Unitary Bijections From Strictly Increasing Functions On The Real Line

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1 Introduction

This short note establishes the fundamental relationship between unitary bijections in L^2 spaces and measure-preserving transformations in ergodic theory. Under bijective C^1 changes of variables on unbounded domains, L^2 norm preservation is achieved by a weighted composition operator whose weight is the square root of the Jacobian g'. This unitary transformation is both necessary and sufficient.

2 Bijective Transformations on Unbounded Domains

Theorem 1. (Bijectivity of Strictly Increasing Functions on Unbounded Domains) *Let $g: I \to \mathbb{R}$ be a strictly increasing function where $I \subseteq \mathbb{R}$ is an unbounded interval. Then g is bijective onto its range $J = g(I)$, and J is also an unbounded interval.*

Proof. Since g is strictly increasing, injectivity is immediate. For any $x_1, x_2 \in I$ with $x_1 < x_2$, one has $g(x_1) < g(x_2)$.

For surjectivity onto $J = g(I)$, let $y \in J$. By definition, there exists $x \in I$ such that $g(x) = y$. The uniqueness of such x follows from injectivity.

To establish that J is unbounded, consider two cases:

1. If $I=(a,\infty)$ or $I=[a,\infty)$ for some $a\in\mathbb{R}$, then as $x\to\infty$, since g is strictly increasing, either $g(x)\to\infty$ or $g(x)$ approaches some finite supremum. If the latter held, then by the intermediate value theorem and strict monotonicity, g would map (a,∞) to some bounded interval, contradicting the strict increase property over an unbounded domain.

2. If $I=(-\infty,b)$ or $I=(-\infty,b]$, a similar argument shows J extends to $-\infty$.

3. If $I=\mathbb{R}$, then J must be unbounded in both directions.

Therefore, $g\colon I\to J$ is bijective with both I and J unbounded intervals. □

Theorem 2. (Differentiable Bijections with Positive Derivative) *Let $g\colon I\to J$ be a C^1 bijection between unbounded intervals $I,J\subseteq\mathbb{R}$ such that $g'(y)>0$ for all $y\in I$ except possibly on a set of measure zero. Then g is a well-defined change of variables for Lebesgue integration.*

Proof. The condition $g'(y)>0$ almost everywhere ensures that g is locally invertible almost everywhere. Since g is already assumed bijective and C^1, the standard change of variables formula applies:

$$\int_J f(x)\,dx=\int_I f(g(y))|g'(y)|\,dy=\int_I f(g(y))\,g'(y)\,dy \tag{1}$$

where the last equality uses $g'(y)>0$ almost everywhere. The points where $g'(y)=0$ form a set of measure zero and do not affect the integral. □

3 L^2 Norm Preservation

Definition 3. (Unitary Change of Variables Operator) *Let $g\colon I\to J$ be a C^1 bijection between unbounded intervals with $g'(y)>0$ almost everywhere. For $f\in L^2(J,dx)$, define the unitary change of variables operator T_g by:*

$$(T_g f)(y)=f(g(y))\sqrt{g'(y)} \tag{2}$$

Theorem 4. (L^2 Norm Preservation for Unbounded Domains) *Under the conditions of Definition 3, the operator $T_g\colon L^2(J,dx)\to L^2(I,dy)$ is an isometric isomorphism. Specifically:*

$$\|T_g f\|_{L^2(I,dy)}=\|f\|_{L^2(J,dx)} \tag{3}$$

Proof. For $f \in L^2(J, dx)$, compute directly:

$$\|T_g f\|_{L^2(I,dy)}^2 = \int_I |f(g(y))\sqrt{g'(y)}|^2 \, dy \tag{4}$$

$$= \int_I |f(g(y))|^2 \, g'(y) \, dy \tag{5}$$

By the change of variables formula from Theorem 2 with $x = g(y)$:

$$\int_I |f(g(y))|^2 g'(y) \, dy = \int_J |f(x)|^2 \, dx = \|f\|_{L^2(J,dx)}^2 \tag{6}$$

Since both I and J are unbounded, the change of variables is justified by approximating with bounded subintervals and applying the monotone convergence theorem.

Therefore:

$$\|T_g f\|_{L^2(I,dy)} = \|f\|_{L^2(J,dx)} \tag{7}$$

The fact that $T_g f \in L^2(I, dy)$ follows immediately from equation (7) and the assumption $f \in L^2(J, dx)$. $\square$

Theorem 5. (Necessity of Square Root Unitary Transformation) *Let $g: I \to J$ be as in Theorem 4. If $\phi: I \to \mathbb{R}^+$ is any measurable function such that $f(g(y))\,\phi(y) \in L^2(I, dy)$ and*

$$\|f(g(\cdot))\,\phi(\cdot)\|_{L^2(I,dy)} = \|f\|_{L^2(J,dx)} \tag{8}$$

for all $f \in L^2(J, dx)$, then $\phi(y) = \sqrt{g'(y)}$ almost everywhere.

Proof. From the norm condition in equation (8):

$$\int_I |f(g(y))|^2 \phi(y)^2 \, dy = \int_J |f(x)|^2 \, dx \tag{9}$$

Using the change of variables $x = g(y)$ on the right side:

$$\int_I |f(g(y))|^2 \phi(y)^2 \, dy = \int_I |f(g(y))|^2 \, g'(y) \, dy \tag{10}$$

This gives:

$$\int_I |f(g(y))|^2 \, (\phi(y)^2 - g'(y)) \, dy = 0 \tag{11}$$

Since this holds for all $f \in L^2(J, d\,x)$ and the composition $f(g(\,\cdot\,))$ generates a dense subspace of $L^2(I, g'(y)\ d\,y)$, the fundamental lemma of calculus of variations implies:

$$\phi(y)^2 = g'(y)\text{almost everywhere} \tag{12}$$

Taking $\phi(y) > 0$, one obtains $\phi(y) = \sqrt{g'(y)}$ almost everywhere. □

4 Conclusion

The results show that L^2 norm preservation under C^1 bijections is realized by the weighted composition operator $T_g\,f = f(g(y))\sqrt{g'(y)}$. The factor $\sqrt{g'}$ is both necessary and sufficient for isometry, linking the change-of-variables formula to unitary structure on L^2.

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