2D reductions of the rotating Boussinesq equations

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1 Preliminaries

The rotating Boussinesq equations are

$$\mathrm{D}_t\boldsymbol{u} + 2\boldsymbol{\Omega}\times\boldsymbol{u} - b\,\hat{\boldsymbol{z}} + \boldsymbol{\nabla}p = \mathrm{F}\boldsymbol{u}\,, \tag{1}$$

$$\mathrm{D}_t b + wN^2 = \mathrm{M}b\,, \tag{2}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\,, \tag{3}$$

where $\mathrm{D}_t \stackrel{\text{def}}{=} \partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$ is the material derivative,

$$2\boldsymbol{\Omega} \stackrel{\text{def}}{=} \underbrace{2\Omega\cos\phi}_{\stackrel{\text{def}}{=}\breve{f}}\,\hat{\boldsymbol{y}} + \underbrace{2\Omega\sin\phi}_{\stackrel{\text{def}}{=}f}\,\hat{\boldsymbol{z}}\,, \tag{4}$$

is the axis around which the Earth rotates, and the F and M are operators that represent dissipative frictional processes and diffusive mixing processes, respectively. If dissipation and diffusion are due to isotropic molecular processes, then $\mathrm{F} = \nu\triangle$ and $\mathrm{M} = \kappa\triangle$, where $\triangle = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the three-dimensional Laplacian.

2 Boussinesq equations non-linearized around two-dimensional flow in x, z

The Boussinesq equations with $\partial_y = 0$ and non-linearized around the two-dimensional flow

$$\boldsymbol{U} = U(z,t)\,\hat{\boldsymbol{x}} + V(x,z,t)\,\hat{\boldsymbol{y}} \tag{5}$$

become

$$\mathrm{D}_t u + w\left(\breve{f} + U_z\right) - fv + p_x = \mathrm{F}u\,, \tag{6}$$

$$\mathrm{D}_t v + u\,(f + V_x) + wV_z = \mathrm{F}v\,, \tag{7}$$

$$\mathrm{D}_t w - \breve{f}u - b + p_z = \mathrm{F}w\,, \tag{8}$$

$$\mathrm{D}_t b + wN^2 = \mathrm{M}b\,, \tag{9}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\,, \tag{10}$$

where the material derivative is now

$$\mathrm{D}_t \stackrel{\text{def}}{=} \partial_t + (u+U)\,\partial_x + w\partial_z\,. \tag{11}$$

In addition to the terms describing advection by the 'mean' background flow, there are three qualitatively new refraction terms wU_z, wV_z, and uV_x that appear in the momentum equations.

3 Hydrostatic Boussinesq equations linearized around two-dimensional flow in x, y

The traditional hydrostatic Boussinesq equations linearized around a two-dimensional mean flow are formed by using the hydrostatic approximation in the vertical component of (1), assuming that $\breve{f} = 0$ and constant $f = f_0$, and using the map $\boldsymbol{u} \mapsto \boldsymbol{U}(x, y, t) + \boldsymbol{u}(x, y, z, t)$, so that (1)–(3) become

$$
\begin{aligned}
u_t + \boldsymbol{U} \cdot \boldsymbol{\nabla} u + \boldsymbol{u} \cdot \boldsymbol{\nabla} U - f_0 v + p_x &= \mathrm{F}_u u \,, &(12)\\
v_t + \boldsymbol{U} \cdot \boldsymbol{\nabla} v + \boldsymbol{u} \cdot \boldsymbol{\nabla} V + f_0 u + p_y &= \mathrm{F}_u v \,, &(13)\\
p_z &= b \,, &(14)\\
b_t + \boldsymbol{U} \cdot \boldsymbol{\nabla} b + w N^2 &= \mathrm{M} b \,, &(15)\\
u_x + v_y + w_z &= 0 \,. &(16)
\end{aligned}
$$

A natural model for the two-dimensional flow

$$
\boldsymbol{U}(x, y, t) = -\psi_y \hat{\boldsymbol{x}} + \psi_x \hat{\boldsymbol{y}} \tag{17}
$$

is that is solves the two-dimensional vorticity equation,

$$
\triangle_h \psi_t + \mathrm{J}\left(\psi, \triangle_h \psi\right) = \mathrm{F}_\psi \left(\triangle_h \psi\right) \,, \tag{18}
$$

Equations (12)–(16) describe the advection and refraction of waves by a two-dimensional flow with $\boldsymbol{U}_z = \psi_z = 0$ and thus no buoyancy field. The linearization neglects the complications of nonlinear wave dynamics and permits a two-dimensionalization of (12)–(16) by projection onto vertical modes. (12)–(16) do not have a cascade to small-scales, in general, and therefore permit inviscid numerical solutions.

3.1 The vertical mode decomposition

We restrict attention to waves with simple vertical structure by projecting (12)–(16) onto the hydrostatic vertical modes $\phi_n(z)$ that solve the eigenproblem

$$
\frac{f_0^2}{N^2} \phi_{nzz} + \lambda_n^{-2} \phi_n = 0 \,, \qquad \text{with} \qquad \phi_n = 0 \quad \text{at} \quad z = -H, 0 \,. \tag{19}
$$

Note that the derivative h_{nz} satisfies $h_{nz} = -\lambda_n^2 \mathrm{L} h_{nz}$. The modal amplitudes of the independent variables $A, \boldsymbol{u}, b, p$ are defined by their weighted projection onto ϕ_n or its derivative ϕ_{nz}, with

$$
\Phi_n \stackrel{\text{def}}{=} \int_{-H}^{0} \Phi \, \phi_{nz} \, \mathrm{d}z \qquad \text{for} \qquad \Phi = (A, u, v, p) \,, \tag{20}
$$

and

$$
b_n \stackrel{\text{def}}{=} \int_{-H}^{0} b \, \phi_n \, \mathrm{d}z \qquad \text{and} \qquad w_n \stackrel{\text{def}}{=} \int_{-H}^{0} \frac{N^2}{\lambda_n^2 f_0^2} w \, \phi_n \, \mathrm{d}z \,. \tag{21}
$$

We assume $A, \boldsymbol{u}, b$, and p satisfy free-slip, rigid-lid homogeneous boundary conditions with $A_z = u_z = v_z = p_z = 0$ and $w = b = 0$ at $z = -H, 0$.

The linearized Boussinesq equations (12)–(16) are processed in similar fashion. We project (12) and (13) onto ϕ_{nz}. We assume we can write, approximately

$$\mathrm{F}_{nu}(u_n) \approx \int_{-H}^{0} \phi_{nz} \mathrm{F}_u(u)\, \mathrm{d}z\,. \tag{22}$$

This approximation does not hold for all linear operators F_u. Notice that if $\mathrm{F} = \nu\triangle$, then

$$\int_{-H}^{0} \phi_{nz} \nu \triangle u\, \mathrm{d}z = \nu \triangle_h u_n + \frac{\kappa_n^2}{f_0^2} \int_{-H}^{0} N^2 \phi_n u_z\, \mathrm{d}z\,, \tag{23}$$

and different modes are therefore coupled by the rightmost term. When N is constant, however, this becomes

$$\int_{-H}^{0} \phi_{nz} \nu \triangle u\, \mathrm{d}z = \nu \triangle_h u_n + \nu \left(\tfrac{n\pi}{H}\right)^2 u_n\,, \tag{24}$$

and the modes separate. With the notation in (22), the horizontal momentum equations

$$u_{nt} - f_0 v_n + p_{nx} = -\boldsymbol{U} \boldsymbol{\cdot} \boldsymbol{\nabla} u_n - \boldsymbol{u}_n \boldsymbol{\cdot} \boldsymbol{\nabla} U + \mathrm{F}_{nu} u_n\,, \tag{25}$$

$$v_{nt} + f_0 u_n + p_{ny} = -\boldsymbol{U} \boldsymbol{\cdot} \boldsymbol{\nabla} v_n - \boldsymbol{u}_n \boldsymbol{\cdot} \boldsymbol{\nabla} V + \mathrm{F}_{nu} v_n\,. \tag{26}$$

We next combine (14)–(16) by projecting (16) onto ϕ_{nz}, integrating by parts once, and using (19) to yield $w_n = -u_{nx} - v_{ny}$. We then use $p_z = b$ to combine (14) and (15) and project the result onto ϕ_n. Similar to (22), we use the notation

$$\mathrm{M}_n p_n \approx -\int_{-H}^{0} \phi_n \mathrm{M} p_z\, \mathrm{d}z\,. \tag{27}$$

Similar to (22), this is not true when N is not constant. When N is constant we have $\mathrm{M}_n p_n = \kappa \triangle_h p_n + \kappa \left(\tfrac{n\pi}{H}\right)^2 p_n$. Finally, integrating by parts and using $w_n = -u_{nx} - v_{ny}$ transforms (15) into

$$p_{nt} + \left(\frac{f_0}{\kappa_n}\right)^2 (u_{nx} + v_{ny}) = -\boldsymbol{U} \boldsymbol{\cdot} \boldsymbol{\nabla} p_n + \mathrm{M}_n p_n\,. \tag{28}$$

The three equations (25)–(28) describe the evolution of hydrostatic, vertical mode-n waves in a two-dimensional flow $\boldsymbol{U} = U\,\boldsymbol{\hat{x}} + V\,\boldsymbol{\hat{y}}$ with $\boldsymbol{U}_z = 0$. The parameter f_0/κ_n is the phase speed of a linear wave with mode-n vertical structure.

A ‘Wave operator form’ of the Boussinesq equations

The component-wise rotating inviscid Boussinesq equations are

$$u_t - fv + \breve{f}w + p_x = -\boldsymbol{u} \boldsymbol{\cdot} \boldsymbol{\nabla} u\,, \tag{29}$$

$$v_t + fu + p_y = -\boldsymbol{u} \boldsymbol{\cdot} \boldsymbol{\nabla} v\,, \tag{30}$$

$$w_t - \breve{f}u - b + p_z = -\boldsymbol{u} \boldsymbol{\cdot} \boldsymbol{\nabla} w\,, \tag{31}$$

$$b_t + wN^2 = -\boldsymbol{u} \boldsymbol{\cdot} \boldsymbol{\nabla} b\,, \tag{32}$$

$$\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{u} = 0\,. \tag{33}$$

We first form the 'oscillation equation' with the combination $\partial_t(31) + (32)$:

$$\left(\partial_t^2 + N^2\right) w - \breve{f} u_t + p_{zt} = -\partial_t \left(\boldsymbol{u} \cdot \boldsymbol{\nabla} w\right) - \boldsymbol{u} \cdot \boldsymbol{\nabla} b \,. \tag{34}$$

The 'divergence equation' follows from $-\partial_x(29) - \partial_y(30)$ and using $u_x + v_y = -w_z$,

$$w_{zt} + f\omega - u f_y - \breve{f} w_x - \triangle_h p = \partial_x \big(\boldsymbol{u} \cdot \boldsymbol{\nabla} u\big) + \partial_y \left(\boldsymbol{u} \cdot \boldsymbol{\nabla} v\right) \,. \tag{35}$$

The vertical vorticity equation is obtained from $\partial_x(30) - \partial_y(29)$,

$$\omega_t - f w_z + v f_y = -\boldsymbol{\nabla}_\perp \cdot \left(\boldsymbol{u} \cdot \boldsymbol{\nabla}\right) \boldsymbol{u} \,. \tag{36}$$

Yes! In the penultimate step we calculate $\partial_z \partial_t(35) - f\partial_z(36)$, yielding

$$\Big[\left(\partial_t^2 + f^2\right) \partial_z^2 - \breve{f} \partial_x \partial_z \partial_t \Big] w - \triangle_h p_{zt} = f_y \partial_z \left(u_t + f v\right) + \partial_z \left(\partial_t \boldsymbol{\nabla} + f \boldsymbol{\nabla}_\perp\right) \cdot \left(\boldsymbol{u} \cdot \boldsymbol{\nabla}\right) \boldsymbol{u}_h \,. \tag{37}$$

Finally, the combination $\triangle_h(34) + (37)$ yields the wave operator form,

$$\begin{aligned} \Big[\triangle \partial_t^2 + f^2 \partial_z + N^2 \triangle_h\Big] w &= \breve{f} \partial_t \left(\triangle_h u + w_{xz}\right) + u_t \breve{f}_{yy} + 2 u_{yt} \breve{f}_y + f_y \partial_z \left(u_t + f v\right) \\ &+ \partial_z \left(\partial_t \boldsymbol{\nabla} + f \boldsymbol{\nabla}_\perp\right) \cdot \left(\boldsymbol{u} \cdot \boldsymbol{\nabla}\right) \boldsymbol{u} - \triangle_h \left(\boldsymbol{u} \cdot \boldsymbol{\nabla} b\right) - \triangle \partial_t \left(\boldsymbol{u} \cdot \boldsymbol{\nabla} w\right) \,. \end{aligned} \tag{38}$$