Spectral Factorization For Stationary Gaussian Processes and RKHS Expansions

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Theorem 1

[Spectral Factorization] *Let $K(t,s)$ be a positive definite stationary kernel. Then there exists a spectral density $S(\omega)$ and spectral factor:*

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega)}\, e^{i\omega t}\, d\omega \tag{1}$$

such that:

$$K(t,s) = \int_{-\infty}^{\infty} h(t+\tau)\overline{h(s+\tau)}\, d\tau \tag{2}$$

[1]

Proof. 1. By Bochner's theorem, since K is positive definite and stationary:

$$K(t-s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{i\omega(t-s)}\, d\omega \tag{3}$$

where $S(\omega) \geq 0$ is the spectral density.

2. Define $h(t)$ as stated. Then:

$$\begin{aligned}\int_{-\infty}^{\infty} h(t+\tau)\overline{h(s+\tau)}\, d\tau &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega_1)}\, e^{i\omega_1(t+\tau)}\, d\omega_1 \\ &\quad \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega_2)}\, e^{-i\omega_2(s+\tau)}\, d\omega_2\, d\tau\end{aligned} \tag{4}$$

3. Rearranging integrals (justified by Fubini's theorem since $S(\omega) \geq 0$):

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{S(\omega_1)\, S(\omega_2)}\, e^{i\omega_1 t}\, e^{-i\omega_2 s} \int_{-\infty}^{\infty} e^{i(\omega_1-\omega_2)\tau}\, d\tau\, d\omega_1\, d\omega_2 \tag{5}$$

4. The inner integral gives $2\,\pi\,\delta\,(\omega_1-\omega_2)$:

$$=\frac{1}{2\,\pi}\int_{-\infty}^{\infty} S(\omega)\,e^{i\omega\,(t-s)}\,d\,\omega = K\,(t-s) \tag{6}$$

□

Lemma 2

[Convolution Properties] *For any orthonormal basis* $\{e_i\}$ *of* $L^2(\mathbb{R})$*, define:*

$$h_i(t)=\int_{-\infty}^{\infty} h\,(t+\tau)\overline{e_i(\tau)}d\,\tau \tag{7}$$

Then:

$$\langle h_i,h_j\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K\,(\tau_1-\tau_2)\overline{e_i(\tau_1)}e_j(\tau_2)\,d\,\tau_1\,d\,\tau_2 \tag{8}$$

Proof. 1. Expand the inner product:

$$\begin{aligned}\langle h_i,h_j\rangle &= \int_{-\infty}^{\infty} h_i(t)\overline{h_j(t)}d\,t\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\,(t+\tau_1)\overline{h\,(t+\tau_2)}\,\overline{e_i(\tau_1)}e_j(\tau_2)\,d\,\tau_1\,d\,\tau_2\,d\,t\end{aligned} \tag{9}$$

2. Change variables $u=t+\tau_1$:

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)\overline{h\,(u-\tau_1+\tau_2)}\,\overline{e_i(\tau_1)}e_j(\tau_2)\,d\,\tau_1\,d\,\tau_2\,d\,u \tag{10}$$

3. The inner integral gives $K\,(\tau_1-\tau_2)$ by definition of h:

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K\,(\tau_1-\tau_2)\overline{e_i(\tau_1)}e_j(\tau_2)\,d\,\tau_1\,d\,\tau_2$$

□

Proposition 3

[Orthonormal Basis Construction] *The normalized functions:*

$$\psi_i(t)=\frac{h_i(t)}{\sqrt{|h_i|^2}} \tag{11}$$

form an orthonormal basis for the RKHS $\mathcal{H}$*.*

Proof. 1. First verify orthonormality. For any i, j:

$$\begin{aligned}\langle \psi_i, \psi_j \rangle &= \left\langle \frac{h_i}{\sqrt{|h_i|^2}}, \frac{h_j}{\sqrt{|h_j|^2}} \right\rangle \\ &= \frac{1}{\sqrt{|h_i|^2 |h_j|^2}} \int_{-\infty}^{\infty} h_i(t) \overline{h_j(t)} dt \\ &= \frac{\langle h_i, h_j \rangle}{\sqrt{|h_i|^2 |h_j|^2}} = \delta_{ij}\end{aligned} \tag{12}$$

2. For completeness, show any $f \in \mathcal{H}$ can be expanded:

$$f(t) = \sum_i \langle f, \psi_i \rangle \, \psi_i(t) \tag{13}$$

3. This follows from completeness of $\{e_i\}$ in L^2 and the fact that h maps L^2 to $\mathcal{H}$. □

Theorem 4

[Kernel Expansion] *The kernel has the expansion:*

$$K(t, s) = \sum_i h_i(t) \, h_i(s) \tag{14}$$

which converges in the RKHS norm.

Proof. 1. Start with normalized basis expansion:

$$K(t, s) = \sum_i |h_i|^2 \, \psi_i(t) \, \psi_i(s) \tag{15}$$

2. Substitute normalized basis functions:

$$\begin{aligned}&= \sum_i |h_i|^2 \frac{h_i(t)}{\sqrt{|h_i|^2}} \frac{h_i(s)}{\sqrt{|h_i|^2}} \\ &= \sum_i h_i(t) \, h_i(s)\end{aligned} \tag{16}$$

3. Expand using convolution definition:

$$= \sum_i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h\,(t + \tau_1) \, h\,(s + \tau_2) \overline{e_i(\tau_1)} e_i(\tau_2) \, d\tau_1 \, d\tau_2 \tag{17}$$

4. By completeness of $\{e_i\}$:

$$=\int_{-\infty}^{\infty} h(t+\tau)\overline{h(s+\tau)}d\tau = K(t,s)$$ □

Corollary 5

[Basis Independence] *The kernel expansion is independent of the choice of orthonormal basis* $\{e_i\}$.

Proof. Let $\{e_i\}$ and $\{f_i\}$ be two orthonormal bases of $L^2(\mathbb{R})$. The expansions:

$$\begin{aligned} K(t,s) &= \sum_i h_i^e(t)\, h_i^e(s) \\ K(t,s) &= \sum_i h_i^f(t)\, h_i^f(s) \end{aligned} \tag{18}$$

where superscripts indicate the basis used, must be equal as they both equal:

$$\int_{-\infty}^{\infty} h(t+\tau)\overline{h(s+\tau)}d\tau \tag{19}$$

by the Kernel Expansion Theorem, 4. □

Bibliography

[1] Harald Cramér. A contribution to the theory of stochastic processes. *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 2:329–339, 1951.