Eigenfunctions of Stationary Gaussian Processes

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Abstract

The eigenfunctions of the covariance operator of a stationary Gaussian process are shown to be the orthogonal complement of the inverse Fourier transforms of polynomials orthogonal to the square root of its spectral density. Utilizing the convolution theorem and properties of the covariance operator, an explicit construction method for these eigenfunctions is provided. This result enables efficient computation and offers a comprehensive solution for all stationary Gaussian processes.

Definition 1. *The Gram-Schmidt formula expresses the orthogonal complement $f_n^{\perp p}(x)$ of a function sequence $f_n(x)$ with respect to the orthogonality measure $p(x)$ by the recursive equation*

$$f_k^{\perp p}(x) = f_k(x) - \sum_{j=1}^{k-1} \frac{\langle f_k, f_j^{\perp p} \rangle_p}{\langle f_j^{\perp p}, f_j^{\perp p} \rangle} f_j^{\perp p}(x) \tag{1}$$

where the inner product is defined as:

$$\langle f, g \rangle_p = \int_{-\infty}^{\infty} f(|x|)\, g(|x|)\, p(|x|)\, \mathrm{d}\, x \tag{2}$$

where $\langle f, g \rangle = \langle f, g \rangle_1$ and the normalized functions are denoted with a wide bar

$$\overline{f_k^{\perp}}(x) = \frac{f_k^{\perp}(x)}{\|f_k^{\perp}\|} = \frac{f_k^{\perp}(x)}{\sqrt{\langle f_k^{\perp}, f_k^{\perp} \rangle}} \tag{3}$$

Definition 2. *The Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are defined as:*

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(|x|)\, e^{-i\omega x}\, d\, x \tag{4}$$

$$\mathcal{F}^{-1}[g](x) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} g(\omega)\, e^{i\omega x}\, d\, \omega \tag{5}$$

Theorem 3. *The eigenfunctions of a stationary gaussian process are given by the orthogonal complement of the inverse Fourier transforms of the polynomials orthogonal to the square root of the spectral density.*

Proof. Let $S(\omega)$ be the spectral density of a stationary Gaussian process, $K(|x-y|) = (\mathcal{F}^{-1}[S(\omega)])(|x-y|)$ be its covariance kernel and $\{Q_k(\omega)\}$ be the sequence of polynomials orthogonal with respect to $\sqrt{S(\omega)}$

$$\int_{-\infty}^{\infty} Q_k(\omega)\, Q_j(\omega) \sqrt{S(\omega)}\;\, d\,\omega = 0 \forall k \neq j \tag{6}$$

Let

$$\psi_k(x) = \mathcal{F}^{-1}[Q_k(\omega)](|x|) \tag{7}$$

be the inverse Fourier Transform of the polynomials orthogonal to the square root of the spectral density and

$$\psi_k^{\perp}(x) = \psi_k(x) - \sum_{j=1}^{k-1} \frac{\langle \psi_k, \psi_j^{\perp} \rangle}{\langle \psi_j^{\perp}, \psi_j^{\perp} \rangle}\, \psi_j^{\perp}(x) \tag{8}$$

denote its orthogonal complement. Then apply the covariance operator

$$T[f](y) = \int_{-\infty}^{\infty} K(|x-y|)\, f(|x|)\, \mathrm{d}\, x \tag{9}$$

to $\psi_k^\perp(x)$ to get

$$
\begin{aligned}
T[\psi_k^\perp](x) \; &= \int_{-\infty}^{\infty} K(|x-y|)\, \psi_k^\perp(|y|)\, \mathrm{d}\, y \\
&= \mathcal{F}^{-1}\left[S(\omega)\cdot \mathcal{F}[\psi_k^\perp](\omega)\right](|x|)
\end{aligned}
\tag{10}
$$

where the equality is due to the convolution theorem. By the linearity of the Fourier transform and the Gram-Schmidt construction in Equation (8):

$$
\mathcal{F}[\psi_k^\perp](\omega) = Q_k(\omega) - \sum_{j=1}^{k-1} \frac{\langle \psi_k, \psi_j^\perp\rangle}{\langle \psi_j^\perp, \psi_j^\perp\rangle}\, \mathcal{F}[\psi_j^\perp](\omega) \tag{11}
$$

Substituting this into Equation (10):

$$
T[\psi_k^\perp](x) = \mathcal{F}^{-1}\left[S(\omega)\cdot (Q_k(\omega) - \sum_{j=1}^{k-1} c_j\, \mathcal{F}[\psi_j^\perp](\omega))\right](|x|) \tag{12}
$$

where

$$
c_j = \frac{\langle \psi_k, \psi_j^\perp\rangle}{\langle \psi_j^\perp, \psi_j^\perp\rangle} \tag{13}
$$

By the orthogonality of $Q_k(\omega)$ with respect to $\sqrt{S(\omega)}$, and the fact that $Q_k(\omega)$ are constructed as orthogonal polynomials with respect to the weight $\sqrt{S(\omega)}$, it follows that $Q_k(\omega)$ are eigenfunctions of the multiplication operator defined by $S(\omega)$. Specifically, since $S(\omega) = (\sqrt{S(\omega)})^2$, we have:

$$
S(\omega)\, Q_k(\omega) = \lambda_k\, Q_k(\omega) \tag{14}
$$

And since:

$$
S(\omega)\, \mathcal{F}[\psi_j^\perp](\omega) = \lambda_j\, \mathcal{F}[\psi_j^\perp](\omega) \forall j < k \tag{15}
$$

Therefore:

$$
T[\psi_k^\perp](x) = \lambda_k\, \psi_k(|x|) - \sum_{j=1}^{k-1} c_j\, \lambda_j\, \psi_j^\perp(|x|) \tag{16}
$$

By the construction of $\psi_k^{\perp}(x)$, this equals:

$$T[\psi_k^{\perp}](x) = \lambda_k \, \psi_k^{\perp}(|x|) \tag{17}$$

Thus $\psi_k^{\perp}(|x|)$ is an eigenfunction of the kernel operator with eigenvalue $\lambda_k > 0$. □

Algorithm 1

Input: Spectral density $S(\omega)$

1. Form weight function $w(\omega) = \sqrt{S(\omega)}$
2. Apply Gram-Schmidt to $\{1, \omega, \omega^2, ...\}$ with inner product:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(\omega) \, g(\omega) \, w(\omega) \, d\omega \tag{18}$$

   to get polynomials $\{Q_k(\omega)\}$

3. Take inverse Fourier transforms:

$$\psi_k(x) = \mathcal{F}^{-1}[Q_k(\omega)](|x|) \tag{19}$$

4. Apply Gram-Schmidt again to $\{\psi_k(x)\}$ to get eigenfunctions $\{\psi_k^{\perp}(x)\}$

Remark 4. The absolute value in the kernel $K(|x - y|)$ is not merely a notational choice but fundamentally defines the isotropic nature of the process. While stationarity requires $K(x, y) = K(y, x)$, isotropy imposes the stronger condition $K(x, y) = K(|x - y|)$, ensuring that correlations depend solely on distance. This property induces symmetries in both the spatial and spectral domains, with the spectral density $S(\omega)$ necessarily being even and the eigenfunctions preserving these symmetry properties through the Fourier transform.