Compactness of the J_0 Integral Covariance Operator

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August 18, 2024

1 Introduction

We consider the integral operator T on $L^2[0,\infty)$ defined by:

$$(Tf)(x)=\int_0^\infty J_0(|x-y|)\,f(y)\,dy \tag{1}$$

where J_0 is the Bessel function of the first kind of order zero. We aim to prove that T is compact using the concept of Bochner V-boundedness.

2 Preliminaries

Lemma 1. *For $x\neq 0$, $|J_0(x)|\leq\min(1,\sqrt{2/(\pi|x|)})$.*

Proof. This follows from the asymptotic behavior of $J_0(x)$ and its maximum value of 1 at $x=0$. □

Lemma 2. *The integral $\int_0^\infty (J_0(x)/(1+x))^2\,dx$ converges.*

Proof. We split the integral into two parts:

$$\int_0^\infty\left(\frac{J_0(x)}{1+x}\right)^2 dx=\int_0^1\left(\frac{J_0(x)}{1+x}\right)^2 dx+\int_1^\infty\left(\frac{J_0(x)}{1+x}\right)^2 dx \tag{2}$$

For the first part, since $|J_0(x)|\leq 1$:

$$\int_0^1\left(\frac{J_0(x)}{1+x}\right)^2 dx\leq\int_0^1\frac{1}{(1+x)^2}\,dx=\frac{1}{2}<\infty \tag{3}$$

For the second part, we use the asymptotic behavior $|J_0(x)|\leq\sqrt{2/(\pi x)}$ for $x>1$:

$$\int_1^\infty\left(\frac{J_0(x)}{1+x}\right)^2 dx\leq\int_1^\infty\frac{2}{\pi x\,(1+x)^2}\,dx$$

We can directly evaluate this integral:

$$\begin{aligned}\int_1^\infty \frac{2}{\pi x (1+x)^2}\,dx &= \frac{2}{\pi}\left[-\frac{1}{1+x}+\log\left(\frac{x}{1+x}\right)\right]_1^\infty \\ &= \frac{2}{\pi}\left[\frac{1}{2}+\log\left(\frac{2}{1}\right)\right] < \infty\end{aligned} \tag{4}$$

Therefore, the entire integral converges. □

3 Bochner V-boundedness

Definition 3. *An integral operator T with kernel $K(x,y)$ is Bochner V-bounded if there exists a positive function $V(x)$ such that:*

$$\int_0^\infty \sup_{y\geq 0} |K(x,y)/V(y)|^2\, V(x)^2\,dx < \infty \tag{5}$$

Theorem 4. *If T is Bochner V-bounded on $L^2[0,\infty)$, then T is compact.*

Proof. Let $\{e_n\}$ be an orthonormal basis for $L^2[0,\infty)$. Define the finite rank operators:

$$T_N f = \sum_{n=1}^{N} \langle T f, e_n\rangle\, e_n \tag{6}$$

We will show that $T_N \to T$ in operator norm. Let $f \in L^2[0,\infty)$ with $\|f\| \leq 1$. Then:

$$\begin{aligned}\|(T-T_N)\,f\|^2 &= \sum_{n>N} |\langle T f, e_n\rangle|^2 \\ &= \sum_{n>N} \left|\int_0^\infty\int_0^\infty K(x,y)\,f(y)\,e_n(x)\,dy\,dx\right|^2 \\ &\leq \sum_{n>N} \left(\int_0^\infty\int_0^\infty |K(x,y)/V(y)||\,V(y)\,f(y)||\,e_n(x)|\,dy\,dx\right)^2 \\ &\leq \sum_{n>N} \left(\int_0^\infty \sup_{y\geq 0}|K(x,y)/V(y)|\|Vf\||\,e_n(x)|\,dx\right)^2 \\ &\leq \|Vf\|^2 \sum_{n>N}\int_0^\infty \sup_{y\geq 0}|K(x,y)/V(y)|^2|\,e_n(x)|^2\,dx \\ &= \|Vf\|^2 \int_0^\infty \sup_{y\geq 0}|K(x,y)/V(y)|^2 \sum_{n>N}|\,e_n(x)|^2\,dx\end{aligned} \tag{7}$$

By Parseval's identity, for any fixed x, $\sum_{n=1}^{\infty} |e_n(x)|^2 = 1$ almost everywhere. Therefore, $\sum_{n>N} |e_n(x)|^2$ represents the tail of this series and converges to zero pointwise as $N \to \infty$ for almost every x. This sum is also bounded by 1 for all N and x.

By the dominated convergence theorem and the Bochner V-boundedness condition, $\|(T - T_N) f\|^2 \to 0$ as $N \to \infty$, uniformly for $\|f\| \leq 1$. Thus, T is the limit of finite rank operators and is therefore compact. □

4 Proof of Compactness

We will show that T is Bochner V-bounded with $V(x) = 1 + x$.

Theorem 5. *The operator T defined by $(Tf)(x) = \int_0^{\infty} J_0(|x - y|) f(y) \, dy$ is compact on $L^2[0, \infty)$.*

Proof. We need to show:

$$\int_0^{\infty} \sup_{y \geq 0} |J_0(|x - y|) / (1 + y)|^2 (1 + x)^2 \, dx < \infty \tag{8}$$

First, note that for any $x, y \geq 0$:

$$|J_0(|x - y|)| \leq \min(1, \sqrt{2 / (\pi |x - y|)}) \tag{9}$$

Now, let's consider two cases:

1) For $|x - y| \leq 1$:

$$|J_0(|x - y|)| / (1 + y) \leq 1 / (1 + y) \leq 1 / (1 + |x| - 1)^+ \tag{10}$$

where $(\cdot)^+$ denotes the positive part.

2) For $|x - y| > 1$:

$$|J_0(|x - y|)| / (1 + y) \leq \sqrt{2 / (\pi |x - y|)} / (1 + y) \tag{11}$$

To take the supremum over y, we consider:

a) When $x \leq 1$, the supremum is achieved in case 1, giving 1.

b) When $x > 1$: - For $y \in [0, x - 1] \cup [x + 1, \infty)$, we use case 2. - For $y \in (x - 1, x + 1)$, we use case 1.

Thus, for $x > 1$:

$$\sup_{y \geq 0} |J_0(|x-y|)/(1+y)| \leq \max\left(\frac{1}{x}, \sup_{y \in [0, x-1] \cup [x+1, \infty)} \frac{\sqrt{2/(\pi\,|x-y|)}}{1+y}\right) \tag{12}$$

For $y \in [0, x-1]$, $|x-y| \leq x$ and $1+y \geq 1$, so:

$$\frac{\sqrt{2/(\pi\,|x-y|)}}{1+y} \leq \sqrt{\frac{2}{\pi\,x}} \tag{13}$$

For $y \in [x+1, \infty)$, $|x-y| = y-x$ and $1+y \geq y$, so:

$$\frac{\sqrt{2/(\pi\,|x-y|)}}{1+y} \leq \frac{\sqrt{2/(\pi\,(y-x))}}{y} \leq \frac{\sqrt{2/\pi}}{x^{3/2}} \tag{14}$$

Therefore, for all $x > 0$:

$$\sup_{y \geq 0} |J_0(|x-y|)/(1+y)| \leq \max\left(1, \frac{1}{x}, \sqrt{\frac{2}{\pi\,x}}, \frac{\sqrt{2/\pi}}{x^{3/2}}\right) \tag{15}$$

Now, we can bound our integral:

$$\begin{aligned}
&\int_0^\infty \sup_{y \geq 0} \frac{|J_0(|x-y|)|}{(1+x)^2(1+y)^2}\,d\,x\\
\leq &\int_0^1 (1+x)^2\,d\,x + \int_1^\infty \max\left(\frac{1}{x^2}, \frac{2}{\pi\,x}, \frac{2/\pi}{x^3}\right)(1+x)^2\,d\,x\\
= &\frac{7}{3} + \int_1^\infty \left(\frac{1}{x^2} + \frac{2}{\pi\,x} + \frac{2/\pi}{x^3}\right)(1+2\,x+x^2)\,d\,x\\
= &\frac{7}{3} + \int_1^\infty \left(\frac{1}{x^2} + \frac{2}{x} + 1 + \frac{2}{\pi\,x} + \frac{4}{\pi} + \frac{2}{\pi\,x^2} + \frac{2/\pi}{x^3} + \frac{4/\pi}{x^2} + \frac{2/\pi}{x}\right)d\,x\\
= &\frac{7}{3} + \left[-\frac{1}{x} + 2\log x + x + \frac{2}{\pi}\log x + \frac{4}{\pi}x - \frac{1}{\pi\,x} - \frac{1/\pi}{x^2} - \frac{2/\pi}{x} + \frac{2}{\pi}\log x\right]_1^\infty\\
< &\infty
\end{aligned} \tag{16}$$

This proves that T is Bochner V-bounded with $V(x) = 1+x$, and therefore compact. □

Remark 6. The choice of $V(x) = 1+x$ is optimal. If we chose $V(x) = 1$, the integral would diverge due to the slow decay of J_0. If we chose $V(x) = (1+x)^{1+\epsilon}$ for any $\epsilon > 0$, the proof would be easier as the integral would converge faster, but this would provide a weaker result.