# Reduced equations for Boussinesq internal waves

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## 1 The YBJ equation

The two prognostic variables in the YBJ equation are wave field amplitude $A$ and quasi-geostrophic potential vorticity $q$. Quasi-geostrophic potential vorticity is related to the quasi-geostrophic streamfunction through the elliptic equation

$$q = \Big( \underbrace{\partial_x^2 + \partial_y^2}_{\stackrel{\text{def}}{=} \triangle_h} + \underbrace{\partial_z \frac{f_0^2}{N^2} \partial_z}_{\stackrel{\text{def}}{=} \mathrm{L}} \Big) \psi \,, \tag{1}$$

where the streamfunction is related to the velocity and buoyancy fields via $\boldsymbol{U} = -\psi_y \hat{\boldsymbol{x}} + \psi_x \hat{\boldsymbol{y}}$ and $B = f_0 \psi_z$. The quasi-geostrophic potential vorticity solves the advection equation

$$q_t + \mathrm{J}\,(\psi, q) = \mathrm{F}_u \triangle_h \psi + \mathrm{M}_b \mathrm{L} \psi \,. \tag{2}$$

The wave field amplitude represents the complexified horizontal velocity via

$$\tilde{u} + \mathrm{i} \tilde{v} = \mathrm{e}^{-\mathrm{i} f_0 t} \mathrm{L} A \,, \tag{3}$$

and solves the YBJ equation

$$\mathrm{L} A_t + \frac{\mathrm{i} f_0}{2} \triangle_h A + \mathrm{J}\,(\psi, \mathrm{L} A) + \mathrm{L} A \left( \tfrac{\mathrm{i}}{2} \triangle_h \psi + \beta y \right) = \mathrm{F}_A \mathrm{L} A \,. \tag{4}$$

### 1.1 In $x, y$ with barotropic $\psi(x, y, t)$

### 1.2 In $x, z$ with stationary $\psi(x, z)$

## A The vertical mode decomposition

The hydrostatic vertical modes $\phi_n(z)$ solve the eigenproblem

$$\frac{f_0^2}{N^2} \phi_{nzz} + \lambda_n^{-2} \phi_n = 0 \,, \qquad \text{with} \qquad \phi_n = 0 \quad \text{at} \quad z = -H, 0 \,. \tag{5}$$

Note that the derivative $h_{nz}$ satisfies $h_{nz} = -\lambda_n^2 \mathrm{L} h_{nz}$. The amplitudes of certain quantities are determined by their weighted projection onto $\phi_n$ or its derivative $\phi_{nz}$. For example,

$$u_n \stackrel{\text{def}}{=} \int_{-H}^{0} \Phi \, \phi_{nz} \, \mathrm{d}z \,, \qquad b_n \stackrel{\text{def}}{=} \int_{-H}^{0} b \, \phi_n \, \mathrm{d}z \qquad \text{and} \qquad w_n \stackrel{\text{def}}{=} \int_{-H}^{0} \frac{N^2}{\lambda_n^2 f_0^2} \, w \, \phi_n \, \mathrm{d}z \,. \tag{6}$$

The proper choice depends on the equation set and chosen boundary conditions.

Some trickiness is associated with the friction operator F. We want to assume, for example, that we can define an operator $\mathrm{F}_n$ such that

$$\mathrm{F}_{nu}(u_n) \approx \int_{-H}^{0} \phi_{nz} \mathrm{F}_u(u)\,\mathrm{d}z\,, \tag{7}$$

for example. This is not always possible, however. Notice that if $\mathrm{F} = \nu\triangle$, then

$$\int_{-H}^{0} \phi_{nz} \nu \triangle u\,\mathrm{d}z = \nu\triangle_h u_n + \frac{\kappa_n^2}{f_0^2}\int_{-H}^{0} N^2 \phi_n u_z\,\mathrm{d}z\,, \tag{8}$$

and different modes are therefore coupled by the rightmost term. When $N$ is constant, however, this becomes

$$\int_{-H}^{0} \phi_{nz} \nu \triangle u\,\mathrm{d}z = \nu\triangle_h u_n + \nu\left(\tfrac{n\pi}{H}\right)^2 u_n\,, \tag{9}$$

and therefore $\mathrm{F}_n = \nu\left(\triangle_h + \left(\tfrac{n\pi}{H}\right)^2\right)$. A similar conundrum is associated with the diffusion operator M.

# A The non-hydrostatic Boussinesq equations

The rotating Boussinesq equations are

$$\mathrm{D}_t\boldsymbol{u} + 2\boldsymbol{\Omega}\times\boldsymbol{u} - b\,\hat{\boldsymbol{z}} + \boldsymbol{\nabla}p = \mathrm{F}\boldsymbol{u}\,, \tag{10}$$

$$\mathrm{D}_t b + wN^2 = \mathrm{M}b\,, \tag{11}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\,, \tag{12}$$

where $\mathrm{D}_t \stackrel{\text{def}}{=} \partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$ is the material derivative,

$$2\boldsymbol{\Omega} \stackrel{\text{def}}{=} \underbrace{2\Omega\cos\phi}_{\stackrel{\text{def}}{=}\breve{f}}\,\hat{\boldsymbol{y}} + \underbrace{2\Omega\sin\phi}_{\stackrel{\text{def}}{=}f}\,\hat{\boldsymbol{z}}\,, \tag{13}$$

is the axis around which the Earth rotates, and the F and M are operators that represent dissipative frictional processes and diffusive mixing processes, respectively. If dissipation and diffusion are due to isotropic molecular processes, then $\mathrm{F} = \nu\triangle$ and $\mathrm{M} = \kappa\triangle$, where $\triangle = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the three-dimensional Laplacian.

# B 'Wave operator form' of the Boussinesq equations

The component-wise rotating inviscid Boussinesq equations are

$$u_t - fv + \breve{f}w + p_x = -\boldsymbol{u}\cdot\boldsymbol{\nabla}u\,, \tag{14}$$

$$v_t + fu + p_y = -\boldsymbol{u}\cdot\boldsymbol{\nabla}v\,, \tag{15}$$

$$w_t - \breve{f}u - b + p_z = -\boldsymbol{u}\cdot\boldsymbol{\nabla}w\,, \tag{16}$$

$$b_t + wN^2 = -\boldsymbol{u}\cdot\boldsymbol{\nabla}b\,, \tag{17}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\,. \tag{18}$$

We first form the 'oscillation equation' with the combination $\partial_t(16) + (17)$:

$$\left(\partial_t^2 + N^2\right) w - \breve{f} u_t + p_{zt} = -\partial_t \left(\boldsymbol{u} \cdot \boldsymbol{\nabla} w\right) - \boldsymbol{u} \cdot \boldsymbol{\nabla} b\,. \tag{19}$$

The 'divergence equation' follows from $-\partial_x(14) - \partial_y(15)$ and using $u_x + v_y = -w_z$,

$$w_{zt} + f\omega - u f_y - \breve{f} w_x - \triangle_h p = \partial_x \big(\boldsymbol{u} \cdot \boldsymbol{\nabla} u\big) + \partial_y \left(\boldsymbol{u} \cdot \boldsymbol{\nabla} v\right)\,. \tag{20}$$

The vertical vorticity equation is obtained from $\partial_x(15) - \partial_y(14)$,

$$\omega_t - f w_z + v f_y = -\boldsymbol{\nabla}_\perp \cdot \left(\boldsymbol{u} \cdot \boldsymbol{\nabla}\right) \boldsymbol{u}\,. \tag{21}$$

Yes! In the penultimate step we calculate $\partial_z \partial_t(20) - f\partial_z(21)$, yielding

$$\Big[\left(\partial_t^2 + f^2\right) \partial_z^2 - \breve{f} \partial_x \partial_z \partial_t\Big] w - \triangle_h p_{zt} = f_y \partial_z \left(u_t + f v\right) + \partial_z \left(\partial_t \boldsymbol{\nabla} + f \boldsymbol{\nabla}_\perp\right) \cdot \left(\boldsymbol{u} \cdot \boldsymbol{\nabla}\right) \boldsymbol{u}_h\,. \tag{22}$$

Finally, the combination $\triangle_h(19) + (22)$ yields the wave operator form,

$$\begin{aligned} \Big[\triangle \partial_t^2 + f^2 \partial_z + N^2 \triangle_h\Big] w = {} & \breve{f} \partial_t \left(\triangle_h u + w_{xz}\right) + u_t \breve{f}_{yy} + 2 u_{yt} \breve{f}_y + f_y \partial_z \left(u_t + f v\right) \\ & + \partial_z \left(\partial_t \boldsymbol{\nabla} + f \boldsymbol{\nabla}_\perp\right) \cdot \left(\boldsymbol{u} \cdot \boldsymbol{\nabla}\right) \boldsymbol{u} - \triangle_h \left(\boldsymbol{u} \cdot \boldsymbol{\nabla} b\right) - \triangle \partial_t \left(\boldsymbol{u} \cdot \boldsymbol{\nabla} w\right)\,. \end{aligned} \tag{23}$$