Proof of the Triangular Factorization of Stationary Gaussian Process Kernels

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1 Theorem: Spectral Representation

Let $K(t-s)$ be a stationary kernel function. Then $K(t-s)$ can be expressed in the form:

$$K(t-s) = \int_{-\infty}^{\infty} h(\tau)\, h(t-\tau-s)\, d\tau \tag{1}$$

where $h(t)$ is defined as:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega)}\, e^{i\omega t}\, d\omega \tag{2}$$

and $S(\omega)$ is the spectral density function.

2 Proof

1. Start with the spectral representation:

$$K(t-s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{i\omega(t-s)}\, d\omega \tag{3}$$

2. Factor $S(\omega)$:

$$K(t-s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega)}\, e^{i\omega t} \sqrt{S(\omega)}\, e^{-i\omega s}\, d\omega \tag{4}$$

3. Define $h(t)$:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega)}\, e^{i\omega t}\, d\omega \tag{5}$$

4. Express $\sqrt{S(\omega)}$:

$$\sqrt{S(\omega)} = \int_{-\infty}^{\infty} h(\tau)\, e^{-i\omega\tau}\, d\tau \quad \text{(by inverse Fourier transform)}$$

5. Substitute into the kernel equation:

$$K\,(t-s) = \frac{\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}h(\tau)\,e^{-i\omega\tau}\,d\,\tau)\,e^{i\omega t}(\int_{-\infty}^{\infty}h(\sigma)\,e^{-i\omega\sigma}\,d\,\sigma)\,e^{-i\omega s}\,d\,\omega}{2\,\pi} \tag{6}$$

6. Apply Fubini's theorem to change the order of integration

$$K\,(t-s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(\tau)\,h(\sigma)\,\frac{\int_{-\infty}^{\infty}e^{i\omega\,(t-\tau-s+\sigma)}\,d\,\omega}{2\,\pi}\,d\,\tau\,d\,\sigma \tag{7}$$

7. Evaluate the inner ω integral by the Fourier integral representation of the delta function:

$$\frac{1}{2\,\pi}\int_{-\infty}^{\infty}e^{i\omega\,(t-\tau-s+\sigma)}\,d\,\omega = \delta\,(t-\tau-s+\sigma) \tag{8}$$

8. Apply the delta function:

$$K\,(t-s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(\tau)\,h(\sigma)\,\delta\,(t-\tau-s+\sigma)\,d\,\tau\,d\,\sigma \tag{9}$$

9. Integrate with respect to σ:

$$\begin{aligned} K\,(t-s) &= \int_{-\infty}^{\infty}h(\tau)\int_{-\infty}^{\infty}h(\sigma)\,\delta\,(t-\tau-s+\sigma)\,d\,\sigma\,d\,\tau \\ &= \int_{-\infty}^{\infty}h(\tau)\,h\,(t-\tau-s)\,d\,\tau \end{aligned} \tag{10}$$

This completes the proof of the spectral representation theorem.