Solving Linearized Rational Expectations Models

Based on Notes by Eric Sims (Spring 2024)

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May 7, 2025

Graduate Macro Theory II

Introduction

- Discrete time dynamic economic models often result in systems of non-linear difference equations.
- A common solution method is log-linearization around a steady state, transforming them into linear difference equations.
- This presentation outlines how to derive policy functions (decision rules) from such linearized systems.
- While tools like Dynare automate this, understanding the underlying mechanics is crucial.

System Setup and Eigenvalue Decomposition

System Representation

Let X_t be an $(n+m) \times 1$ vector of variables in percentage deviations from steady state.

- n: number of "jump" or "forward-looking" variables.
- m: number of "state" or "predetermined" variables.

Partition X_t as: $X_t = \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}$, where $x_{1,t}$ is $n \times 1$ (jumps) and $x_{2,t}$ is $m \times 1$ (states).

The linearized system can be written as:

$$\mathbb{E}_t X_{t+1} = M X_t \tag{1}$$

Or, more explicitly:

$$\mathbb{E}_t \begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = M \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}$$

M is an $(n+m) \times (n+m)$ matrix derived from model parameters. The challenge: We know initial states $x_{2,0}$, but need to find initial jumps $x_{1,0}$ using a terminal (non-explosion) condition.

Eigenvalue-Eigenvector Decomposition

Recall eigenvalues (λ) and eigenvectors (v) satisfy:

$$Mv = \lambda v \implies (M - \lambda I)v = 0 \tag{2}$$

Assume $n + m$ distinct eigenvalues λ_k and eigenvectors v_k. Stacking these: $M[v_1, \dots, v_{n+m}] = [v_1\lambda_1, \dots, v_{n+m}\lambda_{n+m}]$.

Let $\Gamma = [v_1, \dots, v_{n+m}]$ be the matrix of eigenvectors. Let $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_{n+m})$ be the diagonal matrix of eigenvalues. Then:

$$M\Gamma = \Gamma\Lambda \tag{3}$$

$$M = \Gamma\Lambda\Gamma^{-1} \tag{4}$$

Order eigenvalues by modulus: stable ($|\lambda| < 1$) then unstable ($|\lambda| > 1$).

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}$$

Λ_1 is $Q \times Q$ (stable), Λ_2 is $B \times B$ (unstable), with $Q + B = n + m$.

Canonical Form and Solution Principle

Substitute $M = \Gamma\Lambda\Gamma^{-1}$ into the system:

$$\mathbb{E}_t X_{t+1} = \Gamma\Lambda\Gamma^{-1} X_t$$

Pre-multiply by Γ^{-1}:

$$\mathbb{E}_t(\Gamma^{-1} X_{t+1}) = \Lambda(\Gamma^{-1} X_t)$$

Define auxiliary variables $Z_t = \Gamma^{-1} X_t$. The system becomes:

$$\mathbb{E}_t Z_{t+1} = \Lambda Z_t \tag{5}$$

Canonical Form and Solution Principle

Partition $Z_t = \begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix}$ corresponding to Λ_1 ($Q \times 1$) and Λ_2 ($B \times 1$).

$$\mathbb{E}_t \begin{bmatrix} Z_{1,t+1} \\ Z_{2,t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix}$$

Forward iteration gives:

$$\mathbb{E}_t Z_{1,t+T} = \Lambda_1^T Z_{1,t} \to 0 \text{ as } T \to \infty$$
$$\mathbb{E}_t Z_{2,t+T} = \Lambda_2^T Z_{2,t}$$

For non-explosive paths (transversality/feasibility), we require $Z_{2,t} = 0$.

Deriving the Policy Function

Let Γ^{-1} be partitioned:

$$\Gamma^{-1} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

Dimensions: G_{11} is $Q \times n$, G_{12} is $Q \times m$, G_{21} is $B \times n$, G_{22} is $B \times m$.

Then $Z_t = \Gamma^{-1} X_t$ expands to:

$$Z_{1,t} = G_{11} x_{1,t} + G_{12} x_{2,t}$$
$$Z_{2,t} = G_{21} x_{1,t} + G_{22} x_{2,t}$$

The stability condition $Z_{2,t} = 0$ implies:

$$G_{21} x_{1,t} + G_{22} x_{2,t} = 0$$

Solving for $x_{1,t}$ (the jump variables):

$$G_{21} x_{1,t} = -G_{22} x_{2,t}$$

Deriving the Policy Function

If G_{21} is square and invertible (i.e., $B = n$, the number of unstable roots equals the number of jump variables - saddle path stability):

$$x_{1,t} = -G_{21}^{-1} G_{22} x_{2,t} \tag{6}$$

This is the linearized policy function for jump variables in terms of state variables.

Examples

Example: Deterministic Growth Model

Log-linearized system:

$$-\sigma \tilde{C}_t = -\sigma \mathbb{E}_t \tilde{C}_{t+1} + \beta(\alpha - 1) R \mathbb{E}_t \tilde{K}_{t+1}$$
$$\tilde{K}_{t+1} = \frac{1}{\beta} \tilde{K}_t - \frac{C}{K} \tilde{C}_t$$

Here $n = 1$ ($\tilde{C}_t$), $m = 1$ ($\tilde{K}_t$). In $\mathbb{E}_t X_{t+1} = M X_t$ form :

$$\mathbb{E}_t \begin{bmatrix} \tilde{C}_{t+1} \\ \tilde{K}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 - \frac{\beta(\alpha-1)R}{\sigma} \frac{K}{C} & \frac{(\alpha-1)R}{\sigma} \\ -\frac{C}{K} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \tilde{C}_t \\ \tilde{K}_t \end{bmatrix}$$

Example: Deterministic Growth Model

$$M = \begin{bmatrix} 1 - \frac{C}{K}\frac{\beta(\alpha-1)R}{\sigma} & \frac{(\alpha-1)R}{\sigma} \\ -\frac{C}{K} & \frac{1}{\beta} \end{bmatrix}$$

Parameters: $\sigma = 1, \beta = 0.95, \delta = 0.1, \alpha = 0.33$. Imply $K_{ss} = 3.16$, $C_{ss} = 1.146$.

Numerical M:

$$M = \begin{bmatrix} 1.0352 & -0.1023 \\ -0.3625 & 1.0526 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 0.85$ (stable), $\lambda_2 = 1.24$ (unstable). One jump ($\tilde{C}_t$), one unstable root. Saddle path stability holds.

Example: Deterministic Growth Model (Cont.)

Γ^{-1} (after ordering eigenvalues smallest to largest):

$$\Gamma^{-1} = \begin{bmatrix} -1.0759 & -0.5462 \\ 1.0547 & -0.5861 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

So, $G_{21} = [1.0547]$ and $G_{22} = [-0.5861]$. $Z_{1,t} = -1.0759\tilde{C}_t - 0.5462\tilde{K}_t$
$Z_{2,t} = 1.0547\tilde{C}_t - 0.5861\tilde{K}_t$

Set $Z_{2,t} = 0$ (associated with unstable $\lambda_2 = 1.24$):

$$1.0547\tilde{C}_t - 0.5861\tilde{K}_t = 0$$

Policy function for $\tilde{C}_t$:

$$\tilde{C}_t = \frac{0.5861}{1.0547}\tilde{K}_t = 0.5557\tilde{K}_t$$

This is the policy function in log-deviations.

Example: Deterministic Growth Model (Cont.)

To recover levels: $C_t = C_{ss}\exp(\tilde{C}_t)$, $K_t = K_{ss}\exp(\tilde{K}_t)$. Or using approximation $\tilde{X}_t \approx (X_t - X_{ss})/X_{ss}$:

$$C_t \approx (1 - 0.5557)C_{ss} + 0.5557\frac{C_{ss}}{K_{ss}}K_t = 0.4443C_{ss} + 0.5557\frac{C_{ss}}{K_{ss}}K_t$$

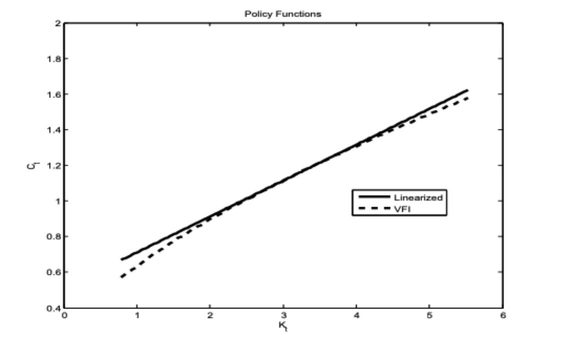

Figure 1: Linearized vs. VFI Policy Function

Example: Stochastic Growth Model

System equations:

$$
\begin{aligned}
C_t^{-\sigma} &= \beta \mathbb{E}_t C_{t+1}^{-\sigma}(\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1-\delta)) \\
K_{t+1} &= A_t K_t^{\alpha} - C_t + (1-\delta) K_t \\
\ln A_t &= \rho \ln A_{t-1} + e_t
\end{aligned}
$$

Example: Stochastic Growth Model

Log-linearized equations:

$$
\begin{aligned}
\mathbb{E}_t \tilde{C}_{t+1} &= \tilde{C}_t + \frac{\beta(\alpha-1)R}{\sigma}\mathbb{E}_t \tilde{K}_{t+1} + \frac{\beta R}{\sigma}\mathbb{E}_t \tilde{A}_{t+1} \\
\tilde{K}_{t+1} &= \frac{1}{\beta}\tilde{K}_t - \frac{C}{K}\tilde{C}_t + K^{\alpha-1}\tilde{A}_t \\
\mathbb{E}_t \tilde{A}_{t+1} &= \rho \tilde{A}_t
\end{aligned}
$$

Variables: $\tilde{C}_t$ (jump, $n = 1$), $\tilde{K}_t, \tilde{A}_t$ (states, $m = 2$). Total $n + m = 3$.

Example: Stochastic Growth Model

The matrix M in $\mathbb{E}_t X_{t+1} = MX_t$ with $X_t = [\tilde{C}_t, \tilde{K}_t, \tilde{A}_t]'$ (after substituting $\mathbb{E}_t \tilde{A}_{t+1} = \rho \tilde{A}_t$ into the Euler):

$$M = \begin{bmatrix} 1 - \frac{\beta(\alpha-1)R}{\sigma}\frac{C}{K} & \frac{(\alpha-1)R}{\sigma} & \frac{\beta R(\rho+(\alpha-1)K^{\alpha-1})}{\sigma} \\ -\frac{C}{K} & \frac{1}{\beta} & K^{\alpha-1} \\ 0 & 0 & \rho \end{bmatrix}$$

Using parameters from before and $\rho = 0.95$, eigenvalues: 0.8512, 0.95, 1.2367. One jump variable ($\tilde{C}_t$), one unstable root (1.2367). Saddle path stability holds.

Example: Stochastic Growth Model (Cont.)

Γ^{-1} (after ordering eigenvalues):

$$\Gamma^{-1} = \begin{bmatrix} -1.0759 & -0.5462 & 3.5671 \\ 0 & 0 & 3.4172 \\ 1.0547 & -0.5861 & -0.6041 \end{bmatrix}$$

The unstable eigenvalue $\lambda_3 = 1.2367$ corresponds to the 3rd row of Γ^{-1}. So $Z_{2,t}$ is the 3rd element of Z_t. States $x_{2,t} = [\tilde{K}_t, \tilde{A}_t]'$. Jump $x_{1,t} = \tilde{C}_t$. G_{21} (for $Z_{2,t}$ row, $x_{1,t}$ column) $= [1.0547]$. G_{22} (for $Z_{2,t}$ row, $x_{2,t}$ columns) $= [-0.5861 \quad -0.6041]$.

Example: Stochastic Growth Model

Set $Z_{2,t} = G_{21} x_{1,t} + G_{22} x_{2,t} = 0$:

$$1.0547\tilde{C}_t + \begin{bmatrix} -0.5861 & -0.6041 \end{bmatrix} \begin{bmatrix} \tilde{K}_t \\ \tilde{A}_t \end{bmatrix} = 0$$

Policy function for $\tilde{C}_t$:

$$\tilde{C}_t = -\frac{1}{1.0547} \begin{bmatrix} -0.5861 & -0.6041 \end{bmatrix} \begin{bmatrix} \tilde{K}_t \\ \tilde{A}_t \end{bmatrix}$$

$$\tilde{C}_t = 0.5557\tilde{K}_t + 0.5728\tilde{A}_t$$

Example: Stochastic Growth Model