

# The Operational Matrix of The Random Wave Process

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## Introduction

Let  $I$  be defined as the inverse Fourier transform

$$I_{m,n}(y) = \int_{-1}^1 {}_2F_1\left(-m, m+1; 1; \frac{1}{2} - \frac{x}{2}\right) {}_2F_1\left(-n, n+1; 1; \frac{1}{2} - \frac{x}{2}\right) e^{ixy} dx \quad (1)$$

where  ${}_2F_1(a, b; c; z)$  is the Gauss hypergeometric function, and  $m, n$  are non-negative integers.

## Hypergeometric Series Expansion

**Lemma 1.** *For any non-negative integer  $p$  and complex numbers  $b, c$  with  $c \notin \{0, -1, -2, \dots\}$ :*

$${}_2F_1(-p, b; c; z) = \sum_{k=0}^p \frac{(-p)_k (b)_k}{(c)_k k!} z^k \quad (2)$$

where  $(a)_k = a(a+1) \cdots (a+k-1)$  is the Pochhammer symbol.

**Proof.** By definition,  ${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k$ . When  $a = -p$  for non-negative integer  $p$ :

$$(-p)_k = (-p)(-p+1) \cdots (-p+k-1) = (-1)^k p(p-1) \cdots (p-k+1) \quad (3)$$

Therefore  $(-p)_k = 0$  for all  $k > p$  since one of the factors becomes zero. Thus the infinite series terminates at  $k = p$ .  $\square$

**Lemma 2.** *For the given integral, applying the series expansion:*

$$I_{m,n}(y) = \sum_{k=0}^m \sum_{l=0}^n \frac{(-m)_k (m+1)_k}{k!} \frac{(-n)_l (n+1)_l}{l!} \int_{-1}^1 \left( \frac{1}{2} - \frac{x}{2} \right)^{k+l} e^{ixy} dx \quad (4)$$

**Proof.** Substituting the series expansions for both hypergeometric functions:

$$I_{m,n}(y) = \int_{-1}^1 \left( \sum_{k=0}^m \frac{(-m)_k (m+1)_k}{k!} \left( \frac{1}{2} - \frac{x}{2} \right)^k \right) \left( \sum_{l=0}^n \frac{(-n)_l (n+1)_l}{l!} \left( \frac{1}{2} - \frac{x}{2} \right)^l \right) e^{ixy} dx \quad (5)$$

The series are finite, so we can interchange summation and integration by Fubini's theorem.  $\square$

## Integral Evaluation

**Lemma 3.** *For non-negative integer  $s$ :*

$$\int_{-1}^1 \left( \frac{1}{2} - \frac{x}{2} \right)^s e^{ixy} dx = \frac{e^{iy}}{2^s} \left[ \frac{\Gamma(s+1, -2iy)}{(-iy)^{s+1}} - \frac{\Gamma(s+1)}{(-iy)^{s+1}} \right] \quad (6)$$

**Proof.** Make the substitution  $u = 1 - x$ . Then  $dx = -du$  and when  $x = -1$ ,  $u = 2$ ; when  $x = 1$ ,  $u = 0$ . Thus:

$$\begin{aligned} \int_{-1}^1 \left( \frac{1}{2} - \frac{x}{2} \right)^s e^{ixy} dx &= \frac{1}{2^s} \int_{-1}^1 (1-x)^s e^{ixy} dx \\ &= \frac{1}{2^s} \int_0^2 u^s e^{iy(1-u)} du \\ &= \frac{e^{iy}}{2^s} \int_0^2 u^s e^{-iuy} du \end{aligned} \quad (7)$$

Let  $v = iuy$ . Then  $du = \frac{dv}{iy}$  and:

$$\begin{aligned} \frac{e^{iy}}{2^s} \int_0^2 u^s e^{-iuy} du &= \frac{e^{iy}}{2^s} \frac{1}{(iy)^{s+1}} \int_0^{2iy} v^s e^{-v} dv \\ &= \frac{e^{iy}}{2^s} \frac{1}{(iy)^{s+1}} [\gamma(s+1, 2iy)] \end{aligned} \quad (8)$$

where  $\gamma(a, z) = \Gamma(a) - \Gamma(a, z)$  is the lower incomplete gamma function.  $\square$

## Double Sum Transformation

**Theorem 4.** *The double sum can be rewritten as:*

$$\sum_{l=0}^{m+n} \Phi_s(m, n) z^l = \sum_{k=0}^m \sum_{l=0}^n \frac{(-m)_k (m+1)_k}{k!} \frac{(-n)_l (n+1)_l}{l!} z^{k+l} \quad (9)$$

where

$$\Phi_s(m, n) = {}_3F_2(-m, m+1, -s; 1, 1-s-n; 1) \frac{(-n)_s (n+1)_s}{s!} \quad (10)$$

**Proof.** For fixed  $s = k + l$  sum over all  $k$  from 0 to  $\min(m, s)$  with  $l = s - k$  and factor out terms independent of  $k$

$$\begin{aligned} \sum_{k=0}^{\min(m, s)} \frac{(-m)_k (m+1)_k}{k!} \frac{(-n)_{s-k} (n+1)_{s-k}}{(s-k)!} &= \frac{(-n)_s (n+1)_s}{s!} \sum_{k=0}^{\min(m, s)} \frac{(-m)_k (m+1)_k (-s)_k}{k! (1)_k (1-s-n)_k} \\ &= {}_3F_2(-m, m+1, -s; 1, 1-s-n; 1) \end{aligned} \quad (11) \quad \square$$

## Final Result

**Theorem 5.** *The integral evaluates to:*

$$I_{m,n}(y) = e^{iy} \sum_{s=0}^{m+n} \frac{\Phi_s(m, n)}{2^s} \left[ \frac{\Gamma(s+1, -2iy) - \Gamma(s+1)}{(-iy)^{s+1}} \right] \quad (12)$$

where  $\Phi_s(m, n)$  is as defined above.

**Proof.** Combining the series expansion from Lemma 2, the integral evaluation from Lemma 3, and the sum transformation from Theorem 1:

$$\begin{aligned} I_{m,n}(y) &= \sum_{k=0}^m \sum_{l=0}^n \frac{(-m)_k (m+1)_k}{k!} \frac{(-n)_l (n+1)_l}{l!} \int_{-1}^1 \left( \frac{1-x}{2} \right)^{k+l} e^{ixy} dx \\ &= \sum_{s=0}^{m+n} \Phi_s(m, n) \cdot \frac{e^{iy}}{2^s} \left[ \frac{\Gamma(s+1, -2iy) - \Gamma(s+1)}{(-iy)^{s+1}} \right] \end{aligned} \quad (13)$$

$\square$