A Positive Definite Modulated Negative Exponential Kernel

by Stephen Andrew Crowley

November 22, 2024

Table of contents

1 Basis for the Modified Riemann-Siegel Theta Function

The focus is on establishing a basis for the modified Riemann-Siegel theta function $\theta^*(t)$, ensuring it retains the desired properties of positive definiteness and monotonicity.

1.1 Key Properties

1. **Basis Construction**: We establish an orthonormal basis for the modified theta function $\theta^*(t)$, ensuring that it maintains the desired properties.

2. **Positive Definiteness**: The modification of $\theta(t)$ to $\theta^*(t)$ through reflection maintains positive definiteness, enabling us to derive a suitable kernel.

3. **Kernel Properties**: The modified theta function can be expressed in terms of this basis, allowing for the study of its properties and behavior.

1.2 Counting Function and Expectation

The expectation of the counting function is given by:

$$\mathbb{E}[N(T)] = \frac{\theta(T)}{\pi} + 1 \tag{1}$$

The full counting function includes the argument term:

$$N(T) = \frac{\theta(T)}{\pi} + \frac{S(T)}{\pi} + 1 \tag{2}$$

where $S(T) = \arg \zeta\left(\frac{1}{2} + i\,T\right)$ can be expressed as:

$$S(T) = \frac{\ln \zeta\left(\frac{1}{2} + i\,T\right) - \ln \overline{\zeta\left(\frac{1}{2} + i\,T\right)}}{2\,i} \tag{3}$$

For the associated kernel function:

$$K(t, s) = e^{-\frac{(\theta^*(t) - \theta^*(s))^2}{2}} \tag{4}$$

1.3 Implementation Steps

1. Establish the complete orthonormal basis for $\theta^*(t)$
2. Verify positive definiteness of the constructed kernel
3. Prove monotonicity properties
4. Develop numerical methods for computation