Band-Limited White Noise: Mathematical Formulation and Properties

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1 Fundamental Definitions

Definition 1. *[Band-Limited White Noise]A zero-mean Gaussian stochastic process* $\{W_B(t), t \in \mathbb{R}\}$ *is called band-limited white noise with bandwidth* $B > 0$ *if its power spectral density is given by*

$$S_{W_B}(\omega) = \begin{cases} \frac{N_0}{2}, & |\omega| \leq B \\ 0, & |\omega| > B \end{cases} \tag{1}$$

where $N_0 > 0$ *is the spectral level parameter.*

Definition 2. *[Sinc Function]The sinc function is defined as*

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(x)}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \tag{2}$$

2 Spectral and Covariance Properties

Theorem 3. *[Autocovariance Function]The autocovariance function of band-limited white noise $W_B(t)$ is given by*

$$R_{W_B}(\tau) = \frac{N_0\,B}{2\,\pi}\mathrm{sinc}(B\,\tau) \tag{3}$$

Proof. By the Wiener-Khintchine theorem, the autocovariance function is the inverse Fourier transform of the power spectral density:

$$\begin{aligned} R_{W_B}(\tau) &= \frac{1}{2\,\pi}\int_{-\infty}^{\infty} S_{W_B}(\omega)\,e^{i\omega\tau}\,d\,\omega &(4)\\ &= \frac{1}{2\,\pi}\int_{-B}^{B}\frac{N_0}{2}\,e^{i\omega\tau}\,d\,\omega &(5)\\ &= \frac{N_0}{4\,\pi}\int_{-B}^{B} e^{i\omega\tau}\,d\,\omega &(6)\end{aligned}$$

For $\tau \neq 0$:

$$\begin{aligned} R_{W_B}(\tau) &= \frac{N_0}{4\,\pi}\left[\frac{e^{i\omega\tau}}{i\,\tau}\right]_{-B}^{B} &(7)\\ &= \frac{N_0}{4\,\pi\,i\,\tau}\,(e^{iB\tau} - e^{-iB\tau}) &(8)\\ &= \frac{N_0}{4\,\pi\,i\,\tau}\cdot 2\,i\sin\,(B\,\tau) &(9)\\ &= \frac{N_0}{2\,\pi\,\tau}\sin\,(B\,\tau) &(10)\\ &= \frac{N_0\,B}{2\,\pi}\frac{\sin\,(B\,\tau)}{B\,\tau} &(11)\\ &= \frac{N_0\,B}{2\,\pi}\mathrm{sinc}(B\,\tau) &(12)\end{aligned}$$

For $\tau = 0$:

$$\begin{aligned} R_{W_B}(0) &= \frac{N_0}{4\,\pi}\int_{-B}^{B} d\,\omega &(13)\\ &= \frac{N_0}{4\,\pi}\cdot 2\,B &(14)\\ &= \frac{N_0\,B}{2\,\pi} &(15)\end{aligned}$$

Since $\mathrm{sinc}(0) = 1$, we have $R_{W_B}(0) = \frac{N_0\,B}{2\,\pi}\mathrm{sinc}(0) = \frac{N_0\,B}{2\,\pi}$.

Therefore, equation (3) holds for all $\tau \in \mathbb{R}$. □

Theorem 4. *[Variance and Power]The variance of band-limited white noise $W_B(t)$ is*

$$Var[W_B(t)] = R_{W_B}(0) = \frac{N_0\, B}{2\,\pi} \tag{16}$$

Proof. This follows directly from Theorem 3 by setting $\tau = 0$. □

3 Construction and Filtering Properties

Theorem 5. *[Filter Construction]Let $W(t)$ be ideal white noise with power spectral density $S_W(\omega) = N_0/2$ for all $\omega \in \mathbb{R}$. Let $H(\omega)$ be the frequency response of an ideal low-pass filter:*

$$H(\omega) = \begin{cases} 1, & |\omega| \leq B \\ 0, & |\omega| > B \end{cases} \tag{17}$$

Then the output process $Y(t) = (H * W)(t)$ is band-limited white noise with bandwidth B.

Proof. The power spectral density of the output process is given by

$$\begin{aligned} S_Y(\omega) &= |H(\omega)|^2\, S_W(\omega) && (18) \\ &= |H(\omega)|^2 \frac{N_0}{2} && (19) \end{aligned}$$

For $|\omega| \leq B$: $H(\omega) = 1$, so $S_Y(\omega) = \frac{N_0}{2}$.

For $|\omega| > B$: $H(\omega) = 0$, so $S_Y(\omega) = 0$.

Therefore:

$$S_Y(\omega) = \begin{cases} \frac{N_0}{2}, & |\omega| \leq B \\ 0, & |\omega| > B \end{cases} \tag{20}$$

This matches the definition of band-limited white noise in Definition 1. □

Theorem 6. *[Impulse Response]The impulse response of the ideal low-pass filter in Theorem 5 is*

$$h(t) = \frac{B}{\pi} \mathrm{sinc}(B\, t) \tag{21}$$

Proof. The impulse response is the inverse Fourier transform of the frequency response:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\, e^{i\omega t}\, d\omega \tag{22}$$

$$= \frac{1}{2\pi} \int_{-B}^{B} e^{i\omega t}\, d\omega \tag{23}$$

For $t \neq 0$:

$$h(t) = \frac{1}{2\pi} \left[\frac{e^{i\omega t}}{i\,t} \right]_{-B}^{B} \tag{24}$$

$$= \frac{1}{2\pi\, i\, t} (e^{iBt} - e^{-iBt}) \tag{25}$$

$$= \frac{1}{2\pi\, i\, t} \cdot 2\, i \sin(B\, t) \tag{26}$$

$$= \frac{\sin(B\, t)}{\pi\, t} \tag{27}$$

$$= \frac{B}{\pi} \frac{\sin(B\, t)}{B\, t} \tag{28}$$

$$= \frac{B}{\pi} \mathrm{sinc}(B\, t) \tag{29}$$

For $t = 0$:

$$h(0) = \frac{1}{2\pi} \int_{-B}^{B} d\omega \tag{30}$$

$$= \frac{1}{2\pi} \cdot 2\, B \tag{31}$$

$$= \frac{B}{\pi} \tag{32}$$

Since $\mathrm{sinc}(0) = 1$, we have $h(0) = \frac{B}{\pi} \mathrm{sinc}(0) = \frac{B}{\pi}$.

Therefore, equation (21) holds for all $t \in \mathbb{R}$. □

4 Spectral Containment Properties

Theorem 7. *[Spectral Support]The band-limited white noise process $W_B(t)$ has spectral support contained in the interval $[-B, B]$.*

Proof. By Definition 1, the power spectral density $S_{W_B}(\omega) = 0$ for $|\omega| > B$. Since the power spectral density completely characterizes the second-order properties of a Gaussian process, all spectral content is contained within $[-B, B]$. □

Corollary 8. *[Sampling Theorem Applicability]The band-limited white noise process $W_B(t)$ satisfies the conditions for the sampling theorem with Nyquist rate $2\,B$.*

Proof. This follows immediately from Theorem 7 and the classical sampling theorem for band-limited signals. □

5 Sampling Theory and Reconstruction

Theorem 9. (Nyquist-Shannon Sampling Theorem) *Let $W_B(t)$ be band-limited white noise with bandwidth B. If $W_B(t)$ is sampled at rate $f_s \geq 2\,B$, then $W_B(t)$ can be perfectly reconstructed from its samples $\{W_B\,(n\,T_s)\}_{n=-\infty}^{\infty}$ where $T_s = 1\,/\,f_s$ is the sampling period.*

Proof. By Theorem 7, $W_B(t)$ has spectral support in $[-B, B]$. The classical sampling theorem applies to any function (or stochastic process) with finite bandwidth. Since $f_s \geq 2\,B$, the sampling rate exceeds the Nyquist rate $f_N = 2\,B$, ensuring no aliasing occurs during sampling. □

Theorem 10. (Whittaker-Shannon Interpolation Formula) *The perfect reconstruction of band-limited white noise $W_B(t)$ from its samples is given by*

$$W_B(t) = \sum_{n=-\infty}^{\infty} W_B\,(n\,T_s)\mathrm{sinc}\left(\frac{t - n\,T_s}{T_s}\right) \tag{33}$$

where $T_s = \pi\,/\,B$ is the Nyquist sampling period.

Proof. The Fourier transform of $W_B(t)$ has support in $[-B, B]$. The sampling operation in frequency domain corresponds to periodic extension of the spectrum with period $2\,\pi\,/\,T_s$. For $T_s = \pi\,/\,B$, we have $2\,\pi\,/\,T_s = 2\,B$, so the periodic extensions do not overlap.

The reconstruction filter is an ideal low-pass filter with cutoff B and impulse response $h(t) = \frac{B}{\pi}\mathrm{sinc}(B\,t)$. Substituting $B = \pi\,/\,T_s$:

$$h(t) = \frac{\pi\,/\,T_s}{\pi}\mathrm{sinc}\left(\frac{\pi\,t}{T_s}\right) = \frac{1}{T_s}\mathrm{sinc}\left(\frac{t}{T_s}\right) \tag{34}$$

The reconstructed signal is:

$$W_B(t) = \sum_{n=-\infty}^{\infty} W_B\,(n\,T_s)\,\delta\,(t - n\,T_s) * h(t) \tag{35}$$

$$= \sum_{n=-\infty}^{\infty} W_B\,(n\,T_s)\,h\,(t - n\,T_s) \tag{36}$$

$$= \sum_{n=-\infty}^{\infty} W_B\,(n\,T_s)\frac{1}{T_s}\mathrm{sinc}\left(\frac{t - n\,T_s}{T_s}\right) \tag{37}$$

For normalized sinc function, this becomes equation (33). $\square$

Lemma 11. (Orthogonality of Sampling Functions) *The sampling functions $\phi_n(t) = \mathrm{sinc}\left(\frac{t - n\,T_s}{T_s}\right)$ form an orthogonal set:*

$$\int_{-\infty}^{\infty} \phi_m(t)\,\phi_n(t)\,d\,t = T_s\,\delta_{mn} \tag{38}$$

where δ_{mn} is the Kronecker delta.

Proof. The Fourier transform of $\phi_n(t)$ is:

$$\Phi_n(\omega) = T_s\,e^{-i\omega n T_s}\,\Pi\left(\frac{\omega\,T_s}{2\,\pi}\right) \tag{39}$$

where $\Pi(\cdot)$ is the rectangular function. By Parseval's theorem:

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi_m(t)\,\phi_n(t)\,d\,t =& \frac{1}{2\,\pi}\int_{-\infty}^{\infty} \Phi_m^*(\omega)\,\Phi_n(\omega)\,d\,\omega & (40)\\
=& \frac{T_s^2}{2\,\pi}\int_{-\pi/T_s}^{\pi/T_s} e^{i\omega(m-n)T_s}\,d\,\omega & (41)\\
=& T_s\,\delta_{mn} & (42)
\end{aligned}$$

$\square$

Theorem 12. (Sampling Theorem for Stochastic Processes) *For band-limited white noise $W_B(t)$ with autocovariance function $R_{W_B}(\tau)$, the samples $\{W_B\,(n\,T_s)\}$ at the Nyquist rate have autocovariance:*

$$R_{W_B}[m,n] = \mathbb{E}\,[W_B\,(m\,T_s)\,W_B\,(n\,T_s)] = R_{W_B}\,((m-n)\,T_s) \tag{43}$$

Proof. Since $W_B(t)$ is a stationary process, its autocovariance depends only on the time difference. For samples taken at times $m\,T_s$ and $n\,T_s$:

$$\begin{aligned}
R_{W_B}[m,n] =& \mathbb{E}\,[W_B\,(m\,T_s)\,W_B\,(n\,T_s)] & (44)\\
=& R_{W_B}\,(m\,T_s - n\,T_s) & (45)\\
=& R_{W_B}\,((m-n)\,T_s) & (46)
\end{aligned}$$

$\square$

Corollary 13. (Sampling Theorem Applicability) *The band-limited white noise process $W_B(t)$ satisfies the conditions for the sampling theorem with Nyquist rate $f_N = 2\,B$ and critical sampling period $T_s = \pi\,/\,B$.*

Proof. This follows immediately from Theorem 7 and Theorems 9-12. ☐

6 Aliasing and Reconstruction Error Analysis

Theorem 14. (Aliasing Error) *If band-limited white noise $W_B(t)$ is sampled at rate $f_s < 2\,B$, the aliasing error power is:*

$$P_{\text{alias}} = \frac{N_0}{2\,\pi} \sum_{k \neq 0} \int_{-B}^{B} 1_{[-\pi f_s, \pi f_s]}\,(\omega + 2\,\pi\,k\,f_s)\,d\,\omega \tag{47}$$

where $1_A(\cdot)$ is the indicator function for set A.

Proof. Sampling at rate f_s creates spectral replicas at frequencies $\omega + 2\,\pi\,k\,f_s$ for integer k. Aliasing occurs when these replicas overlap with the baseband spectrum $[-B, B]$. The aliasing power is the integral of the overlapping spectral components. ☐

Theorem 15. (Reconstruction Mean Square Error) *For band-limited white noise $W_B(t)$ reconstructed from samples using ideal low-pass filtering, the mean square error is zero when $f_s \geq 2\,B$.*

Proof. When the Nyquist criterion is satisfied, the Whittaker-Shannon interpolation formula provides perfect reconstruction. Since the process is band-limited and the sampling rate is sufficient, no information is lost, yielding zero reconstruction error. ☐