A Quadratic Extremal Problem on the Dirichlet Space*

BY STEPHEN D. FISHER

Department of Mathematics, Northwestern University, Evanston, IL 60208, USA

Originally Appeared In Complex Variables, 1995, Vol. 26, pp. 367–380

AMS No. 30D50, 30D45, 30D10, 30C15
Communicated: R. P. Gilbert
(Received October 2, 1993)

Table of contents

It is shown that there is a unique solution F to the problem

$$\lambda = \sup\left\{\mathrm{Re}\int_\Delta F'\,\bar{F}'\,dA\colon \int_\Delta |F'|^2\,dA \le 1\right\} \tag{1}$$

The function F is entire with a number of special properties. The number λ is the reciprocal of the smallest zero of the 0th Bessel function of the first kind.

INTRODUCTION

The Dirichlet space, D, on the open unit disc Δ consists of all analytic functions f

$$f(z) = \sum_{k=1}^{\infty} a_k\,z^k \quad \forall |z| < 1, \quad f(0) = 0 \tag{2}$$

for which the quantity

$$\int_\Delta |f'(z)|^2\,dA(z) = \sum_{k=1}^{\infty} k\,|a_k|^2 =: \|f\|_D^2 \tag{3}$$

*. In memory of Ralph P. Boas, Jr. (1912–1992).

is finite. In connection with a generalization of Harnack's inequality, Boris Korenblum [2] has asked how large the quantity

$$\lambda := \sup_{f \in D} \frac{\mathrm{Re}(\sum_{k=1}^{\infty} a_k\, a_{k+1})}{\sum_{k=1}^{\infty} k|a_k|^2} \tag{4}$$

is and, if possible, to characterize all functions F which attain the value λ in (2). The expression in the numerator in (2) is not a linear function of f but rather quadratic; hence, the title of this paper.

It is simple to show that

$$\sum_i a_k\, a_{k+1} = \int_{\Delta} |F'(z)|^2\, d\, A(z) \tag{5}$$

367