EE 512

DIGITAL SIGNAL PROCESSING

Session 9

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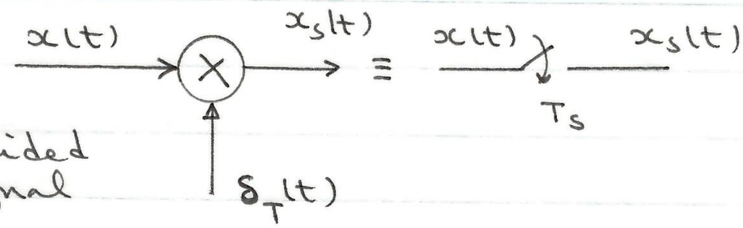
Relation Between Laplace Transform and Z-Transform

Consider a finite energy signal $x(t)$ having a bandlimited spectrum. Assume that this signal is sampled at Nyquist rate with sampling frequency $\omega_s > 2\omega_m$ where ω_m represents the bandlimits of the signal. Then we can write

$$x_s(t) = x(t)\,\delta_T(t)$$



where

$$\delta_T(t) = \sum_{n=0}^{\infty} \delta(t - nT_s)$$

one-sided signal

or

Recall that

$$x_s(t) = x(t)\,\delta(t) + x(t)\,\delta(t - T_s) + \cdots$$

$$= x(0)\,\delta(t) + x(T_s)\,\delta(t - T_s) + \cdots = \sum_{n=0}^{\infty} x(nT_s)\,\delta(t - nT_s)$$

Taking LT of $x_s(t)$ gives

$$X_s(s) = \int_0^{\infty} x_s(t)\, e^{-st}\, dt$$

$$= x(0) + x(T_s)\, e^{-T_s s} + x(2T_s)\, e^{-2T_s s} + \cdots$$

or

$$X_s(s) = \sum_{n=0}^{\infty} \int_0^{\infty} x(\overset{nT_s}{t})\,\delta(t - nT_s)\, e^{-st}\, dt$$

$$= \sum_{n=0}^{\infty} x(nT_s)\, e^{-nT_s s} \qquad \text{LT of the sampled signal}$$

Define $z = e^{T_s s}$ or $s = \frac{1}{T_s} \ln(z)$ then

$$X_s(s)\Big|_{s = \frac{1}{T_s}\ln(z)} \triangleq X(z) = \sum_{n=0}^{\infty} x(nT_s)\, z^{-n} \qquad \text{i.e.}$$

Denoting $x(n) = x(nT_s)$ where the sampling interval T_s is implied yields

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

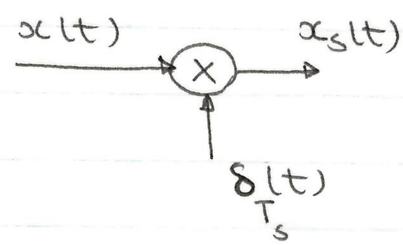
i.e. the LT of a sampled signal yields the z-Transform when $e^{T_s s}$ is replaced by z.

Discrete-Time Fourier Transform and its Relation to Fourier Transform

Recall that

$$X_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s) \quad \text{periodic with period } \omega_s$$

$$x_s(t) = x(t) \cdot \delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s)$$



Taking the FT of $x_s(t)$ gives

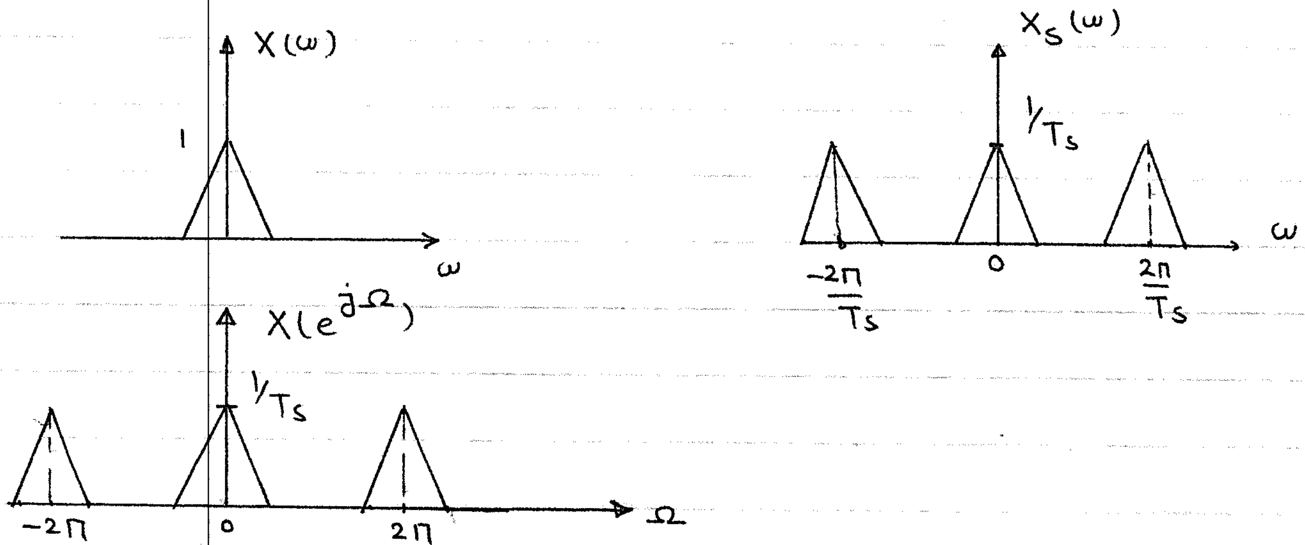
$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j\omega n T_s}$$

This result can also be obtained by $s = j\omega$ in $X_s(s)$. Now, define

$$x(n) = x(nT_s) \quad \text{and} \quad \Omega \triangleq \omega T_s \quad \text{then}$$

$$X(e^{j\Omega}) \triangleq X_s\left(\frac{\Omega}{T_s}\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} \Rightarrow \text{DTFT of } \{x(n)\}$$

Since $X(e^{j\Omega}) = X_s\left(\frac{\Omega}{T_s}\right)$, $X(e^{j\Omega})$ is a frequency scaled version of $X_s(\omega)$ and is periodic in Ω with period 2π. The scaling in this equation is the result of time normalization and $\Omega = \omega T_s$ is consistent with the notion of converting $x_s(t)$ to $x(n)$ requires scaling the time axis by $1/T_s$.



Remarks

1- DTFT can also be interpreted as the restriction of Z-Transform to the unit circle i.e.

$$X(z)\Big|_{z=e^{j\Omega}} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

In order for the unit circle to be part of the ROC for $X(z)$

$$\left| \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n} \right| = \sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

2- The DTFT of a sequence $x(n)$, i.e. $X(e^{j\Omega})$ is periodic with period 2π and so as the Kernel $e^{-j\Omega n}$. This is the major difference between DTFT and FT. The Kernel $e^{-j\Omega n}$ is the DTFT of $\delta(k-n)$ i.e

$$\delta(k-n) \overset{DTFT}{\longleftrightarrow} e^{-j\Omega n}$$

These two signals are orthogonal orthonormal both in time and frequency domains i.e.

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$$d_n(k) = \delta(k-n) \qquad D_n(e^{j\Omega}) = e^{-j\Omega n}$$

then

$$\sum_{k=-\infty}^{\infty} d_m(k)\, d_n(k) = \delta(m-n)$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_m(e^{j\Omega})\,[D_n(e^{j\Omega})]^{*}\, d\Omega = \delta(m-n)$$

3 - Using the orthogonality we establish the inverse DTFT

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\,[D_n(e^{j\Omega})]^{*}\, d\Omega = \sum_{m} x(m)\,\frac{1}{2\pi}\int_{-\pi}^{\pi} D_m(e^{j\Omega})\,[D_n(e^{j\Omega})]^{*}\, d\Omega$$

$$= \sum_{m} x(m)\,\delta(m-n) = x(n)$$

or

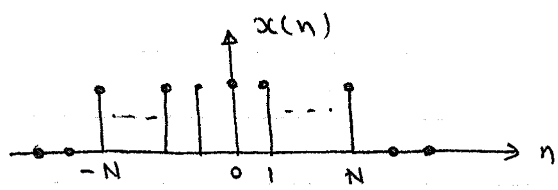
$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega n}\, d\Omega$$

Note that in contrast to FT when the integration is done over an infinite interval, in this case since both $X(e^{j\Omega})$ and $e^{j\Omega n}$ are periodic with period 2π and also $X(e^{j\Omega})\, e^{j\Omega n}$ is periodic with period 2π, the integration is taken in any interval of length 2π.

4 - DTFT applies to nonperiodic signals both of finite and infinite durations.

<u>Example</u>

Consider a rectangular pulse



$$x(n) = \begin{cases} 1 & |n| \le N \\ 0 & \text{elsewhere} \end{cases}$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n} = \sum_{n=-N}^{N} e^{-j\Omega n} = \sum_{n=0}^{N} e^{-j\Omega n} + \sum_{n=0}^{N} e^{j\Omega n} - 1$$

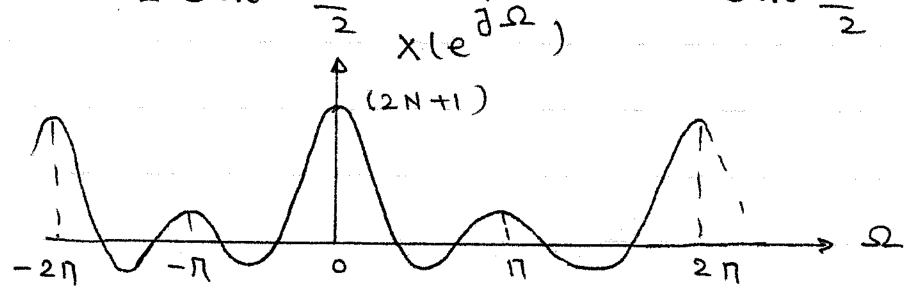
$$X(e^{j\Omega}) = \frac{1 - e^{-j\Omega(N+1)}}{1 - e^{-j\Omega}} + \frac{1 - e^{j\Omega(N+1)}}{1 - e^{j\Omega}} - 1$$

$$= \frac{e^{-j\Omega N} + e^{j\Omega N} - e^{-j\Omega(N+1)} - e^{+j\Omega(N+1)}}{2 - e^{-j\Omega} - e^{j\Omega}}$$

$$= \frac{2\cos\Omega N - 2\cos\Omega(N+1)}{2 - 2\cos\Omega}$$

$$= \frac{2\sin\Omega(N+1/2)\sin\frac{\Omega}{2}}{2\sin^2\frac{\Omega}{2}} = \frac{\sin\Omega(N+1/2)}{\sin\frac{\Omega}{2}}$$

Hospital Rule



This is the discrete-time counterpart of the continuous-time sinc function but this function is periodic with period 2π.

At $\Omega = 0$ $\quad X(e^{j\Omega})\Big|_{\Omega=0} = \dfrac{(N+1/2)\cos\Omega(N+1/2)}{1/2\cos\frac{\Omega}{2}}\Bigg|_{\Omega=0}$

$$= 2N+1 \quad \text{using Hopital's rule.}$$

Also note that $X(e^{j\Omega})$ is the aliased version of $X(\omega)$ and since $X(\omega)$, (sinc function) is unbandlimited $x(t)$ cannot be recovered from $X(e^{j\Omega})$.

Examples

DTFT example (3.22)

Let $X(e^{j\Omega}) = \dfrac{1}{1-a\,e^{-j\Omega}}$

Determine DTFT of (a) $x(2n+1)$, (b) $e^{\pi n/2}\,x(n+2)$

(c) $x(n)\cos(0.3\pi n)$, (d) $x(n) * x(n-1)$

(a) Let $x_1(n) = x(2n+1)$ then

$$X_1(e^{j\Omega}) = \sum_n x_1(n)\,e^{-j\Omega n} = \sum_n x(2n+1)\,e^{-j\Omega n}$$

Change $2n+1 = k \implies n = \frac{k-1}{2}$ hence

$$X_1(e^{j\Omega}) = \sum_k x(k)\,e^{-j\Omega/2\,k}\,e^{-j\Omega/2}$$

$$= e^{-j\Omega/2}\,X(e^{j\Omega/2}).$$

(b) Let $x_2(n) = e^{\pi n/2}\,x(n+2)$ then

$$X_2(e^{j\Omega}) = \sum_n x(n+2)\,e^{\pi n/2}\,e^{-j\Omega n}$$

Change $n+2 = k$ then

$$X_2(e^{j\Omega}) = \sum_k x(k)\,e^{-\pi\frac{(k-2)}{2}}\,e^{-j\Omega(k-2)}$$

$$= -X(e^{j(\Omega+j\pi/2)})\,e^{2j\Omega}$$

(c) Let $x_3(n) = x(n)\cos(0.3\pi n) = \frac{1}{2}(e^{j0.3\pi n} + e^{-j0.3\pi n})\,x(n)$

then

$$X_3(e^{j\Omega}) = \frac{1}{2}\sum_n x(n)\left[e^{-j(\Omega-0.3\pi)n} + e^{-j(\Omega+0.3\pi)n}\right]$$

$$= \frac{1}{2}\left[X(e^{j(\Omega-0.3\pi)}) + X(e^{j(\Omega+0.3\pi)})\right]$$

(d) $X_4(e^{j\Omega}) = X^2(e^{j\Omega})\,e^{-j\Omega}$ Using convolution.

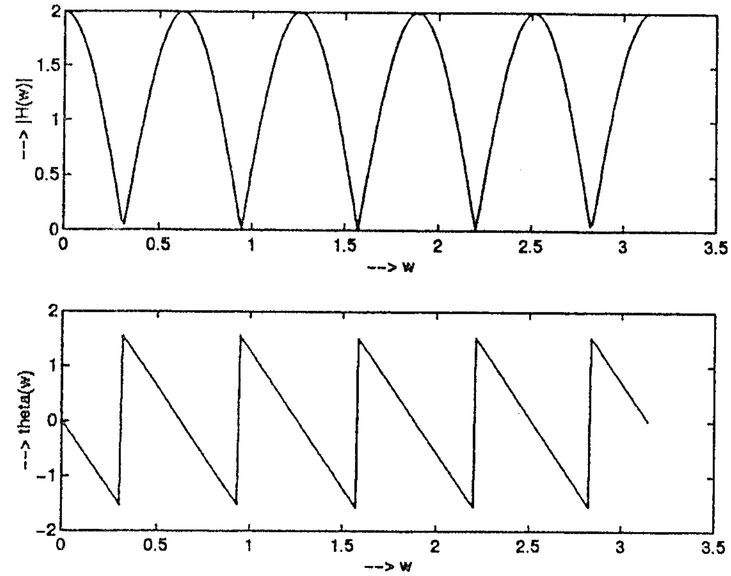


Figure 4.25:

$$\begin{aligned} y_{tr}(n) &= 10e^{\frac{\pi n}{2}}u(n) + 20e^{\frac{\pi(n-2)}{2}}u(n-2) + 10e^{\frac{\pi(n-4)}{2}}u(n-4) \\ &= 10\delta(n) + j10\delta(n-1) + 10\delta(n-2) + j10\delta(n-3) \end{aligned}$$

4.32

(a)

Given FIR

$$\begin{aligned} y(n) &= x(n) + x(n-4) \\ Y(w) &= (1 + e^{-j4w})X(w) \\ H(w) &= (2cos2w)e^{-j2w} \end{aligned}$$

Compute response to $x(n) = \cos \pi/_2 \, n + \cos \pi/_4 \, n$

Refer to fig 4.26.

(b)

$$\begin{aligned} y(n) &= cos\frac{\pi}{2}n + cos\frac{\pi}{4}n + cos\frac{\pi}{2}(n-4) + cos\frac{\pi}{4}(n-4) \\ \text{But } cos\frac{\pi}{2}(n-4) &= cos\frac{\pi}{2}ncos2\pi + sin\frac{\pi}{2}nsin2\pi \\ &= cos\frac{\pi}{2}n \\ \text{and } cos\frac{\pi}{4}(n-4) &= cos\frac{\pi}{4}ncos\pi - sin\frac{\pi}{4}nsin\pi \\ &= -cos\frac{\pi}{4}n \\ \text{Therefore, } y(n) &= 2cos\frac{\pi}{2}n \end{aligned}$$

(c) Note that $H(\frac{\pi}{2}) = 2$ and $H(\frac{\pi}{4}) = 0$. Therefore, the filter does not pass the signal $cos(\frac{\pi}{4}n)$.

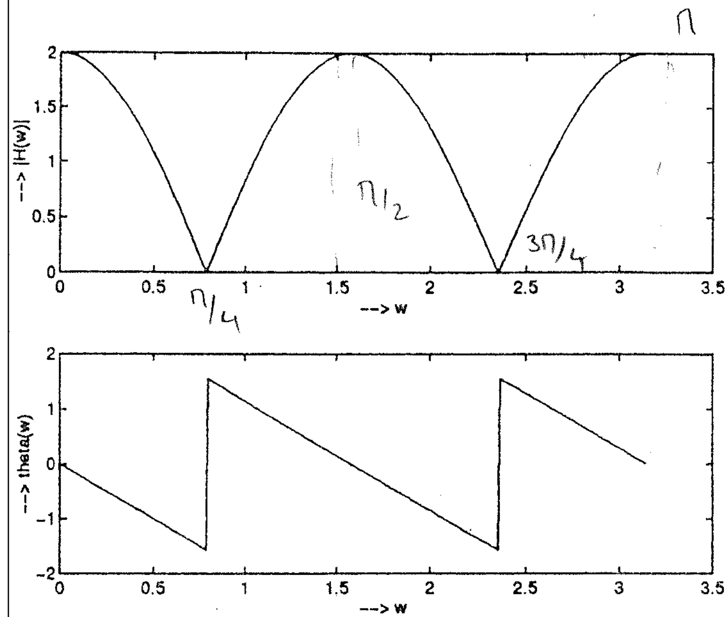


Figure 4.26:

4.33

$$
\begin{aligned}
y(n) &= \frac{1}{2}[x(n) - x(n-2)] \\
Y(w) &= \frac{1}{2}(1 - e^{-j2w})X(w) \\
H(w) &= \frac{1}{2}(1 - e^{-j2w}) \\
&= (sinw)e^{j(\frac{\pi}{2} - w)} \\
H(0) &= 0, H(\frac{\pi}{2}) = 1 \\
\text{Hence, } y_{ss}(n) &= 3cos(\frac{\pi}{2}n + 60^o) \\
y_{tr}(n) &= 0
\end{aligned}
$$

4.34

$x(n) = Acos\frac{\pi}{4}n$
(a) $y(n) = x(2n) = Acos\frac{\pi}{2}n \Rightarrow w = \frac{\pi}{2}$
(b) $y(n) = x^2(n) = A^2cos^2\frac{\pi}{4}n = \frac{1}{2}A^2 + \frac{1}{2}A^2cos\frac{\pi}{2}n$. Hence, $w = 0$ and $w = \frac{\pi}{2}$
(c)

$$
\begin{aligned}
y(n) &= x(n)cos\pi n \\
&= Acos\frac{\pi}{4}ncos\pi n \\
&= \frac{A}{2}cos\frac{5\pi}{4}n + \frac{A}{2}cos\frac{3\pi}{4}n \\
\text{Hence, } w &= \frac{3\pi}{4} \text{ and } w = \frac{5\pi}{4}
\end{aligned}
$$