Chapter 6: Random Processes

In engineering and science, we often encounter problems, which involve random signals that vary with time, e.g., bit stream in a binary communication system (random message), noise, etc.

So far, we have investigated random events and r.v.'s (numerical functions of events). Now consider assigning *time functions* to the outcome of random events.

Definition:

Let s be a possible outcome of an experiment, assign (according to some rule) a time function, $x(t,s)$ to each outcome. Then,

$$x(t,s)\text{: Sample Function}$$

The family of such functions, *X(t,s)*, is called random process (or r.p.), i.e., *X(t,s)* can take on one of many possible time functions $x(t,s_i)$'s. Thus *X(t,s)* is a family or an *ensemble* of time functions $x(t,s_i)$'s.

$$X(t,s)=\underbrace{\{x(t,s_1),x(t,s_2),\ldots\}}_{Ensemble\ Set}$$

Suppressing the '*s*' index, *x(t)* is a sample function of r.p. *X(t)*.

For example, assume that we have repeated the experiment of collecting a speech signal *N* times and every time, due to presence of random noise, we get a new signal $x_i(t)$, then

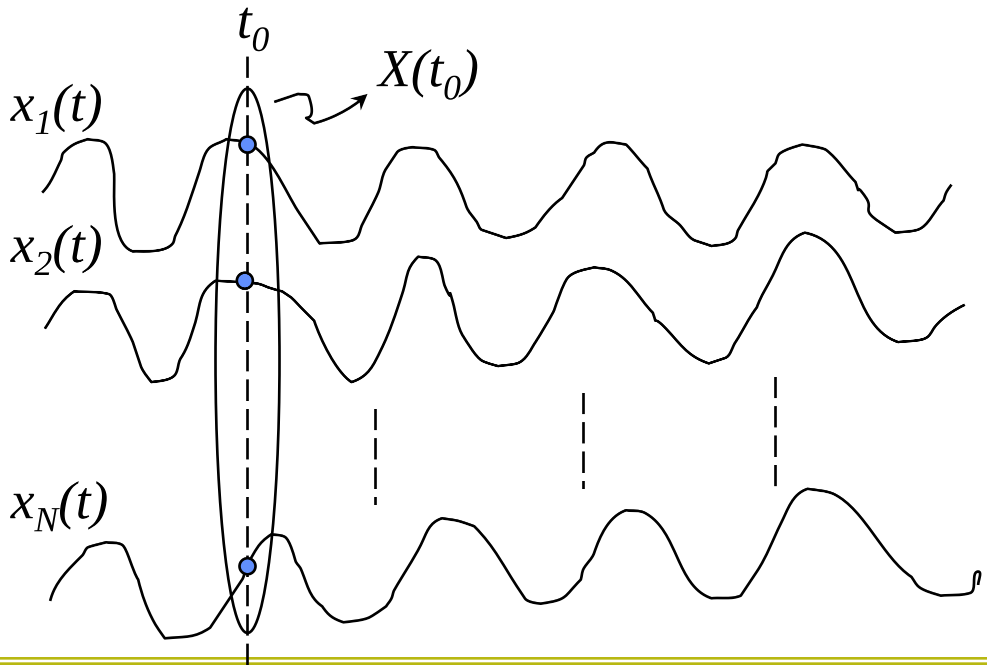
$$X(t)=\underbrace{\{x_1(t),x_2(t),\ldots,x_N(t)\}}_{Ensemble\ Set\ of\ speech\ signals}$$

If '*t*' is fixed, *X* becomes a r.v. i.e. randomness wrt the ensemble set.

Question: How can we describe some statistical measures on *X(t)*? We could look at a particular time, $t_{0,}$ then

$$X(t_0) = \{x_1(t_0), x_2(t_0), \ldots, x_N(t_0)\}$$

is a r.v., which can be described by either PDF or CDF.



Thus, we can define its CDF and PDF, i.e.,

$$\boxed{F_{X(t_0)}\big(x(t_0)\big) = P\big[X(t_0) \le x(t_0)\big]}$$

and

$$\boxed{f_{X(t_0)}\big(x(t_0)\big) = \frac{dF_{X(t_0)}\big(x(t_0)\big)}{dx(t_0)}}$$

The r.p. is said to be statistically defined if the CDF is known for all t_j's. In most cases, this is a lot of information to ask for. Thus, some assumptions need to be made to facilitate the representation of r.p. in practical cases.

Definitions:

Continuous vs. Discrete r.p.:

(1) $X(t)$ is said to be a continuous r.p., if both X and t take Continuous range of values. e.g., Thermal noise in circuits

(2) $X(t)$ is said to be a discrete r.p., if X takes only a finite number of values while t changes continuously.

Continuous vs. Discrete random sequence:

(1) $X(n)$ is said to be a continuous random sequence if X takes continuous range of values while n is discrete.

(2) $X(n)$ is said to be a discrete random sequence, where both time and amplitude take discrete values (*sampled & quantized*).

In what follows, we primarily confine our discussions to the 1st case though the results can also be extended to other cases.

Deterministic (Pseudo-Random) vs. Non-deterministic (True Random):

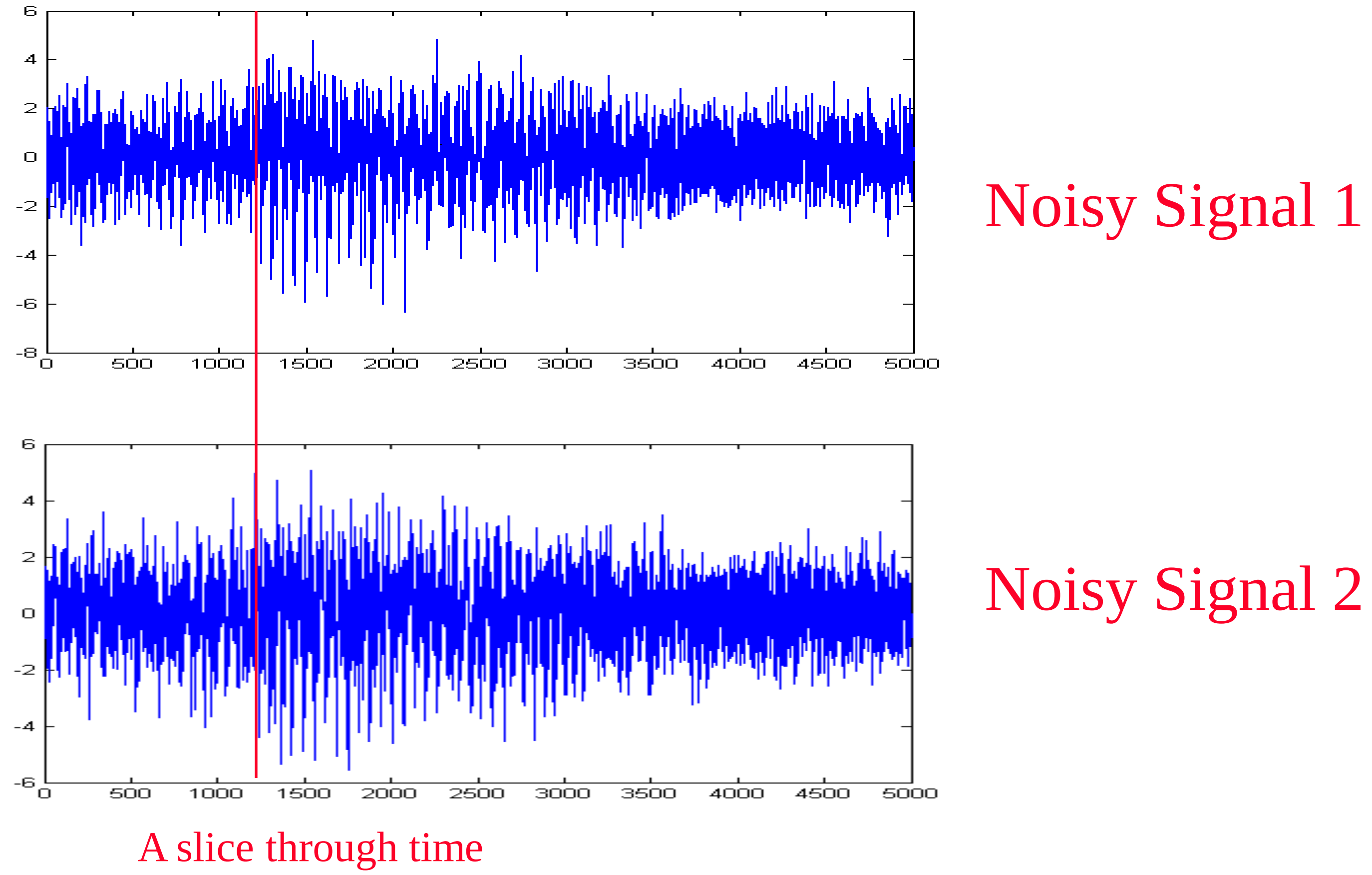
Deterministic: The function that represents each sample function is known. e.g.,

$$X(t) = A\cos(\omega t + \phi)$$

where A, ω are fixed but ϕ is a r.v. with a specified PDF. For every new value of ϕ, a r.p. in the ensemble set is generated, i.e., randomness is over the ensemble set not with respect to time. Similarly, the randomness could be due to A or ω.

Non-deterministic: A totally random signal i.e. nothing is predictable about the r.p.

Two Noisy Measurements of a Speech Signal ('Hello')



Measures on r.p.'s and Averages:

Since a r.p. is an indexed set of r.v.'s, we may likewise characterize the process by statistical averages of the random variables comprising of random process, such averages are called "*Ensemble Averages*".

(1) Ensemble Mean:

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x(t)\, f_{X(t)}(x(t))\, dx(t)$$

Note t is assumed to be fixed.

For discrete case,

$$\boxed{\mu_X(n) = E[X(n)] = \sum_{i=1}^{N} x_i(n) P[X(n) = x_i(n)]}$$

$x_i(n) = i^{\text{th}}$ sample function

In general,

$$\mu_{g(X(t))}(t) = E[g(X(t))] = \int_{-\infty}^{\infty} g(x(t))\, f_{X(t)}(x(t))\, dx(t)$$

(2) Ensemble Auto-Correlation:

Measure of how much *X(t)* varies in time on the average.

Let t_1 and t_2 be two instances in time, then the autocorrelation is defined by,

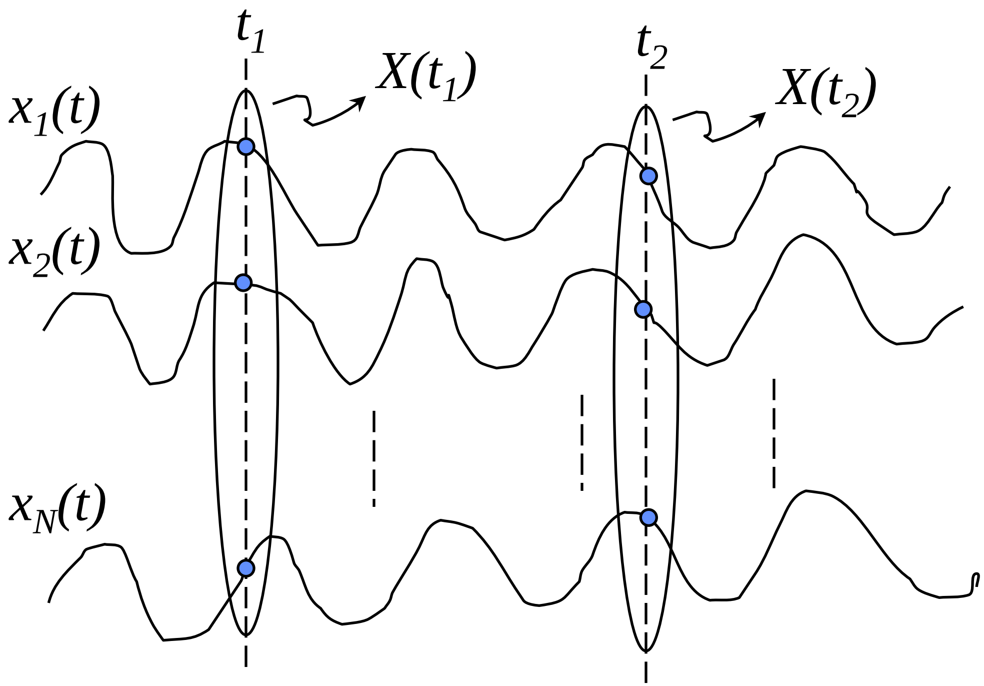
$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$= \int_{-\infty}^{\infty}\int x(t_1)x(t_2)\, f_{X(t_1),X(t_2)}(x(t_1), x(t_2))\, dx(t_1)dx(t_2)$$

For discrete case,

$$R_{XX}(n_1, n_2) = E[X(n_1)X(n_2)]$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{M} x_i(n_1)x_j(n_2)P[X(n) = x_i(n_1), X(n) = x_j(n_2)]$$



(3) Ensemble Auto-Covariance:

Measure of spread from the mean.

$$
\begin{aligned}
C_{XX}(t_1,t_2) &= E[(X(t_1)-\mu_X(t_1))(X(t_2)-\mu_X(t_2))] \\
&= E[X(t_1)X(t_2)] - \mu_X(t_1)\mu_X(t_2) \\
&= R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2) \quad \longleftarrow
\end{aligned}
$$

Measures on two (or more) r.p.'s:

Let $X(t)$ and $Y(t)$ be r.p.'s with ACF's $R_{XX}(t_1,t_2)$ and $R_{YY}(t_1,t_2)$, respectively. The *cross-correlation* between $X(t)$ and $Y(t)$, which gives a measure of dependence between these r.p.'s is,

$$\boxed{R_{XY}(t_1,t_2) = E[X(t_1)Y(t_2)]}$$

Note that: $R_{YX}(t_1,t_2) = E[Y(t_1)X(t_2)] \neq R_{XY}(t_1,t_2)$

If $X(t_1)$ and $Y(t_2)$, independent,

$$R_{XY}(t_1,t_2) = E[X(t_1)]E[Y(t_2)] = \mu_X(t_1)\mu_Y(t_2)$$

for all t_1 and t_2

If $X(t_1)$ and $Y(t_2)$ are orthogonal,

$$R_{XY}(t_1,t_2) = 0 \qquad \text{for all } t_1 \text{ and } t_2$$

The cross-covariance function is

$$C_{XY}(t_1,t_2) = E[(X(t_1)-\mu_X(t_1))(Y(t_2)-\mu_Y(t_2))]$$
$$= R_{XY}(t_1,t_2)-\mu_X(t_1)\mu_Y(t_2)$$

For independent processes,

$$C_{XY}(t_1,t_2)=0 \qquad \text{for all } t_1 \text{ and } t_2$$

Example 1:

The process $X(t) = a\cos(\omega t) + b\sin(\omega t)$ (pseudo r.p.)

where a, b are two independent Gaussian (normal) r.v.'s with

$$E[a] = E[b] = 0, \qquad E[a^2] = E[b^2] = \sigma^2$$

and ω is a constant. Find $R_{XX}(t_1,t_2)$.

Solution: Clearly, r.v.'s $X(t_i)$ are linear combinations of normal r.v.'s, *a*, *b*, thus they are jointly normal (why?) and to determine their statistics, it suffices to find the mean and autocorrelation of $X(t)$.

$$E[X(t)] = E[a]\cos\omega t + E[b]\sin\omega t$$
$$= 0$$

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$
$$= E[\{a\cos\omega t_1 + b\sin\omega t_1\}\{a\cos\omega t_2 + b\sin\omega t_2\}]$$

0, since independent r.v.'s

$$= E[a^2]\cos\omega t_1\cos\omega t_2 + \cancel{E[ab]}\cos\omega t_1\sin\omega t_2$$

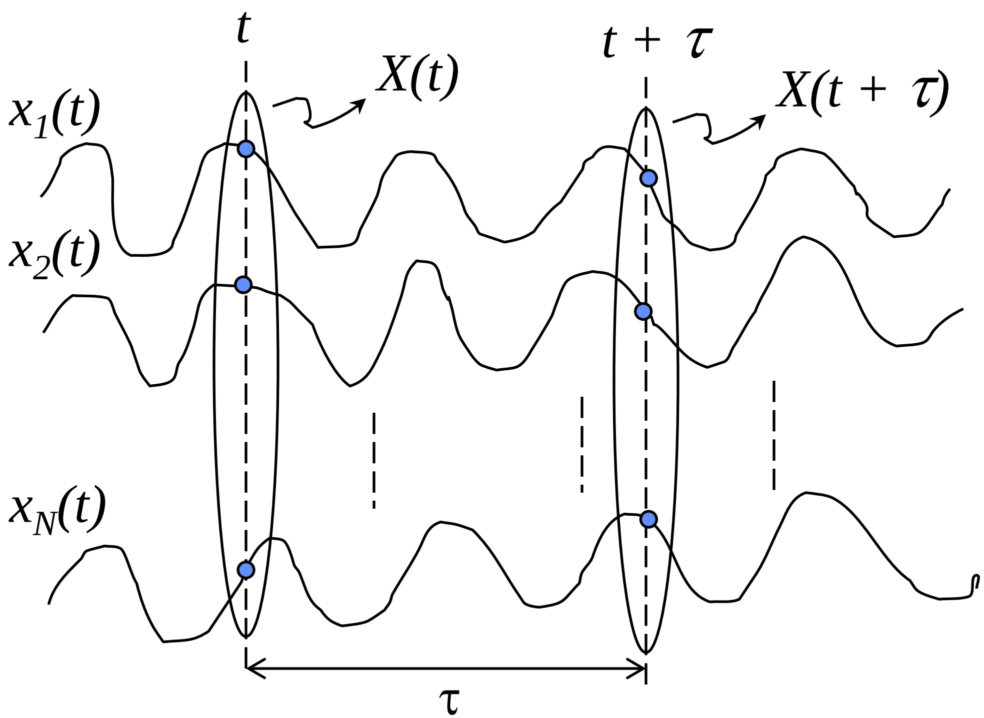
0

$$E[b^2]\sin\omega t_1\sin\omega t_2 + \cancel{E[ba]}\sin\omega t_1\cos\omega t_2$$
$$= \sigma^2\cos\omega t_1\cos\omega t_2 + \sigma^2\sin\omega t_1\sin\omega t_2$$
$$= \sigma^2\cos\omega(t_1 - t_2)$$

Stationary Processes:

(1) Strictly Stationary:

A r.p. is *strictly stationary* if $X(t)$ *and* $X(t+\tau)$ have the same statistics (of all orders) for all τ.



(2) Stationary of order n:

For *stationary process of order n*, the joint PDF and CDF must be invariant under any finite time shift, i.e.

$$f_X(x(t_1),x(t_2),\ldots,x(t_n)) = f_X(x(t_1+\tau),x(t_2+\tau),\ldots,x(t_n+\tau))$$

$$\forall\, t_1,t_2,\ldots t_n \quad \textit{and any } \tau$$

and similarly for the joint CDF.

The r.p. is *strict-sense stationary* if this is true for all orders n.

(3) Wide-Sense Stationary (WSS):

A r.p. is stationary of order n=2 or *wide-sense stationary* if

$$f_X(x(t_1)) = f_X(x(t_1+\tau))$$

and $$f_X(x(t_1),x(t_2)) = f_X(x(t_1+\tau),x(t_2+\tau))$$

It can easily be shown that these conditions reduce to

$$E[X(t)] = \mu_X = const.$$

$$E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

or $$E[X(t_1)X(t_2)] = R_{XX}(t_2 - t_1)$$

… Only dependent on shift, τ and not t.

i.e., the 1st and 2nd order moments are not time dependent.

Also
$$\boxed{\begin{aligned} C_{XX}(\tau) &= E[(X(t)-\mu_X)(X(t+\tau)-\mu_X)] \\ &= R_{XX}(\tau) - \mu_X^2 \end{aligned}}$$

Clearly, $C_{XX}(0) = \sigma_X^2$

Example 2: Given $X(t) = A \cos \pi t$ (pseudo r.p.)

where A is a Gaussian r.v. with $E[A] = 0$ and $E[A^2] = \sigma_A^2$

Find PDF's of $X(0)$ and $X(1)$. Is $X(t)$ stationary in any sense?

Solution:

For $t = 0$, $X(0) = A \Rightarrow E[X(0)] = 0$

$E[X^2(0)] = \sigma_A^2$

Thus, $$f_X(x(0)) = \frac{1}{\sqrt{2\pi}\sigma_A} \exp\left[-\frac{x^2}{2\sigma_A^2}\right] \longleftarrow$$

For $t = 1$, $X(1) = A \cos \pi = -A \Rightarrow E[X(0)] = 0$

$E[X^2(0)] = \sigma_A^2$

Thus again, $$f_X(x(1)) = \frac{1}{\sqrt{2\pi}\sigma_A} \exp\left[-\frac{x^2}{2\sigma_A^2}\right] \longleftarrow$$

$E[X(t)] = E[A] \cos \pi t = 0 =$ fixed.

$E[X^2(t)] = E[A^2] \cos \pi t$

Since $E[X^2(t)]$ is time dependent, r.p. $X(t)$ is NOT *wide–sense stationary*.

Properties of WSS Processes (Important):

(1) Recall that $R_{XX}(\tau) = E[X(t)X(t+\tau)]$

$$R_{XX}(0) = E[X^2(t)] \quad \ldots \text{ Mean squared value.}$$

$$C_{XX}(0) = E[(X(t)-\mu_X)(X(t)-\mu_X)] = \sigma_X^2$$
$$= R_{XX}(0) - \mu_X^2$$

(2) $R_{XX}(\tau) = R_{XX}(-\tau)$ … even symmetry.

Since, $E[X(t)X(t+\tau)] = E[X(t+\tau)X(t)]$

(3) $|R_{XX}(\tau)| \leq R_{XX}(0)$ i.e., ACF is decreasing function of τ and bounded by $R_{XX}(0)$.

(4) If *X(t)* and *Y(t)* are WSS and *jointly WSS*, i.e.,

$$f_{X,Y}(x(t), y(t)) = f_{X,Y}(x(t+\tau), y(t+\tau))$$

then, $R_{XY}(t_1, t_2) = R_{XY}(t_2 - t_1) = R_{XY}(\tau)$

$$C_{XY}(t_1, t_2) = C_{XY}(t_2 - t_1) = C_{XY}(\tau)$$

For WSS and independent r.p.'s,

$$R_{XY}(\tau) = \mu_X \mu_Y$$

$$C_{XY}(\tau) = 0$$

(5) $R_{XY}(\tau) = R_{YX}(-\tau)$

(6) $|R_{XY}(\tau)| \le \left[R_{XX}(0) R_{YY}(0)\right]^{1/2}$

$$|C_{XY}(\tau)| \le \left[C_{XX}(0) C_{YY}(0)\right]^{1/2}$$

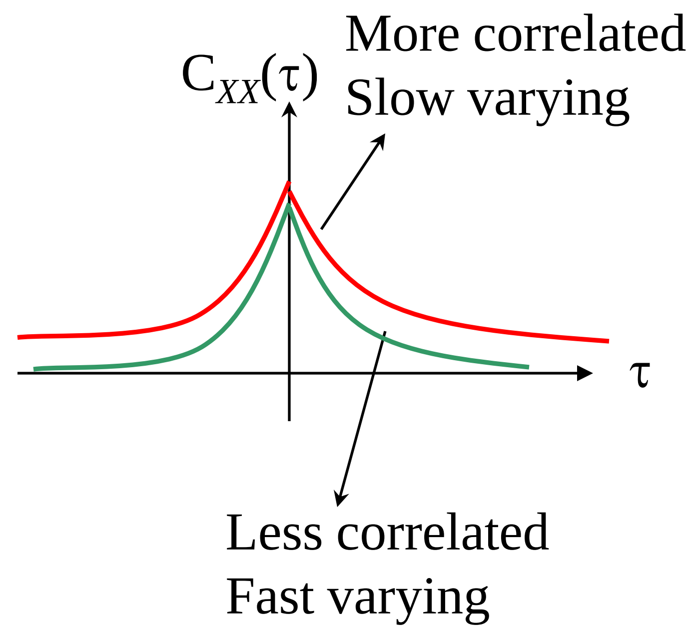
(7) For many r.p.'s, they become less correlated as they become more separated in time. Thus

$$\lim_{\tau \to \infty} R_{XX}(\tau) = \mu_X^2$$

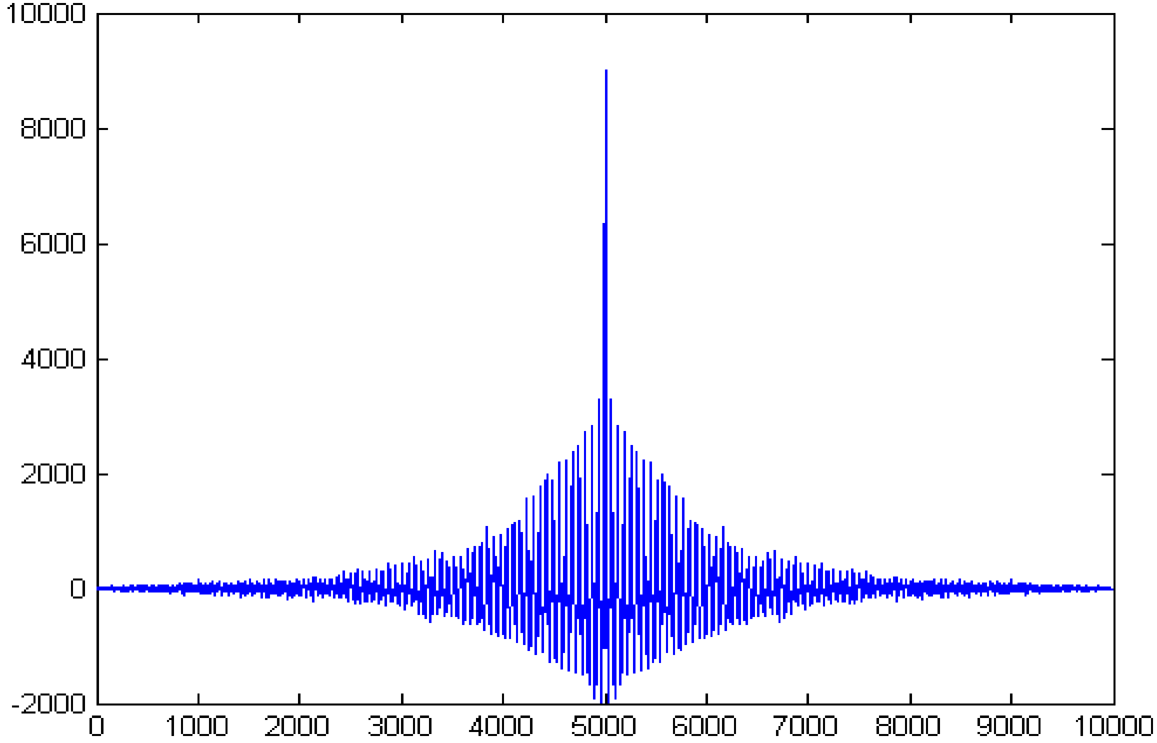
$$\lim_{\tau \to \infty} C_{XX}(\tau) = 0$$

$$\lim_{\tau \to \infty} R_{XY}(\tau) = \mu_X \mu_Y$$

$$\lim_{\tau \to \infty} C_{XY}(\tau) = 0$$



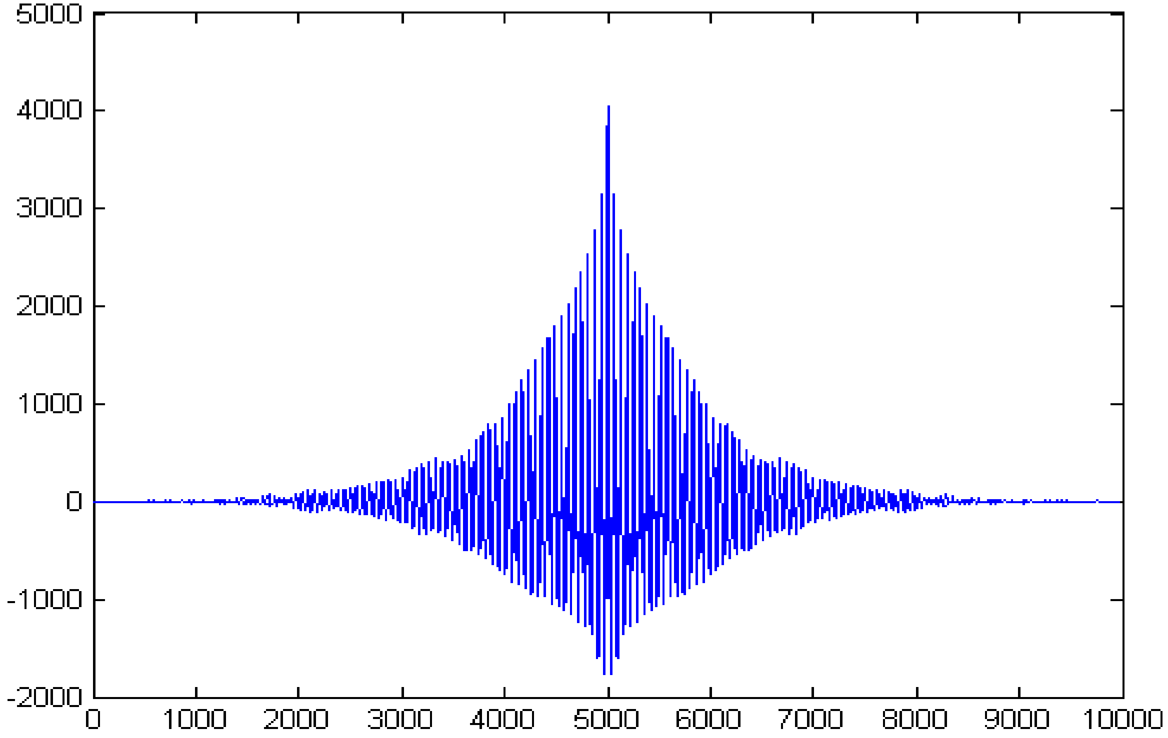
Plot of Auto-Correlation as a Function of lag (τ)



ACF of the Noisy Signal 1

- The large spike at τ =0 is due to white additive noise.
- The exponentially decreasing behavior of this ACF is clearly noticeable.

Plot of Auto-Correlation as a Function of lag (τ)



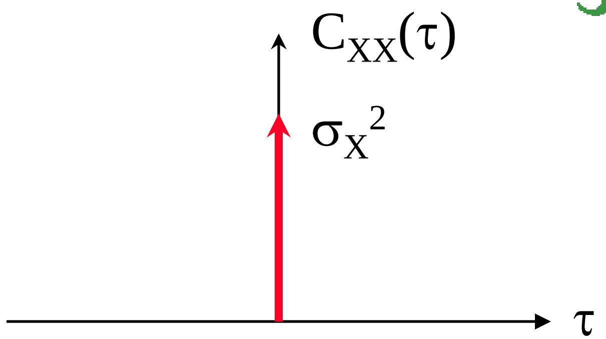
ACF of the Actual Signal-Ensemble average

- The spike at τ =0 (due to white additive noise) does not exist anymore.

Remark:

The r.p. $X(t)$ is said to be "*white*" if

$$C_{XX}(\tau) = \sigma_X^2 \delta(\tau)$$



This is in analogy with *white light*, in which all the frequencies are present.

Example 3:

Given statistically independent, zero mean r.p.'s $X(t)$ and $Y(t)$ with ACF's

$$R_{XX}(\tau) = e^{-|\tau|}$$

$$R_{YY}(\tau) = \cos(2\pi\tau)$$

(a) Find $R_{W_1W_1}(\tau)$ of $W_1(t) = X(t) + Y(t)$
(b) Find $R_{W_2W_2}(\tau)$ of $W_2(t) = X(t) - Y(t)$
(c) Find $R_{W_1W_2}(\tau)$.

Solution:

(a) $R_{W_1W_1}(\tau) = E[W_1(t)W_1(t+\tau)]$

$$= E[(X(t)+Y(t))(X(t+\tau)+Y(t+\tau))]$$

$$= E[X(t)X(t+\tau)] + E[X(t)Y(t+\tau)] +$$

$$E[Y(t)X(t+\tau)] + E[Y(t)Y(t+\tau)]$$

Since statistically independent and zero mean,

$$E[(X(t)Y(t+\tau))] = E[(Y(t)X(t+\tau))] = 0$$

$$R_{XY}(\tau) = R_{YX}(\tau) = 0$$

Thus, $R_{W_1W_1}(\tau) = R_{XX}(\tau) + R_{YY}(\tau)$

$$= e^{-|\tau|} + \cos(2\pi\tau) \longleftarrow$$

(b) $R_{W_2W_2}(\tau) = E[W_2(t)W_2(t+\tau)]$

$$= E[(X(t)-Y(t))(X(t+\tau)-Y(t+\tau))]$$

$$= E[(X(t)X(t+\tau))] - E[(X(t)Y(t+\tau))] -$$

$$E[(Y(t)X(t+\tau))] + E[(Y(t)Y(t+\tau))]$$

$$= R_{XX}(\tau) + R_{YY}(\tau)$$

$$= e^{-|\tau|} + \cos(2\pi\tau) \quad \longleftarrow$$

(c) $R_{W_1W_2}(\tau) = E[W_1(t)W_2(t+\tau)]$

$$= E[(X(t)+Y(t))(X(t+\tau)-Y(t+\tau))]$$

$$= E[(X(t)X(t+\tau))] - E[(X(t)Y(t+\tau))] +$$

$$E[(Y(t)X(t+\tau))] - E[(Y(t)Y(t+\tau))]$$

$$= R_{XX}(\tau) - R_{YY}(\tau)$$

$$= e^{-|\tau|} - \cos(2\pi\tau) \quad \longleftarrow$$

Reading Assignment For Week 11: Sections 6.1-6.3 (Peebles)

Quiz 1 (6 minutes- 2%)

1. Give clear definitions of pseudo and true r.p's and provide an example of each (different than those discussed in class). Can the sample functions in an ensemble set correspond to different 'phenomenon'? Why?

2. Describe the practical importance of auto- and cross-correlation functions and give real-life examples.