A Quadratic Extremal Problem on the Dirichlet Space*

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It is shown that there is a unique solution F to the problem

$$\lambda = \sup \left\{ \mathrm{Re} \int_{\Delta} F' \, \bar{F}' \, d\,A \colon \int_{\Delta} |F'|^2 \, d\,A \leq 1 \right\} \tag{1}$$

The function F is entire with a number of special properties. The number λ is the reciprocal of the smallest zero of the 0th Bessel function of the first kind.

INTRODUCTION

The Dirichlet space, D, on the open unit disc Δ consists of all analytic functions f

$$f(z) = \sum_{k=1}^{\infty} a_k \, z^k \quad \forall |z| < 1, \quad f(0) = 0 \tag{2}$$

*. In memory of Ralph P. Boas, Jr. (1912–1992).

for which the quantity

$$\int_{\Delta} |f'(z)|^2 \, d\,A(z) = \sum_{k=1}^{\infty} k\,|a_k|^2 =: \|f\|_D^2 \tag{3}$$

is finite. In connection with a generalization of Harnack's inequality, Boris Korenblum [2] has asked how large the quantity

$$\lambda := \sup_{f \in D} \frac{\mathrm{Re}(\sum_{k=1}^{\infty} a_k\, a_{k+1})}{\sum_{k=1}^{\infty} k\,|a_k|^2} \tag{4}$$

is and, if possible, to characterize all functions F which attain the value λ in (2). The expression in the numerator in (2) is not a linear function of f but rather quadratic; hence, the title of this paper.

It is simple to show that

$$\sum_i a_k\, a_{k+1} = \int_{\Delta} |F'(z)|^2 \, d\,A(z) \tag{5}$$

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and therefore Korenblum's problem has this alternate form:

$$\lambda = \sup \left\{ \mathrm{Re}\left(\int_{\Delta} F'\, \bar{F}'\, d\,A \right) : \|f\|_D \leq 1 \right\}.$$

We show here that the extremal problem (2) or (4) has a unique solution F, up to multiplication by a constant; moreover, F is an entire function of exponential type with infinitely many zeros, all in the left half-plane, none of which lie in Δ or on the real axis, except for a first-order zero at the origin. Moreover, the number λ is the reciprocal of the smallest positive zero of $J_0(x)$, the 0th Bessel function. Finally,

$$F(z) = C \sum_{n=1}^{\infty} J_n(\lambda)\, z^n,$$

where J_n is the nth Bessel function and C is a certain constant.

The conclusions above are proved in Sections 1 and 2; Section 3 contains a number of results which generalize the extremal problem (2).

1. EXISTENCE AND UNIQUENESS

We begin by establishing simple bounds on λ.

Proposition 1. $\frac{1}{\sqrt{6}} < \lambda \leq \frac{1}{2}$.

Proof. Since $2\,\mathrm{Re}(z\bar{w}) \leq |z|^2 + |w|^2$, we have

$$\begin{aligned} 2\ \mathrm{Re}(a_1\,a_2 + a_2\,a_3 + \cdots) \ & \leq |a_1|^2 + |a_2|^2 + |a_3|^2 + \cdots^2 \\ & = |a_1|^2 + 2|a_2\,|^2 + 3|\,a_3|^2 + \cdots \\ & = \sum\ k\,|a_k| \end{aligned} \tag{6}$$

which implies that $\lambda \leq 1/2$. The lower bound is obtained by the specific choices

$$a_2 = \sqrt{\frac{3}{2}}\,a_1, \quad a_3 = \frac{3}{4}\,a_1, \quad a_4 = a_5 = \cdots = 0 \tag{7}$$

which give

$$\lambda = \frac{(a_1\,a_2 + a_2\,a_3)}{(a_1^2 + 2\,a_2^2 + 3\,a_3^2)} = \frac{\sqrt{\frac{3}{2}} + 3\,(\frac{3}{4})}{1 + 2\,(\frac{3}{2}) + 3\,(\frac{9}{16})} = \frac{\sqrt{\frac{3}{2}} + (\frac{9}{4})}{1 + (\frac{3}{4}) + (\frac{27}{16})} = \frac{\sqrt{1}}{6} \tag{8} \quad \square$$

To prove the existence of a solution, we shall need the following Lemma.

Lemma 2. *Given $\epsilon > 0$, there is an $R_0, 0 < R_0 < 1$, such that*

$$\int_R^{R+1} f\,(r\,e^{it})^r\,d\,t\,d\,r < e||\,f\,||_D^{(5)}, f(0) = 0$$