$$d_k = \frac{(2n-k)!}{2^{n-k}(k!)(n-k)!} \qquad k = 0, 1, 2 \cdots n$$

$B_n(s)$ satisfies the following recursive equation

$$B_n(s) = (2n-1) B_{n-1}(s) + s^2 B_{n-2}(s)$$

with

$$B_0(s) = 1 \quad , \quad B_1(s) = s+1$$

(4) cut-off frequency varies with the order of the filter (disadvantage)

$$\omega_c = d_0^{1/n}$$

Before Bilinear z-transform

IIR Digital Filter Design methods

Indirect methods

1- Impulse Invariant Method:

Impulse response of the digital filter is the sampled version of the impulse response of the analog filter. Assume that the analog filter has

$$H_A(s) = \sum_{i=1}^{n} \frac{A_i}{(s+\alpha_i)}$$

$-\alpha_i$: distinct poles of $H_A(s)$.

$$A_i = (s+\alpha_i) H_A(s) \Big|_{s=-\alpha_i}$$

If all the poles are ~~real and~~ distinct then

$$h_A(t) = \mathcal{L}^{-1}[H_A(s)] = \sum_{i=1}^{n} A_i e^{-\alpha_i t}$$

The impulse response of the digital filter is

$$h_A(t)\Big|_{t=KT} = h_D(KT) = \sum_{i=1}^{n} A_i e^{-\alpha_i KT}$$

The transfer function $H_D(z)$ is

$$H_D(z) = \sum_{K=0}^{\infty} h_D(KT) z^{-K}$$

$$= \sum_{K=0}^{\infty} \sum_{i=1}^{n} A_i e^{-\alpha_i KT} z^{-K} = \sum_{i=1}^{n} A_i \sum_{K=0}^{\infty} (z^{-1} e^{-\alpha_i T})^K$$

$$= \sum_{i=1}^{n} \frac{A_i}{(1 - e^{-\alpha_i T} z^{-1})}$$

Comparison with

$$H_A(s) = \sum_{i=1}^{n} \frac{A_i}{(s+\alpha_i)}$$

gives the mapping as

$$H_D(z) = H_A(s)\Big|_{(s+\alpha_i) \longrightarrow (1 - z^{-1} e^{-\alpha_i T})}$$

Thus $H_D(z)$ can be obtained from $H_A(s)$ without evaluating $h_A(t)$ or $h_D(KT)$.

For stable analog filter $\alpha_i > 0$ for $\forall i \in [1, n]$

The corresponding digital filter has poles at

$z = e^{-\alpha_i T}$ with $|z| < 1$ when $\alpha_i > 0$

i.e. the digital filter is also ~~filter~~ Stable filter.

Using the sampling, the frequency responses are related by

$$H_D(e^{j\Omega}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} H_A\left(\frac{\Omega + 2\pi n}{T}\right) \qquad \Omega: \text{Digital freq.}$$

or since $\omega T = \Omega$ $\qquad \omega$: Analog freq.

$\frac{2\pi}{T} = \omega_s$

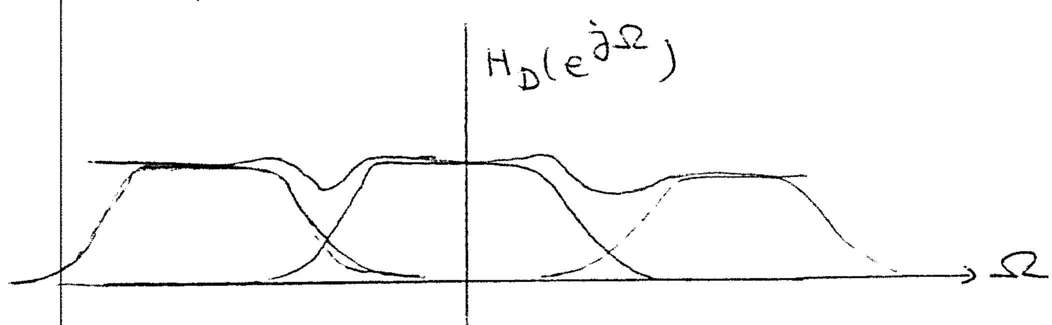
$$\boxed{H_D(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} H_A(\omega + n\omega_s)}$$

If $H_A(\omega) = 0$ when $|\omega| \geq \frac{\pi}{T} = \frac{\omega_s}{2}$ or $\Omega \geq \pi$ i.e. Bandlimited

$H_D(\omega) = \frac{1}{T} H_A(\omega)$

then $H_D(e^{j\Omega}) = \frac{1}{T} H_A\left(\frac{\Omega}{T}\right)$ i.e. no aliasing

Unfortunately $H_A(\omega)$ is not bandlimited in practical cases an Consequently aliasing problem is inevitable



As a result this method is appropriate only for Bandlimit LPF and HPF and BPF would require additional bandlimiting to avoid severe aliasing distortion.

Remarks

1- Let $\alpha_i = \sigma + j\omega$ then

$$z = e^{-\sigma T} e^{-j\omega T}$$

Compare with $z = r e^{-j\Omega}$ $\Rightarrow$ $r = e^{-\sigma T}$ and $\boxed{\Omega = \omega T}$

i.e. linear mapping. (between frequencies)

2- For Complex Conjugate poles one can alternatively use

$$\frac{s+\sigma}{s^2+2\sigma s+\sigma^2+\omega^2} \xrightarrow{\text{mapped}} \frac{1-z^{-1}e^{-\sigma T}\cos\omega T}{1-2z^{-1}e^{-\sigma T}\cos\omega T+z^{-2}e^{-2\sigma T}}$$

$$\frac{\omega}{s^2+2\sigma s+\sigma^2+\omega^2} \longrightarrow \frac{z^{-1}e^{-\sigma T}\sin\omega T}{1-2z^{-1}e^{-\sigma T}\cos\omega T+z^{-2}e^{-2\sigma T}}$$

Example (LPF)

Design a digital filter of Butterworth Type with

$n = 3$, $\omega_s = 15\ \omega_c$

using impulse invariant method.

From the tables we can get the normalized transfer function

$$H_N(s) = \frac{1}{1+2s+2s^2+s^3} = \frac{1}{(s+1)(s^2+s+1)}$$

$$= \frac{A_1}{(s+1)} + \frac{A_2}{(s+\alpha)} + \frac{A_3}{(s+\alpha^*)}$$

where

$$\alpha = \frac{1}{2}(1-j\sqrt{3})$$

Direct Methods (after Bilinear z)

Least Square Inverse Design

This method leads to a set of linear equations

$\{h_D(n)\}$: $n \in [0, L-1]$ Desired Impulse Response (first L Samples)

Filter transfer function

$$H(z) = \frac{b_0}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$ must find a_k's

Generalization is proposed by Shanks and Burrus and Parks

The output of the inverse of $H(z)$ must approximate a unit sample ($\delta(n)$) when the input is $h_D(n)$. If $v(n)$ denotes

Let $v(n)$: the output of inverse system with transfer function $1/H(z)$ then

$$h_D(n) \to \boxed{1/H(z)} \to v(n)$$

$$V(z) = \frac{H_d(z)}{H(z)}$$

Thus we can write

$$b_0 v(n) = h_D(n) - \sum_{r=1}^{N} a_r h_D(n-r)$$

Recall that we require $v(n) \approx \delta(n)$ thus $b_0 = h_D(0)$

and that $v(n)$ be as small as possible for $n > 0$. Thus we

Choose the remaining coeff to minimize

$$E = \sum_{n=1}^{\infty} (v(n))^2$$

$$= \frac{1}{b_0^2}\left[\sum_{n=1}^{\infty} (h_D(n))^2 - 2 \sum_{n=1}^{\infty} h_D(n) \sum_{r=1}^{N} a_r h_D(n-r) + \sum_{n=1}^{\infty}\left[\sum_{r=1}^{N} a_r h(n-r)\right]^2\right]$$

$$\frac{\partial E}{\partial a_i} = 0 \qquad i \in [1, N]$$

$$\sum_{r=1}^{N} a_r \sum_{n=1}^{\infty} h_D(n-r)\, h_D(n-i) = \sum_{n=1}^{\infty} h_D(n)\, h_D(n-i)$$

Define the autocorrelation function

$$\varphi(i,r) = \sum_{n=1}^{\infty} h_D(n-r)\, h_D(n-i)$$

Then the above equation becomes a normal equation

$$\sum_{r=1}^{N} a_r \varphi(i,r) = \varphi(i,0) \qquad i \in [1, N]$$

These equations can be solved using GE or LU factorization.

~~Toeplitz form~~ The correlation matrix is Toeplitz and efficient procedure exist for the solution known as Levinson algorithm

$$a_1 \varphi(i,1) + \cdots + a_N \varphi(i,N) = \varphi(i,0)$$

$$\begin{bmatrix} \varphi(1,1) & \cdots & \varphi(1,N) \\ \varphi(2,1) & \varphi(2,2) & \varphi(2,N) \\ & & \\ & & \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} \varphi(1,0) \\ \vdots \\ \varphi(N,0) \end{bmatrix}$$

then the error introduced

$$E = \sum_{n=1}^{\infty} (e(n))^2 \qquad \text{MSE}$$

we have to selects a_i's such that E is minimum.

i.e. $\frac{\partial E}{\partial a_i} = 0 \qquad \forall\, i \in [1, N]$

$$E = \frac{1}{b_0^2} \sum_{n=1}^{\infty} (h_D(n))^2 - 2 \sum_{n=1}^{\infty} h_D(n) \sum_{r=1}^{N} a_r h_D(n-r)$$

$$+ \sum_{n=1}^{\infty} \left[\sum_{r=1}^{N} a_r h_D(n-r) \right]^2$$

$$\therefore \frac{\partial E}{\partial a_i} = -2 \sum_{n=1}^{\infty} h_D(n)\, h_D(n-i) + 2 \sum_{r=1}^{N} a_r \sum_{n=1}^{\infty} h_D(n-r)\, h_D(n-i)$$

$$= 0$$

Denote

$$\rho(i,r) \triangleq \sum_{n=1}^{\overset{L+r-1}{\infty}} h_D(n-r)\, h_D(n-i)$$

$$\rho(i,0) \triangleq \sum_{n=1}^{\overset{L+r-1}{\infty}} h_D(n)\, h_D(n-i)$$

then

$$\sum_{r=1}^{N} a_r \rho(i,r) = \rho(i,0) \quad \text{—— (*)} \qquad i \in [1, N]$$

using (*) we can compute a_i's.

Direct Method

This is direct design methods for IIR filters.

(1) Least Square Inverse Method

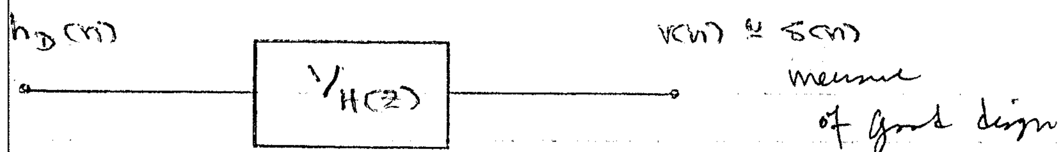
Solving a set of linear eq^{n}

Desired impulse response

Let $\{h_D(n)\}$: 1st L-point of the impulse response $n \in [0, L-1]$

$$H(z) = \frac{b_0}{1 - \sum_{k=1}^{N} a_k z^{-k}} = \sum_{n=0}^{\infty} h(n) z^{-n}$$

objective: find a_k's



$$V(z) = H_D(z) \cdot \frac{1}{H(z)}$$

$$= H_D(z) \cdot \left[\frac{1}{b_0} \left(1 - \sum_{k=1}^{N} a_k z^{-k} \right) \right]$$

take IZT

$$b_0 v(n) = h_D(n) - \sum_{k=1}^{N} a_k h_D(n-k)$$

require that $v(n) \simeq \delta(n) \Rightarrow$ $\underline{b_0 = h_D(0)}$

$\underline{\text{all } v(n),\ n>0 \text{ should be very small.}}$

eg $r=1$

$$\rho(0,1) = \sum_{n=1}^{L} h_D(n-1)\, h_D(n)$$

$$= h_D(0) \cdot h_D(1) + h_D(1)\, h_D(2) + \dots$$

$$\dots + h_D(L-1) \cdot h_D(L)$$

system of linear eq^n can be solved simultaneously to get the values of a_i's.

$$\begin{bmatrix} \rho(1,1) & \rho(1,2) & \dots & \rho(1,N) \\ \rho(2,1) & \rho(2,2) & \dots & \rho(2,N) \\ \vdots & & & \\ \rho(N,1) & \dots & \dots & \rho(N,N) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} \rho(1,0) \\ \rho(2,0) \\ \vdots \\ \rho(N,0) \end{bmatrix}$$

use Levinson-Durbin Algorithm to solve the above.

(2) Linear Programming Approach:

Given the desired squared Magnitude Response for the desired filter $A^2_D(\cancel{\omega})\ \Omega$. Design a filter with $\tilde{A}^2(\cancel{\omega})\ \Omega$ as close as possible to $A^2_D(\cancel{\omega})\ \Omega$ and obtain the relevant coefficient.

$$\tilde{A}^2(\cancel{\omega}\ \Omega) = \tilde{H}(\cancel{\omega}\ \Omega) \cdot \tilde{H}(-\cancel{\omega}\ \Omega) \quad e^{j\Omega}$$

Let
$$\tilde{H}(z) = \frac{\sum_{i=0}^{M} b_i z^{-i}}{\sum_{j=0}^{N} a_j z^{-j}}$$

$$\therefore \tilde{H}(z)\tilde{H}(z^{-1}) = \frac{\left(\sum_{i=0}^{M} b_i z^{-i}\right)\left(\sum_{i=0}^{M} b_i z^{i}\right)}{\left(\sum_{j=0}^{N} a_j z^{-j}\right)\left(\sum_{j=0}^{N} a_j z^{j}\right)}$$

$$= \frac{\sum_{i=-M}^{M} c_i z^{-i}}{\sum_{j=-N}^{N} d_j z^{j}}$$

C_i's and D_j's are expressed in terms of a_i's and b_j's and we have symmetry property

$$c_i = c_{-i}$$
$$d_i = d_{-i}$$

now we can write $\overset{N_2}{A}(\Omega)$ as follows;

$$\overset{N_2}{A}(e^{j\Omega}) = \overset{N}{H}(z)\,\overset{N}{H}(z^{-1})\Big|_{z=e^{j\Omega}}$$

$$= \frac{c_0 + \sum_{i=1}^{M} 2c_i \cos i\Omega}{d_0 + \sum_{i=1}^{N} 2d_i \cos i\Omega}$$

$$\triangleq \frac{N_1(e^{j\Omega})}{D_1(e^{j\Omega})}$$

find filter coeff.

Given $A_D^2(\Omega)$ find $\overset{N}{A}^2(\Omega) = \dfrac{N_1(e^{j\Omega})}{D_1(e^{j\Omega})}$ such that

$$-E(\Omega) \le \overset{N_2}{A}(\Omega) - \overset{N_2}{A}_D(\Omega) \le E(\Omega)$$ — tolerance function

$$-E(\Omega) \le \frac{N_1(e^{j\Omega})}{D_1(e^{j\Omega})} - \overset{N_2}{A}_D(\Omega) \le E(\Omega)$$

$E(\Omega)$: tolerence function

$$+N_1(e^{j\Omega}) - D_1(e^{j\Omega})\left[A_D^2(\Omega) + E(\Omega)\right] \le 0$$

$$-N_1(e^{j\Omega}) + D_1(e^{j\Omega})\left[A_D^2(\Omega) - E(\Omega)\right] \le 0$$

$$N_1(e^{j\Omega}) \ge 0$$

$$D_1(e^{j\Omega}) \ge 0$$

now introduce a dummy variable δ to both equations.

$$-N_1(e^{j\Omega}) - D_1(e^{j\Omega})\left[\tilde{A}_D^2(\Omega) + \epsilon(\Omega)\right] - \delta \leq 0 \quad (*)$$

$$-N_1(e^{j\Omega}) + D_1(e^{j\Omega})\left[\tilde{A}_D^2(\Omega) - \epsilon(\Omega)\right] - \delta \leq 0 \quad (*)$$

solve (*) eq^n's using linear programming techniques.

if $\delta = 0$ you have sol^n
$\delta > 0$ you can still make $\delta = 0$
$\delta < 0$ you have to redesign with compromises $\tilde{A}^2(\Omega)$.