EE 512

Digital Signal Processing

Session 2

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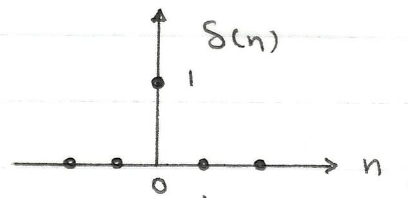
Some important Discrete-Time Signals

The following signals play an important role in digital signal processing.

1- Unit pulse Sequence or Kronecker Delta Function

The unit pulse (or Kronecker delta) sequence $\{\delta(n)\}$ is defined as

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$



This signal plays the same fundamental role in discrete-time theory that the Dirac delta function plays in continuous-time theory. This function satisfies a relation analogous to the shifting property of Dirac delta i.e.

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\, \delta(n-k)$$

Similar To

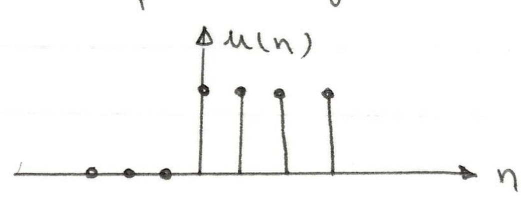
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

Note that the Kronecker delta function does not suffer from the mathematical Complications of that of the Dirac delta. Also this is not a sampled data version of the Dirac delta.

2- Unit step Sequence

The unit step sequence $\{u(n)\}$ is defined by

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Similar to the Continuous-time case

$$u(n) = \sum_{k=0}^{\infty} \delta(n-k) = \sum_{k=-\infty}^{n} \delta(k)$$

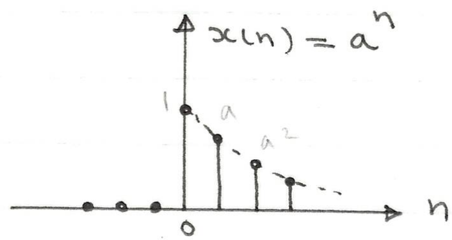
Also we have

$$\delta(n) = u(n) - u(n-1)$$

3- The One-Sided Real Exponential Sequence

Let $0<a<1$ be a real Constant. The sequence

$$x(n) = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases} \qquad 0<a<1$$



This sequence is the discrete-time analog of the Continuous-time exp. function $\{e^{\alpha t}, t \geq 0\}$. In fact we may construct $\{a^n\}_0^\infty$ as a sampled-data version of $\{e^{\alpha t}, t \geq 0\}$ by sampling the latter function every T sec.

$$e^{\alpha t}\Big|_{t=nT} = e^{\alpha T n} = a^n$$

where $a \triangleq e^{\alpha T}$ or $\alpha = \frac{1}{T} \ln a$

For $0<a<1 \implies -\infty < \alpha < 0$

4- The Complex Exponential Sequence

The Complex exponential Sequence is defined by

$$x(n) = e^{j\Omega n}, \forall n \qquad \Omega: \text{Frequency of Complex EXP.}$$

This function is also obtained from sampling the Continuous-time function $\{e^{j\omega t}, \forall t\}$ i.e

$$e^{j\omega t}\Big|_{t=nT} = e^{j\omega nT} = e^{j\Omega n}$$

where $\Omega = \omega T$

Periodicity:

Consider an arbitrary Complex sequence $\{x(n)\}$. The sequence is said to be periodic if

$$x(n) = x(n+N) \qquad \forall n$$

and N is the smallest positive integer constant for which the equality holds.

periodicity of $e^{j\Omega n}$ in n requires that

$$e^{j\Omega(n+N)} = e^{j\Omega n}\, e^{j\Omega N}$$

which is equal to $e^{j\Omega n}$ if

$$\Omega N = 2\pi m \implies N = \frac{2\pi m}{\Omega}$$

or $\dfrac{\Omega}{2\pi} = \dfrac{m}{N}$

i.e. $e^{j\Omega n}$ is periodic only when $\frac{\Omega}{2\pi}$ is a rational number. This is in contrast to the continuous-time case where there is no constraint on ω.

Now assume that the above condition is satisfied i.e.

$$\Omega = \frac{2\pi m}{N}$$

then the fundamental period N_o of $e^{jm(\frac{2\pi}{N})n}$ is the smallest integer which gives

$$x(n) = x(n+N_o) \quad \text{or} \quad e^{jm(\frac{2\pi}{N})(n+N_o)} = e^{jm(\frac{2\pi}{N})n}$$

which requires

$$m\left(\frac{2\pi}{N}\right)N_o = 2\pi K$$

or

$$m\left(\frac{N_o}{N}\right) = K \qquad K \in I$$

If gcd (m, N) is the greatest common Divisor of m and N, then

$$m = m' \gcd(m, N) \quad \text{and} \quad N = N' \gcd(m, N)$$

Note that m', N' have no common factors (prime), hence

$$m'\left(\frac{N_o}{N'}\right) = K \implies N_o = N' \quad (\text{smallest case})$$

or

$$N_o = N / \gcd(m, N)$$

If m and N have no common factor the fundamental period $N_o = N$ and the fundamental frequency is $\frac{2\pi}{N} = \frac{\Omega}{m}$.

Remark

1- In the continuous-time case signal $e^{j\omega t}$ are all distinct for distinct values of ω. In discrete-time case $e^{j\Omega n}$ are not distinct, since

$$e^{j(\Omega \pm 2\pi K)n} = e^{j\Omega n} e^{\pm j 2\pi K n} = e^{j\Omega n}$$

Thus, we need to consider only an interval of length 2π in which to choose Ω.

$$0 \leq \Omega \leq 2\pi \quad \text{or} \quad -\pi \leq \Omega \leq \pi$$

$e^{j\omega t}$	$e^{j\Omega n}$
Distinct for distinct values of ω.	Identical for Exp. at frequencies separated by 2π.
periodic for any choice of ω	periodic only if $\Omega = \frac{2\pi m}{N}$ for some integers $N > 0$ and m
Fundamental frequency ω	Fundamental frequency $\frac{\Omega}{m}$ †
Fundamental period $\omega = 0$: undefined $\omega \neq 0$ $\frac{2\pi}{\omega}$	Fundamental period † $\Omega = 0$: undefined $\Omega \neq 0$: $m\left(\frac{2\pi}{\Omega}\right)$
†: means m, N do not have any factors in common.	

Differences Between $e^{j\omega t}$ and $e^{j\Omega n}$