# Injectively Time-Changed Stationary Processes

**Definition 1.** *An* injectively time-changed stationary process *is a stochastic process* $\{X(t)\}_{t \in \mathbb{R}}$ *with spectral representation*

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda\theta(t)} \, dZ(\lambda) \tag{1}$$

*where* $\theta: \mathbb{R} \to \mathbb{R}$ *is strictly increasing,* $\theta \in C^1(\mathbb{R})$*, and* $\{Z(\lambda)\}_{\lambda \in \mathbb{R}}$ *is an orthogonal increment process with* $E[|dZ(\lambda)|^2] = F(d\lambda)$*.*

**Proposition 2.**

$$X(t) = \int_{-\infty}^{\infty} A(t, \lambda) \, e^{i\lambda t} \, dZ(\lambda) \tag{2}$$

*where*

$$A(t, \lambda) = e^{i\lambda(\theta(t) - t)} \tag{3}$$

**Proof.**

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda\theta(t)} \, dZ(\lambda) \tag{4}$$

$$= \int_{-\infty}^{\infty} e^{i\lambda(\theta(t) - t)} \, e^{i\lambda t} \, dZ(\lambda) \tag{5}$$

□

**Theorem 3.** *1.* $E[|X(t)|^2] = \int_{-\infty}^{\infty} F(d\lambda) < \infty$

*2.* $Cov(X(s), X(t)) = \int_{-\infty}^{\infty} e^{i\lambda(\theta(s) - \theta(t))} \, F(d\lambda)$

**Proof.**

$$E[|X(t)|^2] = \int_{-\infty}^{\infty} |e^{i\lambda\theta(t)}|^2 \, F(d\lambda) = \int_{-\infty}^{\infty} F(d\lambda) \tag{6}$$

$$\mathrm{Cov}(X(s), X(t)) = \int_{-\infty}^{\infty} e^{i\lambda\theta(s)} \overline{e^{i\lambda\theta(t)}} F(d\lambda) \tag{7}$$

$$= \int_{-\infty}^{\infty} e^{i\lambda(\theta(s) - \theta(t))} \, F(d\lambda) \tag{8}$$

□

**Theorem 4.** *$X(t)$ is stationary if and only if $\theta(t) = t + c$ for some $c \in \mathbb{R}$.*

**Proof.** ($\Leftarrow$) If $\theta(t) = t + c$:

$$\mathrm{Cov}(X(s), X(t)) = \int_{-\infty}^{\infty} e^{i\lambda c} e^{-i\lambda c} F(d\lambda) = \int_{-\infty}^{\infty} F(d\lambda) \tag{9}$$

($\Rightarrow$) Stationarity requires $\theta(s) - \theta(t) = g(s - t)$. Differentiating: $\theta'(s) = g'(s - t)$. Both sides constant implies $\theta'(t) = k$, so $\theta(t) = k t + c$. Covariance depending only on $s - t$ requires $k = 1$. □

**Definition 5.** $\Delta(t) := \theta(t) - t$

**Proposition 6.**

1. *$\Delta'(t) = \theta'(t) - 1$*
2. *$A(t, \lambda) = e^{i\lambda\Delta(t)}$*
3. *Instantaneous frequency: $\frac{d}{dt}[\lambda\,\theta(t)] = \lambda\,\theta'(t)$*

**Theorem 7.** *If $\theta$ has inverse $\psi$ and $F(d\lambda) = f(\lambda)\,d\lambda$:*

$$f(\lambda) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(\psi(u))\, e^{-i\lambda u} \frac{du}{\psi'(u)} \tag{10}$$

**Proof.** Substitution $u = \theta(t)$:

$$X(\psi(u)) = \int_{-\infty}^{\infty} e^{i\mu u}\, dZ(\mu) \tag{11}$$

Standard inversion formula applies with measure transformation factor $\frac{1}{\psi'(u)}$. □

**Theorem 8.** *If $|\theta(s) - \theta(t)| \to \infty$ as $|t - s| \to \infty$ and $F$ is absolutely continuous:*

$$\lim_{|t-s| \to \infty} Cov(X(s), X(t)) = 0 \tag{12}$$

**Proof.** Riemann-Lebesgue lemma applied to

$$\mathrm{Cov}(X(s), X(t)) = \int_{-\infty}^{\infty} e^{i\lambda(\theta(s) - \theta(t))} F(d\lambda) \tag{13}$$ □

**Corollary 9.** *[Band-Limited Case] When $F$ has support in $[-B, B]$:*

$$X(t) = \int_{-B}^{B} e^{i\lambda\theta(t)}\, dZ(\lambda) \tag{14}$$