Reduced equations for weakly nonlinear Boussinesq flows

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1 The quasi-geostrophic equation

Quasi-geostrophic flow is a small amplitude, slowly-evolving, hydrostatic and small-aspect-ratio mode of motion in rotating stratified flow. Its evolution is much slower than a rotation period due to small departures from a linear, static balance betwen velocity and pressure. The evolution of quasi-geostrophic flow is governed by the advection of quasi-geostrophic potential vorticity, which is related to its advecting streamfunction through the elliptic equation

$$q = \Big(\underbrace{\partial_x^2 + \partial_y^2}_{\stackrel{\text{def}}{=} \triangle_h} + \underbrace{\partial_z \frac{f_0^2}{N^2} \partial_z}_{\stackrel{\text{def}}{=} \mathrm{L}} \Big) \psi \,, \tag{1}$$

With the rotation rate $f = f_0 + \beta y$, quasi-geostrophic potential vorticity solves the advection equation

$$q_t + \mathrm{J}\left(\psi, q\right) + \beta \psi_x = \mathrm{F}\left(\triangle_h \psi\right) + \mathrm{M}\left(\mathrm{L}\psi\right) \,. \tag{2}$$

A projection of both (1) and (2) onto the vertical modes ϕ_{nz} yields

$$q_n = \left(\triangle_h - \lambda_n^{-2}\right) \psi_n \,, \qquad \text{and} \qquad q_{nt} + \beta \psi_{nx} + \int \phi_{nz} \mathrm{J}\left(\psi, q\right) \, \mathrm{d}z = \mathrm{F}_n \left(\triangle_h \psi_n\right) - \lambda_n^{-2} \mathrm{M}_n \left(\psi_n\right) \,. \tag{3}$$

It is not true in general that the vertical mode projection commutes with the operators F and M, but we assume this here for simplicity.

1.1 Two-layer quasi-geostrophic flow with tracers

The two-layer equations for potential vorticities q_1, q_2, and streamfunctions ψ_1, ψ_2, in layers 1 and 2 consist of a potential vorticity conservation equation for each layer,

$$q_{1t} + \mathrm{J}\left(\psi_1, q_1\right) + \beta \psi_{1x} = \mathrm{F}_1 \left(\triangle_h \psi_1\right) \,, \tag{4}$$

$$q_{2t} + \mathrm{J}\left(\psi_2, q_2\right) + \beta \psi_{2x} = \mathrm{F}_2 \left(\triangle_h \psi_2\right) \,, \tag{5}$$

where typically the layer-1 friction operator is an isotropic viscous-like term of the form

$$\mathrm{F}_1 = (-1)^{n/2-1} \nu \triangle_h^{n/2} \,, \tag{6}$$

and the layer-2 friction operator includes both a viscous term and a term for linear drag, so that

$$\mathrm{F}_2 = -r \triangle_h + (-1)^{n/2-1} \nu \triangle_h^{n/2} \,. \tag{7}$$

The potential vorticity in each layer depends both on the layerwise relative vorticity, as well as the ‘shear’ $\psi_1 - \psi_2$, according to

$$q_1 = \triangle_h \psi_1 - F_1 \left(\psi_1 - \psi_2\right) \tag{8}$$

$$q_2 = \triangle_h \psi_2 + F_2 \left(\psi_1 - \psi_2\right) \,. \tag{9}$$

The Fourier transform of (8) and (9) can be rearranged into

$$\begin{bmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \end{bmatrix} = -\frac{1}{\mathrm{det} M_2} \begin{bmatrix} K^2 + F_2 & F_1 \\ F_2 & K^2 + F_1 \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \hat{q}_2 \end{bmatrix} , \tag{10}$$

where M_2 is a matrix and it's determinant $\mathrm{det} M_2$ is

$$\mathrm{det} M_2 = K^2 \left(K^2 + F_1 + F_2 \right) . \tag{11}$$

We also include layerwise tracer distributsion c_1 and c_2, whose conservation is governed by

$$c_{1t} + \mathrm{J} \left(\psi_2, c_1 \right) = \kappa \left(c_1 - c_2 \right) + \mathrm{M} c_1 , \tag{12}$$

$$c_{2t} + \mathrm{J} \left(\psi_2, c_2 \right) = \kappa \left(c_2 - c_1 \right) + \mathrm{M} c_2 , \tag{13}$$

κ is the vertical diffusion coefficient.

1.2 n-layer quasi-geostrophic flow

2 The YBJ equation

The two prognostic variables in the YBJ equation are wave field amplitude A and quasi-geostrophic potential vorticity q. The wave field amplitude represents the complexified horizontal velocity via

$$\tilde{u} + \mathrm{i} \tilde{v} = \mathrm{e}^{-\mathrm{i} f_0 t} \mathrm{L} A , \tag{14}$$

and solves the YBJ equation

$$\mathrm{L} A_t + \frac{\mathrm{i} f_0}{2} \triangle_h A + \mathrm{J} \left(\psi, \mathrm{L} A \right) + \mathrm{L} A \left(\tfrac{\mathrm{i}}{2} \triangle_h \psi + \beta y \right) = \mathrm{F} \left(\mathrm{L} A \right) . \tag{15}$$

2.1 In x, y with barotropic $\psi(x, y, t)$

When $\psi = \psi(x, y, t)$ is barotropic and does not depend on z, the YBJ equation (14) can be projected onto vertical modes to yield

$$A_{nt} - \frac{\mathrm{i} f_0 \lambda_n^2}{2} \triangle_h A_n + \mathrm{J} \left(\psi, A_n \right) + A_n \left(\tfrac{\mathrm{i}}{2} \triangle_h \psi + \beta y \right) = \mathrm{F}_n \left(A_n \right) . \tag{16}$$

The quasi-geostrophic streamfunction ψ obeys the ordinary two-dimensional turbulence equation,

$$\triangle_h \psi_t + \mathrm{J} \left(\psi, \triangle_h \psi \right) = \mathrm{F} \left(\triangle_h \psi \right) . \tag{17}$$

2.2 In y, z with stationary $\psi(y, z)$

When $\psi(y, z)$ does not depend on y and therefore does not evolve, and N is constant, the YBJ equation reduces to

$$A_{zzt} + \frac{\mathrm{i} N^2}{2 f_0} A_{yy} + \left(\tfrac{\mathrm{i}}{2} \psi_{yy} + \beta y \right) A_{zz} = \mathrm{F} \left(A_{zz} \right) . \tag{18}$$

Because $q_t = 0$ when $\psi_x = q_x = 0$, the PV equation is solved automatically and ψ does not evolve.

3 The hydrostatic wave equation

The hydrostatic wave equation for a wave field with frequency σ is

$$\begin{aligned}
&\mathrm{E}A_t + \mathrm{J}\left(\psi, \mathrm{E}A\right) + \mathrm{i}\alpha\sigma\mathrm{D}A + \mathrm{J}\left(A, \mathrm{D}\psi\right) \\
&\quad - \tfrac{2\mathrm{i}\sigma}{f_0^2}\Big[\mathrm{J}\left(\psi_x, \mathrm{i}\sigma A_x - f_0 A_y\right) + \mathrm{J}\left(\psi_y, \mathrm{i}\sigma A_y + f_0 A_x\right)\Big] \\
&\qquad + \tfrac{\mathrm{i}\sigma}{f_0}\left[\boldsymbol{\nabla}_h \cdot \left(\mathrm{D}\psi \boldsymbol{\nabla}_h A\right) - \mathrm{D}\left(\tfrac{\alpha f_0^2}{N^2}\psi_z A_z\right) + \partial_z\left(\tfrac{\alpha f_0^2}{N^2}\psi_z \mathrm{D}A\right)\right] = 0\,.
\end{aligned} \tag{19}$$

where the operators E and D are defined

$$\mathrm{E} \stackrel{\text{def}}{=} \frac{\alpha}{2}\left[\triangle_h + (4+3\alpha)\,\mathrm{L}\right] \qquad \text{and} \qquad \mathrm{D} \stackrel{\text{def}}{=} \triangle_h - \alpha\mathrm{L}\,. \tag{20}$$

3.1 In x, y with barotropic $\psi(x, y, t)$

$$\begin{aligned}
&\mathrm{E}_n A_{nt} + \mathrm{i}\alpha\sigma\mathrm{D}_n A_n + \mathrm{J}\left(\psi, \mathrm{E}_n A_n\right) + \mathrm{J}\left(A_n, \triangle_h\psi\right) + \tfrac{\mathrm{i}\sigma}{f_0}\boldsymbol{\nabla}_h \cdot \left(\triangle_h\psi\boldsymbol{\nabla}_h A_n\right) \\
&\quad - \tfrac{2\mathrm{i}\sigma}{f_0^2}\left[\mathrm{J}\left(\psi_x, \mathrm{i}\sigma A_{nx} - f_0 A_{ny}\right) + \mathrm{J}\left(\psi_y, \mathrm{i}\sigma A_{ny} + f_0 A_{nx}\right)\right] = \mathrm{F}_n\left(A_n\right)\,,
\end{aligned} \tag{21}$$

where ν_A is the hyperviscosity applied to A_n, and the mode-wise operators E_n and D_n are

$$\mathrm{E}_n = \frac{\alpha}{2}\Big[\triangle_h - \lambda_n^{-2}\left(4+3\alpha\right)\Big] \qquad \text{and} \qquad \mathrm{D}_n = \triangle_h + \alpha\lambda_n^{-2}\,. \tag{22}$$

Equation (20) describes the horizontal propagation of a mode-n wave field with amplitude $A_n(x, y, t)$ through two-dimensional turbulence with streamfunction ψ. The arbitrary stratification profile $N(z)$ enters (20) via the eigenvalue λ_n^{-2} determined by (22).

3.2 In y, z with stationary $\psi(y, z)$

A The vertical mode decomposition

The hydrostatic vertical modes $\phi_n(z)$ solve the eigenproblem

$$\frac{f_0^2}{N^2}\phi_{nzz} + \lambda_n^{-2}\phi_n = 0\,, \qquad \text{with} \qquad \phi_n = 0 \quad \text{at} \quad z = -H, 0\,. \tag{23}$$

Note that the derivative h_{nz} satisfies $h_{nz} = -\lambda_n^2 \mathrm{L} h_{nz}$. The amplitudes of certain quantities are determined by their weighted projection onto ϕ_n or its derivative ϕ_{nz}. For example,

$$u_n \stackrel{\text{def}}{=} \int_{-H}^{0} \Phi\,\phi_{nz}\,\mathrm{d}z\,, \qquad b_n \stackrel{\text{def}}{=} \int_{-H}^{0} b\,\phi_n\,\mathrm{d}z \qquad \text{and} \qquad w_n \stackrel{\text{def}}{=} \int_{-H}^{0} \frac{N^2}{\lambda_n^2 f_0^2}\,w\,\phi_n\,\mathrm{d}z\,. \tag{24}$$

The proper choice depends on the equation set and chosen boundary conditions.

Some trickiness is associated with the friction operator F. We want to assume, for example, that we can define an operator F_n such that

$$\mathrm{F}_{nu}(u_n) \approx \int_{-H}^{0} \phi_{nz}\mathrm{F}_u(u)\,\mathrm{d}z\,, \tag{25}$$

for example. This is not always possible, however. Notice that if $\mathrm{F} = \nu\triangle$, then

$$\int_{-H}^{0} \phi_{nz} \nu \triangle u \, \mathrm{d}z = \nu \triangle_h u_n - \frac{1}{(f_0\lambda_n)^2} \int_{-H}^{0} N^2 \phi_n u_z \, \mathrm{d}z \,, \tag{26}$$

and different modes are therefore coupled by the rightmost term. When N is constant, however, this becomes

$$\int_{-H}^{0} \phi_{nz} \nu \triangle u \, \mathrm{d}z = \nu \triangle_h u_n + \nu \left(\tfrac{n\pi}{H}\right)^2 u_n \,, \tag{27}$$

and therefore $\mathrm{F}_n = \nu \left(\triangle_h + \left(\tfrac{n\pi}{H}\right)^2 \right)$. A similar conundrum is associated with the diffusion operator M.

B The non-hydrostatic Boussinesq equations

The rotating Boussinesq equations are

$$\mathrm{D}_t \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} - b\,\hat{\boldsymbol{z}} + \boldsymbol{\nabla} p = \mathrm{F}\boldsymbol{u} \,, \tag{28}$$
$$\mathrm{D}_t b + wN^2 = \mathrm{M}b \,, \tag{29}$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \,, \tag{30}$$

where $\mathrm{D}_t \stackrel{\text{def}}{=} \partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla}$ is the material derivative,

$$2\boldsymbol{\Omega} \stackrel{\text{def}}{=} \underbrace{2\Omega \cos\phi}_{\stackrel{\text{def}}{=} \breve{f}} \, \hat{\boldsymbol{y}} + \underbrace{2\Omega \sin\phi}_{\stackrel{\text{def}}{=} f} \, \hat{\boldsymbol{z}} \,, \tag{31}$$

is the axis around which the Earth rotates, and the F and M are operators that represent dissipative frictional processes and diffusive mixing processes, respectively. If dissipation and diffusion are due to isotropic molecular processes, then $\mathrm{F} = \nu\triangle$ and $\mathrm{M} = \kappa\triangle$, where $\triangle = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the three-dimensional Laplacian.

C 'Wave operator form' of the Boussinesq equations

The component-wise rotating inviscid Boussinesq equations are

$$u_t - fv + \breve{f}w + p_x = -\boldsymbol{u} \cdot \boldsymbol{\nabla} u \,, \tag{32}$$
$$v_t + fu + p_y = -\boldsymbol{u} \cdot \boldsymbol{\nabla} v \,, \tag{33}$$
$$w_t - \breve{f}u - b + p_z = -\boldsymbol{u} \cdot \boldsymbol{\nabla} w \,, \tag{34}$$
$$b_t + wN^2 = -\boldsymbol{u} \cdot \boldsymbol{\nabla} b \,, \tag{35}$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \,. \tag{36}$$

We first form the 'oscillation equation' with the combination $\partial_t(33) + (34)$:

$$\left(\partial_t^2 + N^2\right) w - \breve{f} u_t + p_{zt} = -\partial_t \left(\boldsymbol{u} \cdot \boldsymbol{\nabla} w\right) - \boldsymbol{u} \cdot \boldsymbol{\nabla} b \,. \tag{37}$$

The 'divergence equation' follows from $-\partial_x(31) - \partial_y(32)$ and using $u_x + v_y = -w_z$,

$$w_{zt} + f\omega - uf_y - \breve{f}w_x - \triangle_h p = \partial_x\big(\boldsymbol{u}\cdot\boldsymbol{\nabla}u\big) + \partial_y\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}v\right)\,. \tag{38}$$

The vertical vorticity equation is obtained from $\partial_x(32) - \partial_y(31)$,

$$\omega_t - fw_z + vf_y = -\boldsymbol{\nabla}_\perp\cdot\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u}\,. \tag{39}$$

Yes! In the penultimate step we calculate $\partial_z\partial_t(37) - f\partial_z(38)$, yielding

$$\Big[\left(\partial_t^2 + f^2\right)\partial_z^2 - \breve{f}\partial_x\partial_z\partial_t\Big]w - \triangle_h p_{zt} = f_y\partial_z\left(u_t + fv\right) + \partial_z\left(\partial_t\boldsymbol{\nabla} + f\boldsymbol{\nabla}_\perp\right)\cdot\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u}_h\,. \tag{40}$$

Finally, the combination $\triangle_h(36) + (39)$ yields the wave operator form,

$$\begin{aligned}\Big[\triangle\partial_t^2 + f^2\partial_z + N^2\triangle_h\Big]w &= \breve{f}\partial_t\left(\triangle_h u + w_{xz}\right) + u_t\breve{f}_{yy} + 2u_{yt}\breve{f}_y + f_y\partial_z\left(u_t + fv\right)\\ &+ \partial_z\left(\partial_t\boldsymbol{\nabla} + f\boldsymbol{\nabla}_\perp\right)\cdot\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u} - \triangle_h\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}b\right) - \triangle\partial_t\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}w\right)\,.\end{aligned} \tag{41}$$