Unitary Time Changes of Stationary Processes Yield Oscillatory Processes and a Functional Framework Toward a Hilbert–Pólya Construction

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September 16, 2025

Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are square-integrable over compact sets. Applying U_θ to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot\theta(t)}\; e^{i\lambda\theta(t)}$ and evolutionary spectrum $d\,F_t(\lambda) = \dot\theta(t)\; d\,F(\lambda)$. It is proved that sample paths of any non-degenerate second-order stationary process almost surely lie in $L^2_{\mathrm{loc}}(\mathbb{R})$, making the operator applicable to typical realizations. A zero-localization measure $\mu(dt) = \delta(Z(t))\,|Z'(t)|\,dt$ induces a Hilbert space $L^2(\mu)$ on the zero set of an oscillatory process Z, and the multiplication operator $(L\,f)(t) = t\,f(t)$ has pure point, simple spectrum equal to the zero set of Z. This produces a concrete operator scaffold consistent with a Hilbert–Pólya-type viewpoint.

Table of contents

TODO: add zero counting function   and expected zero counting function!!!!!

1 Function Spaces and Unitary Time Change

1.1 σ-compact sets and local L^2

Definition 1. [σ-compact sets] *A subset $U \subseteq \mathbb{R}$ is σ-compact if*

$$U = \bigcup_{n=1}^{\infty} K_n \tag{1}$$

with each K_n compact.

Definition 2. [Locally square-integrable functions] *Define*

$$L_{\mathrm{loc}}^2(\mathbb{R}) := \left\{ f\colon \mathbb{R} \to \mathbb{C}\colon \int_K |f(t)|^2 \, dt < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \tag{2}$$

Remark 3. Every bounded measurable set in $\mathbb{R}$ is compact or contained in a compact set; hence $L_{\mathrm{loc}}^2(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

1.2 Unitary time-change operator

Definition 4. [Unitary time-change] *Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with $\dot{\theta}(t) > 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero. The function θ maps σ-compact sets to σ-compact sets. Define, for f measurable,*

$$(U_\theta f)(t) = \sqrt{\dot{\theta}(t)} \, f(\theta(t)) \tag{3}$$

Proposition 5. [Inverse map] *The inverse map is given by*

$$(U_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{4}$$

which is well-defined almost everywhere on every σ-compact set.

Proof. Since $\dot{\theta}(t) = 0$ only on sets of measure zero, and θ^{-1} maps sets of measure zero to sets of measure zero (as absolutely continuous bijective functions preserve measure-zero sets), the denominator $\sqrt{\dot{\theta}(\theta^{-1}(s))}$ is positive almost everywhere. The expression is therefore well-defined almost everywhere on every σ-compact set, which suffices for defining an element of $L^2_{\text{loc}}(\mathbb{R})$. $\qquad\square$

Theorem 6. [Local unitarity on σ-compact sets] *For every σ-compact set $C \subseteq \mathbb{R}$ and $f \in L^2_{\text{loc}}(\mathbb{R})$,*

$$\int_C |(U_\theta f)(t)|^2 \, dt = \int_{\theta(C)} |f(s)|^2 \, ds \qquad (5)$$

Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\text{loc}}(\mathbb{R})$.

Proof. Let $f \in L^2_{\text{loc}}(\mathbb{R})$ and let U be any σ-compact set. The local L^2-norm of $U_\theta f$ over C is:

$$
\begin{aligned}
\int_C |(U_\theta f)(t)|^2 \, dt &= \int_C \left| \sqrt{\dot{\theta}(t)} \, f(\theta(t)) \right|^2 dt \\
&= \int_C \dot{\theta}(t) |f(\theta(t))|^2 \, dt
\end{aligned}
\qquad (6)
$$

Since θ is absolutely continuous and strictly increasing, applying the change of variables $s = \theta(t)$ gives

$$ds = \dot{\theta}(t) \, dt \qquad (7)$$

almost everywhere. Since θ maps σ-compact sets to σ-compact sets, as t ranges over C, $s = \theta(t)$ ranges over $\theta(C)$, which is σ-compact. Therefore:

$$\int_C \dot{\theta}(t) \, |f(\theta(t))|^2 \, dt = \int_{\theta(C)} |f(s)|^2 \, ds \qquad (8)$$

To verify that U_θ^{-1} is indeed the inverse, we compute explicitly. For any $f \in L^2_{\text{loc}}(\mathbb{R})$:

$$
\begin{aligned}
(U_\theta^{-1} U_\theta f)(s) &= \left(U_\theta^{-1} \sqrt{\dot{\theta}(s)} \, f(\theta(s)) \right)(s) \\
&= \frac{\sqrt{\dot{\theta}(\theta^{-1}(s))}}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \, f(\theta(\theta^{-1}(s))) \\
&= f(s)
\end{aligned}
\qquad (9)
$$

since $\theta(\theta^{-1}(s)) = s$. Similarly, for any $g \in L^2_{\text{loc}}(\mathbb{R})$:

$$
\begin{aligned}
(U_\theta U_\theta^{-1} g)(t) &= \sqrt{\dot{\theta}(t)} \, (U_\theta^{-1} g)(\theta(t)) \\
&= \frac{\sqrt{\dot{\theta}(t)}}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} \, g(\theta^{-1}(\theta(t))) \\
&= \frac{\sqrt{\dot{\theta}(t)}}{\sqrt{\dot{\theta}(t)}} \, g(t) \\
&= g(t)
\end{aligned}
\tag{10}
$$

since $\theta^{-1}(\theta(t)) = t$. Therefore

$$
U_\theta U_\theta^{-1} = U_\theta^{-1} U_\theta = I \tag{11}
$$

on $L^2_{\text{loc}}(\mathbb{R})$. □

Theorem 7. *[Unitarity on $L^2(\mathbb{R})$]* $U_\theta : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *is unitary:*

$$
\int_{\mathbb{R}} |(U_\theta f)(t)|^2 \, dt = \int_{\mathbb{R}} |f(s)|^2 \, ds \tag{12}
$$

and U_θ^{-1} *is its inverse.*

Proof. For $f \in L^2(\mathbb{R})$, we have:

$$
\int_{\mathbb{R}} |(U_\theta f)(t)|^2 \, dt =
$$

$$
\int_{\mathbb{R}} \dot{\theta}(t) |f(\theta(t))|^2 \, dt
$$

$$
\tag{13}
$$

By the change of variables

$$
s = \theta(t) \tag{14}
$$

with

$$
ds = \dot{\theta}(t) \, dt \tag{15}
$$

4

, and since $\theta \colon \mathbb{R} \to \mathbb{R}$ is bijective:

$$\int_{\mathbb{R}} \dot{\theta}(t) \, |f(\theta(t))|^2 \, dt = \int_{\mathbb{R}} |f(s)|^2 \, ds \tag{16}$$

The inverse relationship follows from the same computation as in Theorem 6, applied globally. $\square$

2 Oscillatory Processes (Priestley)

Definition 8. *[Oscillatory process] Let F be a finite nonnegative Borel measure on $\mathbb{R}$. For each $t \in \mathbb{R}$, let $A_t \in L^2(F)$ be the gain function and*

$$\varphi_t(\lambda) = A_t(\lambda) \, e^{i\lambda t} \tag{17}$$

be the corresponding oscillatory function then an oscillatory process *is a stochastic process which can be represented as*

$$\begin{aligned} Z(t) \ &= \int_{\mathbb{R}} \varphi_t(\lambda) \, \mathrm{d} \, \Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda) \, e^{i\lambda t} \, \mathrm{d} \, \Phi(\lambda) \end{aligned} \tag{18}$$

where Φ is a complex orthogonal random measure with spectral measure F, that is,

$$\mathbb{E}[\Phi \, (d\,\lambda) \, \overline{\Phi \, (d\,\mu)}] = \delta \, (\lambda - \mu) \, d \, F(\lambda) \tag{19}$$

and corresponding covariance kernel

$$\begin{aligned} R_Z(t,s) \ &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \int_{\mathbb{R}} A_t(\lambda) \, \overline{A_s(\lambda)} \, e^{i\lambda(t-s)} \, d\,F(\lambda) \\ &= \int_{\mathbb{R}} \phi_t(\lambda) \overline{\phi_s(\lambda)} \mathrm{d} \, F(\lambda) \end{aligned} \tag{20}$$

Remark 9. [Real-valuedness] Z is real-valued if and only if

$$A_t(-\lambda) = \overline{A_t(\lambda)} \tag{21}$$

for F-a.e. λ, equivalently

$$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \tag{22}$$

for F-a.e. λ. TODO: this needs to be recast as a theorem

Theorem 10. *[Existence] If F is finite and $(A_t)_{t \in \mathbb{R}}$ is measurable in t with*

$$\int_{\mathbb{R}} |A_t(\lambda)|^2 \, dF(\lambda) < \infty \forall t \in \mathbb{R} \tag{23}$$

then there exists a complex orthogonal random measure Φ with spectral measure F such that

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda) \, e^{i\lambda t} \, \Phi(d\lambda) \tag{24}$$

is well-defined in $L^2(\Omega)$ and has covariance R_Z as in (20) above.

Proof. The proof proceeds by constructing the stochastic integral using the standard extension procedure. First, the integral is defined for simple functions of the form

$$g(\lambda) = \sum_{j=1}^{n} c_j \mathbf{1}_{E_j}(\lambda) \tag{25}$$

where $\{E_j\}$ are disjoint Borel sets with $F(E_j) < \infty$ and $c_j \in \mathbb{C}$:

$$\int_{\mathbb{R}} g(\lambda) \, \Phi(d\lambda) := \sum_{j=1}^{n} c_j \, \Phi(E_j) \tag{26}$$

For such simple functions, the isometry property holds:

$$
\begin{aligned}
\mathbb{E}\left[\left| \int_{\mathbb{R}} g(\lambda) \, \Phi(d\lambda) \right|^2 \right] &= \mathbb{E}\left[\left| \sum_{j=1}^{n} c_j \, \Phi(E_j) \right|^2 \right] \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \bar{c}_k \mathbb{E}[\Phi(E_j)\overline{\Phi(E_k)}] \\
&= \sum_{j=1}^{n} |c_j|^2 \, F(E_j \\
&= \int_{\mathbb{R}} |g(\lambda)|^2 \, dF(\lambda)
\end{aligned}
\tag{27}
$$

Since simple functions are dense in $L^2(F)$, the integral is extended by continuity to all $g \in L^2(F)$. For each t, since

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{28}$$

and $A_t \in L^2(F)$, $\varphi_t \in L^2(F)$ holds. Therefore

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda)\, \Phi(d\lambda) \tag{29}$$

is well-defined in $L^2(\Omega)$. The covariance is computed as:

$$
\begin{aligned}
R_Z(t,s) \quad &= \mathbb{E}[Z(t)\overline{Z(s)}]\\
&= \mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda)\, \Phi(d\lambda) \int_{\mathbb{R}} \overline{\varphi_s(\mu)}\, \overline{\Phi(d\mu)}\right]\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\mu)}\, \mathbb{E}\left[\Phi(d\lambda)\overline{\Phi(d\mu)}\right]\\
&= \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)}\, dF(\lambda)\\
&= \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)}e^{i\lambda(t-s)}\, dF(\lambda)
\end{aligned}
\tag{30} \ \square
$$

3 Stationary Processes and Time Change

3.1 Stationary processes

Definition 11. [Cramér representation] *A zero-mean stationary process X with spectral measure F admits the sample path representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, \Phi(d\lambda) \tag{31}$$

which has covariance

$$R_X(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)}\, dF(\lambda) \tag{32}$$

3.2 Stationary → oscillatory via U_θ

> **Theorem 12.** *[Time change yields oscillatory process] Let X be zero-mean stationary as in Definition 11. For θ as in Definition 4, define*
>
> $$\begin{aligned} Z(t) \ &= (U_\theta X)(t) \\ &= \sqrt{\dot\theta(t)} \ X(\theta(t)) \end{aligned} \tag{33}$$
>
> *Then Z is oscillatory with oscillatory function*
>
> $$\varphi_t(\lambda) = \sqrt{\dot\theta(t)} \ e^{i\lambda\theta(t)} \tag{34}$$
>
> *, gain function*
>
> $$A_t(\lambda) = \sqrt{\dot\theta(t)} \ e^{i\lambda(\theta(t)-t)} \tag{35}$$
>
> *, and covariance*
>
> $$R_Z(t,s) = \int_{\mathbb{R}} \sqrt{\dot\theta(t)\,\dot\theta(s)} \ e^{i\lambda(\theta(t)-\theta(s))} \, dF(\lambda) \tag{36}$$

Proof. Applying the unitary time change operator to the spectral representation of $X(t)$:

$$\begin{aligned} Z(t) \ &= (U_\theta X)(t) \\ &= \sqrt{\dot\theta(t)} \ X(\theta(t)) \\ &= \sqrt{\dot\theta(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)} \, d\Phi(\lambda) \\ &= \int_{\mathbb{R}} \sqrt{\dot\theta(t)} \ e^{i\lambda\theta(t)} \, d\Phi(\lambda) \\ &= \int_{\mathbb{R}} \varphi_t(\lambda) \, d\Phi(\lambda) \end{aligned} \tag{37}$$

where

$$\varphi_t(\lambda) = \sqrt{\dot\theta(t)} \ e^{i\lambda\theta(t)} \tag{38}$$

To verify this constitutes an oscillatory representation according to Definition 8, $\varphi_t(\lambda)$ has the form $A_t(\lambda)\,e^{i\lambda t}$:

$$\begin{aligned} \varphi_t(\lambda) \ &= \sqrt{\dot\theta(t)} \ e^{i\lambda\theta(t)} \\ &= \sqrt{\dot\theta(t)} \ e^{i\lambda(\theta(t)-t)} \ e^{i\lambda t} \\ &= A_t(\lambda) \ e^{i\lambda t} \end{aligned} \tag{39}$$

where

$$A_t(\lambda) = \sqrt{\dot\theta(t)}\, e^{i\lambda(\theta(t)-t)} \tag{40}$$

Since $\dot\theta(t) \geq 0$ almost everywhere and $\dot\theta(t) = 0$ only on sets of measure zero, $A_t(\lambda)$ is well-defined almost everywhere. Moreover, $A_t \in L^2(F)$ for each t since:

$$
\begin{aligned}
\int_{\mathbb{R}} |A_t(\lambda)|^2 \, dF(\lambda) &= \int_{\mathbb{R}} \left| \sqrt{\dot\theta(t)}\, e^{i\lambda(\theta(t)-t)} \right|^2 dF(\lambda) \\
&= \int_{\mathbb{R}} \dot\theta(t) |e^{i\lambda(\theta(t)-t)}|^2 \, dF(\lambda) \\
&= \dot\theta(t) \int_{\mathbb{R}} dF(\lambda) \\
&= \dot\theta(t)\, F(\mathbb{R}) < \infty
\end{aligned}
\tag{41}
$$

where $|e^{i\alpha}| = 1$ for all real α is used. The covariance is computed as:

$$
\begin{aligned}
R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\
&= \mathbb{E}\left[\sqrt{\dot\theta(t)}\, X(\theta(t)) \sqrt{\dot\theta(s)}\, \overline{X(\theta(s))} \right] \\
&= \sqrt{\dot\theta(t)\,\dot\theta(s)}\; \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \\
&= \sqrt{\dot\theta(t)\,\dot\theta(s)}\; R_X\left(\theta(t)-\theta(s)\right) \\
&= \sqrt{\dot\theta(t)\,\dot\theta(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} \, dF(\lambda)
\end{aligned}
\tag{42} \quad \square
$$

Corollary 13. *[Evolutionary spectrum]* *The evolutionary spectrum is*

$$
\begin{aligned}
dF_t(\lambda) &= |A_t(\lambda)|^2 \, dF(\lambda) \\
&= \dot\theta(t)\, dF(\lambda)
\end{aligned}
\tag{43}
$$

Proof. By definition of the evolutionary spectrum and using the gain function from Theorem 12:

$$
\begin{aligned}
dF_t(\lambda) &= |A_t(\lambda)|^2 \, dF(\lambda) \\
&= \left| \sqrt{\dot\theta(t)}\, e^{i\lambda(\theta(t)-t)} \right|^2 dF(\lambda) \\
&= \dot\theta(t) |e^{i\lambda(\theta(t)-t)}|^2 \, dF(\lambda) \\
&= \dot\theta(t)\, dF(\lambda)
\end{aligned}
\tag{44}
$$

since

$$|e^{i\alpha}| = 1 \, \forall a \in \mathbb{R} \tag{45} \quad \square$$

9

3.3 Covariance operator conjugation

Proposition 14. *[Operator conjugation]* *Let*

$$(T_K f)(t) := \int_{\mathbb{R}} K(|t-s|) \ f(s) \ ds \tag{46}$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h} \ dF(\lambda) \tag{47}$$

Define the transformed kernel

$$K_\theta(s,t) := \sqrt{\dot\theta(t)\,\dot\theta(s)} \ K(|\theta(t)-\theta(s)|) \tag{48}$$

and corresponding integral covariance operator

$$(T_{K_\theta} f)(t) := \int_{\mathbb{R}} K_\theta(s,t) \ f(s) \ ds \tag{49}$$

Then

$$T_{K_\theta} = U_\theta \ T_K \ U_\theta^{-1} \tag{50}$$

on $L^2_{\mathrm{loc}}(\mathbb{R})$.

Proof. For any $g \in L^2_{\mathrm{loc}}(\mathbb{R})$, the integral operator is transformed from coordinates (r,w) to coordinates (t,s) by applying both coordinate transformations

$$r = \theta(t) \tag{51}$$

and

$$w = \theta(s) \tag{52}$$

by simultaneous substitution with Jacobians

$$dr = \dot\theta(t) \ dt \tag{53}$$

and

$$dw = \dot\theta(s) \ ds \tag{54}$$

The operator T_K in (r, w) coordinates is:

$$(T_K f)(r) = \int_{\mathbb{R}} K(|r - w|) \, f(w) \, dw \tag{55}$$

Under the simultaneous transformation $r = \theta(t)$ and $w = \theta(s)$:

$$
\begin{aligned}
((U_\theta T_K U_\theta^{-1}) \, g)(t) \; &= \sqrt{\dot{\theta}(t) \, \dot{\theta}(s)} \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|) \, (U_\theta^{-1} g)(\theta(s)) \frac{\dot{\theta}(s)}{\sqrt{\dot{\theta}(s)}} \, ds \\
&= \sqrt{\dot{\theta}(t) \, \dot{\theta}(s)} \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|) \frac{g(s)}{\sqrt{\dot{\theta}(s)}} \sqrt{\dot{\theta}(s)} \, ds \\
&= \sqrt{\dot{\theta}(t) \, \dot{\theta}(s)} \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|) \, g(s) \, ds \\
&= \int_{\mathbb{R}} \sqrt{\dot{\theta}(t) \, \dot{\theta}(s)} \, K(|\theta(t) - \theta(s)|) \, g(s) \, ds \\
&= \int_{\mathbb{R}} K_\theta(t, s) \, g(s) \, ds \\
&= (T_{K_\theta} g)(t)
\end{aligned} \tag{56}
$$

where

$$K_\theta(t, s) = \sqrt{\dot{\theta}(t) \, \dot{\theta}(s)} \, K(|\theta(t) - \theta(s)|) \tag{57}$$

Therefore

$$T_{K_\theta} = U_\theta T_K U_\theta^{-1} \tag{58} \quad \square$$

4   Sample Paths Live in L^2_{loc}

> **Theorem 15. [Sample paths in $L^2_{\text{loc}}(\mathbb{R})$]** *Let $\{X(t)\}_{t \in \mathbb{R}}$ be a second-order stationary process with*
>
> $$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \tag{59}$$
>
> *then, almost surely, every sample path $t \mapsto X(\omega, t)$ belongs to $L^2_{\text{loc}}(\mathbb{R})$.*

Proof. Fix any bounded interval $[a, b]$ and consider the random variable

$$Y_{[a,b]} := \int_a^b X(t)^2 \, dt \tag{60}$$

By stationarity and Fubini's theorem:

$$
\mathbb{E}[Y_{[a,b]}] = \mathbb{E}\left[\int_a^b X(t)^2 \, dt\right] = \int_a^b \mathbb{E}[X(t)^2] \, dt
$$
$$
= \int_a^b \sigma^2 \, dt
$$
$$
= \sigma^2 (b-a) < \infty
\tag{61}
$$

Equivalently, invoking the evolutionary spectrum

$$
dF_t(\lambda) = \dot{\theta}(t) \, dF(\lambda)
\tag{62}
$$

it can be seen that the oscillatory process can be expressed as

$$
Z(t) = (U_\theta X)(t)
\tag{63}
$$

whose (ensemble) variance is given by

$$
\mathbb{E}[Z(t)^2] = \int dF_t(\lambda)
$$
$$
= \dot{\theta}(t) \int dF(\lambda)
$$
$$
= \dot{\theta}(t) \, \sigma^2
\tag{64}
$$

Therefore for any bounded interval $[a,b]$:

$$
\mathbb{E}\left[\int_a^b Z(t)^2 \, dt\right] = \int_a^b \dot{\theta}(t) \, \sigma^2 \, dt
$$
$$
= \sigma^2 \left[\theta(b) - \theta(a)\right] < \infty
\tag{65}
$$

By Markov's inequality, for any $M > 0$:

$$
P\left(Y_{[a,b]} > M\right) \leq \frac{\mathbb{E}[Y_{[a,b]}]}{M} = \frac{\sigma^2 (b-a)}{M}
\tag{66}
$$

Taking $M \to \infty$, the conclusion is

$$
P\left(Y_{[a,b]} < \infty\right) = 1
\tag{67}
$$

, i.e., almost surely the sample path is square-integrable on $[a,b]$. Since $\mathbb{R}$ is the countable union of bounded intervals:

$$
\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]
\tag{68}
$$

12

by countable subadditivity of probability:

$$P\left(\bigcap_{n=1}^{\infty}\left\{\int_{-n}^{n}X(t)^2\,dt<\infty\right\}\right)=1 \tag{69}$$

Now let K be any compact set. Then K is bounded, so

$$K\subseteq[-N,N] \tag{70}$$

for some N. Therefore:

$$\int_{K}X(t)^2\,dt\leq\int_{-N}^{N}X(t)^2\,dt<\infty \tag{71}$$

almost surely. This holds for every compact set K, so almost surely every sample path lies in $L_{\mathrm{loc}}^2(\mathbb{R})$. $\qquad\square$

5 Zero Localization and Hilbert–Pólya Scaffold

5.1 Zero localization measure

Definition 16. [Zero localization measure] *Let Z be real-valued with $Z\in C^1(\mathbb{R})$ having only simple zeros*

$$Z(t_0)=0\Rightarrow\dot{Z}(t_0)\neq 0 \tag{72}$$

Define, for Borel $B\subset\mathbb{R}$,

$$\mu(B)=\int_{\mathbb{R}}\mathbf{1}_B(t)\,\delta(Z(t))\,|\dot{Z}(t)|\,dt \tag{73}$$

Theorem 17. [Atomicity on the zero set] *For every $\phi\in C_c^{\infty}(\mathbb{R})$,*

$$\int_{\mathbb{R}}\phi(t)\,\delta(Z(t))\,|\dot{Z}(t)|\,dt=\sum_{t_0:Z(t_0)=0}\phi(t_0) \tag{74}$$

hence

$$\mu(t)=\sum_{t_0:Z(t_0)=0}\delta_{t_0}(t) \tag{75}$$

Proof. Since all zeros of Z are simple and $Z \in C^1(\mathbb{R})$, by the inverse function theorem each zero t_0 is isolated. Near each zero t_0, Z is locally monotonic, so the one-dimensional change of variables formula for the Dirac delta can be applied.

Specifically, near t_0 where $Z(t_0) = 0$ and $\dot{Z}(t_0) \neq 0$, locally

$$Z(t) = (t - t_0)\, \dot{Z}(t_0) + O((t - t_0)^2) \tag{76}$$

holds. The distributional identity for the Dirac delta under smooth changes of variables gives:

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0) = 0} \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|} \tag{77}$$

Therefore:

$$
\begin{aligned}
\int_{\mathbb{R}} \phi(t)\, \delta(Z(t))\, |\dot{Z}(t)|\, dt &= \int_{-\infty}^{\infty} \phi(t)\, |\dot{Z}(t)| \sum_{t_0 : Z(t_0) = 0} \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|}\, dt \\
&= \sum_{t_0 : Z(t_0) = 0} \int_{\mathbb{R}} \phi(t) \frac{|\dot{Z}(t)|\, \delta\,(t - t_0)}{|\dot{Z}(t_0)|}\, dt \\
&= \sum_{t_0 : Z(t_0) = 0} \frac{|\dot{Z}(t_0)|}{|\dot{Z}(t_0)|}\, \phi(t_0) \\
&= \sum_{t_0 : Z(t_0) = 0} \phi(t_0)
\end{aligned}
\tag{78}
$$

This shows that μ is the discrete measure

$$\mu = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0} \tag{79}$$

assigning unit mass to each zero. $\qquad\square$

5.2 Hilbert space on zeros and multiplication operator

Definition 18. [Hilbert space on the zero set] *Let $\mathcal{H} = L^2(\mu)$ with inner product*

$$\langle f, g \rangle = \int f(t)\overline{g(t)}\, \mu\,(dt) \tag{80}$$

14

Proposition 19. [Atomic structure] *Let*

$$\mu = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0} \tag{81}$$

then

$$\mathcal{H} \cong \left\{ f \colon \{t_0 \colon Z(t_0) = 0\} \to \mathbb{C} \colon \sum_{t_0 : Z(t_0) = 0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \tag{82}$$

with orthonormal basis $\{e_{t_0}\}_{t_0 : Z(t_0) = 0}$ *where*

$$e_{t_0}(t_1) = \delta_{t_0 t_1} \tag{83}$$

Proof. By the atomic form of μ, for any $f \in L^2(\mu)$:

$$\|f\|_{\mathcal{H}}^2 = \int |f(t)|^2 \, \mu(dt) \tag{84}$$

$$= \int |f(t)|^2 \sum_{t_0 : Z(t_0) = 0} \delta_{t_0}(dt) \tag{85}$$

$$= \sum_{t_0 : Z(t_0) = 0} |f(t_0)|^2 \tag{86}$$

This shows the isomorphism with ℓ^2. The functions e_{t_0} defined by

$$e_{t_0}(t_1) = \delta_{t_0}(t_1) \tag{87}$$

satisfy the relations

$$\begin{aligned}
\langle e_{t_0}, e_{t_1} \rangle &= \int e_{t_0}(t) \overline{e_{t_1}(t)} \mathrm{d}\mu(t) \\
&= \sum_{t : Z(t) = 0} \delta_{t_0}(t) \, \delta_{t_1}(t) \\
&= \delta_{t_0}(t_1) \\
&= \delta_{t_1}(t_0)
\end{aligned} \tag{88}$$

so they form an orthonormal set. Thus, any $f \in \mathcal{H}$ can be written as

$$f = \sum_{t_0 : Z(t_0) = 0} f(t_0) \, e_{t_0} \tag{89}$$

proving they form a basis. $\qquad \square$

Definition 20. *[Multiplication operator] Define the linear operator*

$$L : \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H} \tag{90}$$

by

$$(L f)(t) = t f(t) \tag{91}$$

on the support of μ with domain

$$\mathcal{D}(L) := \left\{ f \in \mathcal{H} : \int |t \, f(t)|^2 \, \mu(dt) < \infty \right\} \tag{92}$$

Theorem 21. *[Self-adjointness and spectrum] L is self-adjoint on $\mathcal{H}$ and has pure point, simple spectrum*

$$\sigma(L) = \overline{\{t \in \mathbb{R} : Z(t) = 0\}} \tag{93}$$

with eigenvalues $\lambda = t_0$ for each zero t_0 and corresponding eigenvectors e_{t_0}.

Proof. First, self-adjointness is verified. For $f, g \in \mathcal{D}(L)$:

$$
\begin{aligned}
\langle L f, g \rangle &= \int (L f)(t) \overline{g(t)} \mathrm{d}\mu(t) \\
&= \int t \, f(t) \overline{g(t)} \, \mathrm{d}\mu(t) \\
&= \int f(t) \overline{t \, g(t)} \, \mathrm{d}\mu(t) \\
&= \int f(t) \overline{(L \, g)(t)} \mathrm{d}\mu(t) \\
&= \langle f, L g \rangle
\end{aligned}
\tag{94}
$$

Thus L is symmetric and acts as

$$(L f)(t_0) = t_0 f(t_0) \tag{95}$$

for each t_0 in the atomic representation where

$$Z(t_0) = 0 \tag{96}$$

This is unitarily equivalent to the diagonal operator on ℓ^2 with diagonal entries

$$\{t_0 : Z(t_0) = 0\} \tag{97}$$

Such diagonal operators are self-adjoint. For the spectrum calculation:

$$L\, e_{t_0} = t_0\, e_{t_0} \forall \{t_0\colon Z(t_0) = 0\} \tag{98}$$

holds, so each t_0 is an eigenvalue of L with eigenvector e_{t_0} and since $\{e_{t_0}\}$ forms an orthonormal basis, L has pure point spectrum. The spectrum of a diagonal operator equals the closure of the set of diagonal entries, hence

$$\sigma(L) = \overline{\{t_0\colon Z(t_0) = 0\}} \tag{99}$$

The eigenvalues are simple. $\qquad\square$

Remark 22. [Operator scaffold] The construction

$$\text{stationary } X \xrightarrow{\ U_\theta\ } \text{oscillatory } Z \xrightarrow{\ \mu = \delta(Z)|\dot{Z}|\,dt\ } L^2(\mu) \xrightarrow{\ L\colon t\cdot\ } (L, \sigma(L)) \tag{100}$$

produces a concrete self-adjoint operator whose eigenvalues equal the zero set of Z and whose spectrum equals the closure of the zero set, determined by the choice of time-change θ and spectral measure F. This provides an explicit realization consistent with Hilbert–Pólya approaches to encoding arithmetic information in operator spectra.

6 Appendix: Regularity and Simple Zeros

Definition 23. [Regularity and simplicity] *Assume $Z \in C^1(\mathbb{R})$ and every zero is simple:*

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \tag{101}$$

Lemma 24. [Local finiteness and delta decomposition] *Under Definition 23, zeros are locally finite and*

$$\delta(Z(t)) = \sum_{t_0\colon Z(t_0)=0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \tag{102}$$

whence

$$\mu = \sum_{t_0\colon Z(t_0)=0} \delta_{t_0} \tag{103}$$

Proof. Since $Z \in C^1(\mathbb{R})$ and $\dot{Z}(t_0) \neq 0$ at each zero t_0, the inverse function theorem implies that Z is locally invertible near each zero. Specifically, there exists a neighborhood U_{t_0} of t_0 such that $Z|_{U_{t_0}}$ is strictly monotonic and invertible.

This implies zeros are isolated: if $Z(t_0) = 0$ and $\dot{Z}(t_0) \neq 0$, then there exists $\epsilon > 0$ such that $Z(t) \neq 0$ for $0 < |t - t_0| < \epsilon$. Therefore zeros are locally finite (finitely many in any bounded interval).

For the distributional identity, the one-dimensional change of variables formula for the Dirac delta is considered. If $g : I \to \mathbb{R}$ is C^1 on interval I with $\dot{g}(x) \neq 0$ for all $x \in I$, then

$$\delta(g(x)) = \sum_{x_0 : g(x_0) = 0} \frac{\delta(x - x_0)}{|\dot{g}(x_0)|} \tag{104}$$

Applying this locally around each zero t_0 of Z, and since zeros are isolated, the local results can be patched together to obtain the global identity:

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0) = 0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \tag{105}$$

Consequently:

$$
\begin{aligned}
\mu(dt) &= \delta(Z(t)) |\dot{Z}(t)| \, dt \\
&= \sum_{t_0 : Z(t_0) = 0} \frac{|\dot{Z}(t)|}{|\dot{Z}(t_0)|} \delta(t - t_0) \, dt \\
&= \sum_{t_0 : Z(t_0) = 0} \delta_{t_0}(dt)
\end{aligned}
\tag{106}
$$

where the last equality uses the fact that

$$\frac{|\dot{Z}(t_0)|}{|\dot{Z}(t_0)|} = 1 \tag{107}$$

when evaluating at $t = t_0$. $\qquad\square$