Inversion of Bandlimited Stationary and Oscillatory Processes

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1 Inversion of Bandlimited Stationary Processes

Theorem 1. *[Stationary Inversion] Let $X(t)$ be a stationary Gaussian process with spectral support in $[-\Omega,\Omega]$. Each sample path $X(\cdot,\omega)$ is an entire function of exponential type Ω and satisfies*

$$X(t,\omega)=\int_{-\Omega}^{\Omega} e^{i\mu t}\, d\,\Phi_\omega(\mu) \tag{1}$$

Then for each $\lambda\in[-\Omega,\Omega]$,

$$d\,\Phi_\omega(\lambda)=\lim_{T\to\infty}\int_{-T/2}^{T/2} X(t,\omega)\ e^{-i\lambda t}\ \frac{\sin(Tt/2)}{\pi t}\ d\,t \tag{2}$$

Proof. Substitute the representation

$$\int_{-T/2}^{T/2} X(t,\omega)\, e^{-i\lambda t}\,\frac{\sin(Tt/2)}{\pi t}\, d\,t=\int_{-T/2}^{T/2}\left(\int_{-\Omega}^{\Omega} e^{i\mu t}\, d\,\Phi_\omega(\mu)\right) e^{-i\lambda t}\,\frac{\sin(Tt/2)}{\pi t}\, d\,t \tag{3}$$

Exchange integration:

$$=\int_{-\Omega}^{\Omega}\left(\int_{-T/2}^{T/2} e^{i(\mu-\lambda)t}\,\frac{\sin(Tt/2)}{\pi t}\, d\,t\right) d\,\Phi_\omega(\mu) \tag{4}$$

Define

$$K_T(\mu-\lambda) := \int_{-T/2}^{T/2} e^{i(\mu-\lambda)t} \frac{\sin(Tt/2)}{\pi t}\, dt \tag{5}$$

Then

$$\int_{-T/2}^{T/2} X(t,\omega)\, e^{-i\lambda t} \frac{\sin(Tt/2)}{\pi t}\, dt = \int_{-\Omega}^{\Omega} K_T(\mu-\lambda)\; d\,\Phi_\omega(\mu) \tag{6}$$

As $T \to \infty$, one has

$$\lim_{T\to\infty} K_T(\mu-\lambda) = \begin{cases} 1, & \mu=\lambda \\ 0, & \mu\neq\lambda \end{cases} \tag{7}$$

Therefore the limit isolates $d\,\Phi_\omega(\lambda)$. □

2 Inversion of Bandlimited Oscillatory Processes

Theorem 2. *[Oscillatory Process Inversion] Let*

$$Y(t,\omega) = \int_{-\Omega}^{\Omega} a_\mu(t)\; e^{i\mu t}\; d\,\Phi_\omega(\mu) \tag{8}$$

where each $a_\mu(t)$ is bounded and analytic. Then

$$d\,\Phi_\omega(\lambda) = \lim_{T\to\infty} \int_{-T/2}^{T/2} \overline{a_\lambda(t)}\, Y(t,\omega)\; e^{-i\lambda t}\; \frac{\sin(Tt/2)}{\pi t}\; dt \tag{9}$$

Proof. Substitute the representation:

$$\int_{-T/2}^{T/2} \overline{a_\lambda(t)} \left(\int_{-\Omega}^{\Omega} a_\mu(t)\, e^{i\mu t}\, d\,\Phi_\omega(\mu) \right) e^{-i\lambda t} \frac{\sin(Tt/2)}{\pi t}\, dt \tag{10}$$

Exchange order:

$$= \int_{-\Omega}^{\Omega} \left(\int_{-T/2}^{T/2} a_\mu(t) \overline{a_\lambda(t)} e^{i(\mu-\lambda)t} \frac{\sin(Tt/2)}{\pi t}\, dt \right) d\,\Phi_\omega(\mu) \tag{11}$$

Define

$$K_T^{(\lambda,\mu)} := \int_{-T/2}^{T/2} a_\mu(t) \overline{a_\lambda(t)} e^{i(\mu-\lambda)t} \frac{\sin(Tt/2)}{\pi t}\, dt \tag{12}$$

For $\mu = \lambda$,

$$K_T^{(\lambda,\lambda)} = \int_{-T/2}^{T/2} |a_\lambda(t)|^2 \frac{\sin(Tt/2)}{\pi t}\, dt \to 1 \tag{13}$$

For $\mu \neq \lambda$,

$$K_T^{(\lambda,\mu)} = \int_{-T/2}^{T/2} a_\mu(t)\overline{a_\lambda(t)} e^{i(\mu-\lambda)t} \frac{\sin(Tt/2)}{\pi t}\, dt \to 0 \tag{14}$$

Thus the limit recovers $d\,\Phi_\omega(\lambda)$. □

2.1 The Oscillatory Subclass of Unitarily Time-Changed Stationary Processes

Theorem 3. *[Subclass: Monotone Time Change] Let* $m\colon \mathbb{R} \to \mathbb{R}$ *be* C^1 *and strictly increasing. Define*

$$Y(t) = \sqrt{m'(t)}\ X(m(t)) \tag{15}$$

where

$$X(u) = \int_{-\Omega}^{\Omega} e^{i\lambda u}\, d\,\Phi_\omega(\lambda) \tag{16}$$

Then

$$\tilde{Y}(u) = \frac{Y(m^{-1}(u))}{\sqrt{m'(m^{-1}(u))}} = X(u) \tag{17}$$

and

$$d\,\Phi_\omega(\lambda) = \lim_{T\to\infty} \int_{-T/2}^{T/2} \tilde{Y}(u)\ e^{-i\lambda u}\ \frac{\sin(Tu/2)}{\pi u}\, du \tag{18}$$

Proof. For $u = m(t)$,

$$Y(m^{-1}(u)) = \sqrt{m'(m^{-1}(u))}\ X(u) \tag{19}$$

Dividing,

$$\frac{Y(m^{-1}(u))}{\sqrt{m'(m^{-1}(u))}} = X(u) \tag{20}$$

Therefore

$$\tilde{Y}(u) = X(u) \tag{21}$$

Since $X(u)$ admits

$$X(u) = \int_{-\Omega}^{\Omega} e^{i\lambda u}\, d\,\Phi_\omega(\lambda) \tag{22}$$

the inversion integral gives $d\,\Phi_\omega(\lambda)$. □

2.1.1 Hardy Z Random Wave Example

Corollary 4. *[Hardy Z Random Wave via Riemann–Siegel θ] Let $X(t)$ be the stationary Gaussian random wave with*

$$R(\tau) = J_0(\tau), \qquad S(\lambda) = \frac{1}{\pi\sqrt{1-\lambda^2}}, \quad \lambda \in [-1, 1] \tag{23}$$

Define

$$\theta(t) = \Im\Big(\log\Gamma\Big(\tfrac{1}{4} + \tfrac{i\,t}{2}\Big)\Big) - \frac{t}{2}\log\pi \tag{24}$$

This function is C^1 and strictly increasing. Define

$$Z(t) = \sqrt{\theta'(t)}\ X(\theta(t)) \tag{25}$$

Then

$$\tilde{Z}(u) = \frac{Z(\theta^{-1}(u))}{\sqrt{\theta'(\theta^{-1}(u))}} = X(u) \tag{26}$$

and

$$d\,\Phi_\omega(\lambda) = \lim_{T\to\infty}\int_{-T/2}^{T/2} \tilde{Z}(u)\,e^{-i\lambda u}\,\frac{\sin(T u/2)}{\pi\,u}\,d\,u \tag{27}$$

Proof. For $u = \theta(t)$,

$$Z(\theta^{-1}(u)) = \sqrt{\theta'(\theta^{-1}(u))}\,X(u) \tag{28}$$

So

$$\frac{Z(\theta^{-1}(u))}{\sqrt{\theta'(\theta^{-1}(u))}} = X(u) \tag{29}$$

Since

$$X(u) = \int_{-1}^{1} e^{i\lambda u}\,d\,\Phi_\omega(\lambda) \tag{30}$$

the inversion integral recovers $d\,\Phi_\omega(\lambda)$. □