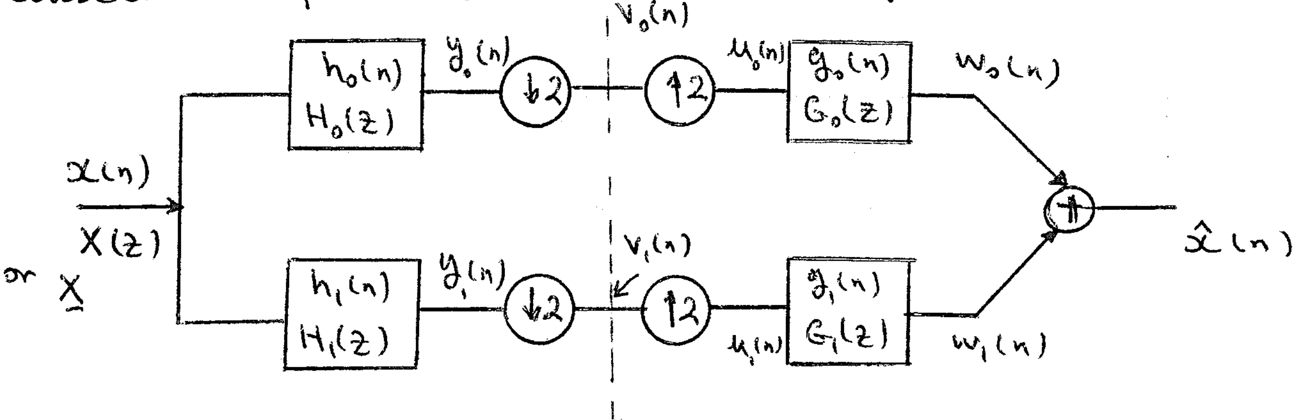
Z ~~(or Modulation)~~ Domain Analysis of Filter Banks

Consider the following two-channel filter bank



we can write

$$Y_0(z) = H_0(z)\,X(z) \quad , \quad Y_1(z) = H_1(z)\,X(z)$$

$$V_0(z) = \frac{1}{2}\left[Y_0(z^{1/2}) + Y_0(-z^{1/2})\right] \quad , \quad V_1(z) = \frac{1}{2}\left[Y_1(z^{1/2}) + Y_1(-z^{1/2})\right]$$

$$U_0(z) = V_0(z^2) = \frac{1}{2}\left[Y_0(z) + Y_0(-z)\right]$$

$$U_1(z) = V_1(z^2) = \frac{1}{2}\left[Y_1(z) + Y_1(-z)\right]$$

$$W_0(z) = U_0(z)\,G_0(z) \qquad \text{Channel 1}$$

$$W_1(z) = U_1(z)\,G_1(z) \qquad \text{Channel 2}$$

Combining all of these Eqs yields

$$W_0(z) = \frac{1}{2}\,G_0(z)\left[H_0(z)X(z) + H_0(-z)\,X(-z)\right]$$

$$W_1(z) = \frac{1}{2}\,G_1(z)\left[H_1(z)\,X(z) + H_1(-z)\,X(-z)\right]$$

And $\hat{X}(z) = W_0(z) + W_1(z)$

$$= \frac{1}{2}\left[G_0(z) \;\; G_1(+z)\right] \underbrace{\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}}_{H_m(z)} \underbrace{\begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}}_{X_m(z)}$$

$H_m(z)$: Analysis Modulation Matrix

or

$$\hat{X}(z) = \frac{1}{2}\overbrace{\left[G_0(z)H_0(z) + G_1(z)H_1(z)\right]}^{\text{Distortion}} X(z) + \frac{1}{2}\underbrace{\left[G_0(z)H_0(-z) + G_1(z)H_1(-z)\right]}_{\text{Aliasing}} X(-z)$$

Aliased term

For perfect reconstruction (PR) aliasing term must be zero and there should be no distortion (only maybe delay by l due to the filters) i.e.

$$G_0(z)H_0(z) + G_1(z)H_1(z) = 2z^{-l} \qquad \text{PR}$$

$$G_0(z)H_0(-z) + G_1(z)H_1(-z) = 0 \qquad \text{Aliasing Cancellation}$$

or alternatively in matrix form

$$\begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix} \underbrace{\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}}_{H_m(z)} = \begin{bmatrix} 2z^{-l} & 0 \end{bmatrix}$$

Solving for $G_0(z)$ and $G_1(z)$ gives (transpose and multiply by $H_m^{-1}(z)$).

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \frac{\begin{bmatrix} H_1(-z) & -H_1(z) \\ -H_0(-z) & H_0(z) \end{bmatrix}}{\Delta(z)} \begin{bmatrix} 2z^{-l} \\ 0 \end{bmatrix}$$

where $\Delta(z) \triangleq H_0(z)H_1(-z) - H_0(-z)H_1(z)$

or simply

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \frac{2z^{-l}}{\Delta(z)} \begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix}$$

$P_0(z) = G_0(z)H_0(z) = \frac{2z^{-l}}{\Delta(z)} H_0(z)H_1(-$

Also $\Delta(z) = -\Delta(-z)$

$P_1(z) = G_1(z)H_1(z) = \frac{-2z}{E(z)}$, $H_0(-z)H_1(z) = -P_0(-z)$
for ℓ: odd $\Longrightarrow$ Thus PR becomes $P_0(z) - P_0(-z) = 2z^{-\ell}$

Note for FIR Solution $\Delta(z) = 2z^{-\ell}$.

If we choose $G_0(z) = H_1(-z)$ $\quad z \to -z$ $\quad LP \to HP$, $HP \to LP$
alternating sign

$g_0(n) = (-1)^n h_0(n)$ $\quad G_1(z) = -H_0(-z)$

then the alias cancellation Condition (General biorthogonal Condition) is automatically satisfied since

verify

$$G_0(z)H_0(-z) + G_1(z)H_1(-z) = H_1(-z)H_0(-z) - H_0(-z)H_1(-z) = 0$$

In addition, we also satisfy $\Delta(z) = 2z^{-\ell}$.

since $\Delta(z) = H_0(z)\underbrace{H_1(-z)}_{G_0(z)} \underbrace{-H_0(-z)}_{G_1(z)} H_1(z) = 2z^{-\ell}$

i.e. PR Condition.

Now to design a two-channel filter bank with PR condition define

correlation fcn between h_0 and g_0

$$P_0(z) \triangleq G_0(z)H_0(z) \quad \text{LP part}$$

$$P_1(z) \triangleq G_1(z)H_1(z) \quad \text{HP part}$$

obviously $P_0(-z) = \underbrace{G_0(-z)}_{H_1(z)}\underbrace{H_0(-z)}_{-G_1(z)}$

$$= -P_1(z)$$

Thus PR condition becomes

Let $p(z) = z^{\ell}P_0(z)$, ℓ: odd

$$\boxed{P_0(z) - P_0(-z) = 2z^{-\ell}} \Rightarrow p(z) + p(-z) = 2$$

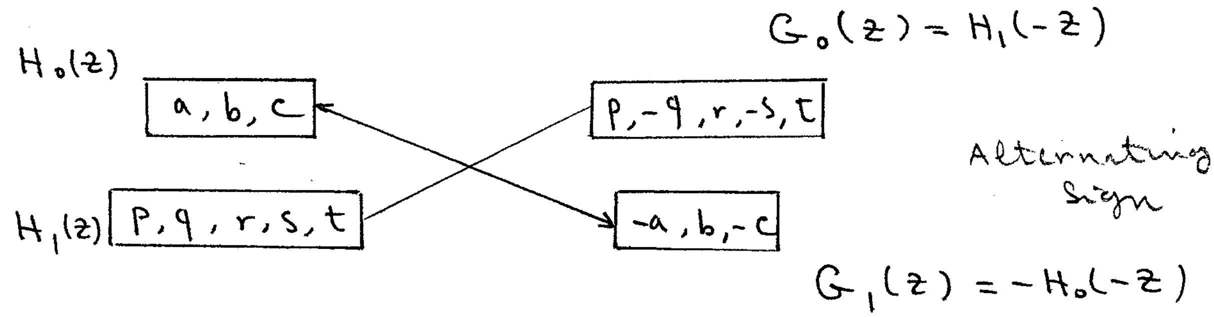
your text Eq.

i.e. all even powers of $p(z)$ are zero except the constant term which is one.

Design Steps

1 - Design $P_0(z)$ (LP) to satisfy PR condition

2 - Factor $P_0(z) = G_0(z)H_0(z)$, then use
$G_0(z) = H_1(-z)$, $\quad G_1(z) = -H_0(-z)$



Remarks

1- Let $z \longrightarrow -z$ in PR condition

$$P_0(-z) - P_0(z) = 2(-z)^{-\ell}$$

i.e. the left side is an odd function of z and thus ℓ has to be odd. To satisfy the PR condition all the odd powers of z in $P_0(z)$ should have zero coefficients except $z^{-\ell}$ which should have coeff. one. (Halfband condition – $P_0(z)$ has odd # of coeff with center $=1$).

e.g.

$$P_0(z) = \frac{1}{16}(-1 + 9z^{-2} + 16z^{-3} + 9z^{-4} - z^{-6})$$

with center term $z^{-3} = +z^{-\ell}$ satisfies $P_0(z) - P_0(-z) = 2z^{-\ell}$ condition. or $P(z) = \frac{1}{16}(-z^3 + 9z + 16 + 9z^{-1} - z^{-3})$
$P(z) + P(-z) = 2$

2- The choice $P_0(z) = (1+z^{-1})^{2p}/_{2^{2p}}\, Q(z)$ is especially interesting ($Q(z)$ is a polynomial of order $2p-2$), since it provides maximally flat response at $\Omega = \pi$ due to max. # of zeros ($2p$) at $z=-1$. These filters (Daubechies) are called "binomial" or "maxflat" filters.

All derivatives $P_0^{(k)}(z)$ vanishes at $z=-1$, $0 \le k \le 2p-1$

~~As will be shown later,~~ factorizing $P_0(z)$ into $G_0(z)H_0(z)$ can either generate linear phase filter or orthogonal filters but not both.

In the above example $Q(z) = -z^{-2} + 4z^{-1} - 1$

$P_0(z) = (1+z^{-1})^4\, Q(z)$ where $Q(z)$ has option $H_0(z) = \frac{1}{8}(-1+2z^{-1}+6z^{-2}+2z^{-3}-z^{-4})$ and $G_0(z) = \frac{1}{2}(1+2z^{-1}+z^{-2})$

roots at $c = 2-\sqrt{3}$ and $1/c = 2+\sqrt{3}$.

3- The PR condition gives us a way to select G_0, and G_1. In addition, $H_0(z)$ is obtained from the factorization or other information. $H_1(z)$ is still to be chosen using LP $\rightarrow$ HP mapping. Two choices are

$$\begin{cases} H_1(z) = H_0(-z) & \text{i.e. alternating sign} \\ H_1(z) = (-z)^{-(N-1)} H_0(-z^{-1}) & \text{alternating flip.} \end{cases}$$

or can be chosen using general biorthogonal condition $H_1(-z) = G_0(z)$.

The 1st choice gives the Quadrature Mirror Filter (QMF) of Croisier - Estaban - Galand (1976) as

$H_1(e^{j\Omega}) = H_0(e^{j(\Omega+\pi)})$ i.e. mirror image on the unit circle. For this choice

$$H_1(z) = H_0(-z) \qquad \text{Alternating sign}$$

and

$$G_0(z) = H_0(z)$$

$$G_1(z) = -H_1(z) = -H_0(-z)$$

Thus, the PR condition $\Rightarrow G_0(z)H_0(-z) + G_1(z)H_1(-z) = 0$ $\Rightarrow H_0(z)H_0(-z) - H_0(-z)H_0(z) = 0$ i.e. no aliasing

$$G_0(z)H_0(z) + G_1(z)H_1(z) = 2z^{-l}$$

becomes

Quadrature: both the filter and mirror image are squared $\leftarrow$

$$H_0^2(z) - H_1^2(z) = H_0^2(z) - H_0^2(-z) = 2z^{-l}$$

i.e. $H_0^2(z)$ should have exactly one odd power z^{-l}. This cannot be satisfied for FIR filters except the Haar filters for which $H_0(z) = (1+z^{-1})/\sqrt{2}$

$$\frac{1}{2}(1+2z^{-1}+z^{-2}) - \frac{1}{2}(1-2z^{-1}+z^{-2}) = 2z^{-1}$$

For the 2nd choice (Smith–Barnwell 1984–86 and Mintzer 1985)

$$H_1(z) = (-z)^{-(N-1)} H_0(-z^{-1}) \qquad N: \text{even}$$

then

$$P_0(z) = G_0(z) H_0(z) = \underbrace{H_1(-z)}_{\text{or } -G_1(-z)} H_0(z)$$
$$= z^{-(N-1)} H_0(z^{-1}) H_0(z)$$

Define $P(z) \triangleq z^{\ell} P_0(z)$ with $\ell = N-1$ to center the filters, then the PR condition becomes

$$P_0(z) - P_0(-z) = 2 z^{-\ell}$$

$P_1(z) = -P_0(-z)$
$= G_1(z) H_1(z)$

or $\quad P(z) + P(-z) = 2$

$P(-z) = H_1(z^{-1}) H_1(z)$

Which in this case is

$$H_0(z^{-1}) H_0(z) + H_0(-z^{-1}) H_0(-z) = 2$$

In frequency domain

$$|H_0(e^{j\Omega})|^2 + |H_0(e^{j(\Omega+\pi)})|^2 = 2$$

Smith – Barnwell Condition

and similarly for H_1. This means that the filter and its modulated version are "power complementary".

The condition $P(z) = H_0(z) H_0(z^{-1})$ corresponds to the "spectral factorization" of a halfband filter.

Thus, the alternating flip which guarantees the orthogonality between H_0 and H_1 together with the symmetric factorization of $P(z)$ generate orthonormal filter banks with PR.

The flattest $P(z)$ leads to the Daubechies wavelets.

4- Recall that for linear phase FIR we had

$$H(z^{-1}) = \pm z^{(N-1)} H(z)$$

i.e. zeros of $H(z^{-1})$ are also zeros of $H(z)$. This obviously contradicts with the condition for orthonormality i.e. the spectral factorization.
i.e. $H(z)$ min phase, $H(z^{-1})$: max phase.

Thus, there is no orthonormal linear phase solution with real FIR filters except in some trivial cases such as Haar filters. This will be proved later more rigorously.

5- In a biorthogonal linear phase filter bank (2-channel), the analysis filters can be

(a) both symmetric, of odd length (equal or differ by multiples of 2)

(b) one symmetric and the other antisymmetric, of even length (equal or differ by even multiples of 2)

To see this consider product $P_0(z) = G_0(z) H_0(z)$
$= H_0(z) H_1(-z)$ which has to have odd # of coefficients with center one being one.

6- A similar "synthesis modulation matrix $G_m(z)$ can be defined by $z \to -z$ in

$$[G_0(z) \quad G_1(z)] \underbrace{H_m(z)}_{\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}} = [2z^{-\ell} \quad 0]$$

$G_0(z)H_0(z) + G_1(z)H_1(z) = 2z^{-\ell}$
$G_0(z)H_0(-z) + G_1(z)H_1(-z) = 0$

i.e. $[G_0(-z) \quad G_1(-z)]\, H_m(z) = [0 \quad 2(-z)^{-\ell}]$

or $\quad G_m(z)\, H_m(z) = \begin{bmatrix} 2z^{-\ell} & 0 \\ 0 & 2(-z)^{-\ell} \end{bmatrix}$ where $G_m(z) \triangleq \begin{bmatrix} G_0(z) & G_1(z) \\ G_0(-z) & G_1(-z) \end{bmatrix}$

If we center the filter coeff around zero (i.e. symmetric or anti-symmetric), $H_i(z) = H_i(z^{-1})$ then

$$G_m(z)\, H_m(z) = 2I \quad \leftarrow$$

$p(z)$: z-transform of ACF of $h_0(n)$

Example 1

Consider LPF $\quad H_0(z) = \frac{1}{4}(1+z^{-1})^2 = \frac{1}{4}(1+2z^{-1}+z^{-2})$

This filter is symmetric and has two roots at $z=-1$ or $\Omega = \pi$ and is a short filter. However, since it is not even length it is not orthonormal
wrt even shift

$$\tilde{H}_0 = \frac{1}{4}\begin{bmatrix} \vdots & \vdots & \ddots & & \\ \cdots 1 & 2 & 1 & & \\ 0 & 0 & 1 & 2 & 1 \\ & \ddots & & \ddots & \ddots \end{bmatrix}$$

$\tilde{H}_0 \tilde{H}_0^t \neq I$

$\sum_{n=0}^{2} h_0^2(n) \neq 1$

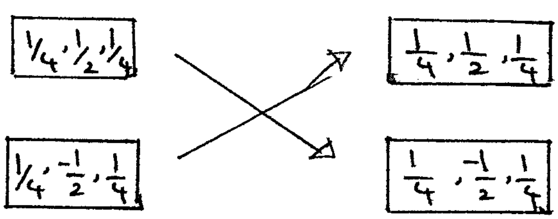
also $\sum_{n=0}^{2} h_0(n+m)\, h_0(n-2\ell) \neq 0 \quad \forall m, \ell$

Let us choose $\quad H_1(z) = H_0(-z)$ i.e. alternating sign to generate the HPF as

$$H_1(z) = \frac{1}{4}(1-2z^{-1}+z^{-2}) \qquad \text{i.e. QMF of Croisier – Estaban – Galand}$$

Then $\quad G_0(z) = H_1(-z) = H_0(z)$

and $\quad G_1(z) = -H_0(-z) = -H_1(z)$



In this case $\quad P_0(z) = G_0(z)H_0(z) = H_0^2(z)$

$$= \frac{1}{16}(1+z^{-1})^4 = \frac{1}{16}(1+4z^{-1}+6z^{-2}+4z^{-3}+z^{-4})$$

which obviously violates the condition in Remark 1. Consequently

$$P_0(z) - P_0(-z) = 2z^{-\ell}$$

$$\Rightarrow \quad H_0^2(z) - H_0^2(-z) = 2z^{-\ell}$$

This filter bank does not work!

does not satisfy as pointed out in Remark 3.

$$P_0(z) - P_0(-z) = 2z^{-3}$$

Example 2

Now Consider $\quad P_0(z) = \frac{1}{16}(-1 + 9z^{-2} + 16z^{-3} + 9z^{-4} - z^{-6})$

which satisfies the PR condition. Then factorize

into $\quad P_0(z) = G_0(z)\, H_0(z) \quad$ with

$$H_0(z) = \frac{1}{8}(-1 + 2z^{-1} + 6z^{-2} + 2z^{-3} - z^{-4})$$

$$G_0(z) = \frac{1}{4}(1 + 2z^{-1} + z^{-2})$$

or $\quad H_0(z) = \frac{1}{8}(1+z^{-1})^2(-1 + 4z^{-1} - z^{-2})$ Roots at $z=-1\,(2)$ $z_{1,2} = 2 \pm \sqrt{3}$

$\quad G_0(z) = \frac{1}{4}(1+z^{-1})^2$ Roots at $z=-1\,(2)$

$\boxed{\frac{1}{8}(-1,2,6,2,-1)}$ $\qquad\qquad$ $\boxed{\frac{1}{4}(1,2,1)}$

5/3 Filter Bank

$\boxed{\frac{1}{4}(1,-2,1)}$ $\qquad\qquad$ $\boxed{\frac{1}{8}(1,2,-6,2,1)}$

$$H_1(z) = G_0(-z) = \frac{1}{4}(1 - 2z^{-1} + z^{-2})$$

$$G_1(z) = -H_0(-z) = \frac{1}{8}(1 + 2z^{-1} - 6z^{-2} + 2z^{-3} + z^{-4})$$

This is a good choice since filters are

(a) Short , (b) Satisfy PR , (c) Have two zeros at

$z=-1$, $\Omega=\pi$ i.e. smooth , (d) linear Phase (symmetr

(e) integer coefficients, (f) good choice for

Compression. But they are biorthogonal.