A Quadratic Extremal Problem on the Dirichlet Space*

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It is shown that there is a unique solution F to the problem

$$\lambda = \sup \left\{ \mathrm{Re} \int_\Delta F' \bar{F}' \, d\, A \colon \int_\Delta |F'|^2 \, d\, A \leq 1 \right\} \tag{1}$$

The function F is entire with a number of special properties. The number λ is the reciprocal of the smallest zero of the 0th Bessel function of the first kind.

INTRODUCTION

The Dirichlet space, D, on the open unit disc Δ consists of all analytic functions f

$$f(z) = \sum_{k=1}^{\infty} a_k \, z^k \quad \forall |z| < 1, \quad f(0) = 0 \tag{2}$$

*. In memory of Ralph P. Boas, Jr. (1912–1992).

for which the quantity

$$\int_{\Delta} |f'(z)|^2 \, dA(z) = \sum_{k=1}^{\infty} k |a_k|^2 =: \|f\|_D^2 \tag{3}$$

is finite. In connection with a generalization of Harnack's inequality, Boris Korenblum [2] has asked how large the quantity

$$\lambda := \sup_{f \in D} \frac{\mathrm{Re}(\sum_{k=1}^{\infty} a_k \, a_{k+1})}{\sum_{k=1}^{\infty} k |a_k|^2} \tag{4}$$

is and, if possible, to characterize all functions F which attain the value λ in (2). The expression in the numerator in (2) is not a linear function of f but rather quadratic; hence, the title of this paper.

It is simple to show that

$$\sum_i a_k \, a_{k+1} = \int_{\Delta} |F'(z)|^2 \, dA(z) \tag{5}$$

and therefore Korenblum's problem has this alternate form:

$$\lambda = \sup \left\{ \mathrm{Re}\left(\int_{\Delta} F' \, \bar{F}' \, dA \right) : \|f\|_D \leq 1 \right\} \tag{6}$$

We show here that the extremal problem (2) or (4) has a unique solution F, up to multiplication by a constant; moreover, F is an entire function of exponential type with infinitely many zeros, all in the left half-plane, none of which lie in Δ or on the real axis, except for a first-order zero at the origin. Moreover, the number λ is the reciprocal of the smallest positive zero of $J_0(x)$, the 0th Bessel function. Finally,

$$F(z) = C \sum_{n=1}^{\infty} J_n(\lambda) \, z^n \tag{7}$$

where J_n is the nth Bessel function and C is a certain constant.

The conclusions above are proved in Sections 1 and 2; Section 3 contains a number of results which generalize the extremal problem (2).

1. EXISTENCE AND UNIQUENESS

We begin by establishing simple bounds on λ.

Proposition 1. $\frac{1}{\sqrt{6}} < \lambda \leq \frac{1}{2}$.

Proof. Since $2\,\mathrm{Re}(z\bar{w}) \leq |z|^2 + |w|^2$, we have

$$\begin{aligned} 2\ \mathrm{Re}(a_1\,a_2 + a_2\,a_3 + \cdots) &\leq |a_1|^2 + |a_2|^2 + |a_3|^2 + \cdots^2 \\ &= |a_1|^2 + 2|a_2\,|^2 + 3|\,a_3|^2 + \cdots \\ &= \sum\,k\,|a_k| \end{aligned} \tag{8}$$

which implies that $\lambda \leq 1/2$. The lower bound is obtained by the specific choices

$$a_2 = \sqrt{\frac{3}{2}}\,a_1, \quad a_3 = \frac{3}{4}\,a_1, \quad a_4 = a_5 = \cdots = 0 \tag{9}$$

which give

$$\lambda = \frac{(a_1\,a_2 + a_2\,a_3)}{(a_1^2 + 2\,a_2^2 + 3\,a_3^2)} = \frac{\sqrt{\frac{3}{2}} + 3\,(\frac{3}{4})}{1 + 2\,(\frac{3}{2}) + 3\,(\frac{9}{16})} = \frac{\sqrt{\frac{3}{2}} + (\frac{9}{4})}{1 + (\frac{3}{4}) + (\frac{27}{16})} = \frac{\sqrt{1}}{6} \tag{10}$$ □

To prove the existence of a solution, we shall need the following Lemma.

Lemma 2. *Given $\epsilon > 0$, there is an $R_0, 0 < R_0 < 1$, such that*

$$\int_R^{R+1} f\,(r\,e^{it})^r\,d\,t\,d\,r < e||\,f\,||_D^{(5)}, f(0) = 0 \tag{11}$$

Proof. Let $f(z) = \sum_{k=1}^{\infty} c_k\,z^k$. Then

$$\begin{aligned} \frac{1}{\pi}\int_R^1\int_0^{2\pi} |f'\,(r\,e^{i\theta})|^2 d\,\theta\ r\ d\,r &= \sum_{k=1}^{\infty}\,|c_k|^2 \frac{1 - R^{2k+2}}{k+1} \\ &= \sum_{k=1}^{\infty}\,(k|c_k|^2)\left(\frac{1 - R^{2k+2}}{k\,(k+1)}\right) \\ &\leq \|\,f\,\|_D^2 \left(\sum_{k=1}^{\infty}\,\frac{1 - R^{2k+2}}{k\,(k+1)}\right) \end{aligned} \tag{12}$$

Here we used the simple inequality

$$k|c_k|^2 \leq \|\,f\,\|_D^2, \quad k = 1, 2, \ldots. \tag{13}$$

The expression

$$\sum_{k=1}^{\infty} \frac{(1-R^{2k+2})}{k\,(k+1)} \tag{14}$$

goes to zero monotonically as R increases to 1. We are done. ■

Theorem 1. A solution to (4) exists.

Proof. Let f_k be a sequence with $f_k(0)=0$, $\|f_k\|_D=1$, and

$$\mathrm{Re}\left(\int_\Delta f_k' \, f_k' \, d\,A\right) \rightarrow \lambda \tag{15}$$

We may assume that f_k converges weakly in the Hilbert space D to a function $F, F(0)=0$, $\|F\|_D \leq 1$. This implies that $f_k' \rightarrow F'$ uniformly on compact subsets of Δ, and also that $f_k \rightarrow F$ uniformly on compact subsets of Δ. Thus,

$$\left|\int_\Delta f_k' \, f_k' \, d\,A - \int_\Delta F' \, F' \, d\,A\right| \leq \left|\int_\Delta (f_k' - F') \, f_k' \, d\,A\right| + \left|\int_\Delta (F' \, f_k' - F' \, F') \, d\,A\right| \tag{16}$$

The second term goes to zero since $f_k' \rightarrow F'$ weakly in D. The first term is no larger than

$$||\, f_k'||_D \, ||\, f_k - F\,||_{L^2} = ||\, f_k - F\,||_{L^2} \tag{17}$$

The latter goes to zero as $k \rightarrow \infty$, since

$$||\, f_k - F\,||_{L^2} = \sqrt{\left(\int_{|z|<R} |f_k - F\,|^2 \, d\,A + \int_{R<|z|<1} |f_k - F\,|^2 \, d\,A + \int_{R<|z|<1} |F\,|^2 \, d\,A\right)}$$

This completes the proof.