Special Representations of Weakly Harmonizable Processes

by Derek K. Chang
Department of Mathematics
California State University
Los Angeles, California 90032 M. M. Rao
Department of Mathematics
University of California
Riverside, California 92521

Abstract

This paper contains an account of orthogonal and general series as well as moving-average representations of weakly harmonizable processes. Also oscillatory and periodically correlated second order processes, their relation to harmonizable classes and an operator representation are presented. Further, supports of the spectral functions of weakly harmonizable processes are considered.

Table of contents

1 Introduction

The purpose of this article is to consider some special types of representations for second order processes related to different classes of harmonizable families. These include the series, moving average, and oscillatory representations. To make the statements precise, let us start with their standard formulations.

Thus if (Ω,Σ,P) is a probability space and $L_0^2(\Omega,\Sigma,P)$, or $L_0^2(P)$, is the space of (equivalence classes of) scalar random variables on Ω with means zero and finite variances, then $X\colon T\to L_0^2(P)$, a curve, is called a second order process denoted also as $\{X_t, t\in T\}$. In what follows $T=\mathbb{R}$ (or $\mathbb{Z}$), unless specified differently. Let $r(\cdot,\cdot)\colon(s,t)\mapsto E(X_s X_t)$, called the covariance function on $\mathbb{R}\times\mathbb{R}$, be continuous, where

$$E(X_s X_t)=(X_s,X_t)=\int_\Omega X_s X_t\, dP \forall (s,t)\in T\times T \tag{1}$$

Recall that the process $\{X_t, t\in\mathbb{R}\}$ is then termed (i) (weakly) stationary if $r(s,t)=\tilde{r}(s-t)$ so that by the classical Bochner theorem

$$r(s,t)=\int_{\mathbb{R}} e^{i(s-t)\lambda} F(d\lambda) \tag{2}$$

for a unique positive bounded nondecreasing function F; (ii) strongly harmonizable if

$$r(s,t)=\int_{\mathbb{R}}\int_{\mathbb{R}} e^{is\lambda-it\mu} F(d\lambda,d\mu) \tag{3}$$

for a positive definite $F\colon\mathbb{R}\times\mathbb{R}\to\mathbb{C}$ of bounded variation in the sense of Vitali; and (iii) weakly harmonizable if (3) holds in which F is of the (weaker) bounded variation in Fréchet's sense. (See [2] or [11] on a discussion of these variations and a need for a new integral in the case of (iii) extending Lebesgue's classical concept.)

Some related classes are also of interest. Thus $\{X_t, t\in\mathbb{R}\}$ is (a) periodically correlated if $r(s+\alpha,t+\alpha)=r(s,t)$ for some fixed $\alpha>0$ and all $s,t\in\mathbb{R}$ (α is termed a period); (b) oscillatory if the covariance is representable as (the Lebesgue integral)

$$r(s,t)=\int_{\mathbb{R}} e^{i(s-t)\lambda} a_g(\lambda)\overline{a_t(\lambda)} F(d\lambda) \tag{4}$$

where again F is a positive nondecreasing function, defining a σ-finite measure, and $\{a_s, s\in\mathbb{R}\}$ is a square integrable (for F) collection of scalar functions on $\mathbb{R}$; and (c) Karhunen class if in (4) the factor $e^{i(s-t)\lambda}$ is absent. In all these cases the real or complex F is called the *spectral (measure)* function of the process. There are a few other second order classes of interest in applications, but we shall defer them at this point.

Under some reasonable conditions it is possible to present several series representations of these processes and also certain forms of moving averages for them. Since both of these are useful in many applications, we consider them for harmonizable families which extend the stationary classes.

2 Factorizable Spectral Functions

It is of interest to start with a subclass of harmonizable processes. They were isolated for a special study in [8]. Thus if $\{X_t, t\in\mathbb{R}\}$ is a weakly (or strongly) harmonizable process with $F(\cdot,\cdot)$ as its spectral measure function, then it is termed a *factorizable spectral measure* (f.s.m.) class if $F(s,t)=M(s)\overline{M(t)}$. [Note that if $F(s,t)=M(s)\overline{N(t)}$ is assumed, it is not really different since the positive definiteness of F implies that $N(s)=c\,M(s)$, for some $c\geq 0$, $s\in\mathbb{R}$, so that $M=N$ can be taken as a general form.] This class has some nice structural properties of which the following will be used here. It is stated for the index $T=\mathbb{R}$, but a similar fact holds for $T=\mathbb{Z}$.

Proposition 1. If $\{X_t, t\in\mathbb{R}\}\subset L_0^2(P)$ is a weakly harmonizable process, then the following are equivalent:

(i) the X_t-process is of f.s.m. class,

(ii) the covariance function r of the process is factorizable, in that

$$r(s,t)=f(s)\overline{f(t)} \quad \text{for some continuous function } f\colon\mathbb{R}\to\mathbb{C} \tag{5}$$

In either of these equivalent cases, the process is necessarily strongly harmonizable and the f of (ii) is then the Fourier transform of some scalar measure μ on $\mathbb{R}$ (so that $f=\hat{\mu}$).

Proof: Suppose (i) holds. Then $F(A,B)=M(A)\overline{M(B)}$. Since F is a bimeasure, this implies that $M(\cdot)$ is a scalar σ-additive set function on the Borel σ-algebra $\mathcal{B}$ of $\mathbb{R}$. Thus it is a scalar measure (hence bounded). Then the covariance function r can be written as (cf. (3)):

$$\begin{aligned} r(s,t) &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{is\lambda-it\mu}\, M(d\lambda)\overline{M(d\mu)} \\ &= \int_{\mathbb{R}} e^{is\lambda}\, M(d\lambda)\cdot\left(\int_{\mathbb{R}} e^{it\mu}\overline{M(d\mu)}\right) \\ &= f(s)\overline{f(t)} (=\overline{M(s)\, M(t)^{-1}}) \end{aligned} \tag{6}$$

by using a form of the Fubini theorem. So (ii) holds.

Suppose, on the other hand, that (ii) holds. Then $r(s,t)=f(s)\overline{f(t)}$, for some f. Since the process is harmonizable, r can be expressed as (3). For the weakly harmonizable case, the integral there must be in the sense of M. Morse and W. Transue. Even in this general case, one can use a suitable form of the inversion formula as follows. (See [13], Thm. 3.2 without proof; and [11], Thm. 8.2 with a detailed and different argument.) Thus

$$F(A,B)=\lim_{0<T_1,T_2\to\infty}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2}\frac{e^{-i\lambda s_1}\,e^{-i\lambda_1 s_2}}{-i\,s}\cdot\frac{e^{i\lambda_2 t}\,e^{i\lambda_1 t}}{i\,t}\, r(s,t)\,d\,s\,d\,t \tag{7}$$

where $A=(\lambda_1,\lambda_2)$, $B=(\lambda_1',\lambda_2')$ are intervals of $\mathbb{R}$ such that $F(\{\lambda_1\},\{\lambda_1'\})=0=F(\{\lambda_2\},\{\lambda_2'\})$. Substituting $r(s,t)=f(s)\overline{f(t)}$ in (5), this reduces to

$$F(A,B)=M(A)\overline{M(B)}\text{for a suitable } M(\cdot).$$

Since F is a bimeasure, this shows that $M(\cdot)$ is σ-additive on the class of all such intervals which are also continuity intervals of $M(\cdot)$. But then the standard theory says that $M(\cdot)$ has a unique σ-additive extension to be a scalar measure on B, so that the process is of f.s.m. class and (i) obtains.

When these equivalent conditions hold, $F(A,B)=M(A)\times\overline{M(B)}$, and hence has a unique extension to be a bounded Borel (scalar) measure on the Borel sets of $\mathbb{R}\times\mathbb{R}$. But this means (3) is a Lebesgue integral and the process is strongly harmonizable. This implies that the f.s.m. class $\subseteq$ strongly harmonizable class. The above result illuminates an interesting connection between the harmonizable and the periodically correlated classes. While introducing and analyzing the latter class, Gladyšev [3] has shown that every periodically correlated sequence $\{X_n,-\infty<n<\infty\}$ is necessarily strongly harmonizable. However, a continuous parameter periodically correlated process need not even be weakly harmonizable. This also follows from Proposition 1. Indeed let $f\colon\mathbb{R}\to\mathbb{C}$ be a periodic continuous function having only a conditionally convergent Fourier series. For instance such a function may be constructed as follows. Let $\varphi\colon[0,2\pi]\to\mathbb{R}$ be the classical Cantor function and define $\tilde{f}\colon[0,2\pi]\to\mathbb{R}$ by the equation $\tilde{f}(x)=\varphi(x)-(x/2\pi)$, and extend it to $\mathbb{R}$ by periodicity. If this extended function is f, then it will be a candidate. Let $r(s,t)=f(s)\overline{f(t)}$. Then $r(\cdot,\cdot)$ is clearly a covariance function of a periodically correlated Gaussian process with mean zero. But it cannot be weakly harmonizable, since otherwise by the above proposition $f=\hat{M}$ for a scalar measure which must then have an absolutely convergent Fourier series. This fact was first noted for an example in [4]. It is also not hard to construct examples of periodically correlated weakly harmonizable processes that are not strongly harmonizable by considering a bimeasure F in (3) which determines a σ-finite but nonfinite measure on $\mathbb{R}\times\mathbb{R}$. One such simple example is also given in [4]. *The point of this paragraph is to draw the attention of the reader to a significant distinction between certain discrete and continuous parameter harmonizable processes in their structures.*

3 Series Representations

Turning to some other aspects, we recall that every (weakly or strongly) harmonizable process is of Karhunen class (cf. [2], Cor. 7.6, and also p. 77, Eqs. (85) and (86) for a second proof). One of these proofs actually uses a series representation in its demonstration. There are several such expansions and we shall present a "natural" series representation of (weakly) harmonizable processes using its spectral domain properties, and another type later on.

Consider a weakly harmonizable process $\{X_t,t\in\mathbb{R}\}$ with its spectral measure function F. Then the integral in (3) relative to F is defined and termed a "strict" MT-integral in [2], and denoted

$$r(s,t)=\int_{\mathbb{R}^2}^{*} e^{is\lambda-it\mu}\,F(d\lambda,d\mu) \tag{8}$$

and it reduces to the standard Lebesgue integral when F has finite Vitali variation. The spectral domain of the process is denoted as:

$$L_*^2(F)=\{f\text{measurable: }\int_{\mathbb{R}^2}^{*} f(\lambda)\overline{f(\mu)}F(d\lambda,d\mu)=(f,f)_F<\infty\}.$$

Evidently $e_s\in L_*^2(F)$ and $(e_s,e_t)_F=r(s,t)$ where $e_s(\lambda)=e^{is\lambda}$.

It is easy to verify that $(\cdot,\cdot)_F$ is a semi-inner product and that $L^2_*(F)$ is a vector space. However, it is also true but nontrivial that $\{L^2_*(F), (\,\cdot,\cdot)_F\}$ is a complete (semi-)inner product space; and the proof uses several results of the structure theory of harmonizable processes (cf. [2], p. 76). Even in the strongly harmonizable case the completeness proof is nontrivial. Thus if $\mathcal{N} = \{f\colon (f, f)_F = 0\}$ and $L^2(F) = L^2_*(F)/\mathcal{N}$ is the quotient space, let $\tilde{f}_i = f_i + \mathcal{N}, i = 1, 2$. Set

$$(\tilde{f}_1, \tilde{f}_2)_F = (f_1, f_2)_F \tag{9}$$

Then $\{L^2(F), (\,\cdot,\cdot)_F\}$ is a Hilbert space and if $\mathcal{H}(X) = \mathrm{sp}\{X_t, t\in\mathbb{R}\} \subset L^2_0(P)$ is the closed linear span, then the correspondence $\tau\colon \tilde{f} \mapsto \int_{\mathbb{R}} f\, d\, Z (\in \mathcal{H}(X))$, sets up an isometric isomorphism between $L^2(F)$ and $\mathcal{H}(X)$. Here $Z(\cdot)$ is the stochastic measure representing the X_t process so that

$$X_t = \int_{\mathbb{R}} e^{it\lambda}\, Z\,(d\,\lambda) \forall t \in \mathbb{R} \tag{10}$$

where the integral is taken in the Dunford-Schwartz sense (cf. [2]). Since the covariance function r of (6) is continuous, it is known from prior results that $\mathcal{H}(X)$ is separable and, due to the isometry noted above, so is $L^2(F)$. Let $\{f_n, n\geq 1\} \subset L^2(F)$ be any complete orthonormal sequence. If $\tilde{f}_n = f_n + \mathcal{N}$ and

$$\xi_n = \int_{\mathbb{R}} f_n(\lambda)\, Z\,(d\,\lambda) \quad (=\tau(\tilde{f}_n)),$$

then $\{\xi_n, n\geq 1\} \subset \mathcal{H}(X)$ is also a complete orthonormal sequence. With this setup we can establish the following result which extends a similar one for the strongly harmonizable case in [1]. However, the latter authors assumed implicitly the completeness of $L^2(F)$, and this property was not established until recently.

Proposition 2. Let $\{X_t, t\in\mathbb{R}\}$ be a weakly harmonizable process with $L^2(F)$ as its spectral domain. If $\{\tilde{f}_n, n\geq 1\}$ is a complete orthonormal set in $L^2(F)$ and $\{\xi_n, n\geq 1\}$ the corresponding set in $\mathcal{H}(X) = \mathrm{sp}\{X_t, t\in\mathbb{R}\} \subset L^2_0(P)$, then

$$X_t = \sum_{n=1}^{\infty} a_n(t)\, \xi_n, \quad t\in\mathbb{R} \tag{11}$$

where $a_n(t) = (e_t, f_n)_F, n\geq 1, \tilde{f}_n = f_n + \mathcal{N}$.

Proof: Since $\tau\colon L^2(F) \to \mathcal{H}(X)$ is an isometric isomorphism and $\tau(\tilde{f}_n) = \xi_n$, one has

$$(\xi_n, \xi_m) = (\tau(\tilde{f}_n), \tau(\tilde{f}_m)) = (f_n, f_m)_F = \delta_{mn}. \tag{12}$$

Expanding X_t in a Fourier series in $\mathcal{H}(X)$, we get (11) immediately with the coefficients $a_n(t)$ given by

$$a_n(t) = (X_t, \xi_n) = (X_t, \tau(\tilde{f}_n)) = (\tau^*(X_t), f_n)_F \tag{13}$$

with τ^* being the adjoint of

$$\tau = (e_t, f_n)_F \tag{14}$$

since $\tau^*(X_t) = e_t$. This is precisely the assertion. ■

It is natural to ask for a Karhunen-Loève type expansion (cf. [5], p. 64) for harmonizable processes. However, the covariance function $r: \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ of such a process, being bounded and continuous, is not necessarily square integrable on $\mathbb{R} \times \mathbb{R}$, and here one needs to alter slightly the above type series representation. But with such a modification one can present a result for a somewhat general class including all harmonizable families as follows.

Let $\{X_t, t \in T\} \subset L_0^2(P), T \subset \mathbb{R}$, be a process with a bounded continuous covariance function r. Let

$$\tilde{r}(s,t) = \sqrt{\frac{r(s,t)}{(1+s^2)\,(1+t^2)}} \tag{15}$$

Then $\tilde{r}$ is also continuous and positive definite. The first being obvious, the second property is seen from the fact that for any continuous $\varphi: \mathbb{R} \to \mathbb{C}$ with compact support, one has

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \tilde{r}(s,t)\, \varphi(s) \overline{\varphi(t)} d\,s\,d\,t = \int_{\mathbb{R}} \int_{\mathbb{R}} r(s,t) \tilde{\varphi}(s) \overline{\tilde{\varphi}(t)} d\,s\,d\,t \geq 0 \tag{16}$$

since r is positive definite and $\frac{\varphi(s)}{(1+s^2)\;12}$ is integrable. If $\tilde{X}_t = X_t / \sqrt{1+t^2}$, then $\tilde{X}_t \in L_0^2(P)$, $E\,(X_s \tilde{X}_t) = \tilde{r}(s,t)$, and what is decisive, $\tilde{r}$ is square integrable on $\mathbb{R} \times \mathbb{R}$, if r is bounded. In this case the operator A defined by

$$(A\,f)(s) = \int_{\mathbb{R}} \tilde{r}(s,t)\; f(t)\,d\,t \tag{17}$$

is completely continuous on the standard Lebesgue space $L^2(\mathbb{R})$, and has a complete set of (orthonormal) eigenfunctions corresponding to its eigenvalues $\lambda_i \geq 0$ (cf. [12], p. 243) so that $A\,\varphi_i = \lambda_i\,\varphi_i$. We then have the following representation:

Proposition 3. Let $\{X_t, t \in T\} \subset L_0^2(P)$ be a process with a bounded continuous covariance function r. [In particular, this is automatic if the process is weakly or strongly harmonizable.] If $\tilde{r}: (s,t) \mapsto r(s,t) / [(1+s^2)\,(1+t^2)]^{\frac{1}{2}}$ and $\lambda_i (\geq 0)$ are its eigenvalues and φ_i are the corresponding normalized eigenfunctions, then $\{\xi_n, n \geq 1\}$ defined (as a Bochner or stochastic integral) by

$$\xi_n = \lambda_n^{-\frac{1}{2}} \int_{\mathbb{R}} X_t\, \varphi_n(t)\, (1+t^2)^{-\frac{1}{2}}\, d\,t \tag{18}$$

form a complete orthonormal set in $\mathcal{H}(X) = \mathrm{sp}\{X_t, t \in T\} \subset L_0^2(P)$, and one has

$$X_t = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\; \varphi_n(t)\, (1+t^2)^{\frac{1}{2}}\, \xi_n \tag{19}$$

the series converging in $\mathcal{H}(X)$-mean.

Proof: If $\tilde{X}_t = X_t / \sqrt{1+t^2}$, then $E\,(X_s\,\tilde{X}_t) = \tilde{r}(s,t)$ and using (12) and the classical Hilbert-Schmidt theory, we get

$$\tilde{r}(s,t) = \sum_{i=1}^{\infty} \lambda_i\,\varphi_i(s)\overline{\varphi_i(t)} \forall \lambda_i \geq 0 \tag{20}$$

the series converging in mean (and uniformly also). If we take $F(\cdot)$ to be the function which increases by jumps of size λ_i at $i \geq 0$ and vanishing on the negative line, and $a_s(v) = \varphi_v(s)$ for $v = i$, and $=0$ elsewhere in (4), without the exponential term, then (15) can be expressed as

$$\tilde{r}(s,t) = \int_{\mathbb{R}} a_s(v)\overline{a_t(v)} F\,(d\,v) \forall s,t \in T \times T \tag{21}$$

so that $\{\tilde{X}_t, t \in T\}$ is a Karhunen process relative to $\{a_s(\,\cdot), s \in T\}$. Then it may be represented as

$$\tilde{X}_t = \int_{\mathbb{R}} a_t(v)\,Z\,(d\,v) \forall t \in T \tag{22}$$

where $Z(\cdot)$ is a stochastic measure with orthogonal values in $L_0^2(P)$, and $E\,(Z(A)\,Z(B)) = F\,(A \cap B)$ (cf. [5], p. 59; or [2], p. 55). If now we set $Z(\{i\}) = \xi_i \sqrt{\lambda_i}$ where ξ_n is given by (13), then (17) becomes

$$\tilde{X}_t = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\,\varphi_n(t)\,\xi_n \tag{23}$$

the series converging in $\mathcal{H}(X)$. Since $X_t = \tilde{X}_t\,(1+t^2)^{\frac{1}{2}}$, this gives (14). ∎

Remark: If T is a compact set, then no factor such as $[(1+s^2)\,(1+t^2)]^{-\frac{1}{2}}$ is needed in the above development, since the classical Mercer theorem applies and yields the expansion (15) for r. In the noncompact case, Mercer's theorem is no longer available and some such positive factor seems necessary to invoke the Hilbert-Schmidt result.

4 A Moving Average Representation

Another expression used in the stationary case is the moving average representation. Following the classical case (cf. [5], p. 79) one terms a Karhunen type process $\{X_t, t \in \mathbb{R}\} \subset L_0^2(P)$ to have a moving average representation if it is of the form

$$X_t = \int_{\mathbb{R}} g\,(t-u)\,Z\,(d\,u) \forall t \in \mathbb{R}$$

where $Z(\cdot)$ is a stochastic measure and $\tilde{g}\,(t - \cdot) \in L^2(F)$ for each t. Here F is the spectral measure such that $F(A,B) = E(Z(A)\overline{Z(B)})$. If $Z(\cdot)$ has orthogonal values in $L_0^2(P)$ then this reduces to the classical (stationary) case when F is moreover the Lebesgue measure. Here we consider again that $Z(\cdot)$ is an arbitrary stochastic measure in $L_0^2(P)$, and find conditions on g in order that the X_t-process is weakly harmonizable.

The following strong conclusion is somewhat unexpected.

Theorem 4. Let $\{X_t, t \in \mathbb{R}\}$ be a process of the form (18) with $g = \hat{f}$, where $f \in L^1(\mathbb{R})$ and Z is a stochastic measure with values in $L_0^2(P)$. Then the process is strongly harmonizable and its spectral measure is absolutely continuous relative to the planar Lebesgue measure.

Proof: Since $g = \hat{f}$ in (18), and hence g is bounded, it follows that the integral is well defined as a Dunford-Schwartz integral, and some of it properties will be used. Thus, since a form of the Fubini theorem is available here (cf. [11], p. 317 for detailed reasons), one has

$$\begin{aligned} X_t &= \int_{\mathbb{R}} \hat{f}\,(t-u)\, Z\,(d\,u) \\ &= \int_{\mathbb{R}} Z\,(d\,u) \left(\int_{\mathbb{R}} e^{i(t-u)\lambda}\, f(\lambda)\, d\,\lambda \right) \\ &= \int_{\mathbb{R}} e^{it\lambda}\, f(\lambda) \left(\int_{\mathbb{R}} e^{-iu\lambda}\, Z\,(d\,u) \right) d\,\lambda \\ &= \int_{\mathbb{R}} e^{it\lambda}\, f(\lambda) \tilde{Y}(\lambda)\, d\,\lambda, \end{aligned} \tag{24}$$

where $\{\tilde{Y}(\lambda), \lambda \in \mathbb{R}\}$ is a weakly harmonizable process by ([2], Prop. 6.4), and the integral now becomes a Bochner integral. Let $\tilde{Z}: A \mapsto \int_A \tilde{Y}(t)\, f(t)\, d\,t, A \in \mathcal{B}$. Then $\tilde{Z}$ is a stochastic measure on the Borel sets $\mathcal{B}$ of $\mathbb{R}$, and one has

$$\begin{aligned} E(\tilde{Z}(A)\overline{\tilde{Z}(B)}). &= \int_A \int_B E(\tilde{Y}(s)\bar{Y}(t)) \cdot f(s)\overline{f(t)} d\,s\,d\,t \\ &= \int_A \int_B r_Y(s,t)\overline{f(s)}\, f(t)\, d\,s\,d\,t \\ &= \nu(A,B) \qquad \text{(say)} \end{aligned} \tag{25}$$

Here r_Y is the covariance function of the $\tilde{Y}$-process and is bounded. So $\nu(\cdot,\cdot)$ has finite Vitali variation, and (19) can be written, with the properties of the Bochner integral, as

$$X_t = \int_{\mathbb{R}} e^{it\lambda}\, \tilde{Z}(d\,\lambda) \forall \quad t \in \mathbb{R}.$$

Consequently $\{X_t, t \in \mathbb{R}\}$ is strongly harmonizable.

Finally we may calculate the covariance function of X_t as

$$\begin{aligned} r(s,t) &= E(X_s \overline{X_t}) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{is\lambda - it\lambda'}\, \nu(d\,\lambda, d\,\lambda') \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{is\lambda - it\lambda'}\, r_Y(\lambda,\lambda')\overline{f(\lambda)}\, f(\lambda')\, d\,\lambda\, d\,\lambda' \end{aligned} \tag{26}$$

Thus $\nu(\cdot,\cdot)$ is absolutely continuous relative to the Lebesgue measure on $\mathbb{R}\times\mathbb{R}$, being a Borel measure. ■

Remarks: 1. It is easily shown that $|\int_{\mathbb{R}} X_t\, h(t)\, d\, t|_2 \leq K_0 \cdot \|h\|_\infty$, for each $h \in L^1(\mathbb{R})$ and this implies the (V-boundedness and so equivalently) weak harmonizability of the X_t-process (cf. [11], Thm. 4.2). But the above stronger conclusion needs additional work, and the given demonstration seems to be the shortest.

2. If $Z(\cdot)$ has orthogonal increments with an absolutely continuous spectral measure (i.e., the stationary case), then one can prove a converse of the above result. In any event it is an extension of a classical theorem due to Karhunen (cf. [5], p. 83), and is a considerable strengthening of ([1], Thm. 4).

5 Operator Representations

We have defined the Karhunen class and the oscillatory one in Section 1. Both are of second order. The following elementary observation is useful for the ensuing work.

Proposition 5. *The class of oscillatory processes $\{X_t, t\in T\}\subset L_0^2(P)$ and the class of Karhunen processes indexed by T where $T=\mathbb{R}$ or $\mathbb{Z}$, coincide.*

Setting $g_s(\lambda)=e^{is\lambda}\, a_s(\lambda)$ in (4), $s\in T$, and $\lambda\in\hat{T}$ where $\hat{T}=\mathbb{R}$ or $(0,2\,\pi]$ accordingly as $T=\mathbb{R}$ or $\mathbb{Z}$ respectively, in their definitions, the result follows immediately.

This simple identification enables an operator theoretical representation of mean continuous Karhunen processes indexed by $\mathbb{R}$ or $\mathbb{Z}$. We present this form and connect it with the (weakly) harmonizable class. This helps in gaining a better insight of these families. We shall deduce it from reformulations of essentially known results. Hereafter "operator" means "linear operator." **Theorem 6.** *Let $\{X_t, t\in T\}\subset L_0^2(P)$ be a mean continuous process. Then it is representable as*

$$X_t = A_t\, U_t\, Y_0, \quad t\in T \quad (T=\mathbb{R}\ or\ \mathbb{Z}) \tag{27}$$

for some point Y_0 in $\mathcal{H}(X)=\mathrm{sp}\{X_t, t\in T\}\subset L_0^2(P)$, where A_t is a densely defined closed operator in $\mathcal{H}(X)$, for each $t\in T$, and $\{U_s, s\in T\}$ is a weakly continuous unitary group of operators in $\mathcal{H}(X)$ which commutes with each $A_t, t\in T$. If, moreover, the process is weakly harmonizable, then there is an enlarged Hilbert space $\mathcal{K}\supset\mathcal{H}(X)$, operators A_t: $\mathcal{K}\to\mathcal{H}(X)$ and $\{U_s, s\in T\}$ a weakly continuous unitary group on $\mathcal{K}$, $Y_0\in\mathcal{K}$, such that $\{A_t, t\in T\}$ restricted to $\mathcal{H}(X)$ is a weakly continuous contractive positive definite family satisfying $A_0=$ identity on $\mathcal{H}(X)$. [Here positive definiteness means that for any $h_i\in\mathcal{H}(X), t_i\in T$, $i=1,\ldots,n$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(A_{t_i-t_j}h_i, h_j)\geq 0\forall \quad n\geq 1 \tag{28}$$

On the other hand a process defined by (21) is always weakly harmonizable if only $\{A_t, t\in T\}$ is a weakly continuous positive definite contractive family of operators there, with $A_0=$ identity, as above.

Proof: This result is obtained from several known, but deep, facts and we include its essential details for completeness.

Since the process is mean continuous, $\mathcal{H}(X)$ is separable and so $\{X_t, t \in T\}$ is of Karhunen class relative to a family $\{g_t, t \in \hat{T}\}$ and a Borel measure μ. By the preceding proposition, replacing g_t' with $e^{it(\cdot)} g_t$, we may express this as ($\hat{T}$ denoting the dual group of T):

$$X_t = \int_{\hat{T}} e^{it\lambda} g_t(\lambda) Z(d\lambda) \forall t \in T \tag{29}$$

where $Z(\cdot)$ is an orthogonally valued σ-additive set function in $\mathcal{H}(X)$ from the Borel σ-algebra of $\hat{T}$. Here $E(Z(A)\overline{Z(B)}) = \mu(A \cap B)$ and $\int_{\hat{T}} |g_t(\lambda)|^2 \mu(d\lambda) < \infty$. But then the Y_t-process defined by

$$Y_t = \int_{\hat{T}} e^{it\lambda} Z(d\lambda) \forall t \in T \tag{30}$$

is stationary; so, as is well known, there is a weakly continuous group of unitary operators $\{U_t, t \in T\}$ on $\mathcal{H}(X)$ such that $Y_t = U_t Y_0$. Also by the spectral theorem for this family of operators (cf. [12], p. 281 and p. 383)

$$U_t = \int_{\hat{T}} e^{it\lambda} E(d\lambda) \forall t \in T \tag{31}$$

where $\{E(\cdot), \mathcal{B}\}$ is the resolution of the identity of the $\{U_t, t \in T\}$, with $\mathcal{B}$ as the Borel σ-algebra of $\hat{T}$. Thus $Z(A) = E(A) Y_0, A \in \mathcal{B}$. Now define

$$A_t = \int_{\hat{T}} g_t(\lambda) E(d\lambda) \tag{32}$$

Since $\mu(\mathcal{B}) = (E(B) Y_0, E(B) Y_0)$ and $\int_{\hat{T}} |g_t(\lambda)|^2 \mu(d\lambda) < \infty$, it follows from standard results of the functional calculus in Hilbert space that A_t is a closed densely defined operator in $\mathcal{H}(X)$ and its domain contains $\{Y_s, s \in T\}$ for each t. Since U_t and $E(B)$ commute for all t and B, we deduce that A_t and $E(B), B \in \mathcal{B}$ commute. It then follows that A_t and $\{U_s, s \in T\}$ commute for each t (cf. [12], p. 351) and one has

$$\begin{aligned} A_t U_t Y_0 &= A_t \left(\int_{\hat{T}} e^{itv} E(dv) Y_0 \right) \\ &= \int_{\hat{T}} g_t(\lambda) E(d\lambda) \left(\int_{\hat{T}} e^{itv} E(dv) Y_0 \right) \\ &= \int_{\hat{T}} e^{it\lambda} g_t(\lambda) E(d\lambda) Y_0 \\ &= \int_{\hat{T}} e^{it\lambda} g_t(\lambda) Z(d\lambda) = X_t \end{aligned} \tag{33}$$

by (25) and a property of the spectral integral(22). Thus the representation (21) holds.

On the other hand, if X_t is given by (21), with A_t and $\{U_s, s\in T\}$ commuting for each $t\in T$, so that A_t and $\{E(B), B\in\mathcal{B}\}$ also commute, one concludes from a deep theorem of von Neumann and F. Riesz (cf. [12], p. 351 again, and the footnote there) that A_t is a function φ_t of U_t. It follows with the spectral theorem that

$$A_t = \varphi_t(U_t) = \int_{\hat{T}} \varphi_t(\lambda)\, E\,(d\,\lambda) \tag{34}$$

Consequently one has

$X_t = A_t\,(U_t\,Y_0) = \int_{\hat{T}} \varphi_t(\lambda)\, E\,(d\,\lambda) \int_{\hat{T}} e^{itv}\, E\,(d\,v)\, Y_0, = \int_{\hat{T}} e^{it\lambda}\, \varphi_t(\lambda)\, E\,(d\,\lambda)\, Y_0,$ by a property of the spectral integral, $= \int_{\hat{T}} e^{it\lambda}\, \varphi_t(\lambda)\, Z\,(d\,\lambda).$

If we set $g_t'(\lambda) = e^{it\lambda}\, \varphi_t(\lambda)$, $\mu(B) = (E(B)\,Y_0, E(B)\,Y_0)$, then the above stated properties imply that $g_t' \in L^2(\hat{T}, B, \mu)$, so that the process $\{X_t, t\in T\}$ is of Karhunen class.

For the last part since each weakly harmonizable process is of Karhunen class, it admits the representation (21), but now with special properties for $\{A_t, t\in T\}$. Let $Q_t = A_t\,U_t$. Then by a known result (cf. [11], p. 330), $\{Q_t, t\in T\}$ must be positive definite, weakly continuous and contractive. Since $\{U_t, t\in T\}$ is a unitary group which is weakly continuous, it follows that $\{A_t, t\in T\}$ must also be contractive and weakly continuous. To see that it is also positive definite, let $\{h_1, \ldots, h_n\} \subset \mathcal{H}(X)$ be an arbitrary set and $t_1, \ldots, t_n$ be any points of the additive group $T, n \geq 1$. Then letting $v_i = U_{t_i}^*\, h_i \in \mathcal{H}(X)$, we have

$$\begin{aligned}
0 \;&\leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} (Q_{t_i - t_j}\, v_i, v_j) && \text{since } Q_t \text{is positive definite} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} (A_{t_i - t_j}\, U_{t_i}^*\, h_i, U_{t_j}^*\, h_j) && \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} (U_{t_i}^*\, A_{t_i - t_j}\, h_i, U_{t_j}^*\, h_j) && \text{since } U_t^* \text{ and } A_s \text{ also commute as do } U_t \text{ and } U_s \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} (A_{t_i - t_j}\, h_i, h_j) &&
\end{aligned} \tag{35}$$

This establishes the asserted property of the $\{A_t, t\in T\}$. Note that from the fact [11] that Q_0 on $\mathcal{H}(X)$ is also identity, we obtain that $A_0\,U_0 = A_0$ is identity as well.

In the opposite direction the result depends on an important theorem of Grothendieck. If $A_t\colon \mathcal{K} \to \mathcal{H}(X)$ has the stated properties, then $Q_t = A_t\,U_t$ is positive definite and satisfies the hypothesis of the converse part of ([11], p. 330). Hence a process defined by (21) is weakly harmonizable. This finishes the essential details of all the assertions. ■

Remarks: 1. As mentioned before, the first part in an equivalent form is already given in [9], and the last part in [11]. Both results obviously depend on some deep facts in abstract analysis. The simple Proposition 5 is used in an essential way in obtaining the representation (21) which thus gives an operator representation of Karhunen processes indexed by T. The same result extends if T is replaced by a locally compact abelian group when $\mathcal{H}(X)$ is assumed separable in addition.

2. If in (4) we set $s=t$, then

$$\mathrm{var} X_t = r(t,t) = \int_{\mathbb{R}} |a_t(\lambda)|^2 F(d\lambda) = \int_{\mathbb{R}} \mu_t(d\lambda) \tag{36}$$

where $\mu_t(A)=\int_A |a_t(\lambda)|^2 F(d\lambda)$ (taking $T=\mathbb{R}$ for simplicity). This function, depending on t, is termed an "evolving spectrum" in time, by Priestley [10]. Since $g_t(\lambda)=e^{it\lambda}a_t(\lambda)$ can be oscillatory for each t, the process X_t which necessarily has the representation (cf. [2], p. 55):

$$X_t = \int_{\mathbb{R}} e^{it\lambda} a_t(\lambda) Z(d\lambda) \forall \quad t\in\mathbb{R} \tag{37}$$

is termed an oscillatory process. In [10], the a_t-family is normalized so that $a_0(\lambda)=1$, all λ, based on some natural reasons motivated by applications. (It was further assumed that $a_t(\cdot)$ is the Fourier transform of some $H_\lambda(\cdot)$ of bounded variation such that $|H_\lambda(0)|$ has a maximum.) The point of (29) is that $Z(\cdot)$ has orthogonal values and the form suggests a parallel development of the ideas (and interpretations) of the global stationary theory, locally at t if $a_t(\cdot)$ is slowly varying. In any case, for the general representation theory one need not place these restrictions, but the specializations can be useful in estimation and inference theory itself, as indicated in [10].

3. As Proposition 5 also implies, the representation (21) of a Karhunen process is not unique. The preceding sections show that several other (series type) representations are possible.

6 Spectral Support Sets

Finally we consider briefly the support sets of spectral bimeasure functions of the (subclass) periodically correlated weakly harmonizable processes. Also as implied by Proposition 1, a clear distinction emerges between the continuous and discrete parameter cases. This may be anticipated from the work of [3] and [4].

The result to be given here is a generalization of one for the strongly harmonizable case found in [6]. First we need to state the concept of "support" precisely. A set $S_\beta\subset\mathbb{R}\times\mathbb{R}$ is called the *support* of a bimeasure β, if it consists of points $(x,y)\in\mathbb{R}\times\mathbb{R}$ such that for each neighborhood where $\mu_t(A)=\int_A |a_t(\lambda)|^2 F(d\lambda)$ (taking $T=\mathbb{R}$ for simplicity). This function, depending on t, is termed an "evolving spectrum" in time, by Priestley [10]. Since $g_t(\lambda)=e^{it\lambda}a_t(\lambda)$ can be oscillatory for each t, the process X_t which necessarily has the representation (cf. [2], p. 55):

$$X_t = \int_{\mathbb{R}} e^{it\lambda} a_t(\lambda) Z(d\lambda) \forall t\in\mathbb{R} \tag{38}$$

is termed an oscillatory process. In [10], the a_t-family is normalized so that $a_0(\lambda) = 1,$ all λ, based on some natural reasons motivated by applications. (It was further assumed that $a_t(\cdot)$ is the Fourier transform of some $H_\lambda(\cdot)$ of bounded variation such that $|H_\lambda(0)|$ has a maximum.) The point of (29) is that $Z(\cdot)$ has orthogonal values and the form suggests a parallel development of the ideas (and interpretations) of the global stationary theory, locally at t if $a_t(\cdot)$ is slowly varying. In any case, for the general representation theory one need not place these restrictions, but the specializations can be useful in estimation and inference theory itself, as indicated in [10].

3. As Proposition 5 also implies, the representation (21) of a Karhunen process is not unique. The preceding sections show that several other (series type) representations are possible.

7 Spectral Support Sets

Finally we consider briefly the support sets of spectral bimeasure functions of the (subclass) periodically correlated weakly harmonizable processes. Also as implied by Proposition 1, a clear distinction emerges between the continuous and discrete parameter cases. This may be anticipated from the work of [3] and [4].

The result to be given here is a generalization of one for the strongly harmonizable case found in [6]. First we need to state the concept of "support" precisely. A set $S_\beta \subset \mathbb{R} \times \mathbb{R}$ is called the *support* of a bimeasure β, if it consists of points $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that for each neighborhood $U_1 \times U_2$ of (x, y), the variation function $|\beta|(U_1, U_2) \neq 0$. Thus S_β is the smallest closed set off of which β vanishes identically. We can now state the desired extension in the following: $U_1 \times U_2$ of (x, y), the variation function $|\beta|(U_1, U_2) \neq 0$. Thus S_β is the smallest closed set off of which β vanishes identically. We can now state the desired extension in the following:

Proposition 7. *Let $\{X_t, t \in \mathbb{R}\} \subset L_0^2(P)$ be a weakly harmonizable and periodically correlated process with period $\alpha > 0$ so that its covariance function r satisfies $r(s + \alpha, t + \alpha) = r(s, t)$ for all $(s, t) \in \mathbb{R} \times \mathbb{R}$. Then the support of the spectral bimeasure F of the process is contained in*

$$S_F = \{(\lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R} \colon \lambda_1 - \lambda_2 = 2\pi k / \alpha, k \in \mathbb{Z}\} \tag{39}$$

Conversely if the support of the bimeasure F of a weakly harmonizable process lies in S_F of (30), then it is periodically correlated.

Proof: Since a discrete parameter periodically correlated process is always strongly harmonizable (cf. [3]) and the strong case was treated in [6], we only need to consider the continuous parameter weakly harmonizable case. We establish the result using an approximation procedure.

From the structure theory it is long known that each weakly harmonizable process X_t is a limit in $L_0^2(P)$ of a sequence of strongly harmonizable processes X_t^n, uniformly for t in compact subsets (cf. e.g., [11], Thm. 4.4, and reference to related works of others). This implies in particular that, if r_n and r are the covariance functions of the approximant and the given processes, then

$$\lim_{n\to\infty} r_n(s,t) = r(s,t) \tag{40}$$

uniformly for $(s,t)\in K\times K$ where $K\subset\mathbb{R}$ is compact. Now since r is periodic, with period α, we assert that r_n is also periodic with period α, for all large enough n. To see this, suppose it is not true. Then there is an $\epsilon>0$, and a point $(s,t)\in\mathbb{R}\times\mathbb{R}$, such that

$$\liminf_{n\to\infty} |r_n(s+k\,\alpha, t+k\,\alpha) - r_n(s,t)| > \epsilon \tag{41}$$

for some $k\in\mathbb{Z}-\{0\}$. But since $r(s+k\,\alpha, t+k\,\alpha) = r(s,t)$ and (31) holds uniformly on the two point (hence compact) set $K=\{(s,t),(s+k\,\alpha,t+k\,\alpha)\}$ we get from (31)

$$\begin{aligned} 0 &< \epsilon \\ &< |r_n(s+k\,\alpha,t+k\,\alpha) - r_n(s,t)| \\ &\le |r_n(s+k\,\alpha,t+k\,\alpha) - r(s,t)| + |r(s+k\,\alpha,t+k\,\alpha) - r(s,t)| + |r(s,t) - r_n(s,t)| \\ &\to 0 \end{aligned} \tag{42}$$

for large enough n, giving a contradiction. Hence we can take X_t^n to be also periodically correlated in addition.

Let F_n and F be the spectral measures of r_n and r. We assert that $F_n(A,B)\to F(A,B)$ for any Borel sets A,B. Since there is no Helly-Bray type result, we cannot use the representation (3) directly to obtain this result. The following alternative argument can be given in its place. Indeed, the X_t^n-processes can be obtained, to satisfy (31), as

$$X_t^n = \sum_{k=1}^{n} \varphi_k\,(X_t,\varphi_k) \forall n\ge 1 \tag{43}$$

where $\{\varphi_k, k\ge 1\}$ is a complete orthonormal system of the separable space $\mathcal{H}(X)$ of the X_t-process. Then $X_t^n\in\mathcal{H}(X)$ and $X_t^n\to X_t$ in $L_0^2(P)$-mean, uniformly for t in compact sets of $\mathbb{R}$. Let $\ell_k\colon Y\to(Y,\varphi_k), Y\in\mathcal{H}(X)$, be the linear functional on $\mathcal{H}(X)$. If ζ_n and Z are the representing stochastic measures of X_t^n- and X_t-processes, then (cf. [11], p. 319) (33) gives

$$\zeta_n(\cdot) = \sum_{k=1}^{n} \varphi_k\,\ell_k(Z(\cdot)), \quad \text{and } F_n(A,B) = (\zeta_n(A),\zeta_n(B)) \tag{44}$$

In particular, since $\ell_k(Z(A)) = (Z(A),\varphi_k)$, it follows from the Parseval's equation that

$$Z(A) = \lim_{n\to\infty} \zeta(A), \quad \text{in mean} \tag{45}$$

This immediately gives

$$F(A,B) = (Z(A), Z(B)) = \lim_{n\to\infty} (\zeta_n(A), \zeta_n(B)) = \lim_{n\to\infty} F_n(A,B) \tag{46}$$

Conversely, if the support of F is in S_F, since (35) always holds irrespective of periodicity, then for all large enough n, the support of F_n, S_{F_n}, satisfies $S_{F_n} \subset S_F$. But then by the converse part of the result of [6], X_t^n is periodically correlated and strongly harmonizable. Since (31) holds, it follows that r is periodic so that X_t-process is also periodically correlated, as asserted. ■

Acknowledgement

This research was supported, in part, by the ONR contract No. N00014-84-K-0356 (mod P00004).

References

1. S. Cambanis and B. Liu, "On harmonizable stochastic processes," *Information and Control* 17(1970), 183–202.

2. D. K. Chang and M. M. Rao, "Bimeasures and nonstationary processes," in: *Real and Stochastic Analysis*, Wiley, New York, 1986, 7–118.

3. E. G. Gladyshev, "Periodically correlated random sequences," *Soviet Math.* 2(1962), 385–388.

4. E. G. Gladyshev, "Periodically and almost-periodically correlated random processes with continuous time parameter," *Theor. Prob. and Appl.* 8(1963), 173–177.

5. U. Grenander, *Abstract Inference*, Wiley, New York, 1981.

6. H. L. Hurd, "Representation of harmonizable periodically correlated processes and their covariances," *Technical Report #182*, Center for Stochastic Processes, Univ. of North Carolina, Chapel Hill, 1987.

7. K. Karhunen, "Über lineare Methoden in der Wahrscheinlichkeitsrechnung," *Ann. Acad. Sci. Fenn. Ser AI Math.* 37(1947), 3–79.

8. J. P. Kelsh, *Linear Analysis of Harmonizable Time Series*, Ph.D thesis, Univ. of Calif., Riverside, 1978.