

GSP-Calculus

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Abstract.

1 Abstract Global Sequence Protocol Language

1.1 Syntax

We assume the following countable sets of vertex names \mathbb{V} ranged over by $v; v_0; v_1, \dots$; We shall use α, β to denote sequence of updates \mathbb{U} ranged over by $u^v, u_0^{v_0}, u_1^{v_1}, \dots$; Let $\mathcal{U} = u_0^{v_0} \cdot u_1^{v_1} \cdot \dots \cdot u_n^{v_n}$. For the sake of clarity, we shall only use vertices as update's decoration whenever it would be necessary.

Let ρ, σ be sequence defined as $\varepsilon \mid [\alpha] \cdot \rho$. We use ε to denote the empty sequence, i.e., the sequence of length 0.

Let ρ and α be sequence, we write \setminus as relative complement of ρ in α , i.e. $\rho \setminus \alpha = [u_i \mid u_i \in \rho \wedge u_i \notin \alpha]$.

Definition 1.1 (Global Sequence Protocol Language). *The set of tgsp-calculus expressions is defined by the following syntax:*

$$\begin{array}{ll}
 \text{(SYSTEM)} & N := C \parallel S \\
 \text{(CLIENT)} & C := 0 \mid \{P, k, \rho, \alpha, \sigma, j\} \mid C \parallel C \quad \text{where } k \text{ and } j \in \mathbb{N} \\
 \text{(STATE)} & S := \varepsilon \mid S \cdot \rho \\
 \text{(PROGRAM)} & P := \text{update}(u); P \mid \text{let } x = \text{read}(v); P \mid \text{pull}(); P \mid \text{push}(); P \mid \\
 & \quad \text{let } x = \text{confirmed}(); P \mid \text{while}(\text{cond}) \text{ do } P
 \end{array}$$

We model the system N as several clients interacting with a message queue S . This message queue will represent the program state. We formalize an i^{th} client as a tuple $\{P, k, \rho, \alpha, \sigma, j\}_i$ where P , called Program, is a set of programming primitives that allow programmer to interact with the distributed store by invoking of actions on objects, a index k which denotes the maximum index of update that reading in S , a sequence of updates sent but unconfirmed their reception ρ , a transaction buffer α , a sequence of all the updates sent by the client to the RTOB σ and finally j which represents a amount of received updates.

1.2 Operational Semantics

The operational semantics of *tgsp-calculus* is defined by a labeled transition system over well-formed terms, up-to the structural congruence.

The labeled transition system considers the following actions:

$$\alpha ::= \tau \mid read(r) \mid update(u) \mid pull() \mid push() \mid confirmed()$$

(T-READ)

$$\frac{rvalue(r, flatten(S[0..k-1] \cdot \rho) \cdot \alpha) = v}{\{let\ x = read(r); P, k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S \xrightarrow{read(r)} \{P[x \mapsto v], k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S}$$

(T-UPDATE)

$$\{update(u); P, k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S \xrightarrow{update(u)} \{P, k, \rho, \alpha \cdot u, \sigma, j\}_i \parallel C \parallel S$$

(T-PUSH)

$$\{push(); P, k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S \xrightarrow{push()} \{P, k, \rho \cdot [\alpha], \epsilon, \sigma \cdot [\alpha], j\}_i \parallel C \parallel S$$

(T-PULL)

$$\{pull(); P, k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S \xrightarrow{pull()} \{P, k+j, \rho \setminus S[k..k+j], \alpha, \sigma, 0\}_i \parallel C \parallel S$$

(T-CONFIRMED)

$$\{let\ x = confirmed(); P, k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S \xrightarrow{confirmed()} \{P[x \mapsto \rho \neq [] \vee \alpha \neq \epsilon], k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S$$

(T-WHILE-TRUE)

$$\frac{\neg(cond \downarrow) = false}{\{while(cond) \mathbf{do} P, k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S \xrightarrow{\tau} \{P; while(cond) \mathbf{do} P, k, \rho, \alpha, \sigma, j+1\}_i \parallel C \parallel S}$$

(T-WHILE-FALSE)

$$\frac{\neg(cond \downarrow) = true}{\{while(cond) \mathbf{do} P; Q, k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S \xrightarrow{\tau} \{Q, k, \rho, \alpha, \sigma, j+1\}_i \parallel C \parallel S}$$

(T-RECEIVE)

$$\frac{k+j+1 \leq |S|}{\{P, k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S \xrightarrow{\tau} \{P, k, \rho, \alpha, \sigma, j+1\}_i \parallel C \parallel S}$$

(T-PROCESS)

$$\{P, k, \rho, \alpha, [\beta] \cdot \sigma, j\}_i \parallel C \parallel S \xrightarrow{\tau} \{P, k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S \cdot \beta$$

The meaning of τ is standard. A *tgsp-calculus* program is given by a Labeled Transition System (LTS) before introducing. Transitions for systems are labeled by actions α as usual.

Rule (T-READ) states that a client i can perform an action $read(r)$, substituting each occurrence of x in P with the value v . Notice that v is got from the queue message and the client's internal queues. Rule (T-UPDATE) describes effects performed by an update

operation. The update u is saving in the temporal queue α . After (T-PUSH) the content of the transaction queue α is moved to the pending queue, ρ , and the sent queue, σ . Rule (T-PULL) increases the amount of updates that client i can read in the message queue S . This increase is given by the amount of update receiving. The pending queue removes those updates which belong between index k and the amount of update receiving j in S . Besides, j is restarted in 0. (T-CONFIRMED) sets x in false whether updates are in pending queue or transaction queue, true in otherwise. Rule (T-WHILE-TRYE) and (T-WHILE-FALSE) are standard. Rule (T-RECEIVE) states that a counter j is incremented (by 1). Finally, the last one rule, (T-PROCESS), moves updates from sent queue to the message queue.

Notation. Let f and g be a partial function, we define the update operator $[-]$ such that $dom(f[g]) = dom(f) \cup dom(g)$ and

$$f[g](x) = \begin{cases} f(x) & \text{if } x \notin dom(g) \wedge x \in dom(f) \\ g(x) & \text{if } x \in dom(g) \\ \perp & \text{Otherwise} \end{cases}$$

We write $[x_1 \mapsto y_1, \dots, x_n \mapsto y_n]$ for the partial function f such that $dom(f) = \{x_1, \dots, x_n\}$ and $f(x_i) = y_i$; $A \setminus B$ to denote the usual difference of sets.

2 Implementation of GSP

The GSP model introduced in the previous section is an abstract protocol which does not consider implementation details. Problems like space explosion caused by the growing of the update sequence have to allow for implementing these protocol.

Instead of storing update sequence, the implementation to present reduce such sequence to in two different objects. *State* when the update sequence reduced represents a prefix of the global queue message or *Delta* when the update sequence reduced represents a interval or segment of the global queue message.

Let \mathcal{N} be a set of clients' name ranged over by b, b_1, b_2, \dots ; $GSPrefix$ be a set over sets $State \times \mathcal{N} \rightarrow \mathbb{N}$, $GSSegment$ be a set operation over sets $Delta \times \mathcal{N} \rightarrow \mathbb{N}$ and $Round$ the set over sets $\mathcal{N} \times \mathbb{N} \times Delta$; θ is a fresh element of $GSSegment$.

Furthermore, we assume the following parametric functions defined over the abstract data types above borrowed from[1]:

```

const    initialstate : State
function read        : Read  $\times$  State  $\rightarrow$  Value
function apply       : State  $\times$  Delta*  $\rightarrow$  State
const    emptydelta  : Delta
function append      : Delta  $\times$  Update  $\rightarrow$  Delta
function reduce      : Delta*  $\rightarrow$  Delta

```

Let *State* be a set of state ranged over by s, s_1, s_2, \dots ; We shall use δ to denote sequence of *Delta* ranged over by $d, d_0, d_1 \dots$; Let ρ, σ be sequence defined:

$\rho, \sigma := \varepsilon \mid [\delta] \cdot \rho$

We assume the following countable sets of persisted state $GSPrefix$ ranged over by ps, ps_1, \dots ;

Let in_s be a partial function in $\mathcal{N} \rightarrow \mathbb{N} \times Delta$

Let out_s be a partial function in $\mathcal{N} \rightarrow GSsegment^* \cup GSPrefix$

2.1 Auxiliar Function

Before describing the implementation of GSP, we will introduce the following auxiliars functions which will be used when we define the operational semantic.

```

append :: GSsegment × Round* → GSsegment
append(<δ, mr>, ε) = <δ, mr>
append(<δ, mr>, <i, n0, δ0>:rs) = append(<reduce(δ · δ0 · ε), mr[b0 ↦ n0]>,
rs)

apply :: GSPrefix × GSsegment → GSPrefix
apply(<ps, mr>, <δ, mr'>) = <apply(ps, δ · ε), mr[mr']>

receivedrounds :: (N → <N × Delta>*) → Round*
receivedrounds(⊥) = ε
receivedrounds(b → <n0, δ0> · f) = <b, n0, δ0> · receivedrounds(f)

curstate :: State × Round* × Delta × Delta → State
curstate(ps, p, pb, tb) = apply(ps, getdeltas(p) · pb · tb)

getdeltas :: Round* → Delta*
getdeltas(ε) = ε
getdeltas(<n0, δ0> · δ) = δ0 · getdeltas(δ)

remove :: Delta* → Delta*
remove(rs, ε) = ε
remove(rs, xs) = (foldr(\x rec ys → h (filter (/=x)ys)) id)xs rs

```

The functions **reduce** and **apply** are abstract and depend on the data model used.

2.2 Syntax

Definition 2.1 (Implementation of GSP). *The syntax of clients and server is given by the following grammar*

```

(SYSTEM)   N ::= 0 ∣ S ∥ C
(SERVER)   S ::= 0 ∣ {ps, ins, outs}
(CLIENTS)  C ::= ∅ ∣ {P, <State, ρ, δ, GSsegment ∪ GSPrefix, N, inc, outc>} ∣ C ∥ C
(PROGRAM)  P ::= 0 ∣ let x = read(r); P ∣ update(u); P ∣ push(); P ∣ pull(); P

```

The implementation of a system is a server and clients interacting concurrently. A server will be represented with a persisted state denoted by ps , input messages in_s and output messages out_s . We will refer to clients, as tuple of program P and set of states and sequences E . A program client stores a state ($E.k$), a pending queue ($E.p$), i.e., updates sent to the server without confirmation of their reception, a push buffer ($E.pb$) which holds updates that were pushed by the client but have not been sent to the server, a transaction buffer ($E.tb$) which holds updates for sending with another ones, a receive buffer ($E.GSSegment \cup GSPrefix$) for updates which were sent by the server, the number of round sent ($E.n$), a input message queue ($E.in_c$) and a output message queue ($E.out_c$).

2.3 Operational Semantics

The operational semantics of *gsp-calculus* is defined by a labeled transition system over well-formed terms, up-to the structural congruence.

The labeled transition system considers the following actions:

$$\alpha ::= \tau \mid read(r) \mid update(u) \mid pull() \mid push() \mid confirmed()$$

SERVER

$$\begin{array}{c}
\text{(DROP-CONN)} \\
\frac{b_i \in in_s \quad b_i \in out_s}{\{ps, in_s, out_s\} \xrightarrow{\dagger(b_i)} \{ps, in_s \setminus b_i, out_s \setminus b_i\}} \\
\\
\text{(CRASH-AND-RECOVER)} \\
\{ps, in_s, out_s\} \xrightarrow{\ddagger} \{ps, \perp, \perp\} \\
\\
\text{(BATCH)} \\
\frac{rs = receivedRounds(in_s) \quad gs = append(\theta, rs) \quad ps' = \mathbf{apply}(ps, gs.delta)}{\{ps, in_s, out_s\} \xrightarrow{\tau} \{ps', in_s, notify(dom(out_s), out_s, gs)\}} \\
\\
\text{(ACCEPT-CONN)} \\
\frac{b_i \notin in_s \quad b_i \notin out_s}{\{ps, in_s, out_s\} \xrightarrow{\oplus(b_i)} \{ps, in_s, out_s[b_i \mapsto ps]\}}
\end{array}$$

COMMUNICATION

$$\begin{array}{c}
\text{(COMM SERVER-CLIENT)} \\
\frac{out_s(i) = gs \cdot gss}{\{ps, in_s, out_s\} \parallel \{P, E\}_i \xrightarrow{\tau} \{ps, in_s, out_s[i \mapsto gss]\} \parallel \{P, E[in_c \mapsto E.in_c \cup \{gs\}]\}_i} \\
\\
\text{(COMM CLIENT-SERVER)} \\
\frac{out_c = \langle i, n_0, \delta_0 \rangle \cdot rs}{\{ps, in_s, out_s\} \parallel \{P, E\}_i \xrightarrow{\tau} \{ps, in_s[i \mapsto \langle i, n_0, \delta_0 \rangle], out_s\} \parallel \{P, E[out_c \mapsto rs]\}_i}
\end{array}$$

CLIENTS

$$\begin{array}{c}
\text{(READ)} \\
\frac{\mathbf{read}(r, \text{curstate}(k, p, pb, tb)) = v}{\{let\ x = read(r); P, E\}_i \xrightarrow{read(r)} \{P[x \mapsto v], E\}_i} \\
\\
\text{(UPDATE)} \\
\{update(u); P, E\}_i \xrightarrow{update(u)} \{P, E[tb \mapsto \mathbf{append}(E.tb, u)]\}_i \\
\\
\text{(PUSH)} \\
\{push(); P, E\}_i \xrightarrow{push()} \{P, E[pb \mapsto \mathbf{reduce}(E.pb \cdot E.tb); tb \mapsto \epsilon; n \mapsto E.n + 1]\}_i \\
\\
\text{(PULL)} \\
\frac{E.in_c(i) \neq \perp \quad E.out_c(i) \neq \perp \quad |E.rb| > 0}{\{pull(); P, E\}_i \xrightarrow{pull()} \{P, E[k \mapsto \mathbf{reducestate}(E.k, E.rb); rb \mapsto \epsilon; E.p \mapsto remove(E.p, E.rb)]\}_i} \\
\\
\text{(CONFIRMED)} \\
\{let\ x = confirmed(); P, E\}_i \xrightarrow{confirmed()} \{P[x \mapsto E.p = \epsilon \vee E.pb = \epsilon \vee E.tb = \epsilon \mapsto \mathbf{true}], E\}_i \\
\\
\text{(WHILE-TRUE)} \\
\frac{\neg(cond \downarrow) = true}{\{while(cond) \mathbf{do}\ P, E\}_i \xrightarrow{\tau} \{P; while(cond) \mathbf{do}\ P, E\}_i} \\
\\
\text{(WHILE-FALSE)} \\
\frac{\neg(cond \downarrow) = true}{\{while(cond) \mathbf{do}\ P; Q, E\}_i \xrightarrow{\tau} \{Q, E\}_i} \\
\\
\text{(SEND)} \\
\{P, E\}_i \xrightarrow{send()} \{P, E[p \mapsto E.p \cdot E.pb; pb \mapsto \epsilon; out_c(i) \mapsto out_c(i) \cdot \langle i, E.n, E.pb \rangle]\}_i \\
\\
\text{(RECEIVE)} \\
\frac{E.in_c = gs_0 \cdot gs_t}{\{P, E\}_i \xrightarrow{receive()} \{P, E[rb \mapsto E.rb \cdot gs_0.gssegment; in_c(i) \mapsto gs_t]\}_i}
\end{array}$$

notify :: $\mathcal{N}^* \times (\mathcal{N} \rightarrow GSSegment^* \cup GSPrefix) \times GSSegment$

notify($\{b\}$, out_s , gs) = $out_s(b) = gss \cdot gs$

notify($b \cdot bs$, out_s , gs) = $out_s(b) = gss \cdot gs \wedge notify(bs, out_s, gs)$

Esto no me convencí,
discutir si hay una
manera mejor

Rule (DROP-CONN) removes from the server's queues the client called b_i . Rule (CRASH-AND-RECOVER) leaves undefined the server's queues and preserves the server's persistent state. Rule (ACCEPT-CONN) show how to add a new connection and send to its the persistent state. Last rule from (SERVER) is Rule (BATCH), the most interesting

of this group. Its has three hypothesis, the first one, is responsible for receiving rounds from the queue of in-messages. Second one, let θ be a empty segment, it gives back a delta object who represents the combination of numbers of rounds into a single object. Finally, this object is applied to the persistent state. As result, the persistent state is putted into the queues messages-out.

There are two (COMMUNICATIONS) rules. Rule (COMM-SERVER-CLIENT) when the server has a message for client i^{th} , this is removed from the server's queue message-out and is putted into the queue message-in from client i^{th} . Rule (COMM-CLIENT-SERVER) states when a round from client i^{th} is left into server queue message-in.

Rule (READ) gives the result of performing a lecture on messages queues from the client. Rule (UPDATE) adds an update to the transaction buffer. Rule (PUSH) leaves into push buffer a delta object resulted of reducing the push buffer with transaction buffer. The transaction buffer is cleaned and the number of rounds sent is incremented by one. Rule (PULL) shows when a persisted state from a client is modified. The hypothesis are that a channel have been accepted, i.e., these must be defined for client i^{th} besides the receive buffer should have an element at least. (CONFIRMED) computes the states of the internal queues, i.e., if these has any element. Rules (WHILE-TRUE) and (WHILE-FALSE) are standard. Rule (SEND) creates a new round setting who is the client (i^{th}), how many rounds client (i^{th}) has sent and content from the push buffer. Finally rule (RECEIVE) move out segments from the client's queue message-in to receive buffer.

3 Equivalence

We have introduced an abstract GSP protocol and a robust streaming server-client implementation of GSP. There exists a relation represented by \triangleleft [ref:] which relates state and delta objects to the update sequences.

- On $\Delta \times \text{Update}^*$, let \triangleleft be the smallest relation such that (1) $\text{emptydelta} \triangleleft []$, and (2) $d \triangleleft a$ implies $\text{append}(d, u) \triangleleft a \cdot u$ for all updates u , and (3) $d_1 \triangleleft a_1 \dots d_n \triangleleft a_n$ implies $\text{reduce}(d_1 \dots d_n) \triangleleft a_1 \dots a_n$ and (4) $d_1 \triangleleft a_1 \dots d_n \triangleleft a_n$ implies $\text{remove}(d_1 \dots d_n, d_n) \triangleleft a_1 \dots a_{n-1}$.
- On $\text{State} \times \text{Update}^*$, let \triangleleft be the smallest relation such that (1) $\text{initialstate} \triangleleft []$, and (2) $s \triangleleft a \wedge d_1 \triangleleft a_1 \wedge \dots \wedge d_n \triangleleft a_n$ implies $\text{apply}(s, d_1 \dots d_n) \triangleleft a \cdot a_1 \dots a_n$.

El (4) lo agregamos nosotros. Se aclara? Cómo?

Next theorem states one of most important result of the paper, saying that abstract GSP protocol and the implementation of GSP are weak bisimulation equivalent (or weakly bisimilar).

Theorem 3.1. *Let A be a system from abstract GSP Protocol, defined as then both system are weakly bisimilar, written $A \approx C$, if $(A, C) \in \mathcal{R}$, where \mathcal{R} be a binary relation defined as $\{(\|i \in I_{\{0..n\}} A_i \| S_A), (\|i \in I_{\{0..n\}} C_i \| S_C) \mid \forall i \in \mathbb{N}, \forall r : \text{Read}, A_i = \{P_i, k_i, \rho_i, \alpha_i, \sigma_i, j_i\} \wedge C_i = \{P_i, E_i\}\}$ and the conjunction of the following properties:*

1. $rvalue(r, \text{flatten}(S_A[0..k-1] \cdot \rho) \cdot \alpha) = \text{read}(r, \text{curstate}(E_i.k, E_i.p, E_i.pb, E_i.tb))$
2. $S_C.ps \triangleleft S_A$

Ac' debería decir que R cumple las propiedades de weak bisimulation. Cómo lo escribo mejor?

3. $E_i.tb \triangleleft [\alpha_i]$
4. $E_i.rb[0].delta \triangleleft S_A[k_i] \dots E_i.rb[j-1].delta \triangleleft S_A[k_i+j-1]$
5. $E_i.known \triangleleft S_A[0 \dots k_i-1]$
6. $E_i.p \cdot E_i.pb \triangleleft \rho_i$
7. For all natural l , such that $1 \leq l < |in_c|$ then $E_i.in_c[l] \triangleleft S_A[k+j-1+l]$
8. $k+j+1 \leq |S_A| \Leftrightarrow |E_i.in_c| + |S_C.out_s| > 0$
9. For all update sequence $\beta \in \sigma_i$ then $\beta \in E_i.out_c \cup S_C.in_s$
10. For all natural m , such that $0 \leq m < |out_s(i)| - 1$ then $out_s(i)[m] \triangleleft S_A[k+j+|in_c|+m]$

Proof. The proof follows by induction on the length of the derivation \rightarrow^* . We use \tilde{A} and \tilde{C} to denote clients, except client i^{th} , interacting concurrently, i.e., $\parallel_{j \in I_{\{0 \dots n\}-i}} A_j$ or C_j respectively.

- **n=0.** Then $A = \{0, \emptyset, \emptyset, \emptyset, \emptyset, 0\}_i \parallel \tilde{A} \parallel S_A$ and $C = \{E, initialstate\}_i$. In particular,
 1. The operation *rvalue* in the abstract language is performing on empty sequences, hence, the result is undefined. Moreover the parametric function *read* tries to get a result on a *initialstate*, then the result returned is undefined.
 2. It is easy to see, due to property (1) of \triangleleft between States and Updates.
 3. It is guaranteed by property (1) of \triangleleft between Deltas and Updates.
 4. Analogous to property 2.
 5. Analogous to property 3.
 6. Precondition of implication is *false*.
 7. Precondition of both implications are *false*.
 8. $\emptyset \in \emptyset$.
- **n=k+1.** $\forall (A, C) \in \mathcal{R}$
 - **rule (T-READ).** If $A \xrightarrow{read(r)}_i A'$, then by rule (T-READ), A must be the following term: $\{let\ x = read(r); P, k, \rho, \alpha, \sigma, j\}_i \parallel \tilde{A} \parallel S_A$, therefore, the client i^{th} is only one who changes. So that, A' will be $\{P[x \mapsto v], k, \rho, \alpha, \sigma, j\}_i \parallel \tilde{A} \parallel S_A$. As (A, C) belongs \mathcal{R} then C must be $\{let\ x = read(r); P, E\}_i \parallel \tilde{C} \parallel S_C$, and the read transition is the only one that C could perform. Then, looking at the rule, $C' = \{P[x \mapsto v], E\}_i \parallel \tilde{C} \parallel S_C$. We prove that $(A', C') \in \mathcal{R}$ if ten properties introduced are keeping.
 1. It is easy to see, since P is the same in both clients and the value got back by each read operations is equals because of $(A, C) \in \mathcal{R}$ then property 1 is worth.
 2. to 9. Do not change.
 - **rule (T-UPDATE).** If $A \xrightarrow{update(u)}_i A'$, then by rule (T-UPDATE), A must be the following term: $\{update(u); P, k, \rho, \alpha, \sigma, j\}_i \parallel \tilde{A} \parallel S_A$, therefore, the client i^{th} is only one who changes. So that, A' will be $\{P, k, \rho, \alpha \cdot u, \sigma, j\}_i \parallel \tilde{A} \parallel S_A$. We know that (A, C) belongs \mathcal{R} , then C must be $\{update(u); P, E\}_i \parallel \tilde{C} \parallel S_C$, and the update transition is the only one that C could perform. Then, $C' = \{P, E[tb \mapsto \mathbf{append}(E.tb, u)]\}_i \parallel \tilde{C} \parallel S_C$. We prove that $(A', C') \in \mathcal{R}$ if ten properties introduced are keeping.

1. States $rvalue(r, flatten(S_A[0..k_i-1] \cdot \rho_i) \cdot \alpha_i) = \mathbf{read}(r, \mathbf{apply}(E_i.k, E_i.p \cdot E_i.pb \cdot E_i.tb))$ however after applying rules, the transaction buffers change. Then, performing a read action in abstract GSP, $rvalue(r, flatten(S_A[0..k_i-1] \cdot \rho_i) \cdot \alpha_i \cdot u)$ and the implementation, $\mathbf{read}(r, \mathbf{apply}(E_i.k, E_i.p \cdot E_i.pb \cdot \mathbf{append}(E_i.tb, u)))$. We rename $\alpha \cdot u$ by α' and $\mathbf{append}(E_i.tb, u)$ by $E_i.tb'$. If we show that $\forall u : \mathbf{Update}$ such that $\mathbf{append}(E_i.tb, u) \triangleleft \alpha \cdot u$, we will prove that $(A', C') \in \mathcal{R}$. By property 3, we know that $E_i.tb \triangleleft [\alpha_i]$, using the property of relation between Delta and Update, $\mathbf{append}(E_i.tb, u) \triangleleft \alpha \cdot u$.
2. to 9. Do not change.
- **rule (T-PUSH).** If $A \xrightarrow{push()}_i A'$, then by rule (T-PUSH), A must be the following term: $\{push(); P, k, \rho, \alpha, \sigma, j\}_i \parallel \tilde{A} \parallel S_A$, therefore, the client i^{th} is only one who changes. So that, A' will be $\{P, k, \rho \cdot [\alpha], \alpha \cdot u, \sigma \cdot [\alpha], j\}_i \parallel \tilde{A} \parallel S_A$. We know that (A, C) belongs \mathcal{R} , then C must be $\{push(); P, E\}_i \parallel \tilde{C} \parallel S_C$, and the push transition is the only one that C could perform. Then, after applying the rule, $C' = \{P, E[pb \mapsto \mathbf{reduce}(E.pb \cdot E.tb); tb \mapsto \varepsilon; n \mapsto E.n + 1]\}_i \parallel \tilde{C} \parallel S_C$. We prove that $(A', C') \in \mathcal{R}$ if ten properties introduced are keeping.
 1. We have to prove that after push transitions, the operations $rvalue$ and $read$ get back the same value. Then $rvalue(r, flatten(S_A[0..k_i-1] \cdot \rho_i) \cdot \alpha_i) = rvalue(r, flatten(S_A[0..k_i-1] \cdot (\rho_i \cdot [\alpha_i])) \cdot \varepsilon)$ by Lemma XX besides $\mathbf{read}(r, curstate(E_i.k, E_i.p, E_i.pb, E_i.tb)) = \mathbf{read}(r, curstate(E_i.k, E_i.p, \mathbf{reduce}(E.pb \cdot E.tb), \varepsilon))$ by Lemma YY.
 2. It does not change.
 3. $E_i.tb = \varepsilon$ and $\alpha_i = \varepsilon$. Using the property of the relation \triangleleft , $\varepsilon \triangleleft \varepsilon$
 4. It does not change.
 5. It does not change.
 6. States $E_i.p \cdot E_i.pb \triangleleft \rho_i$ besides by 3. $E_i.tb \triangleleft [\alpha_i]$. Finally, applying the property of the relation \triangleleft , $\mathbf{reduce}(E_i.p \cdot E_i.pb \cdot E_i.tb) \triangleleft \rho_i \cdot [\alpha_i]$.
 7. to 9. do not change.
- **rule (T-PULL).** If $A \xrightarrow{pull()}_i A'$, then by rule (T-PULL), A must be the following term: $\{pull(); P, k, \rho, \alpha, \sigma, j\}_i \parallel \tilde{A} \parallel S_A$, therefore, the client i^{th} is only one who changes. So that, A' will be $\{P, k + j, \rho \setminus S[k..k+j], \alpha, \sigma, 0\}_i \parallel \tilde{A} \parallel S_A$. Due to (A, C) belongs \mathcal{R} , then C must be $\{pull(); P, E\}_i \parallel \tilde{C} \parallel S_C$, and the pull transition is the only one that C could perform. Then, after applying the rule, $C' = \{P, E[k \mapsto \mathbf{reducestate}(E.k, E.rb); rb \mapsto \varepsilon; E.p \mapsto \mathbf{remove}(E.p, E.rb)]\}_i \parallel \tilde{C} \parallel S_C$. We prove that $(A', C') \in \mathcal{R}$ if ten properties introduced are keeping.
 1. We have to prove that after push transitions, the operations $rvalue$ and $read$ get back the same value. Then $rvalue(r, flatten(S_A[0..k_i+j-1] \cdot (\rho \setminus S[k..k+j])) \cdot \alpha_i) = \mathbf{read}(r, curstate(\mathbf{reducestate}(E.k, E.rb), \mathbf{remove}(E.p, E.rb), E.pb, E.tb)) = \mathbf{read}(r, \mathbf{apply}(\mathbf{reducestate}(E.k, E.rb), \mathbf{remove}(E.p, E.rb) \cdot E.pb \cdot E.tb))$. On the one hand, property 5 states that $E_i.known \triangleleft S_A[0 \dots k_i-1]$, on the other hand by property 4, $E_i.rb[0].delta \triangleleft S_A[k_i] \dots E_i.rb[j-1].delta \triangleleft S_A[k_i+j-1]$, after, using the property of the relation \triangleleft between state and updates, we know that $\mathbf{apply}(E_i.k, E_i.rb[0].delta \dots E_i.rb[j-1].delta) \triangleleft S_A[0 \dots k_i-1] \cdot \dots \cdot S_A[k_i+j-1]$. As you note, the left side is $\mathbf{reducestate}$'s definition, then $\mathbf{reducestate}(E.k, E.rb)u \triangleleft S_A[0 \dots k_i+j-1]$. Next, property by

6., we know that $E_i.p \cdot E_i.pb \triangleleft \rho_i$, if we apply the extra axiom of \triangleleft , we will get that $\text{remove}(E_i.p \cdot E_i.pb, E_i.rb[0].\text{delta}) \triangleleft \rho \setminus S[k]$, so that, $\text{remove}(E_i.p \cdot E_i.pb, E_i.rb) \triangleleft \rho \setminus S[k..k+j]$. Therefore it is guaranteed.

2. It does not change.
3. It does not change.
4. $E_i.rb$ is ε and j is 0, therefore it property is guaranteed.
5. It does not change.
6. We know that $E_i.p \cdot E_i.pb \triangleleft \rho_i$, if we apply the extra axiom of \triangleleft , we will get that $\text{remove}(E_i.p \cdot E_i.pb, E_i.rb[0].\text{delta}) \triangleleft \rho \setminus S[k]$, so that, $\text{remove}(E_i.p \cdot E_i.pb, E_i.rb) \triangleleft \rho \setminus S[k..k+j]$.
7. to 9. do not change.

- **rule (T-CONFIRMED)** If $A \xrightarrow{\text{confirmed}()}_i A'$, then by rule (T-CONFIRMED), A must be the following term: $\{let\ x = confirmed(); P, k, \rho, \alpha, \sigma, j\}_i \parallel \tilde{A} \parallel S_A$, therefore, the client i^{th} is only one who changes. So that, A' will be $\{P[x \mapsto \rho \neq [] \vee \alpha \neq \varepsilon], k, \rho, \alpha, \sigma, j\}_i \parallel \tilde{A} \parallel S_A$. As (A, C) belongs \mathcal{R} then C must be $\{let\ x = confirmed(); P, E\}_i \parallel \tilde{C} \parallel S_C$, and the read transition is the only one that C could perform. Then, looking at the rule, $C' = \{P[x \mapsto E.p = \varepsilon \vee E.pb = \varepsilon \vee E.tb = \varepsilon \mapsto \text{true}], E\}_i \parallel \tilde{C} \parallel S_C$. We prove that $(A', C') \in \mathcal{R}$ if ten properties introduced are keeping.

1. Both clients has the same program P . By 6. we know that the value of $E.p \cdot E.pb$ is related to ρ , thereby, when $E.p$ or $E.pb$ have any elements then ρ also has it. Finally, $E_i.tb$ has elements iff α_i also has by property 3.
2. to 9. Do not change.

- **rule (T-RECEIVE)** If $A \xrightarrow{\tau}_i A'$, then by internal rule (T-RECEIVE), A must be the following term: $\{P, k, \rho, \alpha, \sigma, j\}_i \parallel \tilde{A} \parallel S_A$, therefore, the client i^{th} is only one who changes. So that, A' will be $\{P, k, \rho, \alpha, \sigma, j+1\}_i \parallel C \parallel S$. As (A, C) belongs \mathcal{R} then C must be $\{P, E\}_i \parallel \tilde{C} \parallel S_C$, such that after internal transitions τ , it will become C' . By Hypothesis of (T-RECEIVE, we know that $k+j+1 \leq |S_A|$, then, by Property 8., $|E_i.in_c| + |S_C.out_s| > 0$. We will consider two cases:

- * $|E_i.in_c| > 0$, then $|E_i.in_c|$ has a *GsSegment* at least, i.e., $E.in_c = gs_0 \cdot gs_l$. If C took the internal rule (RECEIVE), we will get the following C' termn, $\{P, E[rb \mapsto E.rb \cdot gs_0.gssegment]; in_c \mapsto gs_l\}_i$. Now, we will check if (A', C') belongs to \mathcal{R} .

1. The inputs term in read operation have not changed.
2. It does not change.
3. It does not change.
4. Term j was incremented by 1. Then, we should prove that $E_i.rb[0].\text{delta} \triangleleft S_A[k_i] \dots E_i.rb[(j+1)-1].\text{delta} \triangleleft S_A[k_i + (j+1) - 1]$ is guaranteed. By Property 4., we only should prove $E_i.rb[j].\text{delta} \triangleleft S_A[k_i + j_i]$ allow (A', C') to belong to \mathcal{R} . It worths by Property 7., in particular with $l = 1$.
5. It does not change.
6. It does not change.
7. It is easy to see such that $\neg in_c' \prec |in_c|$, then we use Property 7.
8. to 9. They do not change.

- $|S_C.out_s| > 0$, then $|S_C.out_s|$ has a *GsSegment* at least, i.e., $S_C.out_s(i) = gs \cdot gss$. Then, by internal rule (COMM-SERVER-CLIENT) we get $out_s[i \mapsto gss]$ and $E[in_c \mapsto E.in_c \cup \{gs\}]$. Then, by internal rule (RECEIVE), we get a term who has $|E_i.in_c| > 0$. The rest of the prove is equivalent to above case.
- **rule (T-PROCESS)** If $A \xrightarrow{\tau}_i A'$, then by rule (T-PROCESS), A must be the following term: $\{P, k, \rho, \alpha, [\beta] \cdot \sigma, j\} \parallel \tilde{A} \parallel S_A$, therefore, the client i^{th} is only one who changes. So that, A' will be $\{P, k, \rho, \alpha, \sigma, j\} \parallel C \parallel S \cdot \beta$. Property 9 states that for all β such that $\beta \in \sigma_i$ then $\beta \in E_i.out_c \cup S_C.in_s$. Analyzing cases:
 - * If $\beta \in E_i.out_c$ then, by rule (COMM-CLIENT-SERVER), $C \xrightarrow{\tau}_i C'$, so that, $C' = \{P, E[out_c \mapsto rs]\}_i \parallel \tilde{C} \parallel \{ps, in_s[i \mapsto \langle i, n_0, \delta_0 \rangle], out_s\}_C$ and $\delta_0 \triangleleft \beta$. Performing a new internal action by rule (BATCH), i.e., $C' \xrightarrow{\tau}_i C''$, then $C'' = \{apply(ps, d), in_s, notify(dom(out_s), out_s, gs)\} \parallel \tilde{C} \parallel \{P, E\}_i$. We have to prove that (A', C'') belongs \mathcal{R} .
 1. Theirs terms do not change.
 2. Our hypothesis is $S_C.ps \triangleleft S_A$ besides $\delta_0 \triangleleft \beta$, then using the property of the relation \triangleleft between state and updates, we will get that $apply(S_C.ps, \delta_0) \triangleleft S_A \cdot \beta$.
 3. It does not change.
 4. It does not change.
 5. It does not change.
 6. It does not change.
 7. It does not change.
 8. S_A has incremented by one, so that, by Hypothesis, $k + j + 1 \leq |S_A| < |S_A| + 1$, besides, $\Leftrightarrow |E_i.in_c| + |S_C.out_s| > 0$ because of neither $E_i.in_c$ nor $S_C.out_s$ have changed.
 9. By Hypothesis, for all element in σ , those belongs to $E_i.out_c \cup S_C.in_s$, in particular, σ without a element keeps these guarantee.
 - * If $\beta \in E_i.in_s$ then, applying rule (BATCH), we can see that we are at above case.

We have prove that if A perform an action, C also can perform an action, the terms after both transitions are in \mathcal{R} . Now, we will prove the opposite side. When C perform an action, then A can perform an action and the news terms are in \mathcal{R} .

- **n=0**. Then $C = \{E, initialstate\}_i$ and $A = \{0, \emptyset, \emptyset, \emptyset, 0\}_i \parallel \tilde{A} \parallel S_A$. It is analogous to the base case previously proved.
- **n=k+1**. $\forall (C, A) \in \mathcal{R}$
 - **rule (COMM-SERVER-CLIENT)**. If $C \xrightarrow{\tau}_i C'$, then by rule (COMM-SERVER-CLIENT), C' must be the following term: $\{ps, in_s, out_s[i \mapsto gss]\} \parallel \{P, E[in_c \mapsto E.in_c \cup \{gs\}]\}_i$, therefore, the client i^{th} at C , is the only one who changed. We will prove that $(C', A) \in \mathcal{R}$ if ten properties introduced are keeping.
 1. to 6. Their terms do not changed.
 7. $l' = (|E_i.in_c| + 1)$, so that, we have to prove that $E_i.in_c[l'] \triangleleft S_A[k + j - 1 + l']$, however, it easy to see because by property 10. when $m = 0$ then $out_s(i)[0] \triangleleft S_A[k + j + |in_c|]$.

8. It is easy to see that $(|E_i.in_c| + 1) + (|S_C.out_s| - 1) > 0$.
 9. It does not change.
 10. It is easy to see, because, Property 10 is guaranteed with $0 \leq l$ therefore $1 \leq l$.
- **rule (COMM-CLIENT-SERVER).** If $C \xrightarrow{\tau}_i C'$, then by rule (COMM-CLIENT-SERVER), C' must be the following term: $\{ps', in_s, notify(dom(out_s), out_s, gs)\}$, therefore, the client i^h is the only one who has changed. We will prove that $(C', A) \in \mathcal{R}$ if the ten properties introduced are keeping.
 1. to 8. Their terms do not changed.
 9. It is easy to see, because, we have exchanged a sequence from out_c to in_s .
 10. It does not change.
 - **rule (BATCH).** If $C \xrightarrow{\tau}_i C'$, then by rule (BATCH), C' must be the following term: $\{ps, in_s, out_s[i \mapsto gss]\} \parallel \{P, E[in_c \mapsto E.in_c \cup \{gs\}]\}_i$, therefore, the client i^h at C , is the only one who changed. Hence (C, A) belongs to \mathcal{R} then A after of internal transition τ is $\{P, k, \rho, \alpha, \sigma, j\} \parallel S_C \parallel S \cdot \beta$. We will prove that $(C', A') \in \mathcal{R}$ if the ten properties introduced are keeping.
 1. Their terms do not change.
 2. Property 9 states that for all β such that $\beta \in \sigma_i$ then $\beta \in E_i.out_c \cup S_C.in_s$. In particular, we consider $\beta \in S_C.in_s$, then there exists δ_o such that $\delta_o \triangleleft \beta$, besides by Property 2., $S_C.ps \triangleleft_{SA}$, so that using the property of the relation \triangleleft between state and updates, we will get that $apply(S_C.ps, \delta_o) \triangleleft_{SA} \cdot \beta$.
 3. It does not change.
 4. It does not change.
 5. It does not change.
 6. It does not change.
 7. It does not change.
 8. $S_C.out_s$ is updated with the new persisted state however its size does not change. By Hypothesis 9, $k + j + 1 \leq |S_A| < |S_A| + 1$.
 9. It does not change.
 10. It does not change.

4 Consistency Guarantees

We shall introduce a series of store-level consistency guarantees and then we shall show which are captured by application written in GSP. We start identifying three kinds of relations between actions of update and read:

Session Order relates whatever pair of actions from the same client, indicating the program order. It is a total order on actions.

Visibility relates Updates with Reads. It is used to indicate is if an action of Update is visible for an action of Read.

Arbitration relates Updates with Updates. It is used to resolve update conflicts. It is a total order on actions of update.

We extend the GSP language with a new term which capture the relations amount operation in our system.

$$(ENVIRONMENT) \quad E ::= \{N, OP, SO, VIS, AR\}$$

Let E , a new term, where N represents our system introduced in Definition 1.1, OP is a mapping of vertices to actions, SO is a session order relation defined from vertices to relations of vertices $\mathbb{V} \times (\mathbb{V} \times \mathbb{V})$ and VIS, AR are visibility and arbitration relation.

Notation. Given a session order relation so from client i and a vertex v , we shall write $so \triangleright_i v$ meaning that $(\mathcal{V}, \mathcal{R}) \triangleright_i v = (\mathcal{V} \cup \{v\}, \mathcal{R} \cup \{(x, v)/x \in \mathbb{V}\})$. We shall refer to an update action on the queue message as $S[n]_i$. The arbitration relation AR is defined as $\{(v, w)/\{v \mapsto S[m]_h\} \in OP \wedge \{w \mapsto S[n]_i\} \in OP \wedge m < n\}$. A transition $\xrightarrow{\alpha}_i$ denotes the fact that action α is performed by client i .

The following operational semantic allow to understand how working the environment when the actions are executed.

$$(E-READ) \quad \frac{N \xrightarrow{read(r)}_i N' \quad v \notin dom(OP) \quad VIS' = VIS \cup \{(x, v)/\{x \mapsto update(u)_h\} \in OP \wedge u^x \in S[0..k-1] \cdot p \cdot [\alpha]\}}{\{N, OP, SO, VIS, AR\} \xrightarrow{read(r)}_i \{N', OP \cup \{v \mapsto read(r)\}, SO \triangleright_i v, VIS', AR\}}$$

$$(E-UPDATE) \quad \frac{N \xrightarrow{update(u^v)}_i N' \quad v \notin dom(OP)}{\{N, OP, SO, VIS, AR\} \xrightarrow{update(u^v)}_i \{N', OP \cup \{v \mapsto update(u)\}, SO \triangleright_i v, VIS, AR\}}$$

4.1 Ordering Guarantees

We now prove what ordering guarantees are assured by GSP language and what do not.

First, we prove a useful lemma:

Lemma 4.1. *Let u an update action, S a message queue, p_i and α_i a pending queue and transaction queue from the client i , if $\{N, \emptyset, \emptyset, \emptyset\} \rightarrow^* \{N, OP, SO, VIS, AR\}$ then $\{x \mapsto update(u)_i\} \in OP \Rightarrow u^x \in S[0..k_i-1] \cdot p_i \cdot [\alpha_i]$*

Proof. The proof follows by induction on the length of the derivation \rightarrow^* .

- **n=0.** Then OP is \emptyset , so that antecedent is false, then the preposition is true.
- **n=k+1.** Then $\{N, \emptyset, \emptyset, \emptyset\} \rightarrow^n \{N, OP, SO, VIS, AR\} \rightarrow_i \{x \mapsto update(u)_i\} \in OP \Rightarrow u^x \in S[0..k_i-1] \cdot p_i \cdot [\alpha_i]$. We proceed by case analysis on the last transition:
 - **rule (E-READ).** As x is an update operation then it must not be v , so that $\{x \mapsto update(u)_i\} \in OP$, then by inductive hypothesis $u^x \in S[0..k_i-1] \cdot p_i \cdot [\alpha_i]$.
When $N \xrightarrow{read(r)}_i N'$, k_i, p_i, α_i do not change.

- **rule (E-UPDATE).** There are two possibilities:
 - * $x \neq v$. Then we can use inductive hypothesis, so that it is easy to see that if $u^x \in S[0..k_i - 1] \cdot \rho_i \cdot [\alpha_i]$ then $u^x \in S[0..k_i - 1] \cdot \rho_i \cdot [\alpha_i] \cdot u_i^v$ too.
 - * $x = v$. When $N \xrightarrow{\text{update}(u^v)}_i N'$, $u^v \in \alpha'_i$ (with α'_i transaction queue in N') because $\alpha'_i = \alpha_i \cdot u^v$. It is immediate to note that $u^x \in S[0..k'_i - 1] \cdot \rho'_i \cdot [\alpha'_i]$.
- **rule (E-PUSH).** As op do not change, then by inductive hypothesis, $u^x \in S[0..k_i - 1] \cdot \rho_i \cdot [\alpha_i] \equiv u^x \in S[0..k_i - 1] \cdot (\rho_i \cdot [\alpha_i]) \cdot \varepsilon$. When $N \xrightarrow{\text{push}()}_i N'$, $k'_i = k_i$, $\rho'_i = \rho_i \cdot [\alpha_i]$ and $\alpha'_i = \varepsilon$.
- **rule (E-PULL).** As op do not change, then by inductive hypothesis, $u^x \in S[0..k_i - 1] \cdot \rho_i \cdot [\alpha_i]$. We should prove that it is equivalent to $u^x \in S[0..k_i - 1 + j_i] \cdot \rho_i \setminus S[k_i..k_i + j_i] \cdot [\alpha_i]$. There are two interesting cases to consider:
 - * If $u^x \in \rho_i \wedge u^x \notin S[k_i..k_i + j_i]$, then $u^x \in \rho'_i$.
 - * If $u^x \in \rho_i \wedge u^x \in S[k_i..k_i + j_i]$, then $u^x \notin \rho'_i$ but $u^x \in S[k_i..k_i + j_i]$.

The proof for the remaining cases is by inductive hypothesis because op , k_i , ρ_i and α_i do not change.

Theorem 4.1 (READ MY WRITES).

Let SO_R the second component of the relation SO and VIS a visibility relation, if $\{N_0, \emptyset, \emptyset, \emptyset, \emptyset\} \rightarrow^* \{N, \text{op}, \text{so}, \text{vis}, \text{ar}\}$ then $\text{SO}_R \cap (\mathbb{U} \times \mathbb{R}) \subseteq \text{VIS}$

Proof. The proof follows by induction on the length of the derivation \rightarrow^* .

- **n=0.** In particular op and vis are \emptyset , so that $\emptyset \subseteq \emptyset$.
 - **n=k+1.** Then $\{N_0, \emptyset, \emptyset, \emptyset, \emptyset\} \rightarrow^n \{N, \text{op}, \text{so}, \text{vis}, \text{ar}\} \xrightarrow{\alpha}_i \{N', \text{op}', \text{so}', \text{vis}', \text{ar}\}$. We proceed by definition:
 - $\text{SO}_R' = \text{SO}_R \cup \{(w, v) / \{w \mapsto \text{read}(r)\} \vee \{w \mapsto \text{update}(u)_j\}\}$. Applying the intersection $(\mathbb{U} \times \mathbb{R})$, we shall obtain $\text{SO}_R \cup \{(w, v) / \{w \mapsto \text{update}(u)_j\}\}$.
 - $\text{VIS}' = \text{VIS} \cup \{(x, v) / \{x \mapsto \text{update}(u)_h\} \in \text{op} \wedge u^x \in S[0..k - 1] \cdot \rho \cdot [\alpha]\}$.
- By inductive hypothesis, $\text{SO}_R \cup (\mathbb{U} \times \mathbb{R}) \subseteq \text{vis}$. We only have to prove that $\{(w, v) / \{w \mapsto \text{update}(u)_j\} \in \text{op}\} \subseteq \{(x, v) / \{x \mapsto \text{update}(u)_h\} \in \text{op} \wedge u^x \in S[0..k - 1] \cdot \rho \cdot [\alpha]\}$. When $j = h$ and $w = x$, we can use Lemma 4.1. So that, we have proved that $\text{SO}_R' \cap (\mathbb{U} \times \mathbb{R}) \subseteq \text{VIS}'$.

Theorem 4.2 (MONOTONIC READ). Let SO_R the second component of the relation SO and VIS a visibility relation, if $\{N_0, \emptyset, \emptyset, \emptyset, \emptyset\} \rightarrow^* \{N, \text{op}, \text{so}, \text{vis}, \text{ar}\}$ then $(\text{VIS}; \text{SO}_R) \cap (\mathbb{U} \times \mathbb{R}) \subseteq \text{VIS}$

Proof. The proof follows by induction on the length of the derivation \rightarrow^* .

- **n=0.** In particular SO_R and VIS are \emptyset , so that $\emptyset \subseteq \emptyset$.
- **n=k+1.** Then $\{N_0, \emptyset, \emptyset, \emptyset, \emptyset\} \rightarrow^n \{N, \text{op}, \text{so}, \text{vis}, \text{ar}\} \xrightarrow{\alpha}_i \{N', \text{op}', \text{so}', \text{vis}', \text{ar}\}$. Let R be a composition of relations. We shall say that if $(x, y) \in R$ iff $\exists y \in \mathbb{V}$ such that $(x, y) \in \text{VIS}' \wedge (y, z) \in \text{SO}_R'$. We have to prove that $(x, z) \in \text{VIS}'$. We proceed by case analysis on the last transition:
 - **rule (E-READ).** We know that v is fresh, therefore, v have not be neither x nor y because there exists z such that, z happens after from x and y . There are only two possibilities:

- * $z = v$. As $(y, z) \in \text{SO}_R$, then y and z are from the same client, called i . We are only interested in relations of $\text{update} \times \text{read}$. We know that v is associated to an read action besides x have to be an update action. As $(x, y) \in \text{VIS}'$ then y is an read action, i.e., $\{y \mapsto \text{read}(r)_i\} \in \text{OP}$. By Lemma xx, there exists an update u such that $u^x \in S[0..k_i - 1] \cdot \rho_i \cdot [\alpha_i]$. Substituting w by v in VIS' , we prove that $(x, z) \in \text{VIS}'$. item $z \neq v$. This case follows immediately by inductive hypothesis.
- **rule (E-UPDATE)**. Visibility relation does not change, i.e., $\text{VIS} = \text{VIS}'$. Let v be a vertex associated an update action, then $\text{SO}_R' = \text{SO}_R \cup \{(w, v) / \{w \mapsto \text{read}(r)\} \vee \{w \mapsto \text{update}(u)_j\}\}$. As we are only interested in relations of $\text{update} \times \text{read}$, then $(\text{VIS}', \text{SO}_R') \cap (\mathbb{U} \times \mathbb{R}) \equiv (\text{VIS}, \text{SO}_R) \cap (\mathbb{U} \times \mathbb{R})$. By inductive hypothesis we can prove that $(\text{VIS}', \text{SO}_R') \cap (\mathbb{U} \times \mathbb{R}) \subseteq \text{VIS} \subseteq \text{VIS}'$. The proof for the remaining cases follow are not interesting because the relations does not change.

Theorem 4.3 (NO CIRCULAR CAUSALITY).

Let SO_R the second component of the relation SO and VIS a visibility relation, if $\{N_0, \emptyset, \emptyset, \emptyset, \emptyset\} \rightarrow^* \{N, \text{OP}, \text{SO}, \text{VIS}, \text{AR}\}$ then **acyclic** $(\text{SO}_R \cup \text{VIS})^+$.

Proof. Since VIS is acyclic and $\text{SO}_R \cap (\mathbb{U} \times \mathbb{R}) \subseteq \text{VIS}$ by Theorem 4.1, we have to prove that VIS^+ is acyclic. In particular, the transitive closure of an acyclic graph is the reachability relation of the directed acyclic graph and a strict partial order.

Theorem 4.4 (CAUSAL VISIBILITY).

Let SO_R the second component of the relation SO and VIS a visibility relation, $(\text{SO}_R \cup \text{VIS})^+ \cap (\mathbb{U} \times \mathbb{R}) \subseteq \text{VIS}$.

Proof. The proof follows by induction on the number of union sets between SO_R and VIS . Then, $\bigcup_{n=1}^{\infty} (\text{SO}_R \cup \text{VIS})^n \cap (\mathbb{U} \times \mathbb{R}) \subseteq \text{VIS}$.

- **n=0**. This means that SO_R and VIS are \emptyset , so that $\emptyset \subseteq \emptyset$.
- **n=k+1**. Suppose that we have proved that the number of union sets $< k+1$. Now, we have to prove that: $\forall (a, b) \mid (a, b \in \mathbb{V} \Rightarrow (a, b) \in (\text{SO}_R \cup \text{VIS})^{k+1} \cap (\mathbb{U} \times \mathbb{R})) \Rightarrow (a, b) \in \text{VIS}$.
 Assume $(a, x_1), (x_1, x_2), \dots, (x_{k-1}, x_k), (x_k, b)$ are relations from $(\text{SO}_R \cup \text{VIS})^{k+1}$. Then $(a, x_1), (x_1, x_2), \dots, (x_{k-1}, x_k)$ are relations from $(\text{SO}_R \cup \text{VIS})^k$. By the induction hypothesis, $(a, x_k) \in (\text{SO}_R \cup \text{VIS})^k$, and we also have $(x_k, b) \in (\text{SO}_R \cup \text{VIS})$. Thus by the definition of $(\text{SO}_R \cup \text{VIS})^{k+1}$, $(a, b) \in (\text{SO}_R \cup \text{VIS})^{k+1}$. Conversely, assume $(a, b) \in (\text{SO}_R \cup \text{VIS})^{k+1} = (\text{SO}_R \cup \text{VIS})^k \circ (\text{SO}_R \cup \text{VIS})$. Then there is a vertex $c \in \mathbb{V}$ such that $(a, c) \in (\text{SO}_R \cup \text{VIS})^k$ and $(c, b) \in (\text{SO}_R \cup \text{VIS})$. We are only interested when a is an update action and b a read action. It is because of the intersection with $(\text{Update} \times \text{Read})$. We have two possible cases:
 - c is a read action. It means that (c, b) only can be in SO_R because VIS requires that c will be an update action. In particular, if $(c, b) \in \text{SO}_R$, they belong to the same client.
 - * if $(a, c) \in \text{VIS}$ and $(c, b) \in \text{SO}_R$, by Theorem 4.2, $(a, b) \in \text{VIS}$.

- * if $(a, c) \in \text{SO}_R$ then by Theorem 4.1, $(a, c) \in \text{VIS}$. Then, it is analogous to the previous case.
- c is an update action. It means that (a, c) only can be in SO_R because VIS requires that c will be an read action. In particular, if $(a, c) \in \text{SO}_R$, they belong to the same client.
 - * if $(a, c) \in \text{SO}_R$ and $(c, b) \in \text{VIS}$, by Theorem Monotonic Writes (Falta probar!), $(a, b) \in \text{VIS}$.
 - * if $(a, c) \in \text{VIS} \dots \text{VER}$.

Theorem 4.5 (CAUSAL ARBITRATION).

Let SO_R the second component of the relation SO , VIS a visibility relation and AR an arbitration relation then $(\mathbb{U} \times \mathbb{U}) \cap (\text{SO}_R \cup \text{VIS})^+ - \text{SO}_R \subseteq \text{AR}$.

Proof. The proof follows by induction on the number of union sets between SO_R and VIS . Then, $(\mathbb{U} \times \mathbb{U}) \cap \bigcup_{n=1}^{\infty} (\text{SO}_R \cup \text{VIS})^n - \text{SO}_R \subseteq \text{AR}$.

- $n=0$. This means that SO_R , VIS and AR are \emptyset , so that $\emptyset \subseteq \emptyset$.
- $n=k+1$. Suppose that we have proved that the number of union sets $< k+1$. Now, we have to prove that: $\forall (a, b) \mid (a, b \in \mathbb{V} \Rightarrow (a, b) \in (\mathbb{U} \times \mathbb{U}) \cap \bigcup_{n=1}^{\infty} (\text{SO}_R \cup \text{VIS})^{k+1} - \text{SO}_R \Rightarrow (a, b) \in \text{AR}$.
Assume $(a, b) \in (\text{SO}_R \cup \text{VIS})^{k+1} = (\text{SO}_R \cup \text{VIS})^k \circ (\text{SO}_R \cup \text{VIS})$. Then there is a vertex $c \in \mathbb{V}$ such that $(a, c) \in (\text{SO}_R \cup \text{VIS})^k$ and $(c, b) \in (\text{SO}_R \cup \text{VIS})$.
We are only interested when a and b are an update actions. It is because of the intersection with $(\text{Update} \times \text{Update})$. We have two possible cases:
 - c is a read action. It means that (c, b) only can be in SO_R because VIS requires that c will be an update action. In particular, if $(c, b) \in \text{SO}_R$, they belong to the same client. **COMPLETAR**
 - c is an update action. **COMPLETAR**

Example 4.1 (Consistent Prefix). Consider the following system built-up from two different client:

$$E = \{ \{ \text{update}(aList.add('a')); \text{push}();, 0, \epsilon, \epsilon, \epsilon, 0 \}_1 \parallel \{ \text{update}(aList.add('b')); \text{push}(); \text{let } v = \text{read}(x);, 0, \epsilon, \epsilon, \epsilon, 0 \}_2 \parallel \epsilon, \emptyset, \emptyset, \emptyset, \emptyset \}$$

Environment has a system N with two clients and a message queue S without updates. Relation sets $\text{OP}, \text{SO}, \text{VIS}, \text{AR}$ are empty, i.e., there were not an execution in N captured by the relation sets. The update action add a string to an object called $aList$ which has not elements. In this state, C_1 and C_2 can perform their update actions. E may non-deterministically choose to do $\text{update}(aList.add('a'))$ or $\text{update}(aList.add('b'))$. If the first communication takes place over $\text{update}(aList.add('a'))$, then the system evolves as follows:

$$E \xrightarrow{\text{update}(aList.add('a'))^{v_0}} \{ \{ \text{push}();, 0, \epsilon, [aList.add('a')], \epsilon, 0 \}_1 \parallel \{ \text{update}(aList.add('b')); \text{push}(); \text{let } x = \text{read}(aList);, 0, \epsilon, \epsilon, \epsilon, 0 \}_2 \parallel \epsilon, \emptyset, \{ \{ v_0 \}, \emptyset \}, \emptyset, \emptyset \}$$

The action $update(aList.add('a'))$ of the client 1 will be identified by vertex v_0 by rule E-UPDATE. In particular, the update action will be left into the transactional queue of Client 1 besides the new vertex will be added to vertices in so. Now, Client 1 performs $push()$:

$$\begin{aligned} E & \xrightarrow{push()}_1 \{ \{0, 0, [aList.add('a')], \epsilon, [aList.add('a')], 0 \}_1 \parallel \\ & \{ update(aList.add('b')); push(); let x = read(aList);, 0, \epsilon, \epsilon, 0 \}_2 \parallel \epsilon, \\ & \emptyset, \{ (\{v_0\}, \emptyset) \}, \emptyset, \emptyset \} \end{aligned}$$

When it happens, $aList.add('a')$ is moved to the sent queue and pending queue. At this moment, Client 2 can perform an update action however Client 1 will realize an internal action which is given by E-PROCESS.

$$\begin{aligned} E & \xrightarrow{\tau}_1 \{ \{0, 0, [aList.add('a')], \epsilon, \epsilon, 0 \}_1 \parallel \\ & \{ update(aList.add('b')); push(); let x = read(aList);, 0, \epsilon, \epsilon, 0 \}_2 \parallel \\ & aList.add('a'), \emptyset, \{ (\{v_0\}, \emptyset) \}, \emptyset, \emptyset \} \end{aligned}$$

Analogously, Client 2 realize its actions leaving to environment evolves as below:

$$\begin{aligned} E & \xrightarrow{update(aList.add('b'))}_2 \{ \{0, 0, [aList.add('a')], \epsilon, \epsilon, 0 \}_1 \parallel \\ & \{ push(); let x = read(aList);, 0, \epsilon, [aList.add('b')], \epsilon, 0 \}_2 \parallel \\ & aList.add('a'), \emptyset, \{ (\{v_0, v_1\}, \{ (v_0, v_1) \}) \}, \emptyset, \emptyset \} \\ \\ & \xrightarrow{push()}_2 \{ \{0, 0, [aList.add('a')], \epsilon, \epsilon, 0 \}_1 \parallel \\ & \{ let x = read(aList);, 0, [aList.add('b')], \epsilon, [aList.add('b')], 0 \}_2 \parallel \\ & aList.add('a'), \emptyset, \{ (\{v_0, v_1\}, \{ (v_0, v_1) \}) \}, \emptyset, \emptyset \} \\ \\ & \xrightarrow{\tau}_2 \{ \{0, 0, [aList.add('a')], \epsilon, \epsilon, 0 \}_1 \parallel \\ & \{ let x = read(aList);, 0, [aList.add('b')], \epsilon, \epsilon, 0 \}_2 \parallel \\ & aList.add('a') \cdot [aList.add('b')], \emptyset, \{ (\{v_0, v_1\}, \{ (v_0, v_1) \}) \}, \emptyset, \{ (v_0, v_1) \} \} \\ \\ & \xrightarrow{read(aList)}_2 \{ \{0, 0, [aList.add('a')], \epsilon, \epsilon, 0 \}_1 \parallel \\ & \{ 0[x \mapsto 'b'], 0, [aList.add('b')], \epsilon, \epsilon, 0 \}_2 \parallel aList.add('a') \cdot [aList.add('b')], \\ & \emptyset, \emptyset, \emptyset \} \end{aligned}$$

Note that AR is modified because the message queue has two elements. The value returned will be the list with a only element, $['b']$, because of the internal action associated to the rule E-RECEIVE did not perform it. Then, if the rule E-RECEIVE were performed, the value of the list would be $['a', 'b']$. It is because the content of the pending queue is removed when the server left their message.

The guarantee Consistent Prefix, which rules is $(AR; VIS) \subseteq AR; VIS$, states that if we see an result from a client in a particular order, we will never see this result in a different order.

5 Extending GSP

As illustrated in above examples, there are situations in which a environment does not ensures consistency guarantees as Consistent Prefix or Single Order. The nature of such cases is caused by having no transactions as the usual databases.

The remaining of this section is proposing a environment that ensures these consistency guarantees. We start by refining the environment in GSP. We will distinguish a new relation:

Return Before relates updates actions. It indicates the ordering of non-overlapping operations.

$$\begin{array}{ll}
 \text{(ENVIRONMENT)} & E ::= \{N, \text{OP}, \text{SO}, \text{VIS}, \text{AR}, \text{RB}, \text{TO}, \text{TC}\} \\
 \text{(TRANSACTIONS WITH OVERLAPPING)} & \text{TO} ::= \emptyset \mid w^j \mapsto \mathcal{N}, \text{TO} \text{ with } w \in \mathbb{V} j \in \mathbb{N} \\
 \text{(TRANSACTIONS CLOSED)} & \text{TC} ::= \varepsilon \mid w^j \cdot \text{TC} \text{ with } w \in \mathbb{V} j \in \mathbb{N}
 \end{array}$$

The term TO relates vertex from a client to set of vertex which have just not finished; TC denotes transaction closed.

5.1 Operational Semantic

We assume the following countable sets of transactional vertices names $[\mathbb{V}]$ ranged over by $[v]; [v_0]; [v_1], \dots$;

The labeled transition system for the extension of GSP considers the following actions β :

$$\beta ::= \alpha \mid \dagger \mid \ddagger \mid [\alpha]$$

These labels allow system to perform an update synchronous. Label \dagger stands for the begining of a transaction, and \ddagger the end of it.

ENVIRONMENT

(E-START-SYNC)

$$\frac{N \xrightarrow{\text{sync_update}(u^v)}_i N' \quad [v^i] \notin \text{dom}(\text{TO})}{\{N, \text{OP}, \text{SO}, \text{VIS}, \text{AR}, \text{RB}, \text{TO}, \text{TC}\} \xrightarrow{\dagger}_i \{N', \text{OP}, \text{SO}, \text{VIS}, \text{AR}, \text{RB}, \text{TO}; v^i \mapsto \text{dom}(\text{TC}), \text{TC}\}}$$

(E-END-SYNC)

$$\frac{N \xrightarrow{\ddagger(u^{[v]})}_i N' \quad [v^i] \in \text{dom}(\text{TO})}{\{N, \text{OP}, \text{SO}, \text{VIS}, \text{AR}, \text{RB}, v^i \mapsto \mathcal{N}, \text{TO}, \text{TC}\} \xrightarrow{\ddagger([v^i])}_i \{N', \text{OP}, \text{SO}, \text{VIS}, \text{AR}, \text{RB}', \text{TO}, \text{TC} \cdot v^i\}}$$

where $\text{RB}' = \text{RB} \cup \{(x, v) \mid \forall 0 \leq m, n < |S|, x \mapsto S[m] \wedge v \mapsto S[n] \wedge m \leq n \wedge x \notin \text{TC}(v)\}$

PROGRAM

(T-SYNC-UPDATE)

$$\{sync_update(u); P, k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S \xrightarrow{sync_update(u^v)}_i \{[update(u); flush()]; P, k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S$$

(T-TRAN)

$$\frac{\{P, k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S \xrightarrow{\alpha}_i \{P', k', \rho', \alpha', \sigma', j'\}_i \parallel C \parallel S}{\{[P]; Q, k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S \xrightarrow{[\alpha]}_i \{[P']; Q, k', \rho', \alpha', \sigma', j'\}_i \parallel C \parallel S}$$

(T-FLUSH)

$$\{[flush()]; P, k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S \xrightarrow{\tau}_i \{[while(cond) \mathbf{do} Q]; P, k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S$$

(T-END-SYNC)

$$\{[0]; P, k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S \xrightarrow{\ddagger(u^{[v]})}_i \{P, k, \rho, \alpha, \sigma, j\}_i \parallel C \parallel S$$