Monotonic Time Changes of Stationary Processes Are Oscillatory Processes

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Abstract

This paper establishes that monotonic time changes of stationary Gaussian processes are oscillatory processes. The transformation of a stationary process through a monotonic function $\theta(t)$ produces an oscillatory process with kernel $K(t,s) = K_0(|\theta(t) - \theta(s)|)\sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}$ and gain function $A(t, \lambda) = \exp(i\,\lambda\,(\theta(t) - t))\sqrt{\dot{\theta}(t)}$ for any stationary kernel K_0. The expected number of zeros over $[0,T]$ is exactly $\sqrt{-\ddot{K}_0(0)} \cdot (\theta(T) - \theta(0))$. The Hardy Z-function is demonstrated to be a realization of the oscillatory process with kernel $J_0(|\theta(t) - \theta(s)|)\sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}$.

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1 Introduction

Monotonic time transformations of stationary Gaussian processes are oscillatory processes[4, 5]. This fundamental result demonstrates that the transformation preserves essential spectral structure while introducing oscillatory behavior through the phase modulation induced by the time change function $\theta(t)$. The investigation centers on three key aspects: the relationship between eigenfunctions of the covariance operators, the preservation of normalization and eigenvalues under transformation, and the expected number of zeros of the resulting processes.

2 Oscillatory Processes

Definition 1. *[Priestley's Oscillatory Process][4] An oscillatory process $X(t)$ is defined by the spectral representation:*

$$X(t) = \int_{-\infty}^{\infty} A(t,\lambda)\, d\,Z(\lambda) \tag{1}$$

where $A(t,\lambda)$ is the amplitude function, $d\,Z(\lambda)$ is an orthogonal random measure with

$$\mathbb{E}[|d\,Z(\lambda)|^2] = d\,F(\lambda) \tag{2}$$

and $F(\lambda)$ is the integrated spectrum.

Definition 2. *[Evolutionary Spectrum][4] The evolutionary spectrum of an oscillatory process is defined as:*

$$f(t,\lambda) = |A(t,\lambda)|^2 f(\lambda) \tag{3}$$

where

$$f(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda} F(\lambda) \tag{4}$$

is the spectral density of the orthogonal random measure.

Definition 3. *[Oscillatory Function][3] The oscillatory function $\phi_t(\lambda)$ associated with the process is given by:*

$$\phi_t(\lambda) = e^{i\lambda t} A(t,\lambda) \tag{5}$$

Remark 4. The characterization of oscillatory processes through their envelope properties has been extensively studied[1, 2], providing insights into their crossing statistics and envelope behavior.

3 Time Change Functions

Definition 5. *Let $\mathcal{T}$ denote the class of time change functions $\theta: \mathbb{R} \to \mathbb{R}$ that are:*

1. *Strictly monotonically increasing: $\theta(s) < \theta(t)$ for all $s < t$*
2. *Twice continuously differentiable with $\dot{\theta}(t) > 0$*

4 Main Results

Theorem 6. *[Time-Changed Kernel] For any stationary kernel $K_0(|t-s|)$ and time change function $\theta \in \mathcal{T}$, the time-changed kernel is:*

$$K(t,s) = K_0(|\theta(t) - \theta(s)|) \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}$$

Proof. Consider a stationary process $Y(u)$ with kernel $K_0(|u-v|)$ and eigenfunctions $\psi_n(u)$ satisfying:

$$\int_{-\infty}^{\infty} K_0(|u-v|)\,\psi_n(v)\,dv = \lambda_n\,\psi_n(u) \tag{6}$$

The time-changed process $X(t) = Y(\theta(t))$ requires eigenfunctions $\phi_n(t)$ satisfying:

$$\int_{-\infty}^{\infty} K(t,s)\,\phi_n(s)\,ds = \lambda_n\,\phi_n(t) \tag{7}$$

Let $\phi_n(t) = \psi_n(\theta(t))\sqrt{\dot{\theta}(t)}$. The normalization condition requires:

$$\int_{-\infty}^{\infty} |\phi_n(t)|^2 dt = \int_{-\infty}^{\infty} |\psi_n(\theta(t))|^2\,\dot{\theta}(t)\,dt \tag{8}$$

Under the substitution $u = \theta(t)$, $du = \dot{\theta}(t)\,dt$:

$$\int_{-\infty}^{\infty} |\psi_n(u)|^2\,du = 1 \tag{9}$$

Therefore $\phi_n(t)$ are properly normalized. Substituting into the eigenvalue equation:

$$\int_{-\infty}^{\infty} K(t,s)\,\psi_n(\theta(s))\sqrt{\dot{\theta}(s)}\,ds = \lambda_n\,\psi_n(\theta(t))\sqrt{\dot{\theta}(t)} \tag{10}$$

Dividing by $\sqrt{\dot{\theta}(t)}$:

$$\int_{-\infty}^{\infty} \frac{K(t,s)}{\sqrt{\dot{\theta}(t)}}\,\psi_n(\theta(s))\sqrt{\dot{\theta}(s)}\,ds = \lambda_n\,\psi_n(\theta(t)) \tag{11}$$

Under the change of variables $v = \theta(s)$, $dv = \dot{\theta}(s)\,ds$:

$$\int_{-\infty}^{\infty} \frac{K(t,\theta^{-1}(v))}{\sqrt{\dot{\theta}(t)}}\,\psi_n(v)\sqrt{\dot{\theta}(\theta^{-1}(v))}\,\frac{dv}{\dot{\theta}(\theta^{-1}(v))} = \lambda_n\,\psi_n(\theta(t))$$

This simplifies to:

$$\int_{-\infty}^{\infty} \frac{K(t,\theta^{-1}(v))}{\sqrt{\dot{\theta}(t)\,\dot{\theta}(\theta^{-1}(v))}}\,\psi_n(v)\,dv = \lambda_n\,\psi_n(\theta(t)) \tag{12}$$

Setting $u = \theta(t)$, this becomes:

$$\int_{-\infty}^{\infty} \frac{K(\theta^{-1}(u),\theta^{-1}(v))}{\sqrt{\dot{\theta}(\theta^{-1}(u))\,\dot{\theta}(\theta^{-1}(v))}}\,\psi_n(v)\,dv = \lambda_n\,\psi_n(u) \tag{13}$$

For this to equal the original eigenvalue equation, we require:

$$\frac{K(\theta^{-1}(u),\theta^{-1}(v))}{\sqrt{\dot{\theta}(\theta^{-1}(u))\,\dot{\theta}(\theta^{-1}(v))}} = K_0(|u-v|) \tag{14}$$

Therefore:

$$K(\theta^{-1}(u), \theta^{-1}(v)) = K_0(|u-v|)\sqrt{\dot{\theta}(\theta^{-1}(u))\,\dot{\theta}(\theta^{-1}(v))} \tag{15}$$

Setting $t=\theta^{-1}(u)$ and $s=\theta^{-1}(v)$:

$$K(t,s) = K_0(|\theta(t)-\theta(s)|)\sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \qquad \square$$

Corollary 7. *[Eigenvalue Invariance] The eigenvalues $\{\lambda_n\}$ of the time-changed kernel's covariance operator are identical to those of the original kernel K_0's covariance operator.*

Theorem 8. *[Oscillatory Process Generation] Any Gaussian process with time-changed kernel $K(t,s) = K_0(|\theta(t)-\theta(s)|)\sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}$ where $\theta \in \mathcal{T}$ and K_0 is any stationary kernel is an oscillatory process with amplitude function:*

$$A(t,\lambda) = e^{i\lambda(\theta(t)-t)}\sqrt{\dot{\theta}(t)} \tag{16}$$

Proof. The oscillatory function is defined as:

$$\phi_t(\lambda) = \exp(i\,\lambda\,t)\,A(t,\lambda) \tag{17}$$

where $A(t,\lambda) = \exp(i\,\lambda\,(\theta(t)-t))\sqrt{\dot{\theta}(t)}$.

Computing $\phi_t(\lambda)$:

$$\begin{aligned}\phi_t(\lambda) =& \exp(i\,\lambda\,t)\exp(i\,\lambda\,(\theta(t)-t))\sqrt{\dot{\theta}(t)} && (18)\\ =& \exp(i\,\lambda\,t)\exp(i\,\lambda\,\theta(t))\exp(-i\,\lambda\,t)\sqrt{\dot{\theta}(t)} && (19)\\ =& \exp(i\,\lambda\,\theta(t))\sqrt{\dot{\theta}(t)} && (20)\end{aligned}$$

The process admits the spectral representation:

$$X(t) = \int_{-\infty}^{\infty} \exp(i\,\lambda\,\theta(t))\sqrt{\dot{\theta}(t)}\,d\,Z(\lambda)$$

This establishes the oscillatory nature with the specified amplitude function. $\square$

Theorem 9. *[Expected Zero-Counting Function] Let $\theta \in \mathcal{T}$ and let $K_0(\cdot)$ be any positive-definite, stationary covariance function, twice differentiable at 0. Consider the centered Gaussian process with covariance:*

$$K(t,s) = K_0(|\theta(t)-\theta(s)|)\sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \tag{21}$$

Then the expected number of zeros in $[0,T]$ is:

$$\mathbb{E}[N([0,T])] = \sqrt{-\ddot{K}_0(0)}\cdot(\theta(T)-\theta(0)) \tag{22}$$

Proof. By the Kac-Rice formula:

$$\mathbb{E}[N([0,T])] = \int_0^T \sqrt{-\lim_{s \to t} \frac{\partial^2}{\partial t\, \partial s} K(s,t)}\, d t \tag{23}$$

Let

$$u = \theta(t) - \theta(s) \tag{24}$$

Then

$$\frac{\partial}{\partial t} K(t,s) = \sqrt{\dot{\theta}(t)\dot{\theta}(s)} \left[\dot{K}_0(|u|)\, \mathrm{sgn}(u)\, \dot{\theta}(t) + \frac{\ddot{\theta}(t)}{2\, \dot{\theta}(t)} K_0(|u|) \right] \tag{25}$$

Therefore:

$$\mathbb{E}[N([0,T])] = \int_0^T \sqrt{-(-\ddot{K}_0(0)\, \dot{\theta}(t)^2)}\, d t \tag{26}$$

$$= \sqrt{-\ddot{K}_0(0)} \int_0^T \dot{\theta}(t)\, d t \tag{27}$$

$$= \sqrt{-\ddot{K}_0(0)} \cdot (\theta(T) - \theta(0)) \tag{28}$$

□

Theorem 10. *[Spectral Inversion Formula] The orthogonal random measure satisfies:*

$$d Z(\lambda) = \lim_{T \to \infty} \frac{1}{2 T} \int_{-T}^{T} X(t) \exp(-i \lambda \theta(t)) \sqrt{\dot{\theta}(t)}\, d t \cdot d \lambda$$

Proof. From the spectral representation $X(t) = \int_{-\infty}^{\infty} \exp(i \lambda \theta(t)) \sqrt{\dot{\theta}(t)}\, d Z(\lambda)$, multiply both sides by $\exp(-i \mu \theta(t)) \sqrt{\dot{\theta}(t)}$ and integrate:

$$\int_{-T}^{T} X(t) \exp(-i \mu \theta(t)) \sqrt{\dot{\theta}(t)}\, d t = \int_{-\infty}^{\infty} \int_{-T}^{T} \exp(i (\lambda - \mu) \theta(t))\, \dot{\theta}(t)\, d t\ d Z(\lambda)$$

Under the change of variables $u = \theta(t)$, $d u = \dot{\theta}(t)\, d t$:

$$= \int_{-\infty}^{\infty} \int_{\theta(-T)}^{\theta(T)} \exp(i (\lambda - \mu) u)\, d u\ d Z(\lambda)$$

$$= \int_{-\infty}^{\infty} \frac{\exp(i (\lambda - \mu) \theta(T)) - \exp(i (\lambda - \mu) \theta(-T))}{i (\lambda - \mu)} d Z(\lambda)$$

As $T \to \infty$, this approaches $2 \pi \delta(\lambda - \mu)$ times the increment of Z over an interval containing λ, yielding the inversion formula. □

5 Hardy Z-Function Realization

Theorem 11. *[Hardy Z-Function as Oscillatory Process] The Hardy Z-function*

$$Z(t) = e^{i\theta(t)}\,\zeta\left(\frac{1}{2} + i\,t\right)$$

is a realization of the oscillatory process with kernel:

$$K(t,s) = J_0(|\theta(t) - \theta(s)|)\sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}$$

and amplitude function:

$$A(t,\lambda) = \exp\left(i\,\lambda\,(\theta(t) - t)\right)\sqrt{\dot{\theta}(t)}$$

Proof. The Hardy Z-function exhibits the spectral representation:

$$Z(t) = \int_{-1}^{1} \exp\left(i\,\lambda\,\theta(t)\right)\sqrt{\dot{\theta}(t)}\,d\,Z(\lambda)$$

The orthogonal random measure satisfies:

$$\mathbb{E}[|d\,Z(\lambda)|^2] = \frac{d\,\lambda}{\sqrt{1-\lambda^2}}$$

corresponding to the integrated spectrum $F(\lambda) = \arcsin(\lambda)$.

The spectral density is:

$$f(\lambda) = \frac{d}{d\,\lambda}\arcsin(\lambda) = \frac{1}{\sqrt{1-\lambda^2}}$$

The evolutionary spectrum becomes:

$$f(t,\lambda) = |A(t,\lambda)|^2\,f(\lambda) = \dot{\theta}(t)\cdot\frac{1}{\sqrt{1-\lambda^2}}$$

The orthogonal random measure can be expressed as:

$$d\,Z(\lambda) = \lim_{T\to\infty}\frac{1}{2\,T}\int_{-T}^{T} Z(t)\exp\left(-i\,\lambda\,\theta(t)\right)\sqrt{\dot{\theta}(t)}\,d\,t\cdot d\,\lambda$$

This establishes the Hardy Z-function as a realization of the oscillatory process with the specified kernel structure. □

6 Conclusion

Monotonic time changes of stationary Gaussian processes are oscillatory processes in the sense of Priestley's theory[4] The transformation preserves eigenvalue structure while introducing oscillatory behavior through phase modulation. The expected number of zeros scales as $\sqrt{-\ddot{K}_0(0)}\cdot(\theta(T) - \theta(0))$, providing a direct connection between the time change function and the zero-counting statistics. The Hardy Z-function serves as a canonical realization of such processes, demonstrating the deep connection between number theory and stochastic process theory.

Bibliography

[1] A. M. Hasofer. A uniqueness problem for the envelope of an oscillatory process. *Journal of Applied Probability*, 16(4):822–829, December 1979.

[2] A. M. Hasofer and P. Petocz. The envelope of an oscillatory process and its upcrossings. *Advances in Applied Probability*, 10(4):711–716, 1978.

[3] V. Mandrekar. A characterization of oscillatory processes and their prediction. *Proceedings of the American Mathematical Society*, 32(1):280–284, 1972.

[4] Maurice B Priestley. Evolutionary spectra and non-stationary processes. *Journal of the Royal Statistical Society: Series B (Methodological)*, 27(2):204–229, 1965.

[5] MB Priestley. Power spectral analysis of non-stationary random processes. *Journal of Sound and Vibration*, 6(1):86–97, 1967.