The residue of $X(z)z^{n-1}$ at a given pole at $z = z_i$ say $\mathrm{Res}_i|_{z=z_i}$ can be evaluated using

$$\mathrm{Res}_i|_{z=z_i} = \frac{d^{m-1}}{dz^{m-1}}\left[\frac{(z-z_i)^m}{(m-1)!} X(z) z^{n-1}\right]\Bigg|_{z=z_i}$$

m: order of the pole at $z = z_i$

For example

$m = 1$ $\quad \mathrm{Res}_i|_{z=z_i} = (z-z_i)X(z)z^{n-1}\Big|_{z=z_i}$

$m = 2$ $\quad \mathrm{Res}_i|_{z=z_i} = \dfrac{d}{dz}\left[(z-z_i)^2 X(z) z^{n-1}\right]\Big|_{z=z_i}$

Example

Find IZT of

$$X(z) = \frac{1}{(z-1)(z-0.5)} \quad \text{thus} \quad X(z)z^{n-1} = \frac{z^{n-1}}{(z-1)(z-0.5)}$$

When $n = 0$, simple poles at $z = 0, 0.5$ and 1.

When $n \geq 1$, simple poles at $z = 0.5$ and 1.

Thus, these cases should be treated separately.

part 1

$n = 0 \Rightarrow X(z)z^{n-1} = \dfrac{1}{z(z-1)(z-0.5)}$

Residue theorem gives

$\mathrm{Res}|_{z=0} = z\,X(z)z^{-1}\Big|_{z=0} = 2$

$\mathrm{Res}|_{z=1} = (z-1)X(z)z^{-1}\Big|_{z=1} = 2$

$\mathrm{Res}|_{z=0.5} = (z-0.5)X(z)z^{-1}\Big|_{z=0.5} = -4$

Thus

$$x(0) = 2+2-4 = 0$$

or use initial value theorem

$$x(0) = \lim_{z \to \infty} X(z) = 0$$

Part 2

$n \geq 1$, then

$$x(n) = \text{Res}\Big|_{z=1} + \text{Res}\Big|_{z=0.5}$$

$$\text{Res}\Big|_{z=1} = (z-1)\, X(z)\, z^{n-1} \Big|_{z=1} = 2$$

$$\text{Res}\Big|_{z=0.5} = (z-0.5)\, X(z)\, z^{n-1} \Big|_{z=0.5} = -2\,(0.5)^{n-1}$$

Thus,

$$x(n) = 2 - 2\,(0.5)^{n-1} \qquad , n \geq 1$$

Combine the two results

$$x(n) = \begin{cases} 0 & n = 0 \\ 2(1-(0.5)^{n-1}) , & n \geq 1 \end{cases}$$

Solution of Difference Equation Using Z-Transform

Consider an LSI System described by difference equation

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{\ell=0}^{M} b_\ell\, x(n-\ell)$$

Taking Z-transform of both sides and using the shifting property we obtain

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) + \sum \text{initial condition terms} = \left(\sum_{\ell=0}^{M} b_\ell z^{-\ell}\right) X(z)$$

we assume that $x(n)=0$ for $\forall\, n<0$, thus

$$Y(z) = \underbrace{\frac{\left(\sum_{\ell=0}^{M} b_\ell z^{-\ell}\right) X(z)}{\sum_{k=0}^{N} a_k z^{-k}}}_{Y_p(z)} + \underbrace{\frac{\sum \text{IC terms}}{\sum_{k=0}^{N} a_k z^{-k}}}_{Y_h(z)}$$

The 1st term in the right corresponds to the response when all initial conditions are zero i.e. the zero-state response or particular solution; the 2nd term relates to the zero-input response or the homogeneous solution.

Example

Consider the previous example and this time use z-transform i.e.

$$y(n) - \tfrac{3}{2}\, y(n-1) + \tfrac{1}{2}\, y(n-2) = x(n)$$

For $x(n) = 1 + 3^{-n}$, $y(-2) = 0$, $y(-1) = 2$

Taking z-transform of both sides and using shifting we obtain

$$Y(z) - \tfrac{3}{2} z^{-1}\left(Y(z) + y(-1) z\right) + \tfrac{1}{2} z^{-2}\left(Y(z) + y(-1) z + \underbrace{\cancel{y(-2)}}_{0} z^2\right) = X(z)$$

$$X(z) = \frac{z}{z-1} + \frac{z}{z-1/3} = \frac{2z(z-2/3)}{(z-1)(z-1/3)}$$

$$(1 - 3/_2 z^{-1} + 1/_2 z^{-2}) Y(z) = X(z) + 3 - z^{-1}$$

or $$Y(z) = \frac{z^2 X(z)}{(z^2 - 3/_2 z + 1/_2)} + \frac{3z^2 - z}{(z^2 - 3/_2 z + 1/_2)}$$

$$= \frac{z\left[3(z-1)(z-1/_3)^2 + 2z^2(z-2/_3)\right]}{(z-1)^2(z-1/_2)(z-1/_3)}$$

Expand $\frac{Y(z)}{z}$ as

$$\frac{Y(z)}{z} = \frac{A_1}{(z-1/_2)} + \frac{A_2}{(z-1/_3)} + \frac{B_1}{(z-1)} + \frac{B_2}{(z-1)^2}$$

$$A_1 = (z-1/_2)\frac{Y(z)}{z}\bigg|_{z=1/2} = -3$$

$$A_2 = (z-1/_3)\frac{Y(z)}{z}\bigg|_{z=1/3} = 1$$

$$B_2 = (z-1)^2\frac{Y(z)}{z}\bigg|_{z=1} = 2$$

$$B_1 = \frac{d}{dz}\left[(z-1)^2\frac{Y(z)}{z}\right]\bigg|_{z=1} = 7$$

Thus

$$Y(z) = \frac{-3z}{(z-1/_2)} + \frac{z}{(z-1/_3)} + \frac{7z}{(z-1)} + \frac{2z}{(z-1)^2}$$

or

$$y(n) = (-3)2^{-n} + 3^{-n} + 7 + 2n \quad , \forall n \geq 0$$

Example 2

Given

$$y(n+2) + 3y(n+1) + 2y(n) = x(n)$$

Solve for $y(n)$ when $x(n) = \delta(n)$, $y(0) = 1$, $y(1) = -1$.

Take z-transform and use the shifting property

$$z^2[Y(z) - y(0) - y(1)z^{-1}] + 3z[Y(z) - y(0)] + 2Y(z) = 1$$

Thus

$$(z^2 + 3z + 2)Y(z) = z^2 + 2z + 1$$

$$H(z) = \frac{1}{(z+1)(z+2)}$$

$$Y(z) = \frac{(z+1)^2}{(z+1)(z+2)} = \frac{z+1}{(z+2)}$$

$$\frac{H(z)}{z} = \frac{0.5}{z} - \frac{1}{(z+1)} + \frac{0.5}{(z+2)}$$

Expand $\dfrac{Y(z)}{z} = \dfrac{(z+1)}{z(z+2)} = \dfrac{A_1}{z} + \dfrac{A_2}{(z+2)}$

$$h(n) = 0.5\,\delta(n) - (-1)^n + 0.5(-2)^n \quad n \geq 0$$

$$A_1 = z\frac{Y(z)}{z}\Big|_{z=0} = 1/2, \quad A_2 = (z+2)\frac{Y(z)}{z}\Big|_{z=-2} = 1/2$$

$$Y(z) = 1/2 + \frac{1/2\, z}{z+2} \Rightarrow y(n) = \frac{1}{2}\delta(n) + 1/2(-2)^n$$

Using Residue theorem

$$Y(z)z^{n-1} = \frac{z^{n-1}(z+1)}{(z+2)}$$

a) $n = 0 \Rightarrow Y(z)z^{n-1} = \dfrac{(z+1)}{z(z+2)}$

$$y(0) = \mathrm{Res}|_{z=0} + \mathrm{Res}|_{z=-2} = 1/2 + 1/2 = 1$$

b) $n \geq 1 \Rightarrow y(n) = \mathrm{Res}|_{z=-2}$

$$\mathrm{Res}|_{z=-2} = (z+2)Y(z)z^{n-1}\Big|_{z=-2} = 1/2(-2)^n \quad n \geq 1$$

Transfer Function

An alternative way to represent the I/o relation is via the transfer function. Recall that

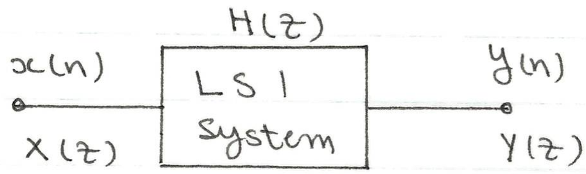
$$Y(z) = \frac{\left(\sum_{\ell=0}^{M} b_\ell z^{-\ell}\right) X(z)}{\sum_{k=0}^{N} a_k z^{-k}} + \frac{\sum \text{IC terms}}{\sum_{k=0}^{N} a_k z^{-k}}$$

If all the initial conditions are zero then

$$H(z) \triangleq \frac{Y(z)}{X(z)} = \frac{\sum_{\ell=0}^{M} b_\ell z^{-\ell}}{\sum_{k=0}^{N} a_k z^{-k}}$$

is the transfer function for the LSI system.

$$H(z) = \frac{Z\{\text{output Sequence}\}}{Z\{\text{input Sequence}\}}$$



$$= \frac{Y(z)}{X(z)}$$

If $x(n) = \delta(n) \Rightarrow X(z) = 1$

then $Y(z) = H(z)$

i.e. the transfer function is the ZT of the impulse response or unit pulse response of the system. Thus

$$H(z) = Z\{h(n)\} = \sum_{n=0}^{\infty} h(n) z^{-n} \quad \text{for causal systems.}$$

Special Cases

1- $N=0$ i.e Nonrecursive or FIR systems, we have

$$H(z) = \sum_{\ell=0}^{M} b_\ell z^{-\ell}$$

or

$$H(z) = \sum_{\ell=0}^{M} h_\ell z^{-\ell}$$

All-zero system
(except poles at $z=0$)

2- $M=0$ i.e. All-pole recursive or IIR system

$$H(z) = \frac{b_0}{\sum_{k=0}^{N} a_k z^{-k}}$$

All-pole system
(except zeros at $z=0$)

Definition:

A rational transfer function is called "proper" when the order of numerator polynomial is less than or equal to the order of denominator polynomial. polynomials must be arranged as function of z not z^{-1}. If the order of the numerator polynomial is strictly less than that of the denominator (in z) then the transfer function is called "strictly proper". Improper transfer function lead to noncausal systems.

Example

The transfer function of an LSI system is given by

$$H(z) = \frac{z - z^{-1}}{z^{-2} + 1}$$

Determine whether or not the system is causal.

$$H(z) = \frac{z^3 - z}{z^2 + 1}$$

Using long division

$$H(z) = z - 2z^{-1} + 2z^{-3} - 2z^{-5} + \cdots$$

Thus, comparing this with

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

gives $h(-1)=1$, $h(0)=0$, $h(1)=-2$, $h(2)=0$

$h(3)=2$, $h(4)=0$ $h(5)=2$, - - - .

or $h(2n+1)=(-1)^{n+1}\,2$ $n \geq 0$

$h(2n)=0$

Since $h(n) \neq 0$ for $\forall n<0$ $\Longrightarrow$ System is noncausal

Alternatively we can form the relevant difference equation

$$H(z)=\frac{Y(z)}{X(z)}=\frac{z^3-z}{z^2+1}$$

i.e. $y(n+2)+y(n)=x(n+3)-x(n+1)$

$y(n+2)$ depends on future input $x(n+3)$ i.e. noncausal.

Stability of LSI Systems

A system is said to be BIBO (bounded-input, bounded output) Stable if a bounded input sequences implies the output sequence is also bounded. Since LSI systems are characterized by their unit pulse sequence, the property of BIBO stability must depend only on $\{h(n)\}$.

Theorem 1

An LSI system is BIBO stable iff

$$S \triangleq \sum_{K=-\infty}^{\infty} |h(K)| < \infty$$

i.e. $h(n)$ is absolutely summable.

Proof :

If $h(n)$ is absolutely summable and $|x(n)|<M$ it can be shown

that
$$|y(n)| = \left|\sum_{k=-\infty}^{\infty} h(k)\,x(n-k)\right| \leq M \sum_{k=-\infty}^{\infty}|h(k)| < \infty$$

i.e $y(n)$ is bounded. To prove the Converse let assume $S=\infty$, then a bounded input can be found which gives an unbounded output. As an example, let

$$x(n) = \begin{cases} \dfrac{h^*(-n)}{|h(-n)|} & h(n) \neq 0 \\ 0 & h(n) = 0 \end{cases}$$

which is bounded, then y at $n=0$ is

$$y(0) = \sum_{k=-\infty}^{\infty} h(k)\,x(-k) = \sum_{k=-\infty}^{\infty} \frac{|h(k)|^2}{|h(k)|} = S$$

i.e. y is unbounded.

Theorem 2:

An LSI System is BIBO Stable iff all the poles of the transfer function lie inside the unit circle in the z-plane.

Proof

To see this let factorize the numerator and denominator polynomials to give

$$H(z) = \frac{A\prod_{i=1}^{M}(z-z_i)}{\prod_{j=1}^{N}(z-P_j)} \qquad z_i : \text{zeros}, \quad P_j : \text{poles}$$

If the system is causal i.e. $H(z)$ is proper then using PFE

$$H(z) = \frac{A_1 z}{(z-P_1)} + \frac{A_2 z}{(z-P_2)} + \cdots \frac{A_N z}{(z-P_N)}$$

Each $\dfrac{A_i z}{(z-P_i)} \xrightarrow{\mathcal{Z}^{-1}} A_i P_i^n$

For stability $|P_i| < 1$ for $\forall i \in [1, N]$
i.e. poles are within the unit circle. If a pole is outside the unit circle $|P_i| > 1$ then the relevant term in the impulse response goes to ∞ when $n \to \infty$ and hence $h(n)$ will not be absolutely summable.

Remarks

1- FIR systems inherently benefit from BIBO stability and they are always stable.

2- IIR systems require stability considerations.

3- If a system is BIBO stable the ROC of $H(z)$ includes the unit circle. Furthermore, if the system is also causal the ROC will include the unit circle and the entire z-plane outside the unit circle, including $z = \infty$.