Pushforward and Pullback Operations in Measure Theory

1 Introduction

Pushforward and pullback operations constitute essential techniques in measure theory, enabling the transfer of measures and functions between measurable spaces via measurable transformations. These constructs are fundamentally connected to the behavior of σ-algebras and the preservation of measurability under mapping, providing indispensable tools for analysis, probability theory, and related fields.

2 Preliminaries

Definition 1. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, meaning X and Y are sets, and $\mathcal{A}, \mathcal{B}$ are σ-algebras of subsets of X and Y, respectively. A map $f: X \to Y$ is said to be* measurable *if for all $B \in \mathcal{B}$, the preimage $f^{-1}(B) \in \mathcal{A}$.*

Let μ be a measure on $(X, \mathcal{A})$ and ν a measure on $(Y, \mathcal{B})$.

3 The Pushforward (Image) Measure

Definition 2. *[Pushforward Measure] Let $f: X \to Y$ be a measurable function and $(X, \mathcal{A}, \mu)$ a measure space. The* pushforward *(or image) measure $f_* \mu$ on $(Y, \mathcal{B})$ is defined by*

$$(f_* \mu)(B) = \mu(f^{-1}(B)) \forall B \in \mathcal{B} \tag{1}$$

Theorem 3. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $f: X \to Y$ a measurable function, and $(Y, \mathcal{B})$ a measurable space. Then $f_* \mu$ is a measure on $(Y, \mathcal{B})$.*

Proof. It suffices to verify the properties of a measure:

1. *Non-negativity*:

$$(f_* \mu)(B) = \mu(f^{-1}(B)) \geq 0 \forall B \in \mathcal{B} \tag{2}$$

   since μ is a measure.

2. *Null empty set*:

$$(f_* \mu)(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0 \tag{3}$$

3. *Countable additivity*: Let $(B_n)_{n\in\mathbb{N}} \subset \mathcal{B}$ be pairwise disjoint. Then

$$\begin{aligned}(f_*\,\mu)\left(\bigcup_{n=1}^{\infty} B_n\right) &= \mu\left(f^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right)\right)\\ &= \mu\left(\bigcup_{n=1}^{\infty} f^{-1}(B_n)\right)\\ &= \sum_{n=1}^{\infty} \mu(f^{-1}(B_n))\\ &= \sum_{n=1}^{\infty} (f_*\,\mu)(B_n)\end{aligned} \tag{4}$$

where the third equality uses measurability of f and the fact that preimages preserve unions and disjointness.

Thus, $f_*\,\mu$ is a measure. □

Remark 4. If μ is a probability measure, then so is $f_*\,\mu$. In this context, $f_*\,\mu$ describes the distribution of the random variable f induced by μ.

4 The Pullback Operation for Measurable Functions

The pullback operation allows the transfer of functions and, in more elaborate contexts, measures across measurable spaces.

Definition 5. *[Pullback of a Function] Let $f: X \to Y$ be a measurable function and $g: Y \to \mathbb{R}$ a $\mathcal{B}$-measurable function. The* pullback *of g along f, denoted $f^*\,g$, is defined by*

$$f^*\,g := g \circ f, \quad x \mapsto g(f(x)) \tag{5}$$

for $x \in X$.

Theorem 6. *If $g: Y \to \mathbb{R}$ is $\mathcal{B}$-measurable and $f: X \to Y$ is $\mathcal{A}$-$\mathcal{B}$-measurable, then $f^*\,g = g \circ f$ is $\mathcal{A}$-measurable.*

Proof. Let $B \in \mathcal{B}(\mathbb{R})$, the Borel σ-algebra on $\mathbb{R}$. Then

$$(f^*\,g)^{-1}(B) = \{x \in X: g(f(x)) \in B\} = f^{-1}(g^{-1}(B)) \tag{6}$$

Since g is $\mathcal{B}$-measurable, $g^{-1}(B) \in \mathcal{B}$. Since f is $\mathcal{A}$-$\mathcal{B}$-measurable, $f^{-1}(g^{-1}(B)) \in \mathcal{A}$. Thus, $f^*\,g$ is $\mathcal{A}$-measurable. □

5 Pullback of a Measure: Theoretical Caveat

Generally, the pullback of a measure via a function is not always well defined. In particular, given a measure ν on $(Y, \mathcal{B})$ and a measurable $f: X \to Y$, the set function

$$\mu(A) := \nu(f(A)) \forall A \in \mathcal{A} \tag{7}$$

is not, in general, a measure. Issues arise due to the failure of countable additivity unless f is injective or further structure is present.

Remark 7. A legitimate pullback of measures (under the name *inverse image measure*) exists in the context of differentiable manifolds, or via the theory of signed measures and distributions, but not in general for arbitrary measure spaces.

6 The Push-Pull Formula (Change of Variables)

Theorem 8. *[Pushforward and Integration (Change of Variables)] Let $(X, \mathcal{A}, \mu)$ be a measure space, $(Y, \mathcal{B})$ a measurable space, and $f: X \to Y$ a measurable map. Let $g: Y \to [0, +\infty]$ be $\mathcal{B}$-measurable. Then*

$$\int_Y g \, d(f_* \mu) = \int_X g \circ f \, d\mu \tag{8}$$

Proof. Consider first $g = 1_B$ for $B \in \mathcal{B}$. Then

$$\int_Y 1_B \, d(f_* \mu) = (f_* \mu)(B) = \mu(f^{-1}(B)) = \int_X 1_{f^{-1}(B)} \, d\mu = \int_X (1_B \circ f) \, d\mu \tag{9}$$

By linearity and monotone convergence, the result extends to all non-negative $\mathcal{B}$-measurable functions g. $\square$

7 Interrelationships and Further Properties

Proposition 9. *The assignments $f \mapsto f_*$ and $g \mapsto f^* g$ are functorial in the sense that*

1. *For measurable maps $f: X \to Y$ and $g: Y \to Z$, $(g \circ f)_* = g_* \circ f_*$ as assignments on measures.*

2. *For measurable maps $f: X \to Y$ and $g: Y \to Z$, $(g \circ f)^* = f^* \circ g^*$ as assignments on functions.*

Proof.

1. Let μ be a measure on X. For $C \in \mathcal{C}$ (where $\mathcal{C}$ is a σ-algebra on Z):

$$\begin{aligned} ((g\circ f)_* \mu)(C) &= \mu((g\circ f)^{-1}(C)) \\ &= \mu(f^{-1}(g^{-1}(C))) \\ &= (f_* \mu)(g^{-1}(C)) \\ &= (g_* f_* \mu)(C) \end{aligned} \tag{10}$$

2. Let $h: Z \to \mathbb{R}$. Then for $x \in X$,

$$\begin{aligned} (g\circ f)^* h(x) &= h(g(f(x))) \\ &= (g^* h)(f(x)) \\ &= f^* (g^* h)(x) \end{aligned} \tag{11}$$ □

8 Conclusion

Pushforward and pullback operations are pivotal in connecting the structures of different measurable spaces via measurable functions. The pushforward provides a mechanism to transfer measures in a functorial and natural way, while the pullback appropriately lifts functions, preserving measurability. The integration change-of-variables formula encapsulates the deep relationship between these operations, forming a cornerstone of modern analysis.