Macroeconomics II.
Problem Set I.

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1. Suppose that you have a random variable, x_t, which obeys the following MA(4):

$$x_t = \epsilon_t + 0.9\epsilon_{t-1} + 0.7\epsilon_{t-2} + 0.4\epsilon_{t-3} - 0.1\epsilon_{t-4}$$

$$\epsilon_t \sim N(0,1)$$

The mean and variance of x_t are given by:

$$E(x_t) = 0$$

$$var(x_t) = \left(1 + 0.9^2 + 0.7^2 + 0.4^2 + (-0.1)^2\right) = 2.47$$

2. After simulating the process and dropping the first 100 observations, the mean and variance of the simulated $\tilde{x_t}$are given by:

$$E(\tilde{x_t}) = -0.0051$$

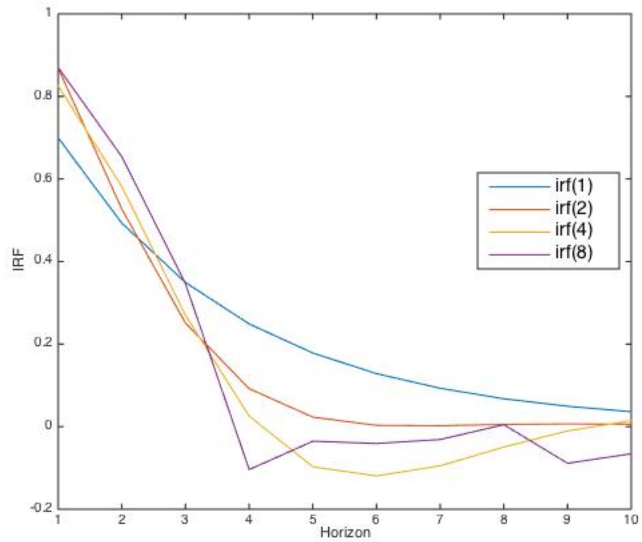
$$var(\tilde{x_t}) = 2.4544$$

As expected, the simulated mean and variance is similar to the theoretical values.

3. Now, I estimate the MA(4) coefficients by using the IRFs of an AR(p) process. I use three different lag lengths: p=1, p=2, p=4 and p=8. The IRFs are constructed up to horizon H=10.

	AR(1)	AR(2)	AR(4)	AR(8)
h=1	0.7012	0.8694	0.8275	0.8709
h=2	0.4946	0.5292	0.5825	0.6545
h=3	0.3510	0.2536	0.2736	0.3468
h=4	0.2505	0.0944	0.0307	-0.0976
h=5	0.1797	0.0245	-0.0940	-0.0339
h=6	0.1295	0.0039	-0.1186	-0.0388
h=7	0.0938	0.0029	-0.0963	-0.0329
h=8	0.0683	0.0057	-0.0521	-0.0042
h=9	0.0499	0.0070	-0.0125	-0.0960
h=10	0.0366	0.0063	0.0137	-0.0762

As can be observed, as p increases, the coefficients of the first 4 horizons of the IRFs approach to the true values of the original MA(4). This is explained by the fact that the IRF is essentially a plot of the coefficients of the MA representation of the time series.



4. **The Hodrick-Prescott Filter**

(a) The HP filter idea is to separate a time-series into a trend and a cyclical component. It is a constrained minimization. What the objective function is doing is to minimize squared deviations and reducing the variation of the trend by imposing a penalty, λ. In this case, the constraint tries to reduce variation of the trend.

(b) If $\lambda = 0$,

$$\min_{\tau_t} \sum_{t=1}^{T} (y_t - \tau_t)^2$$

The FOC yield to:

$$y_t = \tau_t$$

And therefore, the trend is equal to the time series.

(c) If $\lambda = \infty$,

$$\min_{\tau_t} \sum_{t=1}^{T} (y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} \Big((\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1}) \Big)^2$$

Becomes,

$$\min_{\tau_t} \sum_{t=1}^{T} (y_t - \alpha t)^2 + \lambda \sum_{t=2}^{T-1} \Big((\alpha(t+1) - \alpha t) - (\alpha t - \alpha(t-1)) \Big)^2$$

$$\min_{\tau_t} \sum_{t=1}^{T} (y_t - \alpha t)^2$$

And the FOC are given by:

$$y_t = \alpha t$$

Which is linear.

(d) If $0 < \lambda < \infty$,

$$\min_{\tau_t} \sum_{t=1}^{T} (y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} \Big((\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1}) \Big)^2$$

The FOC are given by,

$$t = 1: \quad y_1 = (1 + \lambda)\, \tau_1 - 2\lambda \tau_2 + \lambda \tau_3$$

$$t = 2: \quad y_2 = -2\lambda\tau_1 + (1+5\lambda)\,\tau_2 - 4\lambda\tau_3 + \lambda\tau_4$$
$$t = 3: \quad y_3 = \lambda\tau_1 - 4\lambda\tau_2 + (1+6\lambda)\,\tau_3 - 4\lambda\tau_4 + \lambda\tau_5$$
$$t = t: \quad y_t = \lambda\tau_{t-2} - 4\lambda\tau_{t-1} + (1+6\lambda)\,\tau_t - 4\lambda\tau_{t+1} + \lambda\tau_{t+2}$$
$$t = T-1: \quad y_{T-1} = \lambda\tau_{T-3} - 4\lambda\tau_{T-2} + (1+5\lambda)\,\tau_{T-1} - 2\lambda\tau_T$$
$$t = T: \quad y_T = \lambda\tau_{T-2} - 2\lambda\tau_{T-1} + (1+\lambda)\,\tau_T$$

Using this set of equations, we can write the system as,

$$\mathit{\Lambda}\mathbf{T} = \mathbf{Y}$$

where the trend will be given by,

$$\mathbf{T} = \mathit{\Lambda}^{\mathbf{-1}}\mathbf{Y}$$

(e) Using quarterly, seasonally adjusted data on US real GDP from 1947q1 to 2014q3, the standard deviation of the cycle is 0.016. The trend and cyclical components of the real GDP are: