EE 512

Digital Signal Processing

Session 4

September 10, 1992

M. Azimi 800-525-4950, ext. 7956

Colorado State University

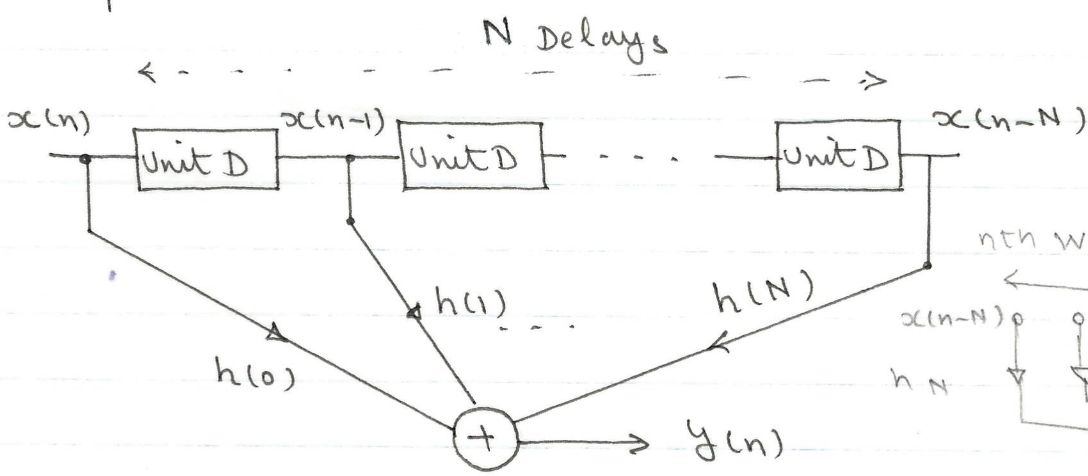
Finite Impulse Response (FIR) and Infinite Impulse Response (IIR) Systems

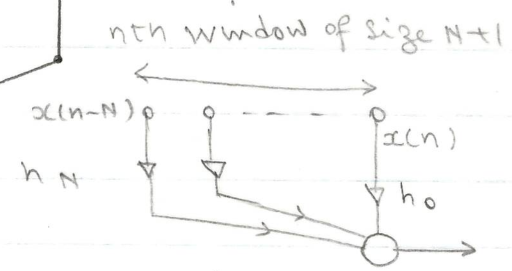
If $h(n)$ is of finite duration the system is referred to as FIR system otherwise for the general case when $h(n)$ is of infinite extent the system is called IIR.

For an FIR system with $\quad h(n) = 0 \quad n > N, \; n < 0$

we have
$$y(n) = \sum_{k=0}^{N} h(k)\, x(n-k)$$

The block diagram of such system containing unit delay, multipliers and adders is shown below





moving average Process

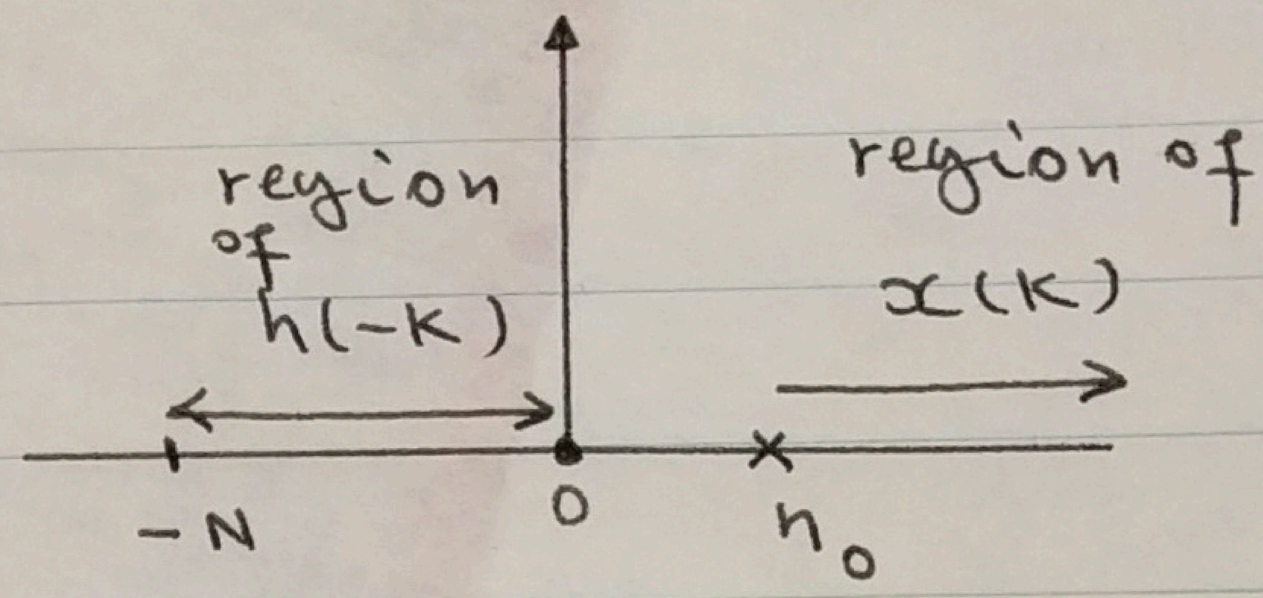
Example (Problem 3)

Evaluate the output $y(n) = h(n) * x(n)$ where

$$h(n) = \begin{cases} \alpha^n & 0 < n \leq N \\ 0 & \text{elsewhere} \end{cases} \qquad \text{FIR System}$$

$$x(n) = \begin{cases} \beta^{n-n_0} & n_0 \leq n \\ 0 & n < n_0 \end{cases}$$

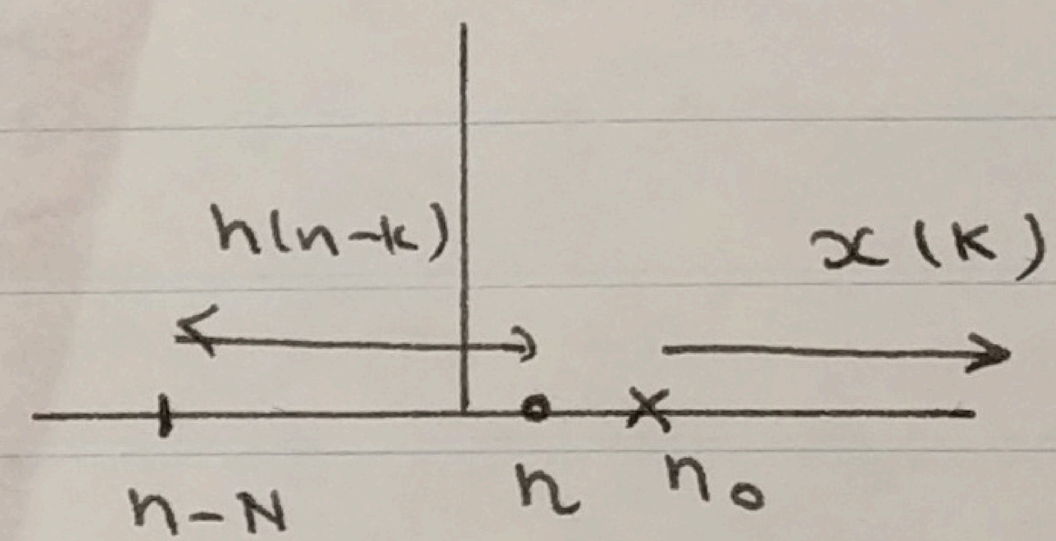
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$



There are three regions to consider.

Region 1

For $n < n_o$, $y(n) = 0$ No overlap



Region 2

For $n_o \leq n \leq n_o + N + 1$

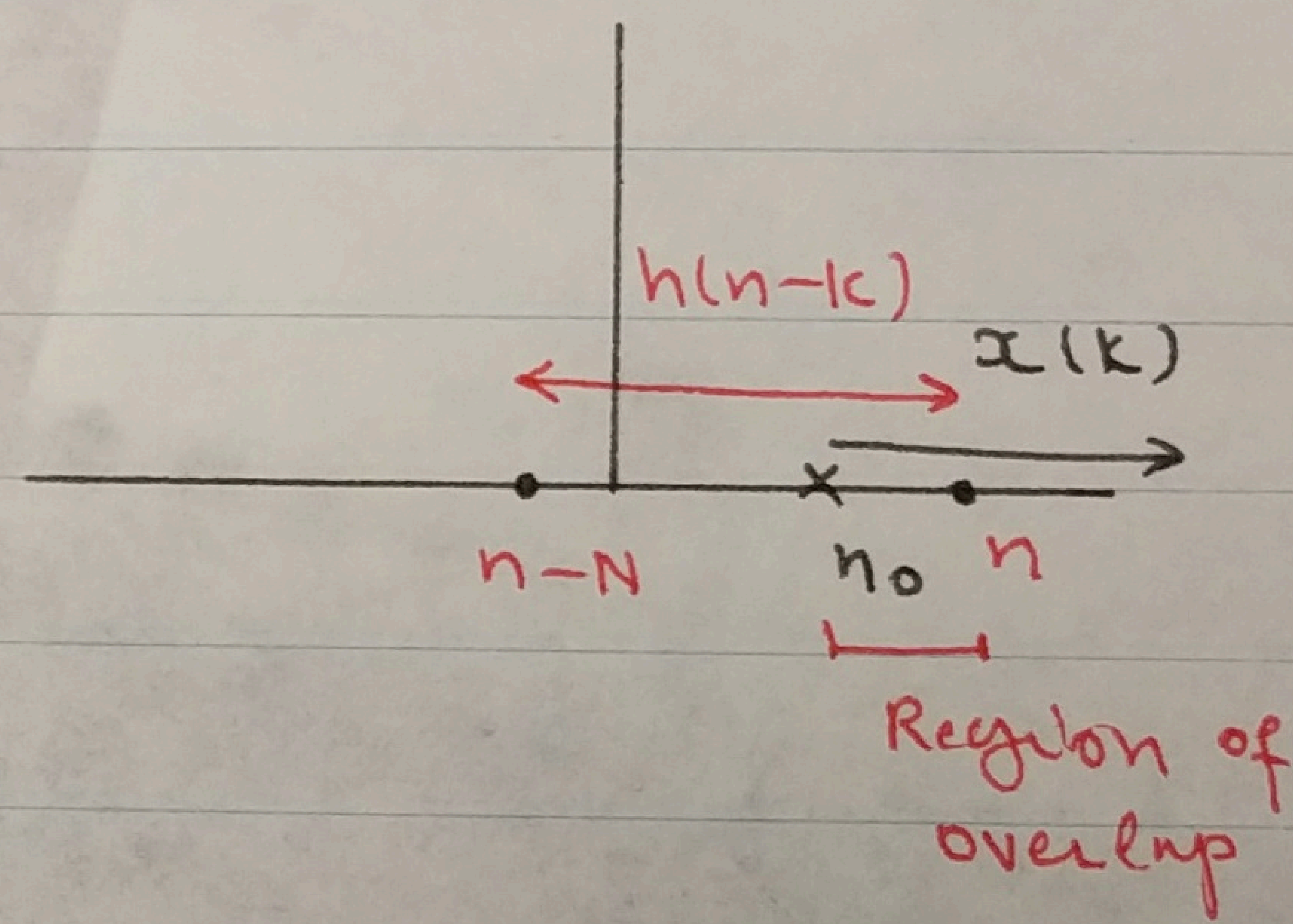
$$y(n) = \sum_{k=n_o}^{n} \alpha^{n-k} \beta^{k-n_o}$$

$$= \alpha^n \beta^{-n_o} \sum_{k=n_o}^{n} (\beta \alpha^{-1})^k$$

$$= \alpha^n \beta^{-n_o} \left[\frac{(\beta\alpha^{-1})^{n_o} - (\beta\alpha^{-1})^{n+1}}{1-\beta\alpha^{-1}} \right]$$

$$\sum_{r=0}^{n} a k^r = a \frac{(1-k^{n+1})}{1-k}$$

$$= \frac{\alpha^{n-n_o+1} - \beta^{n-n_o+1}}{\alpha - \beta} \quad , \alpha \neq \beta$$

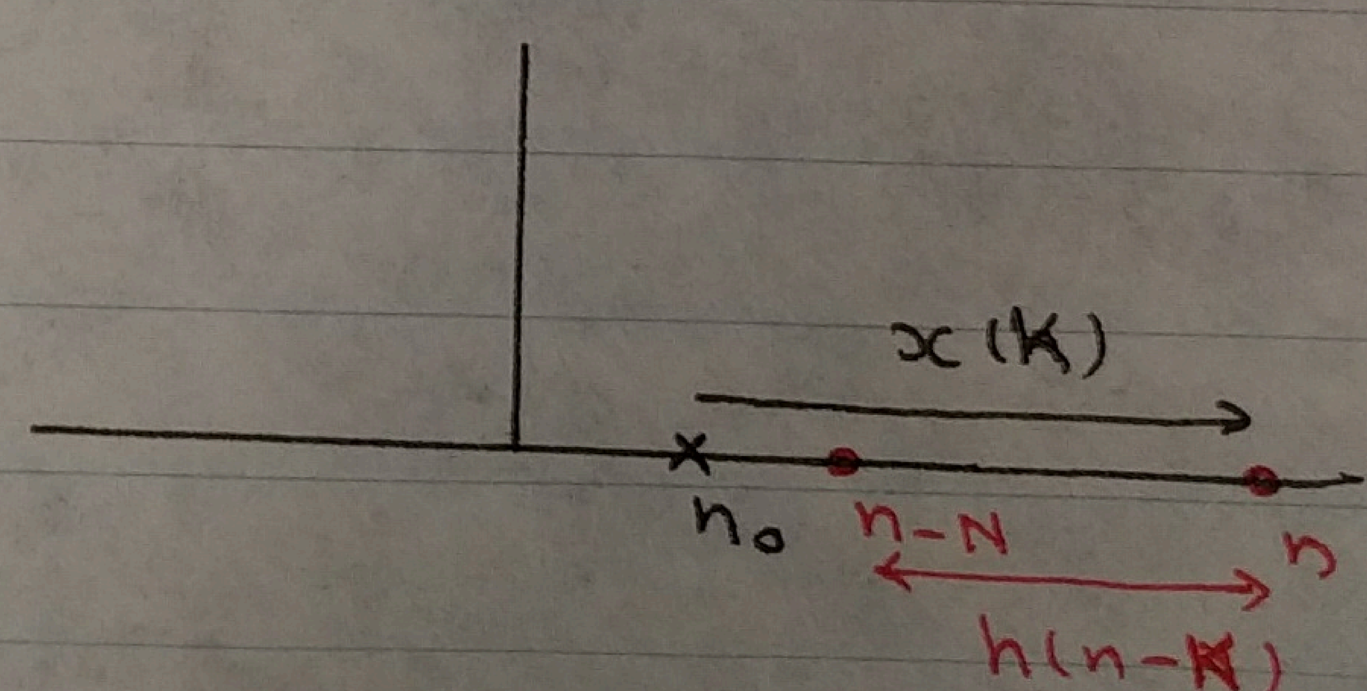


If $\alpha = \beta \Rightarrow y(n) = \alpha^{n-n_o} \sum_{k=n_o}^{n} 1 = (n-n_o+1)\, \alpha^{n-n_o}$

Region 3

For $n \geq n_o + N$

$$y(n) = \sum_{k=n-N}^{n} \alpha^{n-k} \beta^{k-n_o} = \alpha^n \beta^{-n_o} \sum_{k=n-N}^{n} (\beta\alpha^{-1})^k$$



$$= \alpha^n \beta^{-n_0} \left[\frac{(\beta\alpha^{-1})^{n-N} - (\beta\alpha^{-1})^{n+1}}{1-\beta\alpha^{-1}} \right]$$

$$= \beta^{n-n_0-N} \left(\frac{\alpha^{N+1} - \beta^{N+1}}{\alpha-\beta} \right) , \qquad \alpha \neq \beta$$

If $\alpha = \beta$

$$y(n) = \sum_{k=n-N}^{n} \alpha^{n-n_0} = (N+1)\, \alpha^{n-n_0}$$

Linear Constant Coefficient Difference Equation

An LTI or LSI discrete-time system can be described alternatively by a constant coefficient difference equation of form

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{\ell=0}^{M} b_\ell\, x(n-\ell) \qquad N: \text{ order of Diff. Eq.}$$

or

$$y(n) = \underbrace{\sum_{\ell=0}^{M} b_\ell\, x(n-\ell)}_{\text{past and present Input samples}} - \underbrace{\sum_{k=1}^{N} a_k\, y(n-k)}_{\text{past output samples}} , \quad a_0 = 1$$

Recursive Equations

Recursive application of this equation can generate the required output samples. In general, the difference equation (similar to differential equation) does not uniquely specify the I/O of an LSI System. There is a family of solutions gives $h(-1) = 1$, $h(0) = 0$, $h(1) = -2$, $h(2) = 0$

$h(3) = 2$, $h(4) = 0$, $h(5) = 2$, - - - .

or $h(2n+1) = (-1)^{n+1} 2 \quad n \geq 0$

$h(2n) = 0$

Since $h(n) \neq 0$ for $\forall n < 0 \Rightarrow$ System is noncausal

Alternatively we can form the relevant difference equation

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^3 - z}{z^2 + 1}$$

i.e. $y(n+2) + y(n) = x(n+3) - x(n+1)$

$y(n+2)$ depends on future input $x(n+3)$ i.e. noncausal.

Stability of LSI Systems

A system is said to be BIBO (bounded-input, bounded output) stable if a bounded input sequences implies the output sequence is also bounded. Since LSI systems are characterized by their unit pulse sequence, the property of BIBO stability must depend only on $\{h(n)\}$.

Example

Let $h(n) = \begin{cases} a^n & n \geq 0 \text{ neven} \\ 0 & \text{otherwise} \end{cases}$

Is stable ?

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0, \text{ even}}^{\infty} |a|^n = \sum_{K=0}^{\infty} |a|^{2K} = \frac{1}{1-|a|^2}$$

Stable if $|a| < 1$

Theorem 1

An LSI system is BIBO stable iff

$$S \triangleq \sum_{K=-\infty}^{\infty} |h(K)| < \infty$$

i.e. $h(n)$ is absolutely summable.

Proof :

If $h(n)$ is absolutely summable and $|x(n)| \leq M$ it can be shown that $$|y(n)| = \left| \sum_{K=-\infty}^{\infty} h(K)\, x(n-K) \right| \leq M \sum_{K=-\infty}^{\infty} |h(K)| < \infty$$

i.e $y(n)$ is bounded. To prove the converse let assume $S = \infty$, then a bounded input can be found which gives an unbounded output. As an example, let

$$x(n) = \begin{cases} \dfrac{h^*(-n)}{|h(-n)|} & h(n) \neq 0 \\ 0 & h(n) = 0 \end{cases}$$

which is bounded, then y at $n=0$ is

$$y(0) = \sum_{K=-\infty}^{\infty} h(K)\, x(-K) = \sum_{K=-\infty}^{\infty} \frac{|h(K)|^2}{|h(K)|} = S$$

i.e. y is unbounded.

Theorem 2:

An LSI system is BIBO stable iff all the poles of the transfer function lie inside the unit circle in the z-plane.

Proof

To see this let factorize the numerator and denominator polynomials to give

$$H(z) = \frac{A \prod_{i=1}^{M} (z - z_i)}{\prod_{j=1}^{N} (z - p_j)} \qquad z_i : \text{zeros} \;,\; p_j : \text{poles}$$

If the system is causal i.e. $H(z)$ is proper then using PFE

$$H(z) = \frac{A_1 z}{(z-p_1)} + \frac{A_2 z}{(z-p_2)} + \cdots \frac{A_N z}{(z-p_N)}$$

Each $\dfrac{A_i z}{(z-p_i)} \xrightarrow{Z^{-1}} A_i\, p_i^{\,n}$

Example

Consider the 1st order difference equation

$$y(n) = a\,y(n-1) + x(n)$$

Let $x(n) = \delta(n)$ and assume initial rest conditions

$$h(n) = 0 \qquad n < 0$$

$$h(0) = a\,h(-1) + 1 = 1$$

$$h(1) = a$$

$$\vdots$$

$$h(n) = a\,h(n-1) = a^n$$

Thus $\quad h(n) = a^n u(n) \qquad$ causal and stable for $|a| < 1$

To obtain a different solution let $x(n) = \delta(n)$ but assume

$$y(n) = 0 \qquad n > 0$$

$$y(n-1) = \frac{1}{a}\left[\,y(n) - x(n)\,\right]$$

or $\quad y(n) = \dfrac{1}{a}\left[\,y(n+1) - x(n+1)\,\right]$

$$h(n) = 0 \qquad n > 0 \qquad \text{Anticausal}$$

$$h(0) = \frac{1}{a}\left[h(1) - x(1)\right] = 0$$

$$h(-1) = \frac{1}{a}\left[h(0) - x(0)\right] = -a^{-1}$$

$$h(-2) = \frac{1}{a}\left[h(-1) - x(-1)\right] = -a^{-2}$$

$$\vdots$$

$$h(n) = \frac{1}{a}\,h(n+1) = -a^n$$

or $\quad h(n) = -a^n u(-n-1) \qquad$ Noncausal and stable if $|a| > 1$

i.e for the same Diff. Eq. we obtain different solutions.

The True Solution requires additional information to evaluate the solution. This can be

(1) the solution that is causal and the initial values are specified,

(2) the solution that is stable, i.e. $y(n) \to 0$ as $|n| \to \infty$, but the solution may be noncausal.

Special Cases

1- For $N = 0$ $\quad y(n) = \sum_{\ell=0}^{M} b_\ell \, x(n-\ell)$ Nonrecursive filter (All-Zero) Moving Average process

we get the convolution sum for FIR system

$$h(n) = \begin{cases} b_n & n = 0, 1 \ldots M \\ 0 & \text{otherwise} \end{cases}$$

As a result, an FIR system is also a nonrecursive system.

2- For $M = 0$ we get

$$y(n) = b_0 x(n) - \sum_{k=1}^{N} a_k \, y(n-k)$$

Recursive filter (All-pole) AR or Autoregressive Process

3- General case $M \neq 0$, $N \neq 0$ refers to a general recursive filter or Autoregressive moving Average (ARMA) process for statistical modelling.

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September 15, 1992

M. Azimi 800-525-4950, ext. 7956

Colorado State University