Chapter 7: Spectral Density

Although Fourier transform does not exist for r.p.'s (infinite energy), the auto-correlation and cross-correlation functions are non-periodic and energy signals. Thus, for these functions, Fourier transform does exist.

Review on Fourier Transform:

Definition: A deterministic non-periodic signal x(t) is said to be "*energy-signal*" if and only if,

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt < \infty$$

Fourier transform of a non-periodic energy signal x(t) is

$$\boxed{F\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt}$$

where $X(\omega)$: Fourier Transform (FT) of $x(t)$ or frequency spectrum of $x(t)$.

ω : Continuous variable representing frequency in rad / sec, where $\omega = 2\pi f$ … (f in Hz.)

$e^{-j\omega t}$: Basis function for FT representation.

x(t) can be recovered uniquely from its FT via the inverse FT, i.e.,

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega}$$

Thus, there is a one-to-one correspondence,

$$x(t) \overset{F}{\longleftrightarrow} X(\omega)$$

Time domain

Frequency domain

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Remarks and Properties:

(1) FT, X(ω) is a complex function in ω having amplitude and phase, i.e.,

$$X(\omega) = |X(\omega)|\, e^{j\phi(\omega)}$$

where |X(ω)| : Amplitude spectrum

ϕ(ω) : Phase spectrum

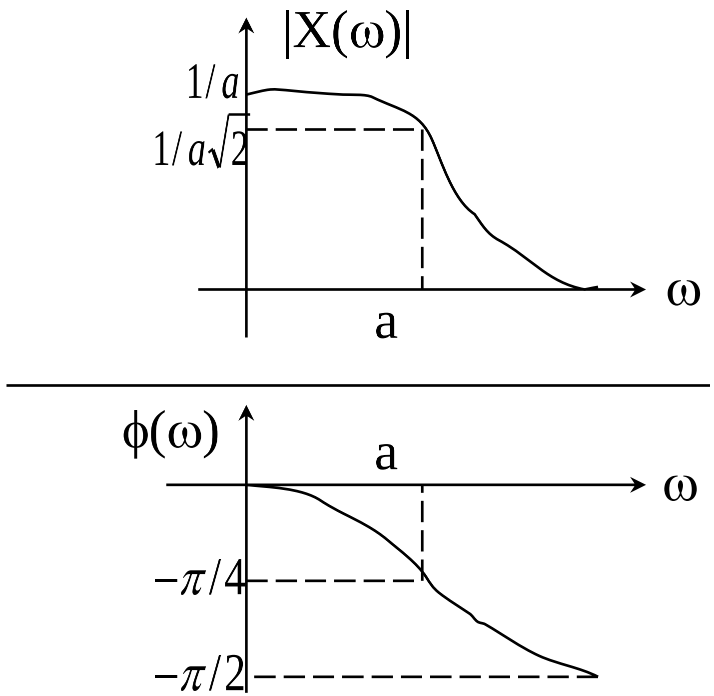
Example 1: Let x(t) = e^{-at} u(t), then

$$X(\omega) = \int_0^\infty e^{-at} e^{-j\omega t} dt$$

$$= -\frac{1}{a+j\omega} e^{-(a+j\omega)t}\Big|_0^\infty = \frac{1}{a+j\omega}$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$ and

$$\phi(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$



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(2) X(ω) has conjugate symmetry, i.e.,

$$X^*(\omega) = X(-\omega)$$

$$\Rightarrow |X(\omega)| = |X(-\omega)| \quad \text{… Symmetrical w.r.t. vertical axis}$$

$$\phi(\omega) = -\phi(\omega) \quad \text{… Anti-symmetric}$$

(3) Parseval's Theorem : Energy is preserved under FT operation.

$$E = \int_{-\infty}^{\infty} x^2(t)dt \quad \text{Energy in time domain}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \frac{1}{\pi}\int_{0}^{\infty} |X(\omega)|^2 d\omega \quad \text{Energy in frequency domain}$$

$|X(\omega)|^2$: Energy spectrum

More general case:

Inner Product in time/Freq

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)Y^*(\omega)d\omega$$

(4) Linearity:

$$ax(t)+by(t)\xleftrightarrow{F}aX(\omega)+bY(\omega)$$

(5) Scaling:

$$x(at)\xleftrightarrow{F}\frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

|a| > 1 → Compression in Time (more rapid)
Expansion in Frequency (more high frequencies)

|a| < 1 → Expansion in Time (slow varying)
Compression in Frequency (more low frequencies)

This property is used in data transmission to reduce the required BW of the receiver.

For a = – 1 → Time reflection

$$x(-t)\xleftrightarrow{F}X(-\omega)$$

$$X(-\omega)=|X(\omega)|e^{-j\phi(\omega)}$$ i.e., only phase is affected (-).

(6) Time shift:

$$x(t-t_0) \xleftrightarrow{F} e^{-j\omega t_0} X(\omega)$$

No change in amplitude as

$$\left| e^{-j\omega t_0} X(\omega) \right| = \left| X(\omega) \right|$$

But, $\quad \angle e^{-j\omega t_0} X(\omega) = \phi(\omega) - \omega t_0$

(7) Frequency shift:

$$x(t)e^{j\omega_0 t} \xleftrightarrow{F} X(\omega - \omega_0)$$

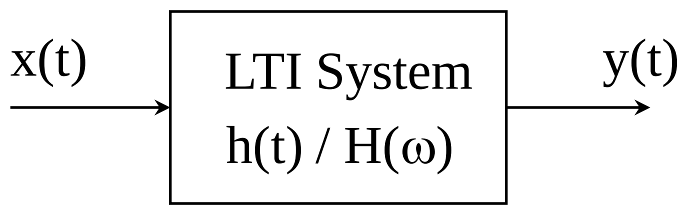
This is used in Amplitude Modulation (AM).

(8) Convolution in Time:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

Then, $Y(\omega) = X(\omega)H(\omega), \qquad y(t) = F^{-1}[Y(\omega)]$

Function H(ω) = Y(ω) / X(ω), is the "*transfer function*" of the system with input x(t) and output y(t).



$$h(t) = F^{-1}[H(\omega)]$$

H(ω) is also called frequency response of the LTI system.

$$H(\omega) = |H(\omega)|\, e^{j\theta(\omega)}$$

where |H(ω)| : Magnitude response
θ(ω) : Phase response

These give the behavior of the system at any particular frequency.

(9) Convolution in Frequency:

$$x(t).y(t) \xleftrightarrow{F} \frac{1}{2\pi} X(\omega) * Y(\omega)$$

This is used in sampling in time.

(10) Differentiation in time:

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega\, X(\omega)$$

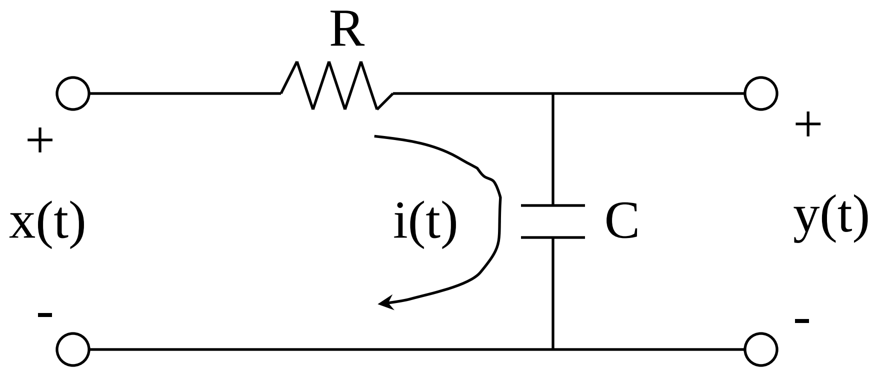
(11) Integration in time:

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(\omega)$$

provided X(ω) / ω has a bounded limit at ω = 0

See Appendices D & E for properties and Fourier transforms of some typical signals.

Example 2: Consider an RC circuit as shown. Find h(t) for this circuit.



Write KVL,

$$R\,i(t) + \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt = x(t)$$

Take FT of both the sides,

$$R\,I(\omega) + \frac{1}{j\omega C} I(\omega) = X(\omega)$$

$$\Rightarrow I(\omega) = \frac{j\omega C\, X(\omega)}{1 + j\omega RC}$$

Also,

$$y(t) = \frac{1}{C} \int_{-\infty}^{t} i(t)\, dt$$

or

$$Y(\omega) = \frac{1}{j\omega C} I(\omega) = \frac{X(\omega)}{1 + j\omega RC}$$

Thus,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 + j\omega RC} = \frac{1/RC}{1/RC + j\omega}$$

Using the FT table,

$$\boxed{h(t) = \frac{1}{RC} e^{-t/RC} u(t)}$$

RC = τ = Time constant of RC circuit

Power Spectrum or Spectral Density Function (SDF):

Let X(t) be a WSS r.p. with ACF $R_{XX}(\tau)$, then SDF is

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

and
$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$$

Thus,
$$R_{XX}(\tau) \xleftrightarrow{F} S_{XX}(\omega)$$

Wiener-Khintchine Relations

Properties:

(1) $S_{XX}(\omega)$ is real, and $S_{XX}(0) \geq 0$

(2) Since $R_{XX}(\tau)$ is real, $S_{XX}(-\omega) = S_{XX}(\omega)$, i.e., symmetrical.

(3) $$S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau) d\tau$$

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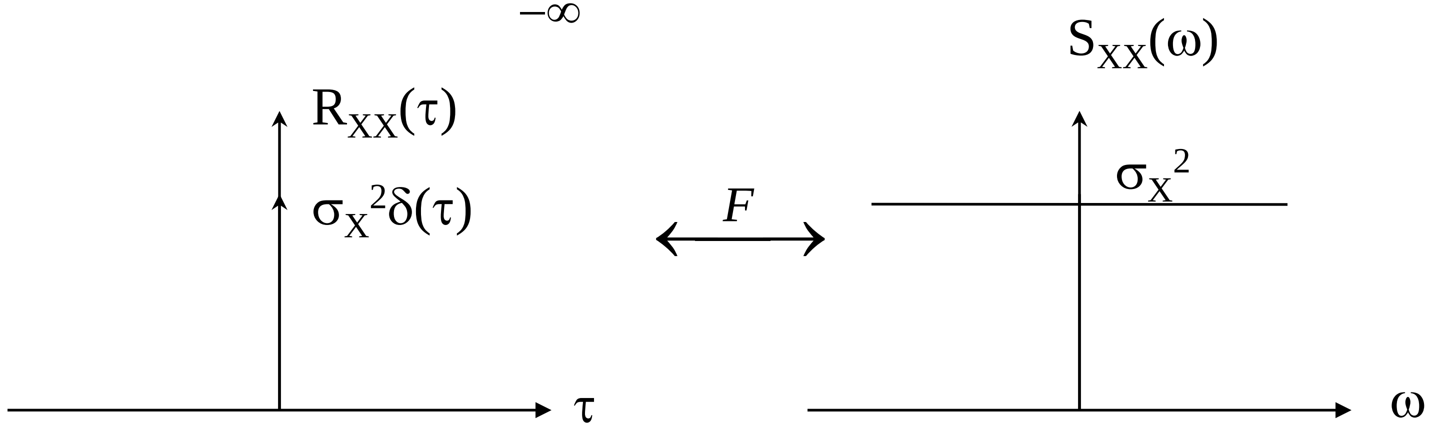
$$(4)\quad \sigma_X^2 = R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega$$

Remark:

For a "white" noise process, $R_{XX}(\tau) = \sigma_X^2\,\delta(\tau)$

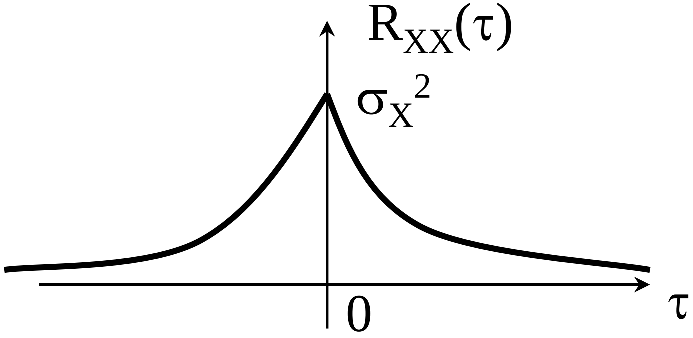
Thus, $$S_{XX}(\omega) = \int_{-\infty}^{\infty} \sigma_X^2\,\delta(\tau)\,e^{-j\omega\tau}\,d\tau = \sigma_X^2$$



i.e. contains all the frequencies with equal contribution.

Example 3: Random process X(t), which is WSS has an ACF given by,

$$R_{XX}(\tau) = \sigma_X^2 e^{-|\tau|}$$



Find SDF.

Solution:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau = \sigma_X^2 \int_{-\infty}^{\infty} e^{-|\tau|} e^{-j\omega\tau} d\tau$$

$$= \sigma_X^2 \int_{-\infty}^{0} e^{+\tau} e^{-j\omega\tau} d\tau + \sigma_X^2 \int_{0}^{\infty} e^{-\tau} e^{-j\omega\tau} d\tau$$

$$= \sigma_X^2 \left[\int_{-\infty}^{0} e^{+\tau} e^{-j\omega\tau} d\tau + \int_{0}^{\infty} e^{-\tau} e^{-j\omega\tau} d\tau \right]$$

$$= \sigma_X^2 \left[\frac{1}{1 - j\omega} + \frac{1}{1 + j\omega} \right] = \frac{2\sigma_X^2}{1 + \omega^2} \longleftarrow$$

Example 4: A WSS r.p. X(t) has an SDF given by,

$$S_{XX}(\omega) = \frac{8}{(\omega^2 + 9)^2}$$

Find its ACF.

Solution: Let us rewrite $S_{XY}(\omega)$ as

$$S_{XX}(\omega) = G(\omega).G(\omega) = \left[\frac{\sqrt{8}}{(\omega^2 + 9)}\right]^2$$

Now, using the convolution property, we have

$$R_{XX}(\tau) = g(\tau) * g(\tau) = \int g(\xi) g(\tau - \xi) d\xi$$

But, from Table E.1. (page 434), we have

$$G(\omega) = \left[\frac{\sqrt{8}}{(\omega^2 + 9)}\right] \xleftrightarrow{F^{-1}} g(\tau) = \frac{\sqrt{8}}{6} e^{-3|\tau|}$$

Thus,

$$R_{XX}(\tau) = \frac{8}{36}\int_{-\infty}^{\infty} e^{-3|\xi|}e^{-3|\tau-\xi|}d\xi$$

$$= \frac{8}{36}\int_{-\infty}^{0} e^{3\xi}e^{-3(\tau-\xi)}d\xi + \frac{8}{36}\int_{0}^{\tau} e^{-3\xi}e^{-3(\tau-\xi)}d\xi + \frac{8}{36}\int_{\tau}^{\infty} e^{-3\xi}e^{3(\tau-\xi)}d\xi$$

$$= \frac{2}{9}e^{-3\tau}(\tau + 1/3), \qquad \tau \geq 0$$

Since $R_{XX}(-\tau) = R_{XX}(\tau)$, then

$$R_{XX}(\tau) = \frac{2}{9}e^{-3|\tau|}(|\tau| + 1/3) \qquad \longleftarrow$$

Cross Power Spectrum:

Consider two r.p.'s X(t) and Y(t) with cross-correlation function $R_{XY}(\tau)$. We assume that X(t), Y(t) are individually and jointly WSS. Then, *cross power spectrum* is defined by,

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau$$

and

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\omega\tau} d\omega$$

Properties:

(1) $S_{XY}(\omega) = S_{YX}(-\omega) = S^{*}_{YX}(\omega)$.

(2) The real part of $S_{XY}(\omega)$ or $S_{YX}(\omega)$ has even symmetry while the imaginary part has odd symmetry.

$$\mathrm{Re}\{S_{XY}(\omega)\} = \mathrm{Re}\{S_{XY}(-\omega)\} = \mathrm{Re}\{S_{YX}(\omega)\} = \mathrm{Re}\{S_{YX}(-\omega)\}$$

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and

$$\mathrm{Im}\{S_{XY}(\omega)\} = -\mathrm{Im}\{S_{XY}(-\omega)\} = \mathrm{Im}\{S_{YX}(-\omega)\} = -\mathrm{Im}\{S_{YX}(\omega)\}$$

(3) If X(t) and Y(t) are orthogonal, then $S_{XY}(\omega)= S_{YX}(\omega)=0$.

(4) $$R_{XY}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)d\omega$$

Example 5: If X(t) is stationary and Y(t) = A + B X(t)

with A, B real constants. Find R_{YY} , R_{XY} , S_{YY} and S_{XY}.

Solution:

$$\begin{aligned} R_{YY}(\tau) &= E[Y(t)Y(t+\tau)] \\ &= E[(A+BX(t))(A+BX(t+\tau))] \\ &= E\left[A^2 + ABX(t) + ABX(t+\tau) + B^2 X(t)X(t+\tau)\right] \end{aligned}$$

$$R_{YY}(\tau) = A^2 + 2AB\mu_X + B^2 R_{XX}(\tau) \quad \longleftarrow$$

$$S_{YY}(\omega) = F\{R_{YY}(\tau)\}$$
$$= \left(A^2 + 2AB\mu_X\right)2\pi\delta(\omega) + B^2 S_{XX}(\omega) \quad \longleftarrow$$

$$R_{XY}(\tau) = E\left[X(t)Y(t+\tau)\right]$$
$$= E\left[X(t)\left(A + BX(t+\tau)\right)\right] = E\left[AX(t) + AB\,X(t)X(t+\tau)\right]$$
$$= A\mu_X + AB\,R_{XX}(\tau) \quad \longleftarrow$$

$$S_{XY}(\omega) = F\{R_{XY}(\tau)\}$$
$$= A\mu_X\, 2\pi\delta(\omega) + AB\,S_{XX}(\omega) \quad \longleftarrow$$

Example 6: Let Y(t) = X(t) + N(t) be a r.p. representing an observed signal and X(t) be a WSS signal, while N(t) is a zero mean white Gaussian noise process with variance σ_N^2 independent of signal. Find expression for S_{YY} and S_{XY}.

Solution:

$$R_{YY}(\tau) = E[Y(t)Y(t+\tau)] = E[(X(t)+N(t))(X(t+\tau)+N(t+\tau))]$$

$$= R_{XX}(\tau) + \sigma_N^2 \delta(\tau)$$

Thus,

$$S_{YY}(\omega) = F[R_{XX}(\tau)] = S_{XX}(\omega) + \sigma_N^2$$

Also,

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = E[X(t)(X(t+\tau)+N(t+\tau))]$$

$$= R_{XX}(\tau)$$

$$S_{XY}(\omega) = F[R_{XY}(\tau)] = S_{XX}(\omega)$$

Reading Assignment For Week 13: Sections 7.1-7.3 (Peebles)