

Metric Entropy and Compactness Properties of Gaussian Processes

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Definition 1

For a Gaussian process $\{X(t): t \in T\}$, the **canonical metric** d is defined as:

$$d(s, t) := \sqrt{\mathbb{E}[(X(s) - X(t))^2]} \quad (1)$$

This metric reflects the expected distance between the process values at any two points $s, t \in T$.

Definition 2

The **spectral radius** R of the covariance operator K associated with a Gaussian process is defined as the square of the largest eigenvalue λ_1 of K :

$$R := \lambda_1^2 \quad (2)$$

The spectral radius indicates the maximum variance contributed by the process in the direction of the first eigenfunction.

Definition 3

The **covering number** $N(T, d, \varepsilon)$ is the minimum number of points needed to cover the space T within distance ε using the canonical metric d . An upper bound for the covering number is given by:

$$N(T, d, \varepsilon) \leq \min \{n \in \mathbb{N}: \lambda_{n+1}^2 \leq \varepsilon\} \quad (3)$$

where $\{\lambda_k\}$ are the eigenvalues of the covariance operator, ordered in decreasing order.

Definition 4

The **metric entropy** is the logarithm of the covering number:

$$\log N(T, d, \varepsilon) \quad (4)$$

which measures the complexity of the set T in the canonical metric d at scale $\varepsilon > 0$.

Definition 5

The *metric entropy integral* is defined as:

$$\int_0^R \log N(T, d, \varepsilon) d\varepsilon \quad (5)$$

where $R := \lambda_1^2$ is the spectral radius. This integral quantifies the total complexity of covering the metric space (T, d) as ε varies from R to 0.

Theorem

Theorem 6

Let $\{X(t): t \in T\}$ be a Gaussian process with covariance operator K having eigenvalues $\{\lambda_k\}$. If the eigenvalues satisfy $\lambda_k \rightarrow 0$ as $k \rightarrow \infty$, then the metric entropy integral

$$\int_0^R \log N(T, d, \varepsilon) d\varepsilon < \infty \quad (6)$$

is finite, indicating that the space is relatively compact in the canonical metric d .

Proof

Given $\lambda_k \rightarrow 0$, for any $\varepsilon > 0$, there exists a finite set of indices such that $\lambda_k^2 > \varepsilon$. Thus, the covering number $N(T, d, \varepsilon)$ is finite for any $\varepsilon > 0$.

The metric entropy integral is:

$$\begin{aligned} \int_0^R \log N(T, d, \varepsilon) d\varepsilon &= \sum_{k=1}^{N(T, d, R)} \log(k) (\lambda_k^2 - \lambda_{k+1}^2) \\ &= \sum_{\lambda_{k+1}^2 \leq \varepsilon} \log(k) (\lambda_k^2 - \lambda_{k+1}^2) \\ &= \sum_{\lambda_{k+1} \leq \sqrt{\varepsilon}} \log(k) (\lambda_k^2 - \lambda_{k+1}^2) \end{aligned} \quad (7)$$

Both $\log k$ and $(\lambda_k^2 - \lambda_{k+1}^2)$ are finite, thus the sum is finite, ensuring that the metric entropy integral is finite, implying relative compactness.