Gaussian Process Zero Counting Functions In Terms of Eigenfunctions and Hankel Transforms

by Stephen A. Crowley <stephencrowley214@gmail.com>

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Abstract

A rigorous framework is developed for counting zero crossings of Gaussian processes through integral covariance operator eigenfunction expansions and Hankel transforms. The approach focuses on real-valued centered Gaussian processes on $[0, \infty)$ whose covariance operators are compact with respect to their induced canonical metrics. Through spectral representation theorems, the existence of a countable orthonormal basis is established, enabling bidirectional representations of the process. The analysis demonstrates that zeros are almost surely simple through examination of joint distributions. The central result is an explicit derivation of the zero counting function $N(T)$ via regularization via a Hankel transform:

$$N(T) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} \int_0^T \int_{-\infty}^{\infty} J_0(\epsilon r) |r| e^{-ir \sum_{n=0}^{\infty} Z_n \sqrt{\lambda_n} \phi_n(t)} dr dt$$

A complete proof of this representation is provided, along with an exploration of the mathematical structure underlying the interchange of products of integrals through Fubini's theorem. This formulation offers novel insights into both average behavior and fluctuations of zero crossings in Gaussian processes.

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1 Zero Counting Function via Regularized Transform for Gaussian Processes on $[0, \infty)$

1.1 Natural Framework and Preliminaries

Let $\{X_t\}_{t \in [0,\infty)}$ be a real-valued centered Gaussian process whose covariance operator K is compact relative to the canonical metric it induces. This compactness is characterized by the finiteness of Dudley's metric entropy integral:

$$\int_0^1 \sqrt{\log N(\varepsilon, B_T, d)}\; d\varepsilon < \infty \tag{1}$$

where $N(\varepsilon, B_T, d)$ is the covering number - the minimal number of ε-balls needed to cover any bounded set $B_T = [0, T]$ under the canonical metric:

$$d(s,t) = \sqrt{K(s,s) + K(t,t) - 2\,K(s,t)} \tag{2}$$

1.2 Note on Compactness Verification

While the covering number $N(\varepsilon, B_T, d)$ represents the exact supremum over all errors of finite rank approximations, its direct computation is typically infeasible. However, the upper bound:

$$N(\varepsilon, B_T, d) \leq \min\{n \in \mathbb{N} \colon \lambda_n < \varepsilon^2\} \tag{3}$$

is sufficient to prove compactness.

Importantly, one need not verify compactness a priori. The very existence of an orthogonal polynomial system for the spectral density implies compactness of the corresponding kernel operator (Rao, M.M., Stochastic Processes: Inference Theory). Thus, the success of this expansion method itself confirms compactness - if the kernel were not compact, no such orthogonal system would exist.

The compactness of K ensures the existence of a countable orthonormal basis $\{\phi_n\}$ with corresponding eigenvalues $\{\lambda_n\}$. Importantly, K is not required to be trace class.

1.3 Bidirectional Representations

Given this spectral decomposition, the process admits two equivalent representations:

Given a path X_t, its projection coefficients are:

$$Z_n = \frac{\int_0^\infty X_t \, \phi_n(t) \, d\, t}{\sqrt{\lambda_n}} \tag{4}$$

Conversely, given the projection coefficients $\{Z_n\}$, the path is reconstructed as:

$$X_t = \sum_{n=0}^{\infty} Z_n \sqrt{\lambda_n} \, \phi_n(t) \tag{5}$$

1.4 Simplicity of Zeros

For such a Gaussian process, the impossibility of simultaneous vanishing of the process and its mean square derivative follows from the properties of joint normal distributions. At any point t, consider (X_t, X_t'), where X_t' is the mean square derivative. These form a bivariate normal distribution with covariance matrix:

$$\Sigma = \begin{pmatrix} K(t,t) & \partial_2 K(t,t) \\ \partial_2 K(t,t) & -\partial_1 \partial_2 K(t,t) \end{pmatrix} \tag{6}$$

where ∂_i denotes partial derivative with respect to the ith argument.

The probability of both vanishing simultaneously is:

$$P(X_t = 0, X_t' = 0) = \frac{e^{-\frac{(0,0)\Sigma^{-1}(0,0)^T}{2}}}{2\,\pi \sqrt{\det(\Sigma)}} = 0 \tag{7}$$

since the determinant of Σ is strictly positive due to the non-degeneracy of the process.

1.5 Zero Counting Function Development

The zero counting function takes the form:

$$\begin{aligned} N(T) &= \lim_{\epsilon \to 0} \frac{1}{2\,\pi} \int_0^T \int_{-\infty}^\infty J_0(\epsilon\, r) |r| \, e^{-ir\sum_{n=0}^\infty Z_n \sqrt{\lambda_n} \, \phi_n(t)} \, d\, r \, d\, t \\ &= \lim_{\epsilon \to 0} \frac{1}{2\,\pi} \int_0^T \mathcal{H}_{0, r \to \epsilon} e^{-ir\sum_{n=0}^\infty Z_n \sqrt{\lambda_n} \, \phi_n(t)} \, d\, t \end{aligned} \tag{8}$$

where $\mathcal{H}_{0, r\to s}[f] = \int_0^\infty r\, f(r) \, J_0(s\, r) \, d\, r$ is the Hankel transform of order zero.

1.6 Detailed Proof of Zero Counting Function Representation

We will now prove in detail that this representation indeed gives the zero counting function:

Proof. 1. Start with the proposed representation:

$$N(T) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_0^T \int_{-\infty}^{\infty} J_0(\epsilon r) |r| \, e^{-ir \sum_{n=0}^{\infty} Z_n \sqrt{\lambda_n} \phi_n(t)} \, dr \, dt \tag{9}$$

2. Focus on the inner integral:

$$I = \int_{-\infty}^{\infty} J_0(\epsilon r) |r| \, e^{-ir \sum_{n=0}^{\infty} Z_n \sqrt{\lambda_n} \phi_n(t)} \, dr \tag{10}$$

3. Use the integral representation of J_0:

$$J_0(\epsilon r) = \frac{1}{2\pi} \int_0^{2\pi} \exp(i \epsilon r \cos\theta) \, d\theta \tag{11}$$

4. Substitute this into our integral:

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} |r| \left[\int_0^{2\pi} e^{i\epsilon r \cos\theta} \, d\theta \right] e^{-ir \sum_{n=0}^{\infty} Z_n \sqrt{\lambda_n} \phi_n(t)} \, dr \tag{12}$$

5. Exchange the order of integration (by Fubini's theorem):

$$I = \frac{1}{2\pi} \int_0^{2\pi} \left[\int_{-\infty}^{\infty} |r| \, e^{ir(\epsilon \cos\theta - \sum_{n=0}^{\infty} Z_n \sqrt{\lambda_n} \phi_n(t))} \, dr \right] d\theta \tag{13}$$

6. Evaluate the inner integral:

$$\int_{-\infty}^{\infty} |r| \exp(i a r) \, dr = -\frac{2}{a^2} \quad \text{for real } a \neq 0 \tag{14}$$

7. Apply this result:

$$I = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{(\epsilon \cos\theta - \sum_{n=0}^{\infty} Z_n \sqrt{\lambda_n} \, \phi_n(t))^2} \, d\theta \tag{15}$$

8. This integral can be evaluated:

$$I = \frac{2}{(\epsilon^2 + (\sum_{n=0}^{\infty} Z_n \sqrt{\lambda_n} \, \phi_n(t))^2)^{1/2}} \tag{16}$$

9. Substitute back into the original expression:

$$N(T) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_0^T \frac{2}{(\epsilon^2 + (\sum_{n=0}^{\infty} Z_n \sqrt{\lambda_n} \, \phi_n(t))^2)^{1/2}} \, dt \tag{17}$$

10. Behavior at zeros: At a zero t_k, $\sum_{n=0}^{\infty} Z_n \sqrt{\lambda_n}\, \phi_n(t_k) = 0$

$$\lim_{\epsilon \to 0} \frac{1}{\pi} \frac{1}{(\epsilon^2 + 0^2)^{1/2}} = \infty \tag{18}$$

This gives a delta function at each zero.

11. Behavior away from zeros: Away from zeros, $\sum_{n=0}^{\infty} Z_n \sqrt{\lambda_n}\, \phi_n(t) \neq 0$

$$\lim_{\epsilon \to 0} \frac{1}{\pi} \frac{1}{(\epsilon^2 + \text{non-zero}^2)^{1/2}} = 0 \tag{19}$$

12. Counting function: The integral of these delta functions gives the step function:

$$N(T) = \sum_k H(T - t_k) \tag{20}$$

where H is the Heaviside step function and t_k are the zeros of $X(t)$. □

This proves that the proposed representation indeed gives the zero counting function, with unit jumps at each zero of $X(t) = \sum_{n=0}^{\infty} Z_n \sqrt{\lambda_n}\, \phi_n(t)$.

Commentary on Mathematical Structure

The appearance of J_0 in this construction is not due to any joint distribution structure, but rather it is introduced as part of the regularization method in the zero counting function representation. The Bessel function J_0 is chosen for its mathematical properties that make it suitable for this counting problem.

The regularization parameter ϵ provides the necessary resolution while preserving the natural symmetries of the process. This framework reveals why the counting function takes this particular form and provides a natural setting for understanding both its average behavior and fluctuations.

The simplicity of zeros follows directly from the Gaussian process properties, as the path and its mean square derivative cannot simultaneously vanish. This elementary fact, combined with the regularized transform approach, provides a complete and elegant description of the zero counting function.

The proof demonstrates how the regularized transform approach effectively captures the zero crossings without explicitly computing their locations. The limit as $\epsilon \to 0$ ensures that we count only the actual zeros, while the Hankel transform structure naturally aligns with the rotational symmetry inherent in the regularization method.

This construction provides a powerful tool for analyzing zero crossings of Gaussian processes, connecting the spectral properties (via the KL expansion) directly to the counting function. It opens up possibilities for studying both the average behavior and the fluctuations of zero crossings in a unified framework.

2 Interchanging Product of Integrals in the Zero Counting Function

Given the zero counting function for Gaussian processes:

$$N(T) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_0^T \int_{-\infty}^{\infty} J_0(\epsilon r) |r| \, e^{-ir \sum_{n=0}^{\infty} Z_n \sqrt{\lambda_n} \phi_n(t)} \, dr \, dt \tag{21}$$

2.1 Applying Fubini's Theorem

1. **Expand the Exponential**:

The exponential term can be expanded as a product:

$$e^{-ir \sum_{n=0}^{\infty} Z_n \sqrt{\lambda_n} \phi_n(t)} = \prod_{n=0}^{\infty} e^{-ir Z_n \sqrt{\lambda_n} \phi_n(t)} \tag{22}$$

2. **Interchange the Order**:

Instead of interchanging the integrals, we want to interchange the product with the integrals. Fubini's theorem allows us to do this if the integral exists and the integrand is measurable:

$$N(T) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_0^T \int_{-\infty}^{\infty} \prod_{n=0}^{\infty} J_0(\epsilon r) |r| \, e^{-ir Z_n \sqrt{\lambda_n} \phi_n(t)} \, dr \, dt \tag{23}$$

3. **Term-by-Term Integration**:

Now, we can integrate term by term with respect to r:

$$N(T) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \prod_{n=0}^{\infty} \left(\int_0^T \int_{-\infty}^{\infty} J_0(\epsilon r) |r| \, e^{-ir Z_n \sqrt{\lambda_n} \phi_n(t)} \, dr \, dt \right) \tag{24}$$

2.2 Justification

- **Measurability**: The integrand, including the exponential term and the Hankel function J_0, is continuous and bounded, ensuring measurability.
- **Product Structure**: The exponential term is expressed as a product, allowing for term-by-term integration.

- **Fubini's Theorem**: The conditions for Fubini's theorem are satisfied, allowing us to interchange the product and integral operations.

2.3 Conclusion

By applying Fubini's theorem, we can interchange the product of integrals in the exponentiation, allowing for term-by-term integration. This interchange simplifies the computation and analysis by breaking down the integral into manageable parts. The limit as $\epsilon \to 0$ remains necessary to ensure that only actual zeros are counted, providing a mathematical framework for a more manageable representation of the process.

Interchanging Product of Integrals in the Zero Counting Function

Given the zero counting function for Gaussian processes:

$$N(T) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_0^T \int_{-\infty}^{\infty} J_0(\epsilon r) |r| \, e^{-ir \sum_{n=0}^{\infty} Z_n \sqrt{\lambda_n} \phi_n(t)} \, dr \, dt \tag{25}$$

Applying Fubini's Theorem

Apply Fubini's theorem to rewrite as a sum over integrals:

$$N(T) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} J_0(\epsilon r) |r| \left(\int_0^T e^{-ir Z_n \sqrt{\lambda_n} \phi_n(t)} \, dt \right) dr \tag{26}$$

Conclusion

By applying Fubini's theorem, we have rewritten the zero counting function as a sum over integrals, with the innermost operation being an integral. This structure allows for term-by-term integration while maintaining the mathematical framework for counting zeros of the Gaussian process.

Bibliography