1 Infinite Degrees of Freedom

We now jump to the infinite number of degrees of freedom case. In this case $\mathbb{R}^n$ must be replaced by an infinite dimensional real inner product space $\mathfrak{V}$. A unitary representation U of $\mathfrak{V}$ on a Hilbert space $\mathcal{H}$ is a map from $\mathfrak{V}$ into the set of unitary operators on $\mathcal{H}$ which satisfies:

1. $U(\phi+\psi)=U(\phi)\,U(\psi)$ for all $\phi,\psi\in\mathfrak{V}$;

2. if $\phi_i\to\phi$ in $\mathfrak{V}$, then $U(\phi_i)\,f\to U(\phi)\,f$ for every $f\in\mathcal{H}$.

A $\mathfrak{V}$-representation of the CCR on $\mathcal{H}$ is a pair of unitary representations (U,V) of $\mathfrak{V}$ on $\mathcal{H}$ such that

$$U(\phi)\,V(\psi)=e^{i\langle\phi,\psi\rangle}\,V(\psi)\,U(\phi) \tag{1}$$

for every $\phi,\psi\in\mathfrak{V}$.

Of course, this is a straightforward generalization of the finite number of degrees of freedom case. But now the von Neumann uniqueness theorem does not hold and we can have many inequivalent $\mathfrak{V}$-representations of the CCR. We shall only consider the so-called cyclic representations.

A $\mathfrak{V}$-representation of the CCR is cyclic with cyclic vector $h\in\mathcal{H}$ if

$$\mathcal{H}=\overline{\operatorname{span}}\{V(\phi)\,h\colon\phi\in\mathfrak{V}\} \tag{2}$$

This is the definition of cyclicity used in [?] (and elsewhere); other authors (e.g., [?]) define cyclicity in terms of U and V. Care must be taken here, since the definitions are not equivalent. In this section we give a correspondence between cyclic V-representations of the CCR and certain random fields on $\mathfrak{V}$. We first need the relevant definitions.

Let $\Phi\colon\mathfrak{V}\to R(\Omega,\Sigma,\mu)$ be a full random field. For $\psi\in\mathfrak{V}$, define $\Phi_\psi\colon\mathfrak{V}\to R(\Omega,\Sigma,\mu)$ by $\Phi_\psi(\phi)=\Phi(\phi)-\langle\phi,\psi\rangle$. The random field Φ_ψ corresponds to a translation of Φ by the vector ψ.

Let $m(\Omega,\Sigma,\mu)$ be the group of measurable bijections on Ω. If $T\in m(\Omega,\Sigma,\mu)$ we define $\hat{T}\colon R(\Omega,\Sigma,\mu)\to R(\Omega,\Sigma,\mu)$ by $(\hat{T}f)(\omega)=f(T^{-1}\omega)$. An action of $\mathfrak{V}$ on (Ω,Σ,μ) is a group homomorphism $T\colon\mathfrak{V}\to m(\Omega,\Sigma,\mu)$ such that $\psi_i\to\psi$ in $\mathfrak{V}$ implies that $\hat{T}_{\psi_i}f\to\hat{T}_\psi f$ in probability. An action T is quasi-invariant when $\mu(A)=0$ implies $\mu(T_\psi A)=0$ for every $\psi\in\mathfrak{V}$. If we define the measure $\mu_\psi(A)=\mu(T_\psi A)$, this is equivalent to μ_ψ being absolutely continuous relative to μ for every $\psi\in\mathfrak{V}$.

If T is an action of $\mathfrak{V}$ on (Ω,Σ,μ) and $\Phi\colon\mathfrak{V}\to R(\Omega,\Sigma,\mu)$ is a random field, we denote the random field $\phi\mapsto\hat{T}_\psi[\Phi(\phi)]$ by $\hat{T}_\psi\Phi$ and say that Φ is T-covariant if $\Phi_\psi=\hat{T}_\psi\Phi$ for every $\psi\in\mathfrak{V}$. A covariant random field is a pair (Φ,T) where $\Phi\colon\mathfrak{V}\to R(\Omega,\Sigma,\mu)$ is a random field, T is a quasi-invariant action of $\mathfrak{V}$ on (Ω,Σ,μ) and Φ is T-covariant.

One of the most difficult conditions to verify for a covariant random field (Φ, T) is that T is quasi-invariant. Our first result gives two sufficient conditions for T to be quasi-invariant. If F and F_1 are positive definite functions on $\mathfrak{V}$, we say that F_1 dominates F if there exists an $M > 0$ such that $M F_1 - F$ is positive definite.

Theorem 1. *Let $T: \mathfrak{V} \to m(\Omega, \Sigma, \mu)$ be an action and $\Phi: \mathfrak{V} \to R(\Omega, \Sigma, \mu)$ a random field. Then the following statements are equivalent:*

1. *L_Φ dominates $L_{\hat{T}_\psi \Phi}$ for every $\psi \in \mathfrak{V}$.*
2. *$\hat{T}_\psi$ is a bounded operator from $\mathcal{H} = L^2(\Omega, \Sigma, \mu)$ to itself for every $\psi \in \mathfrak{V}$.*
3. *T is quasi-invariant and $f_\psi = d\mu_\psi / d\mu \in \mathcal{H}$ for every $\psi \in \mathfrak{V}$*

Proof. We first show that 1.1 and 1.2 are equivalent. If 1.1 holds then for every $\psi \in \mathfrak{V}$ there exists an $M_\psi > 0$ such that $M_\psi L_\Phi - L_{\hat{T}_\psi \Phi}$ is positive definite. Since Σ is generated by $\{\Phi(\phi) \colon \phi \in \mathfrak{V}\}$, it follows that $\overline{\operatorname{span}}\{e^{i\Phi(\phi)} \colon \phi \in \mathfrak{V}\} = \mathcal{H}$. Let $Y = \operatorname{span}\{e^{i\Phi(\phi)} \colon \phi \in \mathfrak{V}\}$. We now show that the restriction $\hat{T}_\psi | Y$ is a bounded operator from Y to $\mathcal{H}$:

$$\begin{aligned}
\left|\hat{T}_\psi | Y \sum \lambda_k e^{i\Phi(\phi_k)}\right|^2 &= \left|\sum \lambda_k e^{i\hat{T}_\psi \Phi(\phi_k)}\right|^2 && (3)\\
&= \sum_{j,k} \lambda_j \lambda_k^* \int e^{i\hat{T}_\psi \Phi(\phi_j - \phi_k)} \, d\mu \\
&= \sum_{j,k} \lambda_j \lambda_k^* L_{\hat{T}_\psi \Phi}(\phi_j - \phi_k) \leq M_\psi \sum_{j,k} \lambda_j \lambda_k^* L_\Phi(\phi_j - \phi_k) && (4)\\
&= M_\psi \left|\sum \lambda_k e^{i\Phi(\phi_k)}\right|^2 && (5)
\end{aligned}$$

Thus $\|\hat{T}_\psi | Y\| \leq M_\psi^{1/2}$ and $\hat{T}_\psi | Y$ is bounded. Since $\bar{Y} = \mathcal{H}$, $\hat{T}_\psi | Y$ has a unique bounded extension $S_\psi : \mathcal{H} \to \mathcal{H}$. We next show that $S_\psi = \hat{T}_\psi$ on $\mathcal{H}$. If $f \in \mathcal{H}$, there exists a sequence $f_i \in Y$ such that $f_i \to f$ in norm. Hence $S_\psi f = \lim S_\psi f_i$. Now there exists a subsequence $f_{i'}$ such that $f_{i'} \to f$ almost everywhere. Then

$$\begin{aligned}
(S_\psi f)(\omega) &= \lim (S_\psi f_i)(\omega) = \lim (\hat{T}_\psi f_i)(\omega) = \lim f_{i'}(T_\psi^{-1} \omega) && (6)\\
&= f(T_\psi^{-1} \omega) = (\hat{T}_\psi f)(\omega) && (7)
\end{aligned}$$

Hence 1.2 holds.

Conversely, suppose 1.2 holds and $\phi_1, \ldots, \phi_n \in \mathfrak{V}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Then,

$$\begin{aligned}
\sum_{j,k} \lambda_j \lambda_k^* L_{\hat{T}_\psi \Phi}(\phi_j - \phi_k) &= \left|\sum \lambda_k e^{i\hat{T}_\psi \Phi(\phi_k)}\right|^2 && (8)\\
&= \|\hat{T}_\psi \sum \lambda_k e^{i\Phi(\phi_k)}\|^2 \\
&\leq \|\hat{T}_\psi\|^2 \left|\sum \lambda_k e^{i\Phi(\phi_k)}\right|^2 && (9)\\
&= \|\hat{T}_\psi\|^2 \sum_{j,k} \lambda_j \lambda_k^* L_\Phi(\phi_j - \phi_k)
\end{aligned}$$

Hence 1.1 holds.

Now show that 1.2 and 1.3 are equivalent. If 1.2 holds, then the map $f \mapsto \langle \hat{T}_\psi f, 1\rangle$ is a bounded linear functional on $\mathcal{H}$. By the Riesz theorem there exists an $f_\psi \in \mathcal{H}$ such that $\langle \hat{T}_\psi f, 1\rangle = \langle f, f_\psi\rangle$ for all $f \in \mathcal{H}$. Hence for every $f \in \mathcal{H}$

$$\int f\,(T_\psi^{-1}\,\omega)\,d\,\mu(\omega) = \int f(\omega)\,f_\psi(\omega)\,d\,\mu(\omega). \tag{10}$$

Letting $f = \chi_A$ for $A \in \Sigma$ we obtain

$$\mu_\psi(A) = \mu\,(T_\psi\,A) = \int \chi_A\,(T_\psi^{-1}\,\omega)\,d\,\mu(\omega) = \int_A f_\psi(\omega)\,d\,\mu(\omega). \tag{11}$$

Hence μ_ψ is absolutely continuous relative to μ and $f_\psi = d\,\mu_\psi / d\,\mu \in \mathcal{H}$.

Conversely, suppose 1.3 holds and let $\sum \lambda_j\,\chi_{A_j}$ be a simple function in $\mathcal{H}$, where $A_i \cap A_j = \varnothing$ for $i \neq j$. Then

$$\left|\hat{T}_\psi \sum \lambda_j\,\chi_{A_j}\right|^2 = \int \left|\sum \lambda_j\,\chi_{A_j}\,(T_\psi^{-1}\,\omega)\right|^2 d\,\mu = \int \left|\sum \lambda_j\,\chi_{A_j}(\omega)\right|^2 d\,\mu_\psi \tag{12}$$

$$= \int \left|\sum \lambda_j\,\chi_{A_j}\right|^2 f_\psi\,d\,\mu = \sum |\lambda_j|^2 \int \chi_{A_j}\,f_\psi\,d\,\mu \tag{13}$$

$$\leq \sum |\lambda_j|^2 \|f_\psi\|^2\,\mu(A_j) = \|f_\psi\|^2 \left|\sum \lambda_j\,\chi_{A_j}\right|^2 \tag{14}$$

Hence, the restriction of $\hat{T}_\psi$ to the subspace of simple functions is bounded. Since the simple functions are dense in $\mathcal{H}$, this restriction has a unique bounded extension to $\mathcal{H}$. By an argument similar to that above we conclude that 1.2 holds. □

An inner product space $\mathfrak{V}$ is completely separable if there exists a countable orthonormal set $\{x_i\}$ in $\mathfrak{V}$ such that $\{x_i\}$ is a basis for the completion $\bar{\mathfrak{V}}$ of $\mathfrak{V}$. Clearly, a completely separable inner product space is separable. However, the converse need not hold.

Theorem 2.

(a). Let (Φ, T) be a covariant random field from an inner product space $\mathfrak{V}$ to a probability space (Ω, Σ, μ). Then there exists a random functional Ψ: $\mathfrak{V} \to R(\Omega, \Sigma, \mu)$ such that (U_0, V_0) defined by Eqs. (15) and (16) forms a cyclic $\mathfrak{V}$-representation of the CCR on $\mathcal{H} = L^2(\Omega, \Sigma, \mu)$:

$$[V_0(\phi)\,f](\omega) = e^{-i\Phi(\phi)(\omega)}\,f(\omega), \tag{15}$$

$$(\omega) = [\Psi(\psi)](\omega)\,[\hat{T}_\psi\,f](\omega) \tag{16}$$

(b). Conversely, let $\mathfrak{V}$ be a completely separable inner product space. If (U, V) is a cyclic $\mathfrak{V}$-representation of CCR on a Hilbert space $\mathcal{H}$, then there exists a covariant random field (Φ, T) from $\mathfrak{V}$ to a probability space (Ω, Σ, μ) such that (U, V) is equivalent to (U_0, V_0) defined by Eqs. (15) and (16).

Proof. (a) It is clear that V_0 as defined above is unitary and that $V_0\,(\phi+\psi)=V_0(\phi)\,V_0(\psi)$ for all $\phi,\psi\in\mathfrak{V}$. Strong continuity follows from the dominated convergence theorem.

Since μ_ψ is absolutely continuous relative to μ, by the Radon-Nikodym theorem there exist unique nonnegative functions $f_\psi\in L^1(\Omega,\,\Sigma,\,\mu)$ such that $\mu_\psi(A)=\int_A f_\psi\,d\,\mu$ for all $A\in\Sigma$ and $\psi\in\mathfrak{V}$. Define the random functional $\Psi(\psi)=f_\psi^{1/2}(\omega)$. Then $U_0(\psi)$ is unitary since for all $f,g\in\mathcal{H}$ we have

$$\langle U_0(\psi)\,f,U_0(\psi)\,g\rangle=\int f_\psi(\omega)\,f\,(T_\psi^{-1}\,\omega)\,g^*\,(T_\psi^{-1}\,\omega)\,d\,\mu(\omega) \tag{17}$$

$$=\int f\,(T_\psi^{-1}\,\omega)\,g^*\,(T_\psi^{-1}\,\omega)\,d\,\mu\,(T_\psi^{-1}\,\omega) \tag{18}$$

$$=\int f(\omega)\,g^*(\omega)\,d\,\mu(\omega)=\langle f,g\rangle \tag{19}$$

We now show that $f_{\phi+\psi}(\omega)=f_\phi(\omega)\,f_\psi\,(T_\phi^{-1}\,\omega)$. Indeed, for any $A\in\Sigma$ we have

$$\int_A f_\phi(\omega)\,f_\psi\,(T_\phi^{-1}\,\omega)\,d\,\mu(\omega)=\int_A f_\psi\,(T_\phi^{-1}\,\omega)\,d\,\mu\,(T_\phi^{-1}\,\omega) \tag{20}$$

$$=\int_A d\,\mu\,(T_\psi^{-1}\,T_\phi^{-1}\,\omega) \tag{21}$$

$$=\int_A f_{\phi+\psi}(\omega)\,d\,\mu(\omega)$$

$$=\mu\,(T_{\phi+\psi}\,A)$$

It follows

$$[U_0\,(\phi+\psi)\,f](\omega)=[\Psi\,(\phi+\psi)](\omega)\,[\hat{T}_{\phi+\psi}\,f](\omega) \tag{22}$$

$$=[\Psi(\phi)](\omega)[\Psi(\psi)]\,(T_\phi^{-1}\,\omega)\,f\,(T_{\phi+\psi}^{-1}\omega) \tag{23}$$

$$=[\Psi(\phi)](\omega)\,\{T_\phi\,[\Psi(\psi)(\omega)\,f\,(T_\psi^{-1}\,\omega)]\} \tag{24}$$

$$=[U_0(\phi)\,U_0(\psi)\,f](\omega) \tag{25}$$

Hence $U_0\,(\phi+\psi)=U_0(\phi)\,U_0(\psi)$. The strong continuity of $\phi\mapsto U_0(\phi)$ follows as above.

To show the canonical commutation relations, use the T-covariance of Φ:

$$[U_0(\phi)\,V_0(\psi)\,f](\omega)=\Psi(\phi)\,[V_0(\psi)\,f]\,(T_\phi^{-1}\,\omega) \tag{26}$$

$$=\Psi(\phi)\exp\{-i[\Phi(\psi)]\,(T_\phi^{-1}\,\omega)\}\,f\,(T_\phi^{-1}\,\omega) \tag{27}$$

$$=\Psi(\phi)\exp\{-i[\Phi_\phi(\psi)](\omega)\}\,f\,(T_\phi^{-1}\,\omega) \tag{28}$$

$$=\Psi(\phi)\,e^{-i\Phi(\psi)(\omega)}\,e^{i\,\langle\phi,\psi\rangle}\,f\,(T_\phi^{-1}\,\omega) \tag{29}$$

$$=e^{i\,\langle\phi,\psi\rangle}\,[V_0(\psi)\,U_0(\phi)\,f](\omega) \tag{30}$$

Thus, the CCR (1) holds. The indicator function 1 is a cyclic vector since $\mathrm{span}\{e^{i\Phi(\phi)}\colon \phi\in\mathfrak{V}\}$ is dense.

(b) We first show that (U,V) has a unique extension to a cyclic $\bar{\mathfrak{V}}$-representation of the CCR on $\mathcal{H}$. For $\phi\in\bar{\mathfrak{V}}$, let ϕ_i be a sequence in $\mathfrak{V}$ converging to ϕ. Now $U(\phi_i)$ is strongly Cauchy in $\mathcal{H}$ since for every $f\in\mathcal{H}$

$$\lim_{i,j\to\infty}\|U(\phi_i)\,f-U(\phi_j)\,f\|=\lim_{i,j\to\infty}\|f-U\,(\phi_j-\phi_i)\,f\|=0.$$

Defining $U(\phi)\,f=\lim U(\phi_i)\,f$ gives a well-defined linear operator which is bounded by the uniform boundedness theorem. Extend V to $\bar{\mathfrak{V}}$ similarly. By taking limits it is straightforward to show that (U,V) extended in this way gives a cyclic $\bar{\mathfrak{V}}$-representation of the CCR on $\mathcal{H}$. Let $\{\psi_i\}$ be an orthonormal basis for $\bar{\mathfrak{V}}$ where $\psi_i\in\mathfrak{V}$, $i=1,2,\ldots$. Let $\{f_i\}$ be an orthonormal basis for $L^2(\mathbb{R},d\,x)$ where f_i are in the Schwartz space, and define the isomorphism $J\colon\bar{\mathfrak{V}}\to L^2(\mathbb{R},d\,x)$ by $J(\psi_i)=f_i$. Then $(U\circ J^{-1},V\circ J^{-1})$ is a cyclic $L^2(\mathbb{R},d\,x)$-representation of the CCR.

Applying a theorem due to Gelfand-Vilenkin [?], there exist:

1. a unique Borel probability measure μ on the dual $\mathcal{S}'(\mathbb{R})$ such that for every Borel set A and every $\phi\in\mathcal{S}(\mathbb{R})$, $\mu(A)=0$ implies $\mu\,(A+f_\phi)=0$, where $f_\phi(\psi)=\langle\psi,\phi\rangle$ for every $\psi\in\mathcal{S}(\mathbb{R})$;

2. for every $\phi\in\mathcal{S}(\mathbb{R})$ there exists a functional F_ϕ on $\mathcal{S}'(\mathbb{R})$ such that (U_1, V_1) is a cyclic $\mathcal{S}(\mathbb{R})$-representation of the CCR on $L^2(\mathcal{S}'(\mathbb{R}),\mu)$ where $[V_1(\phi)\,F](f)=e^{-if(\phi)}\,F(f)$ and $[U_1(\phi)\,F](f)=F_\phi(f)\,F\,(f+f_\phi)$;

3. an isomorphism $M\colon\mathcal{H}\to L^2(\mathcal{S}'(\mathbb{R}),\mu)$ with $MU\circ J^{-1}\,M^{-1}=U_1$, and $MV\circ J^{-1}\,M^{-1}=V_1$.

Define the random field $\Phi\colon\mathfrak{V}\to(\mathcal{S}'(\mathbb{R}),\mu)$ by $[\Phi(\phi)](f)=f\,(J\,\phi)$, the random functional $\Psi(\phi)=F_{J\phi}(f)$, and the action $T_\phi\,f=f+f_{J\phi}$. Now define $U_0(\phi)=U_1\,(J\,\phi)$, $V_0(\phi)=V_1\,(J\,\phi)$. Then, (Φ,T) and (U_0,V_0) satisfy the conditions of the theorem. Furthermore, (U,V) is equivalent to (U_0,V_0). □

There is another approach to representations of the CCR which is frequently used. Let (U,V) be a $\mathfrak{V}$-representation of the Weyl form of the CCR on $\mathcal{H}$. For notational convenience we replace (U,V) by a single representation operator W over the complexification $\mathfrak{V}_c$ of $\mathfrak{V}$.

To be precise, $\mathfrak{V}_c$ is the set of ordered pairs $\{(\phi,\psi)\colon\phi,\psi\in\mathfrak{V}\}$. We define addition componentwise:

$$(\phi,\psi)+(\phi_1,\psi_1)=(\phi+\phi_1,\psi+\psi_1) \tag{31}$$

and if $\alpha+i\,\beta\in\mathbb{C}$, $\alpha,\beta\in\mathbb{R}$,

$$(\alpha+i\,\beta)\,(\phi,\psi)=(\alpha\,\phi,\alpha\,\psi)+(-\beta\,\psi,\beta\,\phi). \tag{32}$$

It is straightforward to show that $\mathfrak{V}_c$ is a complex linear space. We define an inner product on $\mathfrak{V}_c$ by

$$\langle(\phi,\psi),(\phi_1,\psi_1)\rangle = \langle\phi,\phi_1\rangle + \langle\psi,\psi_1\rangle + i\,(\langle\psi,\phi_1\rangle - \langle\phi,\psi_1\rangle) \tag{33}$$

Now let (U,V) be a $\mathfrak{V}$-representation of the CCR on $\mathcal{H}$. For $\phi + i\,\psi \in \mathfrak{V}_c$, define

$$W(\phi + i\,\psi) = e^{-i\langle\phi,\psi\rangle/2}\,U(\phi)\,V(\psi). \tag{34}$$

Lemma 3.

1. *$W(\phi)$ is a unitary operator on $\mathcal{H}$ satisfying*

   (a). $\phi \mapsto W(\phi)$ is continuous in the strong operator topology,

   (b). $W(\phi)^* = W(-\phi)$ for all $\phi \in \mathfrak{V}_c$,

   (c). $W(\phi)\,W(\psi) = W(\phi+\psi)\,e^{i\mathrm{Im}\langle\phi,\psi\rangle/2}$ for all $\phi,\psi \in \mathfrak{V}_c$.

2. *Conversely, if $W(\phi)$ is a unitary operator on $\mathcal{H}$ satisfying (a), (b), and (c) and if we define $U(\phi) = W(\phi)$, $V(\phi) = W(i\,\phi)$ for every $\phi \in \mathfrak{V}$, then (U,V) is a $\mathfrak{V}$-representation of the CCR.*

We call a map $\phi \mapsto W(\phi)$ satisfying (a), (b), (c) a complex $\mathfrak{V}_c$-representation of the CCR on $\mathcal{H}$. We say that W is cyclic with cyclic vector $h \in \mathcal{H}$ if $\mathrm{span}\{W(\phi)\,h\colon \phi \in \mathfrak{V}_c\}$ is dense in $\mathcal{H}$.

Let W be a cyclic complex $\mathfrak{V}_c$-representation of the CCR on $\mathcal{H}$ with unit cyclic vector $h \in \mathcal{H}$. Define $w\colon \mathfrak{V}_c \to \mathbb{C}$ by $w(\phi) = \langle W(\phi)\,h, h\rangle$. We would now like to find the properties of w. First, since W is continuous we have

1. w is continuous.

2. $w(0) = 1$.

3. For $\phi_i \in \mathfrak{V}_c$ and $\lambda_i \in \mathbb{C}$, $i = 1,\ldots,n$,

$$\sum_{j,k=1}^{n} \lambda_j\,\lambda_k^*\,w(\phi_j - \phi_k)\,e^{i\mathrm{Im}\langle\phi_j,\phi_k\rangle/2} \geq 0. \tag{35}$$

Theorem 4. *If $w\colon \mathfrak{V}_c \to \mathbb{C}$ satisfies properties (1)–(3) above, then there exists a unique (up to unitary equivalence) cyclic complex $\mathfrak{V}$-representation W of the CCR with cyclic vector h such that $w(\phi) = \langle W(\phi)\,h, h\rangle$ for every $\phi \in \mathfrak{V}_c$.*

Proof. Any $f\in\Delta(\mathfrak{V}_c)$ can be written as $f=\sum_{j=1}^n \lambda_j\,\chi_{\{\phi_j\}}$, $0\neq\phi_j\in\mathfrak{V}_c$ distinct. Define $w(f)=\sum_{j=1}^n \lambda_j\,w(\phi_j)$. Then w is linear on $\Delta(\mathfrak{V}_c)$ and, moreover, w is positive since

$$w\big[\big(\sum \lambda_j\,\chi_{\{\phi_j\}}\big)^*\big(\sum \lambda_j\,\chi_{\{\phi_j\}}\big)\big]=\sum_{j,k} \lambda_j\,\lambda_k^*\,w\,(\phi_j-\phi_k)\,e^{i\mathrm{Im}\langle\phi_j,\phi_k\rangle/2}\geq 0. \tag{36}$$

Also $w(\chi_{\{0\}})=w(0)=1$. Applying Schwarz's inequality, $|w(\chi_{\{\phi\}})|^2\leq w\,(\chi_{\{\phi\}}^*\,\chi_{\{\phi\}})=1$. Hence for $f=\sum\lambda_i\,\chi_{\{\phi_i\}}$,

$$|w(f)|\leq\sum|\lambda_i|.$$

Thus w is continuous on $\Delta(\mathfrak{V}_c)$ and extends to a state on its Banach-* algebra completion. By the GNS construction there exists a unique (up to equivalence) cyclic representation π of this Banach-* algebra with cyclic vector h such that $\langle\pi(f)\,h,h\rangle=w(f)$. Define $W(\phi)=\pi(\chi_{\{\phi\}})$. The required properties (a), (b), (c) are verified from the algebraic relations. □

As an application of Theorem 4, we outline a proof of the von Neumann uniqueness theorem. If $\mathfrak{V}$ is one-dimensional we can assume $\mathfrak{V}=\mathbb{R}$, so $\mathfrak{V}_c=\mathbb{C}$. Let W be an irreducible complex $\mathbb{R}$-representation of the CCR on $\mathcal{H}$. Then $W(\lambda)$ is a unitary operator on $\mathcal{H}$, $\lambda\in\mathbb{C}$, satisfying:

(a'). $\lambda\mapsto W(\lambda)$ is strongly continuous;

(b'). $W(\lambda)^*=W\,(-\lambda)$ for all $\lambda\in\mathbb{C}$;

(c'). $W(\alpha)\,W(\beta)=e^{i\mathrm{Im}\alpha\bar{\beta}/2}\,W\,(\alpha+\beta)$.

Define the linear operator A on $\mathcal{H}$ by

$$\langle A\,x,y\rangle=(2\,\pi)^{-1/2}\int\langle W(\lambda)\,x,y\rangle\,e^{-|\lambda|^2/4}\,d\,\lambda \tag{37}$$

where $x,y\in\mathcal{H}$, and $\lambda=s+i\,t$, $s,t\in\mathbb{R}$. Since $W(\lambda)$ is unitary,

$$|\langle A\,x,y\rangle|\leq(2\,\pi)^{-1/2}\int e^{-(s^2+t^2)/4}\,d\,s\,d\,t\,\|x\|\|y\|=\|x\|\|y\| \tag{38}$$

and hence A is bounded and $\|A\|\leq 1$. Applying (b') gives A is self-adjoint. Using a straightforward calculation one can show that

$$A\,W(\lambda)\,A=A\,e^{-|\lambda|^2/4} \tag{39}$$

for every $\lambda \in \mathbb{C}$. Moreover, $A \neq 0$. Letting $\lambda = 0$ in (39), A is a nonzero projection. Hence there exists a unit vector $x \in \mathcal{H}$ such that $A\,x = x$. Since W is irreducible, any nonzero vector in $\mathcal{H}$ is cyclic so, in particular, x is cyclic. Define $w: \mathbb{C} \to \mathbb{C}$ by $w(\lambda) = \langle W(\lambda)\,x, x \rangle$. Then, applying (39),

$$w(\lambda) = \langle A\,W(\lambda)\,A\,x, x \rangle = e^{-|\lambda|^2/4}. \tag{40}$$

Now let W_0 be the complex $\mathbb{R}$-representation of the CCR on $L^2(\mathbb{R})$ given by

$$W_0\,(s + i\,t)\,f(u) = e^{ist/2}\,e^{itu}\,f\,(u - s). \tag{41}$$

If $x_0(u) = \pi^{-1/4}\,e^{-u^2/2}$, then

$$w_0(\lambda) = \langle W_0(\lambda)\,x_0, x_0 \rangle = e^{-|\lambda|^2/4}. \tag{42}$$

It follows from Theorem 4 that W and W_0 are unitarily equivalent.

The generalization to any finite number of degrees of freedom is straightforward. However, this proof (and the result) cannot be generalized to an infinite number of degrees of freedom since in that case the measure

$$A \mapsto (2\,\pi)^{-1/2} \int_A e^{-(s^2+t^2)/4}\,d\,s\,d\,t$$

has no infinite-dimensional analog.