EE 512

Digital Signal Processing

Session 8

September 24, 1992

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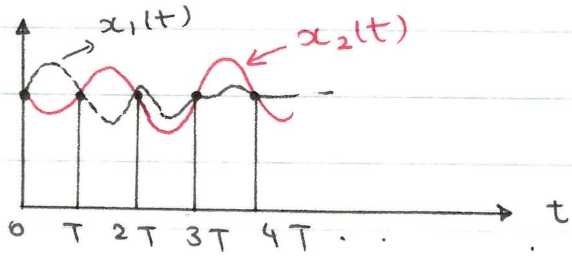
Sampling of Continuous-time Signals

In general, there are infinite number of signals that can generate a given set of samples.



However, if the signal is bandlimited and if the samples are taken sufficiently close together in relation to the highest frequency present in the signal, then the samples uniquely specify the signal and the signal can be reconstructed. Bandlimited signals occur in practical cases such as in telephony (4 KHZ) and television signals (4 MHZ).

The Conversion from Continuous-time to discrete-time signal has great utilities in practice. When transmitting signals through a channel, the time space between the samples of one signal can allow to accommodate without interference the samples of different signals. This process is known as "time-division multiplexing". Moreover, the processing of discrete-time signals offers more flexibility in hardware and software implementation efficiency, accuracy, and simplicity in structure.

Bandlimited Signals

A function $x(t)$ is called bandlimited if it has no spectral content above a certain max. frequency ω_m (f_m in HZ)

$$X(\omega) = 0 \qquad |\omega| > \omega_m$$

ω_m : bandlimits of the signal.

Sampling vs Replication

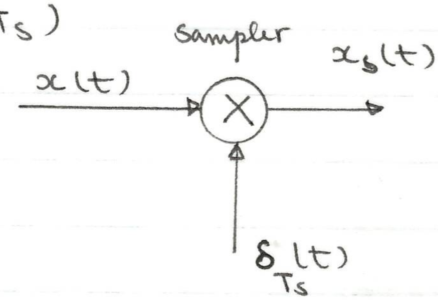
The FT of an arbitrary sampled signal is a scaled, periodic replica of the FT of the original signal. Let $x(t)$ be an arbitrary signal. The ideal signal sampling function is a 1-D infinite array of Dirac delta functions i.e.

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad , \quad T_s: \text{ Sampling period}$$

$$\omega_s = \frac{2\pi}{T_s} : \text{ Sampling frequency}$$

The sampled signal is

$$x_s(t) = x(t) \cdot \delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s)$$



Knowing that

$$\mathcal{F}\{\delta_{T_s}(t)\} = \frac{2\pi}{T_s} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s) = \omega_s\, \delta_{\omega_s}(\omega)$$

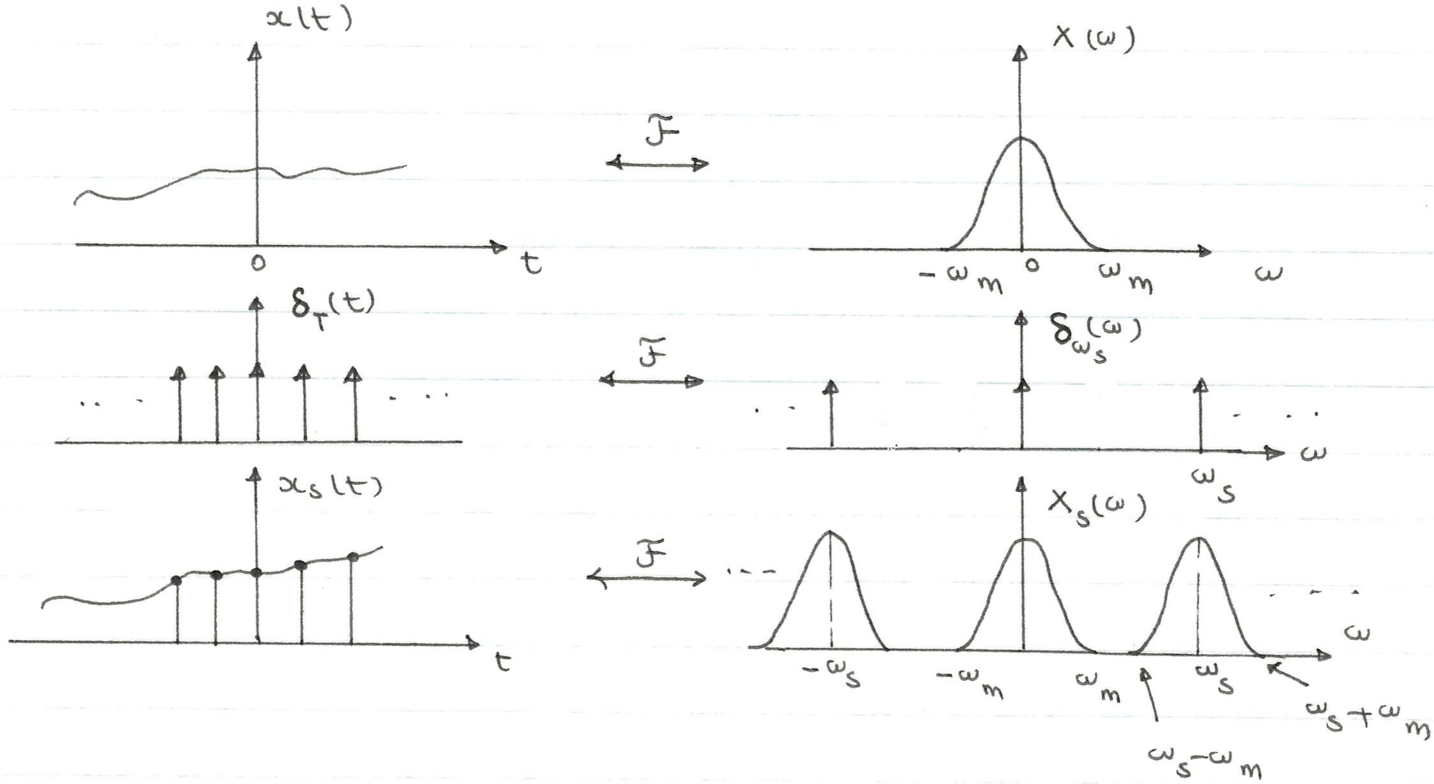
and using the convolution theorem in frequency domain we have

$$X_s(\omega) = \frac{1}{2\pi}\left[X(\omega) * \omega_s\, \delta_{\omega_s}(\omega)\right]$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} X(\xi)\, \delta(\omega - n\omega_s - \xi)\, d\xi$$

$$X_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

Thus, the FT of the sampled signal is a scaled, periodic replica of the FT of the original signal.



Sampling Theorem and Reconstruction

From the uniqueness of FT, the spectrum of the original signal can be recovered from that of the sampled signal $x_s(t)$ by interpolation.

If $\quad \omega_s - \omega_m > \omega_m \quad$ or $\quad \omega_s > 2\omega_m$

there will be no overlap between the shifted replicas and $X(\omega)$ can be recovered from $X_s(\omega)$ by a LPF with frequency response

$$H(\omega) = \begin{cases} T_s & |\omega| \leqslant \omega_c \\ 0 & \text{elsewhere} \end{cases} \qquad \omega_m \leqslant \omega_c \leqslant \omega_s - \omega_m$$

i.e. $\quad X(\omega) = X_s(\omega)\, H(\omega)$

The lower bound on sampling rate i.e. $2\omega_m$ is called "Nyquist frequency" or "Nyquist rate". Its reciprocal is called "Nyquist interval".

If the sampling rate is greater than the Nyquist rate and the signal is bandlimited Complete reconstruction is possible and the reconstructed signal is given by interpolation formula.

Let

$$H(\omega) = \begin{cases} T_s & |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

then $h(t) = T_s \frac{\omega_c}{\pi} \operatorname{Sinc}\left(\frac{\omega_c t}{\pi}\right)$ $\qquad h(t) = \operatorname{Sinc}\left(\frac{\pi t}{T}\right)$ if $\omega_c = \omega_m = \frac{\omega_s}{2}$

and $$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, T_s \frac{\omega_c}{\pi} \operatorname{Sinc}\left[\frac{\omega_c(t-nT_s)}{\pi}\right]$$

For $\omega_c = \omega_m = \frac{\omega_s}{2}$ we have

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \operatorname{Sinc}\left[\frac{\omega_m(t-nT_s)}{\pi}\right]$$

<u>Remarks</u>

1- The above equation is an infinite order interpolation required to reconstruct the Continuous-time function $x(t)$ from its samples $x(nT_s)$.

2- Every bandlimited signal $x(t)$ has a series expansion given by the interpolation formula.

3. The basis functions for this representation are

$$\varphi_m(t) = \operatorname{Sinc}\left[\frac{\omega_c(t-mT_s)}{\pi}\right]$$

and
$$a_m = x(mT_s)$$

Although these functions are orthogonal, they are not complete that is an arbitrary function may not be represented by the expansion.

Undersampling and Aliasing Effects

If the sampling frequency is below the Nyquist rate i.e.

$$\omega_s < 2\omega_m$$

then the periodic replication of $X(\omega)$ will overlap resulting in a distorted spectrum $X_s(\omega)$ from which $X(\omega)$ can not be recovered. In this case, the upper frequencies in $X(\omega)$ get reflected into the lower frequencies in $X_s(\omega)$. These frequencies are called "foldover frequencies" and the phenomenon is known as "Aliasing". Aliasing may be avoided by lowpass filtering the signal first so that its bandlimits is less than one half of the sampling frequency. The spectrum of an undersampled signal is

$$X_s(\omega) = \frac{1}{T_s}\left[X(\omega) + E(\omega)\right]$$

where $E(\omega) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} X(\omega - k\omega_s)$: The repeated components

Effects of artifacts due to Aliasing

If we pass the undersampled signal through a LPF

$$H(\omega) = \begin{cases} T_s & |\omega| < \omega_c \\ 0 & \text{elsewhere} \end{cases}$$

Let $\omega_c = \frac{\omega_s}{2}$

then

$$x_r(t) = x(t) + e(t)$$

where $e(t) = \frac{1}{2\pi}\int_{-\omega_s/2}^{\omega_s/2} E(\omega)\, e^{j\omega t}\, d\omega$

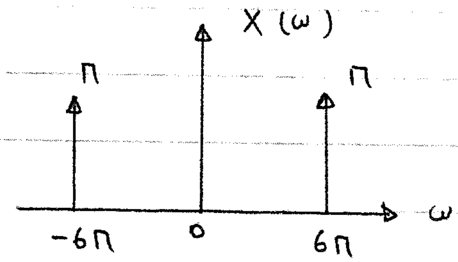
due to aliasing

represents the aliasing error artifact in the reconstructed signal.

Example

A signal $x(t) = \cos 6\pi t$ is sampled at $T_s = 0.2$ sec. Find the reconstructed signal $x_r(t)$.

Clearly $x(t)$ is bandlimited since



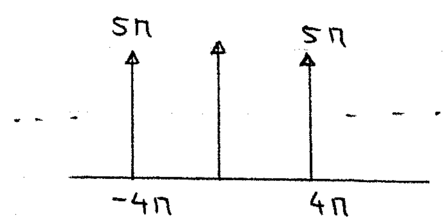
$$X(\omega) = \pi[\delta(\omega - 6\pi) + \delta(\omega + 6\pi)]$$

$$X(\omega) = 0 \qquad |\omega| > 6\pi$$

i.e. $\left.\begin{aligned} \omega_m &= 6\pi \\ \omega_s &= \frac{2\pi}{T_s} = \frac{2\pi}{0.2} = 10\pi \end{aligned}\right\} \Rightarrow \omega_s < 2\omega_m$ undersampling

$$X_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

$$= 5\pi \sum_{n=-\infty}^{\infty} [\delta(\omega - 6\pi - 10\pi n) + \delta(\omega + 6\pi + 10\pi n)]$$



Let the LPF have a cutoff frequency

$\omega_c = \frac{\omega_s}{2} = 5\pi$ i.e.

$$H(\omega) = \begin{cases} 1/5 & |\omega| \leq 5\pi \\ 0 & \text{otherwise} \end{cases}$$

then

$$X_r(\omega) = \pi[\delta(\omega - 4\pi) + \delta(\omega + 4\pi)]$$

or

$$x_r(t) = \cos 4\pi t$$

This shows that any frequency component in the original signal which is above $\frac{\omega_s}{2}$ by $\Delta\omega$ is reproduced or (aliased) as a frequency component at $\frac{\omega_s}{2} - \Delta\omega$. In this example, the frequency component 6π is above $\frac{\omega_s}{2} = 5\pi$ by π and will be