Macroeconomics II.
Problem Set II.

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1. **Comparing solution techniques for a Deterministic Growth Model.**

   The equilibrium of the economy can be described as the solution to the following social planner's problem:

$$\max_{c_t,\, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

$$s.t \quad k_{t+1} = k_t^{\alpha} - c_t + (1-\delta)k_t$$

   (a) Using the L'hopital rule, we can prove that as $\sigma \rightarrow 1$, $u(c_t) \rightarrow \ln(c_t)$,

$$\lim_{(1-\sigma)\to 0} \frac{c_{1t}^{1-\sigma} - 1}{1-\sigma} = \lim_{(1-\sigma)\to 0} c_t^{1-\sigma} \ln(c_t) = \ln(c_t)$$

   (b) The social planner's problem can be written as a dynamic programming problem,

$$v(k) = \max_{k_{t+1}} \left[u\left(\frac{c^{1-\sigma} - 1}{1-\sigma} \right) + \beta * V(k') \right]$$

$$s.t \quad k' = k^{\alpha} - c + (1-\delta)k$$

$$k_0 \ given$$

   where k_t is a state variable and c_t is a control variable.

   (c) This problem can be written as,

$$v(k) = \max_{k'} \left[\left(\frac{(k^{\alpha} - k' + (1-\delta)k)^{1-\sigma} - 1}{1-\sigma} \right) + \beta V(k') \right]$$

   FONC:

$$\frac{\partial V}{\partial k'} = -\left[\left((k^{\alpha} - k' + (1-\delta)k)^{-\sigma} \right) \right] + \beta V'(k') = 0$$

   Using the Benveniste-Sheinkman envelope theorem,

$$V'(k) = \frac{\partial U}{\partial k} = \left((k^{\alpha} - k' + (1-\delta)k)^{-\sigma} \right) \times \left(\alpha k^{\alpha-1} + (1+\delta) \right)$$

   which means that,

$$V'(k') = \left((k'^{\alpha} - k'' + (1-\delta)k')^{-\sigma} \right) \times \left(\alpha k'^{\alpha-1} + (1+\delta) \right)$$

   Therefore,

$$\left[\left((k^{\alpha} - k' + (1-\delta)k)^{-\sigma} \right) \right] = \beta \left[\left((k'^{\alpha} - k'' + (1-\delta)k')^{-\sigma} \right) \times \left(\alpha k'^{\alpha-1} + (1+\delta) \right) \right]$$

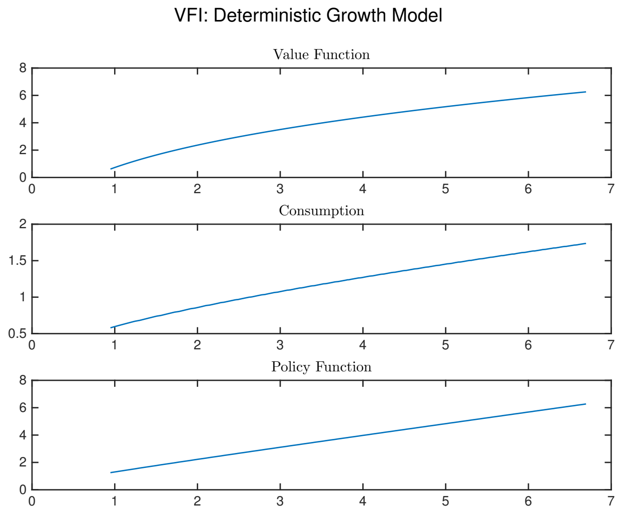


Figure 1: Deterministic Growth Model

(d) Steady State:

$$k^* = \left(\frac{\alpha}{\frac{1}{\beta} - (1-\delta)}\right)^{\frac{1}{1-\alpha}}$$
$$c^* = (k^*)^\alpha - \delta k^*$$

(e) Solving numerically, I find the value and policy functions represented in figure (1).

(f) The same problem written as a Lagrangian,

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma}\right) + \lambda\left(k_t^\alpha + (1-\delta)\, k_t - k_{t+1}\right)$$

FONC:

$$\lambda_t = \lambda_{t+1}\left(\alpha k_{t+1}^{\alpha-1} + (1-\delta)\right)$$
$$\lambda_t = \beta^t c_t^{-\sigma}$$
$$\lambda_{t+1} = \beta^{t+1} c_{t+1}^{-\sigma}$$

Therefore, the Euler equation:

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma}\left(\left(\alpha k_{t+1}^{\alpha-1} + (1-\delta)\right)\right)$$

Transversality condition,

$$\lim_{t\to\infty} \beta^t u'(c_t)\, k_t = 0$$

Which states that the marginal benefit of any over (under) savings of capital is zero when t approaches to infinity. In other words, it rules out the possibility of saving too much capital.

(g) The Log-linearized first order conditions around the steady state are:

$$\sigma \tilde{c}_{t+1} = \sigma \tilde{c}_t + \beta\left(\alpha\left(\alpha-1\right)k\right)$$
$$\tilde{k}_{t+1} = \frac{1}{\beta}\tilde{k}_t - \frac{c^*}{k^*}\tilde{c}_t$$

Which yields to the following VAR(1) system,

$$\begin{bmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 - \frac{\beta\alpha(\alpha-1)(k^*)^{\alpha-1}}{\sigma}\frac{c^*}{k^*} & \frac{\alpha(\alpha-1)(k^*)^{\alpha-1}}{\sigma} \\ \frac{1}{\beta} & -\frac{c^*}{k^*} \end{bmatrix} \begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \end{bmatrix}$$

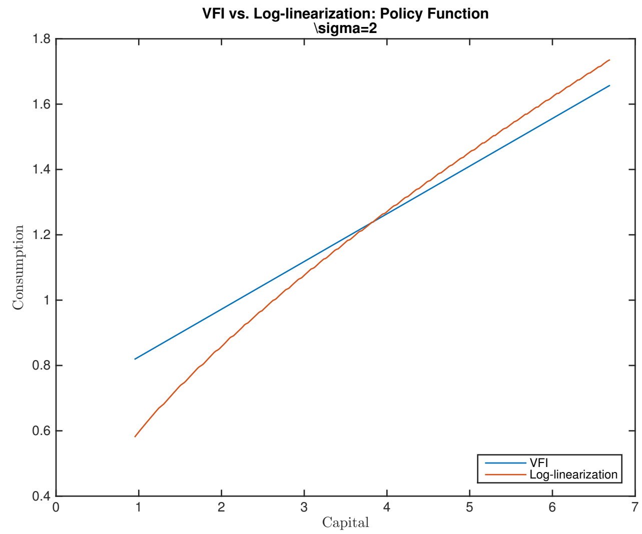
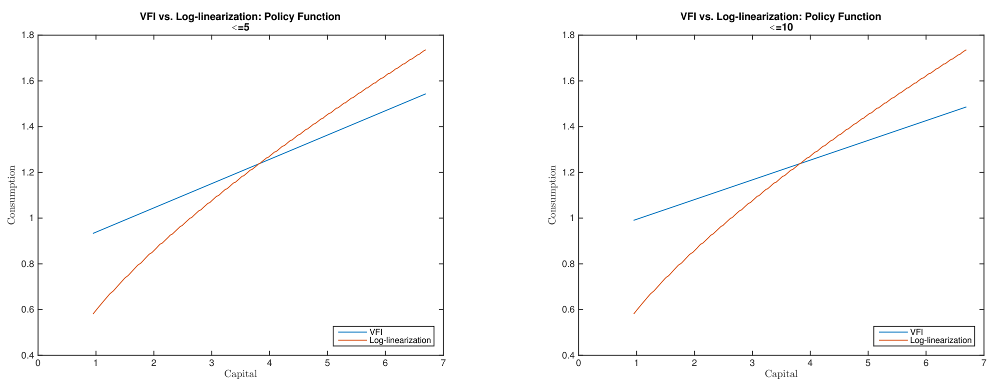


Figure 2: VFI vs. Log-linearization: Policy Function.



(a) $\sigma = 5$

(b) $\sigma = 10$

Figure 3: VFI vs Log-Linearization: Different σ

(h) The numerical policy function for the log-linear system is: $c_t = 0.6102 * k_t$

(i) Figure (2) Comparing the linearized policy function and the value function iteration shows that the approximation is accurate around the steady state capital. However, as the capital moves away from the steady state level, the linearization loses accuracy.

(j) When increasing the coefficient of relative risk aversion, the concavity of the utility function changes dramatically. This implies that the accuracy of the linearization around the steady state also decreases dramatically. Figure (3)shows the cases for $\sigma = 5$ and $\sigma = 10$.

2. **Comparing Impulse Responses: VFI vs. Linearization.**

In this question, I compare the impulse response functions obtained through VFI and log-linearization of the basic stochastic growth model. The equations characterizing the equilibrium are:

$$\frac{1}{c_t^{\sigma}} = \beta E_t \left[\frac{1}{c_t^{\sigma}} \left(\alpha A_{t+1} k_{t+1}^{\alpha - 1} + (1 - \delta) \right) \right]$$

$$k_{t+1} = A_t k_t^{\alpha} - c_t + (1 - \delta) k_t$$

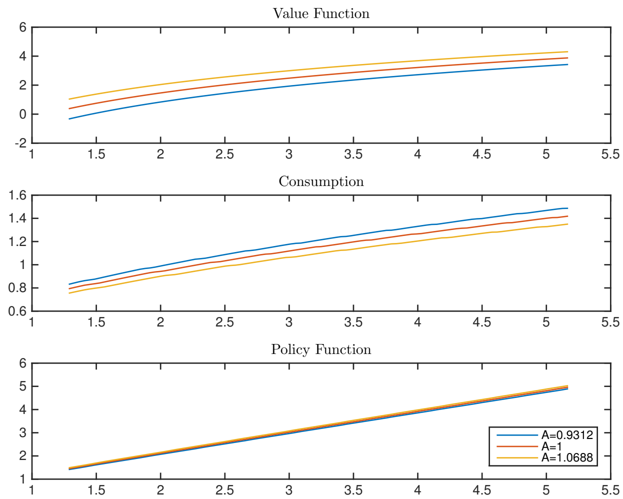


Figure 4: VFI: Stochastic Growth Model. Value and Policy Functions

$$A_t = (1-\rho) + \rho A_{t-1} + \epsilon_t$$

(a) The Bellman equation is:

$$V(A,\ K) = \max_{k'} \left(AK^{\alpha} - k' + (1+\delta)\, K \right)^{\sigma} + \beta E\left[V\left(A',\, k'\right) \right]$$

The model was estimated for the following parameters: $\sigma = 2$, $\alpha = 0.33$, $\delta = 0.1$, $\sigma_{\epsilon} = 0.01$

(b) The value and policy functions are shown in figure(4) . Each line correspond to a possible state of A. For simplicity, only those corresponding to A=1, 4 and 7 were depicted. In total, A can take seven possible values.

(c) Stochastic impulse response functions. Figure (5) shows the responses for capital, consumption, output, investment and productivity. The IRFs were computed after 10000 simulations. The simulated shock consisted in a change from A=1 to A=1.0229. As expected, a permanent positive shock in productivity produces a temporary boost in consumption, investment and output.

(d) **Log-linear Stochastic Growth Model.** Now, I estimated the same model but using log-linearization. Figure (5) shows the responses for each of the interest variables. By comparing figures (5) and (6) it is possible to notice that results produced by the VFI and the linearization are very similar in terms of direction. However, when analyzing the jump in consumption estimated by the VFI is smaller than that predicted by the linearization. Consequently, since the output jumps about the same in both methods, investment's response in the linearization is smaller than in the VFI.

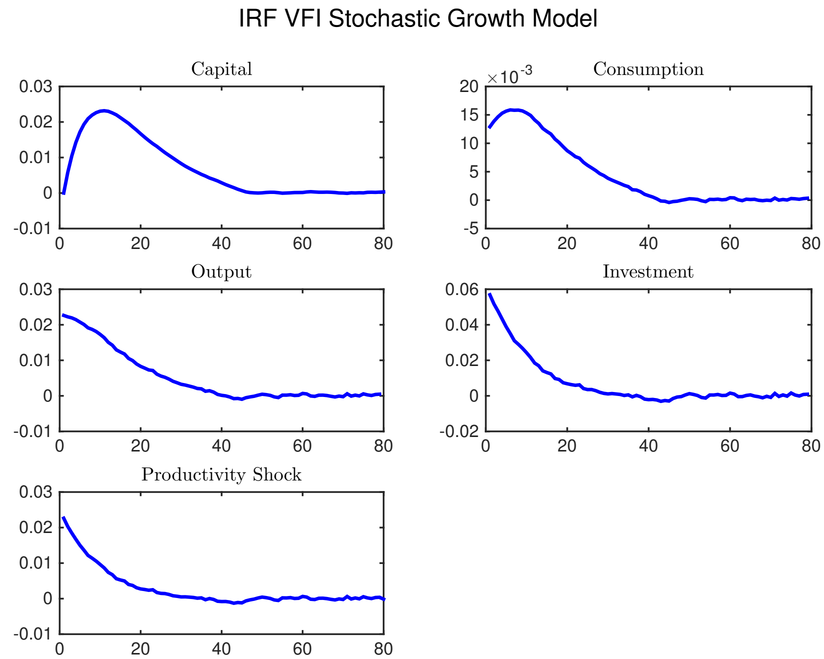


Figure 5: IRFs Stochastic Value Function Iteration.

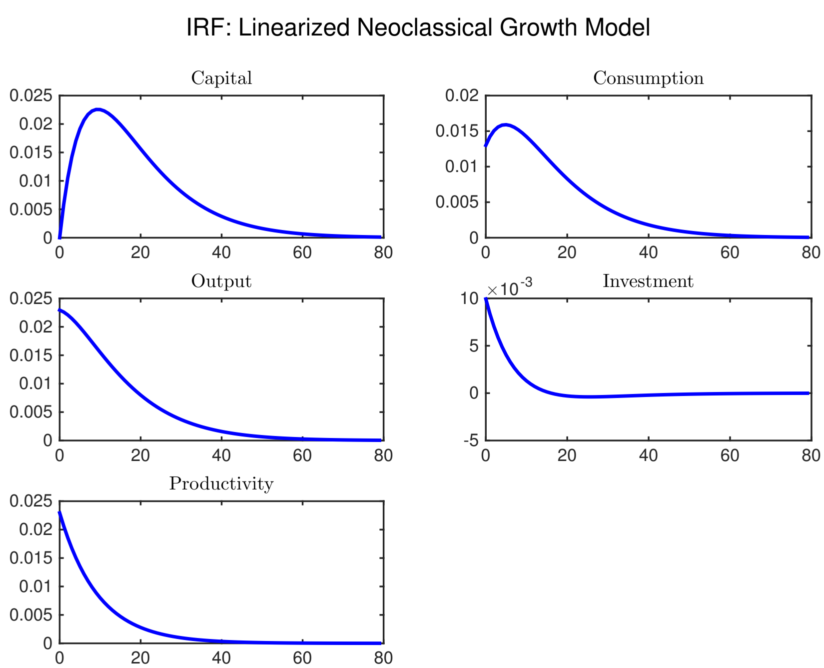


Figure 6: IRFs linearized Stochastic Growth Model.