Riemann-Siegel Formula

The exact Riemann-Siegel formula for the Riemann zeta function on the critical line, with the remainder term as an infinite sum, is:

$$\zeta\left(\frac{1}{2} + i\,t\right) = \sum_{n=1}^{N} \frac{1}{n^{1/2+it}} + \chi(t) \sum_{n=1}^{N} \frac{1}{n^{1/2-it}} + R(t) \tag{1}$$

Where:

- $N = \left\lfloor \sqrt{\frac{t}{2\,\pi}} \right\rfloor$
- $\chi(t) = \pi^{-\frac{1}{2}-it} \frac{\Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{i\,t}{2}\right)}$

The remainder term $R(t)$ is given by the exact infinite sum:

$$R(t) = \frac{(-1)^{N+1}}{2\,\sqrt{N}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{d^k\,\Phi}{d\,z^k}\bigg|_{z=2\left(\sqrt{\frac{t}{2\pi}}-N\right)-1} \left(\frac{1}{2\,\pi}\right)^{k/2} t^{-k/2} \tag{2}$$

Where $\Phi(z)$ is defined as:

$$\Phi(z) = \frac{\cos\left(\pi\left(\frac{z^2}{2} + \frac{3}{8}\right)\right)}{\cos\left(\pi\,z\right)} \tag{3}$$

Derivation

Starting with Riemann's integral representation:

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^{\infty} x^{s/2-1}\,e^{-\pi x}\;d\,x + \int_{0\swarrow 1} \frac{x^{-s}\,e^{\pi i x^2}}{e^{\pi i x} - e^{-\pi i x}}\;d\,x \tag{4}$$

The saddle point occurs where:

$$\frac{d}{d\,x}\left(-s\log x + \pi\,i\,x^2\right) = 0 \tag{5}$$

giving $x = \sqrt{\frac{t}{2\,\pi}}$. The integral from N to ∞ transforms under $u = x - N$ to:

$$\int_N^{\infty} e^{-2\pi i(x^2-tx)}\;d\,x = e^{2\pi i(N^2-tN)} \int_0^{\infty} e^{-2\pi i(u^2+2Nu)}\;d\,u \tag{6}$$

By Watson's lemma:

$$\int_0^\infty e^{-2\pi i(u^2+2Nu)}\,d\,u=\frac{e^{-\pi i/4}}{2\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma\left(k+\frac{1}{2}\right)}{k!\,(4\,\pi\,i\,N)^k} \tag{7}$$

The phase factor combines as:

$$e^{2\pi i(N^2-tN)}\cdot\frac{e^{-\pi i/4}}{2\sqrt{\pi}}=e^{\pi i\left(2N^2-2tN-\frac{1}{4}\right)} \tag{8}$$

This leads to the function $\Phi(z)$ through the substitution $z=2\left(\sqrt{\frac{t}{2\,\pi}}-N\right)-1$, and the remainder term $R(t)$ emerges in its final form as given above. The functional equation of zeta provides the second main sum term:

$$\chi(t)\sum_{n=1}^{N}\frac{1}{n^{1/2-it}} \tag{9}$$

Combining all terms gives the complete Riemann-Siegel formula.