# Haken Manifolds: A Comprehensive Overview

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# 1 Haken Manifolds: A Comprehensive Overview

## 1.1 Definition and Basic Properties

A Haken manifold, named after mathematician Wolfgang Haken, is a fundamental concept in 3-manifold topology. To fully understand Haken manifolds, we need to break down several key terms:

1. **Compact space**: A topological space $X$ that is closed and bounded. Mathematically, for every open cover $\{U_\alpha\}$ of $X$, there exists a finite subcover. In the context of manifolds, this means the manifold has finite extent and includes its boundary.

2. **$\mathbf{P}^2$-irreducible manifold**: A 3-manifold $M$ is $\mathrm{P}^2$-irreducible if:

   - Every embedded 2-sphere $S^2 \subset M$ bounds a 3-ball $B^3 \subset M$, i.e., $S^2 = \partial B^3$.

   - $M$ does not contain any two-sided projective planes.

   This property ensures that the manifold doesn't have any "trivial" pieces that could be easily removed.

3. **Incompressible surface**: A properly embedded surface $S$ in a 3-manifold $M$ is incompressible if the induced homomorphism on fundamental groups is injective:

$$i_*\colon \pi_1(S) \hookrightarrow \pi_1(M)$$

where $i: S \hookrightarrow M$ is the inclusion map. Intuitively, this means that any closed curve on $S$ that can be contracted to a point in $M$ can also be contracted to a point within $S$ itself.

4. **Sufficiently large**: A 3-manifold is sufficiently large if it contains a properly embedded, two-sided, incompressible surface.

**Definition 1.** *[Haken Manifold] A Haken manifold is a compact, $P^2$-irreducible 3-manifold that is sufficiently large. In the orientable case, which is often the focus of study, a Haken manifold is a compact, orientable, irreducible 3-manifold containing an orientable, incompressible surface.*

## 1.2 Historical Context and Development

Wolfgang Haken introduced the concept of Haken manifolds in 1961. His work was part of a broader effort to understand and classify 3-manifolds, which had been a central problem in topology since the early 20th century.

Key developments in the theory of Haken manifolds include:

1. **Haken's Hierarchy (1962)**: Haken proved that Haken manifolds possess a hierarchy, where they can be decomposed into 3-balls along incompressible surfaces. This property is crucial for many proofs involving Haken manifolds.

2. **Waldhausen's Work (1968)**: Friedhelm Waldhausen proved several fundamental results about Haken manifolds, including their topological rigidity and the solvability of the word problem for their fundamental groups.

3. **Jaco-Oertel Algorithm (1984)**: William Jaco and Ulrich Oertel developed an algorithm to determine if a given 3-manifold is Haken.

4. **Thurston's Geometrization (1982)**: William Thurston's geometrization theorem for Haken manifolds was a crucial step in his broader geometrization program, which revolutionized our understanding of 3-manifolds.

5. **Virtually Haken Conjecture (Proved 2012)**: Ian Agol proved the virtually Haken conjecture, which states that every compact, irreducible 3-manifold with infinite fundamental group is virtually Haken (i.e., has a finite cover that is Haken).

## 1.3 Haken Hierarchy in Detail

The Haken hierarchy is a fundamental tool in the study of Haken manifolds. Here's a more detailed explanation of how it works:

1. Start with a Haken manifold $M$.

2. Find an incompressible surface $S \subset M$.

3. Cut $M$ along $S$ to obtain a new manifold $M' = M \setminus N(S)$, where $N(S)$ is a regular neighborhood of $S$.

4. $M'$ is again a Haken manifold (unless it's a collection of 3-balls).

5. Repeat the process with $M'$, finding another incompressible surface and cutting along it.

6. Continue this process until you're left with a collection of 3-balls.

Mathematically, we can express this as a sequence:

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_n$$

where each $M_i$ is obtained from $M_{i-1}$ by cutting along an incompressible surface, and $M_n$ is a disjoint union of 3-balls.

This hierarchy allows for inductive proofs on Haken manifolds. Many properties can be proven by:

- Showing they hold for 3-balls
- Proving that if they hold for the pieces after cutting along an incompressible surface, they hold for the original manifold

## 1.4 Applications and Significance

Haken manifolds have numerous important applications in 3-manifold topology:

1. **Homeomorphism Problem**: Haken's work led to an algorithm for determining whether two Haken manifolds are homeomorphic. Given Haken manifolds $M$ and $N$, there exists an algorithm to decide if $M \cong N$.

2. **Recognition Problem**: The Jaco-Oertel algorithm solves the recognition problem for Haken manifolds. Given a 3-manifold $M$, there exists an algorithm to decide if $M$ is Haken.

3. **Topological Rigidity**: Waldhausen's proof of topological rigidity for Haken manifolds shows that they are completely determined by their fundamental groups. Formally, if $f\colon M \to N$ is a homotopy equivalence between Haken manifolds, then $f$ is homotopic to a homeomorphism.

4. **Geometrization**: Thurston's geometrization theorem for Haken manifolds was a crucial step in the proof of the Poincaré conjecture and the geometrization conjecture. It states that every Haken 3-manifold can be decomposed into geometric pieces.

5. **Word Problem**: The solvability of the word problem for fundamental groups of Haken manifolds has implications in group theory and computational topology. For a Haken manifold $M$, there exists an algorithm to decide if a word $w \in \pi_1(M)$ represents the identity element.

## 1.5 Examples of Haken Manifolds

Let's explore some examples of Haken manifolds in more detail:

1. **Compact, irreducible 3-manifolds with positive first Betti number**:

   - The first Betti number $b_1(M) = \mathrm{rank} H_1(M; \mathbb{Z})$ is the rank of the first homology group.

   - A positive first Betti number implies the existence of a non-trivial map $f: M \to S^1$, which can be used to construct an incompressible surface.

2. **Surface bundles over the circle**:

   - These are 3-manifolds formed by taking a surface $S$ and an interval $I = [0, 1]$, then identifying $(x, 0)$ with $(f(x), 1)$ for some homeomorphism $f$ of $S$.

   - Mathematically, $M_f = (S \times I) / \sim$, where $(x, 0) \sim (f(x), 1)$.

   - The surface $S$ provides a natural incompressible surface in this construction.

3. **Link complements**:

   - The complement of a link $L$ in $S^3$ is often a Haken manifold.

   - Denoted as $S^3 \setminus N(L)$, where $N(L)$ is a tubular neighborhood of $L$.

   - Seifert surfaces for the link components often provide incompressible surfaces.

4. **Most Seifert fiber spaces**:

   - Seifert fiber spaces are 3-manifolds that admit a decomposition into circles in a particularly nice way.

   - Many Seifert fiber spaces contain incompressible tori, making them Haken.

5. **Handlebodies of genus $g > 0$**:

   - These are obtained by attaching $g$ 1-handles to a 3-ball.

- They contain incompressible surfaces (e.g., properly embedded disks).

## 1.6 Advanced Topics and Recent Developments

1. **Virtual Haken Conjecture**:

    - Proved by Ian Agol in 2012

    - States that every compact, irreducible 3-manifold $M$ with infinite fundamental group is virtually Haken, i.e., there exists a finite cover $\tilde{M} \to M$ such that $\tilde{M}$ is Haken.

    - The proof uses a combination of techniques from hyperbolic geometry, group theory, and 3-manifold topology

2. **Relationship to Hyperbolic Geometry**:

    - Many Haken manifolds admit hyperbolic structures, i.e., Riemannian metrics of constant sectional curvature $-1$.

    - Thurston's geometrization theorem for Haken manifolds was a key step in understanding this relationship

3. **Normal Surface Theory**:

    - Normal surfaces, introduced by Haken, are a key tool in algorithms involving Haken manifolds

    - They provide a finite way to describe essential surfaces in a 3-manifold

    - A normal surface intersects each tetrahedron in a triangulation in a finite number of prescribed triangle and quadrilateral types

4. **Mapping Class Groups**:

    - Johannson (1979) proved that atoroidal, anannular, boundary-irreducible Haken 3-manifolds have finite mapping class groups

    - For such a manifold $M$, $\mathrm{MCG}(M) = \mathrm{Homeo}(M)/\mathrm{Homeo}_0(M)$ is finite

    - This result ties into the rigidity properties of hyperbolic 3-manifolds

5. **Connections to Quantum Topology**:

    - Haken manifolds play a role in the study of quantum invariants of 3-manifolds

- The hierarchical structure of Haken manifolds can sometimes be used to compute or analyze these invariants
- For example, the Witten-Reshetikhin-Turaev invariants can often be computed recursively using the Haken hierarchy

## 1.7 Open Questions and Future Directions

While much is known about Haken manifolds, there are still open questions and areas of active research:

1. **Effective Algorithms**: Improving the efficiency of algorithms for recognizing and analyzing Haken manifolds. Can we find polynomial-time algorithms for problems currently solved in exponential time?

2. **Quantitative Aspects**: Understanding quantitative aspects of the Haken hierarchy, such as the number of steps needed to decompose a manifold. Is there a relationship between this number and other invariants of the manifold?

3. **Generalized Haken Manifolds**: Exploring generalizations of Haken manifolds to higher dimensions or different categories of manifolds. What would be the appropriate definition of a "Haken $n$-manifold" for $n > 3$?

4. **Connections to Other Areas**: Further investigating the relationships between Haken manifolds and other areas of mathematics, such as geometric group theory and low-dimensional dynamics. Can techniques from Haken manifolds be applied to problems in these areas?

5. **Computational Topology**: Developing practical software tools based on the theory of Haken manifolds for studying 3-manifolds computationally. Can we create efficient implementations of algorithms for normal surface theory and the Haken hierarchy?

The study of Haken manifolds continues to be a rich and active area of research in topology, with connections to diverse areas of mathematics and potential applications in theoretical physics and computer science.