Design of Non-recursive or FIR Filters

Described by

$$y(n) = \sum_{k=0}^{N} h(k)\, x(n-k)$$

$h(k)$: impulse response sequence or coefficient of the filter.

Advantages

(1) Inherently stable
(2) They can provide linear phase characteristics, speech processing and data processing
(3) Quantization effects can be made small
(4) They can be implemented in parallel / pipeline fashions using array processors.

Disadvantages

(1) For a sharp cut-off performance a large order filter is required.
(2) They need "fractional delays" in some cases. (cannot be realized by physical devices)

Frequency Response of FIR Filter

$h(n)$: impulse response $n \in [0, N-1]$

$$h(n) = 0 \quad \forall\, n \geq N,\ n < 0$$

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$ Transfer Function.

N-1 zeros, and (N-1) poles at z=0

Let $z = e^{j\Omega}$ $\qquad H(e^{j\Omega}) = H(e^{j(\Omega+2\pi\ell)})$ $\quad \forall \ell \in \mathbb{Z}$

$$H(e^{j\Omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\Omega n} = \text{DTFT}\{h(n)\}$$

$z = e^{j\Omega}\big|_{\Omega = \frac{2\pi k}{N}}$

$$\Omega = \frac{2\pi k}{N}$$

$$\text{DFT}\{h(n)\} = H(k) = \sum_{n=0}^{N-1} h(n)\, e^{-j\frac{2\pi k n}{N}}$$

We can use IDFT to get the result. ($\because$ FIR filter).
Following cases are considered

Case 1: Symmetric

$$h(n) = h(N-1-n)$$

(a) N: odd

$$H(e^{j\Omega}) = \underbrace{e^{-j\Omega\left(\frac{N-1}{2}\right)}}_{\text{phase}} \underbrace{\sum_{n=0}^{\left(\frac{N-1}{2}\right)} \alpha(n) \cos \Omega n}_{\text{magnitude}}$$

where $\alpha(n) = 2h\left[\left(\frac{N-1}{2}\right) - n\right]$ $\qquad n = 1, 2, \ldots, \frac{N-1}{2}$

$$\alpha(0) = h\left(\frac{N-1}{2}\right)$$

This has linear phase characteristics.

$$\phi(\Omega) = -\Omega\left(\frac{N-1}{2}\right)$$

group delay $\tau_g = -\dfrac{d\phi(\Omega)}{d\Omega} = \left(\dfrac{N-1}{2}\right)$.

$= \text{integer}$.

(b) N. Even

$$H(e^{j\Omega}) = e^{-j\Omega\left(\frac{N-1}{2}\right)} \sum_{n=1}^{\frac{N}{2}} \beta(n) \cos[\Omega(n-1/2)]$$

$$\beta(n) = 2h\left(\frac{N}{2}-n\right) \quad ; \quad n = 1, 2, \ldots, \frac{N}{2}$$

At $\Omega = \pi$, $H(e^{j\Omega}) = 0$ i.e. HPF cannot be approximated

Case 2: Antisymmetric

$$h(n) = -h(N-1-n)$$

$$h\left(\frac{N-1}{2}\right) = 0$$

(a) N: odd

$$H(e^{j\Omega}) = e^{-j\Omega\left(\frac{N-1}{2}\right)} e^{j\frac{\pi}{2}} \sum_{n=1}^{\left(\frac{N-1}{2}\right)} \gamma(n) \sin \Omega n$$

At $\Omega = 0$ and $\Omega = \pi$ $|H(e^{j\Omega})| = 0$ Hilbert transform + differentiator

$$\gamma(n) = 2h\left[\left(\frac{N-1}{2}\right)-n\right] \quad , \quad n = 1, 2, \ldots, \frac{N-1}{2}$$

(b) N: Even

$$H(e^{j\Omega}) = e^{-j\Omega\left(\frac{N-1}{2}\right)} e^{j\frac{\pi}{2}} \sum_{n=1}^{N/2} \delta(n) \sin[\Omega(n-\tfrac{1}{2})]$$

$$\delta(n) = 2h\left(\frac{N}{2}-n\right) \quad n = 1, 2, \ldots, \frac{N}{2}$$

At $\Omega = 0 \Rightarrow H(e^{j\Omega}) = 0$

$\Omega = \pi \quad H \neq 0$

(D). Design of FIR Filters

There two groups of design methods

Indirect method

(a) Windowing method
(b) Frequency Sampling method

Direct method

(a) Computer Aided Optimization Method
(Remez Exchange Algorithm)

Desired frequency response is $H_D(e^{j\Omega})$

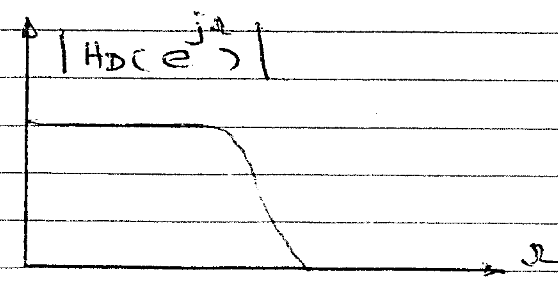
$$h_D(n) = \frac{1}{\cancel{\Omega_s}\;2\pi} \int_{-\frac{\Omega_s}{2}\;-\pi}^{\frac{\Omega_s}{2}\;\pi} H_D(e^{j\Omega})\, e^{j\Omega n}\, \overset{\Omega}{d\cancel{\phi}}$$

$$= \mathrm{IDTFT}\{ H_D(e^{j\Omega}) \}$$

$\mathrm{IDTFT}\{ H_D e^{j\Omega} \}$ is normally of infinite extent !!!

$$H_D(e^{j\Omega}) = \mathrm{DTFT}\{ h_D(n) \}$$

$$= \sum_{n=-\infty}^{\infty} h_D(n)\, e^{-j\Omega n}$$



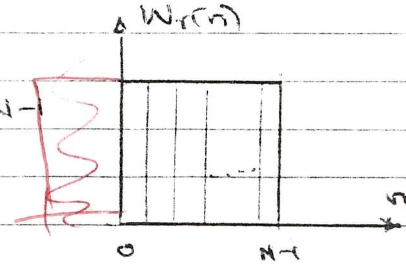
(a) Windowing Methods

(1) Rectangular Windowing

Assume $h_D(n)$ is the impulse response of the desired filter which is of infinite extent, then

$$\overset{N}{h}(n) = h_D(n)\, \omega_r(n)$$

where
$$\omega_r(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$



width of main lobe $= 4\pi/N$

$\omega_r(n)$: rectangular window.

$$W_r(e^{j\Omega}) = \text{DTFT}\{\omega_r(n)\} = \sum_{n=0}^{N-1} e^{-j\Omega n} = \frac{1 - e^{-j\Omega N}}{1 - e^{-j\Omega}}$$

$$= \frac{\sin(\Omega N/2)}{\sin(\Omega/2)}\, e^{-j\Omega(N-1)/2}$$ repetitive Sinc function (see pp 60)

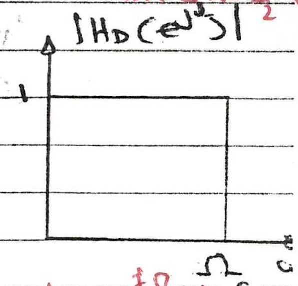
$N \to$ increase main lobe ~~widens~~ narrows

$$|W_r(e^{j\Omega})| = \frac{|\sin \Omega N/2|}{|\sin \Omega/2|}, \quad \phi(\omega) = \begin{cases} -\frac{\Omega(N-1)}{2} & \text{when } \sin\frac{\Omega N}{2} > 0 \\ -\frac{\Omega(N-1)}{2} + \pi & \text{when } \sin\frac{\Omega N}{2} < 0 \end{cases}$$

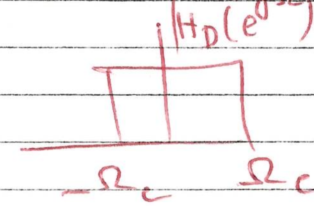
Example

Design a LPF

$$H_D(e^{j\Omega}) = \begin{cases} 1 & |\Omega| \le \Omega_c \\ 0 & |\Omega| > \Omega_c \end{cases}$$



$$h_D(n) = \frac{1}{\Omega_s} \int_{-\Omega_c}^{\Omega_c} e^{j\Omega n}\, d\Omega$$



$$= \frac{1}{\pi n} \sin \Omega_c n$$

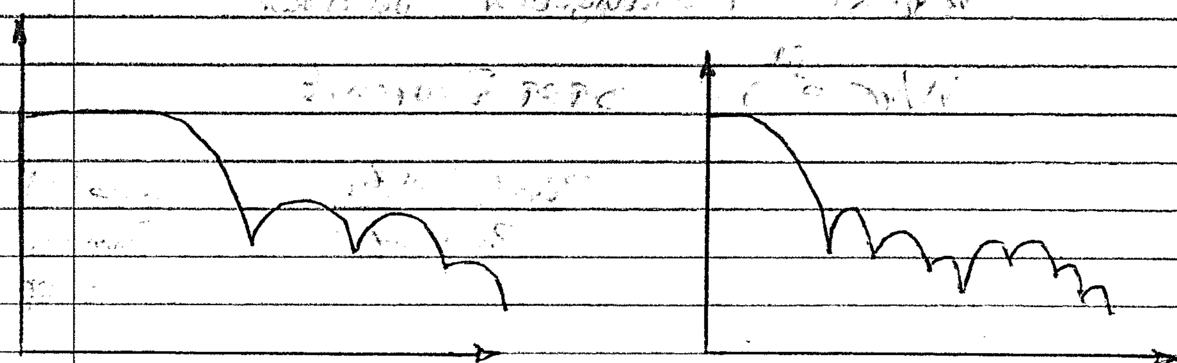
$$= \frac{\Omega_c}{\pi n \Omega_c} \sin \Omega_c n$$

$$= \frac{\Omega_c}{\pi} \operatorname{Sinc} \Omega_c n$$

$$\tilde{h}(n) = h_D(n) \cdot w_r(n)$$

$$\tilde{H}(e^{j\Omega}) = H_D(e^{j\Omega}) * W_R(e^{j\Omega}).$$

Because of the sharp window this will lead to the Gibbs phenomina (ie output contains ripples). # of ripples increase with number of points, but amplitude will remain same.



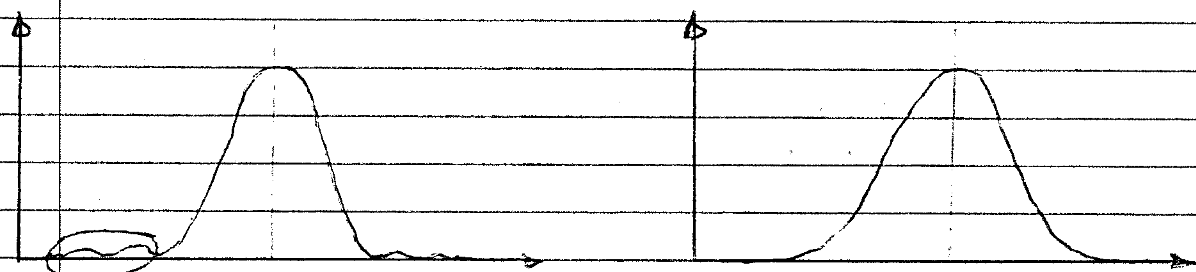
By adding more points we cannot get rid of Gibbs phenomina.

(2) Hann and Hamming Window

$$w_H(n) = \begin{cases} \alpha + (1-\alpha)\cos\dfrac{2\pi n}{N-1} & \text{for } |n| \le \dfrac{N-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$\alpha = 0.5$ Hann window

$\alpha = 0.54$ Hamming window



Hamming window Hann window

oscillations

$$W_H(e^{j\Omega}) = \alpha \frac{\sin(\Omega N/2)}{\sin(\Omega/2)}$$

$$+ \left\{ \frac{1-\alpha \sin[\Omega N/2 - N\pi/(N-1)]}{2\sin[\Omega/2 - \pi(N-1)]} \right\}$$

$$+ \left\{ \frac{1-\alpha \sin[\Omega N/2 + N\pi/(N-1)]}{2\sin[\Omega/2 + \pi(N-1)]} \right\}$$

There windows will greatly reduce the amplitude of the rippler introduced in output by Gibbs phenomona.

(3) Blackman Window

$$w_B(n) = \begin{cases} 0.42 + 0.5\cos\frac{2\pi n}{N-1} + 0.8\cos\frac{4\pi n}{N-1} & \text{for } |n| \le \frac{N-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

(4) Kaiser Window

Optimum in the sense that the energy is concentrated in the main lobe.

$$w_k(n) = \begin{cases} \dfrac{I_0\left[\beta\sqrt{1-(2n/N-1)^2}\right]}{I_0(\beta)} & |n| \le \frac{N-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

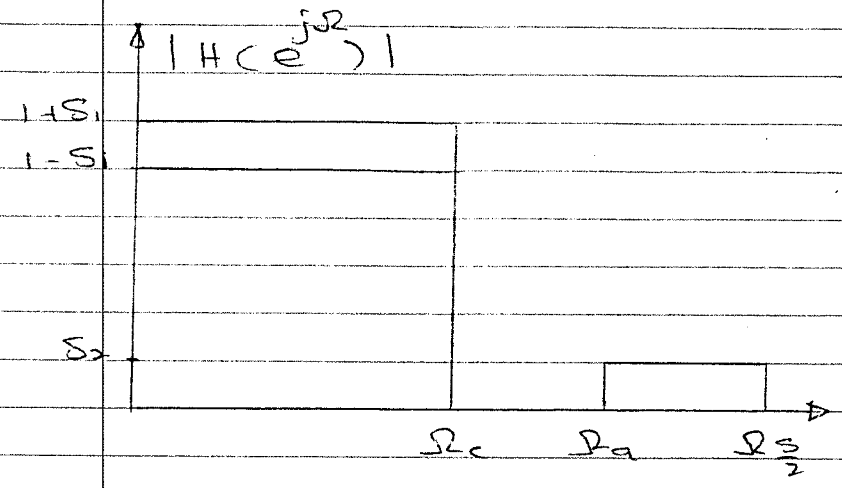
$I_0(\,)$: modified zeroth order Bessel function of 1st kind

β: parameter which determines trade-offs between the main lobe and the peak of the side lobe

$$I_0(x) = 1 + \sum_{m=1}^{\infty}\left[\frac{(x/2)^m}{m!}\right]^2$$

1st fifteen is sufficient for most of Applications

Design Procedure for Kaiser Window



Ω_s - sampling frequency.

Find

(1) The normalized transition BW

$$\Delta\Omega = \frac{\Omega_a - \Omega_c}{\Omega_s}$$

(2) Stopband attenuation

$$A = -20 \log_{10} \delta$$

$$\delta = \min(\delta_1, \delta_2)$$

Assume that $\delta_1 \approx \delta_2$

(3) The order is given by

$$N \simeq \frac{A - 7.95}{14.36\, \Delta\Omega}$$

(4) Shape parameter β is

$$\beta = \begin{cases} 0.1102\,(A - 8.7) & A \geq 50 \\ 0.5842\,(A-21)^{0.4} + 0.07886\,(A-21) & \text{for } 21 < A < 50 \\ 0 & \text{for } A < 21 \end{cases}$$

example:

Design a LPF using Kaiser window to satisfy

$\Omega_c = 2000\pi$ Rad/sec

$\Omega_a = 3000\pi$ Rad/sec

$\Omega_s = 20000\pi$ Rad/sec

$A = 50$ dB

$$\Delta\Omega = \frac{\Omega_a - \Omega_c}{\Omega_s} = 0.05$$

$$N \simeq \frac{A - 7.95}{14.36\,\Delta\Omega} = \frac{50 - 7.95}{14.36\,(0.05)} \simeq 59$$

$\beta = 0.1102\,(50 - 8.7) = 4.55$

$$\boxed{\tilde{H}(z) = z^{-\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\left(\frac{N-1}{2}\right)} \frac{a_n}{2}\left[z^{n} + z^{-n}\right]}$$

Transfer function for all window methods

Transfer function of the designed filter where

$a_0 = W_k(0)\, h_D(0)$

$a_n = 2\, W_k(n)\, h_D(n)$

Disadvantages of Window methods:

(1) It is very difficult to find coefficients $h_D()$ accurately

(2) It cannot get rid of Gibbs phenomena.

* Kaiser window gives the best result of all other windows.

(b) N : Even

$$H(e^{j\Omega}) = e^{-j\Omega\left(\frac{N-1}{2}\right)} e^{j\pi/2} \sum_{n=1}^{N/2} \theta(n) \sin\left[\Omega\left(n-\tfrac{1}{2}\right)\right]$$

Where

$$\theta(n) \triangleq 2h\left(\tfrac{N}{2}-n\right), \quad n \in \left[1, \tfrac{N}{2}\right]$$

In this case at $\Omega = 0$, $H(e^{j\Omega}) = 0$.

Design methods

1- Windowing

2- Frequency Sampling method

3- Computer Aided Design methods.

In ECE 412 notes

Frequency Sampling Method

Idea: Given the frequency response of the desired filter, $H_{Des}(e^{j\Omega})$, sample it to obtain $H(K)$ and then take IDFT of $\{H(K)\}$ to get $\{h(n)\}$

Note that $\{h(n)\}$ and $\{H(K)\}$ are assumed to be finite extent sequences.

Let the transfer function of the FIR filter be

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

We know that

$$H(K) = H(z)\Big|_{z = e^{j\frac{2\pi K}{N}}}$$

Direct Method

Computer Aided Optimization

McClellan - parks

(a) Remez Exchange Algorithm - McCellan and Parks
(b) Linear Programming

Chebyshev Approximation

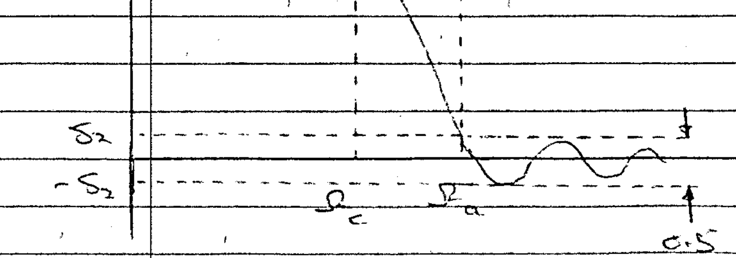
Consider a linear phase FIR filter with N = odd and h(n) symmetric

$$H(e^{j\Omega}) = e^{-j\Omega\frac{(N-1)}{2}} \sum_{n=0}^{\left(\frac{N-1}{2}\right)} \alpha(n) \cos \Omega n$$

$$\alpha(n) = 2\, h\left(\frac{N-1}{2} - n\right) \qquad n \in \left[1, \frac{N-1}{2}\right]$$

$$\alpha(0) = h\left(\frac{N-1}{2}\right)$$

objective: Find $\{h(n)\}$ or $\{\alpha(n)\}$ so that $H(e^{j\Omega})$ has certain desired characteristics.



Ω (normalized frequency) based on sampling frequency Ω_s

δ_1: allowed passband ripple
δ_2: allowed stopband ripple
Ω_c: desired passband edge
Ω_a: desired stopband edge

Define

$$B_P = \{\Omega : 0 \le \Omega \le \Omega_a\}$$

$$B_S = \{\Omega : \Omega_a \le \Omega \le 0.5\} \qquad F = B_p \cup B_s$$

Given $H_D(e^{j\Omega})$ as the desired frequency response and a positive weight function $W(e^{j\Omega})$ then design $\tilde{H}(e^{j\Omega})$
Find $\alpha(n)$

$$|\tilde{H}(e^{j\Omega})| = \left|\sum_{n=0}^{\left(\frac{N-1}{2}\right)} \alpha(n) \cos \Omega n\right|.$$

$$E(e^{j\Omega}) = W(e^{j\Omega}) \left|\tilde{H}(e^{j\Omega}) - H_D(e^{j\Omega})\right|$$

so that

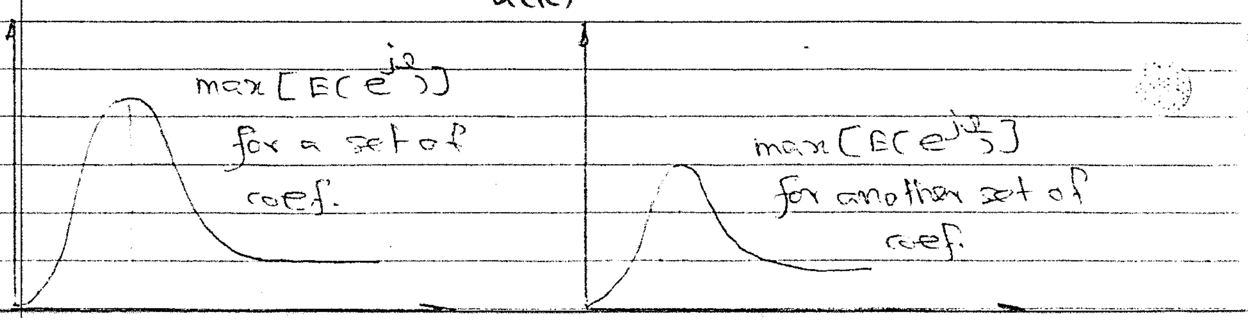
$$\left\| E(e^{j\Omega}) \right\|_\infty = \underset{\Omega \in F}{\mathrm{Max}} \left[W(e^{j\Omega}) \left| \tilde{H}(e^{j\Omega}) - H_D(e^{j\Omega}) \right| \right]$$

is minimized.

$F = B_P \cup B_S$ and

L_∞ norm is defined

$$\left\| E(e^{j\Omega}) \right\|_\infty = \underset{\substack{\{\text{coeff.}\} \\ \alpha(k)}}{\mathrm{Min}} \left[\underset{\Omega \in F}{\mathrm{Max}} \left| E(e^{j\Omega}) \right| \right]$$



Examples for $H_D(e^{j\Omega})$ and $W(e^{j\Omega})$

$$H_D(e^{j\Omega}) = \begin{cases} 1 & \Omega \in B_P \\ 0 & \Omega \in B_S \end{cases}$$

$$W(e^{j\Omega}) = \begin{cases} \cancel{1/K} \; \delta_2/\delta_1 & \Omega \in B_P \\ 1 & \Omega \in B_S \end{cases}$$

$$K = \frac{\delta_1}{\delta_2} \qquad \delta_1 < \delta_2$$

Number of Extrema

For an optimal linear phase FIR filter, the error function has at least $(M+1)$ extrema, where

$$M = \left(\frac{N-1}{2}\right)$$

$$\left|\tilde{H}(e^{j\Omega})\right| = \sum_{n=0}^{M} \alpha'(n)(\cos\Omega)^n$$

$$\frac{d}{d\Omega}\left|\tilde{H}(e^{j\Omega})\right| = -\sin\Omega \sum_{n=1}^{M} n\,\alpha'(n)(\cos\Omega)^{n-1}$$

can show that above f^n has n extremas.

$*$ Alternation Theorem

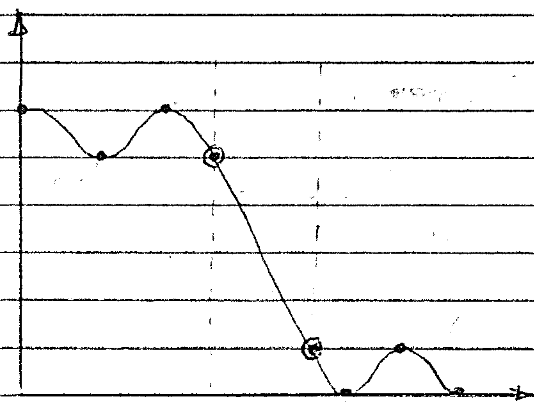
Let $F \in [0, \frac{1}{2}]$. For $\tilde{H}(e^{j\Omega})$ to be the unique best approximation (weighted chebyshev) on F to $H_D(e^{j\Omega})$ it is necessary and sufficient that the error function $E(e^{j\Omega})$ exhibit on F at least $(M+2)$ "alternations".

i.e. $E(e^{j\Omega_i}) = -E(e^{j\Omega_{i-1}}) = \pm \|E\|$

with $\Omega_0 \le \Omega_1 \le \Omega_2 \dots \le \Omega_{M+1}$, $\Omega_i \in F$

where $\|E\| = \max |E(e^{j\Omega})|$ $\Omega \in F$

example:



$M = \frac{N-1}{2}$

$N = 11 \Rightarrow M = 5$

This kind of filters are known as extra ripple filter
$(M+3)$ alternations

$$|\tilde{H}(e^{j\Omega})| = \left|\sum_{n=0}^{5} \alpha(n) \cos \Omega n\right|$$

Remez Exchange Algorithm for optimal FIR filter Design

Assume $(M+2)$ points ($(M+3)$ for extra ripple case) of alternations are approximately known. i.e.

$\{\Omega_i,\ i=0,1,\ldots,M+1\}$ and the error at those points has a magnitude of δ. Then at the ith step of the algorithm,

$$W(e^{j\Omega_i})\left[\tilde{H}(e^{j\Omega_i}) - H_D(e^{j\Omega_i})\right] = (-1)^i \delta$$

$$i=0,1,\ldots,M+1$$

Arranging these equations in matrix and using

$$\tilde{H}(e^{j\Omega}) = \sum_{n=0}^{M} \alpha(n)\cos\Omega n$$

$$\begin{bmatrix} 1 & \cos\Omega_0 & \cos 2\Omega_0 & \ldots & \cos M\Omega_0 & \dfrac{1}{W(e^{j\Omega_0})} \\ 1 & \cos\Omega_1 & \cos 2\Omega_1 & \ldots & \cos M\Omega_1 & \dfrac{-1}{W(e^{j\Omega_1})} \\ \vdots & & & & & \vdots \\ 1 & \cos\Omega_{M+1} & \cos 2\Omega_{M+1} & \ldots & \cos M\Omega_{M+1} & \dfrac{(-1)^{M+1}}{W(e^{j\Omega_{M+1}})} \end{bmatrix} \begin{bmatrix} \alpha(0) \\ \alpha(1) \\ \vdots \\ \alpha(M) \\ \delta \end{bmatrix} = \begin{bmatrix} H_D(e^{j\Omega_0}) \\ H_D(e^{j\Omega_1}) \\ \vdots \\ H_D(e^{j\Omega_{M+1}}) \end{bmatrix}$$

paper: Chebyshev approximation for non-recursive digital filters