Dynamic Programming - Notes

Carlos Rondón-Moreno
Notes based on "The ABCs of RBCs" by George McCandless (2008)

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1 Value Function Iteration

Infinite horizon economies where the nature of the optimization problem does not depend on the period in which they are making decisions.

- What changes from period to period are the initial conditions.
- Solution with recursive techniques.

Policy Function: Mapping from initial conditions, given by the present or the past, to the set of decisions about what to do with the variables we can choose during this period.

State Variables: In some period t, variables whose values are already determined: k_t, A_t, past values of relevant variables.

Control Variables: Variables whose values individuals can choose in period t with the goal of maximizing some objective.

1.1 Robinson Crusoe Model

$$\max_{\{C_t\}} \sum_{i=0}^{\infty} \beta^i u(c_{t+i})$$

$$s.t. \quad \begin{array}{cc} K_{t+1} & = (1-\delta)K_t + i_t \\ i_t & = \quad f(K_t) - c_t \end{array}$$

We can rewrite the system as:

$$V(K_t) = \max_{\{K_s\}_{s=t+1}^{\infty}} \sum_{i=0}^{\infty} \beta^i u\left(f(K_{t+i}) - K_{t+1+i} + (1-\delta)K_{t+i}\right),$$

where the control variable is K_{t+1}.

Value Function ($V(K_t)$): Discounted value of utility when the maximization problem has been solved and K_t is the initial value.

Note that,

$$V(K_t) = \max_{\{K_s\}_{s=t+1}^{\infty}} u\left(f(K_t) - K_{t+1} + (1-\delta)K_t\right) + \max_{\{K_s\}_{s=t+2}^{\infty}} \sum_{i=1}^{\infty} \beta^i u\left(f(K_{t+i}) - K_{t+1+i} + (1-\delta)K_{t+i}\right)$$

Moreover,

$$\max_{\{K_s\}_{s=t+2}^{\infty}} \beta \sum_{i=0}^{\infty} \beta^i u\left(f(K_{t+1+i}) - K_{t+2+i} + (1-\delta)K_{t+1+i}\right) = \beta V(K_{t+1})$$

Therefore,

$$V(K_t) = \max_{\{K_{t+1}\}} u\left(f(K_t) - K_{t+1} + (1-\delta)K_t\right) + \beta V(K_{t+1}) \tag{1}$$

Now, we went from a problem with infinite horizons to one with only one dimension.

Problem: $V(K_{t+1})$ is unknown.

Suppose that $V(\cdot)$ and $V'(\cdot)$ exist:

FONC: $-u'\left(f(K_t) + K_{t+1} + (1-\delta)K_t\right) + \beta V'(K_{t+1}) = 0, \implies V'(K_{t+1})$ is still unknown.

By the Envelope Theorem, $V'(\cdot)$ is simply the partial derivative of u with respect to K_t. That is,

$$V'(K_t) = u'\left(f(K_t) + K_{t+1} + (1-\delta)K_t\right)\left(f'(K_t) + (1-\delta)\right),$$

Iterating forward:

$$V'(K_{t+1}) = u'\left(f(K_{t+1}) + K_{t+2} + (1-\delta)K_{t+1}\right)\left(f'(K_{t+1}) + (1-\delta)\right)$$

Using this into the FONC:

$$\begin{aligned} \beta u'\left(f(K_{t+1}) + K_{t+2} + (1-\delta)K_{t+1}\right)\left(f'(K_{t+1}) + (1-\delta)\right) &= u'(c_t) \\ \iff \quad \beta\left(f'(K_{t+1}) + (1-\delta)\right) &= \frac{u'(c_t)}{u'(c_{t+1})}, \end{aligned}$$

where the last equation is the Euler Equation.

In steady state $c_t = c_{t+1}$, then

$$\frac{1}{\beta} - (1-\delta) = f'(K^*)$$

1.2 The General Case

- X_t: Vector of period t state variables.
- Y_t: Vector of period t control variables.

$$\begin{aligned} &\max_{\{Y_s\}_{s=t+1}^{\infty}} \sum_{s=t}^{\infty} F(X_s, Y_s) \\ &\qquad s.t. \quad X_{s+1} = G(X_s, Y_s), \quad \forall s \geq t \end{aligned}$$

Bellman equation is given by:

$$\begin{aligned} V(X_t) &= \max_{\{Y_t\}} F(X_t, Y_t) + \beta V(X_{t+1}) \\ s.t. &\quad X_{t+1} = G(X_t, Y_t) \end{aligned}$$

Which is the same as:

$$V(X_t) = \max_{\{Y_t\}} F(X_t, Y_t) + \beta V(G(X_t, Y_t)) \tag{2}$$

The solution we are looking for[1]:

$$Y_t = H(X_t),$$

[1] In the R.C model $Y_s = X_{s+1}$, but that is a particular case.

values of Y_t is a function of states, and $H(X_t)$ is the Policy Function. In other words, $H(X_t)$ optimizes the choice of the controls for every permitted value of X_t.

$$V(X_t) = F(X_t, H(X_t)) + \beta V(G(X_t, H(X_t)))$$

Problem: How do we find $H(X_t)$?
FONC:

$$F_Y(X_t, Y_t) + \beta V'(G(X_t, Y_t))G_Y(X_t, Y_t) = 0$$

Note that, $F_Y(X_t, Y_t)$ is a vector of derivatives with respect to the control variables. $V'(G(X_t, Y_t))$ is a vector of derivatives of the Value Function with respect to X_{t+1}, and $G_Y(X_t, Y_t)$ is a vector of derivatives of the constraint with respect to Y_t.

However, $V'(G(X_t, Y_t))$ is still unknown.

Benveniste - Scheinkman Conditions.

If, the following conditions are satisfied:

(1) $X_t \in X$, where X is a convex set with a non-empty interior.

(2) $F(\cdot, \cdot)$ is concave.

(3) $G(\cdot, \cdot)$ is concave, differentiable and invertible.

(4) $Y_t \in Y$, where Y is a convex set with a non-empty interior.

then, the solution to the problem is:

$$V'(X_t) = F_X(X_t, Y_t) + \beta V'(G(X_t, Y_t))(G_Y(X_t, Y_t)) + F_X(X_t, Y_t) + \beta V'(G(X_t, Y_t))G_X(X_t, Y_t)$$

Note the first part is equal to the FONC:

$$V'(X_t) = F_X(X_t, Y_t) + \beta V'(G(X_t, Y_t))G_X(X_t, Y_t)$$

Moreover, if the controls are chosen appropriately, $G_X(X_t, Y_t) = 0$. Because the constraint does not depend on the states. Then,

$$V'(X_t) = F_X(X_t, Y_t)$$

Iterating forward: $V'(X_{t+1}) = F_X(X_{t+1}, Y_{t+1})$. Which yields to:

$$F_Y(X_t, Y_t) + \beta F_X(X_{t+1}, Y_{t+1})G_Y(X_t, Y_t) = 0$$

But note that, $X_{t+1} = G(X_t, Y_t)$.

$$F_Y(X_t, Y_t) + \beta F_X(G(X_t, Y_t), Y_{t+1})G_Y(X_t, Y_t) = 0$$

Which can be solved to find $Y_t = H(X_t)$. Also, using $Y_t = Y_{t+1}$ you could find a steady state solution.

1.3 General Version under Uncertainty

Sequential Problem

$$
\begin{aligned}
V(X_t, Z_t) = \max_{\{Y_s\}_{s=t}^{\infty}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^{s-t} F(X_s, Y_s, Z_s) \\
s.t. \quad X_{s+1} = G(X_s, Y_s, Z_s) \ \forall s \geq t,
\end{aligned}
$$

where X_t are state variables, Y_t are control variables, and Z_t are shocks.

Bellman Equation

$$
\begin{aligned}
V(X_t, Z_t) = \max_{\{Y_t\}} F(X_t, Y_t, Z_t) + \beta \mathbb{E}_t V(X_{t+1}, Z_{t+1}) \\
s.t. \quad X_{t+1} = G(X_t, Y_t, Z_t)
\end{aligned}
$$

FONC:

$$F_Y(X_t, Y_t, Z_t) + \beta \mathbb{E}_t \left[V'(G(X_t, Y_t, Z_t), Z_{t+1}) G_Y(X_t, Y_t, Z_t) \right] = 0$$

As with the deterministic case, the Benveniste-Sheickman Envelope theorem yields to:

$$V_X(X_t, Z_t) = F_X(X_t, Y_t, Z_t) + \beta \mathbb{E}_t \left[V_X(G(X_t, Y_t, Z_t), Z_{t+1}) G_X(X_t, Y_t, Z_t) \right]$$

If controls were chosen appropriately $G_X(X_t, Y_t, Z_t) = 0$. Thus,

$$
\begin{aligned}
V_X(X_t, Z_t) = F_X(X_t, Y_t, Z_t) \\
V_X(X_{t+1}, Z_{t+1}) = F_X(X_{t+1}, Y_{t+1}, Z_{t+1})
\end{aligned}
$$

Using this result, we get the Stochastic Euler Equation:

$$F_Y(X_t, Y_t, Z_t) + \beta \mathbb{E}_t \left[F_X(X_{t+1}, Y_{t+1}, Z_{t+1}) G_Y(X_t, Y_t, Z_t) \right] = 0$$

1.4 Examples

1.4.1 Example 1. Robinson Crusoe Economy

Sequential Problem:

$$
\begin{aligned}
\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
s.t. \quad K_{t+1} = f(K_t) + (1-\delta)K_t - c_t
\end{aligned}
$$

Bellman Equation

$$V(K_t) = \max_{\{K_{t+1}\}} u\left(f(K_t) + (1-\delta)K_t - K_{t+1}\right) + \beta V(K_{t+1})$$

FONC:

$$-u'\left(f(K_t) + (1-\delta)K_t - K_{t+1}\right) + \beta V'(K_{t+1}) = 0$$

Using the Benveniste-Sheickman condition:

$$V'(K_t) = u'(f(K_t) + (1-\delta)K_t - K_{t+2})(f'(K_{t+1}) + (1-\delta))$$

Therefore,

$$u'(f(K_t) + (1-\delta)K_t - K_{t+1}) = \beta u'(f(K_{t+1}) + (1-\delta)K_{t+1} - K_{t+2}) \left[f'(K_{t+1}) + (1-\delta) \right]$$

Which can be written as:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta [f'(K_{t+1}) + (1-\delta)]$$

1.4.2 Example 2. A case with labor.

Sequential Problem:

$$\begin{aligned} &\max_{\{c_t,h_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t,h_t) \\ &s.t. \quad K_{t+1} = i_t + (1-\delta)K_t \\ &\qquad\ \ f(K_t,h_t) = Y_t \geq c_t + i_t \end{aligned}$$

Bellman Equation:

$$V(K_t) = \max_{\{K_{t+1},h_t\}} u(f(K_t,h_t) + (1-\delta)K_t - K_{t+1}, h_t) + \beta V(K_{t+1})$$

Note that $G_X(X_t,Y_t) = 0$, so we can apply Benveniste-Sheickman condition, thus

$$\begin{aligned} \{K_{t+1}\}: \quad -u_{K_{t+1}}(f(K_t,h_t) + (1-\delta)K_t - K_{t+1}, h_t) + \beta V'(K_{t+1}) = 0 \\ \{h_t\}: \quad u_c(\cdot) f_h(K_t,h_t) + u_h(\cdot) = 0 \end{aligned}$$

The B-S condition:

$$\begin{aligned} V'(K_t) &= u_c\left(f(K_t,h_t) + (1-\delta)K_t - K_{t+1}, h_t\right)\left(f_K(K_t,h_t) + (1-\delta)\right) \\ V'(K_{t+1}) &= u_c\left(f(K_{t+1},h_{t+1}) + (1-\delta)K_{t+1} - K_{t+2}, h_{t+1}\right)\left(f_K(K_{t+1},h_{t+1}) + (1-\delta)\right) \end{aligned}$$

$$\begin{aligned} \implies u_c(c_t) = \beta u_c(c_{t+1})(f_K(K_{t+1},h_{t+1}) + (1-\delta)) \\ \implies f_h(K_t,h_t) = -\frac{u_h(\cdot)}{u_c(\cdot)} \end{aligned}$$

Notice that, from the last equation, static variables do not change the dynamic solution.

1.4.3 Example 3. What happens if $G_X(X_t,Y_t) \neq 0$?

Numerical approximation

(1) Initial guess: V^0.

(2) Update the process, such that: $V_1(X_t) = \max_{\{Y_t\}} \left[F(X_t,Y_t) + \beta V^0(G(X_t,Y_t))\right]$.

(3) Repeat: $V_2(X_t) = \max_{\{Y_t\}} \left[F(X_t,Y_t) + \beta V^1(G(X_t,Y_t))\right]$.

(4) Repeat again.

The process arrives to a sequence $\{V_i(X_t)\}_{i=0}^{\infty}$

- Bellman showed that under certain regularity conditions that are often met in economic problems, this sequence converges to $V(X_t)$.
- The repeated calculations of $V(X_t)$ also approximates the policy function.
- The limit of the sequence of Y_t's that are found in the maximization process for each X_t are precisely the values that solve $Y_t = H(X_t)$ for that value of X_t.