Computational Geometry 4th set

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Exercise 1

Prove that any triangulation where every triangle is acute (all angles are less than $\pi/2$) is Delaunay.

Solution

Definitions:

1. Delaunay Validity Condition: For a quadrilateral $ABCD$ with edge AB shared by two triangles, AB is Delaunay-valid if $\theta_A + \theta_B \geq \theta_C + \theta_D$.
2. Quadrilateral Angle Sum: In any planar quadrilateral, $\theta_A + \theta_B + \theta_C + \theta_D = 2\pi$.

Lets consider two adjacent acute triangles ABC and ABD sharing edge AB, forming quadrilateral $ABCD$. All interior angles in both triangles are less than $\frac{\pi}{2}$.

- **1. Angle Sum in Quadrilateral:**

$$\theta_A + \theta_B + \theta_C + \theta_D = 2\pi$$

- **2. Acuteness Implies:**

$$\theta_C \leq \frac{\pi}{2}, \quad \theta_D \leq \frac{\pi}{2}$$

  so,

$$\theta_C + \theta_D \leq \pi$$

- **3. Substitute into Angle Sum:**

$$\theta_A + \theta_B = 2\pi - (\theta_C + \theta_D) \geq 2\pi - \pi = \pi$$

- **4. Validity Condition:** Since $\pi \geq \theta_C + \theta_D$,

$$\theta_A + \theta_B \geq \theta_C + \theta_D$$

  This satisfies the Delaunay validity condition for edge AB.

Every edge in the triangulation satisfies the Delaunay condition. Therefore, any triangulation where every triangle is acute is Delaunay.

Exercise 2

Given a convex polyhedron P in $\mathbb{R}^3$ with n vertices, describe a data structure that can be used to decide in time $O(\log n)$ whether a query point $q \in \mathbb{R}^3$ lies inside P. The data structure must use $O(n)$ space.

Hint: try to reduce the problem to point location.

Solution

To reduce the problem to point location, we will use projection into $\mathbb{R}^2$. First, we will triangulate the polyhedron faces (each polygon of P). This process will take $O(n \log n)$ per polygons(n in total). Afterward, following the idea of Mulmuley's method, we will create a DAG (Directed Acyclic Graph). This will be achieved by projecting the triangulated polygons of P into the plane and building a tree with random insertions.

Theorem: For any S, there exists a Mulmuley's structure on S that requires space $O(n)$ and has query time $O(\log n)$ in the worst case. It can be constructed in expected $O(n \log n)$ time.

The theorem above helps us determine that the construction cost will be $O(n^2 \log n)$.

Query Processing: Once the structure is built, answering whether a query point q lies inside the polyhedron involves projecting q to $\mathbb{R}^2$ and performing point location in $O(\log n)$ time with high probability. After locating the corresponding triangle in the 2D subdivision, we perform a constant-time $O(1)$ check to determine whether q lies above or below the plane defined by that triangle in 3D space. Therefore, the total query time is $O(\log n)$ with high probability and space cost is $O(n)$.

Exercise 3

Consider a set of n sites, $P \subseteq \mathbb{R}^2$. The Euclidean Minimum Spanning Tree (EMST) is any minimum spanning tree of the complete graph with vertices the points in P and edge weight of edge pq the Euclidean distance $\|p - q\|_2$.

Prove that every edge of a EMST is an edge in every Delaunay Triangulation of P.

Hint: Consider the maximum empty circle of the midpoint of pq for every $pq \in$ EMST.

Hint 2: You may need to refresh the cycle property of MST, from your algorithms course.

Solution

Lets consider $|pq| = \|p - q\|_2$ for any $p, q \in \mathbb{R}^2$. We will show by contradiction that if pq is an edge of the EMST, then pq must also be an edge in the Delaunay triangulation of P.

Suppose that pq is an edge of the EMST but is not an edge in the Delaunay triangulation. By definition, this means there is no empty circle passing through both p and q that contains no other points of P in its interior.

Let r be the midpoint of pq. Consider the largest possible circle centered at r that does not contain any points of P in its interior. The radius of this circle is $|pq|/2$, and both p and q lie on its boundary.

Since pq is not a Delaunay edge, there must exist at least one other point $k \in P$ (with $k \neq p, q$) that also lies on the boundary of this circle.

From the Pythagorean theorem, since r is the midpoint and both $|rp|$ and $|rk|$ are equal to the radius $|pq|/2$, we derive:

$$|kp|^2 = |rp|^2 + |rk|^2 = \left(\frac{|pq|}{2}\right)^2 + \left(\frac{|pq|}{2}\right)^2 = \frac{|pq|^2}{2}$$

Thus,

$$|kp| = \frac{|pq|}{\sqrt{2}} < |pq|$$

This means the edge kp is strictly shorter than pq.

Cycle property of minimum spanning trees: in any cycle, the edge with the largest weight cannot be part of the MST.

If we remove pq from the EMST, the tree splits into two components. Since k is on the circle, it must be in one of these components. Connecting k and p (or k and q) with the shorter edge kp (or kq) would yield a new spanning tree with smaller total length, contradicting the minimality of the EMST.

Therefore, our assumption must be false, and every edge of the EMST must indeed be a Delaunay edge.

Exercise 4

Let S be a finite set. Consider the lattice of 2^S as a directed graph, where the vertices are the elements of 2^S and for any two vertices $A, B \subseteq S$, there is an edge $A \to B$ iff $B = A \cup \{s\}$ for some $s \notin A$.

Assume that some edges are marked, and the number of marked incoming edges to a node A are a function of $|A|$.

Given a path from $\emptyset$ to S, define X_i to be the indicator variable of event "edge i on the path is marked". If all paths from $\emptyset$ to S are equiprobable, prove that all X_i are mutually independent.

Hint: the sketch of the proof is in the slides

Solution

Following the proof from Lecture 7 page 23/24.

Let $\sigma_i \in \binom{[n]}{i}$ denote any subset of i elements from $[n]$.
We have to demonstrate that the probability of the event

$$X_1 = a_1 \wedge X_2 = a_2 \wedge \cdots \wedge X_i = a_i$$

remains unchanged despite of which particular subset σ_i is reached after i steps. In other words, the joint distribution of $(X_1, \ldots, X_i)$ does not depend on the specific subset of size i or the sequence of choices that led there.

$$P(X_i = 1 \mid X_{i-1}, \ldots, X_1) = P(X_i = 1)$$

The proof sketch suggests to prove this statement by induction.

- **Base case:** When $i = 1$, the process begins at the empty set $\emptyset$, and the first element σ_1 is picked uniformly at random from S. Since the marking procedure relies solely on the step index (i.e., the cardinality of the current set $|A|$), the random variable X_1 does not depend on which particular element was chosen. Therefore, X_1 is independent of the choice of σ_1.
- **Inductive step:** Suppose the statement holds for all indices less than i. Now, consider the event $X_1 = a_1 \wedge \cdots \wedge X_i = a_i$. Any subset σ_i of size i can be constructed by adding some element $s \in \sigma_i$ to a subset $\sigma_{i-1} = \sigma_i \setminus \{s\}$ of size $i-1$. Because every possible sequence of insertions is equally likely, each pair (s, σ_i) with $s \in \sigma_i$ occurs with the same probability. Thus, the value of X_i depends only on the element s added and the current step i, and is unaffected by the particular path taken to reach σ_{i-1}.

Consequently, the probability of observing $X_1 = a_1, \ldots, X_i = a_i$ does not depend on the specific subset σ_i. This means X_i is independent of $(X_1, \ldots, X_{i-1})$. By the principle of induction, it follows that $X_1, \ldots, X_n$ are all mutually independent.

Exercise 5

1. Consider the ordered binary tree over a set $S \subseteq \mathbb{N}$, built by repeated insertion, where the elements of S are inserted by a permutation picked uniformly at random. Prove that the height of the tree is $O(\log n)$ w.h.p.

   Hint: follow the logic of Mulmuley's structure

2. Prove that the variant of QuickSort where the pivot is selected uniformly at random each time, terminates in time $O(n \log n)$ w.h.p.

   Hint: use the claim of part 1.

Solution

The proof of the above result relies on the following lemma:

Lemma

Given S, and query point q, for any $\lambda > 0$, let C_q be the length of the search path of q in Mulmuley's structure. Then,

$$\Pr\left[C_q \geq 3\lambda \ln(n+1)\right] \leq \frac{1}{(n+1)^{\lambda \ln 1.25 - 1}}.$$

1. When building a binary search tree by inserting elements in a completely random order (uniformly at random), the path you take to place each element inside the tree is the same as the path you would use to search for that element later on. Because the height of the tree is just the longest such path from the root to any node, any result about the length of search paths in a randomly built tree also applies to the height of the tree itself. Thus we can apply the same bound to the height of the tree as in any search path. Lets consider a point in S and H_n the height of the node that corresponds to the point that we considered. Applying the lemma above we get:

   $$\Pr\left[H_n \geq 3\lambda \ln(n+1)\right] \leq \frac{1}{(n+1)^{\lambda \ln 1.25 - 1}}.$$

   By choosing the right λ we derive to the following:

   $$\Pr\left[H_n \geq a \log n\right] \leq \frac{1}{n^x}, \quad a, x \geq 0$$

   Therefore, with high probability, the height H_n of a binary search tree built by inserting n elements in a uniformly random order is $O(\log n)$.

2. The variant of Quicksort that selects a pivot uniformly at random uses the same reasoning as in the previous analysis. We observe that the height of the recursion tree in Quicksort behaves similarly to the randomly constructed BST from the earlier discussion, as both split the input based on random choices at each step. Since we proved that the height of such a tree is $O(\log n)$ with high probability (w.h.p.), and each level of the recursion process partitions subarrays in $O(n)$ time, this variant of Quicksort achieves $O(nlogn)$ time complexity w.h.p.