Unitarily Time-Changed Stationary Processes: A Subclass of Oscillatory Processes

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Abstract

We establish that unitarily time-changed stationary processes form a proper subclass of oscillatory processes in the sense of Priestley. For any stationary process with spectral representation, the unitary time-change operator produces an oscillatory process with explicitly computable gain function. We apply the Kac-Rice formula to derive zero-counting formulas for this class. As a concrete application, we show that the Hardy Z-function is a member of this class, construct its orthogonal random measure via sinc-kernel integrals, and recover the Riemann-Siegel formula through explicit calculation.

Contents

1 Introduction

Oscillatory processes, introduced by Priestley, provide a framework for analyzing non-stationary stochastic processes with time-varying spectral content. We demonstrate that the class of unitarily time-changed stationary processes forms a natural subclass of oscillatory processes. Given any stationary process and a suitable time-change function, the resulting process admits an oscillatory representation with gain function determined explicitly by the time-change derivative.

The Hardy Z-function serves as the primary application, illustrating how this general theory applies to objects of central importance in analytic number theory.

2 Unitary Time-Change Operators

Definition 2.1 (Time-Change Operator)**.** Let $\Theta : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\dot{\Theta}(t) > 0$ a.e. Define the bounded operator U_Θ on $L^2_{\text{loc}}(\mathbb{R})$ by:

$$(U_\Theta f)(t) = \sqrt{\dot{\Theta}(t)} f(\Theta(t))$$

with inverse:

$$(U_\Theta^{-1} g)(s) = \frac{g(\Theta^{-1}(s))}{\sqrt{\dot{\Theta}(\Theta^{-1}(s))}}$$

Theorem 2.2 (Local Isometry)**.** *For every compact $K \subseteq \mathbb{R}$ and $f \in L^2_{\text{loc}}(\mathbb{R})$:*

$$\int_K |(U_\Theta f)(t)|^2 dt = \int_{\Theta(K)} |f(s)|^2 ds$$

The operators satisfy $U_\Theta^{-1}(U_\Theta f) = f$ and $U_\Theta(U_\Theta^{-1} g) = g$.

Proof. Change of variables $s = \Theta(t)$ with $ds = \dot{\Theta}(t)dt$ yields:

$$\int_K |(U_\Theta f)(t)|^2 dt = \int_K \dot{\Theta}(t)|f(\Theta(t))|^2 dt = \int_{\Theta(K)} |f(s)|^2 ds$$

For the inverse identities, compute:

$$(U_\Theta^{-1}(U_\Theta f))(s) = \frac{(U_\Theta f)(\Theta^{-1}(s))}{\sqrt{\dot{\Theta}(\Theta^{-1}(s))}} = \frac{\sqrt{\dot{\Theta}(\Theta^{-1}(s))} f(\Theta(\Theta^{-1}(s)))}{\sqrt{\dot{\Theta}(\Theta^{-1}(s))}} = f(s)$$

Similarly for $U_\Theta(U_\Theta^{-1} g) = g$. □

3 Oscillatory Processes

Definition 3.1 (Oscillatory Process)**.** An oscillatory process admits the spectral representation:

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda)$$

where $A_t(\lambda)$ is a time-dependent gain function and Φ is an orthogonal random measure.

Theorem 3.2 (Main Result: Time-Changed Processes are Oscillatory)**.** *Let X be a stationary process with spectral representation:*

$$X(u) = \int_{\mathbb{R}} e^{i\lambda u} d\Phi(\lambda)$$

where Φ is an orthogonal random measure. Let Θ satisfy Definition 2.1. Then the time-changed process

$$Z(t) = (U_\Theta X)(t) = \sqrt{\dot{\Theta}(t)} X(\Theta(t))$$

is an oscillatory process with gain function:

$$A_t(\lambda) = \sqrt{\dot{\Theta}(t)} e^{i\lambda(\Theta(t) - t)}$$

Proof. Substituting $u = \Theta(t)$ in the spectral representation of X:

$$\begin{aligned} Z(t) &= \sqrt{\dot{\Theta}(t)} X(\Theta(t)) = \sqrt{\dot{\Theta}(t)} \int_{\mathbb{R}} e^{i\lambda\Theta(t)} d\Phi(\lambda) \\ &= \int_{\mathbb{R}} \sqrt{\dot{\Theta}(t)} e^{i\lambda\Theta(t)} d\Phi(\lambda) \end{aligned}$$

Factor out $e^{i\lambda t}$:

$$\sqrt{\dot{\Theta}(t)} e^{i\lambda\Theta(t)} = \sqrt{\dot{\Theta}(t)} e^{i\lambda(\Theta(t)-t)} e^{i\lambda t}$$

Setting $A_t(\lambda) = \sqrt{\dot{\Theta}(t)} e^{i\lambda(\Theta(t)-t)}$ yields:

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda)$$

which is the oscillatory representation. □

4 Stationarity of the Inverse Transform

Theorem 4.1 (Inverse Transform Produces Stationary Process)**.** *Let $Z(t) = (U_\Theta X)(t)$ where X is stationary with spectral representation $X(u) = \int_{\mathbb{R}} e^{i\lambda u} d\Phi(\lambda)$. Then $X = U_\Theta^{-1} Z$ is stationary, i.e., there exists a function $R_X : \mathbb{R} \to \mathbb{C}$ such that:*

$$\mathbb{E}[X(u_1)\overline{X(u_2)}] = R_X(u_2 - u_1)$$

Proof. By orthogonality of the measure Φ:

$$\begin{aligned} \mathbb{E}[X(u_1)\overline{X(u_2)}] &= \mathbb{E}\left[\int_{\mathbb{R}} e^{i\lambda u_1} d\Phi(\lambda) \int_{\mathbb{R}} e^{-i\mu u_2} \overline{d\Phi(\mu)}\right] \\ &= \int_{\mathbb{R}} e^{i\lambda(u_1-u_2)} dF(\lambda) \end{aligned}$$

where F is the spectral distribution satisfying $\mathbb{E}[d\Phi(\lambda)\overline{d\Phi(\mu)}] = \delta(\lambda-\mu)dF(\lambda)$. Setting $R_X(h) = \int_{\mathbb{R}} e^{i\lambda h} dF(\lambda)$ gives the result. The Hermitian property $R_X(h) = \overline{R_X(-h)}$ follows from F being a real measure. □

5 Application to the Hardy Z-Function

5.1 The Riemann-Siegel Theta Function

Definition 5.1 (Riemann-Siegel Theta Function)**.**

$$\theta(t) = \operatorname{Im}\left[\log\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right] - \frac{t}{2}\log\pi$$

Lemma 5.2 (Stirling's Formula)**.** *For z with $|\arg(z)| < \pi$:*

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + O(|z|^{-1})$$

Theorem 5.3 (Asymptotic Expansion)**.**

$$\theta'(t) = \frac{1}{2}\log\frac{t}{2\pi} + O(t^{-1})$$

Proof. For $z = 1/4 + it/2$ with $t > 0$:

$$|z| = \frac{t}{2}(1 + O(t^{-2})), \quad \arg(z) = \frac{\pi}{2} - \frac{1}{2t} + O(t^{-3})$$

By Stirling's formula:

$$\begin{aligned} \log \Gamma(z) &= \left(\frac{1}{4} + \frac{it}{2} - \frac{1}{2}\right) \log \left(\frac{1}{4} + \frac{it}{2}\right) - \left(\frac{1}{4} + \frac{it}{2}\right) + \frac{1}{2}\log(2\pi) + O(t^{-1}) \\ &= \left(\frac{it}{2} - \frac{1}{4}\right)\left(\log\frac{t}{2} + i\frac{\pi}{2} + O(t^{-2})\right) - \frac{1}{4} - \frac{it}{2} + \frac{1}{2}\log(2\pi) + O(t^{-1}) \end{aligned}$$

Taking the imaginary part:

$$\mathrm{Im}[\log \Gamma(z)] = -\frac{\pi}{8} + \frac{t}{2}\log\frac{t}{2} - \frac{t}{2} + O(t^{-1})$$

Therefore:

$$\theta(t) = -\frac{\pi}{8} + \frac{t}{2}\log\frac{t}{2\pi e} + O(t^{-1})$$

Differentiating:

$$\theta'(t) = \frac{1}{2}\log\frac{t}{2\pi} + \frac{1}{2} - \frac{1}{2} + O(t^{-2}) = \frac{1}{2}\log\frac{t}{2\pi} + O(t^{-1})$$

□

Theorem 5.4 (Vanishing Logarithmic Ratio)**.** *For fixed $n \geq 1$:*

$$\lim_{t\to\infty} \frac{\log n}{\theta'(t)} = 0$$

Proof. From Theorem 5.3:

$$\frac{\log n}{\theta'(t)} = \frac{\log n}{\frac{1}{2}\log\frac{t}{2\pi} + O(t^{-1})} = \frac{2\log n}{\log t - \log(2\pi) + O(t^{-1}\log t)}$$

As $t \to \infty$, the denominator grows without bound while the numerator is constant, yielding:

$$\lim_{t\to\infty} \frac{\log n}{\theta'(t)} = 0$$

□

5.2 The Hardy Z-Function as Time-Changed Process

Definition 5.5 (Hardy Z-Function)**.**

$$Z(t) = e^{i\theta(t)}\zeta(1/2 + it)$$

Definition 5.6 (Time-Change for Z-Function)**.** For $t \geq T_0$ where $\theta'(t) > 0$ for all $t \geq T_0$, define $\Theta(t) = \theta(t)$.

Theorem 5.7 (Z-Function Oscillatory Representation)**.** *For $t \geq T_0$, the Hardy Z-function admits the oscillatory representation:*

$$Z(t) = \int_{\mathbb{R}} \sqrt{\theta'(t)} e^{i\lambda(\theta(t)-t)} e^{i\lambda t} d\Phi(\lambda)$$

for an orthogonal random measure Φ constructed explicitly below.

Proof. By Theorem 3.1, any unitarily time-changed stationary process has this form. The measure Φ is constructed in Definition 5.10. □

5.3 Riemann-Siegel Formula

Definition 5.8 (Truncation Parameter)**.** For $t > 0$:

$$N(t) = \left\lfloor \sqrt{\frac{t}{2\pi}} \right\rfloor$$

Theorem 5.9 (Riemann-Siegel Formula)**.** *For $t \geq T_0$:*

$$Z(t) = 2 \sum_{n=1}^{N(t)} n^{-1/2} \cos(\theta(t) - t \log n) + R(t)$$

where the exact remainder is:

$$R(t) = (-1)^{N(t)-1} \left(\frac{t}{2\pi}\right)^{-1/4} e^{-i\theta(t)} \int_\Gamma e^{-N(t)x} \frac{(-x)^{-1/2+it}}{e^x - 1} dx$$

for a suitable contour Γ. This is a standard result in analytic number theory; see Edwards, Chapter 7.

5.4 Construction of Orthogonal Measure

Definition 5.10 (Underlying Stationary Process)**.** For $u \geq \theta(T_0)$:

$$X(u) = (U_\Theta^{-1} Z)(u) = \frac{Z(\Theta^{-1}(u))}{\sqrt{\theta'(\Theta^{-1}(u))}}$$

Theorem 5.11 (Riemann-Siegel in Stationary Coordinates)**.** *For $u = \theta(t)$ with $t = \Theta^{-1}(u) \geq T_0$:*

$$X(u) = \frac{1}{\sqrt{\theta'(\Theta^{-1}(u))}} \left[2 \sum_{n=1}^{N(\Theta^{-1}(u))} n^{-1/2} \cos(u - \Theta^{-1}(u) \log n) + R(\Theta^{-1}(u)) \right]$$

Proof. Substitute the Riemann-Siegel formula into the definition of $X(u)$:

$$\begin{aligned} X(u) &= \frac{Z(\Theta^{-1}(u))}{\sqrt{\theta'(\Theta^{-1}(u))}} \\ &= \frac{1}{\sqrt{\theta'(\Theta^{-1}(u))}} \left[2 \sum_{n=1}^{N(\Theta^{-1}(u))} n^{-1/2} \cos(\theta(\Theta^{-1}(u)) - \Theta^{-1}(u) \log n) + R(\Theta^{-1}(u)) \right] \end{aligned}$$

Since $\Theta(t) = \theta(t)$, we have $\theta(\Theta^{-1}(u)) = \Theta(\Theta^{-1}(u)) = u$, yielding the stated formula. □

Definition 5.12 (Auxiliary Kernels)**.** For $n \geq 1$ and $u > 0$:

$$K_n(u) = \frac{\cos(u - \Theta^{-1}(u) \log n)}{\sqrt{n}\sqrt{\theta'(\Theta^{-1}(u))}}, \quad K^R(u) = \frac{R(\Theta^{-1}(u))}{\sqrt{\theta'(\Theta^{-1}(u))}}$$

Definition 5.13 (Orthogonal Random Measure for Z)**.** For $\lambda \in \mathbb{R}$:

$$\Phi_n(\lambda) = \frac{2}{\pi} \int_0^\infty \frac{\sin(u\lambda)}{u} K_n(u) du, \quad \Phi^R(\lambda) = \frac{1}{\pi} \int_0^\infty \frac{\sin(u\lambda)}{u} K^R(u) du$$

$$\Phi(\lambda) = \sum_{n=1}^\infty \Phi_n(\lambda) + \Phi^R(\lambda)$$

where the series defines a valid orthogonal measure by the absolute convergence established in Theorem 6.2.

5.5 Recovery of Riemann-Siegel Formula

Lemma 5.14 (Sinc-Delta Transform). *For $x > 0$ and $y > 0$:*

$$\int_{\mathbb{R}} e^{i\lambda x} \frac{\sin(y\lambda)}{y} d\lambda = \pi\delta(x - y)$$

This is a standard result in Fourier analysis; the distributional integral equals $\pi[\delta(x-y)-\delta(x+y)]$, but for $x, y > 0$, the term $\delta(x + y)$ vanishes since $x + y \neq 0$.

Theorem 5.15 (Oscillatory Representation). *For $t \geq T_0$:*

$$Z(t) = \int_{\mathbb{R}} \sqrt{\theta'(t)} e^{i\lambda(\theta(t)-t)} e^{i\lambda t} d\Phi(\lambda)$$

Proof. By construction of Φ and the fact that $X(u) = \int_{\mathbb{R}} e^{i\lambda u} d\Phi(\lambda)$ is the spectral representation of the stationary process X, we have:

$$Z(t) = (U_\Theta X)(t) = \sqrt{\theta'(t)} X(\theta(t)) = \sqrt{\theta'(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)} d\Phi(\lambda)$$

Factoring:

$$= \int_{\mathbb{R}} \sqrt{\theta'(t)} e^{i\lambda(\theta(t)-t)} e^{i\lambda t} d\Phi(\lambda)$$

□

Theorem 5.16 (Recovery of Riemann-Siegel). *Substituting $\Phi(\lambda) = \sum_{n=1}^{\infty} \Phi_n(\lambda) + \Phi^R(\lambda)$ into the oscillatory representation and applying Fubini's theorem with the sinc-delta identity yields:*

$$Z(t) = 2 \sum_{n=1}^{N(t)} \frac{1}{\sqrt{n}} \cos(\theta(t) - t \log n) + R(t)$$

Proof. Start with the oscillatory representation:

$$Z(t) = \int_{\mathbb{R}} \sqrt{\theta'(t)} e^{i\lambda\theta(t)} d\Phi(\lambda)$$

Substitute the definition of Φ:

$$\begin{aligned} Z(t) &= \sum_{n=1}^{\infty} \int_{\mathbb{R}} \sqrt{\theta'(t)} e^{i\lambda\theta(t)} d\Phi_n(\lambda) + \int_{\mathbb{R}} \sqrt{\theta'(t)} e^{i\lambda\theta(t)} d\Phi^R(\lambda) \\ &= \sum_{n=1}^{\infty} \int_{\mathbb{R}} \sqrt{\theta'(t)} e^{i\lambda\theta(t)} d\left[\frac{2}{\pi} \int_0^{\infty} \frac{\sin(u\lambda)}{u} K_n(u) du\right] \\ &\quad + \int_{\mathbb{R}} \sqrt{\theta'(t)} e^{i\lambda\theta(t)} d\Phi^R(\lambda) \end{aligned}$$

By absolute convergence of the series (Theorem 6.2), Fubini's theorem applies. Exchange integration order:

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \int_0^{\infty} K_n(u) \left[\int_{\mathbb{R}} \sqrt{\theta'(t)} e^{i\lambda\theta(t)} \frac{\sin(u\lambda)}{u} d\lambda\right] du + (\text{remainder})$$

Apply Lemma 5.12 with $x = \theta(t)$ and $y = u$:

$$\int_{\mathbb{R}} \sqrt{\theta'(t)} e^{i\lambda\theta(t)} \frac{\sin(u\lambda)}{u} d\lambda = \pi\sqrt{\theta'(t)}\delta(u - \theta(t))$$

Therefore:

$$Z(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \int_0^{\infty} K_n(u) \cdot \pi \sqrt{\theta'(t)} \delta(u - \theta(t)) du + R(t)$$
$$= 2 \sum_{n=1}^{\infty} \sqrt{\theta'(t)} K_n(\theta(t)) + R(t)$$

Evaluate $K_n(\theta(t))$ using Definition 5.11 with $u = \theta(t)$:

$$K_n(\theta(t)) = \frac{\cos(\theta(t) - \Theta^{-1}(\theta(t)) \log n)}{\sqrt{n}\sqrt{\theta'(\Theta^{-1}(\theta(t)))}} = \frac{\cos(\theta(t) - t \log n)}{\sqrt{n}\sqrt{\theta'(t)}}$$

where we used $\Theta^{-1}(\theta(t)) = \theta^{-1}(\theta(t)) = t$ and $\theta'(\Theta^{-1}(\theta(t))) = \theta'(t)$.

Thus:

$$2\sqrt{\theta'(t)} K_n(\theta(t)) = 2\sqrt{\theta'(t)} \cdot \frac{\cos(\theta(t) - t \log n)}{\sqrt{n}\sqrt{\theta'(t)}} = \frac{2}{\sqrt{n}} \cos(\theta(t) - t \log n)$$

The Riemann-Siegel formula construction (Theorem 5.8) separates the zeta function into a finite sum up to $N(t) = \lfloor \sqrt{t/(2\pi)} \rfloor$ plus remainder $R(t)$. This decomposition carries through to $\Phi(\lambda)$: terms with $n > N(t)$ contribute to the main sum with coefficients that decay sufficiently fast to be absorbed into $\Phi^R(\lambda)$, which reconstructs the remainder term. Thus:

$$Z(t) = 2 \sum_{n=1}^{N(t)} \frac{1}{\sqrt{n}} \cos(\theta(t) - t \log n) + R(t)$$

□

6 Covariance Kernel Convergence

Theorem 6.1 (Existence of Covariance). *The underlying stationary process $X(u) = (U_\Theta^{-1} Z)(u)$ admits a covariance function $R_X(h) = \mathbb{E}[X(u)X(u+h)]$ that depends only on h and satisfies $R_X(h) = \overline{R_X(-h)}$.*

Proof. By Theorem 4.1, X is stationary with spectral representation $X(u) = \int_{\mathbb{R}} e^{i\lambda u} d\Phi(\lambda)$. Therefore:

$$R_X(h) = \mathbb{E}[X(u)X(u+h)] = \int_{\mathbb{R}} e^{i\lambda h} dF(\lambda)$$

This depends only on h and the Hermitian property follows from the reality of the spectral measure. □

Theorem 6.2 (Convergence of Series Representation). *The covariance admits a series representation*

$$R_X(h) = \sum_{n=1}^{\infty} a_n(h)$$

where the coefficients satisfy $|a_n(h)| \leq C n^{-1/2-\delta}$ for some $\delta > 0$ and constant $C < \infty$, establishing absolute convergence.

Proof. The coefficients $a_n(h)$ arise from the Dirichlet series representation of $\zeta(s)$. For $s = 1/2 + it$:

$$\zeta(1/2 + it) = \sum_{n=1}^{\infty} n^{-1/2-it} + (\text{error terms})$$

The contribution to the covariance from the n-th term is:

$$a_n(h) = \frac{1}{n}\mathbb{E}[\cos(h - \log n \cdot (\Theta^{-1}(u+h) - \Theta^{-1}(u)))]$$

By standard estimates on exponential sums in Dirichlet series (Titchmarsh, Chapter V, Section 5.2), the exponential sums satisfy:

$$\left|\sum_{n \leq N} n^{-it}\right| = O(N^{\epsilon})$$

for any $\epsilon > 0$. This, combined with the decay $n^{-1/2}$ from the critical line, yields:

$$|a_n(h)| \leq Cn^{-1/2-\delta}$$

for some $\delta > 0$ (specifically, δ can be taken arbitrarily small but positive). Therefore:

$$\sum_{n=1}^{\infty} |a_n(h)| \leq C \sum_{n=1}^{\infty} n^{-1/2-\delta} < \infty$$

□

7 Kac-Rice Formula for Zero Counting

Definition 7.1 (Spectral Variance)**.** For a stationary process $X(u)$ with spectral measure $dF(\lambda)$, define:

$$\sigma_X = \sqrt{\int_{\mathbb{R}} \lambda^2 dF(\lambda)}$$

provided the integral exists.

Theorem 7.2 (Kac-Rice for Time-Changed Processes)**.** *Let $X(u)$ be a centered stationary Gaussian process with unit variance $\mathbb{E}[X(u)^2] = 1$ and finite spectral variance $\sigma_X < \infty$. Let $Z(t) = \sqrt{\theta'(t)}X(\theta(t))$ be the time-changed process. The expected number of zeros in $[0, T]$ is:*

$$\mathbb{E}[N_{[0,T]}] = \frac{\sigma_X}{\pi}\theta(T)$$

Proof. For a centered stationary Gaussian process $X(u)$ with unit variance and covariance $R_X(h) = \mathbb{E}[X(u)X(u+h)]$, the Kac-Rice formula gives:

$$\mathbb{E}[N^X_{[a,b]}] = \frac{1}{\pi}\sqrt{-R''_X(0)}(b-a)$$

Since $R_X(h) = \int_{\mathbb{R}} e^{i\lambda h} dF(\lambda)$, we have:

$$R''_X(0) = -\int_{\mathbb{R}} \lambda^2 dF(\lambda) = -\sigma_X^2$$

Therefore:

$$\mathbb{E}[N^X_{[a,b]}] = \frac{\sigma_X}{\pi}(b-a)$$

Now consider zeros of $Z(t) = \sqrt{\theta'(t)}X(\theta(t))$. The equation $Z(t) = 0$ is equivalent to $X(\theta(t)) = 0$ since $\sqrt{\theta'(t)} > 0$.

The time-change $t \mapsto \theta(t)$ maps the interval $[0, T]$ in t-coordinates to $[0, \theta(T)]$ in u-coordinates. Each zero of $X(u)$ in $[0, \theta(T)]$ corresponds to a unique zero of $Z(t)$ in $[0, T]$ via $u = \theta(t)$, and conversely.

By the unitary property of the transformation (Theorem 2.1), the measure of the zero set is preserved:

$$\mathbb{E}[N^Z_{[0,T]}] = \mathbb{E}[N^X_{[0,\theta(T)]}] = \frac{\sigma_X}{\pi}\theta(T)$$

□

Corollary 7.3 (Zero Density for Hardy Z-Function)**.** *For the Hardy Z-function with normalized underlying stationary process, where normalization $\sigma_X = 1$ is achieved by appropriate rescaling of the spectral measure $F(\lambda)$, the expected number of zeros up to height T is:*

$$\mathbb{E}[N_{[0,T]}] = \frac{\theta(T)}{\pi}$$

Asymptotically, as $T \to \infty$:

$$\mathbb{E}[N_{[0,T]}] \sim \frac{1}{\pi} \cdot \frac{T}{2} \log \frac{T}{2\pi e} = \frac{T}{2\pi} \log \frac{T}{2\pi e}$$

matching the Riemann-von Mangoldt formula.

Proof. From Theorem 5.3, $\theta(T) = -\frac{\pi}{8} + \frac{T}{2}\log(T/(2\pi e)) + O(T^{-1})$. Dividing by π:

$$\frac{\theta(T)}{\pi} = -\frac{1}{8} + \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(T^{-1}) \sim \frac{T}{2\pi}\log\frac{T}{2\pi e}$$

This matches the classical result. □

8 Conclusion

We have established that unitarily time-changed stationary processes form a proper subclass of oscillatory processes. The gain function is determined explicitly by the time-change derivative and the phase shift. For the Hardy Z-function, constructing the orthogonal random measure via sinc-kernel integrals and applying the sinc-delta identity with Fubini's theorem exactly recovers the Riemann-Siegel formula. The Kac-Rice formula yields the expected zero count in terms of the Riemann-Siegel theta function, reproducing classical results from analytic number theory through this probabilistic spectral framework.

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