Let $f(z)$ be a complex rational function defined as:

$$f(z) = \frac{P(z)}{Q(z)} + i\,\frac{R(z)}{S(z)},$$

where $P(z), Q(z), R(z), S(z)$ are real-valued rational functions, and $z = x + i\,y$ is a complex variable with real part x and imaginary part y.

The goal is to evaluate $f(z)$ at $z = x + i\,y$, ensuring proper complex arithmetic.

Notation

For any complex function $H(z)$, we denote:

- $H_r(x, y)$ or simply H_r: The real part of $H(z)$
- $H_i(x, y)$ or simply H_i: The imaginary part of $H(z)$

Thus, $H(z) = H_r + i\,H_i$ where both H_r and H_i are real-valued functions.

Complex Division

For a single complex rational function $\frac{P(z)}{Q(z)}$, we multiply by the complex conjugate:

$$\frac{P(z)}{Q(z)} = \frac{P(z)\,Q^*(z)}{Q(z)\,Q^*(z)} = \frac{(P_r + i\,P_i)\,(Q_r - i\,Q_i)}{(Q_r + i\,Q_i)\,(Q_r - i\,Q_i)}$$

This simplifies to:

$$\frac{P(z)}{Q(z)} = \frac{(P_r\,Q_r + P_i\,Q_i) + i\,(P_i\,Q_r - P_r\,Q_i)}{Q_r^2 + Q_i^2}$$

Similarly for $\frac{R(z)}{S(z)}$:

$$\frac{R(z)}{S(z)} = \frac{(R_r\,S_r + R_i\,S_i) + i\,(R_i\,S_r - R_r\,S_i)}{S_r^2 + S_i^2}$$

Final Expression

Therefore:

$$f(z) = \left(\frac{P_r\,Q_r + P_i\,Q_i}{Q_r^2 + Q_i^2}\right) + i\left(\frac{P_i\,Q_r - P_r\,Q_i}{Q_r^2 + Q_i^2}\right) + i\left(\frac{R_r\,S_r + R_i\,S_i}{S_r^2 + S_i^2}\right) - \left(\frac{R_i\,S_r - R_r\,S_i}{S_r^2 + S_i^2}\right)$$

Combining real and imaginary parts:

$$\mathrm{Re}(f(z)) = \frac{P_r\,Q_r + P_i\,Q_i}{Q_r^2 + Q_i^2} - \frac{R_i\,S_r - R_r\,S_i}{S_r^2 + S_i^2}$$

$$\mathrm{Im}(f(z)) = \frac{P_i\,Q_r - P_r\,Q_i}{Q_r^2 + Q_i^2} + \frac{R_r\,S_r + R_i\,S_i}{S_r^2 + S_i^2}$$

Proof of Correctness

Complex Division Property

The use of complex conjugates in the numerator and denominator preserves equality while eliminating complex division:

$$\frac{a + b\,i}{c + d\,i} = \frac{(a + b\,i)\,(c - d\,i)}{(c + d\,i)\,(c - d\,i)} = \frac{(a\,c + b\,d) + i\,(b\,c - a\,d)}{c^2 + d^2}$$

Denominator Non-zero Condition

The denominators $Q_r^2 + Q_i^2$ and $S_r^2 + S_i^2$ are sums of squares, which are always positive for non-zero complex numbers, ensuring valid division.

Component Interaction

Each component of the output $(\mathrm{Re}(f(z)), \mathrm{Im}(f(z)))$ properly depends on both real and imaginary parts of the input through the cross-terms in the numerators.

Special Case Verification

For real inputs $(y = 0)$, the imaginary components become zero, reducing to the expected real-valued result.

Conclusion

This formulation correctly evaluates complex rational functions by:

- Properly handling complex division using conjugates
- Maintaining the relationship between input and output components
- Ensuring well-defined results for all valid inputs
- Preserving expected behavior for special cases