

New Uniformly Convergent Series for the Bessel Functions of the First Kind of Integer Orders

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Definition 1. Let $j_n(x)$ is the spherical Bessel function of the first kind,

$$\begin{aligned} j_n(z) &= \sqrt{\frac{\pi}{2z}} J_{n+\frac{1}{2}}(x) \\ &= \frac{1}{\sqrt{z}} \left(\sin(z) R_{n, \frac{1}{2}}(z) - \cos(z) R_{n-1, \frac{3}{2}}(z) \right) \end{aligned} \quad (1)$$

where $R_{n,v}(z)$ are the (misnamed) Lommel polynomials [2]

$$R_{n,v}(z) = \frac{\Gamma(n+v)}{\Gamma(v)} \left(\frac{2}{z} \right)^n {}_2F_3 \left(\left[-\frac{n}{2}, \frac{1}{2} - \frac{n}{2} \right]; [v, -n, 1-v-n]; -z^2 \right) \quad (2)$$

where ${}_2F_3$ is a generalized hypergeometric function. The “Lommel polynomials” are actually rational functions of z , not polynomial; but rather “polynomial in $\frac{1}{z}$ ”.

Conjecture 2. The series

$$\begin{aligned} J_0(t) &= \sum_{k=0}^{\infty} \lambda_k \psi_k(t) \\ &= \sum_{k=0}^{\infty} \sqrt{\frac{4n+1}{\pi}} \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} (-1)^n \sqrt{\frac{4n+1}{\pi}} j_{2n}(t) \\ &= \sum_{k=0}^{\infty} \frac{4n+1}{\pi} \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} (-1)^n j_{2n}(t) \end{aligned} \quad (3)$$

converges uniformly for all complex t except the origin where it has a regular singular point where $\lim_{t \rightarrow 0} J_0(t) = 1$.

Conjecture 3. *The eigenfunctions of the stationary integral covariance operator*

$$[T\psi_n](x) = \int_0^\infty J_0(x-y) \psi_n(x) dx = \lambda_n \psi_n(x) \quad (4)$$

are given by

$$\psi_n(y) = (-1)^n \sqrt{\frac{4n+1}{\pi}} j_{2n}(y) \quad (5)$$

and the eigenvalues are given by

$$\begin{aligned} \lambda_n &= \int_{-\infty}^\infty J_0(x) \psi_n(x) dx \\ &= \sqrt{\frac{4n+1}{\pi}} \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} \\ &= \sqrt{\frac{4n+1}{\pi}} (n+1)_{-\frac{1}{2}}^2 \end{aligned} \quad (6)$$

where $(n+1)_{-\frac{1}{2}}^2$ is the Pochhammer symbol(ascending/rising factorial).

Definition 4. *The spectral density of a stationary process is the Fourier tranform of the covariance kernel due to Wiener-Khinchine theorem.*

Definition 5. *Let $S_n(x)$ be the orthogonal polynomials whose orthogonality measure is equal to the spectral density of the process. These polynomials shall be called the spectral polynomials corresponding to the process.*

Example 6. Let the kernel function be given by $K(t, s) = J_0(t - s)$ then identify the orthogonal polynomial sequence associated with the spectral density

$$S(\omega) = \int_{-\infty}^\infty J_0(x) e^{ix\omega} dx = \begin{cases} \frac{2}{\sqrt{1-\omega^2}} & \omega \in (-1, 1) \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

as being twice the orthogonality measure of the Type-I Chebyshev polynomials $T_n(x)$ so that the orthogonal polynomial sequence is identified as

$$S_n(x) = \sqrt{2} T_n(x) \quad (8)$$

so that

$$\int_{-1}^1 S_n(\omega) S_m(\omega) S(\omega) d\omega = \begin{cases} 0 & n \neq m \\ 2\pi & n = m = 0 \\ \pi & n = m \neq 0 \end{cases} \quad (9)$$

Remark 7. If the spectral density does not equal the orthogonality measure of a known set of orthogonal polynomials then such a set can always be generated by applying the Gram-Schmidt process to the monomials so that they are transformed into a set that is orthogonal with respect any given spectral density (of a stationary process).

Definition 8. The sequence $\hat{S}_n(y)$ of Fourier transforms of the spectral polynomials $S_n(x)$ is given by

$$\hat{S}_n(y) = \int_{-1}^1 S_n(x) e^{ixy} dx \quad (10)$$

Example 9. The Fourier transforms of the Chebyshev polynomials are just the usual infinite Fourier transforms with the integration restricted to the range $-1 \dots 1$ since $T_n(x) = 0 \forall x \notin [-1, 1]$. Equivalently, the spectral density function can be extended to take the value 0 outside the interval $[-1, 1]$. The derivation of

$$\begin{aligned} \hat{T}_n(y) &= \int_{-\infty}^{\infty} e^{-ixy} T_n(x) dy = \int_{-1}^1 e^{-ixy} T_n(x) dx \\ &= \int_{-\infty}^{\infty} e^{-ixy} {}_2F_1\left(n, -n \middle| \frac{1}{2} - \frac{x}{2}\right) dx \\ &= \frac{i}{y} (e^{-iy} F_n^+(y) - e^{i(\pi n + y)} F_n^-(y)) \end{aligned} \quad (11)$$

where

$$F_n^{\pm}(y) = {}_3F_1\left(1, n, -n \middle| \frac{\pm iy}{2}\right) \quad (12)$$

can be found in [1].

Definition 10. Let $Y_n(y)$ be the normalized spectral polynomials $S_n(x)$

Example 11. When $K = J_0$ the spectral polynomials are given by

$$S_n(x) = \sqrt{2}T_n(x) \quad (13)$$

so that

$$\begin{aligned} Y_n(y) &= \frac{\hat{T}_n(y)}{|\hat{T}_n|} \\ &= \frac{i}{y} \left(\frac{e^{-iy}F_n^+(y) - e^{i(\pi n+y)}F_n^-(y)}{\sqrt{\frac{4(-1)^n\pi - (2n^2-1)}{4n^2-1}}} \right) \end{aligned} \quad (14)$$

where the L^2 norm of $\hat{T}_n(y)$ is given by

$$\begin{aligned} |\hat{T}_n| &= \sqrt{\int_{-\infty}^{\infty} \hat{T}_n(y)^2 dy} \\ &= \sqrt{\frac{4(-1)^n\pi - (2n^2-1)}{4n^2-1}} \end{aligned} \quad (15)$$

Conjecture 12. *The eigenfunctions of the integral covariance operator (4) are given by the orthogonal complement of the normalized Fourier transforms $Y_n(y)$ of the spectral polynomials (via the Gram-Schmidt process)*

$$\psi_n(y) = Y_n^\perp(y) = Y_n(y) - \sum_{m=1}^{n-1} \frac{\langle Y_n(y), Y_m^\perp(y) \rangle}{\langle Y_m^\perp(y), Y_m^\perp(y) \rangle} Y_m^\perp(y) \quad (16)$$

can be equivalently expressed as

$$\begin{aligned} \psi_n(y) &= (-1)^n \sqrt{\frac{4n+1}{\pi}} j_{2n}(y) \\ &= (-1)^n \sqrt{\frac{4n+1}{\pi}} \sqrt{\frac{\pi}{2y}} J_{2n+\frac{1}{2}}(y) \\ &= (-1)^n \sqrt{\frac{4n+1}{2y}} J_{2n+\frac{1}{2}}(y) \\ &= \sqrt{\frac{4n+1}{4\pi}} \int_{-1}^1 P_{2n}(x) e^{ixy} dx \end{aligned} \quad (17)$$

Remark 13. Since T is compact due to its self-adjointness and convergence of the eigenvalues to 0 it converges uniformly since compactness implies uniform convergence of the

eigenfunctions. TODO: cite/theorems from [3, 3. Reproducing Kernel Hilbert Space of a Gaussian Process]

Theorem 14. *The Bessel function identity*

$$J_0(x - y) = \sum_{k=-\infty}^{\infty} J_k(x) J_k(y) \quad (18)$$

can be used to expression Equation (4) as a series

$$\begin{aligned} [Tf](y) &= \int_0^{\infty} J_0(x - y) f(x) dx \\ &= \int_0^{\infty} \sum_{k=-\infty}^{\infty} J_k(x) J_k(y) f(x) dx \\ &= \sum_{k=-\infty}^{\infty} J_k(y) \int_0^{\infty} J_k(x) f(x) dx \end{aligned}$$

when f is absolutely integrable.

Proof. ...

□

Example 15. 1 Simplifying The Convolution

Apply the addition theorem

$$J_0(x - y) = \sum_{k=-\infty}^{\infty} J_k(x) J_k(y) \quad (19)$$

to the integral covariance operator from Conjecture 14

$$\begin{aligned} [T\psi_n](x) &= \int_0^{\infty} J_0(x - y) \psi_n(y) dy \\ &= \int_0^{\infty} \sum_{k=-\infty}^{\infty} J_k(x) J_k(y) \psi_n(y) dy \\ &= \sum_{k=-\infty}^{\infty} J_k(x) \int_0^{\infty} J_k(y) \psi_n(y) dy \\ &= \sum_{k=-\infty}^{\infty} J_k(x) \int_0^{\infty} J_k(y) (-1)^n \sqrt{\frac{4n+1}{\pi}} j_{2n}(y) dy \end{aligned} \quad (20)$$

Where $\psi_n(y)$ is:

$$\psi_n(y) = (-1)^n \sqrt{\frac{4n+1}{\pi}} j_{2n}(y) = (-1)^n \sqrt{\frac{4n+1}{\pi}} \sqrt{\frac{\pi}{2y}} J_{2n+\frac{1}{2}}(y) \quad (21)$$

Substituting

$$\begin{aligned} \int_0^\infty J_k(y) \psi_n(y) dy &= \int_0^\infty J_k(y) (-1)^n \sqrt{\frac{4n+1}{\pi}} j_{2n}(y) dy \\ &= \frac{\sqrt{4n+1} (-1)^n \sqrt{\pi} \Gamma\left(\frac{k}{2} + n + \frac{1}{2}\right)}{2 \Gamma\left(-n + \frac{k}{2} + \frac{1}{2}\right) \Gamma\left(\frac{k}{2} + n + 1\right) \Gamma\left(n + 1 - \frac{k}{2}\right)} \end{aligned} \quad (22)$$

Now, putting it all back into the expansion for $[T\psi_n](x)$:

$$[T\psi_n](x) = \sum_{k=-\infty}^{\infty} J_k(x) \frac{\sqrt{4n+1} (-1)^n \sqrt{\pi} \Gamma\left(\frac{k}{2} + n + \frac{1}{2}\right)}{2 \Gamma\left(-n + \frac{k}{2} + \frac{1}{2}\right) \Gamma\left(\frac{k}{2} + n + 1\right) \Gamma\left(n + 1 - \frac{k}{2}\right)} \quad (23)$$

Conjecture 16.

$$\sum_{k=0}^{\infty} \psi_k(x)^2 = \frac{1}{\pi} \quad (24)$$

Bibliography

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