

Unitary Time Changes of Stationary Processes Yield Oscillatory Processes

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ , acting on functions that are square-integrable over compact sets. Applying U_θ to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)}$ and evolutionary spectrum $dF_t(\lambda) = \dot{\theta}(t)dF(\lambda)$. It is proved that sample paths of any non-degenerate second-order stationary process almost surely lie in $L^2_{\text{loc}}(\mathbb{R})$, making the operator applicable to typical realizations. A zero-localization measure $d\mu(t) = \delta(Z(t)) |\dot{Z}(t)| dt$ induces a Hilbert space $L^2(\mu)$ on the zero set of an oscillatory process Z , and the multiplication operator $(Lf)(t) = t f(t)$ has simple pure point spectrum equal to the zero crossing set of Z . This produces a concrete operator scaffold consistent with a Hilbert–Pólya-type viewpoint.

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1 Function Spaces

1.1 σ -compact sets and locally square-integrable functions

Definition 1. [σ -compact sets] A subset $U \subseteq \mathbb{R}$ is σ -compact if

$$U = \bigcup_{n=1}^{\infty} K_n \quad (1)$$

with each K_n compact.

Definition 2. [Locally square-integrable functions] Define

$$L^2_{\text{loc}}(\mathbb{R}) := \left\{ f: \mathbb{R} \rightarrow \mathbb{C}: \int_K |f(t)|^2 dt < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \quad (2)$$

Remark 3. Every bounded measurable set in \mathbb{R} is compact or contained in a compact set; hence $L^2_{\text{loc}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

2 Gaussian Processes

A Gaussian process is a ...

2.1 Stationary processes

Definition 4. [Cramér representation] A zero-mean stationary process X with spectral measure F admits the sample path representation

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t} \Phi(d\lambda) \quad (3)$$

which has covariance

$$R_X(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)} dF(\lambda) \quad (4)$$

2.2 Oscillatory Processes

A particularly tractable class of non-stationary Gaussian processes is that of the oscillatory processes as defined by M.B. Priestley in 1965[1].

Definition 5. [Oscillatory process] *link to Priestley 1965* Let F be a finite nonnegative Borel measure on \mathbb{R} . Let

$$A_t \in L^2(F) \forall t \in \mathbb{R} \quad (5)$$

be the gain function and

$$\varphi_t(\lambda) = A_t(\lambda) e^{i\lambda t} \quad (6)$$

be the corresponding oscillatory function then an oscillatory process is a stochastic process which can be represented as

$$\begin{aligned} Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \end{aligned} \quad (7)$$

where Φ is a complex orthogonal random measure with spectral measure F which satisfies the relation

$$d\mathbb{E}[\Phi(\lambda)\overline{\Phi(\mu)}] = \delta(\lambda - \mu) dF(\lambda) \quad (8)$$

and has the corresponding covariance kernel

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \int_{\mathbb{R}} A_t(\lambda) \overline{A_s(\lambda)} e^{i\lambda(t-s)} dF(\lambda) \\ &= \int_{\mathbb{R}} \phi_t(\lambda) \overline{\phi_s(\lambda)} dF(\lambda) \end{aligned} \quad (9)$$

Theorem 6. [Real-valuedness criterion for oscillatory processes] Let Z be an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = A_t(\lambda) e^{i\lambda t} \quad (10)$$

and spectral measure F . Then Z is real-valued if and only if

$$A_t(-\lambda) = \overline{A_t(\lambda)} \quad (11)$$

for F -almost every $\lambda \in \mathbb{R}$, equivalently

$$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \quad (12)$$

for F -almost every $\lambda \in \mathbb{R}$.

Proof. Assume Z is real-valued, i.e.

$$Z(t) = \overline{Z(t)} \quad \forall t \in \mathbb{R} \quad (13)$$

Writing its oscillatory representation,

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \quad (14)$$

and taking the complex conjugate gives

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t} d\overline{\Phi}(\lambda) \quad (15)$$

For a real-valued process, the orthogonal random measure Φ must satisfy

$$d\overline{\Phi}(\lambda) = -d\Phi(\lambda) \quad (16)$$

which ensures that the spectral representation produces real values. Substituting this identity and using the substitution

$$\mu = -\lambda \quad (17)$$

it is shown that

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t} d\Phi(\mu) \quad (18)$$

Since $Z(t) = \overline{Z(t)}$, comparison of the integrands (which are unique elements of $L^2(F)$) yields

$$A_t(\lambda) = \overline{A_t(-\lambda)} \quad \text{for } F\text{-a.e. } \lambda \quad (19)$$

Equivalently, because the oscillatory function (6) is given by

$$\varphi_t(\lambda) = A_t(\lambda) e^{i\lambda t} \quad (20)$$

we have

$$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \quad \text{for } F\text{-a.e. } \lambda \quad (21)$$

Conversely, if

$$A_t(-\lambda) = \overline{A_t(\lambda)} \quad (22)$$

for F -a.e. λ , then the same substitution shows that

$$\overline{Z(t)} = Z(t) \quad \forall t \in \mathbb{R} \quad (23)$$

so Z is real-valued. This completes the proof. \square

Theorem 7. [Existence] *Let F be an absolutely continuous spectral measure and the gain function*

$$A_t(\lambda) \in L^2(F) \forall \mathbb{R} \ni t < \infty \quad (24)$$

be measurable in both time and frequency then the time-dependent spectral density is defined by

$$\begin{aligned} S_t(\lambda) &= \int_{\mathbb{R}} |A_t(\lambda)|^2 dF(\lambda) < \infty \\ &= \int_{\mathbb{R}} |A_t(\lambda)|^2 S(\lambda) d\lambda \end{aligned} \quad (25)$$

and there exists a complex orthogonal random measure Φ with spectral measure F such that for each sample path $\varpi \in \Theta$ in the space of sample paths having given covariance constituting the ensemble denoted Θ

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \quad (26)$$

is well-defined in $L^2(\Omega)$ and has covariance R_Z as in (9) above.

Proof. The proof proceeds by constructing the stochastic integral using the standard extension procedure. First, the integral is defined for simple functions of the form

$$g(\lambda) = \lim_{n \rightarrow \infty} \sum_{j=1}^n c_j \mathbf{1}_{E_j}(\lambda) \quad (27)$$

where $\{E_j\}$ are disjoint Borel sets with $F(E_j) < \infty$ and $c_j \in \mathbb{C}$:

$$\int_{\mathbb{R}} g(\lambda) d\Phi(\lambda) = \lim_{n \rightarrow \infty} \sum_{j=1}^n c_j \Phi(E_j) \quad (28)$$

For simple functions such as this, the isometry property holds:

$$\begin{aligned} \mathbb{E} \left[\left| \int_{\mathbb{R}} g(\lambda) d\Phi(\lambda) \right|^2 \right] &= \mathbb{E} \left[\lim_{n \rightarrow \infty} \left| \sum_{j=1}^n c_j \Phi(E_j) \right|^2 \right] \\ &= \lim_{n \rightarrow \infty} \sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \mathbb{E} [\Phi(E_j) \overline{\Phi(E_k)}] \\ &= \lim_{n \rightarrow \infty} \sum_{j=1}^n |c_j|^2 F(E_j) \\ &= \int_{\mathbb{R}} |g(\lambda)|^2 dF(\lambda) \end{aligned} \quad (29)$$

Since simple functions are dense in $L^2(F)$, the integral is extended by continuity $\forall g \in L^2(F)$ since the oscillatory function (6) is defined by

$$\varphi_t(\lambda) = A_t(\lambda) e^{i\lambda t} \in L^2(F) \forall t \in \mathbb{R} \quad (30)$$

and $A_t \in \cdot$. Therefore

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \quad (31)$$

is well-defined in $L^2(\Omega)$. The covariance is computed as:

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t) \overline{Z(s)}] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \int_{\mathbb{R}} \overline{\varphi_s(\mu)} d\overline{\Phi(\mu)}\right] \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\mu)} d\mathbb{E}[\Phi(\lambda) \overline{\Phi(\mu)}] \\ &= \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\lambda)} dF(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda) \overline{A_s(\lambda)} e^{i\lambda(t-s)} dF(\lambda) \end{aligned} \quad (32) \quad \square$$

3 Unitarily Time-Changed Stationary Processes

3.1 Unitary time-change operator $U_\theta f$

Definition 8. [Unitary time-change] Let the time-scaling function $\theta: \mathbb{R} \rightarrow \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with $\dot{\theta}(t) > 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero. The function θ maps σ -compact sets to σ -compact sets. Define, for f measurable,

$$(U_\theta f)(t) = \sqrt{\dot{\theta}(t)} f(\theta(t)) \quad (33)$$

Proposition 9. [Inversion of Unitary time-change] The inverse of the unitary time-change operator U in Equation (33) is given by

$$(U_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \quad (34)$$

which is well-defined almost everywhere on every σ -compact set.

Proof. Since $\dot{\theta}(t) = 0$ only on sets of measure zero, and θ^{-1} maps sets of measure zero to sets of measure zero because of the fact that absolutely continuous bijective functions preserve measure-zero sets, the denominator $\sqrt{\dot{\theta}(\theta^{-1}(s))}$ is positive almost everywhere. The expression is therefore well-defined almost everywhere on every σ -compact set, which suffices for defining an element of $L^2_{\text{loc}}(\mathbb{R})$. \square

Theorem 10. [Local unitarity on σ -compact sets] For every σ -compact set $C \subseteq \mathbb{R}$ and $f \in L^2_{\text{loc}}(\mathbb{R})$,

$$\int_C |(U_\theta f)(t)|^2 dt = \int_{\theta(C)} |f(s)|^2 ds \quad (35)$$

Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\text{loc}}(\mathbb{R})$.

Proof. Let $f \in L^2_{\text{loc}}(\mathbb{R})$ and let C be any σ -compact set. The local L^2 -norm of $U_\theta f$ over C is:

$$\begin{aligned} \int_C |(U_\theta f)(t)|^2 dt &= \int_C \left| \sqrt{\dot{\theta}(t)} f(\theta(t)) \right|^2 dt \\ &= \int_C \dot{\theta}(t) |f(\theta(t))|^2 dt \end{aligned} \quad (36)$$

Since θ is absolutely continuous and strictly increasing, applying the change of variables $s = \theta(t)$ gives

$$ds = \dot{\theta}(t) dt \quad (37)$$

almost everywhere. Since θ maps σ -compact sets to σ -compact sets, as t ranges over C , $s = \theta(t)$ ranges over $\theta(C)$, which is σ -compact. Therefore:

$$\int_C \dot{\theta}(t) |f(\theta(t))|^2 dt = \int_{\theta(C)} |f(s)|^2 ds \quad (38)$$

To verify that U_θ^{-1} is indeed the inverse, it is seen that:

$$\begin{aligned} (U_\theta^{-1} U_\theta f)(s) &= \left(U_\theta^{-1} \sqrt{\dot{\theta}(s)} f(\theta(s)) \right)(s) \\ &= \frac{\sqrt{\dot{\theta}(\theta^{-1}(s))}}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} f(\theta(\theta^{-1}(s))) \quad \forall f \in L^2_{\text{loc}}(\mathbb{R}) \\ &= f(s) \end{aligned} \quad (39)$$

since

$$\theta(\theta^{-1}(s)) = s \quad (40)$$

and similarly, its also plain to see that:

$$\begin{aligned}
 (U_\theta U_\theta^{-1} g)(t) &= \sqrt{\dot{\theta}(t)} (U_\theta^{-1} g)(\theta(t)) \\
 &= \frac{\sqrt{\dot{\theta}(t)}}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} g(\theta^{-1}(\theta(t))) \quad \forall g \in L^2_{\text{loc}}(\mathbb{R}) \\
 &= \frac{\sqrt{\dot{\theta}(t)}}{\sqrt{\dot{\theta}(t)}} g(t) \\
 &= g(t)
 \end{aligned} \tag{41}$$

since

$$\theta^{-1}(\theta(t)) = t \tag{42}$$

Therefore

$$\begin{aligned}
 (U_\theta U_\theta^{-1} f)(t) &= (U_\theta^{-1} U_\theta f)(t) \\
 &= f(t)
 \end{aligned} \tag{43}$$

on $L^2_{\text{loc}}(\mathbb{R})$. □

3.1.1

Theorem 11. *[Inverse Filter for Unitary Time Transformations] Let $\theta: \mathbb{R} \rightarrow \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\theta'(t) > 0$ almost everywhere. Let $Y(u)$ be a stationary process with unit variance, and define*

$$Z(t) = \sqrt{\dot{\theta}(t)} Y(\theta(t)) \tag{44}$$

as the oscillatory process obtained by the unitary time transformation. Then:

1. *The forward filter kernel is*

$$h(t, u) = \sqrt{\dot{\theta}(t)} \delta(u - \theta(t)) \tag{45}$$

2. *The inverse filter kernel is*

$$g(t, s) = \frac{\delta(s - \theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \tag{46}$$

3. *The composition $(g \circ h)$ recovers the identity:*

$$\begin{aligned}
 Y(t) &= \int_{\mathbb{R}} g(t, s) Z(s) ds \\
 &= \frac{Z(\theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}}
 \end{aligned} \tag{47}$$

Proof. We establish parts 1–3 in sequence.

Part 1. From the definition of $Z(t)$,

$$Z(t) = \sqrt{\dot{\theta}(t)} Y(\theta(t)) = \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} \delta(u - \theta(t)) Y(u) du \quad (48)$$

so

$$h(t, u) = \sqrt{\dot{\theta}(t)} \delta(u - \theta(t)) \quad (49)$$

Part 2. Define

$$g(t, s) = \frac{\delta(s - \theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \quad (50)$$

To verify it is the inverse filter, compute:

$$\int_{\mathbb{R}} g(t, s) Z(s) ds = \int_{\mathbb{R}} \frac{\delta(s - \theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \sqrt{\dot{\theta}(s)} Y(\theta(s)) ds \quad (51)$$

Let $v = \theta(s)$, so $s = \theta^{-1}(v)$ and $ds = \frac{dv}{\dot{\theta}(\theta^{-1}(v))}$. Then

$$= \int_{\mathbb{R}} \frac{\delta(\theta^{-1}(v) - \theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \frac{\sqrt{\dot{\theta}(\theta^{-1}(v))} Y(v)}{\dot{\theta}(\theta^{-1}(v))} dv \quad (52)$$

Simplify:

$$= \int_{\mathbb{R}} \frac{\delta(\theta^{-1}(v) - \theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))} \sqrt{\dot{\theta}(\theta^{-1}(v))}} Y(v) dv \quad (53)$$

Using the identity

$$\delta(\theta^{-1}(v) - \theta^{-1}(t)) = \dot{\theta}(\theta^{-1}(t)) \delta(v - t) \quad (54)$$

we obtain

$$= \int_{\mathbb{R}} \frac{\dot{\theta}(\theta^{-1}(t)) \delta(v - t)}{\sqrt{\dot{\theta}(\theta^{-1}(t))} \sqrt{\dot{\theta}(\theta^{-1}(v))}} Y(v) dv \quad (55)$$

Evaluating at $v = t$,

$$= Y(t) \quad (56)$$

Thus

$$g(t, s) = \frac{\delta(s - \theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \quad (57)$$

is the correct inverse kernel.

Part 3. Substituting Z explicitly yields

$$\begin{aligned}
 Y(t) &= \int_{\mathbb{R}} g(t, s) Z(s) \, ds \\
 &= \int_{\mathbb{R}} \frac{\delta(s - \theta^{-1}(t))}{\sqrt{\theta'(\theta^{-1}(t))}} Z(s) \, ds \\
 &= \frac{Z(\theta^{-1}(t))}{\sqrt{\theta'(\theta^{-1}(t))}}
 \end{aligned} \tag{58}$$

Hence $(g \circ h)(t, u)$ acts as the identity transformation in $L^2(\mathbb{R})$. \square

3.2 Transformation of Stationary \rightarrow Oscillatory Processes via U_θ

Theorem 12. *[Unitary time change yields oscillatory process] Let X be zero-mean stationary as in Definition 4. For scaling function θ as in Definition 8, define*

$$\begin{aligned}
 Z(t) &= (U_\theta X)(t) \\
 &= \sqrt{\dot{\theta}(t)} X(\theta(t))
 \end{aligned} \tag{59}$$

Then Z is a realization of an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \tag{60}$$

gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \tag{61}$$

and covariance

$$\begin{aligned}
 R_Z(t, s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\
 &= \mathbb{E}\left[\sqrt{\dot{\theta}(t)} X(\theta(t)) \sqrt{\dot{\theta}(s)} \overline{X(\theta(s))}\right] \\
 &= \sqrt{\dot{\theta}(t)\dot{\theta}(s)} \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \\
 &= \sqrt{\dot{\theta}(t)\dot{\theta}(s)} R_X(\theta(t) - \theta(s)) \\
 &= \sqrt{\dot{\theta}(t)\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} dF(\lambda)
 \end{aligned} \tag{62}$$

Proof. Applying the unitary time change operator to the spectral representation of $X(t)$:

$$\begin{aligned}
 Z(t) &= (U_\theta X)(t) \\
 &= \sqrt{\dot{\theta}(t)} X(\theta(t)) \\
 &= \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)} d\Phi(\lambda) \\
 &= \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} d\Phi(\lambda) \\
 &= \int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda)
 \end{aligned} \tag{63}$$

where

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \quad (64)$$

To verify this constitutes an oscillatory representation according to Definition 5, $\varphi_t(\lambda)$ has the form $A_t(\lambda) e^{i\lambda t}$:

$$\begin{aligned} \varphi_t(\lambda) &= \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \\ &= \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} e^{i\lambda t} \\ &= A_t(\lambda) e^{i\lambda t} \end{aligned} \quad (65)$$

where

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \quad (66)$$

Since $\dot{\theta}(t) \geq 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of measure zero, $A_t(\lambda)$ is well defined almost everywhere. Moreover, $A_t \in L^2(F)$ for each t since:

$$\begin{aligned} \int_{\mathbb{R}} |A_t(\lambda)|^2 dF(\lambda) &= \int_{\mathbb{R}} \left| \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \right|^2 dF(\lambda) \\ &= \int_{\mathbb{R}} \dot{\theta}(t) |e^{i\lambda(\theta(t)-t)}|^2 dF(\lambda) \\ &= \dot{\theta}(t) \int_{\mathbb{R}} dF(\lambda) \\ &= \dot{\theta}(t) F(\mathbb{R}) < \infty \end{aligned} \quad (67)$$

where $|e^{i\alpha}| = 1$ for all real α is used. The covariance (62) is computed by substituting the spectral representation and applying Fubini's theorem to interchange the order of operations.

(68) \square

Corollary 13. *[Evolutionary spectrum of unitarily time-changed stationary process][1] Link to The evolutionary spectrum, also called the time-varying spectral density, is*

$$\begin{aligned} dF_t(\lambda) &= |A_t(\lambda)|^2 dF(\lambda) \\ &= \dot{\theta}(t) dF(\lambda) \end{aligned} \quad (69)$$

Proof. By definition of the evolutionary spectrum and using the gain function from Theorem 12:

$$\begin{aligned} dF_t(\lambda) &= |A_t(\lambda)|^2 dF(\lambda) \\ &= \left| \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \right|^2 dF(\lambda) \\ &= \dot{\theta}(t) |e^{i\lambda(\theta(t)-t)}|^2 dF(\lambda) \\ &= \dot{\theta}(t) dF(\lambda) \end{aligned} \quad (70)$$

since

$$|e^{i\alpha}| = 1 \forall \alpha \in \mathbb{R} \quad (71) \quad \square$$

3.3 Covariance operator conjugation

Proposition 14. *[Operator conjugation] Let*

$$(T_K f)(t) := \int_{\mathbb{R}} K(|t-s|) f(s) ds \quad (72)$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h} dF(\lambda) \quad (73)$$

Define the transformed kernel

$$K_{\theta}(s, t) := \sqrt{\dot{\theta}(t) \dot{\theta}(s)} K(|\theta(t) - \theta(s)|) \quad (74)$$

then the corresponding integral covariance operator is conjugated $\forall f \in L^2_{\text{loc}}(\mathbb{R})$ by

$$\begin{aligned} (T_{K_{\theta}} f)(t) &= \int_{\mathbb{R}} K_{\theta}(s, t) f(s) ds \\ &= (U_{\theta} T_K U_{\theta}^{-1} f)(t) \end{aligned} \quad (75)$$

Proof. For any $g \in L^2_{\text{loc}}(\mathbb{R})$, compute:

$$\begin{aligned} ((U_{\theta} T_K U_{\theta}^{-1}) g)(t) &= (U_{\theta} (T_K U_{\theta}^{-1} g))(t) \\ &= \sqrt{\dot{\theta}(t)} (T_K U_{\theta}^{-1} g)(\theta(t)) \\ &= \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|) (U_{\theta}^{-1} g)(\theta(s)) \dot{\theta}(s) ds \\ &= \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|) \frac{g(s)}{\sqrt{\dot{\theta}(s)}} \dot{\theta}(s) ds \\ &= \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|) g(s) \sqrt{\dot{\theta}(s)} ds \\ &= \int_{\mathbb{R}} \sqrt{\dot{\theta}(t) \dot{\theta}(s)} K(|\theta(t) - \theta(s)|) g(s) ds \\ &= \int_{\mathbb{R}} K_{\theta}(t, s) g(s) ds \\ &= (T_{K_{\theta}} g)(t) \end{aligned} \quad (76) \quad \square$$

4 The Ensemble of Sample Path Realizations

Question: is this called local integrability? state this more eloquently

Theorem 15. *[Sample paths in $L^2_{\text{loc}}(\mathbb{R})$] Let $\{X(t)\}_{t \in \mathbb{R}}$ be a second-order stationary process with*

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \quad (77)$$

then, almost surely, every sample path $t \mapsto X(\omega, t) \in L^2_{\text{loc}}(\mathbb{R})$.

Proof. Fix any bounded interval $[a, b]$ and consider the random variable

$$Y_{[a,b]} := \int_a^b X(t)^2 dt \quad (78)$$

By stationarity and Fubini's theorem:

$$\begin{aligned} \mathbb{E}[Y_{[a,b]}] &= \mathbb{E}\left[\int_a^b X(t)^2 dt\right] = \int_a^b \mathbb{E}[X(t)^2] dt \\ &= \int_a^b \sigma^2 dt \\ &= \sigma^2(b-a) < \infty \end{aligned} \quad (79)$$

By Markov's inequality, for any $M > 0$:

$$P(Y_{[a,b]} > M) \leq \frac{\mathbb{E}[Y_{[a,b]}]}{M} = \frac{\sigma^2(b-a)}{M} \quad (80)$$

Taking $M \rightarrow \infty$, the conclusion is

$$P(Y_{[a,b]} < \infty) = 1 \quad (81)$$

i.e., almost surely the sample path is square-integrable on $[a, b]$. Since \mathbb{R} is the countable union of bounded intervals:

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n] \quad (82)$$

by countable subadditivity of probability:

$$P\left(\bigcap_{n=1}^{\infty} \left\{\int_{-n}^n X(t)^2 dt < \infty\right\}\right) = 1 \quad (83)$$

Now let K be any compact set. Then K is bounded, so

$$K \subseteq [-N, N] \quad (84)$$

for some N . Therefore:

$$\int_K X(t)^2 dt \leq \int_{-N}^N X(t)^2 dt < \infty \quad (85)$$

almost surely. This holds for every compact set K , so almost surely every sample path lies in $L^2_{\text{loc}}(\mathbb{R})$. \square

5 Zero Localization

The construction

$$\text{stationary } X \xrightarrow{U_\theta} \text{oscillatory } Z \xrightarrow{\mu=\delta(Z)|\dot{Z}| dt} L^2(\mu) \xrightarrow{L:tf(t)} (L, \sigma(L)) \quad (86)$$

produces a self-adjoint operator whose eigenvalues equal the zero set of the realization sample path realization $Z(t)$ from the ensemble of possible sample path functions having the given covariance structure and whose spectrum equals the closure of the zero set, determined by the choice of time-change $\theta(t)$, spectral measure $F(\lambda)$, and complex orthogonal random measure $\Phi(\lambda)$ which uniquely corresponds to a given sample path from the ensemble.

5.1 Zero localization measure

Definition 16. [Zero localization measure] Let Z be real-valued with $Z \in C^1(\mathbb{R})$ having only simple zeros

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \quad (87)$$

Define, for Borel $B \subset \mathbb{R}$,

$$\mu(B) = \int_{\mathbb{R}} \mathbf{1}_B(t) \delta(Z(t)) |\dot{Z}(t)| dt \quad (88)$$

Theorem 17. [Atomicity on the zero set] For every $\phi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \phi(t) \delta(Z(t)) |\dot{Z}(t)| dt = \sum_{t_0: Z(t_0)=0} \phi(t_0) \quad (89)$$

hence

$$\mu(t) = \sum_{t_0: Z(t_0)=0} \delta_{t_0}(t) \quad (90)$$

Proof. Since all zeros of Z are simple and $Z \in C^1(\mathbb{R})$, by the inverse function theorem each zero t_0 is isolated. Near each zero t_0 , Z is locally monotonic, so the one-dimensional change of variables formula for the Dirac delta can be applied. Specifically, near t_0 where $Z(t_0) = 0$ and $\dot{Z}(t_0) \neq 0$, locally

$$Z(t) = (t - t_0) \dot{Z}(t_0) + O((t - t_0)^2) \quad (91)$$

holds. The distributional identity for the Dirac delta under smooth changes of variables gives:

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \quad (92)$$

Therefore:

$$\begin{aligned} \int_{\mathbb{R}} \phi(t) \delta(Z(t)) |\dot{Z}(t)| dt &= \int_{-\infty}^{\infty} \phi(t) |\dot{Z}(t)| \sum_{t_0: Z(t_0)=0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} dt \\ &= \sum_{t_0: Z(t_0)=0} \int_{\mathbb{R}} \phi(t) \frac{|\dot{Z}(t)| \delta(t - t_0)}{|\dot{Z}(t_0)|} dt \\ &= \sum_{t_0: Z(t_0)=0} \frac{|\dot{Z}(t_0)|}{|\dot{Z}(t_0)|} \phi(t_0) \\ &= \sum_{t_0: Z(t_0)=0} \phi(t_0) \end{aligned} \quad (93)$$

This shows that μ is the discrete measure

$$\mu(t) = \sum_{t_0: Z(t_0)=0} \delta_{t_0}(t) \quad (94)$$

assigning unit mass to each zero. \square

5.2 Hilbert space on zeros and multiplication operator

Definition 18. [*Hilbert space on the zero set*] Let $\mathcal{H} = L^2(\mu)$ with inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} d\mu(t) \quad (95)$$

Proposition 19. [*Atomic structure*] Let

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (96)$$

then

$$\mathcal{H} \cong \left\{ f: \{t_0: Z(t_0)=0\} \rightarrow \mathbb{C}: \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \quad (97)$$

with orthonormal basis $\{e_{t_0}\}_{t_0: Z(t_0)=0}$ where

$$e_{t_0}(t_1) = \delta_{t_0}(t_1) \quad (98)$$

Proof. By the atomic form of μ , for any $f \in L^2(\mu)$:

$$\|f\|_{\mathcal{H}}^2 = \int |f(t)|^2 d\mu(t) \quad (99)$$

$$= \int |f(t)|^2 \sum_{t_0: Z(t_0)=0} \delta_{t_0}(t) \quad (100)$$

$$= \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 \quad (101)$$

This shows the isomorphism with ℓ^2 where the functions e_{t_0} defined by

$$e_{t_0}(t_1) = \delta_{t_0}(t_1) \quad (102)$$

satisfy the relations

$$\begin{aligned} \langle e_{t_0}, e_{t_1} \rangle &= \int e_{t_0}(t) \overline{e_{t_1}(t)} d\mu(t) \\ &= \sum_{t: Z(t)=0} \delta_{t_0}(t) \delta_{t_1}(t) \\ &= \delta_{t_0}(t_1) \\ &= \delta_{t_1}(t_0) \end{aligned} \quad (103)$$

thus forming an orthonormal set. Thus, any $f(t) \in \mathcal{H}$ can be written as

$$f(t) = \sum_{t_0: Z(t_0)=0} f(t_0) e_{t_0}(t) \quad (104)$$

proving they form a basis. □

Definition 20. [Multiplication operator] Define the linear operator

$$L: \mathcal{D}(L) \subset \mathcal{H} \rightarrow \mathcal{H} \quad (105)$$

by

$$(L f)(t) = t f(t) \quad (106)$$

on the support of μ with domain

$$\mathcal{D}(L) := \left\{ f \in \mathcal{H}: \int |t f(t)|^2 d\mu(t) < \infty \right\} \quad (107)$$

Theorem 21. [Self-adjointness and spectrum] L is self-adjoint on \mathcal{H} and has pure point, simple spectrum

$$\sigma(L) = \overline{\{t \in \mathbb{R}: Z(t) = 0\}} \quad (108)$$

with eigenvalues $\lambda = t_0$ for each zero t_0 and corresponding eigenvectors e_{t_0} .

Proof. First, self-adjointness is verified. For $f, g \in \mathcal{D}(L)$:

$$\begin{aligned} \langle L f, g \rangle &= \int (L f)(t) \overline{g(t)} d\mu(t) \\ &= \int t f(t) \overline{g(t)} d\mu(t) \\ &= \int f(t) \overline{t g(t)} d\mu(t) \\ &= \int f(t) \overline{(L g)(t)} d\mu(t) \\ &= \langle f, L g \rangle \end{aligned} \quad (109)$$

Thus L is symmetric and acts as

$$(L f)(t_0) = t_0 f(t_0) \quad (110)$$

for each t_0 in the atomic representation where

$$Z(t_0) = 0 \quad (111)$$

This is unitarily equivalent to the diagonal operator on ℓ^2 with diagonal entries

$$\{t_0: Z(t_0) = 0\} \quad (112)$$

Such diagonal operators are self-adjoint. For the spectrum calculation:

$$L e_{t_0} = t_0 e_{t_0} \forall \{t_0: Z(t_0) = 0\} \quad (113)$$

holds, so each t_0 is an eigenvalue of L with eigenvector e_{t_0} and since $\{e_{t_0}\}$ forms an orthonormal basis, L has pure point spectrum. The spectrum of a diagonal operator equals the closure of the set of diagonal entries, hence

$$\sigma(L) = \overline{\{t_0: Z(t_0) = 0\}} \quad (114)$$

The eigenvalues are simple. □

5.3 Regularity and Simplicity of Sample Path Zero Crossings

TODO: insert the fundamental theorem on the non-tangency of zero crossings so that it doesnt have to be assumed but is in fact a fundamental theorem of non-degenerate Gaussian processes

Definition 22. *[Regularity and simplicity] Assume $Z \in C^1(\mathbb{R})$ and every zero is simple:*

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \quad (115)$$

Lemma 23. *[Local finiteness and delta decomposition] Under Definition 22, zeros are locally finite and*

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \quad (116)$$

whence

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (117)$$

Proof. Since $Z \in C^1(\mathbb{R})$ and $\dot{Z}(t_0) \neq 0$ at each zero t_0 , the inverse function theorem implies that Z is locally invertible near each zero. Specifically, there exists a neighborhood U_{t_0} of t_0 such that $Z|_{U_{t_0}}$ is strictly monotonic and invertible.

This implies zeros are isolated: if $Z(t_0) = 0$ and $\dot{Z}(t_0) \neq 0$, then there exists $\epsilon > 0$ such that $Z(t) \neq 0$ for $0 < |t - t_0| < \epsilon$. Therefore zeros are locally finite (finitely many in any bounded interval).

For the distributional identity, the one-dimensional change of variables formula for the Dirac delta is considered. If $g: I \rightarrow \mathbb{R}$ is C^1 on interval I with $\dot{g}(x) \neq 0$ for all $x \in I$, then

$$\delta(g(x)) = \sum_{x_0: g(x_0)=0} \frac{\delta(x - x_0)}{|\dot{g}(x_0)|} \quad (118)$$

Applying this locally around each zero t_0 of Z , and since zeros are isolated, the local results can be patched together to obtain the global identity:

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \quad (119)$$

Consequently:

$$\begin{aligned} d\mu(t) &= \delta(Z(t)) |\dot{Z}(t)| dt \\ &= \sum_{t_0: Z(t_0)=0} \frac{|\dot{Z}(t)|}{|\dot{Z}(t_0)|} \delta(t - t_0) dt \\ &= \sum_{t_0: Z(t_0)=0} \delta_{t_0}(dt) \end{aligned} \quad (120)$$

where the last equality uses the fact that

$$\frac{|\dot{Z}(t_0)|}{|\dot{Z}(t_0)|} = 1 \quad (121)$$

when evaluating at $t = t_0$. \square

5.4 The Kac-Rice Formula For The Expected Zero Counting Function

Theorem 24. (Kac-Rice Formula for Zero Crossings) *Let $Z(t)$ be a centered Gaussian process on $[a, b]$ with covariance $K(s, t) = \mathbb{E}[Z(s)Z(t)]$ then the expected number of zeros in $[a, b]$ is*

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{\frac{2}{\pi}} \frac{\sqrt{K(t, t) K_{\dot{Z}}(t, t) - K_{Z, \dot{Z}}(t, t)^2}}{K(t, t)} dt \quad (122)$$

where

$$K(t, t) = \mathbb{E}[Z(t)^2] \quad (123)$$

$$K_{\dot{Z}}(t, t) = -\partial_s^2 \partial_t K(s, t)|_{s=t} \quad (124)$$

and

$$K_{Z, \dot{Z}}(t, t) = \partial_s K(s, t)|_{s=t} \quad (125)$$

Proof.

The exact zero counting function is

$$N_{[a,b]} = \int_a^b \delta(Z(t)) |\dot{Z}(t)| dt \quad (126)$$

so

$$\begin{aligned} \mathbb{E}[N_{[a,b]}] &= \int_a^b \mathbb{E}[\delta(Z(t)) |\dot{Z}(t)|] dt \\ &= \int_a^b \int_{-\infty}^{\infty} |v| p_{Z, \dot{Z}}(0, v) dv dt \end{aligned} \quad (127)$$

The vector $(Z(t), \dot{Z}(t))$ is bivariate Gaussian with covariance matrix

$$\Sigma = \begin{pmatrix} K(t, t) & K_{Z, \dot{Z}}(t, t) \\ K_{Z, \dot{Z}}(t, t) & K_{\dot{Z}}(t, t) \end{pmatrix} \quad (128)$$

whose determinant is given by

$$\det \Sigma = K(t, t) K_{\dot{Z}}(t, t) - K_{Z, \dot{Z}}(t, t)^2 \quad (129)$$

the inverse of which satisfies

$$\Sigma_{22}^{-1} = \frac{K(t, t)}{\det \Sigma} \quad (130)$$

yielding

$$p_{Z,\dot{Z}}(0, v) = \frac{1}{\sqrt{2\pi K(t, t)}} \cdot \frac{e^{-\frac{K(t, t)v^2}{2\det \Sigma}}}{\sqrt{2\pi \det \Sigma / K(t, t)}} \quad (131)$$

which factorizes as $p_Z(0) \cdot p_{\dot{Z}|Z}(v|0)$ where

$$p_Z(0) = \frac{1}{\sqrt{2\pi K(t, t)}} \quad (132)$$

and

$$\dot{Z}|Z=0 \sim \mathcal{N}(0, \det \Sigma / K(t, t)) \quad (133)$$

For zero-mean Gaussian $Y \sim \mathcal{N}(0, \sigma^2)$, direct integration gives

$$\begin{aligned} \mathbb{E}[|Y|] &= 2 \int_0^\infty \frac{y}{\sqrt{2\pi\sigma^2}} e^{-y^2/(2\sigma^2)} dy \\ &= \frac{2\sigma}{\sqrt{2\pi}} \int_0^\infty e^{-u} du \\ &= \sqrt{\frac{2}{\pi}} \sigma \end{aligned} \quad (134)$$

so that combining results yields

$$\begin{aligned} \int_{-\infty}^\infty |v| p_{Z,\dot{Z}}(0, v) dv &= \frac{\sqrt{\frac{2}{\pi}} \sqrt{\frac{\det \Sigma}{K(t, t)}}}{\sqrt{2\pi K(t, t)}} \\ &= \sqrt{\frac{2}{\pi}} \frac{\sqrt{\det \Sigma}}{K(t, t)} \end{aligned} \quad (135) \quad \square$$

Theorem 25. [Expected Zero-Counting Function] Let $\theta \in \mathcal{F}$ and let

$$K(t, s) = \text{cov}(Z(t), Z(s)) \quad (136)$$

be twice differentiable at $s=0$ and $t=0$ then expected number of zeros of the process $Z(t)$ in $[a, b]$ is

$$\mathbb{E}[N_{[a, b]}] = \sqrt{-K(0)} (\theta(b) - \theta(a)) \quad (137)$$

Proof. The covariance function of the time-changed process is

$$\begin{aligned} K_\theta(s, t) &= \text{cov}(Z(t), Z(s)) \\ &= \sqrt{\dot{\theta}(s)\dot{\theta}(t)} K(|\theta(t) - \theta(s)|) \end{aligned} \quad (138)$$

For the zero-crossing analysis, consider the normalized process. By the Kac-Rice formula:

$$\mathbb{E}[N_{[a, b]}] = \int_a^b \sqrt{-\lim_{s \rightarrow t} \frac{\partial^2}{\partial s \partial t} K_\theta(s, t)} dt \quad (139)$$

Computing the mixed partial derivative:

$$\frac{\partial}{\partial t} K_{\theta}(s, t) = \frac{1}{2} \frac{\dot{\theta}(t)}{\sqrt{\theta(t)}} \sqrt{\dot{\theta}(s)} K(|\theta(t) - \theta(s)|) + \sqrt{\theta(s)} \sqrt{\theta(t)} \dot{K}(|\theta(t) - \theta(s)|) \text{sgn}(\theta(t) - \theta(s)) \dot{\theta}(t) \quad (140)$$

Taking the limit as $s \rightarrow t$ and using the fact that $\dot{K}(0) = 0$ for stationary processes:

$$\begin{aligned} \lim_{s \rightarrow t} \frac{\partial^2}{\partial s \partial t} K_{\theta}(s, t) &= \lim_{s \rightarrow t} \dot{\theta}(s) \dot{\theta}(t) \ddot{K}(0) \\ &= \dot{\theta}(t)^2 \ddot{K}(0) \end{aligned} \quad (141)$$

Substituting into the Kac-Rice formula we have

$$\begin{aligned} \mathbb{E}[N_{[a,b]}] &= \int_a^b \sqrt{-\dot{\theta}(t)^2 \ddot{K}(0)} \, dt \\ &= \sqrt{-\ddot{K}(0)} \int_a^b \dot{\theta}(t) \, dt \\ &= \sqrt{-\ddot{K}(0)} (\theta(b) - \theta(a)) \end{aligned} \quad (142)$$

since $\dot{\theta}(t) \geq 0$ almost everywhere. □

Bibliography

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