Stone's Theorem, Shift Group, and Fourier Transform

Definitions

Shift Group: For $f \in L^2(\mathbb{R})$, define the family of unitary operators $(S_t)_{t\in\mathbb{R}}$ by

$$(S_t f)(x) = f(x+t).$$

Generator of Shift Group: Define $A = \frac{d}{dx}$ on the domain

$$D(A) = \{f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R})\},$$

where f' is in the distributional sense.

Momentum Operator: Define $P = -i\,A = -i\,\frac{d}{dx}$ on the same domain $D(P) = D(A)$.

Fourier Transform:

$$\mathcal{F}[f](\omega) = \hat{f}(\omega) := \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx.$$

Theorems and Proofs

Theorem 1. *[Stone's Theorem Applied to Shift Group] The strongly continuous unitary group (S_t) on $L^2(\mathbb{R})$ has a densely defined skew-adjoint generator $A = \frac{d}{dx}$ such that $S_t = e^{tA}$. The generator satisfies*

$$A f = \lim_{h\to 0} \frac{S_h f - f}{h}$$

in the L^2 topology on the domain $D(A)$.

Proof. Let $f \in D(A)$. Then

$$\frac{S_h f(x) - f(x)}{h} = \frac{f(x+h) - f(x)}{h} \to f'(x)$$

as $h \to 0$ in L^2 norm. Thus, the infinitesimal generator of S_t is $A = \frac{d}{dx}$.

To verify A is skew-adjoint, for $f, g \in D(A)$:

$$\begin{aligned}
\langle A f, g\rangle &= \int_{-\infty}^{\infty} f'(x)\overline{g(x)} d x & (1)\\
&= f(x)\overline{g(x)}|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\overline{g'(x)} d x & (2)\\
&= 0 - \int_{-\infty}^{\infty} f(x)\overline{g'(x)} d x & (3)\\
&= -\langle f, A g\rangle & (4)
\end{aligned}$$

Therefore $A^* = -A$, confirming A is skew-adjoint. □

Theorem 2. *[Relation Between Generators] The shift group is generated by both the skew-adjoint operator $A = \frac{d}{dx}$ and the self-adjoint momentum operator $P = -i A$:*

$$S_t = e^{tA} = e^{-itP}$$

Proof. Since $P = -i A$, we have $-i t P = -i t (-i A) = -i^2 t A = t A$. Therefore:

$$e^{-itP} = e^{tA}$$

For $f \in D(A)$, using the Taylor expansion:

$$e^{tA} f(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n f(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f^{(n)}(x) = f(x+t) = S_t f(x)$$ □

Theorem 3. *[Complex Exponentials Are Eigenfunctions] For any $\omega \in \mathbb{R}$:*

1. $A e^{i\omega x} = i \omega e^{i\omega x}$ *(eigenvalue $i\omega$ for skew-adjoint A)*
2. $P e^{i\omega x} = \omega e^{i\omega x}$ *(eigenvalue ω for self-adjoint P)*

Proof. Direct calculations:

1. $A e^{i\omega x} = \frac{d}{dx} e^{i\omega x} = i \omega e^{i\omega x}$
2. $P e^{i\omega x} = -i \frac{d}{dx} e^{i\omega x} = -i (i\omega) e^{i\omega x} = \omega e^{i\omega x}$ □

Theorem 4. *[Spectral Decomposition via Fourier Transform] Under the Fourier transform $\mathcal{F}$:*

1. *The self-adjoint momentum operator becomes multiplication by ω: $\mathcal{F}[Pf](\omega) = \omega\,\hat{f}(\omega)$*

2. *The shift group becomes multiplication by a phase: $\mathcal{F}[S_t f](\omega) = e^{i\omega t}\,\hat{f}(\omega)$*

Proof. For part 1, if $f \in D(P)$:

$$\mathcal{F}[Pf](\omega) = \int_{-\infty}^{\infty} (-i\,f'(x))\,e^{-i\omega x}\,dx \tag{5}$$

$$= -i\int_{-\infty}^{\infty} f'(x)\,e^{-i\omega x}\,dx \tag{6}$$

Integration by parts (boundary terms vanish):

$$= -i\,[0 + i\,\omega\,\hat{f}(\omega)] = \omega\,\hat{f}(\omega) \tag{7}$$

For part 2:

$$\mathcal{F}[S_t f](\omega) = \int_{-\infty}^{\infty} f(x+t)\,e^{-i\omega x}\,dx \tag{8}$$

Let $u = x + t$, so $x = u - t$, $dx = du$:

$$= \int_{-\infty}^{\infty} f(u)\,e^{-i\omega(u-t)}\,du \tag{9}$$

$$= e^{i\omega t}\int_{-\infty}^{\infty} f(u)\,e^{-i\omega u}\,du = e^{i\omega t}\,\hat{f}(\omega) \tag{10}$$

□

Theorem 5. *[Eigenfunction Property of Shift Group] Complex exponentials are eigenfunctions of the shift group:*

$$S_t\,e^{i\omega x} = e^{i\omega t}\,e^{i\omega x}$$

with eigenvalue $e^{i\omega t}$.

Proof. $$S_t\,e^{i\omega x} = e^{i\omega(x+t)} = e^{i\omega x}\,e^{i\omega t} = e^{i\omega t}\,e^{i\omega x}$$ □

Corollary 6. *[Consistency Check] The eigenvalue relationships are consistent:*

$$S_t\, e^{i\omega x} = e^{tA}\, e^{i\omega x} = e^{t(i\omega)}\, e^{i\omega x} = e^{i\omega t}\, e^{i\omega x}$$

since A has eigenvalue $i\,\omega$ on $e^{i\omega x}$.

Conclusion

Stone's theorem ensures that the shift group (S_t) has a **skew-adjoint generator** $A = \frac{d}{d\,x}$, whose eigenfunctions are the complex exponentials $e^{i\omega x}$ with purely imaginary eigenvalues $i\,\omega$. The related **self-adjoint momentum operator** $P = -i\,A$ has the same eigenfunctions but with real eigenvalues ω.

The Fourier transform provides the spectral decomposition that diagonalizes both operators:

- P becomes multiplication by ω (real eigenvalues)
- S_t becomes multiplication by $e^{i\omega t}$ (unitary eigenvalues on the unit circle)

This mathematical structure underlies all of Fourier analysis: complex exponentials are the fundamental building blocks because they are precisely the functions that transform simply under shifts, making them the natural basis for analyzing translation-invariant systems. The distinction between the skew-adjoint generator A (with imaginary eigenvalues) and the self-adjoint momentum operator P (with real eigenvalues) is crucial for understanding why unitary groups arise from self-adjoint operators via Stone's theorem.