

Unitary Time Changes of Stationary Processes Yield Oscillatory Processes

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ , acting on functions that are locally square-integrable. Applying U_θ to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)}$, evolutionary spectrum $dF_t(\lambda) = \dot{\theta}(t) dF(\lambda)$, and expected zero-counting function $\mathbb{E}[N_{[0,T]}] = N_{\det}([0,T]) + \frac{[\theta(T) - \theta(0)]}{\pi} \sqrt{-\frac{\dot{K}(0)}{K(0)}}$, where $N_{\det}([0, T])$ counts deterministic zeros from critical points of the time-change. The sample paths of any non-degenerate second-order stationary process are locally square integrable, making the unitary time-change operator U_θ applicable to typical realizations. A zero-localization measure $d\mu(t) = \delta(Z(t))|\dot{Z}(t)| dt$ induces a Hilbert space $L^2(\mu)$ on the zero set of each oscillatory process realization $Z(t)$, and the multiplication operator $(L f)(t) = t f(t)$ has simple pure point spectrum equal to the zero crossing set of Z .

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1 Gaussian Processes

Unless otherwise stated, all processes considered will be real-valued.

Theorem 1. *Let $X(u)$ be a real-valued process:*

$$X(u) \in \mathbb{R} \quad \forall u \in \mathbb{R} \quad (1)$$

Then its (complex-valued) random orthogonal spectral measure satisfies

$$d\bar{\Phi}(\lambda) = d\Phi(-\lambda) \quad (2)$$

and the corresponding covariance spectral measure F is even:

$$F(-A) = F(A) \quad (3)$$

Proof. 1. The spectral representation for $X(u)$ is

$$X(u) = \int_{-\infty}^{\infty} e^{i\lambda u} d\Phi(\lambda) \quad (4)$$

Since $X(u)$ is real-valued for each u ,

$$\overline{X(u)} = X(u) \quad (5)$$

On the other hand,

$$\overline{X(u)} = \overline{\int_{-\infty}^{\infty} e^{i\lambda u} d\Phi(\lambda)} \quad (6)$$

$$= \int_{-\infty}^{\infty} \overline{e^{i\lambda u}} d\bar{\Phi}(\lambda) \quad (7)$$

$$= \int_{-\infty}^{\infty} e^{-i\lambda u} d\bar{\Phi}(\lambda) \quad (8)$$

By the substitution $\lambda \mapsto -\mu$,

$$\int_{-\infty}^{\infty} e^{-i\lambda u} d\bar{\Phi}(\lambda) = \int_{-\infty}^{\infty} e^{i\mu u} d\bar{\Phi}(-\mu) \quad (9)$$

So

$$X(u) = \int_{-\infty}^{\infty} e^{i\lambda u} d\Phi(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda u} d\bar{\Phi}(-\lambda) \quad (10)$$

By uniqueness of the spectral measure representation, it follows that

$$d\bar{\Phi}(\lambda) = d\Phi(-\lambda) \quad (11)$$

as (orthogonal) random measures.

2. The covariance function of X is

$$R(u) = \mathbb{E}(X(0)X(u)) = \int_{-\infty}^{\infty} e^{i\lambda u} dF(\lambda) \quad (12)$$

Since $X(u)$ is real-valued, $R(u)$ is real and $R(-u) = R(u)$. Thus,

$$R(-u) = \int_{-\infty}^{\infty} e^{-i\lambda u} dF(\lambda) = \int_{-\infty}^{\infty} e^{i\mu u} dF(-\mu) \quad (13)$$

Equating with $R(u)$,

$$\int_{-\infty}^{\infty} e^{i\lambda u} dF(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda u} dF(-\lambda) \quad (14)$$

for all u . By the uniqueness theorem for Fourier–Stieltjes transforms, this implies

$$dF(\lambda) = dF(-\lambda) \quad (15)$$

Thus for any Borel set A ,

$$F(-A) = F(A) \quad (16)$$

establishing the evenness property. \square

1.1 Definition

Definition 2. (Gaussian process) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T a non-empty index set. A family $\{X_t : t \in T\}$ of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Gaussian process if for every finite subset $\{t_1, \dots, t_n\} \subset T$ the random vector $(X_{t_1}, \dots, X_{t_n})$ is multivariate normal (possibly degenerate). Equivalently, every finite linear combination $\sum_{i=1}^n a_i X_{t_i}$ is either almost surely constant or Gaussian. The mean function is $m(t) := \mathbb{E}[X_t]$ and the covariance kernel is

$$K(s, t) = \text{Cov}(X_s, X_t) \quad (17)$$

For any finite $(t_i)_{i=1}^n \subset T$, the matrix $K_{ij} = K(t_i, t_j)$ is symmetric positive semidefinite, and a Gaussian process is completely determined in law by m and K .

1.2 Stationary processes

Definition 3. [Cramér spectral representation] [1] A zero-mean stationary process X with spectral measure F admits the sample path representation

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t} d\Phi(\lambda) \quad (18)$$

which has covariance

$$R_X(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)} dF(\lambda) \quad (19)$$

1.2.1 Fourier Transform Conventions

Definition 4. [Fourier transform conventions] The forward and inverse Fourier transforms on $L^2(\mathbb{R})$ are defined by

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(u) e^{-i\lambda u} du \quad (20)$$

and

$$f(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda u} d\lambda \quad (21)$$

where the factor $\frac{1}{2\pi}$ is incorporated into the definition of the inverse transform.

1.2.2 Sample Path Realizations

Definition 5. [Locally square-integrable functions] Define

$$L^2_{\text{loc}}(\mathbb{R}) := \left\{ f: \mathbb{R} \rightarrow \mathbb{C} : \int_K |f(t)|^2 dt < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \quad (22)$$

Remark 6. Every bounded measurable set in \mathbb{R} is contained in a compact set; hence $L^2_{\text{loc}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

Theorem 7. [Sample paths in $L^2_{\text{loc}}(\mathbb{R})$] Let $\{X(t)\}_{t \in \mathbb{R}}$ be a second-order stationary process with

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \quad (23)$$

Then almost every sample path lies in $L^2_{\text{loc}}(\mathbb{R})$.

Proof. Fix an arbitrary bounded interval $[a, b] \subset \mathbb{R}$ with $a < b$. Define the random variable

$$Y_{[a, b]} := \int_a^b X(t)^2 dt \quad (24)$$

By Fubini's theorem,

$$\mathbb{E}[Y_{[a, b]}] = \int_a^b \mathbb{E}[X(t)^2] dt = (b - a) \sigma^2 < \infty \quad (25)$$

By Markov's inequality, $\mathbb{P}(Y_{[a, b]} = \infty) = 0$. Thus $Y_{[a, b]} < \infty$ almost surely. Covering compacts by countably many dyadic intervals yields the result: for every compact $K \subset \mathbb{R}$, almost surely $\int_K X(t)^2 dt < \infty$. \square

2 Oscillatory Processes

Definition 8. [Oscillatory process] [2] Let F be a finite nonnegative Borel measure on \mathbb{R} . Let

$$A_t \in L^2(F) \quad \forall t \in \mathbb{R} \quad (26)$$

be the gain function and

$$\varphi_t(\lambda) = A_t(\lambda) e^{i\lambda t} \quad (27)$$

be the corresponding oscillatory function; then an oscillatory process is a stochastic process which can be represented as

$$\begin{aligned} Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \end{aligned} \quad (28)$$

where Φ is a complex orthogonal random measure with spectral measure F which satisfies the relation

$$d\mathbb{E}[\Phi(\lambda)\overline{\Phi(\mu)}] = \delta(\lambda - \mu) dF(\lambda) \quad (29)$$

and has the corresponding covariance kernel

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \int_{\mathbb{R}} A_t(\lambda) \overline{A_s(\lambda)} e^{i\lambda(t-s)} dF(\lambda) \\ &= \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\lambda)} dF(\lambda) \end{aligned} \quad (30)$$

2.1 Triad of Gain Representations

Theorem 9. [Triad of Gain Representations] Let $A_t(\omega)$ a gain function as in (26) then:

$$A_t(\omega) = \int_{-\infty}^{\infty} e^{i\lambda t} dH_{\omega}(\lambda) = \int_{-\infty}^{\infty} h_t(u) e^{i\omega u} du \quad (31)$$

$$h_t(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_t(\omega) e^{-i\omega u} d\omega \quad (32)$$

$$H_{\omega}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_t(\omega) e^{-i\lambda t} dt \quad (33)$$

Proof.

1. To show that the filter $h(t, u)$ is the inverse Fourier transform of the gain $A_t(\lambda)$

$$h_t(u) := \frac{1}{2\pi} \int_{-\infty}^{\infty} A_t(\omega) e^{-i\omega u} d\omega \quad (34)$$

Substitute the definition from (34) into the right-hand side of (32) to yield

$$\begin{aligned} \int_{-\infty}^{\infty} h_t(u) e^{i\omega u} du &= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} A_t(\nu) e^{-i\nu u} d\nu \right) e^{i\omega u} du \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} A_t(\nu) \left(\int_{-\infty}^{\infty} e^{i(\omega-\nu)u} du \right) d\nu \end{aligned} \quad (35)$$

The inner integral is the delta function

$$\int_{-\infty}^{\infty} e^{i(\omega-\nu)u} du = 2\pi \delta(\omega - \nu) \quad (36)$$

therefore,

$$\int_{-\infty}^{\infty} h_t(u) e^{i\omega u} du = \int_{-\infty}^{\infty} A_t(\nu) \delta(\omega - \nu) d\nu = A_t(\omega) \quad (37)$$

which establishes (32).

2. To show that the gain $A_t(\lambda)$ and gain distributions $H_\omega(\lambda)$ are Fourier transform pairs fix $\omega \in \mathbb{R}$. By (31), the function $t \mapsto A_t(\omega)$ is the Fourier transform (in t) of the tempered distribution H_ω :

$$A_t(\omega) = \int_{-\infty}^{\infty} e^{i\lambda t} dH_\omega(\lambda) = \mathcal{F}_\lambda[H_\omega](t) \quad (38)$$

where \mathcal{F}_λ denotes the Fourier transform with respect to λ . Consider the Fourier transform of $A_t(\omega)$ with respect to t :

$$\mathcal{F}_t[A_t(\omega)](\lambda) = \int_{-\infty}^{\infty} A_t(\omega) e^{-i\lambda t} dt \quad (39)$$

Substituting the representation of $A_t(\omega)$,

$$\begin{aligned} \int_{-\infty}^{\infty} A_t(\omega) e^{-i\lambda t} dt &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{i\lambda' t} dH_\omega(\lambda') \right) e^{-i\lambda t} dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{i(\lambda'-\lambda)t} dt \right) dH_\omega(\lambda') \\ &= \int_{-\infty}^{\infty} 2\pi \delta(\lambda' - \lambda) dH_\omega(\lambda') \\ &= 2\pi H_\omega(\lambda) \end{aligned} \quad (40)$$

Thus,

$$\int_{-\infty}^{\infty} A_t(\omega) e^{-i\lambda t} dt = 2\pi H_\omega(\lambda) \quad (41)$$

and rearranging gives

$$H_\omega(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_t(\omega) e^{-i\lambda t} dt \quad (42)$$

as an identity in $\mathcal{S}'(\mathbb{R})$ with respect to the λ -variable. This is precisely (31).

□

Theorem 10. (Duality of Gain Representations) *Let $A_t(\omega)$ be a complex-valued function on $\mathbb{R} \times \mathbb{R}$ satisfying:*

1. *For each fixed $\omega \in \mathbb{R}$, the map $t \mapsto A_t(\omega)$ defines a tempered distribution in the time variable and admits a Fourier representation in the sense that there exists a tempered distribution $H_\omega \in \mathcal{S}'(\mathbb{R})$ with*

$$A_t(\omega) = \langle H_\omega, e^{i\lambda t} \rangle \quad \text{for all } t \in \mathbb{R}.$$

2. *For each fixed $t \in \mathbb{R}$, the function $\omega \mapsto A_t(\omega)$ belongs to $L^2(\mathbb{R})$.*

Then there exist a family of tempered distributions $\{H_\omega\}_{\omega \in \mathbb{R}}$ on \mathbb{R} and a function $h_t(u)$ on $\mathbb{R} \times \mathbb{R}$ such that the following relationships hold simultaneously:

1. $A_t(\omega) = \langle H_\omega, e^{i\lambda t} \rangle$ for all $t \in \mathbb{R}$;
2. $A_t(\omega) = \int_{-\infty}^{\infty} h_t(u) e^{i\omega u} du$ for all $\omega \in \mathbb{R}$, where the integral is taken in the sense of L^2 -limits;
3. $h_t(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_t(\omega) e^{-i\omega u} d\omega$ for almost every $u \in \mathbb{R}$, with convergence in $L^2(\mathbb{R})$;
4. as an identity in $\mathcal{S}'(\mathbb{R})$ with respect to the variable λ ,

$$H_\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_t(\omega) e^{-i\lambda t} dt,$$

i.e.

$$\langle H_\omega, \varphi(\lambda) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_t(\omega) \left(\int_{-\infty}^{\infty} \varphi(\lambda) e^{-i\lambda t} d\lambda \right) dt \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}).$$

Proof. 1. **Proof of (1).** By hypothesis 1, for each fixed ω there exists a tempered distribution $H_\omega \in \mathcal{S}'(\mathbb{R})$ such that

$$A_t(\omega) = \langle H_\omega, e^{i\lambda t} \rangle \quad \text{for all } t \in \mathbb{R}.$$

This is precisely the content of (1).

2. **Proof of (3).** Fix $t \in \mathbb{R}$. Hypothesis 2 gives $A_t(\cdot) \in L^2(\mathbb{R})$. By Plancherel's theorem, there exists a unique $h_t \in L^2(\mathbb{R})$ such that

$$h_t(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_t(\omega) e^{-i\omega u} d\omega$$

holds as an L^2 -identity in u , with the integral understood as the L^2 -limit over symmetric compact intervals in ω . This is exactly (3).

3. **Proof of (2).** For each fixed t , $A_t(\cdot) \in L^2(\mathbb{R})$ and h_t is its inverse Fourier transform in the L^2 sense. Applying the forward Fourier transform in the u -variable to h_t , one obtains

$$A_t(\omega) = \int_{-\infty}^{\infty} h_t(u) e^{i\omega u} du$$

as an L^2 -identity in ω , again with the integral interpreted as the L^2 -limit over symmetric compact intervals. This proves (2).

4. **Proof of (4).** For each fixed ω , hypothesis 1 and (1) say that $A(\omega)$ is the Fourier transform (in t) of H_ω :

$$A_t(\omega) = \langle H_\omega, e^{i\lambda t} \rangle = \int_{-\infty}^{\infty} e^{i\lambda t} dH_\omega(\lambda)$$

in the distributional sense. Taking the Fourier transform in t of both sides and using the usual Fourier inversion formula in $\mathcal{S}'(\mathbb{R})$ yields

$$\int_{-\infty}^{\infty} A_t(\omega) e^{-i\lambda t} dt = 2\pi H_\omega$$

as an identity in $\mathcal{S}'(\mathbb{R})$ with respect to λ . Thus

$$H_\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_t(\omega) e^{-i\lambda t} dt$$

in $\mathcal{S}'(\mathbb{R})$, which is the statement of (4). □

Theorem 11. [Real-valuedness criterion for oscillatory processes] Let Z be an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = A_t(\lambda) e^{i\lambda t} \tag{43}$$

and spectral measure F . Then Z is real-valued if and only if

$$A_t(-\lambda) = \overline{A_t(\lambda)} \tag{44}$$

for F -almost every $\lambda \in \mathbb{R}$, equivalently

$$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \tag{45}$$

for F -almost every $\lambda \in \mathbb{R}$.

Proof. 1. Assume Z is real-valued. Then for all $t \in \mathbb{R}$,

$$Z(t) = \overline{Z(t)} \tag{46}$$

2. From the oscillatory representation (28),

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \tag{47}$$

3. Taking the complex conjugate of both sides of (47),

$$\overline{Z(t)} = \overline{\int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t} d\overline{\Phi(\lambda)} \quad (48)$$

4. For a real-valued process, the orthogonal random measure must satisfy the symmetry property from Theorem 1:

$$d\overline{\Phi(\lambda)} = d\Phi(-\lambda) \quad (49)$$

5. Substituting (49) into (48),

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t} d\Phi(-\lambda) \quad (50)$$

6. Apply the change of variables $\mu = -\lambda$, so $d\Phi(-\lambda) = d\Phi(\mu)$ and $e^{-i\lambda t} = e^{i\mu t}$:

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t} d\Phi(\mu) \quad (51)$$

7. By (46), the right sides of (47) and (51) must be equal:

$$\int_{\mathbb{R}} A_t(\mu) e^{i\mu t} d\Phi(\mu) = \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t} d\Phi(\mu) \quad (52)$$

8. Since the stochastic integral representation is unique in $L^2(F)$, the integrands must be equal F -almost everywhere:

$$A_t(\lambda) = \overline{A_t(-\lambda)} \quad \text{for } F\text{-a.e. } \lambda \quad (53)$$

9. This is equivalent to (44). From (43),

$$\varphi_t(-\lambda) = A_t(-\lambda) e^{-i\lambda t} \quad (54)$$

10. Using (44),

$$\begin{aligned} \varphi_t(-\lambda) &= \overline{A_t(\lambda)} e^{-i\lambda t} \\ &= \overline{A_t(\lambda)} e^{i\lambda t} \\ &= \overline{\varphi_t(\lambda)} \end{aligned} \quad (55)$$

establishing (45).

11. Conversely, assume (44) holds. Reversing the steps from (51) to (46) shows that $\overline{Z(t)} = Z(t)$ for all t , so Z is real-valued. \square

Theorem 12. [Existence of Oscillatory Processes] Let F be an absolutely continuous spectral measure and the gain function

$$A_t(\lambda) \in L^2(F) \quad \forall t \in \mathbb{R} \quad (56)$$

be measurable in both time and frequency; then the time-dependent spectral density is defined by

$$\begin{aligned} S_t(\lambda) &= \int_{\mathbb{R}} |A_t(\lambda)|^2 dF(\lambda) < \infty \\ &= \int_{\mathbb{R}} |A_t(\lambda)|^2 S(\lambda) d\lambda \end{aligned} \quad (57)$$

and there exists a complex orthogonal random measure Φ with spectral measure F such that for each sample path $\omega_0 \in \Omega$

$$Z(t, \omega_0) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda, \omega_0) \quad (58)$$

is well-defined in $L^2(\Omega)$ and has covariance R_Z as in (30).

Proof. 1. Define the space of simple functions on \mathbb{R} : for disjoint Borel sets $\{E_j\}_{j=1}^n$ with $F(E_j) < \infty$ and coefficients $\{c_j\}_{j=1}^n \subset \mathbb{C}$,

$$g(\lambda) = \sum_{j=1}^n c_j \mathbf{1}_{E_j}(\lambda) \quad (59)$$

2. For simple functions, define the stochastic integral

$$\int_{\mathbb{R}} g(\lambda) d\Phi(\lambda) := \sum_{j=1}^n c_j \Phi(E_j) \quad (60)$$

3. Compute the second moment:

$$\begin{aligned} \mathbb{E}\left[\left|\int_{\mathbb{R}} g(\lambda) d\Phi(\lambda)\right|^2\right] &= \mathbb{E}\left[\left|\sum_{j=1}^n c_j \Phi(E_j)\right|^2\right] \\ &= \mathbb{E}\left[\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \Phi(E_j) \overline{\Phi(E_k)}\right] \end{aligned} \quad (61)$$

4. By linearity of expectation,

$$\mathbb{E}\left[\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \Phi(E_j) \overline{\Phi(E_k)}\right] = \sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \mathbb{E}[\Phi(E_j) \overline{\Phi(E_k)}] \quad (62)$$

5. By the orthogonality relation (29), since $E_j \cap E_k = \emptyset$ for $j \neq k$,

$$\mathbb{E}[\Phi(E_j)\overline{\Phi(E_k)}] = \begin{cases} F(E_j) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \quad (63)$$

6. Substituting (63) into (62),

$$\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \mathbb{E}[\Phi(E_j)\overline{\Phi(E_k)}] = \sum_{j=1}^n |c_j|^2 F(E_j) \quad (64)$$

7. The right side of (64) equals

$$\sum_{j=1}^n |c_j|^2 F(E_j) = \int_{\mathbb{R}} |g(\lambda)|^2 dF(\lambda) \quad (65)$$

8. Therefore the isometry property holds for simple functions:

$$\mathbb{E}\left[\left|\int_{\mathbb{R}} g(\lambda) d\Phi(\lambda)\right|^2\right] = \int_{\mathbb{R}} |g(\lambda)|^2 dF(\lambda) \quad (66)$$

9. The space of simple functions is dense in $L^2(F)$. For any $h(\lambda) \in L^2(F)$ and $\epsilon > 0$, there exists a simple function $g(\lambda)$ such that

$$\int_{\mathbb{R}} |h(\lambda) - g(\lambda)|^2 dF(\lambda) < \epsilon \quad (67)$$

10. By the isometry (66) and completeness of $L^2(\Omega)$, the integral extends uniquely by continuity to all $h(\lambda) \in L^2(F)$.

11. Since $A_t \in L^2(F)$ by assumption (56), and $|e^{i\lambda t}| = 1$,

$$\int_{\mathbb{R}} |\varphi_t(\lambda)|^2 dF(\lambda) = \int_{\mathbb{R}} |A_t(\lambda)|^2 dF(\lambda) < \infty \quad (68)$$

so $\varphi_t \in L^2(F)$.

12. Therefore

$$\begin{aligned} Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \end{aligned} \quad (69)$$

is well-defined in $L^2(\Omega)$.

13. To compute the covariance, use the sesquilinearity of the stochastic integral:

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \overline{\int_{\mathbb{R}} \varphi_s(\mu) d\Phi(\mu)}\right] \end{aligned} \quad (70)$$

14. By Fubini's theorem for stochastic integrals,

$$\mathbb{E} \left[\int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \overline{\int_{\mathbb{R}} \varphi_s(\mu) d\Phi(\mu)} \right] = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\mu)} \mathbb{E} [d\Phi(\lambda) \overline{d\Phi(\mu)}] \quad (71)$$

15. Using the orthogonality relation (29),

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\mu)} \delta(\lambda - \mu) dF(\lambda) dF(\mu) = \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\lambda)} dF(\lambda) \quad (72)$$

16. Substituting the definition (27),

$$R_Z(t, s) = \int_{\mathbb{R}} A_t(\lambda) \overline{A_s(\lambda)} e^{i\lambda(t-s)} dF(\lambda) \quad (73)$$

as claimed in (30). \square

Definition 13. [*Forward impulse response*] For an oscillatory process $Z(t) = \int \varphi_t(\lambda) d\Phi(\lambda)$ with oscillatory function $\varphi_t(\lambda)$, define the forward impulse response function

$$h(t, u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_t(\lambda) e^{-i\lambda u} d\lambda \quad (74)$$

That is, $h(t, \cdot)$ is the inverse Fourier transform (in the variable u) of the oscillatory function $\varphi_t(\cdot)$ at time t , with the $\frac{1}{2\pi}$ factor as in Definition 4.

Theorem 14. [*Filter representation via impulse response*] Let X be a zero-mean stationary process with Cramér representation $X(u) = \int e^{i\lambda u} d\Phi(\lambda)$ and spectral measure F , and let Z be an oscillatory process with oscillatory function $\varphi_t(\lambda)$ and the same orthogonal random measure Φ . Then

$$Z(t) = \int_{-\infty}^{\infty} h(t, u) X(u) du \quad (75)$$

where $h(t, u)$ is the forward impulse response of Definition 13.

Proof. 1. Substitute the definitions of $h(t, u)$ and $X(u)$:

$$\int_{-\infty}^{\infty} h(t, u) X(u) du = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_t(\lambda) e^{-i\lambda u} d\lambda \right] \left[\int_{-\infty}^{\infty} e^{i\mu u} d\Phi(\mu) \right] du \quad (76)$$

2. Apply Fubini's theorem:

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_t(\lambda) \left[\int_{-\infty}^{\infty} e^{i(\mu-\lambda)u} du \right] d\lambda d\Phi(\mu) \quad (77)$$

3. The inner integral over u is the Fourier representation of the Dirac delta:

$$\int_{-\infty}^{\infty} e^{i(\mu-\lambda)u} du = 2\pi \delta(\mu - \lambda) \quad (78)$$

4. Substitute:

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_t(\lambda) \cdot 2\pi \delta(\mu - \lambda) d\lambda d\Phi(\mu) \quad (79)$$

5. Simplify:

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_t(\lambda) \delta(\mu - \lambda) d\lambda d\Phi(\mu) \quad (80)$$

6. Apply the sifting property of the delta function:

$$= \int_{-\infty}^{\infty} \varphi_t(\mu) d\Phi(\mu) = Z(t) \quad (81) \quad \square$$

3 Unitarily Time-Changed Stationary Processes

3.1 Unitary Time-Change Operator $U_{\theta} f$

Theorem 15. [Unitary time-change operator U_{θ} and its inverse U_{θ}^{-1}] Let the time-change function $\theta: \mathbb{R} \rightarrow \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with

$$\dot{\theta}(t) > 0 \quad (82)$$

almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero. For f measurable, define

$$(U_{\theta} f)(t) = \sqrt{\dot{\theta}(t)} f(\theta(t)) \quad (83)$$

Its inverse is given by

$$(U_{\theta}^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \quad (84)$$

For every compact set $K \subseteq \mathbb{R}$ and $f \in L^2_{\text{loc}}(\mathbb{R})$,

$$\int_K |(U_{\theta} f)(t)|^2 dt = \int_{\theta(K)} |f(s)|^2 ds \quad (85)$$

Moreover, U_{θ}^{-1} is the inverse of U_{θ} on $L^2_{\text{loc}}(\mathbb{R})$.

Proof. 1. Let $f \in L^2_{\text{loc}}(\mathbb{R})$ and let $K \subset \mathbb{R}$ be compact. From the definition (83),

$$\int_K |(U_{\theta} f)(t)|^2 dt = \int_K \left| \sqrt{\dot{\theta}(t)} f(\theta(t)) \right|^2 dt \quad (86)$$

2. Expanding the square,

$$\int_K \left| \sqrt{\dot{\theta}(t)} f(\theta(t)) \right|^2 dt = \int_K \dot{\theta}(t) |f(\theta(t))|^2 dt \quad (87)$$

3. Since θ is absolutely continuous and strictly increasing, $\theta' = \dot{\theta}$ exists almost everywhere and $\dot{\theta}(t) > 0$ a.e.
4. Apply the change of variables $s = \theta(t)$. Then

$$ds = \dot{\theta}(t) dt \quad (88)$$

5. The inverse function $t = \theta^{-1}(s)$ exists since θ is strictly increasing and bijective.
6. As t ranges over K , the variable $s = \theta(t)$ ranges over $\theta(K)$.
7. Since θ is continuous and K is compact, $\theta(K)$ is compact.
8. Substituting (88) into (87),

$$\int_K \dot{\theta}(t) |f(\theta(t))|^2 dt = \int_{\theta(K)} |f(s)|^2 ds \quad (89)$$

9. This establishes the local isometry (85).
10. To verify U_θ^{-1} is the inverse, compute:

$$(U_\theta^{-1} U_\theta f)(s) = U_\theta^{-1} (U_\theta f)(s) \quad (90)$$

11. By definition (84),

$$U_\theta^{-1} (U_\theta f)(s) = \frac{(U_\theta f)(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \quad (91)$$

12. By definition (83),

$$(U_\theta f)(\theta^{-1}(s)) = \sqrt{\dot{\theta}(\theta^{-1}(s))} f(\theta(\theta^{-1}(s))) \quad (92)$$

13. Since $\theta \circ \theta^{-1} = \text{id}$,

$$f(\theta(\theta^{-1}(s))) = f(s) \quad (93)$$

14. Substituting (92) and (93) into (91),

$$\frac{\sqrt{\dot{\theta}(\theta^{-1}(s))} f(s)}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} = f(s) \quad (94)$$

15. Therefore

$$U_\theta^{-1} U_\theta = \text{id} \quad (95)$$

16. Similarly, compute:

$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)} (U_\theta^{-1} g)(\theta(t)) \quad (96)$$

17. By definition (84),

$$(U_\theta^{-1} g)(\theta(t)) = \frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} \quad (97)$$

18. Since $\theta^{-1} \circ \theta = \text{id}$,

$$g(\theta^{-1}(\theta(t))) = g(t), \quad \theta^{-1}(\theta(t)) = t \quad (98)$$

19. Substituting (98) into (97),

$$\frac{g(t)}{\sqrt{\dot{\theta}(t)}} \quad (99)$$

20. Therefore from (96),

$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)} \cdot \frac{g(t)}{\sqrt{\dot{\theta}(t)}} = g(t) \quad (100)$$

21. Thus

$$U_\theta U_\theta^{-1} = \text{id} \quad (101)$$

22. Combining (95) and (101), U_θ^{-1} is the two-sided inverse of U_θ on $L^2_{\text{loc}}(\mathbb{R})$. \square

3.2 Transformation of Stationary \rightarrow Oscillatory Processes via U_θ

Theorem 16. [Unitary time changes of stationary processes produce oscillatory process] Let X be zero-mean stationary as in Definition 3. For scaling function θ as in Theorem 15, define

$$\begin{aligned} Z(t) &= (U_\theta X)(t) \\ &= \sqrt{\dot{\theta}(t)} X(\theta(t)) \end{aligned} \quad (102)$$

Then Z is a realization of an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \quad (103)$$

gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \quad (104)$$

and covariance kernel

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\left[\sqrt{\dot{\theta}(t)} X(\theta(t)) \overline{\sqrt{\dot{\theta}(s)} X(\theta(s))}\right] \\ &= \sqrt{\dot{\theta}(t)\dot{\theta}(s)} \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \\ &= \sqrt{\dot{\theta}(t)\dot{\theta}(s)} R_X(\theta(t) - \theta(s)) \\ &= \sqrt{\dot{\theta}(t)\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} dF(\lambda) \end{aligned} \quad (105)$$

Proof. 1. From the Cramér representation (18),

$$X(u) = \int_{\mathbb{R}} e^{i\lambda u} d\Phi(\lambda) \quad (106)$$

2. Substituting $u = \theta(t)$ into (106),

$$X(\theta(t)) = \int_{\mathbb{R}} e^{i\lambda\theta(t)} d\Phi(\lambda) \quad (107)$$

3. From the definition (102),

$$Z(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)} d\Phi(\lambda) \quad (108)$$

4. By linearity of the stochastic integral,

$$Z(t) = \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} d\Phi(\lambda) \quad (109)$$

5. Define

$$\varphi_t(\lambda) := \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \quad (110)$$

6. Then (109) becomes

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \quad (111)$$

which is the oscillatory representation (28).

7. To express this in terms of the standard oscillatory function form, define the gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \quad (112)$$

8. Then verify the oscillatory function form (27) factorizes

$$\begin{aligned} \varphi_t(\lambda) &= A_t(\lambda) e^{i\lambda t} \\ &= \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} e^{i\lambda t} \\ &= \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t+t)} \\ &= \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \end{aligned} \quad (113)$$

9. To compute the covariance, use (30):

$$R_Z(t, s) = \mathbb{E}[Z(t)\overline{Z(s)}] \quad (114)$$

10. Substituting (102),

$$R_Z(t, s) = \mathbb{E} \left[\sqrt{\dot{\theta}(t)} X(\theta(t)) \sqrt{\dot{\theta}(s)} X(\theta(s)) \right] \quad (115)$$

11. Since $\dot{\theta}$ is deterministic,

$$R_Z(t, s) = \sqrt{\dot{\theta}(t)} \sqrt{\dot{\theta}(s)} \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \quad (116)$$

12. By stationarity of X , using (19),

$$\mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] = R_X(\theta(t) - \theta(s)) = \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} dF(\lambda) \quad (117)$$

13. Substituting (117) into (116),

$$R_Z(t, s) = \sqrt{\dot{\theta}(t) \dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t) - \theta(s))} dF(\lambda) \quad (118)$$

establishing (105). \square

3.2.1 Time-Varying Filter Representations

Theorem 17. Let $\theta: \mathbb{R} \rightarrow \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\dot{\theta}(t) > 0$ almost everywhere. Let $X(u)$ be a stationary process, and define the oscillatory process obtained by the forward unitary time transformation U_θ

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) = \int_{\mathbb{R}} h(t, u) X(u) du \quad (119)$$

where the forward impulse response function is given by

$$h(t, u) = \sqrt{\dot{\theta}(t)} \delta(u - \theta(t)) \quad (120)$$

For the unitary time-change case, this impulse response is a specialization of the general impulse response from Definition 13. Specifically, with $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)}$, the Fourier-based impulse response

$$h(t, u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} e^{-i\lambda u} d\lambda = \sqrt{\dot{\theta}(t)} \delta(u - \theta(t)) \quad (121)$$

simplifies to the Dirac form via the inverse Fourier transform.

Then likewise the transformation can be reversed by expressing the stationary process as

$$X(u) = (U_\theta^{-1} Z)(u) = \frac{Z(\theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} = \int_{\mathbb{R}} g(u, t) Z(t) dt \quad (122)$$

where the inverse impulse response function is

$$g(u, t) = \frac{\delta(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \quad (123)$$

Proof. 1. Recall the forward unitary transformation from Theorem 15:

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) \quad (124)$$

2. To express this as a convolution integral, note that the Dirac delta function satisfies the sifting property: for any continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$,

$$\int_{\mathbb{R}} f(u) \delta(u - a) du = f(a) \quad (125)$$

for any $a \in \mathbb{R}$.

3. Substituting $f(u) = X(u)$ and $a = \theta(t)$, which is well-defined since θ is bijective and continuous,

$$X(\theta(t)) = \int_{\mathbb{R}} X(u) \delta(u - \theta(t)) du \quad (126)$$

4. Multiplying both sides by $\sqrt{\dot{\theta}(t)}$ and substituting into (124),

$$Z(t) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} X(u) \delta(u - \theta(t)) du = \int_{\mathbb{R}} \left[\sqrt{\dot{\theta}(t)} \delta(u - \theta(t)) \right] X(u) du \quad (127)$$

5. Thus, the forward impulse response function is

$$h(t, u) = \sqrt{\dot{\theta}(t)} \delta(u - \theta(t)) \quad (128)$$

establishing (119).

6. To verify this as a specialization of the Fourier-based impulse response from Definition 13, substitute $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)}$:

$$h(t, u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} e^{-i\lambda u} d\lambda = \sqrt{\dot{\theta}(t)} \frac{1}{2\pi} \int e^{i\lambda(\theta(t)-u)} d\lambda = \sqrt{\dot{\theta}(t)} \delta(u - \theta(t)) \quad (129)$$

by the standard Fourier representation of the Dirac delta.

7. For the inverse transformation, recall from Theorem 15 that

$$X(u) = (U_{\theta}^{-1} Z)(u) = \frac{Z(\theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \quad (130)$$

8. Let $s = \theta^{-1}(u)$, so $u = \theta(s)$ and $Z(\theta^{-1}(u)) = Z(s)$. The sifting property applied to $Z(t)$ with point $\theta^{-1}(u)$ gives

$$Z(\theta^{-1}(u)) = \int_{\mathbb{R}} Z(t) \delta(t - \theta^{-1}(u)) dt \quad (131)$$

9. Substituting into (130),

$$X(u) = \frac{1}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \int_{\mathbb{R}} Z(t) \delta(t - \theta^{-1}(u)) dt = \int_{\mathbb{R}} \left[\frac{\delta(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \right] Z(t) dt \quad (132)$$

10. Thus, the inverse impulse response function is

$$g(u, t) = \frac{\delta(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \quad (133)$$

establishing (122) and (123).

11. To confirm invertibility, substitute (127) into (132). The integral becomes

$$X(u) = \int_{\mathbb{R}} g(u, t) \left[\int_{\mathbb{R}} h(t, v) X(v) dv \right] dt \quad (134)$$

12. By Fubini's theorem, since all measures are positive and the delta functions ensure finite support,

$$X(u) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(u, t) h(t, v) X(v) dv dt \quad (135)$$

13. Integrating the impulse response composition

$$g(u, t) h(t, v) = \frac{\delta(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \cdot \sqrt{\dot{\theta}(t)} \delta(v - \theta(t)) \quad (136)$$

over t results in $t = \theta^{-1}(u)$, so

$$\sqrt{\dot{\theta}(t)} = \sqrt{\dot{\theta}(\theta^{-1}(u))} \quad (137)$$

and

$$\delta(v - \theta(t)) = \delta(v - u) \quad (138)$$

yielding

$$\int_{\mathbb{R}} g(u, t) h(t, v) dt = \delta(v - u) \quad (139)$$

14. Thus, (135) simplifies to

$$\int_{\mathbb{R}} \delta(v - u) X(v) dv = X(u) \quad (140)$$

confirming the transformations are inverses. \square

Corollary 18. *The evolutionary spectrum is*

$$dF_t(\lambda) = \dot{\theta}(t) dF(\lambda) \quad (141)$$

Proof. 1. The evolutionary spectrum is defined by

$$dF_t(\lambda) = |A_t(\lambda)|^2 dF(\lambda) \quad (142)$$

2. From (104),

$$|A_t(\lambda)|^2 = \left| \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \right|^2 \quad (143)$$

3. Since $|e^{i\alpha}| = 1$ for all real α ,

$$|e^{i\lambda(\theta(t)-t)}|^2 = 1 \quad (144)$$

4. Therefore

$$|A_t(\lambda)|^2 = \left(\sqrt{\dot{\theta}(t)} \right)^2 \cdot 1 = \dot{\theta}(t) \quad (145)$$

5. Substituting (145) into (142),

$$dF_t(\lambda) = \dot{\theta}(t) dF(\lambda) \quad (146) \quad \square$$

3.3 Covariance operator conjugation

Proposition 19. Let

$$(T_K f)(t) := \int_{\mathbb{R}} K(|t-s|) f(s) ds \quad (147)$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h} dF(\lambda) \quad (148)$$

Define the transformed kernel

$$K_\theta(s, t) := \sqrt{\dot{\theta}(t) \dot{\theta}(s)} K(|\theta(t) - \theta(s)|) \quad (149)$$

then the corresponding integral covariance operator is conjugated for all $f \in L^2_{\text{loc}}(\mathbb{R})$ by

$$(T_{K_\theta} f)(t) = (U_\theta T_K U_\theta^{-1} f)(t) \quad (150)$$

Proof. 1. From (150), expand the right side:

$$(U_\theta T_K U_\theta^{-1} f)(t) = \sqrt{\dot{\theta}(t)} (T_K U_\theta^{-1} f)(\theta(t)) \quad (151)$$

2. By definition (147),

$$(T_K U_\theta^{-1} f)(\theta(t)) = \int_{\mathbb{R}} K(|\theta(t) - s|) (U_\theta^{-1} f)(s) ds \quad (152)$$

3. By definition (84),

$$(U_\theta^{-1} f)(s) = \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \quad (153)$$

4. Substituting (153) into (152),

$$\int_{\mathbb{R}} K(|\theta(t) - s|) \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} ds \quad (154)$$

5. Apply the change of variables $s = \theta(u)$, so $ds = \dot{\theta}(u) du$ and $\theta^{-1}(s) = u$:

$$\int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \frac{f(u)}{\sqrt{\dot{\theta}(u)}} \dot{\theta}(u) du \quad (155)$$

6. Simplify:

$$\int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \frac{\dot{\theta}(u)}{\sqrt{\dot{\theta}(u)}} f(u) du = \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \sqrt{\dot{\theta}(u)} f(u) du \quad (156)$$

7. Substituting (156) into (151),

$$\sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \sqrt{\dot{\theta}(u)} f(u) du \quad (157)$$

8. Bring the constant inside the integral:

$$\int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} \sqrt{\dot{\theta}(u)} K(|\theta(t) - \theta(u)|) f(u) du \quad (158)$$

9. By definition (149),

$$\sqrt{\dot{\theta}(t)} \sqrt{\dot{\theta}(u)} K(|\theta(t) - \theta(u)|) = K_{\theta}(u, t) \quad (159)$$

10. Therefore

$$\int_{\mathbb{R}} K_{\theta}(u, t) f(u) du = (T_{K_{\theta}} f)(t) \quad (160)$$

establishing (150). \square

4 Zero Localization

Definition 20. Let Z be real-valued with $Z \in C^1(\mathbb{R})$ having only simple zeros

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \quad (161)$$

Define, for Borel $B \subset \mathbb{R}$,

$$\mu(B) = \int_{\mathbb{R}} \mathbf{1}_B(t) \delta(Z(t)) |\dot{Z}(t)| dt \quad (162)$$

Theorem 21. Under the assumptions of Definition 20, zeros are locally finite and one has

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \quad (163)$$

whence

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (164)$$

Proof. 1. For any smooth test function ϕ with compact support, apply the standard change of variables formula for the delta function. Let $\{t_0^{(1)}, t_0^{(2)}, \dots\}$ denote the zeros of Z .

2. By the change of variables formula for distributions,

$$\int_{\mathbb{R}} \phi(t) \delta(Z(t)) dt = \sum_{t_0: Z(t_0)=0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} \quad (165)$$

3. The right side of (165) equals

$$\sum_{t_0: Z(t_0)=0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} = \sum_{t_0: Z(t_0)=0} \int_{\mathbb{R}} \phi(t) \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} dt \quad (166)$$

4. By Fubini's theorem (justified since the sum has locally finite terms due to C^1 regularity and simple zeros),

$$\sum_{t_0: Z(t_0)=0} \int_{\mathbb{R}} \phi(t) \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} dt = \int_{\mathbb{R}} \phi(t) \sum_{t_0: Z(t_0)=0} \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} dt \quad (167)$$

5. Comparing (165) and (167),

$$\int_{\mathbb{R}} \phi(t) \delta(Z(t)) dt = \int_{\mathbb{R}} \phi(t) \sum_{t_0: Z(t_0)=0} \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} dt \quad (168)$$

6. Since ϕ is arbitrary,

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} \quad (169)$$

establishing (163).

7. Substituting (169) into the definition (162),

$$\mu(B) = \int_{\mathbb{R}} \mathbf{1}_B(t) \sum_{t_0: Z(t_0)=0} \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} |\dot{Z}(t)| dt \quad (170)$$

8. By the sifting property of the delta function, $|\dot{Z}(t)|$ evaluated at $t=t_0$ gives $|\dot{Z}(t_0)|$:

$$\int_{\mathbb{R}} \mathbf{1}_B(t) \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} |\dot{Z}(t)| dt = \frac{\mathbf{1}_B(t_0) |\dot{Z}(t_0)|}{|\dot{Z}(t_0)|} = \mathbf{1}_B(t_0) \quad (171)$$

9. Summing over all zeros,

$$\mu(B) = \sum_{t_0: Z(t_0)=0} \mathbf{1}_B(t_0) = \sum_{t_0 \in B: Z(t_0)=0} 1 \quad (172)$$

10. This is precisely the atomic measure

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (173)$$

establishing (164). \square

Definition 22. Let $\mathcal{H} = L^2(\mu)$ be the Hilbert space with inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} d\mu(t) \quad (174)$$

Proposition 23. [Atomic structure] Let

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (175)$$

then

$$\mathcal{H} \cong \left\{ f: \{t_0: Z(t_0) = 0\} \rightarrow \mathbb{C}: \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \quad (176)$$

with orthonormal basis $\{e_{t_0}\}_{t_0: Z(t_0)=0}$ where

$$e_{t_0}(t_1) = \delta_{t_0, t_1} \quad (177)$$

Proof. 1. By (175), μ is a purely atomic measure with atoms at the zero set.

2. For any $f \in L^2(\mu)$, the L^2 norm is

$$\|f\|_{L^2(\mu)}^2 = \int_{\mathbb{R}} |f(t)|^2 d\mu(t) \quad (178)$$

3. Substituting (175),

$$\int_{\mathbb{R}} |f(t)|^2 d\mu(t) = \int_{\mathbb{R}} |f(t)|^2 \sum_{t_0: Z(t_0)=0} \delta_{t_0}(dt) = \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 \quad (179)$$

4. Therefore

$$\|f\|_{L^2(\mu)}^2 = \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 \quad (180)$$

5. This is precisely the ℓ^2 norm on the zero set.

6. Define the map $\Psi: L^2(\mu) \rightarrow \ell^2$ by

$$\Psi(f) = (f(t_0))_{t_0: Z(t_0)=0} \quad (181)$$

7. From (180), Ψ is an isometry:

$$\|\Psi(f)\|_{\ell^2}^2 = \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 = \|f\|_{L^2(\mu)}^2 \quad (182)$$

8. Ψ is surjective: for any sequence $(c_{t_0}) \in \ell^2$, define $f(t) = \sum_{t_0} c_{t_0} \delta_{t_0}(t)$, which is in $L^2(\mu)$.

9. Therefore Ψ is a Hilbert space isomorphism, establishing (176).

10. For the orthonormal basis, define e_{t_0} by (177).

11. Then

$$\langle e_{t_0}, e_{t_1} \rangle = \int_{\mathbb{R}} e_{t_0}(t) \overline{e_{t_1}(t)} d\mu(t) = \sum_{s: Z(s)=0} \delta_{t_0, s} \delta_{t_1, s} = \delta_{t_0, t_1} \quad (183)$$

12. Therefore $\{e_{t_0}\}$ is an orthonormal set.

13. Since every $f \in L^2(\mu)$ can be written as

$$f = \sum_{t_0: Z(t_0)=0} f(t_0) e_{t_0} \quad (184)$$

the set $\{e_{t_0}\}$ is complete, hence an orthonormal basis. \square

Definition 24. [Multiplication operator] Define the linear operator

$$L: \mathcal{D}(L) \subset \mathcal{H} \rightarrow \mathcal{H} \quad (185)$$

by

$$(L f)(t) = t f(t) \quad (186)$$

on the support of μ with domain

$$\mathcal{D}(L) := \left\{ f \in \mathcal{H}: \int |t f(t)|^2 d\mu(t) < \infty \right\} \quad (187)$$

Theorem 25. [Self-adjointness and spectrum] L is self-adjoint on \mathcal{H} and has pure point, simple spectrum

$$\sigma(L) = \overline{\{t \in \mathbb{R}: Z(t) = 0\}} \quad (188)$$

with eigenvalues $\lambda = t_0$ for each zero t_0 and corresponding eigenvectors e_{t_0} .

Proof. 1. For $f, g \in \mathcal{D}(L)$, compute the inner product:

$$\langle L f, g \rangle = \int_{\mathbb{R}} (L f)(t) \overline{g(t)} d\mu(t) \quad (189)$$

2. By definition (186),

$$\int_{\mathbb{R}} t f(t) \overline{g(t)} d\mu(t) \quad (190)$$

3. Since t is real-valued, $\bar{t} = t$, so

$$\int_{\mathbb{R}} t f(t) \overline{g(t)} d\mu(t) = \int_{\mathbb{R}} f(t) \overline{t g(t)} d\mu(t) \quad (191)$$

4. The right side of (191) is

$$\int_{\mathbb{R}} f(t) \overline{(L g)(t)} d\mu(t) = \langle f, L g \rangle \quad (192)$$

5. Therefore

$$\langle L f, g \rangle = \langle f, L g \rangle \quad (193)$$

for all $f, g \in \mathcal{D}(L)$, establishing that L is symmetric.

6. Since L is a multiplication operator on $L^2(\mu)$, it is self-adjoint (by standard functional analysis).

7. To determine the spectrum, compute the action on basis vectors. From (186) and (177),

$$(L e_{t_0})(t) = t e_{t_0}(t) = t \delta_{t_0}(t) \quad (194)$$

8. By the sifting property,

$$t \delta_{t_0}(t) = t_0 \delta_{t_0}(t) = t_0 e_{t_0}(t) \quad (195)$$

9. Therefore

$$L e_{t_0} = t_0 e_{t_0} \quad (196)$$

10. This shows that each t_0 is an eigenvalue with eigenvector e_{t_0} .
 11. Since the $\{e_{t_0}\}$ form a complete orthonormal basis (Proposition 23), the spectrum is pure point.
 12. Each eigenspace is one-dimensional (spanned by e_{t_0}), so the spectrum is simple and given by the closure of the zero set

$$\sigma(L) = \{t_0 : Z(t_0) = 0\} = \overline{\{t \in \mathbb{R} : Z(t) = 0\}} \quad (197) \quad \square$$

4.1 Simplicity of Zeros and Their Expected Counting Function

Theorem 26. (Bulinskaya) Let $X(t)$ be a centered stationary Gaussian process with covariance function $K(h) = \mathbb{E}[X(t) X(t+h)]$ that is twice differentiable at $h=0$ with $K(0) > 0$ and $\ddot{K}(0) < 0$. Then the zero set of X has no accumulation points almost surely. In particular, on any compact interval $[a, b]$, the number of zeros is almost surely finite.

Proof. 1. The twice-differentiability of K at $h=0$ implies that the spectral measure F has finite second moment:

$$\lambda_2 = \int_{\mathbb{R}} \omega^2 dF(\omega) = -\ddot{K}(0) < \infty. \quad (198)$$

2. This finite second moment implies that X has mean-square continuous derivative $\dot{X}(t)$, with

$$\mathbb{E}[\dot{X}(t)^2] = -\ddot{K}(0) = \lambda_2 > 0. \quad (199)$$

3. Since $\dot{X}(t)$ is a non-degenerate centered Gaussian process, it is continuous almost surely and does not vanish identically on any interval.
 4. For any zero t_0 of X (i.e., $X(t_0) = 0$), if $\dot{X}(t_0) \neq 0$, then X crosses transversally through zero at t_0 , making t_0 an isolated zero.
 5. The joint distribution of $(X(t_0), \dot{X}(t_0))$ is bivariate Gaussian with covariance matrix

$$\begin{pmatrix} K(0) & K'(0) \\ K'(0) & -\ddot{K}(0) \end{pmatrix} = \begin{pmatrix} K(0) & 0 \\ 0 & -\ddot{K}(0) \end{pmatrix}, \quad (200)$$

where $K'(0)=0$ by evenness of K . Since X and \dot{X} are uncorrelated Gaussians, they are independent.

6. At any zero t_0 of X , the derivative $\dot{X}(t_0)$ is Gaussian with mean zero and variance $-\ddot{K}(0)>0$, hence

$$\mathbb{P} [\dot{X}(t_0)=0 | X(t_0)=0] = \mathbb{P} [\dot{X}(t_0)=0] = 0. \quad (201)$$

Thus almost surely $\dot{X}(t_0)\neq 0$, making t_0 an isolated zero.

7. Since every zero of X is isolated almost surely, the zero set can have no accumulation points almost surely.
8. On a compact interval $[a, b]$, a set with no accumulation points is finite, completing the proof. \square

Theorem 27. (Expected Zero-Counting Function with Deterministic Atoms)
Let $\theta: \mathbb{R} \rightarrow \mathbb{R}$ be C^1 (continuously differentiable), strictly increasing, and bijective with $\dot{\theta}(t) \geq 0$ for all t and $\dot{\theta}(t) > 0$ almost everywhere. Define the zero-derivative set

$$T_0 := \{t \in \mathbb{R} : \dot{\theta}(t) = 0\} \quad (202)$$

and assume that T_0 is at most countable with no accumulation points on any compact interval. Let X be a centered stationary Gaussian process with spectral measure F and covariance function

$$K(h) = \int_{\mathbb{R}} e^{i\omega h} dF(\omega) \quad (203)$$

twice differentiable at $h=0$ with $\ddot{K}(0) < 0$ and $K(0) > 0$. Define the unitarily time-changed process

$$Z(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) \quad (204)$$

Then Z is a centered Gaussian process with covariance

$$K_Z(t, s) = \sqrt{\dot{\theta}(t) \dot{\theta}(s)} K(\theta(t) - \theta(s)) \quad (205)$$

For any compact interval $[0, T]$, define

$$N_{\det}([0, T]) := \#(T_0 \cap [0, T]) \quad (206)$$

By the assumption on T_0 , $N_{\det}([0, T])$ is finite. The expected number of zeros of Z in $[0, T]$ decomposes as

$$\mathbb{E}[N_{[0, T]}(Z)] = N_{\det}([0, T]) + \frac{\theta(T) - \theta(0)}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}} \quad (207)$$

Proof. 1. Define

$$T_0 := \{t \in \mathbb{R} : \dot{\theta}(t) = 0\}, \quad I_T := [0, T] \setminus T_0 \quad (208)$$

For any $t_0 \in T_0$,

$$Z(t_0) = \sqrt{\dot{\theta}(t_0)} X(\theta(t_0)) = 0 \cdot X(\theta(t_0)) = 0 \quad (209)$$

so each $t_0 \in T_0$ is a deterministic zero of Z on every sample path.

2. By the assumption that T_0 has no accumulation points on compact intervals, $T_0 \cap [0, T]$ is finite. Thus

$$N_{\text{det}}([0, T]) = \#(T_0 \cap [0, T]) < \infty. \quad (210)$$

3. On the complement $I_T = [0, T] \setminus T_0$, the derivative satisfies $\dot{\theta}(t) > 0$, hence

$$Z(t) = 0 \iff X(\theta(t)) = 0, \quad t \in I_T \quad (211)$$

Define

$$Y(t) := X(\theta(t)). \quad (212)$$

The random zero set of Z on $[0, T]$ coincides with the zero set of Y on I_T , and the total zero count decomposes as

$$N_{[0, T]}(Z) = N_{\text{det}}([0, T]) + N_{\text{rand}}([0, T]) \quad (213)$$

where $N_{\text{rand}}([0, T])$ counts zeros of Y in I_T .

4. For $t, s \in \mathbb{R}$,

$$K_Y(t, s) = \mathbb{E}[X(\theta(t)) X(\theta(s))] = K(\theta(t) - \theta(s)) \quad (214)$$

Differentiate with respect to s :

$$\frac{\partial}{\partial s} K_Y(s, t) = -\dot{\theta}(s) \dot{K}(\theta(t) - \theta(s)) \quad (215)$$

Since $K(h)$ is even, $\dot{K}(0) = 0$. Taking $s \rightarrow t$,

$$\lim_{s \rightarrow t} \frac{\partial}{\partial s} K_Y(s, t) = 0 \quad (216)$$

5. The mixed partial derivative is

$$\frac{\partial^2}{\partial s \partial t} K_Y(t, s) = -\dot{\theta}(t) \dot{\theta}(s) \ddot{K}(\theta(t) - \theta(s)) \quad (217)$$

Taking $s \rightarrow t$,

$$\lim_{s \rightarrow t} \frac{\partial^2}{\partial s \partial t} K_Y(t, s) = -\dot{\theta}(t)^2 \ddot{K}(0) \quad (218)$$

Also $K_Y(t, t) = K(0)$.

6. The Kac–Rice zero intensity for Y on I_T is

$$\rho_Y(t) = \frac{1}{\pi} \sqrt{\frac{K_Y(t, t) \cdot \lim_{s \rightarrow t} \frac{\partial^2}{\partial s \partial t} K_Y(t, s) - \left(\lim_{s \rightarrow t} \frac{\partial}{\partial s} K_Y(s, t) \right)^2}{K_Y(t, t)^2}} \quad (219)$$

Substituting,

$$K_Y(t, t) \cdot \lim_{s \rightarrow t} \frac{\partial^2}{\partial s \partial t} K_Y(t, s) - 0^2 = K(0) \cdot (-\dot{\theta}(t)^2 \ddot{K}(0)) = K(0) \dot{\theta}(t)^2 (-\ddot{K}(0)) \quad (220)$$

Therefore,

$$\rho_Y(t) = \frac{1}{\pi} \sqrt{\frac{\dot{\theta}(t)^2 (-\ddot{K}(0))}{K(0)}} = \frac{\dot{\theta}(t)}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}} \quad (221)$$

7. The expected random zero count is

$$\mathbb{E}[N_{\text{rand}}([0, T])] = \int_{I_T} \rho_Y(t) dt \quad (222)$$

Since T_0 is countable (hence Lebesgue measure zero),

$$\mathbb{E}[N_{\text{rand}}([0, T])] = \int_0^T \frac{\dot{\theta}(t)}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}} dt = \frac{\theta(T) - \theta(0)}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}} \quad (223)$$

8. The total zero count factors as

$$N_{[0, T]}(Z) = N_{\text{det}}([0, T]) + N_{\text{rand}}([0, T]) \quad (224)$$

so

$$\mathbb{E}[N_{[0, T]}(Z)] = N_{\text{det}}([0, T]) + \frac{\theta(T) - \theta(0)}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}} \quad (225) \quad \square$$

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