma_X^2 \delta(\tau) = 1/12\ \delta(\tau)$$

Thus, the power spectral density of the input is $S_{XX}(\omega) = \sigma_X^2$

From the realization $H(s) = s^2 - 2s\omega_o + \omega^2{}_o$

$$H(\omega) = (j\omega)^2 - 2j\omega\omega_o + \omega^2{}_o = (-\omega^2 + \omega^2{}_o) + j(-2\omega\omega_o)$$

$$|H(\omega)|^2 = (-\omega^2 + \omega^2{}_o)^2 + (-2\omega\omega_o)^2 = (\omega^2 + \omega^2{}_o)^2$$

Thus, $S_{YY}(\omega) = \sigma_X^2 |H(\omega)|^2 = \sigma_X^2 (\omega^2 + \omega^2{}_o)^2$