# Measure-Preserving Bijective Time Changes of Stationary Gaussian Processes Generates A Subclass of Oscillatory Processes

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**Abstract**

This article establishes that Gaussian processes obtained through measure-preserving bijective unitary time transformations of stationary processes constitute a subclass of oscillatory processes in the sense of Priestley[1]. The transformation $Z(t) = \sqrt{\dot{\theta}(t)}X(\theta(t))$, where $X(t)$ is a realization of stationary Gaussian process and $\theta$ is a strictly increasing $C^1$ differentiable monotonic function, yields an oscillatory process with evolutionary power spectrum $dF_t(\omega) = \dot{\theta}(t)d\mu(\omega)$. An explicit unitary transformation between the input stationary process and the transformed oscillatory process is established, preserving the $L^2$-norm and providing a complete spectral characterization.

## Contents

# 1 Scaling Functions

**Definition 1 (Scaling Functions)** *Let $\mathcal{F}$ denote the set of functions $\theta : \mathbb{R} \to \mathbb{R}$ satisfying*

1. *$\theta$ is absolutely continuous with*
$$\dot{\theta}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\theta(t) \geq 0 \tag{1}$$
*almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero*

2. *$\theta$ is strictly increasing and bijective.*

**Remark 1** The conditions in Definition 1 ensure that $\theta^{-1}$ exists and is absolutely continuous. By the inverse function theorem for absolutely continuous functions,
$$\frac{\mathrm{d}}{\mathrm{d}s}(\theta^{-1})(s) = \frac{1}{\dot{\theta}(\theta^{-1}(s))} \tag{2}$$
for almost all $s$ in the range of $\theta$. The condition that $\dot{\theta}(t) = 0$ only on sets of measure zero ensures that $\frac{1}{\dot{\theta}(\theta^{-1}(s))}$ is well-defined almost everywhere.

# 2 Oscillatory Processes

**Definition 2 (Oscillatory Process)** *A complex-valued, second-order process $\{X(t)\}_{t\in\mathbb{R}}$ is called* oscillatory *if there exist*

1. *a family of oscillatory basis functions $\{\phi_t(\omega)\}_{t\in\mathbb{R}}$ with*
$$\phi_t(\omega) = A_t(\omega)e^{i\omega t} \tag{3}$$
*and a given gain function*
$$A_t(\cdot) \in L^2(\mu) \tag{4}$$

2. *and a complex orthogonal random measure $\Phi(\omega)$ with*
$$E|d\Phi(\omega)|^2 = d\mu(\omega) = S(\omega) \tag{5}$$

*such that*
$$\begin{aligned} Z(t) &= \int_{-\infty}^{\infty} \phi_t(\omega)\, d\Phi(\omega) \\ &= \int_{-\infty}^{\infty} A_t(\omega)e^{i\omega t} d\Phi(\omega) \end{aligned} \tag{6}$$
*All stationary processes are oscillatory with $A_t(\omega) = 1$*

# 3 Stationary Reference Process

Let $\{X(t)\}_{t\in\mathbb{R}}$ be a stationary Gaussian process with continuous spectral representation
$$X(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, d\Phi(\omega) \tag{7}$$
where $\Phi(\omega)$ is an orthogonal-increment process with spectral density
$$E|d\Phi(\omega)|^2 = d\mu(\omega) = S(\omega) =< \text{fourier transform of } K_X > \tag{8}$$
and $\mu$ is a finite measure on $\mathbb{R}$.

# 4 Time-Changed Process

## 4.1 Definition and Unitary Operator

**Definition 3 (Unitary Time-Change Operator)** *For $\theta \in \mathcal{F}$, define the operator $M_\theta : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by*

$$(M_\theta f)(t) = \sqrt{\dot{\theta}(t)}\, f(\theta(t)) \tag{9}$$

**Definition 4 (Unitarily Time-Changed Stationary Process)** *For $\theta \in \mathcal{F}$, apply the unitary time change operator $M_\theta$ from Definition-3 to a realization of a stationary process $X(t)$ from the ensemble $\{X(t)\}$ to define a realization of the unitarily time-changed process*

$$Z(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) \forall t \in \mathbb{R} \tag{10}$$

**Definition 5 (Inverse Unitary Time-Change Operator)** *The inverse operator $M_\theta^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ corresponding to the unitary time-change operator $(M_\theta f)(t)$ defined in Equation-9 is given by*

$$(M_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{11}$$

**Lemma 1 (Well-Definedness of Inverse Operator)** *The operator $M_\theta^{-1}$ in Definition 5 is well-defined $\forall \theta \in \mathcal{F}$.*

**Proof** Since $\dot{\theta}(t) = 0$ only on sets of measure zero by Definition 1, and $\theta^{-1}$ maps sets of measure zero to sets of measure zero (as it preserves absolute continuity), the denominator $\sqrt{\dot{\theta}(\theta^{-1}(s))}$ is positive almost everywhere. The expression in equation (11) is therefore well-defined almost everywhere, which is sufficient for defining an element of $L^2(\mathbb{R})$. □

**Theorem 1 (Unitarity of Transformation Operator)** *The operator $M_\theta$ defined in equation* (9) *is unitary, i.e.,*

$$\int_{\mathbb{R}} |(M_\theta f)(t)|^2\, dt = \int_{\mathbb{R}} |f(s)|^2\, ds \forall f \in L^2(\mathbb{R}) \tag{12}$$

**Proof** Let $f \in L^2(\mathbb{R})$. The $L^2$-norm of $M_\theta f$ is computed as follows:

$$\int_{\mathbb{R}} |(M_\theta f)(t)|^2\, dt = \int_{\mathbb{R}} \left| \sqrt{\dot{\theta}(t)}\, f(\theta(t)) \right|^2\, dt \tag{13}$$

$$= \int_{\mathbb{R}} \dot{\theta}(t)\, |f(\theta(t))|^2\, dt \tag{14}$$

Apply the change of variables $s = \theta(t)$. Since $\theta$ is absolutely continuous and strictly increasing, its Jacobian is given by

$$ds = \dot{\theta}(t)\, dt \tag{15}$$

almost everywhere. As $t$ ranges over $\mathbb{R}$, $s = \theta(t)$ ranges over $\mathbb{R}$ due to the bijectivity of $\theta$. Therefore:

$$\int_{\mathbb{R}} \dot{\theta}(t)\, |f(\theta(t))|^2 \, dt = \int_{\mathbb{R}} |f(s)|^2 \, ds \tag{16}$$

This establishes equation (12). To complete the proof of unitarity, it remains to show that $M_\theta^{-1}$ is indeed the inverse of $M_\theta$. For any $f \in L^2(\mathbb{R})$:

$$\begin{aligned}
(M_\theta^{-1} M_\theta f)(s) &= (M_\theta^{-1}) \left[ \sqrt{\dot{\theta}(\cdot)}\, f(\theta(\cdot)) \right] (s) &(17)\\
&= \frac{\sqrt{\dot{\theta}(\theta^{-1}(s))}\, f(\theta(\theta^{-1}(s)))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} &(18)\\
&= f(s) &(19)
\end{aligned}$$

where the last equality uses $\theta(\theta^{-1}(s)) = s$. Similarly, for any $g \in L^2(\mathbb{R})$:

$$\begin{aligned}
(M_\theta M_\theta^{-1} g)(t) &= \sqrt{\dot{\theta}(t)}\, (M_\theta^{-1} g)(\theta(t)) &(20)\\
&= \sqrt{\dot{\theta}(t)}\, \frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} &(21)\\
&= \sqrt{\dot{\theta}(t)}\, \frac{g(t)}{\sqrt{\dot{\theta}(t)}} &(22)\\
&= g(t) &(23)
\end{aligned}$$

Therefore

$$M_\theta M_\theta^{-1} = M_\theta^{-1} M_\theta = I \tag{24}$$

proving that $M_\theta$ is unitary. □

**Corollary 1 (Measure Preservation)** *The transformation $M_\theta$ preserves the $L^2$-measure in the sense that for any measurable set $A \subseteq \mathbb{R}$*

$$\int_A |(M_\theta f)(t)|^2 \, dt = \int_{\theta(A)} |f(s)|^2 \, ds \tag{25}$$

**Proof** The proof follows the same change of variables argument as in Theorem 1, applied to the characteristic function of the set $A$. □

## 4.2 $L^2$-Norm Preservation

**Theorem 2 (Measure Preservation)** *The transformation defined in equation* (10) *preserves the $L^2$-norm in the sense that*

$$\int_I \operatorname{var}(Z(t))\,dt = \int_{\theta(I)} \operatorname{var}(X(s))\,ds \tag{26}$$

*for any measurable set $I \subseteq \mathbb{R}$.*

**Proof** Using the change of variables $s = \theta(t)$ with $ds = \dot{\theta}(t)\,dt$:

$$\begin{aligned}\int_I \operatorname{var}(X(t))\,dt &= \int_I \operatorname{var}\left(\sqrt{\dot{\theta}(t)}\,X(\theta(t))\right)\,dt &(27)\\ &= \int_I \dot{\theta}(t)\operatorname{var}(X(\theta(t)))\,dt &(28)\\ &= \int_{\theta(I)} \operatorname{var}(X(s))\,ds &(29)\end{aligned}$$

□

## 4.3 Oscillatory Representation

**Theorem 3 (Oscillatory Form)** *The process $\{Z(t)\}$ defined in equation* (10) *is oscillatory with oscillatory functions*

$$\phi_t(\omega) = A_t(\omega)e^{i\omega t} = \sqrt{\dot{\theta}(t)}\,e^{i\omega\theta(t)} \tag{30}$$

*and gain functions*

$$A_t(\omega) = \sqrt{\dot{\theta}(t)}\,e^{i\omega(\theta(t)-t)} \tag{31}$$

**Proof** From the spectral representation (7) of the stationary process $X(t)$:

$$\begin{aligned}X(t) &= \sqrt{\dot{\theta}(t)}X(\theta(t)) &(32)\\ &= \sqrt{\dot{\theta}(t)}\int_{-\infty}^{\infty} e^{i\omega\theta(t)}\,d\Phi(\omega) &(33)\\ &= \int_{-\infty}^{\infty} \sqrt{\dot{\theta}(t)}\,e^{i\omega\theta(t)}\,d\phi(\omega) &(34)\\ &= \int_{-\infty}^{\infty} \phi_t(\omega)\,d\Phi(\omega) &(35)\end{aligned}$$

where

$$\phi_t(\omega) = \sqrt{\dot{\theta}(t)}\,e^{i\omega\theta(t)} \tag{36}$$

To verify this is an oscillatory representation according to Definition 2, express $\phi_t(\omega)$ in the form of a function of the time-dependent gain $A_t(\lambda)$ as required

$$\begin{aligned}
\phi_t(\omega) &= A_t(\omega)e^{i\omega t} \\
&= \sqrt{\dot{\theta}(t)}\, e^{i\omega(\theta(t)-t)}\, e^{i\omega t} \\
&= \sqrt{\dot{\theta}(t)} e^{i\omega(\theta(t)-t+t)} \\
&= \sqrt{\dot{\theta}(t)}\, e^{i\omega\theta(t)}
\end{aligned} \tag{37}$$

where

$$A_t(\omega) = \sqrt{\dot{\theta}(t)}\, e^{i\omega(\theta(t)-t)} \tag{38}$$

.

Since $\dot{\theta}(t) \geq 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of measure zero, the function $A_t(\omega)$ is well-defined almost everywhere. Moreover, $A_t(\cdot) \in L^2(\mu)$ for each $t$ since:

$$\int_{-\infty}^{\infty} |A_t(\omega)|^2 \, d\mu(\omega) = \int_{-\infty}^{\infty} \dot{\theta}(t) \, d\mu(\omega) \tag{39}$$

$$= \dot{\theta}(t) \int_{-\infty}^{\infty} d\mu(\omega)$$

$$= \dot{\theta}(t)\mu(\mathbb{R}) < \infty \tag{40}$$

where the finiteness follows from $\mu$ being a finite measure and $\dot{\theta}(t)$ being finite almost everywhere.

□

## 4.4 Envelope and Evolutionary Spectrum

**Corollary 2 (Evolutionary Spectrum)** *The evolutionary power spectrum is*

$$\begin{aligned}
dF_t(\omega) &= |A_t(\omega)|^2 \, d\mu(\omega) \\
&= \dot{\theta}(t) \, d\mu(\omega)
\end{aligned} \tag{41}$$

**Proof** By Definition 2 and the envelope from Equation 4, the evolutionary power spectrum is:

$$dF_t(\omega) = |A_t(\omega)|^2 \, d\mu(\omega) \tag{42}$$

$$= \left| \sqrt{\dot{\theta}(t)}\, e^{i\omega(\theta(t)-t)} \right|^2 \, d\mu(\omega) \tag{43}$$

$$= \dot{\theta}(t)\, |e^{i\omega(\theta(t)-t)}|^2 \, d\mu(\omega) \tag{44}$$

$$= \dot{\theta}(t) \, d\mu(\omega) \tag{45}$$

since

$$|e^{i\alpha}| = 1 \forall \alpha \in \mathbb{R} \tag{46}$$

□

## 5 Operator Conjugation

**Theorem 4 (Operator Conjugation)** *Let $T_K$ be the integral covariance operator defined by*

$$(T_K f)(t) \quad = \textstyle\int_{-\infty}^{\infty} K(|t-s|)f(s)\,ds \tag{47}$$

*where $K(h)$ is the stationary kernel*

$$K(h) = \int_{-\infty}^{\infty} S(\lambda)e^{i\lambda h}\mathrm{d}\lambda \tag{48}$$

*, and let $T_{K_\theta}$ be the integral covariance operator defined by*

$$\begin{aligned}(T_{K_\theta} f)(t) \quad &= \textstyle\int_{-\infty}^{\infty} K_\theta(s,t)f(s)\,ds \\ &= \textstyle\int_{-\infty}^{\infty} K(|\theta(t)-\theta(s)|)\sqrt{\dot\theta(t)\dot\theta(s)}f(s)\,ds\end{aligned} \tag{49}$$

*for the unitarily time-changed kernel*

$$K_\theta(s,t) = K(|\theta(t)-\theta(s)|)\sqrt{\dot\theta(t)\dot\theta(s)} \tag{50}$$

*. Then*

$$T_{K_\theta} = M_\theta T_K M_\theta^{-1} \tag{51}$$

**Proof** For any $g \in L^2(\mathbb{R})$, compute $(M_\theta T_K M_\theta^{-1} g)(t)$:

$$(M_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\theta'(\theta^{-1}(s))}}, \tag{52}$$

$$(T_K M_\theta^{-1} g)(t) = \int_{-\infty}^{\infty} K(|t-s|)\frac{g(\theta^{-1}(s))}{\sqrt{\dot\theta(\theta^{-1}(s))}}ds. \tag{53}$$

Apply the change of variables $u = \theta^{-1}(s)$, so $s = \theta(u)$ and $ds = \dot\theta(u)du$:

$$(T_K M_\theta^{-1} g)(t) = \int_{-\infty}^{\infty} K(|t-\theta(u)|)\frac{g(u)}{\sqrt{\dot\theta(u)}}\dot\theta(u)du \tag{54}$$

$$= \int_{-\infty}^{\infty} K(|t-\theta(u)|)g(u)\sqrt{\dot\theta(u)}du. \tag{55}$$

Now apply $M_\theta$:

$$(M_\theta T_K M_\theta^{-1} g)(t) = \sqrt{\dot\theta(t)}(T_K M_\theta^{-1} g)(\theta(t)) \tag{56}$$

$$= \sqrt{\dot\theta(t)}\int_{-\infty}^{\infty} K(|\theta(t)-\theta(u)|)g(u)\sqrt{\dot\theta(u)}du. \tag{57}$$

Apply the change of variables $s = \theta(u)$ in the reverse direction:

$$(M_\theta T_K M_\theta^{-1} g)(t) = \int_{-\infty}^{\infty} K(|\theta(t) - \theta(s)|) g(s) ds \tag{58}$$

$$= (T_{K_\theta} g)(t) \tag{59}$$

This establishes the conjugation relation (51). □

# 6 Expected Zero Count

**Theorem 5 (Expected Zero-Counting Function)** *Let $\theta \in \mathcal{F}$ and let*

$$K(\tau) = \operatorname{cov}(X(t), X(\tau)) \tag{60}$$

*be twice differentiable at $\tau = 0$. The expected number of zeros of the process $X_t$ in $[a, b]$ is*

$$\mathbb{E}[N_{[a,b]}] = \sqrt{-\ddot{K}(0)}\,(\theta(b) - \theta(a)) \tag{61}$$

**Proof** The covariance function of the time-changed process is

$$K_\theta(s,t) = \operatorname{cov}(X_s, X_t) = \sqrt{\dot{\theta}(s)\dot{\theta}(t)}\, K(|\theta(t) - \theta(s)|) \tag{62}$$

For the zero-crossing analysis, consider the normalized process. By the Kac-Rice formula:

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{-\lim_{s\to t} \frac{\partial^2}{\partial s \partial t} K_\theta(s,t)}\, dt \tag{63}$$

Computing the mixed partial derivative:

$$\frac{\partial}{\partial t} K_\theta(s,t) = \frac{1}{2} \frac{\ddot{\theta}(t)}{\sqrt{\dot{\theta}(t)}} \sqrt{\theta'(s)} K(|\theta(t) - \theta(s)|) \tag{64}$$

$$+ \sqrt{\dot{\theta}(s)\dot{\theta}(t)} \dot{K}(|\theta(t) - \theta(s)|) \operatorname{sgn}(\theta(t) - \theta(s)) \dot{\theta}(t). \tag{65}$$

Taking the limit as $s \to t$ and using the fact that $\dot{K}(0) = 0$ for stationary processes:

$$\lim_{s\to t} \frac{\partial^2}{\partial s \partial t} K_\theta(s,t) = \dot{\theta}(s)\dot{\theta}(t)\ddot{K}(0) \tag{66}$$

$$= \dot{\theta}(t)^2 \ddot{K}(0) \tag{67}$$

Substituting into the Kac-Rice formula:

$$
\begin{aligned}
\mathbb{E}[N_{[a,b]}] &= \int_a^b \sqrt{-\dot{\theta}(t)^2 \ddot{K}(0)}\, dt && (68) \\
&= \sqrt{-\ddot{K}(0)} \int_a^b \dot{\theta}(t)\, dt && (69) \\
&= \sqrt{-\ddot{K}(0)}\, (\theta(b) - \theta(a)) && (70)
\end{aligned}
$$

Here the second equality uses $\dot{\theta}(t) \geq 0$ almost everywhere. □

# 7 Conclusion

This analysis establishes that Gaussian processes generated by measure-preserving bijective time changes of stationary processes form a well-defined subclass of oscillatory processes. The key contributions include:

1. The rigorous construction of the unitary operator $M_\theta$ and its inverse, with proper treatment of the case where $\dot{\theta}(t) = 0$ on sets of measure zero.
2. The explicit oscillatory representation with envelope function $A_t(\omega) = \sqrt{\dot{\theta}(t)} e^{i\omega(\theta(t)-t)}$.
3. The evolutionary power spectrum formula $dF_t(\omega) = \dot{\theta}(t) d\mu(\omega)$.
4. The operator conjugation relationship $T_{K_\theta} = M_\theta T_K M_\theta^{-1}$.
5. A closed-form expression for the expected zero count in terms of the range of the time transformation.

# References


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