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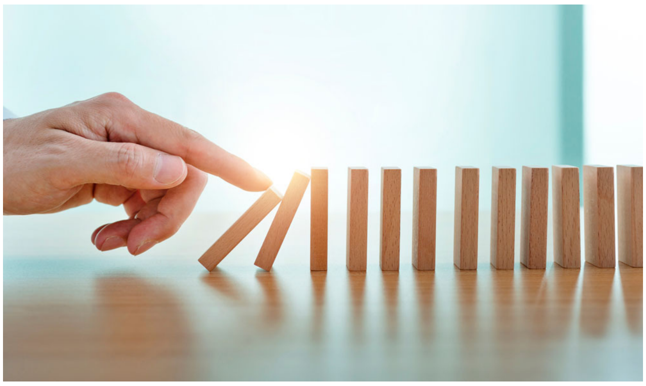

How to build a cross-impact model from first principles: theoretical requirements and empirical results

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Which models of the cross-impacts of trading financial instruments make sense?

1. Introduction

Trading pressure moves prices, a now well-established phenomenon known as *market impact* (Torre 1997, Almgren *et al.* 2005, Bouchaud *et al.* 2018). In fact, market impact has been measured in many independent studies and it is robust across assets, time periods and markets. A more subtle effect is that trading pressure from one asset can move the price of another. This effect has been dubbed *cross-impact*. Cross-impact transmits information across assets and amplifies market shocks. Yet despite its importance, we know much less about this effect, which has been the topic of recent studies (Hasbrouck and Seppi 2001, Pasquariello and Vega 2015, Wang *et al.* 2016, Benzaquen *et al.* 2017, Schneider and Lillo 2019). Many papers incorporate cross-impact in applications but assume that the cross-impact parameters are known (Ekren and Muhle-Karbe 2019, Tsoukalas *et al.* 2019). Thus they require a calibration technique to estimate cross-impact from data.

Empirical studies, whether non-parametric and applied on few instruments (Pasquariello and Vega 2015, Wang *et al.* 2015, 2016, Schneider and Lillo 2019) or parametric and applied on many assets (Benzaquen *et al.* 2017, Min *et al.* 2018) have provided strong evidence for cross-impact. Yet, as observed in Benzaquen *et al.* (2017), data-driven models overfit data. This motivated the authors to pursue market-specific approaches to reduce overfitting. As these approaches rely on properties of the asset class, they may fail elsewhere. For example, Benzaquen *et al.* (2017) found that the first eigenvectors of the stocks return covariance and order flow covariance matrices are roughly aligned. Accurate models on stocks may implicitly rely on this property and fail on markets where it is violated.

While previous empirical studies yield market-specific insights, to our knowledge, there is no study applying cross-impact models on a variety of markets. Therefore, from the literature, we cannot determine whether there exists a universally robust and statistically accurate cross-impact model, or even less ambitiously what properties a cross-impact model should respect.

Theoretical studies have attempted to reduce the universe of possible cross-impact models by constraining their acceptable outcomes. However, a pure no-arbitrage framework as in Alfonsi *et al.* (2016) is not sufficiently restrictive to prescribe

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calibration methodology. While the mean-field framework for optimal execution of Lehalle and Mouzouni (2019) provides one explanation of the many possible phenomena underlying cross-impact, it does not provide a recipe one may use on empirical data. In the optimal market making literature, Bergault *et al.* (2018) finds that the liquidation costs of a market maker when he holds an inventory q is of the form $-q^\top \Lambda q$, where Λ can be estimated in practice. The closest related paper to ours is Rosenbaum and Tomas (2021a). There, the authors characterize suitable cross-impact models within a market where trades are modeled with Hawkes processes. The resulting cross-impact models are well-behaved and can be calibrated, which the authors illustrate on E-Mini S&P 500 futures.

Thus, previous works on cross-impact introduced a given model that satisfied some convenient properties, used a specific theoretical framework to derive a model, or focused purely on empirical data. We proceed differently and use a principled approach. We view a cross-impact as a function of empirical observables and look for reasonable properties, or *axioms*, that they ought to respect.

We introduce four types of axioms. Symmetry axioms ensure models are consistent with dimensional analysis. Abstracting away microstructural effects, we expect any model to satisfy these consistencies. Arbitrage axioms prevent price manipulation in the sense of Gatheral (2010), where agents can use cross-impact to make a profit. These axioms prevent ill-behaved trading strategies in optimal execution problems, like those shown in Alfonsi *et al.* (2016). Fragmentation axioms, generalized from a notion first discussed in Benzaquen *et al.* (2017), guarantee suitable behavior when some combination of instruments have very small price fluctuations. Models which satisfy fragmentation properly aggregate the liquidity of instruments traded on different venues. Finally, cross stability axioms prevent price manipulation of liquid products using illiquid instruments.

Although each axiom treats a separate issue, they are related. We show that symmetry axioms imply arbitrage axioms and relate fragmentation and cross stability axioms.

Axioms enable us to classify models previously introduced in the literature and give perspective on which may work best in a given scenario.

To test these prescriptions, we apply a variety of cross-impact models to different markets and confirm which axioms are critical to explain empirical observations. We find that axioms reduce overfitting and allow us to understand which type of cross-impact model is adapted to a specific market. Furthermore, our results show only two cross-impact models perform well in all markets studied. However, only one prevents arbitrage and is well-behaved when trading both liquid and illiquid instruments. This makes it the ideal model for practical applications, such as optimal execution.

The paper is organized as follows. After introducing some notations used throughout the paper, section 3 lays down axioms and models, highlighting which axiom each model satisfies. Section 4 presents the calibration results of our zoology of models on different markets. We conclude by stressing the main contributions of the paper and discussing open questions and directions for future work in section 5.

2. Notation

Throughout this paper, we write scalars in roman lower cases, vectors in bold lower cases and matrices in roman upper cases. The set of n by n real-valued square matrices is denoted by $\mathcal{M}_n(\mathbb{R})$, the set of orthogonal matrices by $\mathcal{O}_n$, the set of real symmetric positive semi-definite matrices by $\mathcal{S}_n^+(\mathbb{R})$ and the set of real symmetric positive definite matrices by $\mathcal{S}_n^{++}(\mathbb{R})$. Further, given a matrix A in $\mathcal{M}_n(\mathbb{R})$, $A^\top$ denotes its transpose. Given A in $\mathcal{S}_n^+(\mathbb{R})$, we write $A^{1/2}$ for a matrix such that $A^{1/2}(A^{1/2})^\top = A$ and $\sqrt{A}$ for the matrix square root, the unique positive semi-definite symmetric matrix such that $(\sqrt{A})^2 = A$. We write $\ker(M)$ for the null space of a matrix $M \in \mathcal{M}_n(\mathbb{R})$, Π_V for the projector on a linear subspace of $V \in \mathbb{R}^n$ and $\bar{\Pi}_V = \mathbb{I} - \Pi_V$ for the orthogonal projector. Finally, given a vector $\boldsymbol{v} \in \mathbb{R}^n$, we write $\boldsymbol{v} = (v_1, \ldots, v_n)$ and $\mathrm{diag}(\boldsymbol{v})$ for the diagonal matrix with diagonal components the components of $\boldsymbol{v}$.

3. Linear cross-impact models

3.1. Setup

This section introduces the price impact framework in force throughout the paper.

We observe n different assets quoted and traded continuously and measure prices and market orders sent on these assets by all participants. The price of Asset i at time t is noted $p_{i,t}$ so that prices of all assets are summarized in the price vector $\boldsymbol{p}_t = (p_{1,t}, \ldots, p_{n,t})$ while the net (i.e. positive for buy orders and negative for sell orders) volume of market orders sent on Asset i at time t is noted by $v_{i,t}$ so that traded volume of all assets are summarized in the signed volume vector $\boldsymbol{v}_t = (v_{1,t}, \ldots, v_{n,t})$.

To relate signed volumes to price changes, we choose a timescale of reference $\Delta t > 0$ and aggregate prices and traded volumes on this timescale. Thus, we introduce the vector of price changes and order flow imbalances

$$\boldsymbol{\Delta p}_t := \boldsymbol{p}_{t+\Delta t} - \boldsymbol{p}_t \tag{1}$$

$$\boldsymbol{q}_t := \sum_{\tau=t}^{t+\Delta t} \boldsymbol{v}_\tau. \tag{2}$$

In order to ensure mathematical tractability of our construction, we place ourselves in a highly stylized setting. We first discard the influence of past order flows on the current price change.

Assumption 1 *Price changes $\boldsymbol{\Delta p}_t$ are independent of past order flow imbalances $\boldsymbol{q}_{t-1}, \boldsymbol{q}_{t-2}, \ldots$.*

Assumption 1 is quite constraining. In fact, past order flow imbalances are known to play a role (Benzaquen *et al.* 2017, Schneider and Lillo 2019, Wang *et al.* 2017, Rosenbaum and Tomas 2021b, 2021a, Tomas *et al.* 2021). However, it is more prominent when measuring lagged correlations between price changes and imbalances (related to the problem of impact decay). In this study, we focus on equal-time correlations between prices and volumes with the time resolution of $\Delta t = 1$ minute. In this regime, the effect of impact decay is typically much smaller than the price impact due to the latest traded portfolio, hence we adopt for simplicity this working hypothesis. We will comment more on the validity of this assumption below. For tractability, we further assume that order flow imbalances and price changes are linearly related.

ASSUMPTION 2 *Price changes $\boldsymbol{\Delta p}_t$ and order flow imbalances $\boldsymbol{q}_t$ are linearly related, i.e.*

$$\boldsymbol{\Delta p}_t = \Lambda_t \boldsymbol{q}_t + \boldsymbol{\eta}_t, \tag{3}$$

where the $n \times n$ matrix Λ_t is called the cross-impact matrix and $\boldsymbol{\eta}_t = (\eta_{1,t}, \ldots, \eta_{n,t})$ is a vector of zero-mean random variables independent of order flow imbalances $\boldsymbol{q}_t$.

Assumption 2 is justified empirically for small order flow imbalances but the relationship breaks down as the order flow imbalance grows (Patzelt and Bouchaud 2018).

Overall, assumptions 1 and 2 are relevant to describe price impact shortly induced after trading for small portfolios. To assess the validity of assumption 1, we briefly assume prices are affected by past order flow imbalances as in Schneider and Lillo (2019), Alfonsi *et al.* (2016) and Benzaquen *et al.* (2017) so that the price dynamics are

$$\boldsymbol{p}_t = \sum_{s \leq t} G(t-s)\boldsymbol{q}_s + \boldsymbol{\xi}_t,$$

where $G\colon t \mapsto G(t) \in \mathcal{M}_n(\mathbb{R})$ captures the dependence on past order flow and $\boldsymbol{\xi}_t$ is a vector of zero-mean random variables independent of $\boldsymbol{q}$. Then

$$\begin{aligned}\boldsymbol{\Delta p}_t = \boldsymbol{p}_{t+\Delta t} - \boldsymbol{p}_t &= G(0)\boldsymbol{q}_t + \sum_{s<t} (G(t+\Delta t - s) \\ &\quad - G(t-s))\, \boldsymbol{q}_s + \boldsymbol{\xi}_{t+\Delta t} - \boldsymbol{\xi}_t \\ &= G(0)\boldsymbol{q}_t + \boldsymbol{\eta}_t + \sum_{s<t} (G(t+\Delta t - s) - G(t-s))\, \boldsymbol{q}_s,\end{aligned}$$

where $\boldsymbol{\eta}_t := \boldsymbol{\xi}_{t+\Delta t} - \boldsymbol{\xi}_t$ is a vector of zero-mean random variables independent of $\boldsymbol{q}$. Assumptions 1 and 2 lead to equation (3) which accounts for the first two terms in the above equation but ignores the rest. Therefore, we can measure the validity of our approximation by comparing $\mathcal{G}_{ij} := \sum_{s<t} G_{ij}(t+\Delta t - s) - G_{ij}(t-s)$ and $G_{ij}(0)$. For $\Delta t =$ 5 minutes and on stocks, figure 1 of Benzaquen *et al.* (2017) shows $\frac{\mathcal{G}_{ij}}{G_{ij}(0)} \approx 20\%$. On the other hand, figure 3 of Schneider and Lillo (2019) works in transaction time on bonds but a rough estimate for $\Delta t = 1$minute yields $\frac{\mathcal{G}_{ij}}{G_{ij}(0)} \approx 30\%$. This indicates our setup is relevant to capture the salient features of cross impact.

All information about price impact in our setting is encoded in the cross impact matrix Λ_t. To find relevant variables for the cross-impact matrix Λ_t, equation (3) shows that Λ_t should be related to statistics of the random variables $\boldsymbol{\Delta p}_t$ and $\boldsymbol{q}_t$. They must naturally play an important role and motivate the next definition.

DEFINITION 1 (Price and order flow covariances) *We define, respectively, as return covariance, order flow covariance and response the quantities*

$$\begin{aligned}\Sigma_t &:= \mathbb{E}[\boldsymbol{\Delta p}_t \boldsymbol{\Delta p}_t^\top] \\ \Omega_t &:= \mathbb{E}[\boldsymbol{q}_t \boldsymbol{q}_t^\top] \\ R_t &:= \mathbb{E}[\boldsymbol{\Delta p}_t \boldsymbol{q}_t^\top].\end{aligned} \tag{4}$$

The covariances of equation (4) appear very naturally in the context of market microstructure, as they capture simple features of the coupled dynamics of prices and order flows. While Σ_t quantifies the co-variation of returns, Ω_t captures co-trading of different assets, and R_t reflects the average change of asset prices with traded order flow.

If the price changes $\boldsymbol{\Delta p}_t$ and order flow imbalances $\boldsymbol{q}_t$ were zero-mean Gaussian variables, then the covariances introduced in the previous definition would be sufficient statistics and completely characterize Λ. Thus, an important class of functions is those which only depend on these variables. This motivates the following definition.

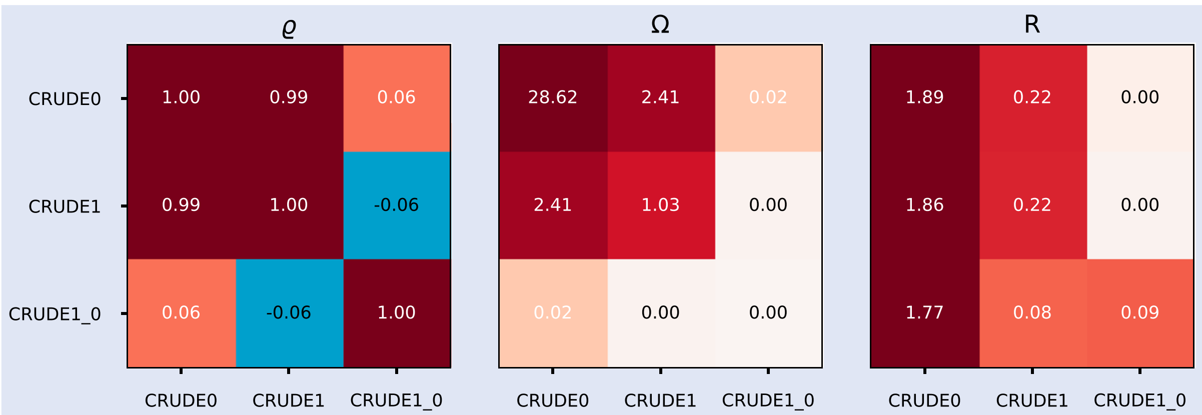

Figure 1. Estimates of ϱ, Ω and R for Crude contracts (in MUSD).
The return correlation matrix ρ (left), order flow covariance matrix Ω (center) and response matrix R (right) were estimated using 2016 data using the procedure described in section 4.1 and computed on the 6th of June 2016. This date represents the typical behavior of these contracts far away from the first notice date, before rolling effects become relevant. To highlight the amount of notional traded, order flow is reported in millions of exchanged dollars according to the average value of each contract on the 6th of June 2016. Though non-nill, order flow covariance of Calendar Spread thus appears small because traded notional is much smaller than on each leg of the futures contract.

DEFINITION 2 (cross-impact model) *A linear, single period cross-impact model (or cross-impact model for short) is a function Λ of the form*

$$\begin{aligned}\Lambda\colon \mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}) &\to \mathcal{M}_n(\mathbb{R})\\ (\Sigma, \Omega, R) &\mapsto \Lambda(\Sigma, \Omega, R),\end{aligned}$$

where we recall that $\mathcal{S}_n^+(\mathbb{R})$ is the space of real, positive semi-definite $n \times n$ matrices, $\mathcal{S}_n^{++}(\mathbb{R})$ is the space of real, strictly positive definite matrices and $\mathcal{M}_n(\mathbb{R})$ the space of $n \times n$ real matrices.

Even if we do not assume price variations nor order flows to be Gaussian random variables, the next assumption focuses on cross-impact matrices which can be expressed solely as a function of Σ, Ω and R.

ASSUMPTION 3 *In the rest of the paper, we assume that the cross-impact matrix Λ_t of equation* (3) *can be written as $\Lambda_t = \Lambda(\Sigma_t, \Omega_t, R_t)$, where Σ_t, Ω_t and R_t are defined in equation* (4) *and Λ is a cross-impact model. Therefore, the price dynamics are*

$$\boldsymbol{\Delta p}_t = \Lambda(\Sigma_t, \Omega_t, R_t)\boldsymbol{q}_t + \boldsymbol{\eta}_t, \tag{5}$$

where $\boldsymbol{\eta}_t = (\eta_{1,t}, \ldots, \eta_{n,t})$ is a vector of zero-mean random variables independent of order flow imbalances $\boldsymbol{q}_t$.

As previously stated, assumption 3 is satisfied if all variables are Gaussian. At time scales around 1 minute, this serves as a reasonable approximation. Before proceeding, we comment on the structure of the trading costs in our setup.

REMARK 1 The simple setting which leads to equation (5) gives a prediction of portfolio trading costs. In particular, if one assumes that the difference between the arrival price and the execution price is given by $\boldsymbol{\Delta p}_t$, the cost incurred after the execution of the portfolio $\boldsymbol{\xi}$ is

$$\mathcal{C}(\boldsymbol{\xi}) = \boldsymbol{\xi}^\top \boldsymbol{\Delta p}_t = \boldsymbol{\xi}^\top \Lambda_t \boldsymbol{\xi} + \boldsymbol{\xi}^\top \Lambda_t \bar{\boldsymbol{q}}_t + \boldsymbol{\xi}^\top \boldsymbol{\eta}_t, \tag{6}$$

where $\bar{\boldsymbol{q}}_t$ is the order flow imbalance due to trades of other market participants. Thus

$$\mathbb{E}[\mathcal{C}(\boldsymbol{\xi})] = \boldsymbol{\xi}^\top \Lambda_t \boldsymbol{\xi} + \boldsymbol{\xi}^\top \Lambda_t \mathbb{E}[\bar{\boldsymbol{q}}_t \mid \boldsymbol{\xi}] + \boldsymbol{\xi}^\top \boldsymbol{\eta}_t,$$

where $\mathbb{E}[\bar{\boldsymbol{q}}_t \mid \boldsymbol{\xi}]$ represents the alignment of the market trades' conditioned to the traded portfolio. This may be non-zero because of herding, where our trades cause other investors to trade. The treatment of this term depends on the trading strategy and is outside of the scope of this paper. Thus, we assume that $\mathbb{E}[\bar{\boldsymbol{q}}_t \mid \boldsymbol{\xi}] = \mathbb{E}[\bar{\boldsymbol{q}}_t] = 0$ so that the average impact costs of trading the portfolio $\boldsymbol{\xi}$ in our setting is $\mathbb{E}[\mathcal{C}(\boldsymbol{\xi})] = \boldsymbol{\xi}^\top \Lambda_t \boldsymbol{\xi}$.

The main purpose of this paper is to find a suitable cross-impact impact model Λ in equation (5) given a set of observations of market data and corresponding statistics Σ_t, Ω_t, R_t. The next section discusses how to choose a proper cross-impact model Λ.

3.2. Axioms

Even though writing down the impact matrix Λ_t as $\Lambda(\Sigma_t, \Omega_t, R_t)$ for some cross-impact model Λ is restrictive, we still have a large number of degrees of freedom for Λ. This is why we propose an axiomatic approach to the calibration of cross-impact models: instead of comparing models solely on empirical fit, we would like to control *ex ante* which properties they satisfy. There are two reasons to do this. First, for applications, it is often preferable to establish theoretical guarantees about the properties satisfied by a cross-impact model. Second, the risk of overfitting in data is considerably lowered when the number of free parameters is reduced.

3.2.1. Symmetries. The first properties we review involve the dimensional consistency of the models. First, the cross-impact model should adapt to the ordering of the assets. This yields the following axiom.

AXIOM 1 (Permutational invariance) *A cross-impact model Λ is permutation-invariant if, for any permutation matrix P and $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$,*

$$\Lambda(P\Sigma P^\top, P\Omega P^\top, PRP^\top) = P\Lambda(\Sigma, \Omega, R)P^\top.$$

In the Gaussian case where second order statistics are diagonal, returns and order flows of different assets are independent. The cross-impact model should then respect this property. This motivates the following axiom.

AXIOM 2 (Direct invariance) *A cross-impact model Λ is direct-invariant if, for any $\boldsymbol{\sigma}, \boldsymbol{\omega} \in \mathbb{R}_+^n$ $\boldsymbol{r} \in \mathbb{R}^n$,*

$$\begin{aligned}&\Lambda(\mathrm{diag}(\boldsymbol{\sigma})^2, \mathrm{diag}(\boldsymbol{\omega})^2, \mathrm{diag}(\boldsymbol{r}))\\ &\quad = \sum_{i=1}^n \Lambda(\sigma_i^2 \boldsymbol{e}_i \boldsymbol{e}_i^\top, \omega_i^2 \boldsymbol{e}_i \boldsymbol{e}_i^\top, r_i \boldsymbol{e}_i \boldsymbol{e}_i^\top),\end{aligned}$$

where $\boldsymbol{e}_i$ is the ith element of the canonical basis.

Impact is expressed in currency units. However, the chosen currency should not matter and cross-impact models should adapt accordingly. The next axiom translates this property.

AXIOM 3 (Cash invariance) *A cross-impact model Λ is cash-invariant if, for any $\alpha > 0$, and $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$,*

$$\Lambda(\alpha^2\Sigma, \Omega, \alpha R) = \alpha\Lambda(\Sigma, \Omega, R).$$

Similarly, cross-impact models should account for changes in volume units. For example, stock splits can double the number of outstanding shares and halve their values. However, the long-term behavior of the system should not be affected by this change. Note that this ignores microstructural effects such as tick size and lot rounding. This leads to the following axiom.

AXIOM 4 (Split invariance) *A cross-impact model Λ is split-invariant if, for any diagonal matrix of positive elements $D \in \mathcal{M}_n(\mathbb{R})$ and $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$,*

$$\Lambda(D^{-1}\Sigma D^{-1}, D\Omega D, D^{-1}RD) = D^{-1}\Lambda(\Sigma, \Omega, R)D^{-1}.$$

The profit and loss of traders is invariant under orthogonal transformations (see equation (6)). It is natural to inquire if cross-impact models share this property. As before, this ignores microstructural effects such as exchange trading fees, bid-ask spreads, etc. The following axiom introduces this property.

AXIOM 5 (Rotational invariance) *A cross-impact model Λ is rotation-invariant if, for any real orthogonal matrix $O \in \mathcal{O}_n$ and $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$,*

$$\Lambda(O\Sigma O^\top, O\Omega O^\top, ORO^\top) = O\Lambda(\Sigma, \Omega, R)O^\top.$$

We say of a model which does not satisfy axiom 5 that it has a privileged basis. Note that any cross-impact model which satisfies axioms 4 and 5 is invariant under the action of any non-singular matrix M.

3.2.2. Arbitrage. This family of axioms clarifies what properties a cross-impact model should satisfy for costs to be positive on average. This excludes any price manipulation strategy in the sense of Gatheral (2010). The first axiom involves the static arbitrages could be exploited in our single-period model if the cost of trading a portfolio $\boldsymbol{\xi}$, $\mathcal{C}(\boldsymbol{\xi}) = \boldsymbol{\xi}^\top \Lambda \boldsymbol{\xi}$, was negative along some direction.

AXIOM 6 (Positive semi-definiteness) *The cross-impact model Λ takes values in the space of positive semi-definite matrices.*

The next axiom that we consider involves *dynamic* arbitrages in the spirit of Alfonsi *et al.* (2016) and Gatheral (2010). Even though these arbitrages cannot be exploited in our single-period setup, they would emerge by generalizing our setup to the multi-period setting as shown in Schneider and Lillo (2019). This is why we choose to also consider this class of arbitrages.

AXIOM 7 (Symmetry) *The cross-impact model Λ takes values in the space of symmetric matrices.*

Axioms 6 and 7 together are sufficient to guarantee absence of statistical arbitrages, i.e. round-trip strategies that generate positive profits on average.

3.2.3. Fragmentation. While the previous axioms focused on ruling out strategies with average negative costs, another related issue is the impact of trading assets which have constant prices. For example, consider a stock traded on multiple markets (say, Apple traded on the Nasdaq and on the Bats venues). For a reasonably large interval of time Δt (and abstracting microstructural effects), we expect $p_{\text{Nasdaq}} - p_{\text{Bats}} = 0$. Thus, buying a volume $q = q_{\text{Nasdaq}} + q_{\text{Bats}}$ of Apple stock should yield the same cost no matter how one fragments the q_{Nasdaq} units bought on Nasdaq and the q_{Bats} units bought on Bats. For this reason, this axiom is dubbed *fragmentation invariance*. We distinguish between three different forms of fragmentation invariance. The first, *weak fragmentation invariance*, concerns the price changes given by a cross impact model and is detailed in the next Axiom.

AXIOM 8 (Weak fragmentation invariance) *A cross-impact model Λ is weakly fragmentation invariant if, for any $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$ and $\emptyset \subset V \subseteq \ker \Sigma$,*

$$\Pi_V \Lambda(\Sigma, \Omega, R) = 0,$$

where we recall that Π_V denotes the projector on the linear subspace V.

In practice, if a linear combination of prices is assumed not to fluctuate, weak fragmentation invariance guarantees that impact does not move its price.

REMARK 2 From now on, we will implicitly assume that $\ker(\Sigma) \subseteq \ker(R^\top)$, which is consistent with the interpretation of Σ and R as covariations in the sense of equation (4). This implies that from the point of view of the fragmentation-related axioms, any condition involving the the kernel of Σ will be naturally related to the kernel of $R^\top$ as well.

We obtain a stronger condition if the volume $\boldsymbol{q}$ traded in zero-volatility directions does not influence impact on any asset. This leads to the following Axiom.

AXIOM 9 (Semi-Strong fragmentation invariance) *A cross-impact model satisfies semi-strong fragmentation invariance if, besides satisfying the weak fragmentation invariance axiom 8, for any $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$ and $\emptyset \subset V \subseteq \ker \Sigma$,*

$$\Lambda(\Sigma, \Omega, R)\Pi_V = 0.$$

We can go one step further by ensuring that the cross-impact model itself should also not depend on how these directions are traded by *other* market members. This is *strong fragmentation invariance*, the subject of the next Axiom.

AXIOM 10 (Strong fragmentation invariance) *A cross-impact model Λ is strongly fragmentation invariant if, besides satisfying semi-strong fragmentation invariance (axiom 9), for any $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$ and $\emptyset \subset V \subseteq \ker(\Sigma)$,*

$$\Lambda(\Sigma, \Omega, R) = \Lambda(\bar{\Pi}_V \Sigma \bar{\Pi}_V, \bar{\Pi}_V \Omega \bar{\Pi}_V, \bar{\Pi}_V R \bar{\Pi}_V).$$

3.2.4. Liquidity. Another set of properties control cross-impact models when instruments have different levels of liquidity. Intuitively, price manipulation of liquid products using illiquid instruments should be excluded. We model this by defining a set V of illiquid instruments. We consider the matrix $\bar{\Pi}_V + \epsilon \Pi_V$ that multiplies by $\epsilon \ll 1$ the liquidity of all instruments belonging to V. After multiplication by this matrix, the observables become

$$\begin{aligned}
\Sigma_\varepsilon &:= \Sigma \\
\Omega_\varepsilon &:= (\bar{\Pi}_V + \epsilon \Pi_V)\Omega(\bar{\Pi}_V + \epsilon \Pi_V) \\
R_\varepsilon &:= R(\bar{\Pi}_V + \epsilon \Pi_V)
\end{aligned}$$

We are now ready to formulate liquidity axioms. First, trading illiquid instruments should not lead to large impact on liquid instruments. We would otherwise be able to manipulate the prices of liquid instruments. The converse should be true: we should not be able to manipulate prices of illiquid instruments by trading liquid instruments. This motivates the next axiom.

Axiom 11 (Weak Cross-Stability) *A cross-impact model Λ is weakly cross-stable if, for any $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$ and linear subspace V and using the above notations,*

$$\bar{\Pi}_V \Lambda(\Sigma_\varepsilon, \Omega_\varepsilon, R_\varepsilon)\Pi_V \underset{\epsilon\to 0}{=} O(1) \tag{7}$$

$$\Pi_V \Lambda(\Sigma_\varepsilon, \Omega_\varepsilon, R_\varepsilon)\bar{\Pi}_V \underset{\epsilon\to 0}{=} O(1). \tag{8}$$

We can formulate a stronger cross-stability property. The next axiom formalizes the intuition that impact among liquid assets should be independent of the behavior of illiquid assets.

Axiom 12 (Strong Cross-Stability) *A cross-impact model Λ is strongly cross-stable if, in addition to satisfying weak-cross stability (axiom 11), for any $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$ and linear subspace V and using the above notations,*

$$\begin{aligned}&\bar{\Pi}_V \Lambda\left(\Sigma_\varepsilon, \Omega_\varepsilon, R_\varepsilon\right)\bar{\Pi}_V\\ &\underset{\epsilon\to 0}{\to} \bar{\Pi}_V \Lambda\left(\bar{\Pi}_V\Sigma\bar{\Pi}_V, \bar{\Pi}_V\Omega\bar{\Pi}_V, \bar{\Pi}_V R\bar{\Pi}_V\right)\bar{\Pi}_V\end{aligned}$$

An unresolved question is the effect of trading illiquid instruments on illiquid products. The following axiom deals with this issue.

Axiom 13 (Self-Stability) *A cross-impact model is self-stable if, for any $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$, subspace V and using the above notations,*

$$\Pi_V \Lambda(\Sigma_\varepsilon, \Omega_\varepsilon, R_\varepsilon)\Pi_V \underset{\epsilon\to 0}{=} O(1). \tag{9}$$

Intuitively this property is less desirable than the previous one since it indicates that, even though a product is illiquid ($q \propto \epsilon$, so that one would expect a diverging impact) the predicted cost of trading such product can be finite.

3.2.5. Predicted covariance. Finally, it can be interesting to consider whether a cross-impact model predicts a contribution to the return covariance that is proportional to Σ or not.

Axiom 14 (Return covariance consistency) *A cross-impact model Λ is return covariance consistent if, for any $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$, it satisfies (up to a multiplicative constant):*

$$\Sigma = \Lambda(\Sigma, \Omega, R)\Omega\Lambda(\Sigma, \Omega, R)^\top.$$

This axiom is motivated by the fact that under the model in equation (5), we expect return covariances to be given by

$$\Sigma = \mathbb{E}[\boldsymbol{\Delta p}\boldsymbol{\Delta p}^\top] = \Lambda\Omega\Lambda^\top + \mathbb{E}[\boldsymbol{\eta}\boldsymbol{\eta}^\top],$$

so that if one assumes that the fundamental return covariance is proportional to the predicted one, i.e. $\mathbb{E}[\boldsymbol{\eta}\boldsymbol{\eta}^\top] \propto \Sigma$, one would recover return covariance consistency.

3.3. Critical assessment

We do not give the same plausibility to the axioms listed above.

Among the invariance-related axioms, permutational and cash invariance (axioms 1 and 3) are the most plausible. Indeed, we do not expect a privileged currency or ordering of instruments to express cross impact. Direct invariance (axiom 2) should hold as uncorrelated instruments should not react to their respective flows. Similarly, split invariance (axiom 4) is realistic though it may break on small timescales due to microstructural effects. On the other hand, it would not be surprising to find rotational invariance (axiom 5) violated in real markets. There, the physical basis of product is expected to play a privileged role (e.g. leverage and gross constraints break this symmetry). Overall, these axioms greatly restrict the set of linear cross-impact models, as shown by proposition 4.

Arbitrage-related axioms (axioms 6 and 7) are of great important in applications, in which one might want to exclude the presence of arbitrages by construction. Still, it is an interesting empirical question, though outside of the scope of this paper, to assess whether real markets admit some kind of arbitrage *à la* (Gatheral 2010), and whether these hold when factoring other transaction costs.

Weak fragmentation invariance (axiom 8) is critical. It prevents models from predicting price changes for zero-volatility instruments. Furthermore, it properly aggregates liquidity of an asset traded on multiple venues. For the same reasons, semi-strong and strong fragmentation invariance (axioms 9 and 10) should be of crucial importance.

We believe weak cross-stability (axiom 11) is fundamental. Indeed, it should be impossible to manipulate prices from liquid assets by trading illiquid assets and vice-versa. The stronger version of this axiom (axiom 12) is also expected to hold: liquid instruments should be insensitive to the behavior of illiquid ones. On the other hand, self-stability (axiom 13) does not penalize trading illiquid instruments. Thus, it is undesirable in applications.

Fragmentation and cross-stability are related for split and rotation-invariant cross-impact models. The next proposition shows that fragmentation invariance implies cross-stability properties for continuous cross-impact models.

Proposition 1 *Let Λ be a cross-impact model continuous in the first and third argument which satisfies split and rotational invariance (axiom 4 and 5). Then*

(i) *If Λ satisfies semi-strong fragmentation invariance (axiom 9), then it is weakly cross-stable (axiom 11).*
(ii) *If Λ is strongly fragmentation invariant (axiom 10), then it is strongly cross-stable (axiom 12).*

We prove proposition 1 in appendix A.1. While the converse is not true, the next proposition shows that, given an additional regularity condition, cross-stability implies fragmentation invariance.

Proposition 2 *Let Λ be a cross-impact model continuous in the first and third argument which satisfies split and rotational invariance (axioms 5 and 4). We further assume that, for any* *linear subspace V and using the notations of the previous section,* $\varepsilon^2\Lambda(\Sigma_\varepsilon, \Omega_\varepsilon, R_\varepsilon) \to_{\varepsilon\to 0} 0$. *Then*

(i) *If* Λ *is weakly cross-stable (axiom* 11*), then it satisfies semi-strong fragmentation invariance (axiom* 9*).*
(ii) *If* Λ *is strongly cross-stable (axiom* 12*), then it is strongly fragmentation invariant (axiom* 10*).*

We prove proposition 2 in appendix A.1. A particularly interesting result of proposition 2 and 1 is that for continuous cross-impact models which satisfy the regularity property of proposition 2, fragmentation invariance and cross-stability are equivalent.

There is no *ex-ante* reason for the return covariance consistency (axiom 14) to be true. However, it is worth noting that Proposition 3 shows that only one cross-impact model satisfies this constraint and no-arbitrage axioms (axiom 6 and 7).

3.4. Cross-impact models

Now that we have characterized the desirable properties of cross-impact models, we provide a set of cross-impact models and detail which axioms they satisfy. Their empirical performance will be explored in section 4. We divide these models in two classes; those that are based on the return covariance Σ and those based on the response R.

Before presenting the different cross-impact models, we introduce some notation. For convenience, we will note the price volatility $\boldsymbol{\sigma}_t := (\sqrt{\Sigma_{ii,t}})_{(1\le i\le n)}$, the signed order flow volatility $\boldsymbol{\omega}_t := (\sqrt{\Omega_{ii,t}})_{(1\le i\le n)}$, and the price and flow correlations $\rho_t := \mathrm{diag}(\boldsymbol{\sigma}_t)^{-1}\Sigma_t\mathrm{diag}(\boldsymbol{\sigma}_t)^{-1}$, $\rho_{\Omega,t} := \mathrm{diag}(\boldsymbol{\omega}_t)^{-1}\Omega_t\mathrm{diag}(\boldsymbol{\omega}_t)^{-1}$. Though the price volatility $\boldsymbol{\sigma}_t$ is a familiar quantity, it is worth commenting on the signification of $\boldsymbol{\omega}_t$. As the average of the signed order flow $\mathbb{E}[\boldsymbol{q}_t] = 0$, $\boldsymbol{\omega}_t$ quantifies the fluctuations of the net traded order flow and will thus be used (and often referred to) as a proxy for liquidity.

3.4.1. Return covariance based models. Let us start with the simplest possible linear impact model: one without cross impact.

Definition 3 (`direct` model) *The* `direct` *model is defined for any* $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$ *as*

$$\Lambda_{\mathtt{direct}}(\Sigma, \Omega, R) := \mathrm{diag}(\boldsymbol{\sigma})^{1/2}\mathrm{diag}(\boldsymbol{\omega})^{-1/2}. \tag{10}$$

We now introduce the cross-impact model proposed in Gârleanu and Pedersen (2016), which we dub the `gp` model after the authors.

Definition 4 (`gp` model) *The* `gp` *model is defined for any* $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$ *as*

$$\Lambda_{\mathtt{gp}}(\Sigma, \Omega, R) := c\Sigma, \tag{11}$$

where $c > 0$ *is some constant to be calibrated.*

The `gp` model violates both cash, split and rotational invariance, but satisfies many other axioms. To generalize the `direct` model to the multivariate setting while respecting cash invariance, weak fragmentation invariance and consistency with correlations, a first idea is to use the matrices $\Sigma^{1/2}$ and $\Omega^{-1/2}$. Since $\Omega^{-1/2}\boldsymbol{q}$ is a whitening transformation, this model is referred to as the `whitening` model.

Definition 5 (`whitening` model) *Recall that given* $M \in \mathcal{S}_n^+(\mathbb{R})$, $M^{1/2}$ *indicates a symmetric matrix factorization (i.e.* $M^{1/2}(M^{1/2})^\top = \mathbb{I}$*). The* `whitening` *model*† *is defined, for any* $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$, *as*

$$\Lambda_{\mathtt{whitening}}(\Sigma, \Omega, R) := \Sigma^{1/2}\Omega^{-1/2}. \tag{12}$$

Unfortunately, this estimator does not respect symmetry, positive-definiteness, strong fragmentation invariance or weak cross-stability (axioms 6, 7, 10 and 11). To impose symmetry and strong fragmentation invariance, the `el` model‡ proposed in Mastromatteo *et al.* (2017) is directly expressed in the basis of the return covariance matrix, assuming by construction that the cross-impact matrix commutes with the return covariance matrix.

Definition 6 (`el` model) *The eigenliquidity* (`el`) *model is defined, for any* $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$, *as*

$$\Lambda_{\mathtt{el}}(\Sigma, \Omega, R) := \sum_{a=1}^{n} \boldsymbol{s}_a \frac{\sqrt{\lambda_a}}{(\boldsymbol{s}_a^\top \Omega \boldsymbol{s}_a)^{1/2}} \boldsymbol{s}_a^\top, \tag{13}$$

where we have introduced the eigenvalue decomposition of $\Sigma = \sum_{a=1}^n \boldsymbol{s}_a\lambda_a\boldsymbol{s}_a^\top$.

The `el` model is cross-stable, self-stable (axioms 11–13) and is return covariance inconsistent (axiom 14). The `el` model is connected to the proposed cross-impact model of Min *et al.* (2018) for stocks, which proposes to estimate cross-impact by projecting volatility and liquidity along the direction of ETFs. While not strictly the same because of the required access to proprietary data to calibrate the model proposed in Min *et al.* (2018), we can think of the `el` model as a rough proxy for this model in our setup.

As mentioned above, there is in fact only one model which satisfies all the axioms that we have provided: the so-called multivariate `kyle` model, see Garcia del Molino *et al.* (2020).

Definition 7 (`kyle` model) *The* `kyle` *model is defined, for any* $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$, *as*

$$\Lambda_{\mathtt{kyle}}(\Sigma, \Omega, R) := (\Omega^{-1/2})^\top\sqrt{(\Omega^{1/2})^\top\Sigma\Omega^{1/2}}\,\Omega^{-1/2}. \tag{14}$$

The `kyle` model is similar to the cross-impact mark-to-market adjustment found in Guéant (2017) and Evangelista and Vieira (2018). It plays an important role because it is the only model which satisfies multiple assumptions. The next proposition shows it is the only model which satisfies arbitrage axioms and return covariance consistency.

† The `whitening` model is not independent of the symmetric factorization chosen for Σ and Ω. As convention, we will take the square root obtained by an orthogonal decomposition of each matrix and the square root of their eigenvalues.
‡ The model proposed in Mastromatteo *et al.* 2017 is actually the response-based one, referred later as `r-el`$\star$ model.

PROPOSITION 3 *Let Λ be a symmetric, positive-semidefinite and return covariance consistent cross-impact model (axiom 6–14). Then $\Lambda = \Lambda_{kyle}$ up to a multiplicative constant.*

The proof of proposition 3 is given in appendix A.2. The next proposition further shows that the `kyle` model is also the only return covariance based model which satisfies all symmetry axioms.

PROPOSITION 4 *A return covariance-based cross-impact model Λ that is both split-invariant and rotation-invariant (axiom 5 and 4) can always be written in the form*

$$\Lambda(\Sigma, \Omega) = \mathcal{L}^{-\top} U F(\boldsymbol{\mu}) U^\top \mathcal{L}^{-1},$$

where

$$\Omega = \mathcal{L}\mathcal{L}^\top; \quad \hat{\Sigma} := \mathcal{L}^\top \Sigma \mathcal{L}; \quad U^\top \hat{\Sigma} U := \mathrm{diag}(\boldsymbol{\mu});$$
$$F(\boldsymbol{\mu}) := \Lambda(\mathrm{diag}(\boldsymbol{\mu}), \mathbb{I}).$$

Furthermore, if Λ is cash-invariant and direct-invariant axioms 2 and 3, then $F(\boldsymbol{\mu}) \propto \mathrm{diag}(\boldsymbol{\mu})^{1/2}$ and $\Lambda = \Lambda_{\mathtt{kyle}}$ up to a multiplicative constant.

The proof of proposition 4 is given in appendix A.2.

3.4.2. Response based models. All the models presented above assume that it is possible to relate the effect of the order flow imbalance solely with the return and order flow covariances. However, one could expect the response $R = \mathbb{E}[\boldsymbol{\Delta p q}^\top]$ to be more informative in selecting the effect of liquidity shocks, because it directly captures co-variation of prices with the order flow. First, we can define a response-based direct impact model similar to equation (10).

DEFINITION 8 (`r-direct` model) *The response direct (`r-direct`) model is defined, for any $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$, as*

$$\Lambda_{r\text{-}direct}(\Sigma, \Omega, R) := \mathrm{diag}((R_{ii})_{i=1}^n)\mathrm{diag}(\boldsymbol{\omega})^{-1}.$$

This model corresponds to the maximum likelihood estimator of the cross impact matrix Λ under the constraint $\Lambda_{ij} = 0$ for $i \neq j$. Removing this constraint, one obtains the multiavariate maximum likelihood estimator defined below.

DEFINITION 9 (`ml` model) *The maximum likelihood (`ml`) model is defined, for any (Σ, Ω, R) in $(\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$, as*

$$\Lambda_{ml}(\Sigma, \Omega, R) := R\Omega^{-1}.$$

The `ml` does not satisfy desirable arbitrage or liquidity axioms. Thus, for similar reasons the `el` was introduced, we introduce a `r-el` model, so to have a response-based model satisfying more axioms while coinciding with the `ml` when R and Ω commute.

DEFINITION 10 (`r-el` model) *The response-based eigenliquidity (`r-el`) model is defined, for any (Σ, Ω, R) in $(\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$, as*

$$\Lambda_{r\text{-}el}(\Sigma, \Omega, R) := \sum_a \boldsymbol{s}_a \frac{\boldsymbol{s}_a^\top R \boldsymbol{s}_a}{\boldsymbol{s}_a^\top \Omega \boldsymbol{s}_a} \boldsymbol{s}_a^\top, \tag{15}$$

where $\boldsymbol{s_a}$ are the eigenvectors of Σ.

Finally, we can replicate the construction of the `kyle` estimator in a response-based context to obtain the following model.

DEFINITION 11 (`r-kyle` model) *The response-based Kyle (`r-kyle`) model is defined, for any (Σ, Ω, R) in $(\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$, as*

$$\Lambda_{r\text{-}kyle}(\Sigma, \Omega, R) := (\Omega^{-1/2})^\top \sqrt{(\Omega^{1/2})^\top R \Omega^{-1} R^\top \Omega^{1/2}}\, \Omega^{-1/2}. \tag{16}$$

3.4.3. The $\star$ transformation. Some of the models defined in the previous section (`whitening`, `el`, `r-el`) violate split invariance even though they are well-behaved under rotation. If one is willing to trade one axiom for the other, it is possible to cure the lack of split invariance by introducing a privileged basis. Trading axiom 4 with axiom 5 can be achieved through the following transformation.

DEFINITION 12 (The $\star$ transformation) *Given a cross-impact model Λ, the starred version of Λ, written $\Lambda^\star$, is a cross-impact model defined for any (Σ, Ω, R) in $(\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$ as*

$$\Lambda^\star(\Sigma, \Omega, R) := \mathrm{diag}(\boldsymbol{\sigma})\Lambda(\rho, \Omega^\star, R^\star)\mathrm{diag}(\boldsymbol{\sigma}),$$

where we have defined $\Omega^\star = \mathrm{diag}(\boldsymbol{\sigma})\Omega\mathrm{diag}(\boldsymbol{\sigma})$ and $R^\star = \mathrm{diag}(\boldsymbol{\sigma})^{-1}R\mathrm{diag}(\boldsymbol{\sigma})$.

In practice, the starred version of a cross-impact model applies the original cross-impact model after rescaling all the observables in units of risk via a multiplication by the volatility $\boldsymbol{\sigma}$. Of course, this transformation has no effect on models that satisfy split invariance.

Table 1 summarizes the axioms satisfied by each model. The properties listed above are not independent, and one can easily derive several relations that can provide further intuition on the axioms above, and additionally relate them to some of the models as highlighted in table 2. It is particularly instructive to relate the fragmentation-related axioms to the liquidity-related ones. Proofs of results presented here are given in appendix 1.

4. Empirical results

The focus of the present section is to stress test cross-impact models presented in section 3 on three different asset classes. As each cross-impact model satisfies different axioms, we can infer the importance of each axiom using as a proxy the quality of fit of cross-impact models.

To conduct the empirical analysis, we select the timescale Δt to be one minute in order to avoid microstructural effects

Table 1. Summary of axioms satisfied by different cross-impact model.

	Symmetries					Arbitrage		Fragmentation			Liquidity			Covariances
Model	PI	DI	CI	SI	RI	SA	DA	WFI	SSFI	SFI	WCS	SCS	SS	PCC
direct	✓	✓	✓	✓	✗	✓	✓	✗	✗	✗	✓	✓	✗	✗
gp	✓	✓	✗	✗	✗	✓	✓	✓	✓	✓	✓	✓	✗	✗
whitening	✓	✓	✓	✗	✓	✗	✗	✓	✗	✗	✗	✗	✗	✓
whitening⋆	✓	✓	✓	✓	✗	✗	✗	✓	✗	✗	✗	✗	✗	✓
el	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗
el⋆	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✗
kyle	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓
r-direct	✓	✓	✓	✓	✗	✓	✗	✗	✗	✗	✓	✓	✗	✗
ml	✓	✓	✓	✓	✓	✗	✗	✓	✗	✗	✗	✗	✗	✗
r-el	✓	✓	✓	✗	✓	✗	✓	✓	✓	✓	✓	✓	✓	✗
r-el⋆	✓	✓	✓	✓	✗	✗	✓	✓	✓	✓	✓	✓	✓	✗
r-kyle	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗

Notes: We use the symbol ✓ for axioms that are satisfied and ✗ for axioms that are violated. We use the color ForestGreengreen in order to label a desirable property of the model, redred for an undesirable property of the model. amberYellow is used for properties/models whose violation might not be particularly relevant in order to explain empirical data, although they are interesting to consider. Axioms are grouped by category and the order in which they were presented in the text.

Table 2. Salient relations among the axioms introduced in thepaper.

	Symmetries					Arbitrage		Fragmentation			Liquidity			Covariances
Result	PI	DI	CI	SI	RI	SA	DA	WFI	SSFI	SFI	WCS	SCS	SS	PCC
Proposition 1 (i)				H	H				H		✓			
Proposition 1 (ii)				H	H					H		✓		
Proposition 2 (i)				H	H				✓		H			
Proposition 2 (ii)				H	H					✓		H		
Proposition 3	✓	✓	✓	✓	✓	H	H	✓	✓	✓	✓	✓	✗	H
Proposition 4	✓	H	H	H	H	✓	✓	✓	✓	✓	✓	✓	✗	✓

Notes: The table summarizes the results of different propositions relating axioms together. For a given result, we use the symbol H to denote a condition that holds by hypothesis. On the same row, we mark satisfied axioms using the notation of table 1.

while being small. For a given cross-impact model Λ and basket of instruments, the predicted price change at time t due to the measured order flow imbalance $\boldsymbol{q}_t$ is

$$\widehat{\Delta \boldsymbol{p}}_t := \Lambda(\Sigma_t, \Omega_t, R_t)\boldsymbol{q}_t,$$

where Σ_t, Ω_t, R_t are the covariances defined in equation (4) estimated on a given asset class. To evaluate quality of fit of the cross-impact model Λ, we compare the predicted price changes $\widehat{\Delta \boldsymbol{p}}_t$ to the realized price changes $\Delta \boldsymbol{p}_t$.

To compare impact models, we use three different indicators of performance which emphasize different aspects of prediction errors. All three indicators are parametrized by a symmetric, positive definite matrix $M \in \mathcal{S}_n^+(\mathbb{R})$, $M \neq 0$. Given a realization of the price process $\{\Delta \boldsymbol{p}_t\}_{t=1}^T$ of length T and a corresponding series of predictions $\{\widehat{\Delta \boldsymbol{p}}_t\}_{t=1}^T$, the M-weighted generalized $\mathcal{R}^2$ is defined as

$$\mathcal{R}^2(M) := 1 - \frac{\sum_{1\leq t\leq T}(\Delta \boldsymbol{p}_t - \widehat{\Delta \boldsymbol{p}}_t)^\top M(\Delta \boldsymbol{p}_t - \widehat{\Delta \boldsymbol{p}}_t)}{\sum_{1\leq t\leq T} \Delta \boldsymbol{p}_t^\top M \, \Delta \boldsymbol{p}_t}.$$

To highlight different sources of error, we consider the following choices of M:

(i) $M = I_\sigma := \mathrm{diag}(\boldsymbol{\sigma})^{-1}$, to account for errors relative to the typical deviation of the asset considered. This type of error is relevant for strategies predicting idiosyncratic moves of the constituents of the basket, rather than strategies betting on correlated market moves.

(ii) $M = J_\sigma := (\Sigma_{ii}^{-1/2}\Sigma_{jj}^{-1/2})_{1\leq i,j\leq m}$, to check if the model successfully forecasts the overall direction of all assets. This is relevant for strategies predicting global moves of the constituents of the basket.

(iii) $M = \Sigma^{-1}$, to consider how well the model predicts the individual modes of the return covariance matrix. This would be the relevant error measure for strategies that place a constant amount of risk on the modes of the correlation matrix, leveraging up combinations of products with low volatility and scaling down market direction that exhibit large fluctuations.†

Given $M \in \mathcal{S}_n^+(\mathbb{R})$, $M \neq 0$, we compute scores on empirical data in the following manner. First, we divide data into two subsets of roughly equal length: data from 2016 on the one hand and in 2017 on the other hand. Given data from year X and year Y, we calibrate estimators and cross-impact models on year X and use models to predict price changes in year Y, writing $\mathcal{R}^2_{X\to Y}(M)$ for the average score. In-sample scores are defined as $\mathcal{R}^2_{\mathrm{in}}(M) := \frac{1}{2}(\mathcal{R}^2_{2016\to 2016}(M) +$

† Note that this measure strongly penalizes models violating fragmentation invariance: errors along modes of zero risk should *a-priori* be enhanced by an infinite amount. In this study we have decided to clip the eigenvalues of Σ to a small, non-zero amount equal to 10^{-15}.

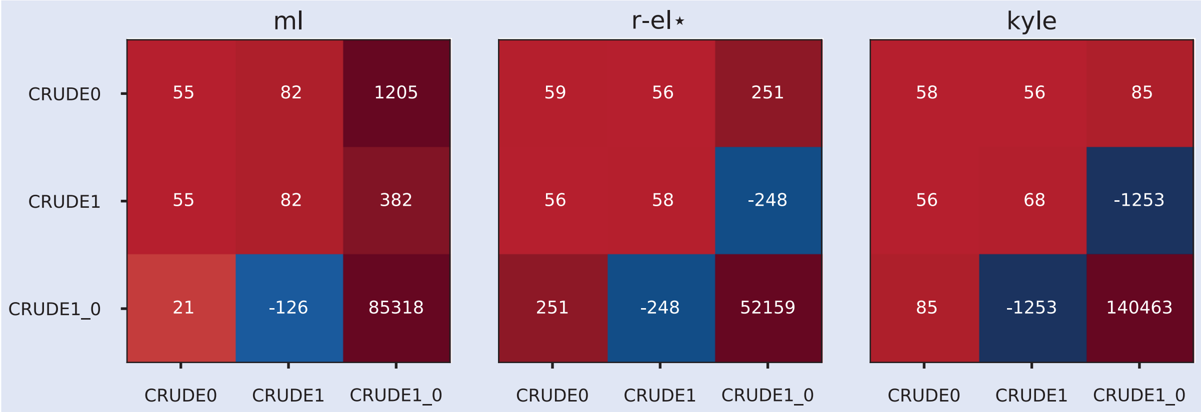

Figure 2. Values of different cross-impact models for Crude contracts.
We report the values of the `ml` (left), `r-el⋆` (center) and `kyle` (right) cross-impact models for the covariances of the 6th of June 2016 obtained using the procedure described in section 4.1. Units are chosen to represent the relative price change in basis points (10^{-4} of the asset price) by hundred million USD worth of contract traded.

Table 3. Goodness of fit of cross-impact models on Crude futures.

	In-sample			Out-sample		
Model	$\mathcal{R}^2_{\text{in}}(I_\sigma)$	$\mathcal{R}^2_{\text{in}}(J_\sigma)$	$\mathcal{R}^2_{\text{in}}(\Sigma^{-1})$	$\mathcal{R}^2_{\text{out}}(I_\sigma)$	$\mathcal{R}^2_{\text{out}}(J_\sigma)$	$\mathcal{R}^2_{\text{out}}(\Sigma^{-1})$
`direct`	0.01 ± 0.01	0.33 ± 0.01	$-\infty$	0.01 ± 0.01	0.33 ± 0.01	$-\infty$
`whitening`	0.03 ± 0.01	0.32 ± 0.01	-0.05 ± 0.02	0.04 ± 0.01	0.32 ± 0.01	-0.05 ± 0.02
`whitening⋆`	0.06 ± 0.01	0.22 ± 0.01	-0.01 ± 0.02	0.06 ± 0.01	0.22 ± 0.01	-0.01 ± 0.02
`el`	0.18 ± 0.01	0.27 ± 0.01	0.07 ± 0.02	0.18 ± 0.01	0.27 ± 0.01	0.07 ± 0.02
`el⋆`	0.18 ± 0.01	0.27 ± 0.01	0.07 ± 0.02	0.18 ± 0.01	0.27 ± 0.01	0.07 ± 0.02
`kyle`	0.35 ± 0.01	0.46 ± 0.01	0.29 ± 0.02	0.35 ± 0.01	0.46 ± 0.01	0.29 ± 0.02
`r-direct`	0.27 ± 0.01	0.40 ± 0.01	$-\infty$	0.27 ± 0.01	0.40 ± 0.01	$-\infty$
`ml`	0.37 ± 0.01	0.45 ± 0.01	0.32 ± 0.02	0.37 ± 0.01	0.45 ± 0.01	0.31 ± 0.02
`r-el`	0.37 ± 0.01	0.46 ± 0.01	0.31 ± 0.02	0.37 ± 0.01	0.46 ± 0.01	0.31 ± 0.02
`r-el⋆`	0.37 ± 0.01	0.46 ± 0.01	0.31 ± 0.02	0.37 ± 0.01	0.46 ± 0.01	0.31 ± 0.02
`r-kyle`	0.22 ± 0.01	0.31 ± 0.01	0.16 ± 0.02	0.22 ± 0.01	0.31 ± 0.01	0.16 ± 0.02

Notes: Goodness of fit was measured using two years of data sampled at a time interval of 1 min. In-sample data was used to calibrate each cross-impact model. Out-of-sample goodness of fit was obtained by applying the calibrated models on never seen before data. We reported as ∞ the scores of models which are numerically infinite, but due to clipping appear finite. The best cross-impact models on this dataset are `kyle`, `r-el` and `ml`.

$\mathcal{R}^2_{2017\to2017}(M)$) while out-of-sample scores are defined as $\mathcal{R}^2_{\text{out}}(M) := \frac{1}{2}(\mathcal{R}^2_{2016\to2017}(M) + \mathcal{R}^2_{2017\to2016}(M))$.

4.1. Crude oil: futures and calendar spreads

In this section, the asset universe is made up of two NYMEX Crude Oil future contracts and the corresponding Calendar Spread contract. The first two contracts (respectively, CRUDE0 and CRUDE1) entail an agreement to buy or sell 1000 barrels of oil either at the next month or at the subsequent month. The Calendar Spread CRUDE1_0 swaps the front month future with the contract settling on the following month.

Structure of ρ, Ω and R Figure 1 reports the estimators of ϱ, Ω and R for the 6th of June 2016. The figure shows ϱ has one zero-volatility direction and one direction of very small fluctuations. Thus models which satisfy fragmentation invariance (axioms 8–10) should give better predictions. On the other hand, Ω highlights the difference in liquidity of our assets. Thus, we should be cautious of models which do not satisfy stability axioms. Indeed, these will not penalize trading directions of small liquidity.

Cross-impact models for Crude oil contracts Figure 2 shows the calibrated `ml`, `r-el⋆` and `kyle` models. Each satisfies weak fragmentation invariance (axiom 8). Therefore, they prevent arbitrage which would trade the physical Calendar Spread contract against the synthetic Calendar Spread (made up of CRUDE0 and CRUDE1). However, `ml` and `kyle` are not self-stable (axiom 13) while the `r-el⋆` model is. This explains why impact from trading the illiquid Calendar Spread is much larger in the `ml` and `kyle` models than in the `r-el⋆` model.

Empirical comparison of models Table 3 shows the scores of cross-impact models on the Crude dataset. First, note that models which do not satisfy weak fragmentation invariance poorly explain idiosyncratic and risk-weighted price changes.† This is because of the small volatility of the Calendar Spread and the presence of a zero-volatility direction. It is therefore more suited to compare all models on the basis

† For example, though the $\mathcal{R}^2(I_\sigma)$ score of the `r-direct` model is small, it explains about 35% of the variance of price changes of CRUDE0, roughly 20% for CRUDE1 but predicts incorrect price changes for the Calendar Spread CRUDE1_0.

of $\mathcal{R}^2(J_\sigma)$. There, cross-impact models slightly improve on direct models (scoring around 46% as compared to 40%). This is surprising: despite the concentration of liquidity in the front month contract and the large correlation between the front and subsequent month contracts, accounting for the off-diagonal elements of Σ and Ω matters. Finally, among cross-impact models, there is little difference between the performance of the `ml`, `r-el`, `r-el⋆` and `kyle` models. However, only the `kyle` model satisfies all stability axioms. This makes it ideal for applications. Overall, this example emphasizes the importance of fragmentation invariance but does not suggest which stability axioms are most relevant.

4.2. *Bonds and indices*

While relevant to illustrate the importance of fragmentation invariance, the previous dataset corresponds to a pathological case where Σ has only one large non-zero eigenvalue, so that cross-impact models give similar results. To circumvent this issue, we look at 10-year US Treasury note futures and the E-Mini S&P500 futures. We collect data from the Chicago Mercantile Exchange and use the first two upcoming maturities of both contracts (respectively called SPMINI and SPMINI3 for E-Mini S&P500 futures and 10USNOTE and 10USNOTE3 for 10-year US treasury notes).

Structure of ρ, Ω and R Figure 3 shows the estimators of ϱ, Ω and R for the 17th of August 2016. Contracts with the same underlying are strongly correlated. Thus, ρ shows 2 by 2 blocks of strongly correlated contracts and an anti-correlation between bonds and futures. Liquidity is heterogeneous as front month contracts are more actively traded. In this configuration, the discriminating factor between models should be stability axioms rather than fragmentation axioms.

Cross-impact models for bonds and indices Figure 4 shows the `ml`, `r-el⋆` and `kyle` models calibrated on bonds and indices. The `r-el` and `kyle` models are weakly cross-stable while the `ml` model is not. Thus `ml` assigns large impact to the less liquid contracts 10USNOTE3 and SPMINI3. Similarly, the self-stability of `r-el` explains the small impact predicted if one trades illiquid contracts. Reassuringly, all models correctly capture the negative index-bonds correlation.

Empirical comparison of models Table 4 shows the scores of cross-impact models on the bonds and indices dataset. In contrast to the previous dataset, Σ has two meaningful modes instead of one. These are the risk-on/risk-off mode (indices minus bonds) and the relative mode (indices plus bonds). In contrast to the previous dataset, there are two meaningful modes instead of one. This explains why cross-impact models outperform direct models to explain idiosyncratic price

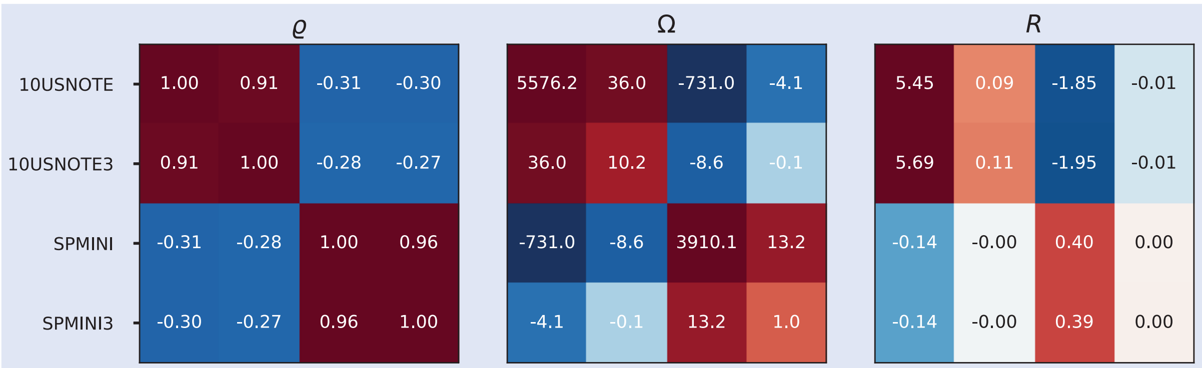

Figure 3. Estimates of ϱ, Ω and R for bonds and indices (in MUSD).
The return correlation matrix ρ (left), order flow covariance matrix Ω (center) and response matrix R (right) were estimated using 2016 data using the procedure described in section 4.2 and computed on the 17th of August 2016. To highlight the amount of notional traded, order flow is reported in millions of exchanged dollars according to the average value of each contract on the 17th of August 2016. Basis points were accounted for, so that one traded unit of the futures contracts entitles the owner to one unit of the underlying.

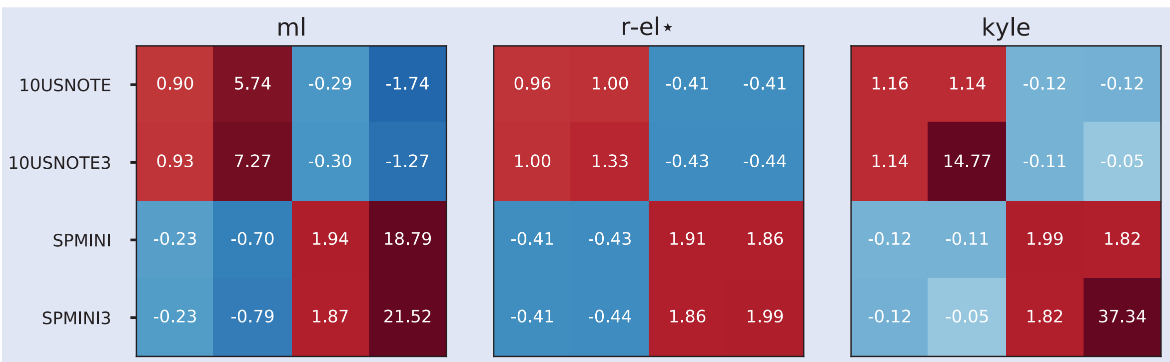

Figure 4. Values of different cross-impact models for bonds and indices.
We report the values of the `ml` (left), `r-el⋆` (center) and `kyle` (right) cross-impact models for the covariances of the 17th of August 2016 obtained using the procedure described in section 4.2. Units are chosen to represent the relative price change in basis points (10^{-4} of the asset price) by hundred million USD worth of contract traded.

Table 4. Goodness of fit of cross-impact models on 10-year US treasury note futures and E-mini futures.

Model	In-sample $\mathcal{R}^2_{\text{in}}(I_\sigma)$	In-sample $\mathcal{R}^2_{\text{in}}(J_\sigma)$	In-sample $\mathcal{R}^2_{\text{in}}(\Sigma^{-1})$	Out-sample $\mathcal{R}^2_{\text{out}}(I_\sigma)$	Out-sample $\mathcal{R}^2_{\text{out}}(J_\sigma)$	Out-sample $\mathcal{R}^2_{\text{out}}(\Sigma^{-1})$
`direct`	-0.11 ± 0.02	0.09 ± 0.02	-7.24 ± 0.21	-0.11 ± 0.02	0.09 ± 0.02	-7.23 ± 0.21
`whitening`	0.03 ± 0.02	-0.09 ± 0.03	-0.37 ± 0.04	0.03 ± 0.02	-0.10 ± 0.03	-0.37 ± 0.04
`whitening`⋆	0.05 ± 0.02	-0.05 ± 0.03	-0.36 ± 0.04	0.04 ± 0.02	-0.05 ± 0.03	-0.36 ± 0.04
`el`	0.19 ± 0.01	0.09 ± 0.02	-0.26 ± 0.03	0.19 ± 0.01	0.09 ± 0.02	-0.26 ± 0.03
`el`⋆	0.02 ± 0.02	-0.21 ± 0.03	-0.37 ± 0.03	0.02 ± 0.02	-0.21 ± 0.03	-0.37 ± 0.03
`kyle`	0.38 ± 0.01	0.29 ± 0.02	0.11 ± 0.03	0.38 ± 0.01	0.29 ± 0.02	0.11 ± 0.03
`r-direct`	0.23 ± 0.01	0.27 ± 0.02	-1.69 ± 0.05	0.23 ± 0.01	0.27 ± 0.02	-1.71 ± 0.05
`ml`	0.40 ± 0.01	0.30 ± 0.02	0.20 ± 0.03	0.40 ± 0.01	0.30 ± 0.02	0.20 ± 0.03
`r-el`	0.38 ± 0.01	0.29 ± 0.02	0.19 ± 0.03	0.38 ± 0.01	0.29 ± 0.02	0.19 ± 0.03
`r-el`⋆	0.27 ± 0.01	0.17 ± 0.02	0.13 ± 0.03	0.27 ± 0.01	0.17 ± 0.02	0.13 ± 0.03
`r-kyle`	0.25 ± 0.01	0.14 ± 0.02	0.07 ± 0.03	0.24 ± 0.01	0.14 ± 0.02	0.07 ± 0.03

Notes: Goodness of fit was measured using two years of data sampled at a time interval of one minute. In-sample data was used to calibrate each cross-impact model. Out-of-sample goodness of fit was obtained by applying the calibrated models on never seen before data. The best cross-impact models on this dataset are `kyle`, `r-el` and `ml`, with scores two times above the goodness of fit of `r-direct`. Starred models perform worse than their non-starred counterpart.

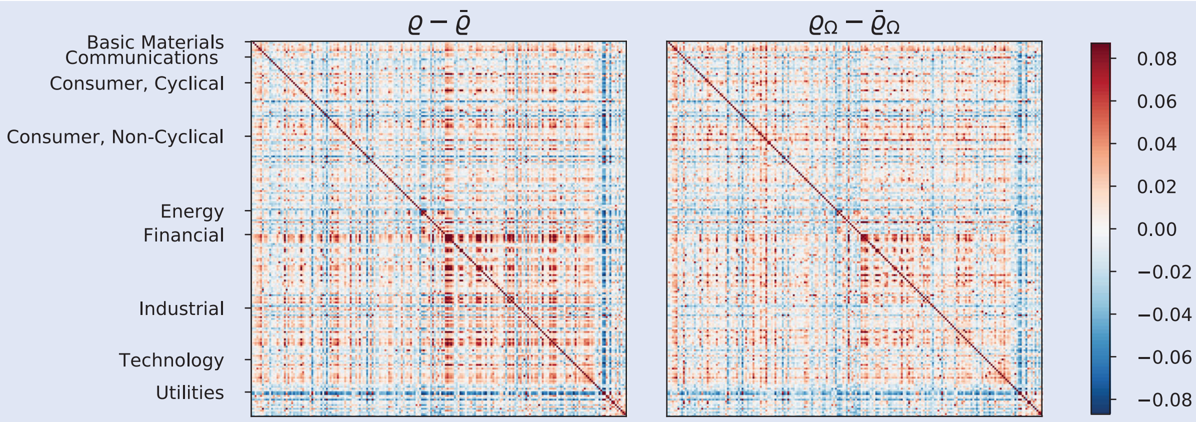

Figure 5. Estimated price and order flow correlation matrices ϱ, ϱ_Ω for stocks.
We represent the return correlation matrix ρ (left), order flow correlation matrix ϱ_Ω (right) estimated on 2016. To highlight the amount of notional traded, order flow is reported in millions of exchanged dollars according to the average value of each contract on the 17th of August 2016. Correlation matrices were represented instead of covariance matrices due to the large volume heterogeneities between stocks. Stocks were grouped by sectors to highlight the blockwise structure of these matrices.

moves. Despite the presence of two illiquid instruments, there is no clear goodness-of-fit difference between models which satisfy different stability.

4.3. Stocks

The previous datasets give us no clear conclusion on the role of stability axioms. In both examples illiquid assets were highly correlated to other liquid assets. This extreme regime of correlations makes it harder to analyze the role of liquidity. To circumvent this issue, we study the behavior of cross-impact models in the low-correlation, many assets regime, using stocks data.

Structure of ϱ, ϱ_Ω and R Figure 5 shows estimators of ϱ, ϱ_Ω. We report correlations instead of covariances to highlight the blockwise structure of these matrices. For the same reasons, R is not shown but presents a bandwise structure one expects from heterogeneity in liquidity. Pairwise price and order flow correlations between assets are small. Thus, improvement of cross-impact models over direct models should be lower than in previous applications. For more details about the structure of the price and volume covariance matrices, see Benzaquen *et al.* (2017).

Cross-impact models for stocks Figure 6 shows the `ml`, `r-el`⋆ and `kyle` models calibrated on the stocks dataset. At first glance, each model appears to present a blockwise structure similar to that of ϱ, ϱ_Ω. However, the `ml` model does not satisfy weak cross-stability and thus predicts large impact on liquid stocks if one trades illiquid stocks. By construction the `r-el`⋆ model weighs most impact on the market mode. Finally, the `kyle` model looks like a symmetrized version of the `r-el`⋆ model.

Empirical comparison of models Table 5 shows the scores of cross-impact models on the stocks dataset. The two previous examples where only a few directions had notable impact contributions. However, the low pair-wise correlation of stocks suggests many directions contribute to impact. Cross-impact models can explain market-wide moves up to twice as well as direct models. This highlights the importance of aggregating liquidity to account for the market

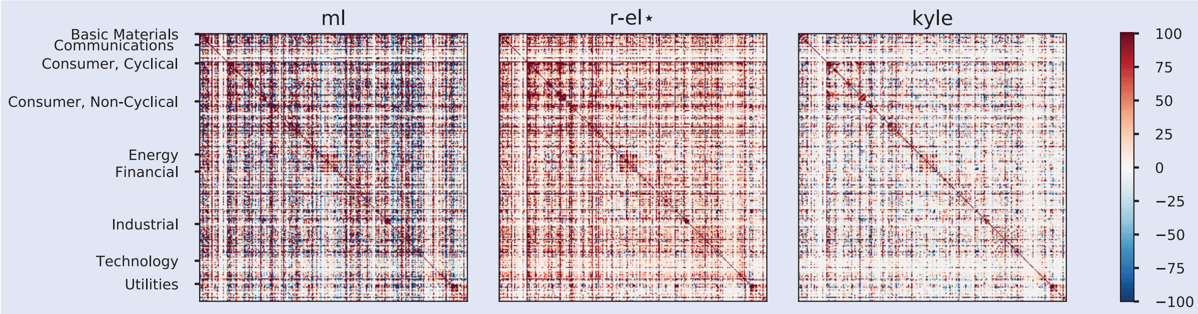

Figure 6. Values of different cross-impact models for stocks.
We report the values of the ml (left), r-el⋆ (center) and kyle (right) cross-impact models for covariances obtained using the procedure described in section 4.3. Units are chosen to represent the relative price change in basis points (10^{-4} of the asset price) by hundred million USD worth of instruments traded.

Table 5. Goodness of fit of cross-impact models on stocks.

	In-sample			Out-sample		
Model	$\mathcal{R}^2_{\text{in}}(I_\sigma)$	$\mathcal{R}^2_{\text{in}}(J_\sigma)$	$\mathcal{R}^2_{\text{in}}(\Sigma^{-1})$	$\mathcal{R}^2_{\text{out}}(I_\sigma)$	$\mathcal{R}^2_{\text{out}}(J_\sigma)$	$\mathcal{R}^2_{\text{out}}(\Sigma^{-1})$
direct	0.038 ± 0.004	0.732 ± 0.006	-0.311 ± 0.004	0.038 ± 0.004	0.732 ± 0.006	-0.293 ± 0.004
whitening	-0.025 ± 0.004	-0.047 ± 0.012	-0.061 ± 0.003	-0.031 ± 0.004	-0.192 ± 0.013	-0.061 ± 0.003
whitening⋆	0.059 ± 0.004	0.277 ± 0.010	-0.056 ± 0.003	0.047 ± 0.004	0.152 ± 0.012	-0.056 ± 0.004
el	-0.631 ± 0.010	-1.770 ± 0.038	-0.262 ± 0.005	-0.642 ± 0.010	-1.785 ± 0.038	-0.260 ± 0.005
el⋆	-0.128 ± 0.008	0.727 ± 0.005	-0.369 ± 0.008	-0.133 ± 0.008	0.701 ± 0.005	-0.360 ± 0.008
kyle	0.343 ± 0.003	0.822 ± 0.003	0.214 ± 0.003	0.336 ± 0.003	0.808 ± 0.004	0.211 ± 0.003
r-direct	0.276 ± 0.004	0.480 ± 0.010	0.180 ± 0.003	0.274 ± 0.004	0.479 ± 0.010	0.180 ± 0.003
ml	0.373 ± 0.003	0.829 ± 0.003	0.215 ± 0.003	0.358 ± 0.003	0.803 ± 0.004	0.208 ± 0.003
r-el	0.257 ± 0.003	0.661 ± 0.005	0.126 ± 0.004	0.249 ± 0.003	0.644 ± 0.005	0.124 ± 0.004
r-el⋆	0.236 ± 0.004	0.753 ± 0.004	0.090 ± 0.004	0.227 ± 0.004	0.733 ± 0.005	0.089 ± 0.004
r-kyle	0.239 ± 0.004	0.788 ± 0.004	0.082 ± 0.004	0.232 ± 0.004	0.776 ± 0.004	0.081 ± 0.004

Notes: Goodness of fit was measured using two years of data sampled at a time interval of one minute. In-sample data was used to calibrate each cross-impact model. Out-of-sample goodness of fit was obtained by applying the calibrated models on never seen before data. The best cross-impact models on this dataset are kyle, r-el and ml.

mode. Naturally, explaining idiosyncratic or risk-weighted price changes is a more challenging task. Nevertheless, cross-impact models (r-el, r-el⋆, kyle, ml) improve r-direct scores by 20 to 30%.

4.3.1. Influence of liquidity on scores.

An interesting feature of our stocks dataset is the heterogeneous liquidity. This allows us to explore the influence of the liquidity of a given stock on the performance of different models. Figure 7 shows the results of this analysis. Consistent with table 5, we find that overall, in score terms, ml¿kyle¿r-direct¿r-el. The r-direct model fares better for liquid stocks, where a larger fraction of variance can be explained by same-stock trades. Surprisingly, the same holds for ml and kyle models. The r-el model stands as an exception. It better explains price moves for stocks which are within the band of typical liquidity, between $\omega_{10\%}$ and $\omega_{90\%}$. This makes sense since the r-el model is self-stable as it aggregates liquidity of all stocks. Thus, though this assumption is justified for stocks of liquidity close to the average, it is violated outside of this zone. The ml and kyle models are not self-stable and better deal with very liquid or illiquid stocks. To further reinforce this point, for stocks of liquidity close to the average in our pool of stocks, the difference scores of the el and kyle models reach a minimum. This is consistent with the fact that in the approximation $\Omega \approx \omega_{50\%}\mathbb{I}$, the two models coincide.

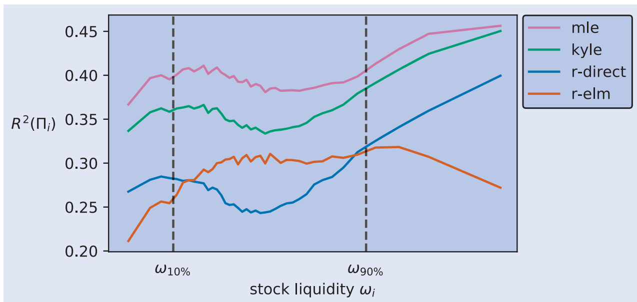

Figure 7. Idiosyncratic scores as a function of liquidity.
For each stock in our dataset, we compute the the in-sample stock--specific scores $\mathcal{R}^2(\Pi_i)$ scores on 2016 data. We then represent the average in-sample stock-specific score as a function of the liquidity ω_i, binning data by ω_i to smooth out noise. Results for the ml (in pink), kyle (in green), r-direct (in blue) and r-el (in orange) models are shown. We have further indicated the 10% and 90% quantiles of liquidity $\omega_{10\%}$ and $\omega_{90\%}$.

Thus, violating self-stability (axiom 13) is key to explain price changes for all ranges of liquidity within a basket of instruments.

4.4. Robustness

The previous section compared the descriptive power of different cross impact models. However, because axioms also

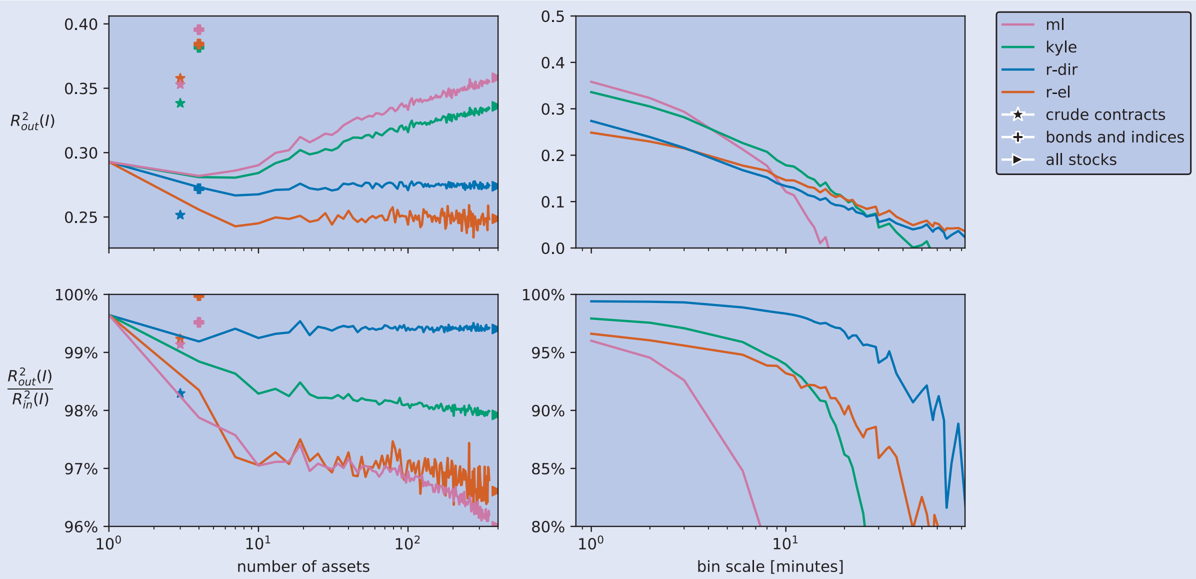

Figure 8. Idiosyncratic score and overfitting as a function of the number of assets and bin timescale.
Left column: average out-of-sample idiosyncratic score $\mathcal{R}^2_{\text{out}}(I_\sigma)$ (top left) and overfitting coefficient $\frac{\mathcal{R}^2_{\text{out}}(I_\sigma)}{\mathcal{R}^2_{\text{in}}(I_\sigma)}$ (bottom left) computed using stocks data. Out-of-sample and in-sample scores were computed by randomly selecting a subset of stocks and computing scores on the given subset, repeating the procedure more when there are fewer stocks are selected than when a large proportion of stocks from our sample is considered. The average score for each models across all samples is then shown. Scores are shown for the `ml` (in pink), the `kyle` (in green), `r-direct` (in blue) and `r-el` (in orange). Stars show results for crude contracts, crosses for bonds and indices and triangles for all 393 stocks of our sample. **Right column:** idiosyncratic scores (top right) and overfitting coefficient (bottom right) as a function of the bin timescale. Scores were computed using the same procedure described in section 4.3, varying the bin parameter from 10 seconds up to around an hour.

constrain cross-impact models, it is interesting to examine how robust different models are when the number of instruments or the time scale increase. In figure 8, we show the out-of-sample score and overfitting coefficient for idiosyncratic price changes for our set of 393 stocks, as a function of the bin timescale and number of instruments.

As expected, the number of degrees of freedom controls the overfitting of different models. This explains why, in terms of overfitting with respect to the number of instruments at the minute timescale, `r-direct` $<$ `kyle` $<$ `ml` $\approx$ `r-el`. In contrast, models overfit less on futures, which suggests that overfitting decreases as the pairwise correlation between instruments increases. Furthermore, out-of-sample idiosyncratic scores for the `ml` and `kyle` model increase with the number of assets. A somewhat surprising result, despite the small pairwise correlation of instruments in our stock dataset and the large number of stocks considered in this study, is that idiosyncratic scores appear to keep increasing for more than 400 assets. This suggests that there is still latent explanatory power in the dataset but only two models manage to extract it.

Focusing on the influence of the bin timescale, there is little overfitting at the minute timescale but it increases with the bin timescale. In particular, the good fit of the `ml` at small timescales quickly breaks down for larger timescales. On the other hand, both the `r-el` and `kyle` models are quite robust up until the 10 minute timescale. At this timescale, we expect the temporal structure of market impact which we ignored to be essential to account for price changes. This highlights the importance of enforcing consistency requirements to reduce overfitting.

5. Conclusion

Let us summarize what we have achieved. Our main objective was to build a principled approach to choose a cross-impact model Λ given a set of empirical observations encoded in the sufficient statistics (Σ, Ω, R). We wanted our cross-impact model to be (i) consistent with market data and (ii) compliant with common sense. To do so, we defined a set of axioms formalizing the notion of 'common sense', helping us to classify cross-impact models on the basis of their properties and to make sense of their performance on empirical data. In all markets studied, our analysis confirms that cross-impact models are well suited to predict execution costs and evaluate liquidity risk of portfolio trades, showing significant improvement compared to impact models in which cross-sectional effects are disregarded (see tables 3–5). However, only the `kyle` and `ml` models perform well on all markets studied, whereas only the `kyle` model prevents arbitrage and is well-behaved when trading both liquid and illiquid instruments. This makes it an ideal model for other applications, such as optimal execution.

Independently of our specific model implementations, our approach also allowed us to establish what properties of cross-impact models are implied by symmetry, and which are the relevant ones in order to explain empirical data. For example:

- Symmetry axioms (axioms 1–5) alone completely characterize return-based cross-impact models (appendix 1).
- Empirical evidence confirms the importance of fragmentation invariance axioms (axioms 8–10) for cross-impact models applied to markets where

some instruments (or linear combination of instruments) display very small price fluctuations, as for calendar spreads (section 4.1).
- Compliance with stability axioms (axioms 11–13) enables models to better explain price moves of instruments in extreme liquidity regimes (see section 4.3 and figure 7).
- Additionally, the reduced number of parameters of models strongly constrained by symmetry (such as the `kyle` model) reduces overfitting—both when increasing the number of instruments in our universe and when reducing the sample size through an increased time bin size (see figure 8).

Even though we have considered a linear, single-period scenario, the ideas introduced in this paper could be generalized to deal with more general cases. Furthermore, the framework can be adapted to deal with derivatives (Tomas *et al.* 2021). Another topic is the generalization of this framework to account for the auto-correlation of the order flow. This question is examined in Rosenbaum and Tomas (2021b) and Rosenbaum and Tomas (2021a) but an axiom-first approach is still lacking. Finally, many questions subsist about the microstructural ingredients from which a cross-impact model should emerge at intermediate time scales: what dynamics of the order book could account for such aggregate price dynamics? This question requires further attention and we hope to examine it in detail from the perspective of zero-intelligence limit order-book models (Mastromatteo *et al.* 2014).

Finally, our results can be used to provide realistic models for cross impact to use in the literature. For optimal trading applications where a cross-impact model needs to be specified (Gârleanu and Pedersen 2016, Ekren and Muhle-Karbe 2019, Lehalle and Mouzouni 2019), it is natural to favor models which satisfy arbitrage axioms to prevent ill behaved trading strategies. When modeling cross impact at the microstructural level (Tsoukalas *et al.* 2019) then arbitrage, fragmentation and liquidity axioms are all important to rule out price manipulation. For each domain, we highlighted which cross-impact models would be good candidates.

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References

Alfonsi, A., Klöck, F. and Schied, A., Multivariate transient price impact and matrix-valued positive definite functions. *Math. Oper. Res.*, 2016, **41**(3), 914–934.

Almgren, R., Thum, C., Hauptmann, E. and Li, H., Direct estimation of equity market impact. *Risk*, 2005, **18**(7), 58–62.

Benzaquen, M., Mastromatteo, I., Eisler, Z. and Bouchaud, J.-P., Dissecting cross-impact on stock markets: An empirical analysis. *J. Statist. Mech. Theory Exp.*, 2017, **2017**(2), 23406.

Bergault, P., Evangelista, D., Guéant, O. and Vieira, D., Closed-form approximations in multi-asset market making, 2018. *arXiv preprint arXiv:1810.04383*.

Bouchaud, J.-P., Bonart, J., Donier, J. and Gould, M., *Trades, Quotes and Prices*, 2018 (Cambridge University Press: Cambridge). 3.

Caballe, J. and Krishnan, M., Imperfect competition in a multi-security market with risk neutrality. *Econometrica (1986–1998)*, 1994, **62**(3), 695.

Ekren, I. and Muhle-Karbe, J., Portfolio choice with small temporary and transient price impact. *Math. Finance*, 2019, **29**(4), 1066–1115.

Evangelista, D. and Vieira, D., New closed-form approximations in multi-asset market making, 2018. *arXiv preprint arXiv:1810.04383*.

Garcia del Molino, L.C., Mastromatteo, I., Benzaquen, M. and Bouchaud, J.-P., The multivariate Kyle model: More is different. *SIAM J. Financ. Math.*, 2020, **11**(2), 327–357.

Gârleanu, N. and Pedersen, L.H., Dynamic portfolio choice with frictions. *J. Econ. Theory.*, 2016, **165**, 487–516.

Gatheral, J., No-dynamic-arbitrage and market impact. *Quant. Finance*, 2010, **10**(7), 749–759.

Guéant, O., Optimal market making. *Appl. Math. Finance*, 2017, **24**(2), 112–154.

Hasbrouck, J. and Seppi, D.J., Common factors in prices, order flows, and liquidity. *J. Financ. Econ.*, 2001, **59**(3), 383–411.

Lehalle, C.-A. and Mouzouni, C., A mean field game of portfolio trading and its consequences on perceived correlations, 2019. *arXiv preprint arXiv:1902.09606*.

Mastromatteo, I., Toth, B. and Bouchaud, J.-P., Agent-based models for latent liquidity and concave price impact. *Phys. Rev. E.*, 2014, **89**(4), 042805.

Mastromatteo, I., Benzaquen, M., Eisler, Z. and Bouchaud, J.-P., Trading lightly: Cross-impact and optimal portfolio execution, 2017. *arXiv preprint arXiv:1702.03838*.

Min, S., Maglaras, C. and Moallemi, C.C., Cross-sectional variation of intraday liquidity, cross-impact, and their effect on portfolio execution, 2018. *arXiv preprint arXiv:1811.05524*.

Pasquariello, P. and Vega, C., Strategic cross-trading in the us stock market. *Rev. Financ.*, 2015, **19**(1), 229–282.

Patzelt, F. and Bouchaud, J.-P., Universal scaling and nonlinearity of aggregate price impact in financial markets. *Phys. Rev. E.*, 2018, **97**(1), 012304.

Rosenbaum, M. and Tomas, M., A characterisation of cross-impact kernels, 2021a. *arXiv preprint arXiv:2107.08684*.

Rosenbaum, M. and Tomas, M., From microscopic price dynamics to multidimensional rough volatility models. *Adv. Appl. Probab.*, 2021b, **53**(2), 425–462.

Schneider, M. and Lillo, F., Cross-impact and no-dynamic-arbitrage. *Quant. Finance*, 2019, **19**(1), 137–154.

Tomas, M., Mastromatteo, I. and Benzaquen, M., Cross impact in derivative markets, 2021. *arXiv preprint arXiv:2102.02834*.

Torre, N., *BARRAMarket ImpactModelHandbook*, 1997 (BARRA Inc.: Berkeley, CA).

Tsoukalas, G., Wang, J. and Gieseckec, K., Dynamic portfolio execution. *Manage. Sci.*, 2019, **65**(5), 2015–2040.

Wang, S., Schäfer, R. and Guhr, T., Price response in correlated financial markets: Empirical results, 2015. *arXiv preprint arXiv:1510.03205*.

Wang, S., Schäfer, R. and Guhr, T., Cross-response in correlated financial markets: Individual stocks. *Euro. Phys. J. B*, 2016, **89**(4), 105.

Wang, S., Neusüß, S. and Guhr, T., Grasping asymmetric information in market impacts, 2017. *arXiv preprint arXiv:1710.07959*.

Appendices

Appendix 1. Models and axioms

This section proves some of the results stated in table 1 and implications between the different axioms.

A.1. Relation between fragmentation and liquidity axioms

In this section, we establish some links between fragmentation and liquidity axioms. To do so, in the rest of this section, we will make use of two kinds of regularised covariance and response matrices. Given a linear subspace V, we first introduce the order flow regularized estimators:

$$\begin{aligned}\Sigma_\varepsilon^q &:= \Sigma\\ \Omega_\varepsilon^q &:= (\bar{\Pi}_V + \epsilon\Pi_V)\Omega(\bar{\Pi}_V + \epsilon\Pi_V)\\ R_\varepsilon^q &:= R(\bar{\Pi}_V + \epsilon\Pi_V).\end{aligned}$$

These correspond to the multiplication of liquidity of instruments in V by ε. Similarly, we introduce the price regularized estimators

$$\begin{aligned}\Sigma_\varepsilon^p &:= (\bar{\Pi}_V + \epsilon\Pi_V)\Sigma(\bar{\Pi}_V + \epsilon\Pi_V)\\ \Omega_\varepsilon^p &:= \Omega\\ R_\varepsilon^p &:= (\bar{\Pi}_V + \epsilon\Pi_V)R.\end{aligned}$$

These correspond to the multiplication of price fluctuations of instruments in V by ε. We begin with a convenient lemma that relates liquidity properties to fragmentation properties.

Lemma A.1 *Let Λ be a split-invariant and rotation-invariant (axiom 4 and 5) cross-impact model and a linear subspace V such that $\emptyset \subset V \subseteq \mathbb{R}^n$. Then, for all $\varepsilon > 0$, we have*

$$\begin{aligned}&\Lambda(\Sigma_\varepsilon^q, \Omega_\varepsilon^q, R_\varepsilon^q)\\ &\quad= \bar{\Pi}_V\Lambda(\Sigma_\varepsilon^p, \Omega_\varepsilon^p, R_\varepsilon^p)\bar{\Pi}_V\\ &\qquad+ \epsilon^{-1}\left[\bar{\Pi}_V\Lambda(\Sigma_\varepsilon^p, \Omega_\varepsilon^p, R_\varepsilon^p)\Pi_V + \Pi_V\Lambda(\Sigma_\varepsilon^p, \Omega_\varepsilon^p, R_\varepsilon^p)\bar{\Pi}_V\right]\\ &\qquad+ \epsilon^{-2}\Pi_V\Lambda(\Sigma_\varepsilon^p, \Omega_\varepsilon^p, R_\varepsilon^p)\Pi_V.\end{aligned}$$

Proof Let $(u_1, \dots, u_k)$ an orthonormal basis for the linear subspace V, where $k = \dim(V)$ and $(u_1, \dots, u_n)$ a completed orthonormal basis on $\mathbb{R}^n$. We write $A := (u_1 \mid u_2 \mid \cdots \mid u_k) \in \mathcal{M}_{n,k}(\mathbb{R})$ and $U := (u_1 \mid u_2 \mid \cdots \mid u_n) \in \mathcal{M}_{n,n}(\mathbb{R})$. Then $\Pi_V = AA^\top$ and $\bar{\Pi}_V = I - AA^\top$. By rotation invariance, we have

$$U\Lambda(\Sigma_\varepsilon^q, \Omega_\varepsilon^q, R_\varepsilon^q)U^\top = \Lambda(U\Sigma_\varepsilon^q U^\top, U\Omega_\varepsilon^q U^\top, UR_\varepsilon^q U^\top).$$

Since $(u_1, \dots, u_n)$ is an orthonormal basis, UA only has non-zero entries along the diagonal. Writing $\widehat{A} := UA$ we can apply split invariance with $D = (I - \widehat{A}\widehat{A}^\top + \varepsilon\widehat{A}\widehat{A}^\top)$ to obtain

$$\begin{aligned}U\Lambda(\Sigma_\varepsilon^q, \Omega_\varepsilon^q, R_\varepsilon^q)U^\top &= D\Lambda(D^{-1}U\Sigma_\varepsilon^q U^\top D^{-1},\\ &\quad DU\Omega_\varepsilon^q U^\top D, D^{-1}UR_\varepsilon^q U^\top D)D.\end{aligned}$$

Straightforward computations show that

$$\begin{aligned}&D^{-1}U\Sigma_\varepsilon^q U^\top D^{-1} = U\Sigma_\varepsilon^p U^\top \quad DU\Omega_\varepsilon^q U^\top D = U\Omega_\varepsilon^p U^\top\\ &D^{-1}UR_\varepsilon^q U^\top D = UR_\varepsilon^p U^\top.\end{aligned}$$

Therefore

$$\begin{aligned}&U\Lambda(\Sigma_\varepsilon^q, \Omega_\varepsilon^q, R_\varepsilon^q)U^\top\\ &\quad= (I - \widehat{A}\widehat{A}^\top + \varepsilon^{-1}\widehat{A}\widehat{A}^\top)\Lambda\left(U\Sigma_\varepsilon^p U^\top, U\Omega_\varepsilon^p U^\top, UR_\varepsilon^p U^\top\right)\\ &\qquad\times(I - \widehat{A}\widehat{A}^\top + \varepsilon^{-1}\widehat{A}\widehat{A}^\top).\end{aligned}$$

Applying rotational invariance once again we get

$$\begin{aligned}&U\Lambda(\Sigma_\varepsilon^q, \Omega_\varepsilon^q, R_\varepsilon^q)U^\top\\ &\quad= U(I - AA^\top)\Lambda\left(\Sigma_\varepsilon^p, \Omega_\varepsilon^p, R_\varepsilon^p\right)(I - AA^\top)U^\top\\ &\qquad+ \varepsilon^{-1}U\Big[(I - AA^\top)\Lambda\left(\Sigma_\varepsilon^p, \Omega_\varepsilon^p, R_\varepsilon^p\right)AA^\top\\ &\qquad+AA^\top\Lambda\left(\Sigma_\varepsilon^p, \Omega_\varepsilon^p, R_\varepsilon^p\right)(I - AA^\top)\Big]U^\top\\ &\qquad+ \varepsilon^{-2}UAA^\top\Lambda\left(\Sigma_\varepsilon^p, \Omega_\varepsilon^p, R_\varepsilon^p\right)AA^\top U^\top.\end{aligned}$$

This finally yields

$$\begin{aligned}&\Lambda(\Sigma_\varepsilon^q, \Omega_\varepsilon^q, R_\varepsilon^q)\\ &\quad= U\bar{\Pi}_V\Lambda\left(\Sigma_\varepsilon^p, \Omega_\varepsilon^p, R_\varepsilon^p\right)\bar{\Pi}_V\\ &\qquad+ \varepsilon^{-1}\left[\bar{\Pi}_V\Lambda\left(\Sigma_\varepsilon^p, \Omega_\varepsilon^p, R_\varepsilon^p\right)\Pi_V + \Pi_V\Lambda\left(\Sigma_\varepsilon^p, \Omega_\varepsilon^p, R_\varepsilon^p\right)\bar{\Pi}_V\right]\\ &\qquad+ \varepsilon^{-2}\Pi_V\Lambda\left(\Sigma_\varepsilon^p, \Omega_\varepsilon^p, R_\varepsilon^p\right)\Pi_V,\end{aligned}$$

which concludes the proof. ∎

In a similar fashion as lemma A.1, one can prove the following Lemma.

Lemma A.2 *Let Λ be a split-invariant and rotation-invariant (axiom 4 and 5) cross-impact model and a subspace V such that $\emptyset \subset V \subseteq \mathbb{R}^n$. Then we have*

$$\begin{aligned}&\Lambda(\Sigma_\varepsilon^p, \Omega_\varepsilon^p, R_\varepsilon^p)\\ &\quad= \bar{\Pi}_V\Lambda(\Sigma_\varepsilon^q, \Omega_\varepsilon^q, R_\varepsilon^q)\bar{\Pi}_V\\ &\qquad+ \epsilon^{-1}\left[\bar{\Pi}_V\Lambda(\Sigma_\varepsilon^q, \Omega_\varepsilon^q, R_\varepsilon^q)\Pi_V + \Pi_V\Lambda(\Sigma_\varepsilon^q, \Omega_\varepsilon^q, R_\varepsilon^q)\bar{\Pi}_V\right]\\ &\qquad+ \epsilon^{-2}\Pi_V\Lambda(\Sigma_\varepsilon^q, \Omega_\varepsilon^q, R_\varepsilon^q)\Pi_V.\end{aligned}$$

Lemma A.1 and A.2 enable us to relate cross-stability to fragmentation invariance. This is the topic of the next proposition.

Proposition A.1 *Let Λ be a cross-impact model continuous in the first and third argument which satisfies split and rotational invariance (axiom 4 and 5). Then*

(i) *If Λ satisfies semi-strong fragmentation invariance (axiom 9), then it is weakly cross-stable (axiom 11).*
(ii) *If Λ is strongly fragmentation invariant (axiom 10), then it is strongly cross-stable (axiom 12).*

Proof We first prove (i). Since the cross-impact model Λ is continuous in the first and third argument and satisfies semi-strong fragmentation invariance we have

$$\Lambda(\Sigma^p_\varepsilon, \Omega^p_\varepsilon, R^p_\varepsilon)\Pi_V \underset{\varepsilon\to 0}{\to} 0$$
$$\Pi_V\Lambda(\Sigma^p_\varepsilon, \Omega^p_\varepsilon, R^p_\varepsilon) \underset{\varepsilon\to 0}{\to} 0.$$

Plugging the above in the results of lemma A.2 yields

$$\bar\Pi_V\Lambda(\Sigma^p_\varepsilon, \Omega^p_\varepsilon, R^p_\varepsilon)\Pi_V = \epsilon^{-1}\bar\Pi_V\Lambda(\Sigma^q_\varepsilon, \Omega^q_\varepsilon, R^q_\varepsilon)\Pi_V \underset{\varepsilon\to 0}{\to} 0$$
$$\Pi_V\Lambda(\Sigma^p_\varepsilon, \Omega^p_\varepsilon, R^p_\varepsilon)\bar\Pi_V = \epsilon^{-1}\Pi_V\Lambda(\Sigma^q_\varepsilon, \Omega^q_\varepsilon, R^q_\varepsilon)\bar\Pi_V \underset{\varepsilon\to 0}{\to} 0\,.$$

Thus Λ is weakly cross-stable. We now prove (ii). Continuity at $\epsilon = 0$ and strong fragmentation invariance yield

$$\bar\Pi_V\Lambda(\Sigma^p_\varepsilon, \Omega^p_\varepsilon, R^q_\varepsilon)\bar\Pi_V \underset{\varepsilon\to 0}{\to} \bar\Pi_V\Lambda(\bar\Pi_V\Sigma\bar\Pi_V, \bar\Pi_V\Omega\bar\Pi_V, \bar\Pi_V R\bar\Pi_V)\bar\Pi_V\,.$$

Plugging the above into the results of lemma A.2 gives

$$\bar\Pi_V\Lambda(\Sigma^q_\varepsilon, \Omega^q_\varepsilon, R^q_\varepsilon)\bar\Pi_V = \bar\Pi_V\Lambda(\Sigma^p_\varepsilon, \Omega^p_\varepsilon, R^p_\varepsilon)\bar\Pi_V \underset{\varepsilon\to 0}{\to} \bar\Pi_V\Lambda(\bar\Pi_V\Sigma\bar\Pi_V, \bar\Pi_V\Omega\bar\Pi_V, \bar\Pi_V R\bar\Pi_V)\bar\Pi_V.$$

This implies that Λ is strongly cross-stable. ∎

Interestingly, the converse of Proposition A.1 does not hold, thus indicating that the fragmentation invariance properties play a more fundamental role than liquidity related axioms. The next proposition shows the converse, provided some additional regularity of the cross impact model.

PROPOSITION A.2 *Let Λ be a cross impact model continuous in the first and third argument which satisfies split and rotational invariance (axiom* 4 *and 5). We further assume that, for every linear subspace V and using the previous notations, $\varepsilon^2\Lambda(\Sigma^q_\varepsilon, \Omega^q_\varepsilon, R^q_\varepsilon) \to_{\varepsilon\to 0} 0$. Then*

(i) *If Λ is weakly cross-stable (axiom* 11*), then it satisfies semi-strong fragmentation invariance (axiom* 9*).*
(ii) *If Λ is strongly cross-stable (axiom* 12*), then it is strongly fragmentation invariant (axiom* 10*).*

Proof We first prove (i). Since the cross-impact model Λ is weakly cross-stable we have

$$\bar\Pi_V\Lambda(\Sigma^q_\varepsilon, \Omega^q_\varepsilon, R^q_\varepsilon)\Pi_V \underset{\varepsilon\to 0}{\to} 0$$
$$\Pi_V\Lambda(\Sigma^q_\varepsilon, \Omega^q_\varepsilon, R^q_\varepsilon)\bar\Pi_V \underset{\varepsilon\to 0}{\to} 0\,.$$

Furthermore, by assumption we have

$$\varepsilon^2\Lambda(\Sigma^q_\varepsilon, \Omega^q_\varepsilon, R^q_\varepsilon) \underset{\varepsilon\to 0}{\to} 0.$$

Plugging the above in the results of lemma A.1 yields

$$\bar\Pi_V\Lambda(\Sigma^q_\varepsilon, \Omega^q_\varepsilon, R^q_\varepsilon)\Pi_V = \varepsilon^{-1}\bar\Pi_V\Lambda(\Sigma^p_\varepsilon, \Omega^p_\varepsilon, R^p_\varepsilon)\Pi_V \underset{\varepsilon\to 0}{\to} 0$$
$$\Pi_V\Lambda(\Sigma^q_\varepsilon, \Omega^q_\varepsilon, R^q_\varepsilon)\bar\Pi_V = \varepsilon^{-1}\Pi_V\Lambda(\Sigma^p_\varepsilon, \Omega^p_\varepsilon, R^p_\varepsilon)\bar\Pi_V \underset{\varepsilon\to 0}{\to} 0$$
$$\varepsilon^2\Pi_V\Lambda(\Sigma^q_\varepsilon, \Omega^q_\varepsilon, R^q_\varepsilon)\Pi_V = \Pi_V\Lambda(\Sigma^p_\varepsilon, \Omega^p_\varepsilon, R^p_\varepsilon)\bar\Pi_V \underset{\varepsilon\to 0}{\to} 0.$$

Combining the above and using continuity in the first and third argument, we obtain

$$\Pi_V\Lambda(\Sigma^p_\varepsilon, \Omega^p_\varepsilon, R^p_\varepsilon) \underset{\varepsilon\to 0}{\to} 0 = \Pi_V\Lambda(\bar\Pi_V\Sigma\bar\Pi_V, \Omega, \bar\Pi_V R)$$
$$\Lambda(\Sigma^p_\varepsilon, \Omega^p_\varepsilon, R^p_\varepsilon)\Pi_V \underset{\varepsilon\to 0}{\to} 0 = \Lambda(\bar\Pi_V\Sigma\bar\Pi_V, \Omega, \bar\Pi_V R)\Pi_V\,.$$

Thus this proves that Λ is semi-strongly fragmentation invariant. We now prove (ii). Continuity at $\epsilon = 0$ and strong cross-stability yield

$$\bar\Pi_V\Lambda(\Sigma^q_\varepsilon, \Omega^q_\varepsilon, R^q_\varepsilon)\bar\Pi_V \underset{\varepsilon\to 0}{\to} \bar\Pi_V\Lambda(\bar\Pi_V\Sigma\bar\Pi_V, \bar\Pi_V\Omega\bar\Pi_V, \bar\Pi_V R\bar\Pi_V)\bar\Pi_V\,.$$

Plugging the above into the results of lemma A.1 gives

$$\bar\Pi_V\Lambda(\Sigma^q_\varepsilon, \Omega^q_\varepsilon, R^q_\varepsilon)\bar\Pi_V = \bar\Pi_V\Lambda(\Sigma^p_\varepsilon, \Omega^p_\varepsilon, R^p_\varepsilon)\bar\Pi_V \underset{\varepsilon\to 0}{\to} \bar\Pi_V\Lambda(\bar\Pi_V\Sigma\bar\Pi_V, \bar\Pi_V\Omega\bar\Pi_V, \bar\Pi_V R\bar\Pi_V)\bar\Pi_V\,.$$

This implies that Λ is strongly fragmentation invariant. ∎

Proposition A.1 and A.2 show that fragmentation and cross-stability axioms are related. Furthermore, for cross impact models which satisfy the regularity property of proposition A.2, the two sets of axioms are equivalent.

A.2. Relations between axioms and models

In this section, we characterize the models which satisfy the axioms introduced in section 3.2. We begin with the following proposition, the proof of which is heavily inspired by Caballe and Krishnan (1994) and Garcia del Molino *et al.* (2020).

PROPOSITION A.3 *Let Λ be a symmetric, positive-semidefinite and return covariance consistent cross impact model (axioms* 6*,* 7 *and* 14*). Then $\Lambda = \Lambda_{kyle}$ up to a multiplicative constant.*

Proof Let Λ be a cross impact model which satisfies axiom 6 and 14 and $(\Sigma, \Omega, R) \in (\mathcal{S}^+_n(\mathbb{R}) \times \mathcal{S}^{++}_n(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$. We assume for convenience that the multiplicative constant in axiom 14 is one. Writing Λ for $\Lambda(\Sigma, \Omega, R)$, and $\mathcal{L}$ for a matrix such that $\Omega = \mathcal{L}\mathcal{L}^\top$, we have

$$\Sigma = \Lambda\Omega\Lambda^\top = \Lambda\mathcal{L}\mathcal{L}^\top\Lambda^\top = (\Lambda\mathcal{L})(\Lambda\mathcal{L})^\top.$$

Thus, by unicity up to a rotation of the square root decomposition, writing $\mathcal{G}$ for a matrix such that $\Sigma = \mathcal{G}\mathcal{G}^\top$, there exists a rotation O such that $\Lambda = \mathcal{G}O\mathcal{L}^{-1}$. Furthermore, since Λ is symmetric,

$$\mathcal{G}O\mathcal{L}^{-1} = (\mathcal{G}O\mathcal{L}^{-1})^\top.$$

Rewriting, we find

$$\mathcal{L}^\top\mathcal{G}O = O^\top\mathcal{G}^\top\mathcal{L},$$

so that the matrix $\mathcal{L}^\top\mathcal{G}O$ is symmetric and satisfies

$$(\mathcal{L}^\top\mathcal{G}O)(\mathcal{L}^\top\mathcal{G}O)^\top = (\mathcal{L}\mathcal{G})(\mathcal{L}^\top\mathcal{G})^\top.$$

Since $(\mathcal{L}^\top\mathcal{G})(\mathcal{L}^\top\mathcal{G})^\top$ is symmetric positive semi-definite, the symmetric square root is unique and

$$\mathcal{L}^\top\mathcal{G}O = \sqrt{(\mathcal{L}^\top\mathcal{G})(\mathcal{L}^\top\mathcal{G})^\top}.$$

Plugging this back into the expression of the cross-impact matrix yields the result:

$$\Lambda = \mathcal{G}O\mathcal{L}^{-1} = \mathcal{L}^{-\top}\sqrt{(\mathcal{L}^\top\mathcal{G})(\mathcal{L}^\top\mathcal{G})^\top}\mathcal{L}^{-1} = \mathcal{L}^{-\top}\sqrt{\mathcal{L}^\top\Sigma\mathcal{L}}\mathcal{L}^{-1}.$$

∎

Hence, there is a single symmetric, positive-semidefinite, correlation-consistent, cross impact model. Given that the fragmentation-related axioms seem so fundamental, one might wonder how many models one can build that satisfy that family of properties. Surprisingly, we find that the class of models enjoying both split invariance and rotational invariance is quite small, as shows the next lemma.

Lemma A.3 *Let Λ be a cross-impact model which satisfies axiom 4 and 5. Then, for all $(\Sigma, \Omega, R) \in (\mathcal{S}_n^+(\mathbb{R}) \times \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}))$, it can be written as*

$$\Lambda(\Sigma, \Omega, R) = \mathcal{L}^{-\top} U \Lambda(U^\top \hat{\Sigma} U, \mathbb{I}, U^\top \hat{R} U) U^\top \mathcal{L}^{-1},$$

where

$$\Omega = \mathcal{L}\mathcal{L}^\top$$
$$\hat{\Sigma} = \mathcal{L}^\top \Sigma \mathcal{L}$$
$$\hat{R} = \mathcal{L}^\top R \mathcal{L}^{-\top}$$

and U is an orthogonal matrix (i.e. $UU^\top = \mathbb{I}$).

Proof The lemma is obtained by applying sequentially rotational invariance, split invariance and again rotational invariance. The first two transformations can be used in order to remove the dependency in Ω as the second argument of the $\Lambda(\Sigma, \Omega, R)$ function. ■

When one discards the influence of the response matrix, the model can further be characterized as shown by the next proposition.

Proposition A.4 *A return covariance based cross-impact model Λ that is both split-invariant and rotation-invariant (axiom 5 and 4) can always be written in the form*

$$\Lambda(\Sigma, \Omega) = \mathcal{L}^{-\top} U F(\boldsymbol{\mu}) U^\top \mathcal{L}^{-1},$$

where

$$\Omega = \mathcal{L}\mathcal{L}^\top; \quad \hat{\Sigma} := \mathcal{L}^\top \Sigma \mathcal{L}; \quad U^\top \hat{\Sigma} U := \mathrm{diag}(\boldsymbol{\mu});$$
$$F(\boldsymbol{\mu}) := \Lambda(\mathrm{diag}(\boldsymbol{\mu}), \mathbb{I}).$$

Furthermore, if Λ is cash-invariant and direct-invariant axiom 2 and 3, then $F(\boldsymbol{\mu}) \propto \mathrm{diag}(\boldsymbol{\mu})^{1/2}$ and $\Lambda = \Lambda_{\mathtt{kyle}}$ up to a multiplicative constant.

Proof For a return covariance-based model, we can simply choose from lemma A.3 to fix U as the rotation that diagonalizes the symmetric matrix $\hat{\Sigma}$, obtaining:

$$U^\top \hat{\Sigma} U = \mathrm{diag}(\boldsymbol{\mu}).$$

This choice implies:

$$\Lambda(\Sigma, \Omega) = \mathcal{L}^{-\top} U \Lambda(\mathrm{diag}(\boldsymbol{\mu}), \mathbb{I}) U^\top \mathcal{L}^{-1},$$

which yields the result of the first part of the proposition. Furthermore, if we assume Λ is cash-invariant and direct-invariant axioms 3 and 2,

$$\Lambda(\mathrm{diag}(\boldsymbol{\mu}), \mathbb{I}) = \sum_{i=1}^{d} \sqrt{\mu_i} \Lambda(\boldsymbol{e_i}\boldsymbol{e_i}^\top, \boldsymbol{e_i}\boldsymbol{e_i}^\top)$$

which yields the `kyle` model up to a constant. ■

The above shows that the only return-based cross-impact model which satisfies all symmetry axioms axiom 1–2 is the `kyle` model. This indicates that data is bound to play a major role in order to select what cross-impact model is deemed to be more suitable in order to describe market microstructure.

A.3. Proof of important properties of the `kyle` *model*

This section is dedicated to showing that the `kyle` model satisfies all the axioms outlined in section 3.2. As the fragmentaion and invariance axioms were discussed in the previous section, the next lemma shows that the model is also cross-stable.

Lemma A.4 *The `kyle` model is strongly cross-stable in the sense of axiom 12 and 13 and is not self-stable in the sense of axiom 13.*

Proof Let V be a linear subspace of $\mathbb{R}^n$) and $\varepsilon > 0$. Note that, writing $\mathcal{G}$ for a matrix such that $\mathcal{G}\mathcal{G}^\top = \Sigma$, for any matrix $\mathcal{L}_\varepsilon$ such that $\mathcal{L}_\varepsilon \mathcal{L}_\varepsilon^\top = \Omega_\varepsilon$, there exists a rotation matrix O_ε such that we have

$$\Lambda_{\mathtt{kyle}} = \mathcal{G} O_\varepsilon \mathcal{L}_\varepsilon^{-1}.$$

However, $\Omega_\varepsilon = (\bar{\Pi}_V + \varepsilon \Pi_V)\Omega(\bar{\Pi}_V + \varepsilon \Pi_V) = (\bar{\Pi}_V + \varepsilon \Pi_V)\mathcal{L}\mathcal{L}^\top (\bar{\Pi}_V + \varepsilon \Pi_V) = [(\bar{\Pi}_V + \varepsilon \Pi_V)\mathcal{L}][(\bar{\Pi}_V + \varepsilon \Pi_V)\mathcal{L}]^\top$. Thus,

$$\begin{aligned}\Lambda_{\mathtt{kyle}} &= \mathcal{G} O_\varepsilon [(\bar{\Pi}_V + \varepsilon \Pi_V)\mathcal{L}]^{-1} \\ &= \mathcal{G} O_\varepsilon \mathcal{L}^{-1} \left(\bar{\Pi}_V + \frac{1}{\varepsilon}\Pi_V\right) \\ &= \mathcal{G} O_\varepsilon \mathcal{L}^{-1} \bar{\Pi}_V + \frac{1}{\varepsilon} \mathcal{G} O_\varepsilon \mathcal{L}^{-1} \Pi_V.\end{aligned}$$

Using the symmetry of the `kyle` model, the above yields:

$$\Lambda_{\mathtt{kyle}} = \bar{\Pi}_V \mathcal{L}^{-\top} O_\varepsilon^\top \mathcal{G}^\top + \frac{1}{\varepsilon} \Pi_V \mathcal{L}^{-\top} O_\varepsilon^\top \mathcal{G}^\top.$$

Thus, we have:

$$\bar{\Pi}_V \Lambda_{\mathtt{kyle}} \Pi_V = \bar{\Pi}_V \mathcal{L}^{-\top} O_\varepsilon^\top \mathcal{G}^\top \Pi_V,$$

and, as $O_\varepsilon^\top$ is an orthogonal matrix:

$$\epsilon^\gamma \bar{\Pi}_V \Lambda_{\mathtt{kyle}} \Pi_V \underset{\varepsilon \to 0}{=} O(1),$$

which proves axiom 11 weak-cross stability. Furthermore,

$$\Pi_V \Lambda_{\mathtt{kyle}} \Pi_V = \frac{1}{\varepsilon} \Pi_V \mathcal{L}^{-\top} O_\varepsilon^\top \mathcal{G}^\top \Pi_V,$$

so that unless $\Pi_V \mathcal{L}^{-\top} O_\varepsilon^\top \mathcal{G}^\top \Pi_V = 0$, we have:

$$||\Pi_V \Lambda_{\mathtt{kyle}} \Pi_V|| = \varepsilon^{-1} ||\Pi_V \mathcal{L}^{-\top} O_\varepsilon^\top \mathcal{G}^\top \Pi_V|| \underset{\varepsilon \to 0}{\to} \infty.$$

Choosing diagonal Σ and Ω such that $\Pi_V \mathcal{L} \neq 0$ and $\mathcal{G}\Pi_V \neq 0$, we see that $\Pi_V \mathcal{L}^{-\top} O_\varepsilon^\top \mathcal{G}^\top \Pi_V = 0$ cannot hold for all Σ, Ω. This shows that `kyle` does not satisfy axiom 13. Finally, notice that by using lemma A.3, one can make Ω appear only in the combination $\mathcal{L}^\top \Sigma \mathcal{L}$, which is insensitive to the components of Ω belonging to the kernel of Σ, which proves strong cross-stability. ■

Appendix 2. Data

This appendix contains details on the datasets and processing used to apply the different models.

A.4. Crude contracts

Description of the dataset. We collected trades and quotes data from January 2016 to December 2017, between 9:30AM to 7:30PM UTC, where most of the trading takes place in our dataset, removing 30 min around the opening of trading hours to mitigate intraday seasonality. After filtering and processing, we have a total of 430 days in our sample (237 in 2016 and 193 in 2017). We highlight below two important features of our pre-processing for the estimation of Σ, Ω and R.

Pre-processing: accounting for non-stationarity. Overall, the front month contract CRUDE0 is by far the most liquid, followed by the subsequent month contract CRUDE1 and the calendar spread CRUDE1_0. However, there are strong seasonal dependencies which are shown in figure A1. For example, the subsequent month contract becomes more liquid as one approaches the maturity of the front month contract. Global estimators of Σ, Ω and R would thus be biased by this varying liquidity $\boldsymbol{\omega}$ ($\boldsymbol{\sigma}$ also appears to follow a non-stationary pattern, but is not shown here). Thus, we

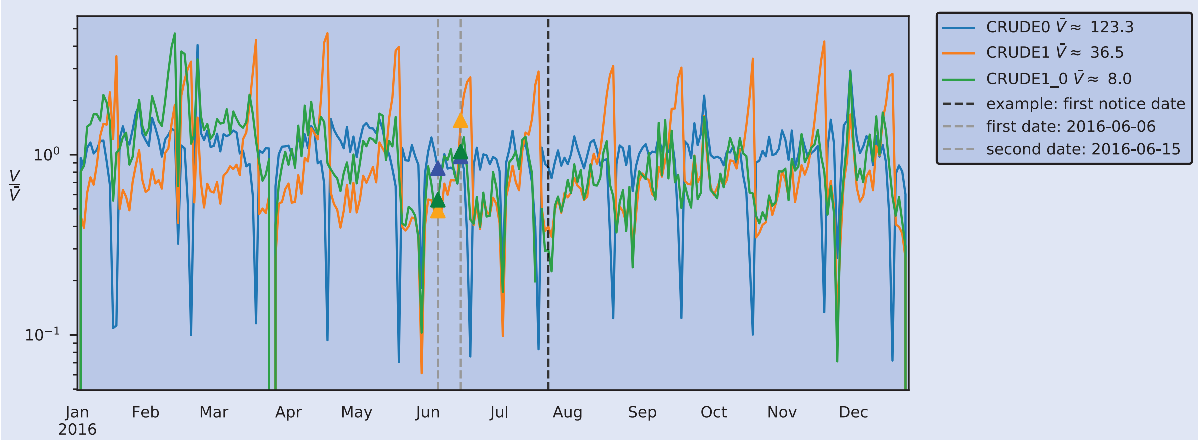

Figure A1. Number of traded NYMEX Crude oil futures and Calendar Spread contracts (in thousands) relative to daily number of traded contracts.
The number of contracts sold relative to the daily average is shown for the front month contract (in blue), the subsequent month (in orange) and the Calendar Spread (in green). The average number of traded NYMEX Crude oil futures and Calendar Spread contracts $\bar{V}$ over 2016 is shown in the upper right corner. Vertical dashed lines show specific dates. An example of first notice date for the front month contract is shown in bold black. After the first notice date, holders of the future contract may ask for physical delivery of the underlying. We also show two dates away from a first notice date: the 6th and 15th of June 2016. Colored triangles show the relative number of contracts exchanged on these dates. Note that the number of contracts is represented in thousands and was not adjusted by the basis point, so that the underlying of each contract is 1000 barrels of oil.

used local (daily) estimators of price volatility $\boldsymbol{\sigma_t}$ and liquidity $\boldsymbol{\omega_t}$, and built local covariance estimators Σ_t and Ω_t by assuming stationarity of the correlations $\varrho = \mathrm{diag}(\boldsymbol{\sigma_t})^{-1}\Sigma_t\mathrm{diag}(\boldsymbol{\sigma_t})^{-1}$ and $\varrho_\Omega = \mathrm{diag}(\boldsymbol{\omega_t})^{-1}\Omega_t\mathrm{diag}(\boldsymbol{\omega_t})^{-1}$. We estimate volatility and liquidity with a simple standard deviation: $\sigma_{i,t}^2 = \langle \Delta p_{i,t}^2 \rangle$ and $\omega_{i,t}^2 = \langle q_{i,t}^2 \rangle$, where the average $\langle \cdot \rangle$ is computed using data on day t.

Pre-processing: cleaning estimators. As illustrated in figure 1, where the structure of Σ, Ω and R are shown for a typical day, one can appreciate that the correlation between the two future contracts CRUDE0 and CRUDE1 is close to one, whereas the correlation with the Calendar Spread contract is very small, due to the small volatility of the fluctuations along the relative mode. Because of these effects, the sign of the Calendar Spread correlations with CRUDE0 and CRUDE1 is non-trivial to estimate: due to microstructural effects, the measured correlation is dominated by tick-size related effects.† In fact, empirical price changes of the Calendar Spread are not given by the difference of price changes of the legs. To solve this issue, we impose the price changes of the Calendar Spread according to the price changes of the futures contracts.

A.5. Bonds and indices

Description of the dataset. We look at 10-year US Treasury note futures and the E-MINI futures. We collect data from the Chicago Mercantile Exchange and use the first two upcoming maturities of both contracts (respectively called SPMINI and SPMINI3 for E-MINI contracts and 10USNOTE and 10USNOTE3 for 10-year US treasury notes). E-Mini futures are quarterly, financially settled contracts with maturities in March, June, September and December. At expiry, the final settlement price of E-MINI futures is a proxy for the S&P500 index using the opening prices of the underlying stocks belonging to the index. Similarly, the 10-year treasury note futures are quaterly, financially settled contracts with maturities in March, June, September and December. At expiry, the final settlement price is volume weighted average price of past trades on the underlying treasury note.‡ We collected trades and quotes data from January 2016 to December 2017, between 9AM to 7PM UTC, where most of the trading takes place in our dataset. After filtering days for which data for one product was missing, we keep a total of 160 days (75 in 2016 and 85 in 2017). We highlight below one important pre-processing step for the estimation of Σ, Ω and R.

Pre-processing: accounting for non-stationarity. The same non-stationary behavior observed for Crude Oil futures contract is observed here. Thus we adopt the same estimation procedure for the local covariance estimators Σ_t and Ω_t by assuming stationarity of the correlations $\varrho = \mathrm{diag}(\boldsymbol{\sigma_t})^{-1}\Sigma_t\mathrm{diag}(\boldsymbol{\sigma_t})^{-1}$ and $\varrho_\Omega = \mathrm{diag}(\boldsymbol{\omega_t})^{-1}\Omega_t\mathrm{diag}(\boldsymbol{\omega_t})^{-1}$.

A.6. Stocks

Description of the dataset. We chose stocks which were in the S&P500 index between January 2016 and December 2017. The resulting universe is made up of with 393 high market cap and liquid stocks. We chose such stocks to build a similar asset universe as in previous studies (Pasquariello and Vega 2015, Wang *et al.* 2015, 2016, Benzaquen *et al.* 2017, Wang *et al.* 2017). We collect trades and quotes data between 2PM and 9:30PM UTC, removing the beginning and end of the trading period to focus on the intraday behavior of liquidity and volatility and circumvent intraday non-stationary issues. We collected trades and quotes data from January 2016 to December 2017, between 2PM and 9:30PM UTC, to focus on the intraday behavior of liquidity and volatility and circumvent intraday non-stationary issues. After filtering days for which data for one product was missing, we keep a total of 302 days (154 in 2016 and 148 in 2017). Some summary characteristics of our sample are presented in table A1. The distribution of stocks in each sector is given in figure A2.

† To test this hypothesis, we estimated the empirical smallest eigenvalue of the covariance matrix for multiple futures contract as a function of relative tick size (not shown). If price changes of the Calendar Spread were given by the legs of the contract, this eigenvalue should be equal to zero. However, we found that as the tick size increases, so does the smallest eigenvalue away from zero. This thus validates our hypothesis and justifies the need for additional processing of futures data.

‡ This is a simplification of the settlement rules to emphasize the expected value of the final settlement price. Further details about the final settlement price of E-MINI futures and 10-year US Treasury Note futures can be found in the CME Rulebook.

Table A1. Summary statistics for our sample of stocks.

	Quantile		
	10%	50%	90%
Relative tick size (in %)	1.6	2.5	4.6
Number of trades per day (in thousands)	5.9	12.6	29.4
Daily turnover (in MUSD)	28.5	56.1	116.2

Pre-processing. To a lesser degree than on the previous datasets, the stock dataset shows non-stationarity in both volatility and liquidity. Thus we adopt the same estimation procedure for the local covariance estimators Σ_t and Ω_t by assuming stationarity of the correlations $\varrho = \mathrm{diag}(\boldsymbol{\sigma_t})^{-1}\Sigma_t\mathrm{diag}(\boldsymbol{\sigma_t})^{-1}$ and $\varrho_\Omega = \mathrm{diag}(\boldsymbol{\omega_t})^{-1}\Omega_t\mathrm{diag}(\boldsymbol{\omega_t})^{-1}$.

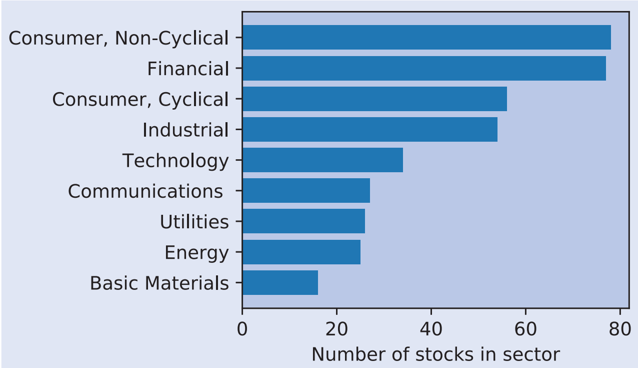

Figure A2. Sector breakdown for the 393 of stocks used in the stocks dataset.