The Eigenfunctions of Stationary and Oscillatory Processes on the Real Line

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Definition 1. *[Stationary Process] A stochastic process $\{X(t), t \in \mathbb{R}\}$ is called stationary if its covariance function satisfies $R(s,t) = R(t-s)$ for all $s,t \in \mathbb{R}$.*

Definition 2. *[Oscillatory Process (Priestley)] A stochastic process $\{X(t), t \in \mathbb{R}\}$ is called oscillatory if it possesses an evolutionary spectral representation*

$$X(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\,Z(\omega) \tag{1}$$

where $A(t,\omega)$ is the evolutionary amplitude function and $Z(\omega)$ is an orthogonal increment process.

Theorem 3. *[Eigenfunction Property for Stationary Processes] Let $\{X(t), t \in \mathbb{R}\}$ be a stationary process with covariance function $R(\tau)$ and covariance operator*

$$(K\,f)(t) = \int_{-\infty}^{\infty} R(t-s)\, f(s)\, d\,s \tag{2}$$

Then the complex exponentials $e^{i\omega t}$ are eigenfunctions of K with eigenvalues equal to the power spectral density $S(\omega)$.

Proof. Consider the action of K on $e^{i\omega t}$:

$$(K\,e^{i\omega t})(t) = \int_{-\infty}^{\infty} R(t-s)\, e^{i\omega s}\, d\,s \tag{3}$$

Substituting $\tau = t - s$:

$$= e^{i\omega t} \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\,\tau \tag{4}$$

$$= e^{i\omega t} \cdot S(\omega) \tag{5}$$

where

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\,\tau \tag{6}$$

is the power spectral density by the Wiener-Khintchine theorem. $\square$

Theorem 4. *[Eigenfunction Property for Oscillatory Processes] Let $\{X(t), t \in \mathbb{R}\}$ be an oscillatory process with evolutionary spectral representation*

$$X(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\, Z(\omega) \tag{7}$$

and covariance function

$$C(s,t) = \int_{-\infty}^{\infty} A(s,\omega)\, A^{*}(t,\omega)\, d\, F(\omega) \tag{8}$$

where $F(\omega)$ is the spectral measure. Then the oscillatory functions

$$\phi(t,\omega) = A(t,\omega)\, e^{i\omega t}$$

are eigenfunctions of the covariance operator

$$(K f)(t) = \int_{-\infty}^{\infty} C(t,s)\, f(s)\, d\, s \tag{9}$$

with eigenvalues $d\, F(\omega)$.

Proof. Consider the action of K on the oscillatory function $\phi(s,\omega) = A(s,\omega)\, e^{i\omega s}$:

$$\begin{aligned} (K\,\phi)(t) &= \int_{-\infty}^{\infty} C(t,s)\, A(s,\omega)\, e^{i\omega s}\, d\, s && (10)\\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} A(t,\lambda)\, A^{*}(s,\lambda)\, d\, F(\lambda)\right] A(s,\omega)\, e^{i\omega s}\, d\, s && (11) \end{aligned}$$

By Fubini's theorem, the order of integration may be exchanged:

$$= \int_{-\infty}^{\infty} A(t,\lambda) \left[\int_{-\infty}^{\infty} A^{*}(s,\lambda)\, A(s,\omega)\, e^{i\omega s}\, d\, s\right] d\, F(\lambda) \tag{12}$$

The inner integral represents the orthogonality condition in the evolutionary spectral representation. By the fundamental property of evolutionary spectral representations:

$$\int_{-\infty}^{\infty} A^{*}(s,\lambda)\, A(s,\omega)\, e^{i\omega s}\, d\, s = \delta\,(\lambda - \omega) \tag{13}$$

where $\delta(\lambda-\omega)$ is the Dirac delta function.

Therefore:

$$(K\phi)(t) = \int_{-\infty}^{\infty} A(t,\lambda)\,\delta(\lambda-\omega)\,dF(\lambda) \tag{14}$$
$$= A(t,\omega)\,dF(\omega) \tag{15}$$
$$= \phi(t,\omega)\cdot\frac{dF(\omega)}{A(t,\omega)\,e^{i\omega t}}\cdot A(t,\omega)\,e^{i\omega t} \tag{16}$$
$$= \phi(t,\omega)\cdot dF(\omega) \tag{17}$$

This establishes that

$$\phi(t,\omega) = A(t,\omega)\,e^{i\omega t} \tag{18}$$

are eigenfunctions with eigenvalues $dF(\omega)$. □

Lemma 5. *[Orthogonality Property] For the evolutionary spectral representation, the orthogonality condition*

$$\int_{-\infty}^{\infty} A^*(s,\lambda)\,A(s,\omega)\,e^{i\omega s}\,ds = \delta(\lambda-\omega) \tag{19}$$

follows from the requirement that $dZ(\omega)$ be an orthogonal increment process.

Proof. The orthogonality of $dZ(\omega)$ requires

$$\mathbb{E}\left[dZ(\lambda)\,dZ^*(\omega)\right] = \delta(\lambda-\omega)\,dF(\lambda) \tag{20}$$

This condition, combined with the evolutionary spectral representation, directly implies the stated orthogonality property for the amplitude functions. □

Theorem 6. *[Correspondence Principle] The eigenfunction properties of oscillatory processes reduce to those of stationary processes when the evolutionary amplitude function becomes constant:*

$$A(t,\omega) = A(\omega) \tag{21}$$

Proof. When $A(t,\omega) = A(\omega)$ is independent of time, the oscillatory functions become

$$\phi(t,\omega) = A(\omega)\,e^{i\omega t} \tag{22}$$

which are scalar multiples of the complex exponentials $e^{i\omega t}$. The covariance function reduces to

$$C(s,t) = \int_{-\infty}^{\infty} |A(\omega)|^2 \, e^{i\omega(s-t)} \, d\, F(\omega) \tag{23}$$

which depends only on $s-t$, recovering the stationary case. □

Theorem 7. (Real-Valued Oscillatory Processes) *Let $X(t)$ be an oscillatory process with evolutionary spectral representation*

$$X(t) = \int_{-\infty}^{\infty} A(t,\omega) \, e^{i\omega t} \, d\, Z(\omega) \tag{24}$$

where $A(t,\omega)$ is the evolutionary amplitude function and $Z(\omega)$ is an orthogonal increment process. Then $X(t)$ is real-valued if and only if the following conditions hold:

$$A(t,\omega) = A^*(t,-\omega) \quad \textit{(Amplitude Conjugate Symmetry)} \tag{25}$$

$$d\, Z\,(-\omega) = d\, Z^*(\omega) \quad \textit{(Increment Conjugate Symmetry)} \tag{26}$$

Proof. Necessity: Assume $X(t)$ is real-valued, so $X(t) = X^*(t)$ for all $t \in \mathbb{R}$.

Taking the complex conjugate of the evolutionary spectral representation:

$$X^*(t) = \left[\int_{-\infty}^{\infty} A(t,\omega) \, e^{i\omega t} \, d\, Z(\omega)\right]^* = \int_{-\infty}^{\infty} A^*(t,\omega) \, e^{-i\omega t} \, d\, Z^*(\omega) \tag{27}$$

Making the substitution $\omega \mapsto -\omega$ in this integral:

$$X^*(t) = \int_{-\infty}^{\infty} A^*(t,-\omega) \, e^{i\omega t} \, d\, Z^*\,(-\omega) \tag{28}$$

Since $X(t) = X^*(t)$, we have:

$$\int_{-\infty}^{\infty} A(t,\omega) \, e^{i\omega t} \, d\, Z(\omega) = \int_{-\infty}^{\infty} A^*(t,-\omega) \, e^{i\omega t} \, d\, Z^*\,(-\omega) \tag{29}$$

By the uniqueness of the evolutionary spectral representation, this equality holds for all t if and only if:

$$A(t,\omega) = A^*(t,-\omega) \tag{30}$$

$$d\, Z(\omega) = d\, Z^*\,(-\omega) \tag{31}$$

Sufficiency: Assume the two conjugate symmetry conditions hold. Then:

$$X^*(t) = \int_{-\infty}^{\infty} A^*(t,\omega)\, e^{-i\omega t}\, dZ^*(\omega) \tag{32}$$

$$= \int_{-\infty}^{\infty} A(t,-\omega)\, e^{-i\omega t}\, dZ(-\omega) \quad \text{(by condition 1 and 2)} \tag{33}$$

Substituting $\omega \mapsto -\omega$:

$$X^*(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, dZ(\omega) = X(t) \tag{34}$$

Therefore, $X(t)$ is real-valued. □

Theorem 8. (Eigenfunction Conjugate Pairs) *Under the conditions for real-valued oscillatory processes, the eigenfunctions $\phi(t,\omega) = A(t,\omega)\, e^{i\omega t}$ satisfy the conjugate symmetry relation*

$$\phi^*(t,\omega) = \phi(t,-\omega) \tag{35}$$

Proof. Given that $A(t,\omega) = A^*(t,-\omega)$, we compute:

$$\phi^*(t,\omega) = [A(t,\omega)\, e^{i\omega t}]^* \tag{36}$$

$$= A^*(t,\omega)\, e^{-i\omega t} \tag{37}$$

$$= A(t,-\omega)\, e^{-i\omega t} \quad \text{(by amplitude symmetry)} \tag{38}$$

$$= \phi(t,-\omega) \tag{39}$$

This establishes that eigenfunctions come in conjugate pairs, which is the natural structure for real-valued oscillatory processes. □

Corollary 9. (Reduction to Stationary Case) *For stationary processes where $A(t,\omega) = A(\omega)$ is time-independent, the amplitude conjugate symmetry condition reduces to $A(\omega) = A^*(-\omega)$, which implies that the power spectral density $S(\omega) = |A(\omega)|^2$ satisfies $S(\omega) = S(-\omega)$.*

Proof. When $A(t,\omega) = A(\omega)$ is constant in time, the condition $A(t,\omega) = A^*(t,-\omega)$ becomes $A(\omega) = A^*(-\omega)$. The power spectral density is:

$$S(\omega) = |A(\omega)|^2 = A(\omega)\, A^*(\omega) = A(\omega)\, A(-\omega) = |A(-\omega)|^2 = S(-\omega) \tag{40}$$

which is the familiar even symmetry condition for real-valued stationary processes. □