Quasi-geostrophic flow with tracers

Greg

1 Preliminaries

More or less according to Vallis (2006), two-layer quasi-geostrophic flow is governed by the equations

$$\tilde{q}_{1t} + \mathrm{J}\big(\tilde{\psi}_1, \tilde{q}_1\big) = \mathrm{F}\big(\triangle_h \tilde{\psi}_1\big) \tag{1}$$
$$\tilde{q}_{2t} + \mathrm{J}\big(\tilde{\psi}_2, \tilde{q}_2\big) = -r\triangle_h \tilde{\psi}_2 + \mathrm{F}\big(\triangle_h \tilde{\psi}_2\big)\,, \tag{2}$$

where the potential vorticities Q_i are defined by

$$\tilde{q}_1 = \triangle_h \tilde{\psi}_1 - F_1\big(\tilde{\psi}_1 - \tilde{\psi}_2\big) + \beta y\,, \tag{3}$$
$$\tilde{q}_2 = \triangle_h \tilde{\psi}_2 + F_2\big(\tilde{\psi}_1 - \tilde{\psi}_2\big) + \beta y\,, \tag{4}$$

and the Burger numbers F_1 and F_2 are defined by $F_i = f_0^2/g'H_i$, where H_i is the height of layer i and g' is the reduced gravity. The number of parameters in the problem are reduced if the deformation 'radius' R and layer depth ratio δ are defined through

$$R^2 \stackrel{\text{def}}{=} \frac{g' H_1 H_2}{f_0^2\,(H_1 + H_2)}\,, \qquad \text{and} \qquad \delta \stackrel{\text{def}}{=} \frac{H_1}{H_2}\,, \tag{5}$$

so that

$$F_1 = \frac{R^{-2}}{1+\delta}\,, \qquad \text{and} \qquad F_2 = \frac{\delta R^{-2}}{1+\delta}\,. \tag{6}$$

We decompose $\tilde{\psi}_i$ into $\tilde{\psi}_i = -U_i y + \psi_i$, so that

$$\tilde{q}_1 = \underbrace{\beta y + F_1\big(U_1 - U_2\big)y}_{\stackrel{\text{def}}{=} Q_1} + \underbrace{\triangle_h \psi_1 - F_1(\psi_1 - \psi_2)}_{\stackrel{\text{def}}{=} q_1}\,, \tag{7}$$

$$\tilde{q}_2 = \underbrace{\beta y - F_2\big(U_1 - U_2\big)y}_{\stackrel{\text{def}}{=} Q_2} + \underbrace{\triangle_h \psi_2 + F_2(\psi_1 - \psi_2)}_{\stackrel{\text{def}}{=} q_2}\,, \tag{8}$$

The potential vorticity conservation equations become

$$q_{1t} + \mathrm{J}\left(\psi_1, q_1\right) + U_1 q_{1x} + \psi_{1x} Q_{1y} = \mathrm{F}\left(\triangle_h \psi_1\right)\,, \tag{9}$$
$$q_{2t} + \mathrm{J}\left(\psi_2, q_2\right) + U_2 q_{2x} + \psi_{2x} Q_{2y} = -r\triangle_h \psi_2 + \mathrm{F}\left(\triangle_h \psi_2\right)\,. \tag{10}$$

2 Streamfunction inversion

The streamfunction-vorticity relationship is usefully expressed in matrix form,

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \triangle_h - F_1 & -F_1 \\ -F_2 & \triangle_h - F_2 \end{bmatrix}}_{\stackrel{\text{def}}{=} \mathsf{M}} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{11}$$

Which, in Fourier space, reads

$$\begin{bmatrix} \hat{q}_1 \\ \hat{q}_2 \end{bmatrix} = - \begin{bmatrix} K^2 + F_1 & F_1 \\ F_2 & K^2 + F_2 \end{bmatrix} \begin{bmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \end{bmatrix} \tag{12}$$

Noting that $\left(K^2 + F_1\right)\left(K^2 + F_2\right) - F_1 F_2 = K^2 \left(K^2 + F_1 + F_2\right)$, the inversion of M in Fourier space yields

$$\mathsf{M}^{-1} = \frac{1}{K^2 \left(K^2 + F_1 + F_2\right)} \begin{bmatrix} -\left(K^2 + F_2\right) & F_1 \\ F_2 & -\left(K^2 + F_1\right) \end{bmatrix} . \tag{13}$$

3 Linear stability analysis

4 Numerics

When we compute the right hand side, we either dealias or use an exponential filter of the form

$$\mathscr{F} = \exp\left\{ -\, d\left[\left(k/k_c\right)^2 + \left(\ell/\ell_c\right)^2 - 1 \right]^{n/2} \right\} . \tag{14}$$