The Spectral Representation for Real-Valued Stationary Processes

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Theorem 1. (Real Spectral Representation for Stationary Processes) *Let* $\{\xi(t), t \in \mathbb{R}\}$ *be a real-valued, zero-mean, second-order stationary process with covariance function* $r(t) = \mathbb{E}[\xi(t)\,\xi(0)]$ *and spectral distribution function* $F(\omega)$*. Then there exist real-valued random measures* $\{U(\omega), \omega \geq 0\}$ *and* $\{V(\omega), \omega \geq 0\}$ *with orthogonal increments such that:*

1. ***Process Representation:***

$$\xi(t) = \int_0^\infty [\cos(\omega t)\; d\,U(\omega) + \sin(\omega t)\; d\,V(\omega)] \tag{1}$$

2. ***Covariance Representation:***

$$r(t) = \int_0^\infty \cos(\omega t)\; d\,F(\omega) \tag{2}$$

3. ***Orthogonality Properties:***

$$\mathbb{E}[U(\omega)] = \mathbb{E}[V(\omega)] = 0 \tag{3}$$

$$\mathbb{E}[d\,U(\omega_1)\; d\,U(\omega_2)] = \mathbb{E}[d\,V(\omega_1)\; d\,V(\omega_2)] = \delta(\omega_1 - \omega_2)\, d\,F(\omega_1) \tag{4}$$

$$\mathbb{E}[d\,U(\omega_1)\; d\,V(\omega_2)] = 0 \quad \textit{for all } \omega_1, \omega_2 \geq 0 \tag{5}$$

Proof.

1. **Construction from Complex Representation:** From the complex spectral representation theorem, there holds

$$\xi(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, d\,\zeta(\omega) \tag{6}$$

where $\zeta(\omega)$ is a complex-valued random measure with orthogonal increments and $\mathbb{E}[d\,\zeta(\omega_1)\, d\overline{\zeta(\omega_2)}] = \delta(\omega_1 - \omega_2)\, d\,F_{two}(\omega)$ for the two-sided spectral distribution function $F_{two}(\omega)$.

2. **Reality Condition:** As $\xi(t)$ is real-valued,

$$\xi(t) = \overline{\xi(t)} = \int_{-\infty}^{\infty} e^{-i\omega t}\, d\overline{\zeta(\omega)} \tag{7}$$

3. **Symmetry Property:** This reality condition requires the spectral random measure to satisfy

$$d\,\zeta\,(-\omega) = d\overline{\zeta(\omega)} \tag{8}$$

for all ω.

4. **Factorization into Real Random Measures:** For $\omega > 0$, define

$$\begin{aligned} d\,U(\omega) &= d\,\zeta(\omega) + d\,\zeta\,(-\omega) = 2\,\Re\,[d\,\zeta(\omega)] && (9)\\ d\,V(\omega) &= i\,[d\,\zeta(\omega) - d\,\zeta\,(-\omega)] = -2\,\Im\,[d\,\zeta(\omega)] && (10)\end{aligned}$$

where $\Re$ and $\Im$ denote the real and imaginary parts.

5. **Derivation of Real Spectral Representation:**

$$\begin{aligned} \xi(t) &= \int_0^\infty e^{i\omega t}\,d\,\zeta(\omega) + \int_0^\infty e^{-i\omega t}\,d\,\zeta\,(-\omega)\\ &= \int_0^\infty e^{i\omega t}\,d\,\zeta(\omega) + \int_0^\infty e^{-i\omega t}\,d\overline{\zeta(\omega)}\\ &= \int_0^\infty [e^{i\omega t} + e^{-i\omega t}]\,\Re\,[d\,\zeta(\omega)] + \int_0^\infty i\,[e^{i\omega t} - e^{-i\omega t}]\,\Im\,[d\,\zeta(\omega)]\\ &= \int_0^\infty 2\cos\,(\omega\,t)\,\Re\,[d\,\zeta(\omega)] - \int_0^\infty 2\sin\,(\omega\,t)\,\Im\,[d\,\zeta(\omega)]\\ &= \int_0^\infty \cos\,(\omega\,t)\,d\,U(\omega) + \int_0^\infty \sin\,(\omega\,t)\,d\,V(\omega) \end{aligned} \tag{11}$$

6. **Orthogonality Verification:** For the real-valued process with one-sided representation, the spectral distribution function $F(\omega)$ is related to the two-sided function by $d\,F(\omega) = 2\,d\,F_{two}(\omega)$ for $\omega > 0$. Since the real and imaginary parts of $d\,\zeta(\omega)$ are orthogonal with equal variances:

$$\mathbb{E}[[\Re\,[d\,\zeta(\omega)]]^2] = \mathbb{E}[[\Im\,[d\,\zeta(\omega)]]^2] = \frac{1}{2}\,d\,F_{two}(\omega) = \frac{1}{4}\,d\,F(\omega) \tag{12}$$

Therefore,

$$\mathbb{E}\,[d\,U(\omega)^2] = \mathbb{E}\,[d\,V(\omega)^2] = 4\cdot\frac{1}{4}\,d\,F(\omega) = d\,F(\omega) \tag{13}$$

7. **Covariance Function:** The covariance is given by

$$\begin{aligned} r(t) &= \mathbb{E}\,[\xi(t)\,\xi(0)]\\ &= \mathbb{E}\left[\left(\int_0^\infty \cos\,(\omega\,t)\,d\,U(\omega) + \sin\,(\omega\,t)\,d\,V(\omega)\right)\int_0^\infty d\,U(\omega')\right]\\ &= \int_0^\infty \cos\,(\omega\,t)\,\mathbb{E}\,[d\,U(\omega)^2] \end{aligned} \tag{14}$$

where cross-terms vanish by orthogonality and the sine term vanishes since $\sin\,(\omega\,\cdot$

$0) = 0$. Using $\mathbb{E}[d\,U(\omega)^2] = d\,F(\omega)$:

$$r(t) = \int_0^\infty \cos(\omega\, t)\, d\,F(\omega) \tag{15}$$ □

Corollary 2. (Physical Interpretation) *In the real spectral representation:*

1. $\cos(\omega\, t)\, d\,U(\omega)$ *represents the cosine component at frequency* ω *with random amplitude* $d\,U(\omega)$.
2. $\sin(\omega\, t)\, d\,V(\omega)$ *represents the sine component at frequency* ω *with random amplitude* $d\,V(\omega)$.
3. $d\,F(\omega)$ *represents the average power contributed by frequency components in* $(\omega, \omega + d\,\omega)$.
4. *The random measures* $U(\omega)$ *and* $V(\omega)$ *are uncorrelated and have equal variance increments.*

Theorem 3. (U and V Random Measures) *For a real-valued stationary process* $\xi(t)$ *with mean-square continuous sample paths and spectral representation*

$$\xi(t) = \int_0^\infty [\cos(\omega\, t)\ d\,U(\omega) + \sin(\omega\, t)\ d\,V(\omega)] \tag{16}$$

the random measures $U(\omega)$ *and* $V(\omega)$ *are given explicitly by:*

1. ***U-process formula:***

$$U(\omega) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{\sin(\omega\, t)}{t}\, \xi(t)\ d\,t \tag{17}$$

2. ***V-process formula:***

$$V(\omega) = -\lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{1 - \cos(\omega\, t)}{t}\, \xi(t)\ d\,t \tag{18}$$

3. ***Incremental form:***

$$U(\omega_2) - U(\omega_1) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{\sin(\omega_2\, t) - \sin(\omega_1\, t)}{t}\, \xi(t)\ d\,t \tag{19}$$

$$V(\omega_2) - V(\omega_1) = -\lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{\cos(\omega_1\, t) - \cos(\omega_2\, t)}{t}\, \xi(t)\ d\,t \tag{20}$$

Proof. The formulas follow from the Fourier inversion theorem applied to the complex spectral measure. Starting from the complex inversion formula:

$$\zeta(\omega) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{1-e^{-i\omega t}}{-i\,t}\,\xi(t)\,d\,t \tag{21}$$

Using the definitions $d\,U(\omega) = 2\,\Re\,[d\,\zeta(\omega)]$ and $d\,V(\omega) = -2\,\Im\,[d\,\zeta(\omega)]$, and the identity $1-e^{-i\omega t} = (1-\cos(\omega\,t)) + i\sin(\omega\,t)$, we obtain:

$$U(\omega) = 2\,\Re\left[\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{1-e^{-i\omega t}}{-i\,t}\,\xi(t)\,d\,t\right] = \lim_{T\to\infty}\frac{1}{\pi}\int_{-T}^{T}\frac{\sin(\omega\,t)}{t}\,\xi(t)\,d\,t \tag{22}$$

$$V(\omega) = -2\,\Im\left[\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{1-e^{-i\omega t}}{-i\,t}\,\xi(t)\,d\,t\right] = -\lim_{T\to\infty}\frac{1}{\pi}\int_{-T}^{T}\frac{1-\cos(\omega\,t)}{t}\,\xi(t)\,d\,t \tag{23}$$

□

Remark 4. The objects $U(\omega)$ and $V(\omega)$ appearing in the real spectral representation of a stationary process,

$$\xi(t) = \int_0^\infty \cos(\omega\,t)\;d\,U(\omega) + \int_0^\infty \sin(\omega\,t)\;d\,V(\omega) \tag{24}$$

are *random measures* (or random set functions) on the frequency axis $[0,\infty)$. Their main property is that their increments over disjoint frequency intervals are orthogonal, i.e., uncorrelated (and independent if Gaussian). The notation $U(\omega)$ denotes the cumulative random measure up to frequency ω:

$$U(\omega) = U([0,\omega]) \qquad V(\omega) = V([0,\omega]) \tag{25}$$

For a stationary process with mean-square continuous sample paths, each sample path uniquely determines the corresponding random measures through the inversion formulas given above.