EE 512

Digital Signal Processing

Session 6

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M. Azimi 800-525-4950, ext. 7956

Colorado State University

3- Multiplication by an exponential Sequence

Let $\mathcal{Z}\{x(n)\} = X(z)$ $\qquad R_1 < |z| < R_2$

then $\mathcal{Z}\{e^{an} x(n)\} = X(e^{-a} z)$ $\qquad |e^{a}| \cdot R_1 < |z| < |e^{a}| \cdot R_2$

If $X(z)$ has a pole at $z = z_1$, then $X(e^{-a} z)$ will have a pole at $z = e^{a} z_1$. If a is complex the scaling corresponds to a rotation in the z-plane.

4- Differentiation

Let $\mathcal{Z}\{x(n)\} = X(z)$ then

$$\mathcal{Z}\{n\, x(n)\} = -z \frac{dX(z)}{dz} \qquad R_1 < |z| < R_2$$

5- Conjugation of a Complex Sequence

$$\mathcal{Z}\{x^{*}(n)\} = X^{*}(z^{*}) \qquad R_1 < |z| < R_2$$

6- Initial Value Theorem

If $x(n) = 0$ for $\forall\, n < 0$, then

$$x(0) = \lim_{z \to \infty} X(z)$$

7- Final Value Theorem

If $x(n) = 0$ for $\forall\, n < 0$ then

$$\lim_{n \to \infty} x(n) = \lim_{z \to 1} \frac{(z-1)}{z} X(z)$$

provided that $(1-z^{-1})X(z)$ does not have any pole on or outside the unit circle, e.g.

$$x(n) = \sin \Omega n \qquad \lim_{n \to \infty} \sin n\Omega = ?$$

But $$\mathcal{Z}\{\sin \Omega n\} = \frac{z \sin \Omega}{z^2 - 2z\cos\Omega + 1}$$

and Final value theorem gives

$$\lim_{z \to 1} \frac{(z-1)}{z} \frac{z \sin \Omega}{(z^2 - 2z\cos\Omega + 1)} = 0$$

The discrepancy is due to the fact that $(1-z^{-1})X(z)$ has poles on the unit circle.

$$z_{1,2} = \cos\Omega \pm \sqrt{\cos^2\Omega - 1} = \cos\Omega \pm j\sin\Omega = e^{\pm j\Omega}$$

8) Convolution

If $$y(n) = x(n) * h(n) = h(n) * x(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} x(n-k)h(k)$$

Then

$$Y(z) = H(z)X(z) \qquad \mathrm{Max}[R_{x^-}, R_{y^-}] < |z| < \mathrm{Min}[R_{x^+}, R_{y^+}]$$

If a pole that borders on the Roc of one of the z-Transforms is cancelled by a zero of the other, then the Roc of $Y(z)$ will be larger.

8 - Complex Convolution Theorem

Let $w(n) = y(n)\,x(n)$

RoC of $X(z)$: $R_{x-} < |z| < R_{x+}$

RoC of $Y(z)$: $R_{y-} < |z| < R_{y+}$

then
$$W(z) = \frac{1}{2\pi j} \oint_{C_2} X(v)\, Y\!\left(\frac{z}{v}\right) v^{-1}\, dv$$

C_2: Closed Contour in the overlap of the RoC's of $X(v)$ and $Y(\frac{z}{v})$.

or alternatively

$$W(z) = \frac{1}{2\pi j} \oint_{C_1} X\!\left(\frac{z}{v}\right) Y(v)\, v^{-1}\, dv$$

C_1: closed Contour in the overlap of the RoC's of $X(\frac{z}{v})$ and $Y(v)$.

The ROC of $W(z)$ is $R_{x-} R_{y-} < |z| < R_{x+} R_{y+}$

Example

Let $x(n) = a^n u(n)$, $y(n) = b^n u(n)$

$$X(z) = \frac{1}{1 - a z^{-1}} \qquad |z| > |a|$$

$$Y(z) = \frac{1}{1 - b z^{-1}} \qquad |z| > |b|$$

using property 9 we have

$$W(z) = \frac{1}{2\pi j} \oint_{C_1} -\frac{(z/a)}{(v - z/a)} \cdot \frac{1}{v - b}\, dv$$

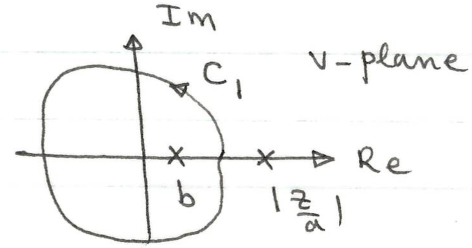
The integrand has two poles $v = b$ and $v = \frac{z}{a}$. The contour of integration must be within the ROC of $Y(v)$ and thus encloses the pole at $v = b$.

$X(z)$ is valid for $|z| > |a|$

Thus $X(\frac{z}{v})$ is valid for $|\frac{z}{v}| > |a|$ or $|\frac{z}{a}| > |v|$

Consequently, the pole $v = \frac{z}{a}$ must lie outside the contour of integration in v. Using Cauchy's Residue theorem we have

$$W(z) = \frac{-z/a}{b - z/a} = \frac{1}{1 - ab\, z^{-1}}$$



This is obtained by evaluating the residue at the pole inside the contour. If we had mistakenly considered the pole at z/a to be inside of contour the result would have become zero.

10- Parseval's Theorem

$$\sum_{n=-\infty}^{\infty} x(n)\, y^*(n) = \frac{1}{2\pi j} \oint_C X(v)\, Y^*\left(\frac{1}{v^*}\right) v^{-1}\, dv$$

C: overlap of the ROC's of $X(v)$ and $Y^*(\frac{1}{v^*})$

If $X(z)$ and $Y(z)$ converge on the unit circle we can choose $v = e^{j\Omega}$ and we get

$$\sum_{n=-\infty}^{\infty} x(n)\, y^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega})\, Y^*(e^{j\Omega})\, d\Omega$$

If $x(n) = y(n)$

$$\sum |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\Omega})|^2\, d\Omega$$

Inverse Z-Transform

The inverse z-transformation can be carried out using one of the following three methods

1- Power Series method

By means of straightforward long-division process, a given $X(z)$ is expressed in form of a power series

$$X(z) = x(0) + x(1)\, z^{-1} + \cdots$$

From which we have find the sequence $\{x(n)\}$. The disadvantage of this method is that it may not give a closed form solution.

Example

$$X(z) = \frac{z^2+z}{z^3-3z^2+3z-1} = \frac{N(z)}{D(z)}$$

Using long-division

$$X(z) = z^{-1} + 4z^{-2} + 9z^{-3} + 16z^{-4} + \cdots$$

Thus

$x(0) = 0$, $x(1) = 1$, $x(2) = 4$

$x(3) = 9$, $x(4) = 16$ which suggests that

$\therefore\ x(n) = n^2$

2- Partial Fraction Expansion

This method is parallel to partial fraction expansion used for inverse Laplace transform, with one minor modification. In this case, we expand $\frac{X(z)}{z}$ instead of $X(z)$, (since $\mathcal{Z}^{-1}\{\frac{A}{z+a}\}$ does not exist in the table).

Example

Given $X(z) = \dfrac{z^2+z}{(z-1)^2}$ determine $x(n)$

Expand $\dfrac{X(z)}{z} = \dfrac{(z+1)}{(z-1)^2} = \dfrac{A_1}{(z-1)} + \dfrac{A_2}{(z-1)^2}$

$$A_2 = (z-1)^2 \frac{X(z)}{z}\Big|_{z=1} = 2$$

$$A_1 = \frac{d}{dz}\left[(z-1)^2 \frac{X(z)}{z}\right]\Big|_{z=1} = 1$$

Thus

$$X(z) = \frac{z}{z-1} + \frac{2z}{(z-1)^2}$$

and

$$x(n) = Z^{-1}\left\{\frac{z}{z-1}\right\} + Z^{-1}\left\{\frac{2z}{(z-1)^2}\right\}$$

$$= 1 + 2n \quad , \quad \forall n \geq 0$$

3- Inversion - Formula method Using Residue Theorem

Let

$$X(z) = Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n} \qquad |z| > R \text{ (ROC)}$$

Then $x(n)$ can be recovered from $X(z)$ by inverse integral formula

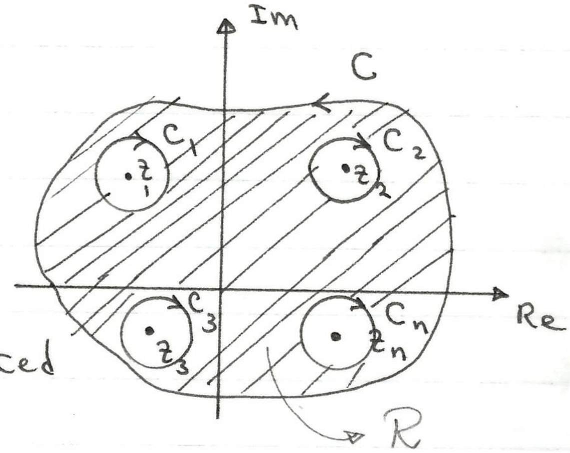
$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

where C is any simple closed curve enclosing $|z| = R$, and $\oint_C$ denotes line or contour integral along C in the counterclockwise direction. The inversion integral may easily be evaluated using Cauchy's Residue theorem.

Cauchy's Residue Theorem

If C is a closed curve and if $f(z)$ is analytic within and on C except at a finite number of singular points in the interior of C that are encircled by small circles $C_1, C_2, \ldots C_n$ then

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz$$



C: Closed Contour, Counterclockwise direction

$C_1, C_2, \ldots C_n$: Small circles enclose $z_1, z_2, \ldots z_n$, Clockwise direction

The integrals on the right are related to the residues of $f(z)$ at various isolated singularities within C. i.e.

$$\oint_C f(z)\,dz = 2\pi j\,(\mathrm{Res}_1 + \mathrm{Res}_2 + \cdots \mathrm{Res}_n)$$

where

$$\mathrm{Res}_i = \frac{d^{m-1}}{dz^{m-1}}\left[\frac{(z-z_i)^m}{(m-1)!} f(z)\right]\Bigg|_{z=z_i}$$

m: order of the pole at $z = z_i$

Inversion by Residue Theorem

Given an $X(z)$, $|z| > R$ (ROC), the corresponding IZT can be found by evaluating the integral using the residue method i.e.

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)\,z^{n-1}\,dz = \sum \left[\text{Residues of } X(z)z^{n-1} \text{ evaluated at poles of } X(z)z^{n-1} \text{ that lie inside } C\right]$$

(C is in the ROC and encircle the origin)