SMAGLR Detector for Change Point Detection in Noisy Observations of Samples from the Wave Equation on Graphs

Christopher Robbiano

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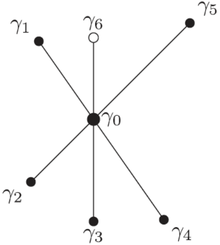
Nondestructive Evaluation of Structures

How can one identify abnormalities in structures before a catastrophic event happens?

Identifying Abnormalities in the Wave Equation on Graphs

Given a tree graph with vertices defined by boundary conditions and edges defined by wave equations, provide a driving source on one exterior vertex and measure the response at all other exterior vertices. Assumption is made that the measured response is contaminated with Gaussian noise.



For a single segment, the network is described as [1][5]

$$u_{tt} - u_{xx} + q(x)u = f(t)g(x) \qquad 0 < x < l,\ 0 < t < T \tag{1}$$

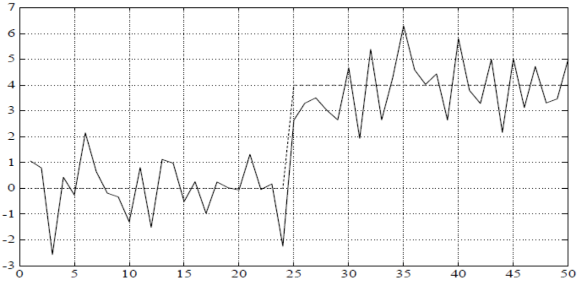
$$u(0,t) = u(l,t) = 0 \qquad 0 < t < T \tag{2}$$

$$u(x,0) = u_t(x,0) = 0 \qquad 0 < x < l \tag{3}$$

f and g are known functions and $u_x(0,t) = \int_0^t f(t-x)g(x)dx = (f * g)(x)$. The measured response is $z(t) = u_x(0,t) + v(t)$, $v(t) \sim \mathcal{N}(0, \sigma_0^2)$.

Change Point Detection

Change point detection seeks to identify a change in the distribution of a signal as fast as possible and provide the time at which the distribution changed[2].



Many different methods for performing change point and sequential detection have been developed.

Modeling Assumptions

Two hypothesis are assumed

$$\mathcal{H}_0 : z(n) = u_x(0,n) + v(n) \qquad \mathcal{H}_1 : \; z(n) = u_x(0,n) + \alpha(n) + v(n) \tag{4}$$

$$\mathcal{H}_0 : z(n) \sim \mathcal{N}(u_x(0,n), \sigma_0^2) \qquad \mathcal{H}_1 : \; z(n) \sim \mathcal{N}(u_x(0,n) + \mu, \sigma_1^2) \tag{5}$$

$u_x(0,n)$, σ_0^2: known
σ_α^2, $\mu(n)$: unknown
$v(n) \sim \mathcal{N}(0, \sigma_0^2)$
$\alpha(n) \sim \mathcal{N}(\mu(n), \sigma_\alpha^2(n))$

In the case where $\sigma_\alpha^2(n) \to 0$, $\alpha(n) \to \mu(n)\delta(n)$ where $\delta(\cdot)$ is the Dirac delta measure.

Only one change point occurs at time t. Changes in variance only decrease SNR while changes in the mean are arbitrary.

The SMAGLR Detector

The SMAGLR detector is named after the following: **S**equential Probability Ratio Test with **M**oving **A**verage using **G**eneralized **L**ikelihood **R**atio detector

Observations are sequential and processed in moving windows[4]

$$\mathbf{P}\big(z(n-k),\ldots,z(n)|\mathcal{H}_j\big) = \prod_{i=n-k}^{n} \mathbf{P}\big(z(i)|\mathcal{H}_j\big) \tag{6}$$

leading to a GLRT

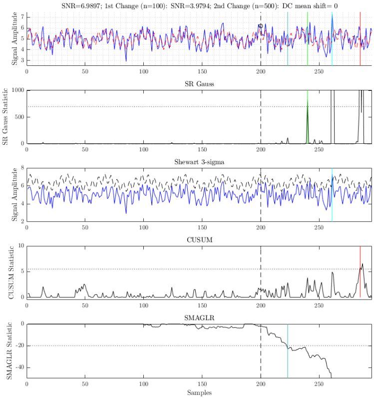
$$\hat{\Lambda}(z(n-k),\ldots,z(n)) = t = \frac{1}{2}\ln\frac{\sigma_0^2}{\hat{\sigma}_1^2} + \frac{1}{\sigma_0^2}\sum_{i=n-k}^{n} z^2(i) - \frac{1}{\hat{\sigma}_1^2}\sum_{i=n-k}^{n}\left(z(i)-\hat{\mu}\right)^2 \gtrless \gamma_1' \tag{7}$$

where $\hat{\mu}$ is a moving average (sample mean) of the observations and $\hat{\sigma}_1^2$ is a moving average of the observation squared (sample variance)[3]

$$\hat{\mu} = \frac{1}{k}\sum_{i=n-k}^{n} z(i) \qquad \hat{\sigma}_1^2 = \frac{1}{k}\sum_{i=n-k}^{n}\big(z(i)-\hat{\mu}\big)^2 \tag{8}$$

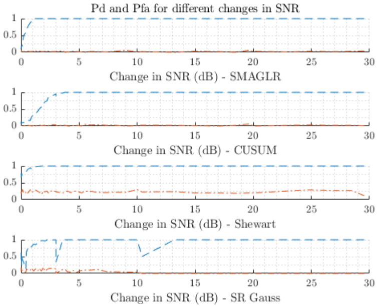
Test Description

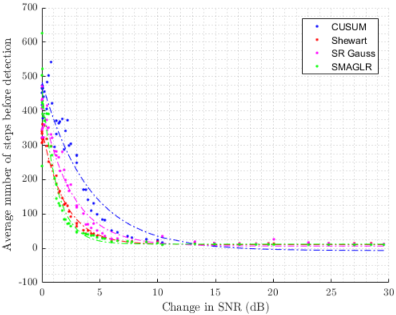
1. 180,000 tests were run comparing four detectors: Shiryaev-Roberts, Shewart 3-sigma, CUSUM and SMAGLR
2. Initial variance $\sigma_0^2 \in [0.001, 1]$, change in variance $\sigma_\alpha^2 \in [0, 1]$, change in mean $\mu \in [-3, 3]$
3. Abrupt changes occurred in the variance, mean, both or neither
4. Metrics calculated were *average detection delay* (ADD), *probability of detection* (P_D) and *probability of false alarm* (P_{FA})



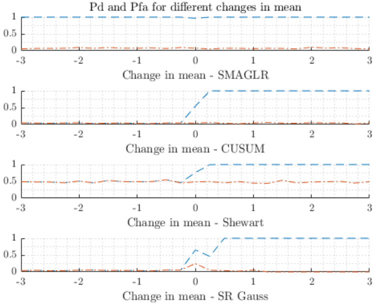
Test Results - Change in Variance

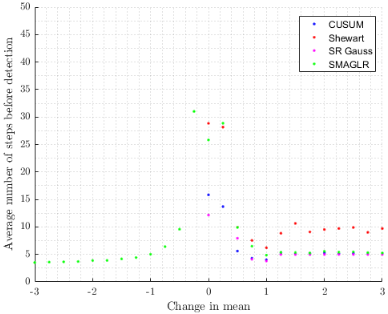
Results were evaluated for the case when there was only an abrupt change in variance OR mean and not both, as a no new insights were gleaned for a change in both.





Test Results - Change in Mean





Future Work

1. Classification of expected detection delay
2. Extension of the detection problem to an arbitrary tree graph
3. Simplify the expression of the detector and classify the distribution under both hypotheses

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