Injectively Time-Changed Stationary Processes: A Spectral Analysis

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1 Introduction

We develop the theory of injectively time-changed stationary processes, which arise from spectral representations of the form

$$X(t) = \int_{-1}^{1} f(\lambda)\, e^{i\lambda(\theta(t) - t)}\, d\lambda \tag{1}$$

where $\theta: \mathbb{R} \to \mathbb{R}$ is strictly increasing and $f \in L^2([-1, 1])$.

Definition 1. *An* injectively time-changed stationary process *is a stochastic process* $\{X(t)\}_{t \in \mathbb{R}}$ *admitting the spectral representation*

$$X(t) = \int_{-1}^{1} e^{i\lambda(\theta(t) - t)}\, dZ(\lambda) \tag{2}$$

where $\theta: \mathbb{R} \to \mathbb{R}$ *is strictly increasing,* $\theta \in C^1(\mathbb{R})$*, and* $\{Z(\lambda)\}_{\lambda \in [-1,1]}$ *is an orthogonal increment process with* $E[|dZ(\lambda)|^2] = F(d\lambda)$ *for some finite measure* F *on* $[-1, 1]$.

2 Fundamental Properties

Theorem 2. *[Spectral Representation] Let $X(t)$ be an injectively time-changed stationary process. Then:*

1. *$X(t)$ is well-defined for all $t \in \mathbb{R}$*
2. *$E[|X(t)|^2] = \int_{-1}^{1} F(d\lambda) < \infty$*
3. *The covariance function satisfies*

$$Cov(X(s), X(t)) = \int_{-1}^{1} e^{i\lambda((\theta(t)-t)-(\theta(s)-s))} F(d\lambda) \tag{3}$$

Proof. (1) Since θ is strictly increasing and continuous, $\theta(t) - t$ is well-defined for all t. The integral converges by the Cauchy-Schwarz inequality:

$$E[|X(t)|^2] = E\left[\left|\int_{-1}^{1} e^{i\lambda(\theta(t)-t)} dZ(\lambda)\right|^2\right] \tag{4}$$
$$= \int_{-1}^{1} F(d\lambda) < \infty \tag{5}$$

(2) Follows immediately from (1).

(3) By orthogonality of increments:

$$\mathrm{Cov}(X(s), X(t)) = E\left[\int_{-1}^{1} e^{i\lambda(\theta(s)-s)} dZ(\lambda) \cdot \overline{\int_{-1}^{1} e^{i\mu(\theta(t)-t)} dZ(\mu)}\right] \tag{6}$$
$$= \int_{-1}^{1} e^{i\lambda((\theta(s)-s)-(\theta(t)-t))} F(d\lambda) \tag{7}$$

□

Theorem 3. *[Non-Stationarity] An injectively time-changed stationary process $X(t)$ is stationary if and only if $\theta(t) = t + c$ for some constant $c \in \mathbb{R}$.*

Proof. ($\Leftarrow$) If $\theta(t) = t + c$, then $\theta(t) - t = c$ and

$$\mathrm{Cov}(X(s), X(t)) = \int_{-1}^{1} F(d\lambda) = \mathrm{Var}(X(0)) \tag{8}$$

which depends only on $|t - s| = 0$, so $X(t)$ is stationary.

($\Rightarrow$) Suppose $X(t)$ is stationary. Then $\operatorname{Cov}(X(s), X(t))$ depends only on $t - s$. From Theorem 1, this requires

$$(\theta(t) - t) - (\theta(s) - s) = g\,(t - s) \tag{9}$$

for some function g. Setting $u = t - s$ and differentiating with respect to t:

$$\theta'(t) - 1 = g'(u) \cdot 1 = g'\,(t - s) \tag{10}$$

Since the left side depends only on t and the right side on $t - s$, both must be constant. Thus $\theta'(t) = 1 + k$ for some constant k, implying $\theta(t) = t + k\,t + c$. For stationarity, we need $g(u) = k\,u$, which requires $k = 0$. Therefore $\theta(t) = t + c$. ☐

3 Warping Deviation Analysis

Definition 4. *The* warping deviation function *is* $\Delta(t) := \theta(t) - t$.

Proposition 5. *[Deviation Properties] Let* $\Delta(t) = \theta(t) - t$ *where* θ *is strictly increasing. Then:*

1. $\Delta'(t) = \theta'(t) - 1$
2. $X(t)$ *is non-stationary unless* $\Delta(t)$ *is constant*
3. *The instantaneous frequency modulation is* $\lambda\,\Delta'(t)$

Proof. (1) and (2) are immediate. For (3), the phase of the spectral component at frequency λ is $\lambda\,(\theta(t) - t) = \lambda\,\Delta(t)$. The instantaneous frequency is

$$\frac{d}{d\,t}\,[\lambda\,\Delta(t)] = \lambda\,\Delta'(t) = \lambda\,(\theta'(t) - 1) \tag{11}$$ ☐

4 Inversion and Reconstruction

Theorem 6. *[Inversion Formula] Let* $X(t)$ *be an injectively time-changed stationary process with spectral measure* F*. If* θ *is invertible, then*

$$F(\{\lambda\}) = \lim_{T \to \infty} \frac{1}{2\,T} \int_{-T}^{T} X(t)\,e^{-i\lambda(\theta(t) - t)}\;d\,t \tag{12}$$

when F *has point masses.*

Proof. For a point mass at λ_0, $X(t) = A\, e^{i\lambda_0(\theta(t)-t)}$ for some constant A. Then:

$$\frac{1}{2T}\int_{-T}^{T} X(t)\, e^{-i\lambda(\theta(t)-t)}\, dt \tag{13}$$

$$= \frac{A}{2T}\int_{-T}^{T} e^{i(\lambda_0-\lambda)(\theta(t)-t)}\, dt \tag{14}$$

As $T \to \infty$, this converges to $A\, \delta_{\lambda_0}(\lambda)$ by the Riemann-Lebesgue lemma when $\lambda \neq \lambda_0$, and to A when $\lambda = \lambda_0$. □

5 Band-Limited Structure

Theorem 7. *[Band-Limited Representation] Every injectively time-changed stationary process with support in* $[-1,1]$ *can be written as*

$$X(t) = \int_{-1}^{1} \hat{f}(\lambda)\, e^{i\lambda(\theta(t)-t)}\, d\lambda \tag{15}$$

where $\hat{f}$ *is the Fourier transform of some* $f \in L^2(\mathbb{R})$.

Proof. Since the spectral measure F has support in $[-1,1]$, we can write $F(d\lambda) = |\hat{f}(\lambda)|^2\, d\lambda$ for some $\hat{f} \in L^2([-1,1])$ by the Radon-Nikodym theorem. The band-limited nature ensures $\hat{f}$ extends to an $L^2(\mathbb{R})$ function that is the Fourier transform of some $f \in L^2(\mathbb{R})$. □

6 Oscillatory Properties

Theorem 8. *[Priestley Oscillatory Characterization] An injectively time-changed stationary process* $X(t)$ *with band-limited spectrum in* $[-1,1]$ *is oscillatory in Priestley's sense if and only if the spectral measure* F *is concentrated away from* $\lambda = 0$.

Proof. Priestley defines oscillatory processes as those whose spectral density is concentrated around non-zero frequencies. Since our process has the form

$$X(t) = \int_{-1}^{1} e^{i\lambda(\theta(t)-t)}\, dZ(\lambda) \tag{16}$$

the oscillatory nature depends on whether F assigns significant mass near $\lambda = 0$. If $F(\{0\}) = 0$ and F is concentrated away from zero, then $X(t)$ exhibits sustained oscillations modulated by the time-change $\theta(t) - t$. □

7 Conclusion

Injectively time-changed stationary processes provide a natural generalization of stationary processes that preserves spectral structure while allowing for non-trivial temporal evolution. The warping deviation $\theta(t) - t$ serves as the fundamental mechanism for introducing non-stationarity while maintaining the interpretability of frequency-domain analysis.