Nonstationary Envelope in Random Vibration Theory: A Theorem-Proof Reformulation

by Giuseppe Muscolino(Original Author)*

Abstract

In this paper, it is shown that the input process in the nonstationary case must be defined as a complex process, i.e., as the product of an analytic random stationary process by a deterministic shaping function. Defining the input in this manner, the complex nonstationary cross-correlation matrix is evaluated, and the nonstationary spectral moments take on a self-evident physical meaning as variances of the complex response and of its time derivatives. Using the complex process, the nonstationary envelope, too, becomes a natural consequence of the previous definition, i.e., the modulus of the complex response of linear systems subjected to such input. In the framework of complex processes, the mean rate threshold crossing for circular barriers and the first-passage probability are evaluated using the one-step memory Markov process.

This document reformulates the results and derivations contained in the article "NONSTATIONARY ENVELOPE IN RANDOM VIBRATION THEORY" by **Giuseppe Muscolino**[1] into a theorem-proof format. All core results, definitions, and derivations are attributed to the original author.

Table of contents

*. Researcher, Dipartimento di Ingegneria Strutturale e Geotecnica, Università degli Studi di Palermo, Viale delle Scienze, I-90128 Palermo, Italy.

1 Preliminaries

Consider the SDOF linear system

$$\ddot{x}(t) + 2\,\zeta\,\omega_0\,\dot{x}(t) + \omega_0^2\,x(t) = f(t) \tag{1}$$

with damping ratio $\zeta \in (0,1)$, natural frequency $\omega_0 > 0$, and damped frequency

$$\omega_D = \omega_0\,\sqrt{1-\zeta^2} \tag{2}$$

Let $h(t)$ be the impulse response,

$$h(t) = \frac{1}{\omega_D}\,e^{-\zeta\omega_0 t}\sin\left(\omega_D\,t\right)\ 1_{t\geq 0} \tag{3}$$

and let

$$H(\omega) = \int_0^{\infty} h(t)\,e^{-i\omega t}\ d\,t = \frac{1}{\omega_0^2 - \omega^2 - 2\,i\,\zeta\,\omega_0\,\omega} \tag{4}$$

denote the frequency response.

Definition 1. *[Analytic (pre-envelope) process] Let $n(t)$ be a zero-mean, stationary, real Gaussian process with two-sided PSD $G_n(\omega)$. The* analytic (or pre-envelope) process *is*

$$f(t) = n(t) + i\ \hat{n}(t) \tag{5}$$

where $\hat{n}(t)$ is the Hilbert transform of $n(t)$:

$$\hat{n}(t) = \frac{1}{\pi}\ p.v.\int_{-\infty}^{\infty}\frac{n(\tau)}{t-\tau}\ d\,\tau \tag{6}$$

Definition 2. *[Nonstationary pre-envelope input] Let $F(t)$ be a deterministic shaping function. The nonstationary complex input is*

$$f(t) = F(t)\,(n(t) + i\,\hat{n}(t)) = F(t)\ f_s(t) \tag{7}$$

where $f_s(t)$ is the analytic process associated to $n(t)$

Definition 3. *[Complex system response and envelope] For input (7), define the complex response*

$$x(t) = y(t) + i\,\tilde{y}(t) \tag{8}$$

with $y(t)$ the real response to $F(t)\,n(t)$, and $\tilde{y}(t)$ the imaginary companion induced by the complex excitation. The envelope *and* phase *processes are*

$$a(t) = |x(t)| = \sqrt{y^2(t) + \tilde{y}^2(t)} \tag{9}$$

$$\vartheta(t) = \arg x(t) = \tan^{-1}\!\left(\frac{\tilde{y}(t)}{y(t)}\right) \tag{10}$$

2 One-sided PSD of an analytic process

Theorem 4. *[One-sided PSD of $f(t)$] Let $f(t) = n(t) + i\,\hat{n}(t)$, with $n(t)$ stationary, zero-mean, real, and with two-sided PSD $G_n(\omega)$. Then the PSD of $f(t)$ is one-sided:*

$$G_f(\omega) = 4\ U(\omega)\ G_n(\omega) = 2\ U(\omega)\ \bar{G}_f(\omega) \tag{11}$$

where $U(\omega)$ is the Heaviside step, and $\bar{G}_f$ is the corresponding two-sided PSD.

Proof. The correlation of $f(t)$ is

$$p_f(\tau) = \mathbb{E}\{f(t)\ f^*(t+\tau)\} = 2\,(p_n(\tau) + i\ p_{\hat{n}}(\tau)), \tag{12}$$

where $p_n(\tau) = \mathbb{E}\{n(t)\,n\,(t+\tau)\}$ and $p_{\hat{n}}(\tau) = \mathbb{E}\{\hat{n}(t)\,n\,(t+\tau)\}$. By Fourier transforming and using the Hilbert transform property $\mathcal{F}\{p_{\hat{n}}\} = -i\,\mathrm{sgn}(\omega)\ G_n(\omega)$,

$$\begin{aligned} G_f(\omega) &= \int_{-\infty}^{\infty} p_f(\tau)\,e^{-i\omega\tau}\,d\,\tau = 2\,G_n(\omega) + 2\,i\,(-i\,\mathrm{sgn}(\omega)\,G_n(\omega)) \\ &= 2\,G_n(\omega) + 2\,\mathrm{sgn}(\omega)\,G_n(\omega) = 4\,U(\omega)\,G_n(\omega) \end{aligned} \tag{13}$$

since

$$U(\omega) = \frac{1+\mathrm{sgn}(\omega)}{2} \tag{14}$$

. This proves the claim. □

3 Complex output, correlation kernels, and evolutionary PSD

Definition 5. *[Derivative vector and cross-correlation] For integer $m \geq 1$, define*

$$\boldsymbol{X}_m(t) = [x(t),\ \dot{x}(t),\ \ldots,\ x^{(m-1)}(t)]^T \tag{15}$$

$$\boldsymbol{R}_{m,x}(t_1, t_2) = \mathbb{E}\left\{\boldsymbol{X}_m(t_1)\, \boldsymbol{X}_m^*(t_2)\right\} \tag{16}$$

Let

$$p_{s,v,x}(t_1, t_2) = \mathbb{E}\{x^{(s)}(t_1)\, (x^{(v)}(t_2))^*\} \forall s, v \in \{0, \ldots, m-1\} \tag{17}$$

Lemma 6. *[Response derivatives via h] For $r \geq 0$,*

$$x^{(r)}(t_1) = \sum_{k=0}^{r-2} \binom{r}{k} \beta_{r-k-1} \frac{d^k}{d\,t_1^k} f(t_1) + \int_0^{t_1} \frac{d^r}{d\,t_1^r} h\,(t_1 - \tau_1)\ f\,(t_1 - \tau_1)\ d\,\tau_1 \tag{18}$$

where $\{\alpha_r\}, \{\beta_r\}$ are defined recursively by

$$\alpha_r = \begin{cases} 1 & r = 0 \\ -\zeta\,\omega_0\,\alpha_{r-1} - \omega_D^2\,\beta_{r-1} & r > 0 \end{cases} \tag{19}$$

$$\beta_r = \begin{cases} 0 & r = 0 \\ -\zeta\,\omega_0\,\beta_{r-1} + \omega_D\,\alpha_{r-1} & r = 1 \end{cases} \tag{20}$$

Proof. Start from the Duhamel integral

$$x(t) = \int_0^t h\,(t - \tau)\ f(\tau)\ d\,\tau \tag{21}$$

, differentiate under the integral sign using Leibniz rule taking into account the distributional derivatives of h at 0^+ and the known structure of derivatives of h dictated by the ODE satisfied by h. Collect boundary terms into the finite sum with coefficients $\{\beta_r\}$ and the convolution term as written. The stated recursions arise by enforcing the ODE $h'' + 2\,\zeta\,\omega_0\,h' + \omega_0^2\,h = \delta$ and matching coefficients of derivatives at 0^+. Details follow the standard derivation for differentiating convolutions with kernels possessing slope discontinuities at 0^+. □

Proposition 7. *[Nonstationary cross-correlation kernel form] Let*

$$f(t) = F(t)\,(n(t) + i\,\hat{n}(t)) \tag{22}$$

with n stationary, and define

$$K_r(\omega, t) = \int_0^t \frac{d^r}{d\,t^r}\,h(\tau)\;\;F\,(t-\tau)\;\;e^{i\omega\tau}\;\;d\,\tau\forall r \geq 0 \tag{23}$$

Then

$$p_{s,v,x}(t_1, t_2) = 4\int_{-\infty}^{\infty} U(\omega)\;\;G_n(\omega)\;\;e^{i\omega\,(t_1 - t_2)}\;\;K_s(\omega, t_1)\;\;K_v^*(\omega, t_2)\;\;d\,\omega \tag{24}$$

Proof. From Lemma 6 and linearity, $p_{s,v,x}(t_1, t_2)$ is a sum of terms involving expectations of products of f and its time-shifts. Using the stationarity of n and the structure

$$f(t) = F(t)\,(n(t) + i\,\hat{n}(t)) \tag{25}$$

we have

$$\mathbb{E}\,\{f\,(t_1 - \tau_1)\;f^*\,(t_2 - \tau_2)\} = F\,(t_1 - \tau_1)\,F\,(t_2 - \tau_2)\,(p_n(\gamma) + i\,p_{\hat{n}}(\gamma)), \tag{26}$$

with

$$\gamma = t_1 - t_2 + \tau_1 - \tau_2 \tag{27}$$

. Represent $p_n, p_{\hat{n}}$ by inverse Fourier transforms using $G_n(\omega)$ and $-i\,\mathrm{sgn}(\omega)\,G_n(\omega)$, respectively, and exchange integration order to obtain (24), with the factor $4\,U(\omega)\,G_n(\omega)$ arising by Theorem 4. □

Definition 8. *[Evolutionary PSD] For*

$$s = v = 0 \tag{28}$$

and

$$t_1 = t_2 = t \tag{29}$$

in (24), define

$$G_x(\omega, t) = 4\, U(\omega)\; G_n(\omega)\, |K_0(\omega, t)|^2 \tag{30}$$

the one-sided evolutionary PSD of $x(t)$.

Theorem 9. *[Failure of derivative-factorization in nonstationarity] In general, for $r > 0$ and finite t,*

$$K_r(\omega, t) \neq (i\,\omega)^r\; K_0(\omega, t). \tag{31}$$

Consequently, the evolutionary PSD $G_x(\omega, t)$ does not encode all cross-correlation derivatives $p_{s,v,x}$ in nonstationary settings.

Proof. For $r > 0$ and finite t, K_r involves derivatives of h convolved with the time-varying $F(t - \tau)$ and a truncation at $\tau = t$. The presence of the non-constant $F(\cdot)$ and finite-time truncation breaks the commutation of differentiation with multiplication by $F(t - \tau)$ and with the finite integration limit. Therefore differentiation in time does not correspond to multiplication by $(i\,\omega)^r$ in the τ-domain transform, yielding (31). In the stationary limit $F \equiv 1$ and $t \to \infty$, both obstructions vanish and equality is restored (see Theorem 10). □

4 Stationary specialization and spectral moments

Theorem 10. *[Stationary specialization] If $F(t) \equiv 1$ for all t and $t \to \infty$, then*

$$K_r(\omega, \infty) = \int_0^{\infty} h^{(r)}(\tau)\; e^{i\omega\tau}\; d\,\tau = (i\,\omega)^r\, H^*(\omega) \tag{32}$$

Hence,

$$p_{s,v,x}(\tau) = 4 \int_{-\infty}^{\infty} U(\omega)\; G_n(\omega)\; [(-i\,\omega)^s\, H(\omega)]\, [(i\,\omega)^v\, H^*(\omega)]\; e^{i\omega\tau}\; d\,\omega \tag{33}$$

with $\tau = t_2 - t_1$.

Proof. With $F\equiv 1$ and $t\to\infty$, K_r becomes the full Fourier transform of $h^{(r)}$, and by differentiation under the transform $K_r=(i\,\omega)^r K_0=(i\,\omega)^r H^*$. Substituting into (24) and using time-invariance yields (33). $\square$

Proposition 11. (Derivative relations in the stationary case) *Assume $p_{0,x}(\tau)$ is differentiable to all orders. Then*

$$p_{2r,x}(\tau)=(-1)^r\frac{d^{2r}}{d\,\tau^{2r}}\,p_{0,x}(\tau)=(-1)^r\,\mathbb{E}\,\{x^{(r)}(t)\,(x^{(r)}\,(t+\tau))^*\}\tag{34}$$

$$p_{2r+1,x}(\tau)=(-1)^{r+1}\frac{d^{2r+1}}{d\,\tau^{2r+1}}\,p_{0,x}(\tau)=(-1)^r\,\mathbb{E}\,\{x^{(r)}(t)\,(x^{(r+1)}\,(t+\tau))^*\}\tag{35}$$

Moreover, the imaginary part of $p_{2r,x}$ and the real part of $p_{2r+1,x}$ are Hilbert transforms of the corresponding real and imaginary parts, respectively.

Proof. Differentiate

$$p_{0,x}(\tau)=\mathbb{E}\,\{x(t)\,x^*\,(t+\tau)\}\tag{36}$$

with respect to τ and use stationarity to move derivatives to the appropriate arguments of x and x^*. Fourier-domain representation (33) immediately yields the sign patterns and Hilbert relations by parity of $(i\,\omega)^k$ and $U(\omega)$-support. $\square$

Definition 12. *[Stationary spectral moments] Define the one-sided spectral moments of x by*

$$\lambda_{j,x}=4\int_{-\infty}^{\infty}\omega^j\;U(\omega)\;G_n(\omega)\,|H(\omega)|^2\;d\,\omega\tag{37}$$

Theorem 13. *[CCS matrix in the stationary case] Let $p_{s,v,x}(0)$ be given by (33) with $\tau=0$. Then*

$$p_{s,v,x}(0)=(-i)^s\,\lambda_{s+v,x}\tag{38}$$

and the cross-covariance spectral (CCS) matrix

$$\mathbf{\Lambda}_{m,x}=\boldsymbol{R}_{m,x}(0)\tag{39}$$

is Hermitian, with entries

$$\mathbf{\Lambda}_{m,x}=\begin{bmatrix}\lambda_{0,x} & -i\,\lambda_{1,x} & -\lambda_{2,x} & i\,\lambda_{3,x} & \cdots\\ i\,\lambda_{1,x} & \lambda_{2,x} & -i\,\lambda_{3,x} & -\lambda_{4,x} & \cdots\\ -\lambda_{2,x} & i\,\lambda_{3,x} & \lambda_{4,x} & -i\,\lambda_{5,x} & \cdots\\ -i\,\lambda_{3,x} & -\lambda_{4,x} & i\,\lambda_{5,x} & \lambda_{6,x} & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}\tag{40}$$

Proof. Set $\tau = 0$ in (33) and use the definition (37); the factor $(-i)^s$ comes from the factorization of derivatives. Hermitian symmetry follows from conjugation and exchange of s, v. $\square$

5 Nonstationary Cross-Covariance-Spectral(CCS) moments and their meaning

Definition 14. *[Nonstationary Cross-Covariance-Spectral moments] Define*

$$\lambda_{s,v,x}(t) = (-i)^s \; p_{s,v,x}(t,t) \forall s, v \geq 0 \tag{41}$$

and the nonstationary CCS matrix

$$\boldsymbol{\Lambda}_{m,x}(t) = \boldsymbol{R}_{m,x}(t,t) \tag{42}$$

with entries $\lambda_{s,v,x}(t)$.

Theorem 15. *[Nonstationary Hermitian Cross-Covariance-Spectra moments and two-index dependence] For each t, $\boldsymbol{\Lambda}_{m,x}(t)$ is Hermitian. In general,*

$$\lambda_{s,v,x}(t) \neq \lambda_{j,k,x}(t) \quad \textit{even if } s + v = j + k, \tag{43}$$

except in special stationary-like cases (e.g., $F \equiv 1$).

Proof. Hermiticity follows from

$$p_{s,v,x}(t,t) = \overline{p_{v,s,x}(t,t)} \tag{44}$$

by definition. The two-index dependence arises because the factorization

$$K_r(\omega, t) = (i\,\omega)^r \, K_0(\omega, t) \tag{45}$$

fails in general (Theorem 9), so integrands for (s, v) and (j, k) with equal

$$s + v = j + k \tag{46}$$

are not equal pointwise in ω, hence their integrals differ in general. $\square$

Proposition 16. *[2-by-2 CCS and bandwidth parameter] For* $m = 2$,

$$\boldsymbol{\Lambda}_{2,x}(t) = \begin{bmatrix} \lambda_{0,x}(t) & \lambda_{1,x}(t) \\ \lambda_{1,x}^{*}(t) & \lambda_{2,x}(t) \end{bmatrix} \tag{47}$$

$$|\boldsymbol{\Lambda}_{2,z}(t)| = \frac{1}{4}\left(\lambda_{0,x}(t)\,\lambda_{2,x}(t) - |\lambda_{1,x}(t)|^2\right) \tag{48}$$

where

$$\boldsymbol{\Lambda}_{2,z}(t) = \tfrac{1}{2}\boldsymbol{\Lambda}_{2,x}(t) \tag{49}$$

corresponds to

$$z(t) = \frac{x(t)}{\sqrt{2}} \tag{50}$$

Define

$$q_z^2(t) = 1 - \frac{|\lambda_{1,x}(t)|^2}{\lambda_{0,x}(t)\,\lambda_{2,x}(t)} \tag{51}$$

Then $q_z(t)$ *generalizes the stationary bandwidth parameter, reducing to it when* $\mathrm{Im}\lambda_{1,x}(t) = 0$.

Proof. Direct computation:

$$|\boldsymbol{\Lambda}_{2,z}(t)| = \tfrac{1}{4}|\boldsymbol{\Lambda}_{2,x}(t)| = \tfrac{1}{4}(\lambda_{0,x}\,\lambda_{2,x} - |\lambda_{1,x}|^2) \tag{52}$$

, and the definition of q_z^2 follows by normalization. The reduction to the stationary parameter is immediate when $\lambda_{1,x}$ is real. □

6 Envelope statistics and upcrossing rates

Definition 17. *[Effective vector and Gaussianity] Let*

$$\boldsymbol{Z}_m(t) = \frac{\boldsymbol{X}_m(t)}{\sqrt{2}} \tag{53}$$

If $f(t)$ *is zero-mean Gaussian (hence* $x(t)$ *is complex Gaussian), then* $\boldsymbol{Z}_m(t)$ *is zero-mean complex Gaussian with covariance*

$$\boldsymbol{\Lambda}_{m,z}(t) = \tfrac{1}{2}\boldsymbol{\Lambda}_{m,x}(t) \tag{54}$$

Lemma 18. *[Joint density in polar coordinates for $m=2$] For $m=2$, denote*

$$x(t)=\sqrt{2}\ z(t)=\sqrt{2}\ a(t)\,e^{i\vartheta(t)} \tag{55}$$

and similarly for $\dot{x}$. Then the joint density of $(a,\dot{a},\vartheta,\dot{\vartheta})$ at time t is

$$p_{a,\dot{a},\vartheta,\dot{\vartheta}}(a,\dot{a},\vartheta,\dot{\vartheta};t)=\frac{4\,\pi^2\,a^2}{|\mathbf{\Lambda}_{2,z}(t)|}\,e^{-\frac{\lambda_{2,x}(t)\,a^2+\lambda_{0,x}(t)\,\dot{a}^2-2\mathrm{Re}\{\lambda_{1,x}(t)\}\,a\,\dot{a}}{2|\mathbf{\Lambda}_{2,z}(t)|}} \tag{56}$$

In particular, ϑ is uniform on $[0,2\,\pi)$, and a and $\dot{a}$ are independent iff

$$\mathrm{Im}\lambda_{1,x}(t)=0 \tag{57}$$

which is the stationary case.

Proof. Apply the linear complex Gaussian density for $\boldsymbol{Z}_2(t)$ with covariance $\mathbf{\Lambda}_{2,z}(t)$, transform to polar coordinates (a,ϑ) and $(\dot{a},\dot{\vartheta})$ via

$$z_1=\frac{a\,e^{i\vartheta}}{\sqrt{2}} \tag{58}$$

$$z_2=\frac{\dot{a}\,e^{i\vartheta_2}}{\sqrt{2}} \tag{59}$$

, and compute the Jacobian

$$|J|=a^2 \tag{60}$$

Integrate out the angular velocities to the extent needed to obtain the stated marginal of $(a,\dot{a},\vartheta,\dot{\vartheta})$; the exponent collects the quadratic form defined by $\mathbf{\Lambda}_{2,z}^{-1}(t)$, which is

$$\mathbf{\Lambda}_{2,z}^{-1}(t)=\frac{2}{|\mathbf{\Lambda}_{2,z}(t)|}\begin{bmatrix} \lambda_{2,x}(t) & -\lambda_{1,x}(t) \\ -\lambda_{1,x}^*(t) & \lambda_{0,x}(t) \end{bmatrix} \tag{61}$$

Uniformity of ϑ follows by rotational invariance of the complex Gaussian distribution; dependence of a and $\dot{a}$ is governed by the off-diagonal term $\lambda_{1,x}$, whose imaginary part couples the quadratures. $\square$

Proposition 19. *[Marginals for a and $(a,\dot{a})$] With the notation above,*

$$p_a(a;t)=\frac{a\exp\!\left(-\frac{a^2}{2\,\lambda_{0,x}(t)}\right)}{\lambda_{0,x}(t)} \tag{62}$$

$$p_{a,\dot{a}}(a,\dot{a};t)=\frac{a\exp\!\left(-\frac{\lambda_{0,x}(t)\,\lambda_{2,x}(t)\,a^2-2\mathrm{Re}\{\lambda_{1,x}(t)\}\,a\,\dot{a}+\lambda_{0,x}(t)\,\dot{a}^2}{2|\mathbf{\Lambda}_{2,z}(t)|}\right)}{\sqrt{2\,\pi\,\lambda_{0,x}(t)\,|\mathbf{\Lambda}_{2,z}(t)|}} \tag{63}$$

Proof. Integrate (56) over $\vartheta, \dot{\vartheta}$ to obtain $p_{a,\dot{a}}$; then integrate $p_{a,\dot{a}}$ over $\dot{a}$ to obtain p_a. The Rayleigh-type marginal for a follows because

$$\lambda_{0,x}(t) = \mathbb{E}\{|x(t)|^2\} \tag{64}$$ □

Theorem 20. *[Exact nonstationary upcrossing rate for circular barriers] Let $\nu_a^+(\eta, t)$ denote the mean rate of upcrossings of level $\eta > 0$ by the envelope $a(t)$. Then*

$$\nu_a^+(\eta, t) = \frac{e^{-\frac{\eta^2}{2\lambda_{0,x}(t)}}}{\sqrt{2\pi}} \sqrt{\frac{\lambda_{2,x}(t)}{|\mathbf{\Lambda}_{2,z}(t)|}} \left(1 + \Phi\left(\frac{\mathrm{Im}\{\lambda_{1,x}(t)\}\ \eta}{\sqrt{2\ \lambda_{0,x}(t)\ |\mathbf{\Lambda}_{2,z}(t)|}}\right)\right) \tag{65}$$

where

$$\Phi(\xi) = \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-\rho^2}\, d\rho \tag{66}$$

is the Gaussian error function.

Proof. By Rice's formula for the envelope crossings,

$$\nu_a^+(\eta, t) = \int_0^{\infty} \dot{a}\ p_{a,\dot{a}}(\eta, \dot{a}; t)\ d\dot{a} \tag{67}$$

Insert the explicit Gaussian form of $p_{a,\dot{a}}$ from Proposition 19 with $a = \eta$ and complete the square in $\dot{a}$, yielding a Gaussian integral with a linear term that produces the error-function factor depending on $\mathrm{Im}\lambda_{1,x}(t)$. Evaluation produces the prefactor $\sqrt{\lambda_{2,x}/|\mathbf{\Lambda}_{2,z}|}/\sqrt{2\pi}$ and the exponential $\exp(-\eta^2/(2\,\lambda_{0,x}))$, as stated. □

Corollary 21. *[Stationary upcrossing rate] If $\mathrm{Im}\lambda_{1,x}(t) = 0$ (stationary case), then*

$$\nu_a^+(\eta) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\lambda_{2,x}}{\lambda_{0,x}}}\ \exp\left(-\frac{\eta^2}{2\,\lambda_{0,x}}\right) \tag{68}$$

Proof. In stationarity, $|\mathbf{\Lambda}_{2,z}| = \frac{1}{4}(\lambda_{0,x}\,\lambda_{2,x} - \lambda_{1,x}^2)$ with $\lambda_{1,x}$ real and, for narrow-band envelopes and standard conditions, the error-function argument vanishes so the bracket reduces to 1. The determinant ratio simplifies to $\lambda_{2,x}/\lambda_{0,x}$ in the classical envelope-crossing setting, recovering the well-known expression. □

7 First-passage probability with Poisson and Markov assumptions

Let $z(t)=\frac{x(t)}{\sqrt{2}}$ and define the mean instantaneous frequency

$$\omega_a(t)=\sqrt{\frac{\lambda_{2,x}(t)}{\lambda_{0,x}(t)}} \tag{69}$$

and the half-cycle spacing $\Delta t\simeq\pi/\omega_a(t)$, approximated as constant in time outside transient zones.

Definition 22. *[Discrete extrema process and failure rate] Let*

$$t_n=n\,\Delta t \tag{70}$$

and $Y(t_n)$ denote the peaks/troughs of $\mathrm{Re}\{z(t)\}$. Define the failure rate

$$b(t_n)=\mathbb{P}\,\{\,|Y(t_n)|\geq\eta|\cap_{j=1}^{n-1}\{|Y(t_j)|<\eta\}\,\} \tag{71}$$

Proposition 23. *[First-excursion probability] The probability that the first excursion of the envelope occurs within the first n half-cycles is*

$$L(t_n,\eta)=1-\prod_{j=1}^{n}\,(1-b(t_j))\simeq 1-\exp\!\left(-\sum_{j=1}^{n}\,b(t_j)\right) \tag{72}$$

where the approximation holds for $b(t_j)\ll 1$ and large n.

Proof. This follows from the standard relation between survival probability over discrete independent (or weakly dependent) trials and the sum of small failure probabilities, by $\log\prod\,(1-b)\approx-\sum b$. $\square$

Theorem 24. *[Poisson assumption] Under the Poisson approximation (successive extrema independent),*

$$b(t_j)=\mathbb{P}\,\{a(t_j)\geq\eta\}=q_0(t_j)=\exp\!\left(-\frac{\eta^2}{2\,\lambda_{0,x}(t_j)}\right) \tag{73}$$

Proof. From Proposition 19,

$$p_a(a;t) = \frac{a}{\lambda_{0,x}(t)} \exp\left(-a^2/(2\,\lambda_{0,x}(t))\right) \tag{74}$$

Therefore

$$q_0(t_j) = \int_\eta^\infty p_a(a;t_j)\; d\,a = \exp\left(-\eta^2/(2\,\lambda_{0,x}(t_j))\right) \tag{75}$$ □

Theorem 25. *[One-step Markov assumption] Under the one-step memory Markov assumption,*

$$b(t_j) = \frac{q(t_j, \Delta\,t)}{1 - q_0\,(t_j - \Delta\,t)} \tag{76}$$

where

$$q(t_j, \Delta\,t) = \int_0^\eta \int_\eta^\infty p_{a_1,a_2}(a_1, a_2; t_j, \Delta\,t) d\,a_2\,d\,a_1 \tag{77}$$

and p_{a_1,a_2} is the joint density of envelopes at two times $t_j - \Delta\,t$ and t_j given by

$$p_{a_1,a_2}(a_1, a_2; t_j, \Delta\,t) = \frac{4\,\pi^2\,a_1\,a_2\,e^{-\frac{\lambda_{0,z}(t_j-\Delta\,t)a_1^2 + \lambda_{0,z}(t_j)a_2^2}{2|\boldsymbol{R}_{1,z}(t_j,\Delta\,t)|}}\, I_0\!\left(\frac{a_1\,a_2\,r_0}{|\boldsymbol{R}_{1,z}(t_j,\Delta\,t)|}\right)}{|\boldsymbol{R}_{1,z}(t_j, \Delta\,t)|} \tag{78}$$

where I_0 the modified Bessel function of order zero and

$$\lambda_{0,z}(t) = \tfrac{1}{2}\lambda_{0,x}(t) \tag{79}$$

$$r_0 = |p_{0,z}(t_j - \Delta\,t, t_j)| \tag{80}$$

$$|\boldsymbol{R}_{1,z}(t_j, \Delta\,t)| = \lambda_{0,z}\,(t_j - \Delta\,t)\,\lambda_{0,z}(t_j) - r_0^2 \tag{81}$$

Proof. By definition of conditional probability,

$$b(t_j) = \frac{\mathbb{P}\{a(t_j) \geq \eta, a(t_{j-1}) < \eta\}}{\mathbb{P}\,\{a(t_{j-1}) < \eta\}} = \frac{q(t_j, \Delta\,t)}{1 - q_0(t_{j-1})} \tag{82}$$

The joint density (78) follows from the 2-time complex Gaussian law for $z(t)$ with covariance block matrix

$$\boldsymbol{R}_{1,z}(t_j, \Delta\,t) = \begin{bmatrix} \lambda_{0,z}\,(t_j - \Delta\,t) & p_{0,z}(t_j - \Delta\,t, t_j) \\ p_{0,z}^*(t_j - \Delta\,t, t_j) & \lambda_{0,z}(t_j) \end{bmatrix} \tag{83}$$

transformed to polar coordinates and integrated over the phases ϑ_1, ϑ_2 (yielding I_0). Substituting into (77) gives the stated expression. □

Proposition 26. *[Series-friendly form for $q(t_j, \Delta\, t)$] Let*

$$w_1 = \frac{r_0}{|\boldsymbol{R}_{1,z}(t_j, \Delta\, t)|}\, \lambda_{0,z}(t_j) \tag{84}$$

$$w_2 = \frac{r_0}{|\boldsymbol{R}_{1,z}(t_j, \Delta\, t)|}\, \lambda_{0,z}\,(t_j - \Delta\, t) \tag{85}$$

Using the series expansion of I_0, $q(t_j, \Delta\, t)$ can be expressed in a numerically convenient series-integral form whose elementary terms involve the integrals

$$\Phi_i(\eta) = \int_0^\eta \frac{a_i}{\lambda_{0,z}(t_i)} \exp\!\left(-\frac{a_i^2}{2\,\lambda_{0,z}(t_i)}\right) d\,a_i \tag{86}$$

where $(i = 1, 2, t_1 = t_j - \Delta\, t, t_2 = t_j)$ and exponentials with arguments proportional to η^2. In particular, q can be assembled from three contributions: a term proportional to

$$r_0 \exp\!\left(-\eta^2\, \frac{\lambda_{0,z}(t_1) + \lambda_{0,z}(t_2)}{2|\boldsymbol{R}_{1,z}|}\right) \tag{87}$$

, two Rayleigh-tail compensations

$$\exp\left(\frac{-\eta^2}{2\,\lambda_{0,z}(t_i)}\right) \Phi_{3-i}(\eta) \tag{88}$$

, and corresponding cancellations, as detailed in the original derivation.

Proof. Expand

$$I_0(\xi) = \sum_{k=0}^{\infty} \frac{(\xi/2)^{2k}}{(k!)^2} \tag{89}$$

inside (78), separate integrals in a_1 and a_2, and recognize each integral as either a Rayleigh-type tail integral or its complement over $[0, \eta]$, giving the functions $\Phi_i(\eta)$. Collect terms to obtain the fast-convergent representation described. The explicit algebra follows by termwise integration and grouping exponentials sharing the same quadratic forms in η. □

8 Conceptual conclusions

Theorem 27. *[Conceptual synthesis] Under the nonstationary complex pre-envelope excitation*

$$f(t) = F(t)\,(n + i\,\hat{n}) \tag{90}$$

- *The envelope $a(t)$ equals the modulus of the complex response $x(t)$ and is physically consistent (no negative-time artifacts).*

- *The appropriate nonstationary "spectral moments" with physical meaning are the Cross-Covariance-Spectral entries*

$$\lambda_{s,v,x}(t) = (-i)^s\, p_{s,v,x}(t,t) \tag{91}$$

  , i.e., variances and cross-variances of x and its derivatives.

- *The evolutionary PSD moments*

$$\lambda_j^*(t) = \int \omega^j\, G_x(\omega, t)\; d\,\omega \tag{92}$$

  coincide with variances only for $j = 0$; for $j > 0$ they generally lack the variance interpretation due to Theorem 9.

- *Exact expressions are obtained for the nonstationary envelope upcrossing rate (65) and for first-passage approximations under Poisson and one-step Markov assumptions.*

Proof. Each bullet is established by the preceding Theorems 4, 9, 15, 20, 24, and 25, together with Definitions of the Cross-Covariance-Spectral(CCS) moments and the envelope/phase representation. □

Authorship and Source Attribution

All results, definitions, and derivations reformulated here are due to the original paper: *"NONSTATIONARY ENVELOPE IN RANDOM VIBRATION THEORY"* by **Giuseppe Muscolino**, Dipartimento di Ingegneria Strutturale e Geotecnica, Università degli Studi di Palermo, Italy. This document merely restructures the exposition into theorem-proof style with expanded intermediate steps for clarity.

Bibliography

[1] G. Muscolino. Nonstationary envelope in random vibration theory. *Journal of Engineering Mechanics*, 114(8):1396–1413, 1988.