1 Convolution and Fourier Transform Discussion

The convolution of two functions $f(x)$ and $g(x)$ over the positive domain can be expressed using the Fourier transform:

$$\begin{aligned}(f*g)(x) &= \int_0^\infty f(x-t)\,g(t)\,dt \\ &= \int_{-\infty}^\infty f(x-t)\,g(t)\,H(t)\,dt \\ &= \mathcal{F}^{-1}\{\mathcal{F}\{f\}\cdot\mathcal{F}\{g\cdot H\}\} \\ &= \mathcal{F}^{-1}\{\mathcal{F}\{f\}\cdot\mathcal{F}\{g\}\cdot\mathcal{F}\{g\}\}\end{aligned} \tag{1}$$

Where:

- $*$ denotes convolution
- $\mathcal{F}$ denotes the Fourier transform
- $\mathcal{F}^{-1}$ denotes the inverse Fourier transform
- $\cdot$ denotes multiplication in the frequency domain
- $H(t)$ is the Heaviside step function, defined as:

$$H(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases} \tag{2}$$

This formulation shows the equivalence between the convolution over the positive domain and the convolution over the entire real line with the Heaviside step function.

The Convolution Theorem states that the convolution of two functions in the time (or spatial) domain is equivalent to the multiplication of their Fourier transforms in the frequency domain, followed by an inverse Fourier transform.

The Fourier transform relationship for this convolution is:

$$\mathcal{F}\{f*(g\cdot H)\} = \mathcal{F}\{f\}\cdot\mathcal{F}\{g\cdot H\} \tag{3}$$

Additionally, we have:

$$(f*(g\cdot H))(x) = ((g\cdot H)*f)(x) \tag{4}$$

And for the Fourier transform of a convolution:

$$\mathcal{F}\{f*(g\cdot H)\} = \mathcal{F}\{f\}\cdot\mathcal{F}\{g\cdot H\} \tag{5}$$

1.1 Translation-Invariant (Stationary) Gaussian Processes

For a stationary Gaussian process, we consider the integral covariance operator T with kernel K. The eigenfunctions $\phi_n(x)$ of this operator satisfy:

$$(T\phi_n)(y) = \int_0^\infty K(x,y)\phi_n(x)\mathrm{d}x = \lambda_n\,\phi_n(y) \tag{6}$$

where λ_n are the corresponding eigenvalues.

We can expand these eigenfunctions in terms of a uniformly convergent orthonormal basis $\{\psi_k(x)\}$ for $L^2(0,\infty)$:

$$\phi_n(x) = \sum_{k=0}^{\infty} c_{n,k}\,\psi_k(x) \tag{7}$$

The expansion coefficients $c_{n,k}$ can be expressed as:

$$c_{n,k} = \frac{\int_{-\infty}^{\infty}\psi_k(x)\,(T\,\phi_n)(x)\,d\,x}{\lambda_n} \tag{8}$$

1.2 Proof of the Expansion Coefficient Formula

Let's prove this formula by substitution and expansion:

1) Start with the eigenvalue equation:

$$(K\,\phi_n)(x) = \lambda_n\,\phi_n(x) \tag{9}$$

2) Multiply both sides by $\psi_k(x)$ and integrate over the entire domain:

$$\int_{-\infty}^{\infty}\psi_k(x)\,(K\,\phi_n)(x)\,d\,x = \lambda_n\int_{-\infty}^{\infty}\psi_k(x)\;\phi_n(x)\,d\,x \tag{10}$$

3) The right-hand side integral is the definition of $c_{n,k}$ due to the orthonormality of $\{\psi_k(x)\}$:

$$\int_{-\infty}^{\infty}\psi_k(x)\,(K\,\phi_n)(x)\,d\,x = \lambda_n\,c_{n,k} \tag{11}$$

4) Rearranging this equation gives us the formula for $c_{n,k}$:

$$c_{n,k} = \frac{1}{\lambda_n}\int_{-\infty}^{\infty}\psi_k(x)\,(K\,\phi_n)(x)\,d\,x \tag{12}$$

5) To verify, let's substitute the expansion of $\phi_n(x)$ into the eigenvalue equation:

$$K\left(\sum_{k=1}^{\infty} c_{n,k}\,\psi_k(x)\right) = \lambda_n \sum_{k=1}^{\infty} c_{n,k}\,\psi_k(x) \tag{13}$$

6) By linearity of K:

$$\sum_{k=1}^{\infty} c_{n,k}\,(K\,\psi_k)(x) = \lambda_n \sum_{k=1}^{\infty} c_{n,k}\,\psi_k(x) \tag{14}$$

7) Multiply both sides by $\psi_j(x)$ and integrate:

$$\sum_{k=1}^{\infty} c_{n,k} \int_{-\infty}^{\infty} \psi_j(x)\,(K\,\psi_k)(x)\,d\,x = \lambda_n\,c_{n,j} \tag{15}$$

8) The left-hand side integral is our formula for $c_{n,k}$ multiplied by λ_n:

$$\sum_{k=1}^{\infty} c_{n,k}\,(\lambda_n\,c_{j,k}) = \lambda_n\,c_{n,j} \tag{16}$$

9) This reduces to an identity, proving that our formula for $c_{n,k}$ satisfies the eigenvalue equation.

Thus, we have proven that the formula for $c_{n,k}$ is correct and consistent with the eigenvalue equation.