11.8a a) $S_n = -1+2-3+\dots+(-1)^n n$

$S_1 = -1$

$S_{n+1} - S_n = -1+2-3+\dots+(-1)^n n + (-1)^{n+1}(n+1) - (-1+2-3+\dots+(-1)^n n) =$

$= (-1)^{n+1}(n+1)$

$$\begin{cases} S_{n+1} = S_n + (-1)^{n+1}(n+1), \ n \geq 2 \\ S_1 = -1 \end{cases}$$

11.8b $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{(n-1)n}$

$S_2 = \frac{1}{(2-1)2} = \frac{1}{2}$

$S_n - S_{n-1} = \frac{1}{1\cdot 2} + \dots + \frac{1}{(n-2)(n-1)} + \frac{1}{(n-1)n} - \left(\frac{1}{1\cdot 2} + \dots + \frac{1}{(n-2)(n-1)}\right) = \frac{1}{(n-1)n}$

$$\begin{cases} S_2 = \frac{1}{2} \\ S_n = \frac{1}{(n-1)n} + S_{n-1}, \ n \geq 3 \end{cases}$$

11.8c $S_n = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \dots + \frac{(-1)^n(n-1)}{n}$

$S_1 = 0 \quad S_2 = \frac{1}{2}$

$S_{n+1} - S_n = \frac{1}{2} - \frac{2}{3} + \dots + \frac{(-1)^n(n-1)}{n} + \frac{(-1)^{n+1}(n-1+1)}{(n+1)} - \left(\frac{1}{2} - \dots + \frac{(-1)^n(n-1)}{n}\right) =$

$= \frac{(-1)^{n+1} \cdot n}{n+1}$

$$\begin{cases} S_{n+1} = S_n + \frac{(-1)^{n+1} \cdot n}{n+1}, \ n \geq 3 \\ S_2 = \frac{1}{2} \end{cases}$$

11. a) $P_n = \prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right)$

$P_2 = \frac{3}{4}$

$$\frac{P_{n+1}}{P_n} = \frac{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right)}{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)} = 1 - \frac{1}{(n+1)^2}$$

$$\begin{cases} P_{n+1} = P_n\left(1 - \frac{1}{(n+1)^2}\right) & n \geq 3 \\ P_2 = \frac{3}{4} \end{cases}$$

b) $P_n = \prod_{i=1}^{n} \left(2 + \frac{1}{i!}\right)$

$P_1 = 3$

$$\frac{P_{n+1}}{P_n} = \frac{\left(2+\frac{1}{1!}\right)\left(2+\frac{1}{2!}\right)\cdots\left(2+\frac{1}{n!}\right)\left(2+\frac{1}{(n+1)!}\right)}{\left(2+\frac{1}{1!}\right)\left(2+\frac{1}{2!}\right)\cdots\left(2+\frac{1}{n!}\right)} = 2 + \frac{1}{(n+1)!}$$

$$\begin{cases} P_{n+1} = P_n\left(2 + \frac{1}{(n+1)!}\right) & n \geq 2 \\ P_1 = 3 \end{cases}$$

c) $P_n = \prod_{i=1}^{n} \frac{i+1}{i+2} \quad P_1 = \frac{2}{3}$

$$\frac{P_{n+1}}{P_n} = \frac{\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n}{n+1}\cdot\frac{n+1}{n+2}}{\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n}{n+1}}$$

$$\begin{cases} P_1 = \frac{2}{3} \\ P_{n+1} = \frac{n+2}{n+3} P_n \end{cases} \quad \left\{P_{n+1} = \frac{n+1}{n+2} P_n\right\} \quad n \geq 2$$

11.7 a) $X_k = \dfrac{x^{2k+1}}{(2k+1)!}$

$X_0 = \dfrac{x^1}{1} = x$

$k \geq 1$:

$$\frac{x_k}{x_{k-1}} = \frac{x^{2k+1}}{(2k+1)!} \cdot \frac{(2k-1)!}{x^{2k-1}} = \frac{x^2 \cdot x^{2k-1}(2k-1)!}{(2k-1)!(2k)(2k+1)\,x^{2k-1}} =$$

$$\begin{cases} X_k = \dfrac{x^2}{(2k+1)(2k)} \cdot X_{k-1}, & k \geq 1 \\ X_0 = x, & k = 0 \end{cases}$$

b) $X_k = \dfrac{(-1)^k x^k}{k}, \quad k \geq 1$

$X_1 = -x$

$$\frac{x_k}{x_{k-1}} = \frac{(-1)^k x^k}{k} \cdot \frac{k-1}{(-1)^k(-1)^{-1} x^{k-1}} = -\left(\frac{k-1}{k}\right) \cdot x$$

$$\begin{cases} x_1 = -x \\ x_k = (x_{k-1}) \cdot \left(-\dfrac{(k-1)}{k} \cdot x\right), & k \geq 2 \end{cases}$$

c) $X_k = \dfrac{(-1)^k x^k}{(k^2+k)!} \qquad X_0 = 1$

$$\frac{x_k}{x_{k-1}} = \frac{(-1)^k x^{k-1} \cdot x \cdot (k^2-k)!}{(k^2+k)! \cdot (-1)^k (-1)^{-1}\, x^{k-1}} = \frac{x\,(k^2-k)!}{-(k^2+k)!}$$

$$\begin{cases} x_k = -\dfrac{x_{k-1} \cdot x \cdot (k^2-k)!}{(k^2+k)!} \\ x_0 = 1 \end{cases}$$

11.7 d $\quad x_k = (k+1)\frac{x^k}{k!}, \quad k \geq 0$

$x_0 = 1$

$$\frac{x_k}{x_{k-1}} = \frac{(k+1)x^{k-1} \cdot x \cdot (k-1)!}{k(k-1)! \; k x^{k-1}} = \frac{k+1}{k^2} \cdot x$$

$$\begin{cases} x_k = \frac{k+1}{k^2} \cdot x \cdot x_{k-1}, \quad k \geq 1 \\ x_0 = 1 \end{cases}$$