Computational Geometry 2nd Set

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Exercise 1

Prove the Meister-Gauss formula for the signed area of a simple k-gon $(p_0, p_1, \ldots, p_{k-1}, p_k)$ (where $p_0 = p_k$), that is,

$$E = \frac{1}{2} \sum_{i=0}^{k-1} p_i \times p_{i+1}.$$

Hint: Prove the formula for $k = 3$ directly from the definition, and use that for the case $k > 3$.

Solution

Prove for $k = 3$

Let the vertices of the triangle be $p_0 = (x_0, y_0)$, $p_1 = (x_1, y_1)$, and $p_2 = (x_2, y_2)$. The signed area of a triangle can be computed using the determinant formula:

$$E = \frac{1}{2} \left| x_0 y_1 + x_1 y_2 + x_2 y_0 - (y_0 x_1 + y_1 x_2 + y_2 x_0) \right|.$$

Rewriting this in terms of summation notation:

$$E = \frac{1}{2} \sum_{i=0}^{2} (x_i y_{i+1} - y_i x_{i+1}),$$

where cyclic indexing ensures that $p_3 = p_0$.

Thus, for $k = 3$, the signed area matches the formula:

$$E = \frac{1}{2} \sum_{i=0}^{k-1} p_i \times p_{i+1}.$$

This proves the formula for a triangle.

Generalize for $k > 3$

For a simple polygon with $k > 3$ vertices, we can divide the polygon into $(k-2)$ triangles. Each triangle formed contributes to the total signed area.

Let us consider a triangle formed by vertices p_0, p_i, and p_{i+1} for $i = 1, 2, \ldots, k-1$. The signed area of each such triangle is given by:

$$E = \frac{1}{2} (x_0 y_i + x_i y_{i+1} + x_{i+1} y_0 - (y_0 x_i + y_i x_{i+1} + y_{i+1} x_0)).$$

Summing over all triangles formed by p_0, p_i, and p_{i+1} for $i = 1, 2, \ldots, k-1$, we obtain:

$$E = \frac{1}{2} \sum_{i=0}^{k-1} (x_i y_{i+1} - y_i x_{i+1}),$$

where cyclic indexing ensures that $p_k = p_0$.

This matches the Meister-Gauss formula:

$$E = \frac{1}{2} \sum_{i=0}^{k-1} p_i \times p_{i+1}.$$

Exercise 2

Prove that there are only five types of convex, regular polyhedra in R^3 (the so-called Platonic solids).

Solution

For any convex polyhedron, Euler's formula holds:

$$V - E + F = 2,$$

where V, E, and F are the number of vertices, edges, and faces, respectively.

For a Platonic solid:

1. Each face is a regular p-sided polygon, and there are F faces.
2. Each vertex is surrounded by q edges (or equivalently q faces).

From these properties:

- The total number of edges is shared between faces:

  $$pF = 2E,$$

  because each edge belongs to two faces.

- Similarly, the total number of edges is shared between vertices:

  $$qV = 2E,$$

  because each edge connects two vertices.

Using these relationships, we can express F and V in terms of E, p, and q:

$$F = \frac{2E}{p}, \quad V = \frac{2E}{q}.$$

Substituting into Euler's formula gives:

$$\frac{2E}{q} - E + \frac{2E}{p} = 2.$$

Simplifying:

$$\frac{1}{q} + \frac{1}{p} = \frac{1}{2} + \frac{1}{E}.$$

Since $E > 0$, the term $\frac{1}{E}$ is positive, so:

$$\frac{1}{q} + \frac{1}{p} > \frac{1}{2}.$$

Both $p \geq 3$ (faces must be polygons with at least three sides) and $q \geq 3$ (at least three edges meet at each vertex). Using the inequality $\frac{1}{q} + \frac{1}{p} > \frac{1}{2}$, we test possible values for p, q:

- For $p = 3, q = 3$:
$$\frac{1}{3} + \frac{1}{3} = \frac{2}{3} > \frac{1}{2}.$$
This corresponds to the tetrahedron.

- For $p = 4, q = 3$:
$$\frac{1}{4} + \frac{1}{3} = \frac{7}{12} > \frac{1}{2}.$$
This corresponds to the cube.

- For $p = 3, q = 4$:
$$\frac{1}{3} + \frac{1}{4} = \frac{7}{12} > \frac{1}{2}.$$
This corresponds to the octahedron.

- For $p = 5, q = 3$:
$$\frac{1}{5} + \frac{1}{3} = \frac{8}{15} > \frac{1}{2}.$$
This corresponds to the dodecahedron.

- For $p = 3, q = 5$:
$$\frac{1}{3} + \frac{1}{5} = \frac{8}{15} > \frac{1}{2}.$$
This corresponds to the icosahedron.

Exclusion of Other Cases

For values of $p, q > 5$, the inequality fails: - For example, if $p = q = 6$:

$$\frac{1}{6} + \frac{1}{6} = \frac{1}{3},$$

which does not satisfy the inequality.

Thus, no other combinations are possible.

Exercise 3

You are given a polygonal area with a boundary of n vertices and with h holes. The area is not necessarily connected. Compute the number of diagonals and the number of triangles for any triangulation.

Solution

To analyze the triangulation of a polygon with holes and disconnected components, we begin by considering each disconnected part separately. Let d represent the number of disconnected components, with each component i containing v_i vertices such that:

$$\sum_{i=1}^{d} v_i = n.$$

For each hole in component i, two bridges are added to isolate the hole. Each bridge introduces 4 new vertices (2 per endpoint), resulting in:

$$v_i' = v_i + 4h_i,$$

where h_i is the number of holes in component i. After separation, each component splits into $h_i + 1$ disjoint simple polygons. Let their vertex counts be $v_{i,0}, v_{i,1}, \ldots, v_{i,h_i}$, satisfying:

$$\sum_{j=0}^{h_i} v_{i,j} = v_i' = v_i + 4h_i.$$

For a simple polygon with k vertices, the triangulation yields $k - 2$ triangles. Applying this to each subdivided polygon in component i:

$$T_i = \sum_{j=0}^{h_i} (v_{i,j} - 2) = (v_i + 4h_i) - 2(h_i + 1).$$

Simplifying:

$$T_i = v_i + 2h_i - 2.$$

Each triangle has 3 edges, but only diagonals are counted once. For component i:

$$D_i = \frac{3T_i - v_i'}{2} = \frac{3(v_i + 2h_i - 2) - (v_i + 4h_i)}{2} = v_i + 2h_i - 3.$$

Summing over all d components:

Total Triangles

$$T_{\text{total}} = \sum_{i=1}^{d} (v_i + 2h_i - 2) = n + 2h - 2d.$$

Total Diagonals

$$D_{\text{total}} = \sum_{i=1}^{d} (v_i + 2h_i - 3) = n + 2h - 3d.$$

Exercise 4

Prove that a polygonal gallery with n vertices and h holes can be guarded by at most $\lfloor \frac{n+2h}{3} \rfloor$ guards.

Solution

First, we triangulate the polygonal gallery with n vertices and h holes. To handle the holes properly, we use the same idea as exercise 3.

We add two bridges per hole, where each bridge connects either the outer boundary to a hole or connects two holes.After bridging, we can triangulate the resulting structure. From Exercise 3, we know that a polygon with n vertices and h holes yields $n + 2h - 2$ triangles in any triangulation.As in the standard gallery problem without holes the vertices of our triangulated polygon can be assigned one of three colors such that no two adjacent vertices share the same color. Let us name these three color classes as *blue*, *red*, and *green*. The key property of this coloring is that every triangle in our triangulation has exactly one vertex from each color class. We now place guards at all vertices belonging to the smallest of the three color classes. Since we have n original vertices plus vertices corresponding to the h holes, we have a total of $n + 2h$ guard positions to consider.

When we distribute these $n + 2h$ positions among three color classes,at least one color class contains at most $\lfloor \frac{n+2h}{3} \rfloor$ vertices.

Formally, if $|blue|$, $|red|$, and $|green|$ represent the number of vertices in each color class, then:

$$|blue| + |red| + |green| = n + 2h$$

Therefore, at least one of $|blue|$, $|red|$, or $|green|$ is at most $\lfloor \frac{n+2h}{3} \rfloor$.

Exercise 5

Present an efficient algorithm which, given a set $P = \{p_1, \ldots, p_n\}$ of n points on the integer grid Z^2, computes the polygon of minimum perimeter that encloses these points, subject to the condition that the sides of this enclosure can be horizontal, vertical, or sloped at $\pm 45°$ (see Figure).
Hint: $\mathcal{O}(n)$ time is possible, but $\mathcal{O}(n \log n)$ time acceptable.
Prove that the enclosure produced by your algorithm has the minimum perimeter. Note that there may generally be many enclosures with the same perimeter, and your algorithm may output any of them.

Solution

We can construct the polygon using the extreme points from set P.

- Points with minimum and maximum x-coordinates.
- Points with minimum and maximum y-coordinates.

- Points farthest above or below diagonal lines with slopes $\pm 45°$:
  - For slope $+45°$: Identify points with maximum and minimum $y - x$ values.
  - For slope $-45°$: Identify points with maximum and minimum $y + x$ values.

These extreme points can be found in a single pass through all n points, which takes $O(n)$ time.

Then using the extreme points we can:

1. Connect these points in clockwise or counterclockwise order to form a polygon.

2. Ensure that all edges conform to the allowed slopes (horizontal, vertical, or $\pm 45°$).

To ensure that all points in P are enclosed we can check each point against the polygon's edges using the winding number algorithm that guarantees at O(n) time. The perimeter of the polygon is calculated by summing up the lengths of its edges. Since there are a constant number of edges, this step takes constant time.

The algorithm guarantees that the resulting polygon has the minimum perimeter because the extreme point define the smallest enclosing region and also the winding number algorithm ensures that no points lies outside the polygon.