On Relativistic Quantum Mechanics and the Mass Operator

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The theory of the electron developed by Bopp, Hönl, and the author is discussed in relation to the relativistic wave equation of Gelfand and Yaglom, with particular regard to the infinite, hermitian representations of its matrix apparatus. It is shown that the ambiguity left in the work of the Russian authors is completely removed in our formulation. Only the form of the mass operator, which depends, in Bopp's interpretation, on the cut-off function in the Born-Bopp variation principle, remains uncertain. It is shown that in the infinite representations the mass operator must necessarily depend on one of the invariants (K) of the momentum tensor, because otherwise its eigenvalues would have a point of accumulation at zero. In order to avoid this possibility, the diagonal transformation of the invariant in question is accomplished, and a necessary condition for the analytic form of the mass operator is established, which considerably restricts its choice.

I

IN various papers, Bopp and the author have put forward the theory of the electron in an attempt to avoid the difficulties of field quantization by a recourse to some unexhausted possibilities of the older theory. In short, the general idea may be characterized as follows. Surely an electron, if its finer interactions with radiation are considered, is a system of an infinity of degrees of freedom. In quantum electrodynamics, as it is commonly used, these degrees are introduced by the coupling of the moving charge with quantized hohlraum states. This procedure is well known to lead to difficulties by divergence of most of the occurring summations over the frequency spectrum. In the classical theory the energy of a point electron is also infinite, but for a charge extended in space it becomes finite. On the other hand, the equation of motion of an extended electron, if developed in powers of the retardation parameter, includes time derivatives up to an infinite order. This corresponds again to an infinite number of degrees of freedom. In this scheme, however, it is easy to restrict it in a controlled way by breaking off the development at a convenient place. Likewise, one may also get a purely point mechanical or, as Bopp puts it, field-mechanical theory of the electron without infinities in quantum mechanics.

Starting with radiation reaction forces, the author was led to promising analogies with Dirac's theory of the electron.[1] Indeed it is possible to explain the independence of momentum and velocity and some more kinematical details in Dirac's theory by the fact that the equation of motion of a radiating point electron is of the third order. Attempts to write the equations of motion in the form of poisson brackets necessitated the introduction of an acceleration dependent rest mass[2] and of a magnetic moment of the particle as an independent variable.[3] After these accomplishments it was possible to quantize the theory by reinterpretation of the poisson brackets as a closed commutator algebra.[4]

A better foundation of the theory, whose connection with the radiation reaction force had become somewhat indistinct, was given by Bopp,[5] who was able to show that a charge by its mere extension in space gets an acceleration-dependent rest mass and third (and higher) time derivatives in its equation of motion even if the irreversible part of the radiation force is discarded. The motion is then conservative and may be derived from a variation principle first given by Born, which leads automatically to the right poisson brackets. Bopp reached exactly the same quantum-mechanical formalism as the author, although his classical equations of motion look rather different. In fact, they differ from ours only by the disappearance of one of the constants of motion, i.e., by a supplementary condition, that does not affect the system of poisson brackets. The details have been discussed elsewhere.[6] There is no doubt that the supplementary condition, which unduly restricts the quantum-mechanical possibilities, should be removed. Meanwhile, after the quantum-mechanical schemes are coincident and self-consistent, it seems reasonable to go forward with them first.

Another support of the present ideas from a very different standpoint was given by Hönl,[7] who tried to describe the spinning electron by the superposition of a (positive) mass and a (positive-negative) mass dipole. A mass dipole in motion has an angular momentum, just as an electric dipole in motion has a magnetic moment. As Bopp has shown, the Hönl theory just coincides with the first approximation of his own. In this interpretation the appearance of a mass dipole, which makes the point of gravity of the particle noncoincident with its mass, corresponds to the fact that, owing to the emission-reabsorption processes, the energy fluctuates around the

[1] W. Wessel, Z. Physik **92**, 407 (1934).
[2] W. Wessel, Naturwiss. **30**, 606 (1942); Ann. Physik (5) **43**, 565 (1943).
[3] W. Wessel, FIAT Report No. 1131; Z. Naturforsch. **1**, 622 (1946).
[4] W. Wessel, Phys. Rev. **76**, 1512 (1949).
[5] F. Bopp, Z. Physik **125**, 615 (1948); Z. Naturforsch. **1**, 196 (1946); **3a**, 564 (1948).
[6] W. Wessel, Z. Naturforsch. **4a**, 645 (1949).
[7] For the present state and bibliography of this work, accomplished partly in collaboration with A. Papapetrou, see H. Hönl, Z. Naturforsch. **3a**, 573 (1948).

charge.[8] Charge fluctuations may happen even when the total charge is zero; hence, the Bopp-Hönl model admits a spin also for noncharged particles, which were a grave difficulty for the original conception of the author.

Regarding the quantum-mechanical aspect, the three preceding theories coincide in the following statements. In addition to being described by its coordinates and momentum components, the electron is characterized by[9]

(a) a six-vector (momentum tensor) M_{ik}, whose components obey the commutation rules

$$[M_{ik}M^{rs}]=i(\delta_i{}^rM_k{}^s-\delta_k{}^rM_i{}^s-\delta_i{}^sM_k{}^r+\delta_k{}^sM_i{}^r); \quad (1.1)$$

(b) two (!) four-vectors, ι_k and κ_k, $k=1.4$, obeying the relations

$$[\iota_iM^{rs}]=i(\delta_i{}^s\iota^r-\delta_i{}^r\iota^s), \quad (1.2)$$

$$[\kappa_iM^{rs}]=i(\delta_i{}^s\kappa^r-\delta_i{}^r\kappa^s), \quad (1.3)$$

$$-[\iota_i\iota^k]=iM_i{}^k=[\kappa_i\kappa^k]; \quad (1.4)$$

(c) two invariants, I and K, connected with the invariants of the momentum tensor by

$$\tfrac{1}{2}M_{ik}M^{ik}=I^2-K^2-1, \quad (1.5)$$

$$\tfrac{1}{2}M_{ik}{}^*M^{ik}=IK, \quad (1.6)$$

from which I commutes with all quantities, whereas K commutes with the M_{ik} only, being related to the ι_k, κ_k by

$$[\iota_i\kappa^k]=-iK\delta_i{}^k,\ [K\iota_k]=i\kappa_k,\ [\kappa^kK]=-i\iota^k. \quad (1.7)$$

Further relations fulfilled by these quantities may be disregarded for the present purpose.

The momentum-energy relation ("wave equation") reads (sum over repeated indices)

$$\iota_kp^k=-m(I,K)c\cdot(I^2+K^2)^{\frac{1}{2}}, \quad (1.8)$$

i.e., the possible energy states are determined by the eigenvalues of cp^4 in this equation, if the momentum components p_1, p_2, p_3 are given. It is a main result of the present theory, that Eq. (1.8) is derived from classical conceptions and not freely postulated as the similar relations in the papers mentioned below. External fields may be introduced in the usual way by their potentials without supplementary conditions. The function $m(I, K)$ is the mass operator, a previously arbitrary function of the character of an invariant, with which this paper will mainly be concerned. The reasons for its arbitrariness are best seen from Bopp's interpretation, where it stems from the arbitrariness of the relativistic "cut-off" function, which determines the "shape" of the electron in the variation principle. The separation of the radical on the right is advised by the fact, that in the classical theory $\iota_k/(I^2+K^2)^{\frac{1}{2}}$ is the four velocity.

II

Mathematically, the foregoing commutation rules establish a so-called Lie algebra. Owing to its derivation from poisson brackets, it is essentially closed; i.e., the commutator $ab-ba$ of two elements a, b leads to an element c belonging to the system. The number of elements is 16 in our scheme; in the classical interpretation of Bopp and Hönl one of them (I) is identically zero. This algebra is connected with the representation theory of the Lorentz group. The commutation rules (1) of the components of the momentum six-vector M_{ik} are just the "nucleus"[10] which "engenders" such representations. In the frame of our commutation relations (1.1)–(1.4), (1.7) the M_{ik} form a sub-algebra. They are known to admit finite and infinite representations. Accordingly, the matrices of the whole system (1.1)–(1.7) may be finite or infinite.

Finite representations have already been studied by many authors from mathematical points of view.[11] They do not always comply with all of the foregoing requirements; for instance, the representations discussed by Bhabha are not always closed. Closed finite representations of the whole system have been derived by Bopp and Bauer.[12] These authors have been able to show that the Dirac and Duffin-Kemmer equations are contained in the present scheme, so that it is now possible to trace these theories back to purely classical concepts.

Further details about the finite representations lie out of the scope of the present paper. Indeed, as Bopp and Bauer have shown, one obtains (for instance) the Dirac and Kemmer equations in their original forms, but no more. Now, since the starting point of the present theory is the finer interaction of the electron with its own field, one should expect also to have effects like the Lamb-Retherford shift included. Indeed, there exist still the infinite representations mentioned above. They have exactly the same physical basis as the finite ones, but involve a much ampler mathematical apparatus; and it is very suggestive to assume that they provide the necessary generalization, for instance, of Dirac's theory. For the physicist accustomed to ordinary quantum mechanics these representations are particularly attractive because they provide infinite, hermitian matrices, partly with continuous spectra, for all physical quantities, especially for the velocity components. It was primarily this feature, and subsequently the important

[8] For a short survey see F. Bopp, Z. angew. Physik **1**, 387 (1949).

[9] For the physical interpretation of the following quantities the reader must be referred to the original papers. Mathematically, they are substitutes for the higher time derivatives in the equation of motion of an electron with a finite extension in space ι_k. The vector is the four velocity but for a factor.

[10] van der Waerden, *Die gruppentheoretische Methode in der Quantenmechanik* (Verlag. Julius Springer, Berlin, 1932); H. Weyl, *The Classical Groups* (Princeton University Press, Princeton, New Jersey, 1946).

[11] Kramers, Belinfante, and Lubanski, Physica **8**, 597 (1941); H. Bhabha, Revs. Modern Phys. **17**, 200 (1945); **21**, 451 (1949); Madhavarao, Thiruvenkatachar, and Venkatachaliengar, Proc. Roy. Soc. (London) **A187**, 385 (1946); H. Hönl and H. Boerner, Z. Naturforsch. **5a**, 353 (1950); K. J. Le Couteur, Proc. Roy. Soc. (London) **A202**, 284, 394 (1950).

[12] F. Bopp and F. L. Bauer, Z. Naturforsch. **4a**, 611 (1949).

fact that they permit the setting up of wave equations with purely positive energy,[4] that led the present author to their investigation. Besides, it is a favorite idea of the author that the infinite dimensionality of the Hilbert space is correlate to the infinite number of degrees of freedom of the moving particle, and it seems an advantageous coincidence to get it imposed by so a natural postulate as hermiteity.

The representation theory of the Lorentz group, i.e., the realization of the rules (1.1), by infinite matrices has been analyzed by Gelfand and Neumark, Harish-Chandra, and Bargmann.[13] Just as the representations of the angular momentum (rotation group) are characterized by one parameter, *viz.*, the value of the total angular momentum, so are the momentum components M_{ik} (Lorentz group) characterized by two parameters, for which the quantities I and K of Eqs. (1.5), (1.6) may be chosen. As long as one is only concerned with the Lorentz group, i.e., with (1.1), these parameters are of a scalar (c-number) character, because they commute with all M_{ik}. In our present scheme I is still scalar (although quantized), but K must be represented by a matrix because it does not commute with the ι_k, κ_k-components, which in turn reproduce the M_{ik} in virtue of Eq. (1.4).

We have already shown,[6] starting from Harish-Chandra's paper, how the necessary generalization may be accomplished. In the meantime, papers of Gelfand and Yaglom[14] became known in this country, which come nearer to our aims in so far as they introduce, besides the M_{ik}, a four-vector L^k, which corresponds (*viz.*, for the infinite representations) to our ι^k. The Russian authors postulate, as usual, the relation (1.8), *n.b.*, with constant right side, demanding no more than relativistic invariance of this equation. It is interesting from the physical standpoint to see that this postulate is not sufficient to determine the representations completely. As the authors show, it leads to our formulas (1.2), where the M_{ik} must be known from Eq. (1.1). Moreover, it is easily seen [see also our formula (3.25)[6]], that once ι^4 (resp. L^0) is gained, the other ι^k, $k=1, 2, 3$, may immediately be constructed by

$$\iota_k = i[\iota^4 M_{4k}]. \tag{2.1}$$

Now, without further information, the component ι^4 remains widely undetermined insofar, as a whole set of constants [$c_{\tau\tau'}$, Eqs. (3.13), (3.14)[14]] may arbitrarily be chosen. In our representations ι^4 is completely determined, *viz.*, equal to $|I|+1$, $|I|+2$, $\cdots$, where I is the minimum spin,[6] and so are all other matrices. Indeed, it is evident from the derivation of our commutation rules that they contain far more physical information than is yielded by the mere principle of invariance. It is exhibited by the existence of another four-vector, κ^k, which in virtue of the relations (1.7) makes the invariant K a matrix, whereas in the work of Gelfand and Yaglom K is still a (complex) number ($=ik_1$) and κ^k does not appear. The question arises, whether these relations (1.7) together with the formulas (1.3) and (1.4), which provide for the closedness of the algebra, make the above choice necessary. In the foregoing paper[6] only its sufficiency has been demonstrated by mere verification. Indeed, it is not very difficult to show, that there exists no other choice. The proof is given in Appendix I.[15]

III

The wave equation (1.8) is now well defined but for the analytic form of $m(I, K)$ on its right side. The origin of this uncertainty has already been mentioned at the end of Sec. I. An analytic dependence exists only on K, because I is discontinuous. For brevity let us write in the following:

$$m(I, K)\cdot(I^2+K^2)^{\frac{1}{2}} = m_I(K). \tag{3.1}$$

As long as K is a scalar, m_I is simply a constant parameter. It is a special feature of our theory, by providing a matrix character of K, to provide a matrix nature of $m_I(K)$, and it is easy to see that m_I must be a matrix in these infinite representations. Indeed, consider a particle in free motion, i.e., with constant momentum vector p_1, p_2, p_3, and chose the system of coordinates so that $p_1=p_2=p_3=0$. The particle will then be "macroscopically" at rest, although its "internal motion" (determined by the ι^k, K, etc., mechanism) persists, and p^4/c will be its rest mass m, so that Eq. (1.8) becomes (note $\iota^4=-\iota_4$):

$$m\iota^4 = m_I(K). \tag{3.2}$$

Now the eigenvalues of ι^4 are of the form $|I|+n$, $n=$integral number; hence, with a constant m_I the eigenvalues of m would be inversely proportional to $|I|+n$, $n=0, 1, 2, \cdots$, i.e., they would have a point of accumulation at $m=0$, an evidently unacceptable consequence. The same thing happens in all examples selected by Gelfand and Yaglom. The seriousness of this argument seems not yet to be recognized, and indeed it could not be as long as the unambiguous character of the infinite representations was not shown. It should be remembered, however, that a similar calamity exists in all finite representations, where the eigenvalues of $(\iota^4)^{-1}$, although, of course, without points

[13] J. M. Gelfand and M. A. Neumark, J. Phys. (U.S.S.R.) **10**, 93 (1946); Bull. Acad. Sci. URSS., Sér. Mat. **11**, 411 (1947); V. Bargmann, Ann. Math. **48**, 568 (1947); Harish-Chandra, Proc. Roy. Soc. (London) **A189**, 372 (1947).

[14] J. M. Gelfand and A. M. Yaglom, Doklady Akad. Nauk SSSR **59**, 655 (1948); Zhur. Eksp. i Teoret. Fiz. **18**, 703 (1948).

[15] For the sake of a complete survey it might seem desirable to derive this result by a direct reduction of the scheme of Gelfand and Yaglom, but this does not seem easily possible, because, in spite of great formal similarities, our representations are not strictly a special case of theirs. We have already shown that the spectrum of our K, that is the ik_1 of the Russian authors, is continuous. (It will be an essential subject of the second half of this paper to accomplish the diagonal transformation of this matrix.) As a consequence, the number of irreducible representations of the Lorentz group involved in the sense of Gelfand and Neumark (labeled by k_0[$=$our I] and k_1) is not countably infinite, in contradiction to a basic assumption made by these authors.

of accumulation, have far too simple intervals to be actually comparable to the very irregular mass distributions realized in nature.

For this reason a constant m_{I} should be excluded. On the other hand, it seems possible, after a whole function of K is available, to represent almost every desired mass spectrum by its appropriate choice; but this, too, is not as simple as it seems. Let us formulate the problem more precisely. Our representation is essentially a ι^4-representation, i.e., ι^4 is diagonal, and K is given in the form $\langle\iota^4|\mathrm{K}|\iota^{4\prime}\rangle$. Being all eigenvalues of ι^4 essentially positive, the matrix $(\iota^4)^{-1}$ can immediately be written down; and so, from Eq. (3.2),

$$m=(\iota^4)^{-1}m_{\mathrm{I}}(\mathrm{K}), \tag{3.3}$$

provided that the matrix $m_{\mathrm{I}}(\mathrm{K})$ is well defined in ι^4-representation. The admissible values of m are the eigenvalues of the matrix on the right side of Eq. (3.3). For a real computation we have to go over from the ι^4- to a K-representation, because, apart from reasons of practicability, only then is the radical in Eq. (3.1) well defined. The operator $(\iota^4)^{-1}$ becomes a symmetrical integral kernel,

$$(\iota^4)^{-1}=\langle\mathrm{K}|(\iota^4)^{-1}|\mathrm{K}'\rangle, \tag{3.4}$$

and Eq. (3.3) goes over, after division by m, into an integral equation of the form

$$\psi_m(\mathrm{K})=(1/m)\int\langle\mathrm{K}|(\iota^4)^{-1}|\mathrm{K}'\rangle m_{\mathrm{I}}(\mathrm{K}')\psi_m(\mathrm{K}')d\mathrm{K}', \tag{3.5}$$

where $\psi_m(\mathrm{K})$ is the eigenfunction belonging to the eigenvalue m of the mass. By a transformation

$$\psi_m(\mathrm{K})=[m_{\mathrm{I}}(\mathrm{K})]^{-\frac{1}{2}}\chi_m(\mathrm{K}), \tag{3.6}$$

Eq. (3.5) is easily brought into symmetrical standard form. The difficulty is now that m appears as the reciprocal eigenvalue of this equation. It is well known that the eigenvalues of a symmetrical integral equation increase towards infinity as long as the kernel is regular;[16] hence, the eigenvalues of m will continue converging to zero unless $m_{\mathrm{I}}(\mathrm{K})$ is chosen so as to make the kernel singular. We will see in the next section, that $\langle\mathrm{K}|(\iota^4)^{-1}|\mathrm{K}'\rangle$ is strongly regular, so that this condition really restricts the choice of $m_{\mathrm{I}}(\mathrm{K})$. At first glance this argument may appear rather artificial, but merely from a physical point of view the mass values occurring in nature look far more like the eigenvalues of a singular than of a regular integral equation; and since the difficulty is of a very general character, it seems worth while to study its consequences thoroughly. For this reason, we compute in the following two sections the $\langle\iota^4|\mathrm{K}\rangle$ transformation function and, with a slight simplification, the operator (3.4) explicitly.

IV

Our first problem will be to bring K into diagonal form. In the ι^4-representation it is given, according to[6] Eq. (3.22), by

$$\langle\iota^4|\mathrm{K}|\iota^{4\prime}\rangle=\tfrac{1}{2}\{[(\iota^4+\sigma)(\iota^4-\sigma-1)]^{\frac{1}{2}}\delta_{\iota,\iota'+1}+[(\iota^4+\sigma+1)(\iota^4-\sigma)]^{\frac{1}{2}}\delta_{\iota,\iota'-1}\}. \tag{4.1}$$

We use here the symbol σ instead of our former κ, to avoid confusion with the components of κ^k. In the following formulas the index 4 of ι^4 will be dropped, for typographical reasons, when ι^4 is used as an index. There will be a unitary matrix $U_{\iota\mathrm{K}}$ depending on σ and I, so that

$$\textstyle\sum_{\iota'\iota''}U_{\mathrm{K}'\iota'}{}^{\dagger}\langle\iota'|\mathrm{K}|\iota''\rangle U_{\iota''\mathrm{K}''}=\langle\mathrm{K}'|\mathrm{K}|\mathrm{K}''\rangle. \tag{4.2}$$

By a left side multiplication with $U_{\iota\mathrm{K}'}$ we obtain from Eq. (4.2), using Eq. (4.1),

$$\tfrac{1}{2}\{U_{\iota+1,\mathrm{K}}[(\iota^4+\sigma+1)(\iota^4-\sigma)]^{\frac{1}{2}}+U_{\iota-1,\mathrm{K}}[(\iota^4+\sigma)(\iota^4-\sigma-1)]^{\frac{1}{2}}\}=U_{\iota,\mathrm{K}}\cdot\mathrm{K}. \tag{4.3}$$

For simplification put

$$U_{\iota\mathrm{K}}=[\Pi(\iota^4-\sigma-1)/\Pi(\iota^4+\sigma)]^{\frac{1}{2}}V_{\iota\mathrm{K}}. \tag{4.4}$$

Insertion into Eq. (4.3) gives

$$(\iota^4-\sigma)V_{\iota+1,\mathrm{K}}+(\iota^4+\sigma)V_{\iota-1,\mathrm{K}}=2\mathrm{K}\cdot V_{\iota,\mathrm{K}}. \tag{4.5}$$

From this recurrence formula the $V_{\iota\mathrm{K}}$ may easily be determined but for a common factor depending on σ alone. The lowest value of ι^4 is $\sigma+1$.[6] To avoid too much formal complication let us henceforth consider only the case $\mathrm{I}=\frac{1}{2}$. The possible values of σ are then[6] $\sigma=\frac{1}{2},\ \frac{3}{2},\ 5/2,\ \cdots\iota^4-1$. Instead of considering these values of σ for a given ι^4 we may as well consider the values

$$\iota^4=\sigma+1+\nu,\quad \nu=0,1,2\cdots \tag{4.6}$$

for a given σ. Putting for brevity

$$2\mathrm{K}=z \tag{4.7}$$

and writing

$$V_{\iota\mathrm{K}}=W_{\iota\sigma}(z)N_{\sigma}{}^{\frac{1}{2}}(z), \tag{4.8}$$

we obtain from Eq. (4.5), for instance, with $\sigma=\frac{1}{2}$

$$\begin{aligned}
&W_{\frac{3}{2}\frac{1}{2}}(z)=1,\quad W_{7/2,\frac{1}{2}}(z)=\tfrac{1}{2}(z^2-3),\\
&W_{\frac{5}{2}\frac{1}{2}}(z)=z,\quad W_{9/2,\frac{1}{2}}(z)=\tfrac{1}{6}(z^3-11z),\\
&W_{11/2,\frac{1}{2}}(z)=(1/24)(z^4-26z^2+45),\\
&W_{13/2,\frac{1}{2}}(z)=(1/120)(z^5-50z^3+309z)
\end{aligned} \tag{4.9}$$

etc.

By the usual methods these polynomials may be shown to be generated, for arbitrary σ, by the function

$$w_{\sigma}(z,t)=e^{z\,\mathrm{arc}\,\tan t}(1+t^2)^{-\sigma-1} \tag{4.10}$$

as the coefficients of the powers of t in its development at $t=0$:

$$w_{\sigma}(z,t)=\sum_{\iota=\sigma+1}^{\infty}W_{\iota\sigma}(z)t^{\iota-\sigma-1}. \tag{4.11}$$

[16] We disregard the possibility of a degeneracy of the kernel, which would also settle the problem, but which seems mathematically very unlikely.

To find the normalizing function $N_\sigma(z)$, we start with

$$\int_{-\infty}^{+\infty} d\mathrm{K} U_{\iota\mathrm{K}} U_{\mathrm{K}\iota'}{}^{\dagger} = \delta_{\iota\iota'}. \quad (4.12)$$

With regard to Eqs. (4.4) and (4.7), (4.8) this means

$$\frac{1}{2}\int_{-\infty}^{+\infty} dz N_\sigma(z) W_{\iota\sigma}(z) W_{\iota'\sigma}(z) = \frac{\Pi(\iota^4+\sigma)}{\Pi(\iota^4-\sigma-1)}\delta_{\iota\iota'}. \quad (4.13)$$

Multiplying $w_\sigma(z, t)$ with a given $N_\sigma(z)\cdot W_{\iota\sigma}(z)$ and integrating, we get from Eqs. (4.11) and (4.10) with regard to Eq. (4.13)

$$t^{\iota-\sigma-1} = \frac{\Pi(\iota^4-\sigma-1)}{\Pi(\iota^4+\sigma)} \cdot \frac{1}{2}\int_{-\infty}^{+\infty} N_\sigma(z) W_{\iota\sigma}(z) dz (1+t^2)^{-\sigma-1} e^{z \arctan t}. \quad (4.14)$$

This formula must be an identity in ι^4, and we may use the smallest value of ι^4, i.e., $\iota^4=\sigma+1$, for the determination of N_σ. The polynomial $W_{\iota\sigma}$ becomes then $W_{\sigma+1,\sigma}=1$, and with the substitution

$$\arctan t = \vartheta; \quad (4.15)$$

formula (4.14) may be written ($|\vartheta|<\pi/2$)

$$\int_{-\infty}^{+\infty} N_\sigma(z) e^{\vartheta z} dz = 2(2\sigma+1)!(\cos\vartheta)^{-2\sigma-2}. \quad (4.16)$$

Inversion gives immediately

$$N_\sigma(z) = 2(2\sigma+1)!(1/2\pi)\int_{-\infty}^{+\infty} e^{-i\vartheta z}(\cosh\vartheta)^{-2\sigma-2} d\vartheta$$

$$= i^{2\sigma+1} \lim_{\epsilon=0} \frac{d^{2\sigma+1}}{d\epsilon^{2\sigma+1}}\left(\frac{\epsilon}{\sinh\epsilon}\right)^{2\sigma+2}$$

$$e^{-i\epsilon z}\begin{cases} \sinh^{-1}\frac{1}{2}\pi z & \sigma \text{ integral}, & (4.17)\\ \cosh^{-1}\frac{1}{2}\pi z & \sigma \text{ half odd integral}. & (4.18)\end{cases}$$

Of course, N_σ is always real. Especially, we have

$$N_0(z) = z/\sinh\tfrac{1}{2}\pi z, \quad N_{\frac{1}{2}}(z) = (1+z^2)/\cosh\tfrac{1}{2}\pi z, \quad N_1(z) = z(1+\tfrac{1}{4}z^2)/3\sinh\tfrac{1}{2}\pi z. \quad (4.19)$$

It is not very difficult to verify Eq. (4.14) for arbitrary ι^4 by insertion of the integral form (4.17) of $N_\sigma(z)$. A test for Eq. (4.19) is that $N_\sigma(z)$ must be finite for $|z|<2\sigma+2$. With formulas (4.4), (4.8), (4.9) [resp. (4.10), (4.11)], and (4.18) the desired unitary transformation is completely established. Owing to the real character of the eigenfunctions, there is simply $U_{\mathrm{K}\iota}{}^\dagger = U_{\iota\mathrm{K}}$.

V

We now go over to the construction of $(\iota^4)^{-1}$, formula (3.4), in K-representation. For simplicity let us make two unimportant restrictions. First let us be content with the simplest case

$$\sigma = \tfrac{1}{2}.$$

It will be seen that the kernel is regular in this case, and there is no reason to believe that it will behave differently for higher σ-values. Furthermore, to study the asymptotic behavior of the eigenvalues of m, it will make no essential difference if we substitute $(\iota^4)^{-1}$ by $(\iota^4-\frac{1}{2})^{-1}$, which may be evaluated in closed form. Owing to the diagonality of ι^4, the transformation corresponding to Eq. (4.2) reduces to a single sum:

$$\langle \mathrm{K}|(\iota^4-\tfrac{1}{2})^{-1}|\mathrm{K}'\rangle = \textstyle\sum_\iota U_{\mathrm{K}\iota}{}^\dagger (\iota^4-\frac{1}{2})^{-1} U_{\iota\mathrm{K}'}. \quad (5.1)$$

The polynomials $W_{\iota\sigma}(z)$ may be represented, after Eqs. (4.10) and (4.11), by derivations:

$$W_{\sigma+1+\nu,\sigma}(z) = \frac{1}{\nu!}\frac{\partial^\nu}{\partial t^\nu} w_\sigma(z, t)\bigg|_{t=0}. \quad (5.2)$$

Thus the series (5.1) has the form (with $\iota^4=\sigma+1+\nu$, $2\mathrm{K}=z$, as before, and $\sigma=\frac{1}{2}$):

$$\langle z/2|(\iota^4-1/2)^{-1}|z'/2\rangle = (N_{\frac{1}{2}}(z)N_{\frac{1}{2}}(z'))^{\frac{1}{2}} \sum_{\nu=0}^{\infty} \frac{\Pi(\nu)(\nu+1)^{-1}}{\Pi(\nu+2)} \times (1/\nu!)^2 \frac{\partial^\nu w_{\frac{1}{2}}(z, t)}{\partial t^\nu}\frac{\partial^\nu w_{\frac{1}{2}}(z', t')}{\partial t'^\nu}\bigg|_{t=t'=0}. \quad (5.3)$$

The summation process is described in Appendix II. As a result, we find (the indices $\frac{1}{2}\frac{1}{2}$ of S refer to $\mathrm{I}=\frac{1}{2}$, $\sigma=\frac{1}{2}$):

$$\langle \mathrm{K}|(\iota^4-\tfrac{1}{2})^{-1}|\mathrm{K}'\rangle = S_{\frac{1}{2}\frac{1}{2}}(\mathrm{K},\mathrm{K}')/(\cosh\pi\mathrm{K}\cdot\cosh\pi\mathrm{K}')^{\frac{1}{2}} \quad (5.4)$$

$$S_{\frac{1}{2}\frac{1}{2}}(z/2, z'/2) = \tfrac{1}{2}\left\{\frac{z-z'}{\sinh\frac{1}{4}\pi(z-z')}\cosh\tfrac{1}{4}\pi(z+z') + \frac{z+z'}{\cosh\frac{1}{4}\pi(z-z')}\sinh\tfrac{1}{4}\pi(z+z') - 2zz'\right\} \times (1+z^2)^{-\frac{1}{2}}(1+z'^2)^{-\frac{1}{2}}. \quad (5.5)$$

Of course, S is symmetrical. Introducing this into Eq. (3.5) and using the transformation (3.6), one finds approximately, allowing for the substitution of ι^4 by $\iota^4-\frac{1}{2}$,

$$\chi_m(\mathrm{K}) \simeq \frac{1}{m}\int_{-\infty}^{+\infty} S_{\frac{1}{2}\frac{1}{2}}(\mathrm{K},\mathrm{K}') \times \left(\frac{m_{\frac{1}{2}}(\mathrm{K})m_{\frac{1}{2}}(\mathrm{K}')}{\cosh\pi\mathrm{K}\cdot\cosh\pi\mathrm{K}'}\right)^{\frac{1}{2}} \chi_m(\mathrm{K}')d\mathrm{K}'. \quad (5.6)$$

It is easily verified that $S_{\frac{1}{2}\frac{1}{2}}(\mathrm{K},\mathrm{K}')$ is finite for all real values of K and K′; hence, the regularity (quadratic integrability) of the kernel in Eq. (5.6) is entirely determined by the relation $m_{\frac{1}{2}}(\mathrm{K})$: $\cosh\pi\mathrm{K}$, and one sees immediately that the integral equation will be regular, i.e., that zero-accumulation of the m-values will occur, unless $m_{\frac{1}{2}}(\mathrm{K})$ has either singularities of at least first order for finite K or else outweighs $\cosh\pi\mathrm{K}$ at infinity. In the second case $m_{\frac{1}{2}}(\mathrm{K})$ would have to behave at infinity like $\mathrm{K}^{\delta}\cosh\gamma\mathrm{K}$ with $\gamma=\pi$, $\delta\geq -1$ or $\gamma>\pi$, δ arbitrary. For $m(\mathrm{I}, \mathrm{K})$, formula (3.1), the same conditions hold with $\delta-1$.

With that we have at least a necessary condition, which allows exclusion of a great variety of functions. We shall not enter here into more details about the presumable form of $m(\mathrm{I}, \mathrm{K})$, in order not to spoil the conclusiveness of the present arguments by considerations of a necessarily more hypothetical character; it may briefly be mentioned[17] that considerations along the lines of those given in reference 3 favor the choice of an operator $m(\mathrm{I}, \mathrm{K})=m_0\cosh[(3\hbar c\mathrm{K}/2e^2)+\mathrm{const}]$. On the other hand, a few words should be added about a formal difficulty, which might cause some trouble: evidently, the operator $m_{\mathrm{I}}(\mathrm{K})$ does not exist in ι^4-representation. This means only that the ι^4-representation is inappropriate and that one has to use others. In the present case, one will either continue working with the K-representation, i.e., with the singular integral equation, or else invert formula (3.2) and start with $[m\mathrm{I}(\mathrm{K})]^{-1}$, whose matrix elements will be excellently convergent in ι^4-representation.[18]

APPENDIX I

The formulas quoted in the following are those of the author (A)[6] and those of Gelfand and Yaglom (GY),[14] second paper. The relations to be proved are the formulas A (3.23), (3.24) for the a_σ which substitute the B_k in GY (2.2″), (2.3″). The last formula is coincident with A (3.13), if $\mathrm{K}=ik_1$ is taken as a scalar. The formulas A (3.11) and GY (2.3′) coincide immediately. Note that our κ, I are the k, k_0 of GY. In the following we write σ instead of κ, with regard to the use in the rest of the present paper.

Starting with A (3.9), which corresponds to GY (2.2″), we obtain in virtue of the relations

$$\iota_k=i[\iota^4\mathrm{M}_{4k}],\quad [\iota^4\iota_k]=i\mathrm{M}_{4k}, \tag{I.1}$$

Eqs. (1.2), (1.4) above, and with A (3.11) and (1.7) above:

$$[\iota^4[\iota^4 a_\sigma]]=a_\sigma,\quad [\iota^4[\iota^4 a_\sigma{}^\dagger]]=a_\sigma{}^\dagger. \tag{I.2}$$

Owing to the diagonality of ι^4, this gives immediately

$$\{(\iota-\iota')^2-1\}\langle\iota|a_\sigma|\iota'\rangle=0, \tag{I.3}$$

and the same relation for $a_\sigma{}^\dagger$ (we omit in the following the index 4 of ι^4 throughout); hence, $\iota=\iota\pm1$, i.e., a_σ and $a_\sigma{}^\dagger$ (hermitian conjugate matrix; note that the a_σ need not be hermitian) must have the forms (an asterisk indicates the complex conjugate quantity),

$$\begin{aligned}\langle\iota|a_\sigma|\iota'\rangle&=\zeta(\sigma,\iota)\delta_{\iota,\iota'+1}+\eta(\sigma,\iota)\delta_{\iota,\iota'-1},\\ \langle\iota|a_\sigma{}^\dagger|\iota'\rangle&=\eta^*(\sigma,\iota')\delta_{\iota,\iota'+1}+\zeta^*(\sigma,\iota')\delta_{\iota,\iota'-1},\end{aligned} \tag{I.4}$$

where the ζ, η are to be determined. We have first to fulfil the first two relations A (3.10), i.e., with regard to A (3.11),

$$\begin{aligned}a_\sigma\mathrm{K}_{\sigma-1}&=\mathrm{K}_\sigma a_\sigma,\\ \mathrm{K}_{\sigma-1}a_\sigma{}^\dagger&=a_\sigma{}^\dagger\mathrm{K}_\sigma.\end{aligned} \tag{I.5}$$

K is given, in ι^4-representation, by A (3.16) in the form

$$\langle\iota|\mathrm{K}|\iota'\rangle=A(\sigma,\iota)\delta_{\iota,\iota'+1}+A^*(\sigma,\iota')\delta_{\iota,\iota'-1}, \tag{I.6}$$

where the A, A^* are determined by A (3.20). They depend on σ by the previously unknown function

$$\rho=\rho(\sigma). \tag{I.7}$$

Substituting (I.4) and (I.6) into (I.5) one obtains, for $(\iota,\iota'+2)$, $(\iota,\iota'-2)$, and (ι,ι') respectively, the three relations

$$\begin{aligned}\zeta(\sigma,\iota)A(\sigma-1,\iota-1)&=\zeta(\sigma,\iota-1)A(\sigma,\iota),\\ \eta(\sigma,\iota)A^*(\sigma-1,\iota+2)&=\eta(\sigma,\iota+1)A^*(\sigma,\iota+1),\\ \zeta(\sigma,\iota)A^*(\sigma-1,\iota)+\eta(\sigma,\iota)A(\sigma-1,\iota+1)&\\ =\eta(\sigma,\iota-1)A(\sigma,\iota)&+\zeta(\sigma,\iota+1)A^*(\sigma,\iota+1)\end{aligned} \tag{I.8}$$

and a similar triple for the conjugate quantities. The η- and ζ-terms on the right side of the third relation (I.8) are reducible by the two foregoing ones to the corresponding terms appearing on the left, so that the $\eta(\sigma,\iota)$ may be expressed by $\zeta(\sigma,\iota)$:

$$\eta(\sigma,\iota)=g(\sigma,\iota)\zeta(\sigma,\iota), \tag{I.9}$$

where

$$g(\sigma,\iota)=\frac{A^*(\sigma-1,\iota+1)\{|A(\sigma-1,\iota)|^2-|A(\sigma,\iota+1)|^2\}}{A(\sigma-1,\iota)\{|A(\sigma,\iota)|^2-|A(\sigma-1,\iota+1)|^2\}}. \tag{I.10}$$

Finally, the $\zeta(\sigma,\iota)$ are determined, in virtue of the relations A (3.13), by the $a_\sigma a_\sigma{}^\dagger$ and $a_\sigma{}^\dagger a_\sigma$. For $(\iota,\iota'\pm2)$ and (ι,ι') one obtains four relations for the $\zeta(\sigma,\iota)$, which must be fulfilled simultaneously and identically in ι by an appropriate choice of the only available function $\rho(\sigma)$. Hence, the problem is considerably overdetermined; and being sure that a solution exists, one has only to show that its determination by any two of the four relations is complete. In order to avoid reproduction of somewhat complicated formulas, this elementary process will better be left to the reader. As a result one obtains formula A (3.21), i.e.,

$$\rho^2=-\sigma(\sigma+1) \tag{I.11}$$

as the only possible solution, and with that the relations A (3.23), (3.24) to be proved. Hence, our representations[6] are completely determined by the full system of commutation relations (1.1)–(1.4), (1.7), if hermitian character is demanded, with the exception, of course, of a unitary transformation, which may change them, for instance, into the form[4] with which we started originally.

The present proof settles also a question raised occasionally by the author regarding the possible existence of positive values of $\mathrm{I}^2+\rho^2$, in the negative sense. Furthermore, it may be mentioned that relation (I.11), which may also be written, because of A (3.18) and (3.6), (3.7),

$$(\kappa^4)^2+\mathrm{K}^2-(\iota^4)^2+\mathrm{M}^2=0 \tag{I.12}$$

represents one of ten algebraic identities of the Dirac algebra, which seem not yet to be noticed.

APPENDIX II

Consider first series (5.3) without the first factor behind the Σ-sign. Its summation may be accomplished in the following way:

$$\begin{aligned}&\sum_{\nu=0}^{\infty}\left(\frac{1}{\nu!}\right)^2\frac{\partial^{2\nu}}{\partial t^\nu\partial t'^\nu}w_\sigma(z,t)w_\sigma(z',t')\\ &\quad=\frac{1}{2\pi}\int_0^{2\pi}d\varphi\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{e^{i\mu\varphi}}{\mu!}\frac{e^{-i\nu\varphi}}{\nu!}\frac{\partial^\mu}{\partial t^\mu}\frac{\partial^\nu}{\partial t'^\nu}w_\sigma(z,t)w_\sigma(z',t')\\ &\quad=\frac{1}{2\pi}\int_0^{2\pi}d\varphi w_\sigma(z,t+e^{i\varphi})w_\sigma(t'+e^{-i\varphi})\end{aligned} \tag{II.1}$$

to be taken with $t=t'=0$. In formula (5.3) this process is rendered more difficult by the factor $\Pi(\nu)/(\nu+1)\Pi(\nu+2)=1/(\nu+1)^2(\nu+2)$. With $(\iota^4)^{-1}$ instead of $(\iota^4-\frac{1}{2})^{-1}$ we would have the still more inconvenient factor $1/(\nu+1)(\nu+\frac{3}{2})(\nu+2)$. The matter would be much simpler, if in (5.3) the quotient of the two Π-functions were inverted, so that we had merely to do with a factor $\Pi(\nu+2)/(\nu+1)\pi(\nu)=\nu+2$ in the numerator, which might easily be generated by differentiation.

Indeed, the determining equation (4.3) is invariant under the substitution $\sigma\rightarrow-\sigma-1$, so that instead of Eq. (4.4) one may put as well

$$U_{\iota\mathrm{K}}=[\Pi(\iota^4+\sigma)/\Pi(\iota^4-\sigma-1)]^{\frac{1}{2}}\bar{V}_{\mathrm{K}\iota}. \tag{II.2}$$

[17] See a forthcoming paper, Z. Naturforsch. **6a** (1951).

[18] There exists also an interesting representation of ι^4 as a difference operator, which makes K diagonal; see another forthcoming paper, Z. Naturforsch. **6a** (1951).

Insertion into Eq. (4.3) now gives

$$(\iota^4+\sigma+1)\bar{V}_{\iota+1,\,\mathrm{K}}+(\iota^4-\sigma-1)\bar{V}_{\iota-1,\,\mathrm{K}}=2\mathrm{K}\cdot\bar{V}_{\iota\mathrm{K}}. \tag{II.3}$$

Of course, the polynomials generated by this formula are identical with those arising from Eq. (4.5) but for a common multiple of $\Pi(\iota^4-\sigma-1)/\Pi(\iota^4+\sigma)$. The advantage is that they can again be expressed in the way of formula (5.2) without further factorials:

$$\bar{V}_{\iota\mathrm{K}}=\bar{W}_{\iota\sigma}(z)N_\sigma^{\frac{1}{2}}(z), \tag{II.4}$$

$$\bar{W}_{\sigma+1+\nu,\,\sigma}(z)=(1/\nu!)\partial^\nu\bar{w}_\sigma(z,t)/\partial t^\nu|_{t=0} \tag{II.5}$$

$$\bar{w}_\sigma(z,t)=(2\sigma+1)(1+t^2)^\sigma\cdot t^{-2\sigma-1}e^{z\,\mathrm{arc}\tan t}\times\int_0^t t^{2\sigma}(1+t^2)^{-\sigma-1}e^{-z\,\mathrm{arc}\tan t}dt. \tag{II.6}$$

For proof write the polynomials (II.5) in the form

$$\bar{W}_{\sigma+1+\nu,\,\sigma}(z)=(1/2\pi i)\oint\bar{w}_\sigma(z,t)dt\cdot t^{-\nu-1}, \tag{II.7}$$

where the integral is taken in the complex t plane along a path inside the unit circle surrounding the point $t=0$. Separating, in the case of $\bar{W}_{\iota+1,\,\sigma}$, a factor $t^{-(2\sigma+\nu+3)}$ instead of $t^{-\nu-1}$ and integrating by parts, it is not very difficult to verify formula (II.6) by substitution into (II.3).

The factors are chosen so that the generating functions $w_\sigma(z,t)$ and $\bar{w}_\sigma(z,t)$ render $W_{\sigma+1,\,\sigma}=\bar{W}_{\sigma+1,\,\sigma}=1$; hence, the normalizing functions N_σ and $\bar{N}_\sigma$ are the same but for a factor that accounts for the inverted factorials in Eqs. (4.4) and (II.2) for $\iota^4=\sigma+1$:

$$\bar{N}_\sigma^{\frac{1}{2}}(z)=[1/(2\sigma+1)!]N_\sigma^{\frac{1}{2}}(z). \tag{II.8}$$

In the case $\sigma=\frac{1}{2}$ formula (II.6) yields

$$\bar{w}_{\frac{1}{2}}(z,t)=2(1+z^2)^{-1}t^{-2}[(1+t^2)^{\frac{1}{2}}e^{z\,\mathrm{arc}\tan t}-(1+zt)]. \tag{II.9}$$

With the help of $\bar{w}_\sigma(z,t)$ one may now set up $(\iota^4-\frac{1}{2})^{-1}$ in the way indicated above. For the case $\sigma=\frac{1}{2}$ it reads

$$\langle z/2|(\iota^4-\tfrac{1}{2})^{-1}|z'/2\rangle=(N_{\frac{1}{2}}(z)N_{\frac{1}{2}}(z'))^{\frac{1}{2}}\sum_{\nu=0}^{\infty}(\nu+2)\times(1/\nu!)^2\frac{\partial^{2\nu}}{\partial t^\nu\partial t'^\nu}\bar{w}_{\frac{1}{2}}(z,t)\bar{w}_{\frac{1}{2}}(z',t')\Big|_{t=t'=0}. \tag{II.10}$$

To calculate it we write first the series (II.1) with $\bar{w}$ instead of w and with a factor $\eta^{\nu+2}$ in its νth term:

$$\sum_{\nu=0}^{\infty}\frac{\eta^{\nu+2}}{(\nu!)^2}\frac{\partial^{2\nu}}{\partial t^\nu\partial t'^\nu}\bar{w}_{\frac{1}{2}}(z,t)\bar{w}_{\frac{1}{2}}(z',t')\Big|_{t=t'=0}=\frac{\eta^2}{2\pi}\int_0^{2\pi}\bar{w}_{\frac{1}{2}}(z,\eta^{\frac{1}{2}}e^{i\varphi})\bar{w}_{\frac{1}{2}}(z',\eta^{\frac{1}{2}}e^{-i\varphi}). \tag{II.11}$$

From this the series in (II.10) is easily obtained by $\partial/\partial\eta$, for $\eta=1$. Of course, $\bar{w}_\sigma(z,t)$ obeys a simple differential equation of first order. A somewhat lengthy integration process, whose details will not be of interest, leads then to the formulas (5.4), (5.5) of the text. The author has checked formula (5.5) by insertion of the first three polynomials (4.9) into the integral equation of the U-functions:

$$U_{\iota\mathrm{K}}=(\iota^4-\tfrac{1}{2})\int_{-\infty}^{+\infty}\langle\mathrm{K}|(\iota^4-\tfrac{1}{2})^{-1}|\mathrm{K}'\rangle U_{\iota\mathrm{K}'}d\mathrm{K}'. \tag{II.12}$$

The evaluation of the integral may be based on the following formula, after a suitable rationalization by a transformation like Eq. (3.6),

$$\frac{1}{4}\int_{-\infty}^{+\infty}\frac{e^{i\epsilon z'}}{\cosh\frac{1}{2}\pi z'}\{\ \}dz'=\left[e^{i\epsilon z}+\frac{iz(1-e^{i\epsilon z}\cosh\epsilon)}{\sinh\epsilon}\right](\cosh\epsilon)^{-2}, \tag{II.13}$$

where the { } indicate the bracketed expression in Eq. (5.5). Developing both sides in powers of ϵ and equating the coefficients of equal powers, one obtains the necessary expressions. Formula (II.13) holds for every real ϵ; it may be evaluated in a complex z' plane by a displacement of the path of integration from the real axis towards $z'=+i\infty$ (for $\epsilon>0$) resp. $z'=-i\infty$ (for $\epsilon<0$) and summation of the residues at the complex zeros of the hyperbolic functions.

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