

# MPC for Group Reconstruction Circuits

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June 12, 2022

## Abstract

In this paper, we present a thing.

## 1 Introduction

Write the introduction

## 2 Background

Throughout this paper, we let  $\mathbb{G}$  denote a group of prime order  $q$ , with generators  $G$  and  $H$ . Let  $\mathbb{F}_q$  denote the scalar field associated with this group, and let  $\mathbb{Z}/(q)$  denote the additive group of elements in this field. We also define  $[n] := [1, \dots, n]$ .

We make heavy use of group homomorphisms throughout this paper. We let

$$\varphi(P_1, \dots, P_m) : \mathbb{A} \rightarrow \mathbb{B}$$

denote a homomorphism from  $\mathbb{A}$  to  $\mathbb{B}$ , parameterized by some public values  $P_1, \dots, P_m$ . Commonly  $\mathbb{A}$  will be a product of several groups  $\mathbb{G}_1, \dots, \mathbb{G}_n$ , in which case we'd write:

$$\varphi(P_1, \dots, P_m)(x_1, \dots, x_n)$$

to denote the application of  $\varphi$  to an element  $(x_1, \dots, x_n)$  of the product group. The public values  $P_i$  are often left implicit.

We often write products  $(x_1, \dots, x_n)$  as a single vector  $\mathbf{x} \in \mathbb{A}^n$ . Operations between these vectors are done element-wise, so we write  $\mathbf{x} + \mathbf{y}$  for  $(x_1 + y_1, \dots, x_n + y_n)$ , as well as  $\mathbf{x} \cdot G$  for  $(x_1 \cdot G, \dots, x_n \cdot G)$ .

### 2.1 Pedersen Commitments

A key component of our scheme are Pedersen commitments [Ped92]. In their basic form, they allow one to commit to a value in  $x \in \mathbb{Z}/(q)$ . This is done by sampling a random  $\alpha \xleftarrow{R} \mathbb{Z}/(q)$ , and forming the commitment:

$$\text{Com}(x, \alpha) := x \cdot G + \alpha \cdot H$$

where  $H$  is a generator of  $\mathbb{G}$ , independent from  $G$ .

This scheme is *perfectly hiding*, because  $\alpha \cdot H$  acts like a random element of  $\mathbb{G}$ , and completely masks  $x \cdot G$ .

On the other hand, this scheme is only *computationally* binding. This is because the discrete logarithm  $H$  with respect to  $G$  must be kept hidden. If the discrete logarithm of  $H$  is known, then it becomes possible to *equivocate*, by finding two different inputs  $(x, \alpha)$  and  $(x', \alpha')$  with equal commitments, i.e.  $\text{Com}(x, \alpha) = \text{Com}(x', \alpha')$ .

In fact, we can more precisely characterize this property: knowing the discrete logarithm of  $H$  is *necessary* in order to be able to equivocate.

**Claim 2.1.** Given two inputs  $(x, \alpha) \neq (x', \alpha')$  such that  $\text{Com}(x, \alpha) = \text{Com}(x', \alpha')$ , it's possible to efficiently compute the discrete logarithm of  $H$ .

The proof is just a matter of algebra:

$$\begin{aligned} x \cdot G + \alpha \cdot H &= x' \cdot G + \alpha' \cdot H \\ (x - x') \cdot G &= (\alpha' - \alpha) \cdot H \\ \frac{(x - x')}{(\alpha' - \alpha)} \cdot G &= H \end{aligned}$$

Thus  $(x - x')/(\alpha' - \alpha)$  is our discrete logarithm.

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### 2.1.1 Vector Pedersen Commitments

It's useful to generalize this scheme to the case of a vector of scalars  $\mathbf{x} \in \mathbb{Z}/(q)^n$ . The randomness becomes a vector of the same size, sampled as  $\boldsymbol{\alpha} \xleftarrow{R} \mathbb{Z}/(q)^n$ . We then define an analogous commitment scheme by doing scalar multiplication element-wise:

$$\text{Com}(\mathbf{x}, \boldsymbol{\alpha}) := \mathbf{x} \cdot G + \boldsymbol{\alpha} \cdot H$$

This generalization is naturally also perfectly hiding, and satisfies an analogous property with regards to equivocation:

**Claim 2.2.** Given two inputs  $(\mathbf{x}, \boldsymbol{\alpha}) \neq (\mathbf{x}', \boldsymbol{\alpha}')$  such that  $\text{Com}(\mathbf{x}, \boldsymbol{\alpha}) = \text{Com}(\mathbf{x}', \boldsymbol{\alpha}')$ , it's possible to efficiently compute the discrete logarithm of  $H$ .

Since the two inputs are different, there must exist an index  $i$  such that  $\mathbf{x}_i \neq \mathbf{x}'_i$  or  $\boldsymbol{\alpha}_i \neq \boldsymbol{\alpha}'_i$ . From here we apply Claim 2.1 with  $(\mathbf{x}_i, \boldsymbol{\alpha}_i)$  and  $(\mathbf{x}'_i, \boldsymbol{\alpha}'_i)$ .

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### 2.1.2 On the Trusted Setup

In theory, Pedersen commitments require a trusted setup, to generate the group elements  $G, H \in \mathbb{G}$ . In practice, we argue that this trusted setup isn't a concern. This is the generator  $G$  is usually part of the specification for the group being used, and because there exist efficient methods to hash into elliptic curves [Ica09]. This reduces the problem of generating  $H$  to that of finding a credibly “unbiased” choice of seed to hash. This can be done in many ways.

One way would be to hash a canonical representation of  $G$  as bytes in order to produce  $H$ . Presumably, the generator  $G$  was not chosen in such a way as to produce an  $H$  with a known discrete logarithm, with that specific method of hashing into the group.

Another method would be to use a public source of randomness, such as public newspapers, lotteries [BDF<sup>+</sup>15], or Cryptographic protocols designed to provide such a service [FIP11].

Now, in practice while we can avoid a trusted setup with these methods, for our security analysis we do actually make use of this setup. Essentially, we prove that the security of our protocol reduces to the hardness of the discrete logarithm problem in  $\mathbb{G}$ , and to do this we need to be able to use an instance of the problem as a setup for the participants in our simulation.

## 2.2 Sigma Protocols

### 2.3 Maurer's $\varphi$ -Proof

In [Mau09], Maurer generalized Schnorr's sigma protocol for knowledge of the discrete logarithm [Sch90] to a much larger class of relations. In particular, Maurer provided a sigma protocol for proving knowledge of the pre-image of a group homomorphism  $\varphi$ . We denote this protocol as a “ $\varphi$ -proof”, and recapitulate the scheme here.

Given a homomorphism  $\varphi : \mathbb{A} \rightarrow \mathbb{B}$ , and a public value  $X \in \mathbb{B}$ , the prover wants to demonstrate knowledge of a private value  $x \in \mathbb{A}$  such that  $\varphi(x) = X$ . The prover does this by means of Protocol 2.1:

**Protocol 2.1:  $\varphi$ -Proof**

Prover	Verifier
knows $x \in \mathbb{A}$	public $X \in \mathbb{B}$
$k \xleftarrow{R} \mathbb{A}$	
$K \leftarrow \varphi(k)$	
	$\xrightarrow{K}$
	$c \xleftarrow{R} \mathbb{Z}/(p)$
	$\xleftarrow{c}$
$s \leftarrow k + c \cdot x$	
	$\xrightarrow{s}$
	$\varphi(s) \stackrel{?}{=} K + c \cdot X$

Here,  $p$  is chosen such that  $\forall B \in \mathbb{B}. p \cdot B = 0$ . In practice, we'll set  $p = q$ , which will work perfectly for the groups we use, which are all products of  $\mathbb{G}$  or  $\mathbb{Z}/(q)$ .

**Claim 2.3.** Protocol 2.1 is a valid sigma protocol.

Completeness follows directly from the fact that  $\varphi$  is a homomorphism.

For the HVZK property, the simulator  $\mathcal{S}(X, c)$  works by generating a random  $s \xleftarrow{R} \mathbb{A}$ , and then setting  $K := \varphi(s) - c \cdot X$ .

Finally, we prove 2-extractability. Given two verifying transcripts  $(K, c, s)$  and  $(K, c', s')$  sharing the first message, we extract a value  $\hat{x}$  satisfying  $\varphi(\hat{x}) = X$  as follows:

$$\begin{aligned}
 \varphi(s) - c \cdot X &= K = \varphi(s') - c' \cdot X \\
 \varphi(s) - \varphi(s') &= c \cdot X - c' \cdot X \\
 \frac{1}{c - c'} \cdot \varphi(s - s') &= X \\
 \varphi\left(\frac{s - s'}{c - c'}\right) &= X
 \end{aligned}$$

Thus, defining  $\hat{x} := (s - s')/(c - c')$ , we successfully extract a valid pre-image.

We conclude that the protocol is a valid sigma protocol.

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Maurer’s protocol can also work even in the case where the order of the groups are not known, but this makes the challenge generation a bit more complicated, and we don’t need this functionality in this work.

## 2.4 UC Security and the Hybrid Model

## 2.5 Ideal Functionalities for Sigma Protocols

### Functionality 2.1: Zero-Knowledge Functionality $\mathcal{F}(\text{ZK}, \varphi)$

A functionality  $\mathcal{F}$  for parties  $P_1, \dots, P_n$ .

On input  $(\text{prove}, \text{sid}, x)$  from  $P_i$ :

$\mathcal{F}$  checks that **sid** has not been used by  $P_i$  before.

$\mathcal{F}$  generates a new token  $\pi$ , and sets  $x_\pi \leftarrow x$ .

$\mathcal{F}$  replies with  $(\text{proof}, \pi)$ .

On input  $(\text{verify}, X, \pi)$ :

$\mathcal{F}$  replies with  $(\text{verify-result}, \varphi(x_\pi) \stackrel{?}{=} X)$ .

## 2.6 Broadcast Functionalities

### Functionality 2.2: Authenticated Broadcast Functionality $\mathcal{C}$

A functionality  $\mathcal{C}$  for parties  $P_1, \dots, P_n$ .

On receiving  $(\text{broadcast-in}, \text{sid}, m)$  from  $P_i$ :

$\mathcal{C}$  checks that **sid** has not been used by  $P_i$  before.

$\mathcal{C}$  sends  $(\text{broadcast-out}, \text{pid}_i, \text{sid}, m)$  to every party  $P_j$ .

### 3 Group Reconstruction Circuits

#### 3.1 Formal Definition

#### 3.2 Normalized Form

### 4 MPC Protocol for GRCs

#### 4.1 Ideal Functionality

**Functionality 4.1:** GRC functionality  $\mathcal{F}(\text{GRC}, \Phi, \mathbf{X}^j, \mathbf{Y}^j)$

A functionality  $\mathcal{F}$  for parties  $P_1, \dots, P_n$ .

After receiving  $(\text{input}, \text{sid}, \mathbf{x}^j, \mathbf{y}^j, \boldsymbol{\alpha}^j, \mathbf{k}^j)$  from every party  $P_j$ :  
 $\mathcal{F}$  checks, for every  $j \in [n]$ , that:

$$\mathbf{X}^j \stackrel{?}{=} \mathbf{x}^j \cdot G$$

$$\mathbf{Y}^j \stackrel{?}{=} \mathbf{y}^j \cdot G + \boldsymbol{\alpha}^j \cdot H$$

$\mathcal{F}$  computes, for each round  $r \in [d]$ :

$$\mathbf{V}_r^j := \varphi_r(\mathbf{V}_1, \dots, \mathbf{V}_{r-1})(\mathbf{x}^j, \mathbf{y}^j, \mathbf{k}^j)$$

$$\mathbf{V}_r := \sum_j \mathbf{V}_r^j$$

$\mathcal{F}$  sends  $(\text{output}, \text{sid}, \mathbf{V}_1^1, \dots, \mathbf{V}_d^n)$  to every party  $P_j$ .

#### 4.2 Protocol

$$\psi_r(\mathbf{x}, \mathbf{y}, \boldsymbol{\alpha}, \mathbf{k}, \boldsymbol{\beta}) := (\varphi_r(\mathbf{x}, \mathbf{y}, \mathbf{j}), \mathbf{x} \cdot G, \text{Commit}(\mathbf{y}, \boldsymbol{\alpha}), \text{Commit}(\mathbf{k}, \boldsymbol{\beta}))$$

**Protocol 4.1: MPC protocol for  $\Phi, \mathbf{X}^j, \mathbf{Y}^j$** 

Each party  $P_j$  has inputs  $\mathbf{x}^j$  and  $\mathbf{y}^j$ , committed to by  $\mathbf{X}^j$  and  $\mathbf{Y}^j$ . They also have decommitments  $\alpha^j$  for  $\mathbf{Y}^j$ . Each party  $P_j$  also has a vector  $\mathbf{k}^j$ , which honest parties will have generated randomly.

**Round 0**

Each party  $P_j$  generates a random vector  $\beta^j$ , and creates a commitment to  $\mathbf{k}^j$  with:

$$\mathbf{K}^j := \text{Commit}(\mathbf{k}^j, \beta^j)$$

$P_j$  sends (`broadcast-in`, `sid`,  $\mathbf{K}^j$ ) to the broadcast functionality  $\mathcal{C}$ .

$P_j$  waits to receive (`broadcast-out`, `sid`,  $\mathbf{K}^i$ ) for each other party  $i$ .

**Round  $r$** 

Each party  $P_j$  computes  $\mathbf{V}_r^j := \varphi_r(\mathbf{V}_1, \dots, \mathbf{V}_{r-1})(\mathbf{x}^j, \mathbf{y}^j, \mathbf{k}^j)$ .

Each party  $P_j$  sends (`prove`, `sid`,  $(\mathbf{x}^j, \mathbf{y}^j, \alpha^j, \mathbf{k}^j, \beta^j)$ ) to  $\mathcal{F}(\text{ZK}, \psi_r)$ , receiving  $\pi_r^j$  in return.

Each party  $P_j$  sends  $(\mathbf{V}_r^j, \pi_r^j)$  to every other party.

After receiving  $(\mathbf{V}_r^i, \pi_r^i)$  from all other parties,  $P_j$  checks, for each  $i$ , that the proof is valid, by sending (`verify`,  $(\mathbf{V}_r^i, \mathbf{X}^i, \mathbf{Y}^i, \mathbf{K}^i)$ ,  $\pi_r^i$ ) to  $\mathcal{F}(\text{ZK}, \psi_r)$ , and aborting if the functionality returns 0.

Each party  $P_j$  then stores each  $\mathbf{V}_r^i$  as part of its output.

**Claim 4.1.** Provided that the discrete logarithm is hard in  $\mathbb{G}$ , Protocol 4.1 securely implements Functionality 4.1, in the hybrid model of universally composable security, given a zk functionality  $\mathcal{F}(\text{ZK}, \varphi)$  (for arbitrary  $\varphi$ ), a broadcast functionality  $\mathcal{C}$ , as well as a common reference string  $(G, H) \in \mathbb{G}^2$ .

**Proof:**

We prove this by constructing a simulator  $\mathcal{S}$  which uses the ideal functionality  $\mathcal{F}(\text{GRC})$  to perfectly simulate an execution of the hybrid protocol against an adversary  $\mathcal{A}$ .

We also work in the common reference string model, where the simulator  $\mathcal{S}$  chooses the bases  $(G, H)$  for the Pedersen commitments.

We use this simulator  $\mathcal{S}$  to construct an adversary against the discrete logarithm game.

Let  $\mathcal{M} \subseteq \mathcal{P}$  be the set of malicious parties, and  $\mathcal{H} \subseteq \mathcal{P}$  be the set of honest parties. Naturally, we have  $\mathcal{H} \cup \mathcal{M} = \mathcal{P}$  and  $\mathcal{H} \cap \mathcal{M} = \emptyset$ .

As an adversary against the discrete logarithm game,  $\mathcal{S}$  receives  $(G, H)$  as an instance of the discrete logarithm problem.

The simulator then proceeds as follows:

$\mathcal{S}$  starts by setting  $(G, H)$  as the common reference string.

**Round 0:**

For each  $j \in \mathcal{H}$ ,  $\mathcal{S}$  samples  $\mathbf{K}^j \xleftarrow{R} \mathbb{G}$ .

For each  $j \in \mathcal{M}$ ,  $\mathcal{S}$  waits to receive  $(\text{broadcast-in}, \text{sid}, \mathbf{K}^j)$ .

$\mathcal{S}$  then sends  $(\text{broadcast-out}, \text{pid}_j, \text{sid}, \mathbf{K}^j)$ , to all parties, for every  $j \in \mathcal{P}$ , emulating  $\mathcal{C}$ .

**Interim:**

$\mathcal{S}$  waits to receive  $(\text{prove}, \text{sid}, (\mathbf{x}^j, \mathbf{y}^j, \boldsymbol{\alpha}^j, \mathbf{k}^j, \boldsymbol{\beta}^j))$  for each malicious  $j \in \mathcal{M}$ , playing the role of  $\mathcal{F}(\text{ZK}, \psi_1)$ .

$\mathcal{S}$  checks, for each  $j$ , that:

$$\mathbf{X}^j \stackrel{?}{=} \mathbf{x}^j \cdot G$$

$$\mathbf{Y}^j \stackrel{?}{=} \mathbf{y}^j \cdot G + \boldsymbol{\alpha}^j \cdot H$$

$$\mathbf{K}^j \stackrel{?}{=} \mathbf{k}^j \cdot G + \boldsymbol{\beta}^j \cdot H$$

otherwise,  $\mathcal{S}$  sets  $\text{bad-values}_1^j \leftarrow 1$ .

$\mathcal{S}$  records the values  $\mathbf{x}^j, \mathbf{y}^j, \boldsymbol{\alpha}^j, \mathbf{k}^j, \boldsymbol{\beta}^j$ , for  $j \in \mathcal{M}$ .

Now, in the real execution against  $\mathcal{F}(\text{GRC})$ , with real honest parties  $P_i$ , for each  $j \in \mathcal{M}$ , the parties  $\mathcal{S}$  controls,  $\mathcal{S}$  sends  $(\text{input}, \text{sid}, \mathbf{x}^j, \mathbf{y}^j, \boldsymbol{\alpha}^j, \mathbf{k}^j)$  to  $\mathcal{F}(\text{GRC})$ .

$\mathcal{S}$  receives  $(\text{output}, \text{sid}, \mathbf{V}_1^1, \dots, \mathbf{V}_d^n)$  in return, and records these values.

**Round  $r$ :**

For each round  $r \in [d]$ ,  $\mathcal{S}$  proceeds as follows:

$\mathcal{S}$  generates a new  $\pi_r^i$  for each  $i \in \mathcal{H}$ , and sends  $(\mathbf{V}_r^i, \pi_r^i)$  to every malicious  $P_j$ , with  $j \in \mathcal{M}$ .

Unless  $r = 1$ ,  $\mathcal{S}$  waits to receive  $(\text{prove}, \text{sid}, \hat{\mathbf{x}}^j, \hat{\mathbf{y}}^j, \hat{\boldsymbol{\alpha}}^j, \hat{\mathbf{k}}^j, \hat{\boldsymbol{\beta}}^j)$  from each malicious  $P_j$ , for  $j \in \mathcal{M}$ , playing the role of  $\mathcal{F}(\text{ZK}, \psi_r)$ .

$\mathcal{S}$  then checks, for each  $j$ , that:

$$\mathbf{X}^j \stackrel{?}{=} \hat{\mathbf{x}}^j \cdot G$$

$$\mathbf{Y}^j \stackrel{?}{=} \hat{\mathbf{y}}^j \cdot G + \hat{\boldsymbol{\alpha}}^j \cdot H$$

$$\mathbf{K}^j \stackrel{?}{=} \hat{\mathbf{k}}^j \cdot G + \hat{\boldsymbol{\beta}}^j \cdot H$$

and sets  $\text{bad-values}_r^j \leftarrow 1$  otherwise.

The first check implies that  $\hat{\mathbf{x}}^j = \mathbf{x}^j$ . If it holds that  $\hat{\mathbf{y}}^j \neq \mathbf{y}^j$  or  $\hat{\mathbf{k}}^j \neq \mathbf{k}^j$ ,



then  $\mathcal{S}$  has found a value  $h$  such that  $h \cdot G = H$ , as shown in [reference previous section](#), and  $\mathcal{S}$  aborts, returning  $h$ .

(Including when  $r = 1$ )  $\mathcal{S}$  generates a new  $\pi_r^j$ , and returns  $(\text{proof}, \pi_r^j)$ , playing the role of  $\mathcal{F}(\text{ZK}, \psi_r)$ .

Concurrently,  $\mathcal{S}$  plays the role of  $\mathcal{F}(\text{ZK}, \psi_r)$ , responding to  $(\text{verify}, (\hat{\mathbf{V}}_r^i, \hat{\mathbf{X}}^i, \hat{\mathbf{Y}}^i, \hat{\mathbf{K}}^i), \pi)$  queries.  $\mathcal{S}$  checks that there exists some  $j \in \mathcal{P}$  such that  $\pi_r^j = \pi$ .  $\mathcal{S}$  then returns:

$$\hat{\mathbf{V}}_r^i \stackrel{?}{=} \mathbf{V}_r^i \wedge \hat{\mathbf{X}}^i \stackrel{?}{=} \mathbf{X}^i \wedge \hat{\mathbf{Y}}^i \stackrel{?}{=} \mathbf{Y}^i \wedge \hat{\mathbf{K}}^i \stackrel{?}{=} \mathbf{K}^i \wedge \text{bad-values}_r^j \neq 1$$

$\mathcal{S}$  then waits to receive  $(\hat{\mathbf{V}}^j, \hat{\pi}_r^j)$  for every malicious party  $P_j$ , with  $j \in \mathcal{M}$ .  $\mathcal{S}$  then checks if the query  $(\text{verify}, (\hat{\mathbf{V}}_r^j, \mathbf{X}^j, \mathbf{Y}^j, \mathbf{K}^j), \hat{\pi}_r^j)$  would yield 1, according to the logic in the section above. (If  $\hat{\pi}_r^j$  doesn't match anything, the check is considered to fail). If this check fails, then  $\mathcal{S}$  simulates every honest  $P_i$  aborting, with  $i \in \mathcal{H}$ , to abort, as if they'd seen an invalid proof themselves.

This concludes the simulation.

If  $\mathcal{S}$  aborts with a value  $h$ , then they've successfully solved an instance of the discrete logarithm problem. Under our assumption that this problem is hard, this happens with negligible probability.

We argue that if  $\mathcal{S}$  does not abort in this way, then the simulation is perfect. For the first round, because pedersen commitments are perfectly hiding, sampling a random  $\mathbf{K}^j$  has an identical distribution as an honest party generating a pedersen commitment. For the rest of the protocol, all of our checks are equivalent to those made by honest parties. This is because the  $\mathbf{V}_j^i$  values are necessarily computed correctly, and use the inputs provided by the parties the adversary  $\mathcal{A}$  controls.

Because our simulator  $\mathcal{S}$  is perfect, and doesn't rewind the adversary  $\mathcal{A}$ , we conclude that our protocol satisfies universally composable security, in the hybrid model.

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### 4.3 Security Analysis

### 4.4 Practical Considerations

## 5 Applications

## 6 Limitations and Further Work

## 7 Conclusion

## References

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