

Quasi-geostrophic flow with tracers

Greg

1 Preliminaries

More or less according to Vallis (2006), two-layer quasi-geostrophic flow is governed by the equations

$$\tilde{q}_{1t} + J(\tilde{\psi}_1, \tilde{q}_1) = F(\Delta_h \tilde{\psi}_1) \quad (1)$$

$$\tilde{q}_{2t} + J(\tilde{\psi}_2, \tilde{q}_2) = -r \Delta_h \tilde{\psi}_2 + F(\Delta_h \tilde{\psi}_2), \quad (2)$$

where the potential vorticities Q_i are defined by

$$\tilde{q}_1 = \Delta_h \tilde{\psi}_1 - F_1(\tilde{\psi}_1 - \tilde{\psi}_2) + \beta y, \quad (3)$$

$$\tilde{q}_2 = \Delta_h \tilde{\psi}_2 + F_2(\tilde{\psi}_1 - \tilde{\psi}_2) + \beta y, \quad (4)$$

and the Burger numbers F_1 and F_2 are defined by $F_i = f_0^2/g'H_i$, where H_i is the height of layer i and g' is the reduced gravity. The number of parameters in the problem are reduced if the deformation ‘radius’ R and layer depth ratio δ are defined through

$$R^2 \stackrel{\text{def}}{=} \frac{g'H_1H_2}{f_0^2(H_1 + H_2)}, \quad \text{and} \quad \delta \stackrel{\text{def}}{=} \frac{H_1}{H_2}, \quad (5)$$

so that

$$F_1 = \frac{R^{-2}}{1 + \delta}, \quad \text{and} \quad F_2 = \frac{\delta R^{-2}}{1 + \delta}. \quad (6)$$

We decompose $\tilde{\psi}_i$ into $\tilde{\psi}_i = -U_i y + \psi_i$, so that

$$\tilde{q}_1 = \underbrace{\beta y + F_1(U_1 - U_2)y}_{\stackrel{\text{def}}{=} Q_1} + \underbrace{\Delta_h \psi_1 - F_1(\psi_1 - \psi_2)}_{\stackrel{\text{def}}{=} q_1}, \quad (7)$$

$$\tilde{q}_2 = \underbrace{\beta y - F_2(U_1 - U_2)y}_{\stackrel{\text{def}}{=} Q_2} + \underbrace{\Delta_h \psi_2 + F_2(\psi_1 - \psi_2)}_{\stackrel{\text{def}}{=} q_2}, \quad (8)$$

The potential vorticity conservation equations become

$$q_{1t} + J(\psi_1, q_1) + U_1 q_{1x} + \psi_{1x} Q_{1y} = F(\Delta_h \psi_1), \quad (9)$$

$$q_{2t} + J(\psi_2, q_2) + U_2 q_{2x} + \psi_{2x} Q_{2y} = -r \Delta_h \psi_2 + F(\Delta_h \psi_2). \quad (10)$$

2 Streamfunction inversion

The streamfunction-vorticity relationship is usefully expressed in matrix form,

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \Delta_h - F_1 & -F_1 \\ -F_2 & \Delta_h - F_2 \end{bmatrix}}_{\stackrel{\text{def}}{=} \mathbf{M}} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \quad (11)$$

Which, in Fourier space, reads

$$\begin{bmatrix} \hat{q}_1 \\ \hat{q}_2 \end{bmatrix} = - \begin{bmatrix} K^2 + F_1 & F_1 \\ F_2 & K^2 + F_2 \end{bmatrix} \begin{bmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \end{bmatrix} \quad (12)$$

Noting that $(K^2 + F_1)(K^2 + F_2) - F_1 F_2 = K^2(K^2 + F_1 + F_2)$, the inversion of \mathbf{M} in Fourier space yields

$$\mathbf{M}^{-1} = \frac{1}{K^2(K^2 + F_1 + F_2)} \begin{bmatrix} -(K^2 + F_2) & F_1 \\ F_2 & -(K^2 + F_1) \end{bmatrix}. \quad (13)$$

3 Two-layer tracers

The two-layer tracer equations are

$$c_{1t} + \mathbf{J}(\psi_1, c_1) + U_1 c_{1x} = -\kappa(c_1 - c_2) + \mathbf{M}c_1, \quad (14)$$

$$c_{2t} + \mathbf{J}(\psi_2, c_2) + U_2 c_{2x} = \kappa(c_1 - c_2) + \mathbf{M}c_2 \quad (15)$$

4 Numerics

When we compute the right hand side, we either dealias or use an exponential filter of the form

$$\mathcal{F} = \exp \left\{ -d \left[(k/k_c)^2 + (\ell/\ell_c)^2 - 1 \right]^{n/2} \right\}. \quad (16)$$