Oscillatory Processes Generated by Unitary Bijective Time Changes of Stationary Gaussian Processes

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Abstract

This article establishes that Gaussian processes obtained via unitary, measure-preserving bijective time transformations of stationary processes are a subclass of oscillatory processes in the sense of Priestley. The central object is the unitary composition operator M_θ (and its inverse), which implements the time change at the Hilbert-space level and conjugates covariance and spectral structures. Comprehensive theorems and proofs are provided for all main statements, including the oscillatory spectral representation, L^2-isometry, evolutionary spectrum, and expected zero formula.

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1 Scaling Functions and Oscillatory Processes

Definition 1. *[Scaling Functions] Let $\mathcal{F}$ denote the set of functions $\theta: \mathbb{R} \rightarrow \mathbb{R}$ such that:*

1. *θ is continuously differentiable and $\theta'(t) > 0$ for all t*

2. *θ is strictly increasing and bijective*

Remark 2. From the inverse function theorem, any $\theta \in \mathcal{F}$ has an everywhere differentiable inverse with $(\theta^{-1})'(s) = 1/\theta'(\theta^{-1}(s))$ for all s in the range of θ.

Definition 3. *[Oscillatory Process] A complex-valued, second-order stochastic process $\{X_t\}_{t \in \mathbb{R}}$ is called* oscillatory *if there exists*

1. *a family of functions $\{\phi_t(\omega)\}$ with $\phi_t(\omega) = A_t(\omega)\, e^{i\omega t}$ where $A_t(\cdot) \in L^2(\mu)$*

2. *a complex orthogonal-increment process $Z(\omega)$ with $E\,|d\,Z(\omega)|^2 = d\,\mu(\omega)$*

such that

$$X_t = \int_{-\infty}^{\infty} \phi_t(\omega)\; d\,Z(\omega) \tag{1}$$

2 Unitary Time-Change Operator and its Inverse

Definition 4. *[Unitary Time-Change Operator] Let $\theta \in \mathcal{F}$. Define the operator M_θ: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by*

$$(M_\theta\, f)(t) := \sqrt{\theta'(t)}\;\; f(\theta(t)) \tag{2}$$

Lemma 5. *[Unitarity of M_θ] The operator M_θ is unitary; that is, for all $f \in L^2(\mathbb{R})$,*

$$\int_{-\infty}^{\infty} |(M_\theta\, f)(t)\,|^2\, d\,t = \int_{-\infty}^{\infty} |\, f(s)|^2\, d\,s \tag{3}$$

Proof. Substitute $s = \theta(t)$, and $d\,s = \theta'(t)\, d\,t$, then:

$$\begin{aligned}
\int_{\mathbb{R}} |(M_\theta\, f)(t)|^2\, d\,t &= \int_{\mathbb{R}} |\sqrt{\theta'(t)}\;\, f(\theta(t))|^2\, d\,t \\
&= \int_{\mathbb{R}} \theta'(t) |\, f(\theta(t))|^2\, d\,t \\
&= \int_{\mathbb{R}} |\, f(s)|^2\, d\,s
\end{aligned}$$

Because θ is bijective and smooth, this covers all of $\mathbb{R}$. □

Definition 6. *[Inverse Operator] The inverse of M_θ is given by*

$$(M_\theta^{-1}\, g)(s) := \frac{g(\theta^{-1}(s))}{\sqrt{\theta'(\theta^{-1}(s))}} \tag{4}$$

Lemma 7. *[Verification of Inverse] For any $f \in L^2(\mathbb{R})$, $M_\theta^{-1} M_\theta f = f$ and $M_\theta M_\theta^{-1} g = g$ for all $g \in L^2(\mathbb{R})$.*

Proof. For f:

$$(M_\theta^{-1} M_\theta f)(s) = \frac{M_\theta f(\theta^{-1}(s))}{\sqrt{\theta'(\theta^{-1}(s))}} = \frac{\sqrt{\theta'(\theta^{-1}(s))}\; f(\theta(\theta^{-1}(s)))}{\sqrt{\theta'(\theta^{-1}(s))}} = f(s)$$

For g:

$$(M_\theta M_\theta^{-1} g)(t) = \sqrt{\theta'(t)} \cdot \frac{g(\theta(\theta^{-1}(\theta(t))))}{\sqrt{\theta'(\theta^{-1}(\theta(t)))}} = g(\theta(t)$$

But under the substitution $s = \theta(t)$, so $g(\theta(t))$ traverses all of L^2 as t traverses $\mathbb{R}$, confirming mutual inverseness. □

3 Oscillatory Representation of Unitary Time-Changed Stationary Processes

Let $\{S_t\}$ be a stationary Gaussian process with continuous spectral representation

$$S_t = \int_{-\infty}^{\infty} e^{i\omega t}\, d\, Z(\omega) \tag{5}$$

For $\theta \in \mathcal{F}$, define the transformed process

$$X_t := \sqrt{\theta'(t)}\; S_{\theta(t)} \tag{6}$$

Theorem 8. *[Oscillatory Spectral Representation] The process defined by (6) has the oscillatory representation*

$$X_t = \int_{-\infty}^{\infty} \sqrt{\theta'(t)}\, e^{i\omega\theta(t)}\, d\, Z(\omega) \tag{7}$$

i.e., with $\phi_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega\theta(t)}$. Moreover, $\phi_t(\omega)$ can be written in the standard form:

$$\phi_t(\omega) = A_t(\omega)\, e^{i\omega t}, \qquad A_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega(\theta(t) - t)} \tag{8}$$

Proof. Substituting (5) into (6):

$$X_t = \sqrt{\theta'(t)}\; S_{\theta(t)} = \sqrt{\theta'(t)} \int_{-\infty}^{\infty} e^{i\omega\theta(t)}\, d\, Z(\omega) = \int_{-\infty}^{\infty} \sqrt{\theta'(t)}\, e^{i\omega\theta(t)}\, d\, Z(\omega).$$

To write in oscillatory standard form:

$$\sqrt{\theta'(t)}\, e^{i\omega\theta(t)} = \sqrt{\theta'(t)}\, e^{i\omega(\theta(t)-t)}\; e^{i\omega t} = A_t(\omega)\, e^{i\omega t} \tag{9}$$

The square-integrability in ω follows because μ is finite and θ' is everywhere positive and finite. □

Theorem 9. *[Evolutionary Spectrum] For the above process, the evolutionary power spectrum at time t is*

$$d\,F_t(\omega) = |A_t(\omega)|^2\, d\,\mu(\omega) = \theta'(t)\, d\,\mu(\omega) \tag{10}$$

Proof. By definition, $d\,F_t(\omega) = |A_t(\omega)|^2\, d\,\mu(\omega)$. Since

$$|A_t(\omega)|^2 = |\sqrt{\theta'(t)}|^2 \cdot |e^{i\omega(\theta(t)-t)}|^2 = \theta'(t)$$

the result follows. □

4 L^2-Norm Preservation by Unitary Time-Change

Theorem 10. *[L^2-Norm Preservation] The unitary time-change preserves the L^2-norms of the stochastic processes: for any measurable $I \subseteq \mathbb{R}$,*

$$\int_I \mathbb{E}[|X_t|^2]\, d\,t = \int_{\theta(I)} \mathbb{E}[|S_s|^2]\, d\,s \tag{11}$$

Proof. Using $X_t = \sqrt{\theta'(t)}\, S_{\theta(t)}$ and stationarity of S:

$$\begin{aligned}
\mathbb{E}[|X_t|^2] &= \theta'(t)\; \mathbb{E}[|S_{\theta(t)}|^2] \\
&= \theta'(t)\; \mathbb{E}[|S_0|^2] \\
&= \theta'(t)\; \sigma^2 \quad \text{(where } \sigma^2 \text{ is constant)}
\end{aligned}$$

Thus,

$$\int_I \mathbb{E}[|X_t|^2]\, d\,t = \sigma^2 \int_I \theta'(t)\, d\,t = \sigma^2 \int_{\theta(I)} d\,s = \int_{\theta(I)} \mathbb{E}[|S_s|^2]\, d\,s$$

□

5 Expected Zero Formula

Theorem 11. *[Expected Zero Count] Let $K(\tau)$ be the covariance function of S_t (assumed twice differentiable at 0), and $\theta \in \mathcal{F}$. The expected number of real zeros of X_t on $[a,b]$ is*

$$\mathbb{E}[N_{[a,b]}] = \sqrt{-\ddot{K}(0)}\ (\theta(b) - \theta(a)) \tag{12}$$

Proof. By the Kac-Rice formula:

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{-\lim_{s \to t} \frac{\partial^2}{\partial s\, \partial t} \mathrm{cov}(X_s, X_t)}\, d\, t \tag{13}$$

For $X_t = \sqrt{\theta'(t)}\, S_{\theta(t)}$,

$$\mathrm{cov}(X_s, X_t) = \sqrt{\theta'(s)\, \theta'(t)}\, K\, (\theta(t) - \theta(s))$$

The relevant limit is:

$$\lim_{s \to t} \frac{\partial^2}{\partial s\, \partial t} \left[\sqrt{\theta'(s)\, \theta'(t)}\, K\, (\theta(t) - \theta(s)) \right] = \theta'(t)^2\, K''(0)$$

since the cross derivatives act on $K\,(\theta(t) - \theta(s))$, and $\theta'(t)$ multiplies through. Therefore,

$$\begin{aligned} \mathbb{E}[N_{[a,b]}] &= \int_a^b \sqrt{-K''(0)\ \theta'(t)^2}\, d\, t = \sqrt{-K''(0)} \int_a^b \theta'(t)\, d\, t \\ &= \sqrt{-K''(0)}\, (\theta(b) - \theta(a)) \end{aligned}$$

□

6 Conclusion

The class of processes $X_t := \sqrt{\theta'(t)}\, S_{\theta(t)}$ forms a subclass of oscillatory processes corresponding to measure-preserving, unitary time changes of stationary Gaussian processes. The unitary composition operator M_θ implements this transformation at the Hilbert space level, preserving all L^2 inner products and yields oscillatory spectral representations with evolving spectra. The zero set and energy properties of the process are determined by the geometry of θ and the spectrum of the underlying stationary process, as proved above.

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