Multi-rate Signal processing

Single-rate vs Multi-rate DSP

- Single-rate: Same Sampling rates at both input and outputs.
- Multi-rate: Different Sampling rates possibly at different points within the overall System

Advantages:

1- Better managements of resources e.g., memory, CPU, etc. hence lower cost.

2- Fractional Sampling rate Conversion. e.g., CD players (44.1KHZ), professional audio systems (48 KHZ), vs digital broadcasting (32 KHZ). worse in video processing.

3- Facilitates Integration of different System with different Bandwidths e.g., CD players with audio DSP (22KHZ) - performs processing in all digital domain.

Disadvantages:

1- Increases System Complexity

2- Aliasing problems - can be avoided.

Basic Elements of Multi-rate Systems

~~Unlike~~ In addition to basic elements of traditional single-rate Systems, i.e. multipliers, adders, and delays elements for multi-rate Systems we have:

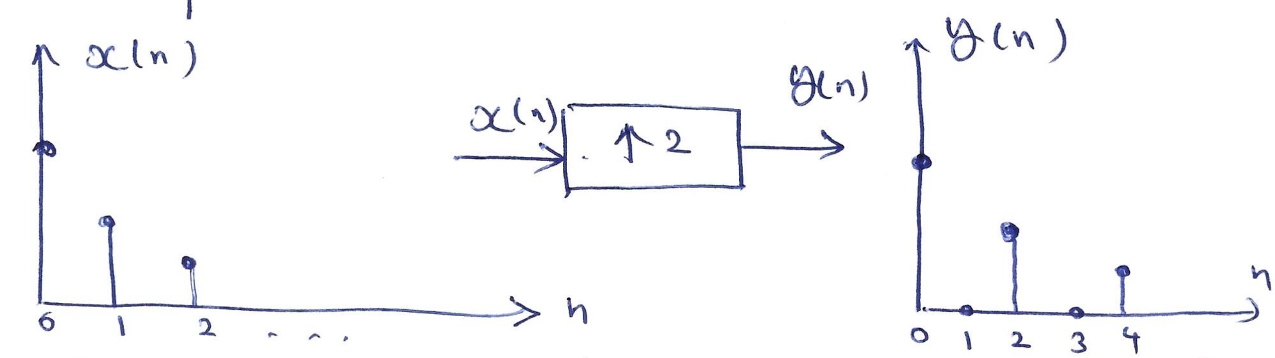
1- M-fold Interpolator / expander / upsampler

performs upsampling by adding M-1 ~~samples~~ zeros between samples i.e.

$$x(n) \longrightarrow \boxed{\uparrow M} \longrightarrow y(n)$$

$$y(n) = \begin{cases} 0 & n \neq lM \\ x\left(\frac{n}{M}\right), & n = 0, \pm M, \pm 2M \ldots \\ & \text{i.e. } n = lM \end{cases}$$

M = 2 example seen below



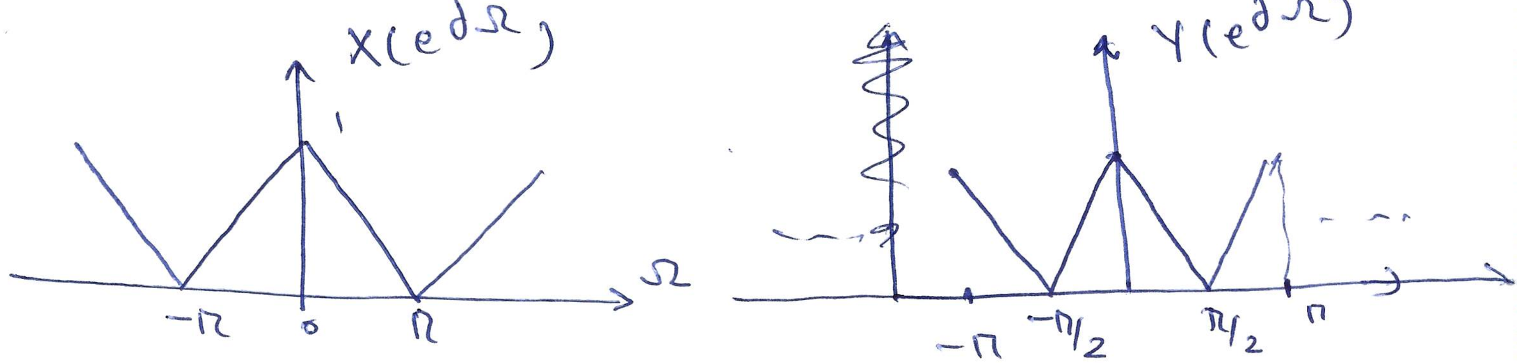
Note:
upsampling does not cause any loss of information.

Let $x(n) \xleftrightarrow{\mathcal{Z}} X(z)$

then $Y(z) = \mathcal{Z}\{y(n)\} = \sum_k x(k)\, z^{-kM} = X(z^M)$

In frequency domain

$$Y(e^{j\Omega}) = X(e^{j\Omega M})$$

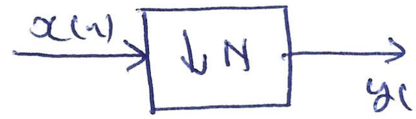


i.e. spectrum contracts by M.

Remark
~~Also~~ Besides the base spectrum, there are spectral "images" caused by interleaving of zeros in the upsampling process. A LPF with cutoff freq. π/M can be used to get rid of these "images".

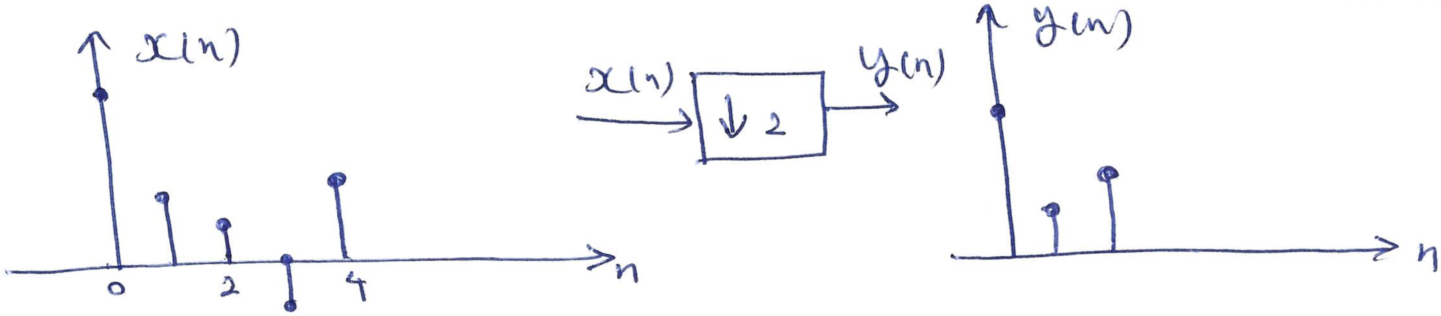
2 - N-fold DownSampler / Decimator

performs downsampling or decimation by keeping one out of N samples

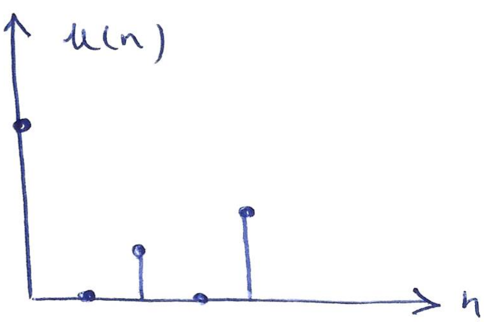


$y(n) = x(nN)$

$N=2$ example shown below



Let us define $u(n)$ as shown



Clearly $u(n)$ is the upsampled version of $y(n)$ i.e.

$u(n) = y(n/N)$ or $U(z) = Y(z^N)$ or

$$Y(z) = U(z^{1/N})$$

On the other hand $u(n)$ is a resampled version of $x(n)$ with sampling function

$$\delta_N(n) = \begin{cases} 1 & \text{when } n = lN \\ 0 & \text{when } n \neq lN \end{cases} \qquad l \in \mathbb{Z}$$

hence

$$u(n) = \delta_N(n)\, x(n)$$

But

$$\delta_N(n) = \frac{1}{N}\sum_{k=0}^{N-1} e^{j\frac{2\pi k n}{N}} = \begin{cases} 1 & \text{if } n = \ell N \\ 0 & \text{if } n \neq \ell N \end{cases}$$

Thus

$$u(n) = x(n)\,\frac{1}{N}\sum_{k=0}^{N-1} e^{-j\frac{2\pi k n}{N}}$$

and

$$U(z) = \sum u(n) z^{-n} = \frac{1}{N}\sum_{k}\sum_{n} x(n)\left(z e^{+j\frac{2\pi k}{N}}\right)^{-n}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X\left(e^{+\frac{2\pi k}{N}} z\right)$$

modulation on the unit circle.

Finally

$$\boxed{Y(z) = U(z^{1/N}) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(e^{+j\frac{2\pi k}{N}} z^{1/N}\right)}$$

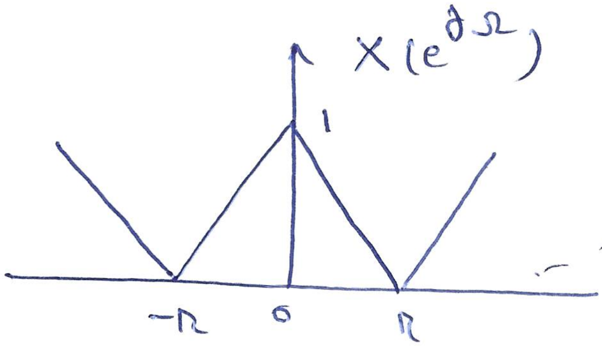
In frequency-domain

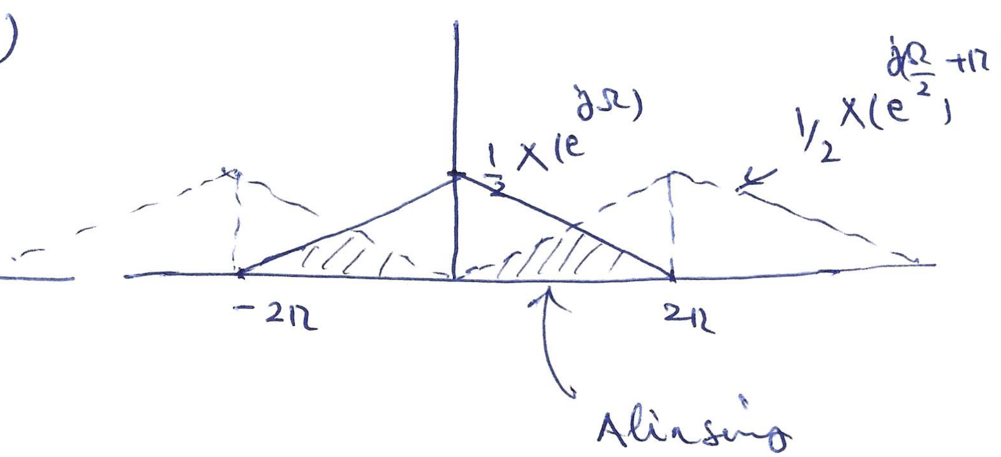
$$Y(e^{j\Omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(e^{j\frac{(\Omega+2\pi k)}{N}}\right)$$

i.e. periodic extension with period $\frac{2\pi}{N}$ (instead of 2π)

For example $N=2$

$$Y(e^{j\Omega}) = \frac{1}{2}\left[X(e^{j\frac{\Omega}{2}}) + X(e^{j(\frac{\Omega}{2}+\pi)})\right]$$



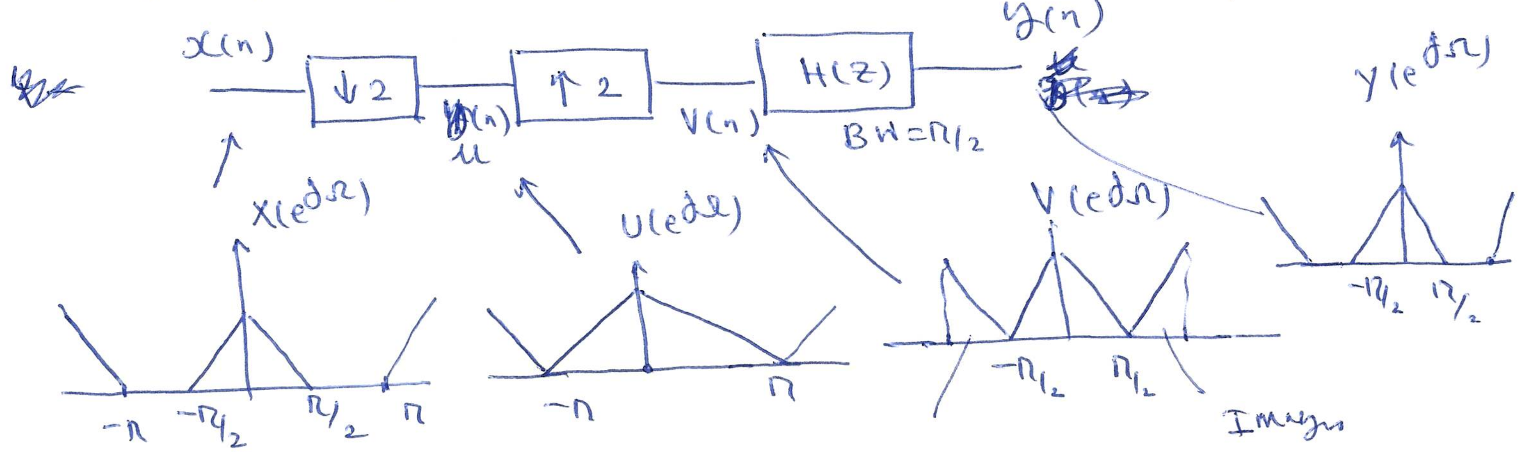


Remarks

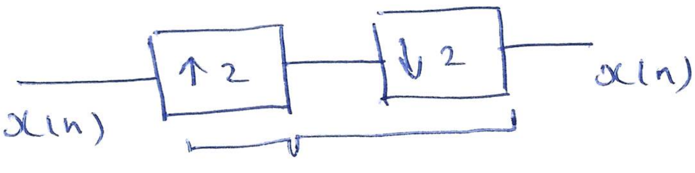
1- Decimation is a lossy operation - cannot be reversed.

2- Although frequencies $-\frac{\pi}{M} < \Omega < \pi/M$ are not affected by aliasing, frequencies outside this region are aliased. To avoid this problem, a prefilter with bandwidth $|\Omega| < \pi/M$ should be used.

3- When no aliasing, one can recover $x(n)$ from $y(n)$ using upsampling followed by filtering to remove the "images" spectral.



4- Note that



$= 1$

Fractional Sampling Rate Conversion and Commutativity

Question: How to change rate by a rational fraction

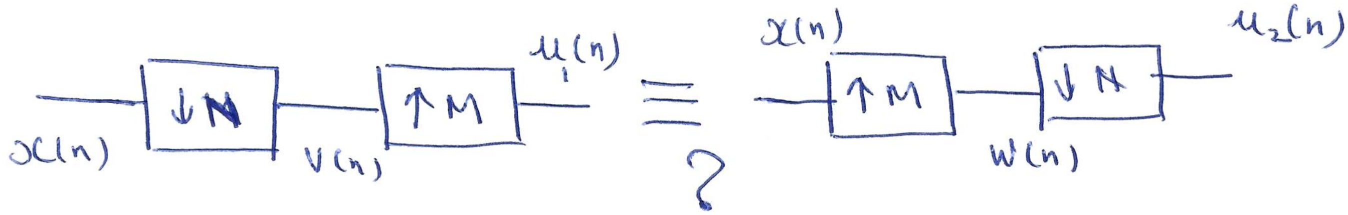
e.g., ~~Audio~~ 44.1 KHz $\longrightarrow$ Audio 48 KHz
CD rate

Approach 1 : Convert to analog and then resample

" 2 : Convert all in digital domain (usually done with filters)

First,

Consider the following scenario



For first one (Left)

$$V(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j\frac{2\pi k}{N}} z^{1/N})$$

and

$$U_1(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j\frac{2\pi k}{N}} z^{-M/N})$$

For second one (right)

$$W(z) = X(z^M) \quad \text{and} \quad U_2(z) = \frac{1}{N} \sum_{k=0}^{N-1} W(e^{j\frac{2\pi k}{N}} z^{1/N})$$

or
$$U_2(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j\frac{2\pi kM}{N}} z^{M/N})$$

These operations are equal (i.e. Commute) when

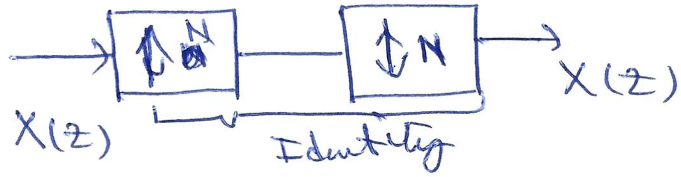
$$e^{j\frac{2\pi kM}{N}} = e^{j\frac{2\pi k}{N}} \Rightarrow kM = nN + k$$

or $kM \bmod N = k$

$\Rightarrow$ M and N are coprime i.e. $\gcd(M,N) = 1$

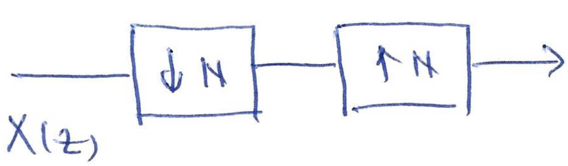
$\Rightarrow$ $mM + nN = 1$ (Bezout Identity)

Special case! If $M = N$



i.e. LSI

Whereas



$$Y(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j\frac{2\pi k}{N}} z^{1})$$

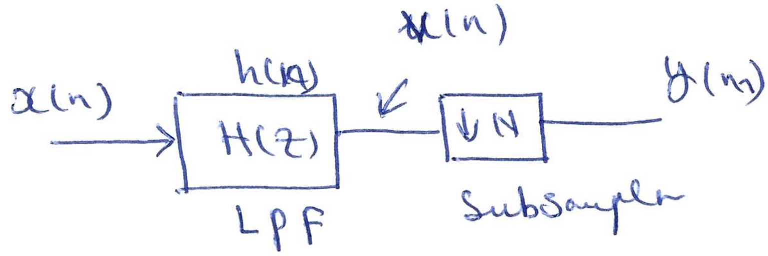
Not LSI

Decimation and Interpolation Filters

A decimator is ~~a~~ typically preceded by a LPF to avoid aliasing while an interpolator is followed by a LPF to ~~avoid~~ suppress the spectral images.

(a) Decimation Filter

Consider



$$v(n) = \sum_{k} h(k)\, x(n-k) = \sum_{k} h(n-k)\, x(k)$$

$$y(m) = v(Nm) = \sum_{k} h(k)\, x(Nm-k) = \sum_{k} h(Nm-k)\, x(k)$$

In z-domain

$$V(z) = H(z)\, X(z)$$

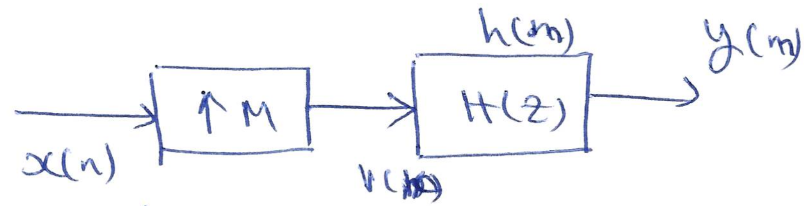
$$Y(z) = \frac{1}{N}\sum_{l=0}^{N-1} V\left(e^{j\frac{2\pi l}{N}} z^{1/N}\right) = \frac{1}{N}\sum_{l=0}^{N-1} HX\left(e^{j\frac{2\pi l}{N}} z^{1/N}\right)$$

or

$$Y(e^{j\Omega}) = \frac{1}{N}\sum_{l=0}^{N-1} HX\left(e^{j\left(\frac{\Omega + 2\pi l}{N}\right)}\right)$$

(b) Interpolation Filter

Consider



$$v(m) = \begin{cases} x(n/L) & n = 0,\ \pm L,\ \pm 2L \ldots \\ 0 & \text{otherwise} \end{cases} \qquad n/L = m$$

$$y(m) = \sum_k v(k)\, h(m-k) = \sum_{k'} x(k')\, h(m-k'L)$$

In ~~the~~ z-domain

$$Y(z) = H(z)\,V(z) \qquad \text{But} \quad V(z) = X(z^M)$$

$$\Rightarrow \quad Y(z) = H(z)\, X(z^M)$$

or

$$Y(e^{j\Omega}) = H(e^{j\Omega})\, X(e^{j\Omega M})$$

Interpolated sequence is more narrowband than the original ($X(e^{j\Omega})$ is periodic w period 2π vs. $X(e^{j\Omega M})$ which is periodic w period $\frac{2\pi}{M}$).

In summary, we have the following input/output relation (time-domain) for

(a) N-fold Decimation filter $\quad y(n) = \sum_k x(k)\, h(nN-k)$

(b) M-fold Interpolation filter $\quad y(n) = \sum_k x(k)\, h(n-kM)$

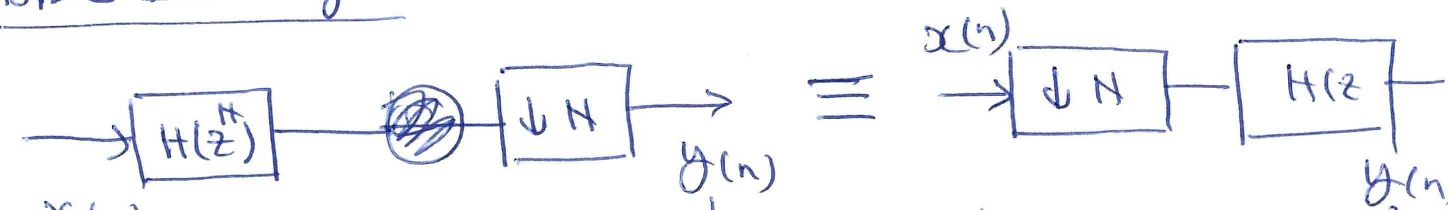
c N/M-fold decimation filter $\quad y(n) = \sum_k x(k)\, h(nN - kM)$

Noble Identity and Interchangeability of Filters and Samplers

The two systems covered:

In previous section, have many inefficiency e.g., in decimation filter, we filter all the data and then throw away M-1 samples. Similarly in the interpolation filter we add M-1 zeros while only one sample is useful in every M samples. Thus, we need more efficient implementation in multi-rate systems. The key is the Noble identities and polyphase implementation.

Noble Identity 1:



$y_1(n) = y_2(n)$

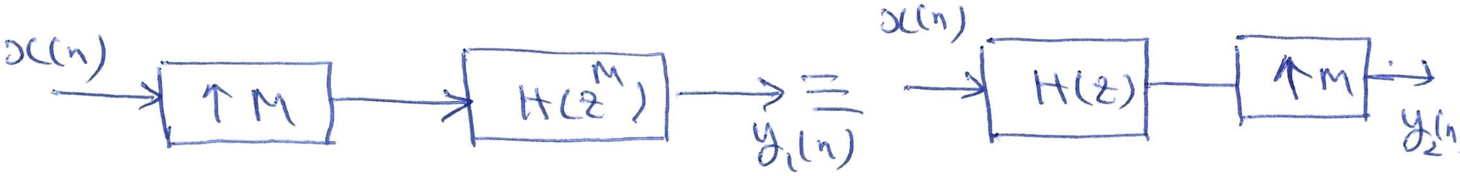
To show

$$Y_1(z) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(e^{\frac{-j2\pi k}{N}} z^{1/N}\right) \underbrace{H\left(\left(e^{\frac{-j2\pi k}{N}} z^{1/N}\right)^N\right)}_{H(z)}$$

$$Y_2(z) = H(z)\,\frac{1}{N}\sum_{k=0}^{N-1} X\left(e^{\frac{-j2\pi k}{N}} z^{1/N}\right)$$

i.e. $Y_1(z) = Y_2(z)$

Noble Identity 2:



$$Y_1(z) = X(z^M)\, H(z^M)$$

$$Y_2(z) = X(z)\,H(z)\Big|_{z \to z^M} = X(z^M)\, H(z^M)$$

Note: In both cases $H(z)$ must be a function of z^N (or z^M) in order to use these identities.

<u>Example</u>

(a) simplify the following system and write $Y(z)$ in terms of $X(z)$.

$$x(n) \longrightarrow \boxed{z^{-4}} - \boxed{\downarrow 2} - \boxed{\uparrow 3} - \boxed{\downarrow 2} \longrightarrow y(n)$$

(b) Find $y(n)$ for $x(n) = \delta(n)$. Is it LSI ?

Since $\gcd(2,3) = 1 \Rightarrow -\boxed{\downarrow 2} - \boxed{\uparrow 3} - \boxed{\downarrow 2} - \equiv -\boxed{\downarrow 4} - \boxed{\uparrow 3} -$

Thus

$$\equiv \quad x(n) - \boxed{z^{-4}} - \boxed{\downarrow 4} - \boxed{\uparrow 3} - y(n)$$

Using identity 1

$$\equiv \quad x(n) - \boxed{\downarrow 4} \underset{v(n)}{-} \boxed{z^{-1}} \underset{u(n)}{-} \boxed{\uparrow 3} \longrightarrow y(n)$$

$$W_4 \triangleq e^{-j\frac{2\pi}{4}}$$

$$V(z) = \frac{1}{4} \sum_{k=0}^{3} X(W_4^k z^{1/4})$$

$$U(z) = z^{-1} V(z)$$

$$Y(z) = U(z^3) = \frac{z^{-3}}{4} \sum_{k=0}^{3} X(W_4^k z^{3/4})$$

(b) For $x(n) = \delta(n) \Rightarrow X(z) = 1 \Rightarrow Y(z) = z^{-3}$

$x(n) = \delta(n-1) \Rightarrow X(z) = z^{-1}$, $V(z) = \frac{1}{4}(z^{-1/4} + j z^{-1/4} - z^{-1/4} - j z^{-1/4}) = 0$

$v(n) = 0 \Rightarrow y(n) = 0$

i.e. Not LSI (LSV)