Generalization of the Wiener-Khintchine Theorem to Nonstationary Processes

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Received November 30, 1953

Abstract

The Wiener-Khintchine theorem which connects the autocorrelation function and the power spectrum of a stationary time series by means of a Fourier cosine transform is well known. It is the purpose of this paper to generalize the theorem to deal with nonstationary time series.

1   Introduction

In a recent paper[1] the author gave a generalization of the Wiener-Khintchine theorem which connected the time dependent power spectra and correlation functions of the outputs of a set of linear filters which were excited at $t = 0$ by suddenly applying stationary random noise. In this paper we shall give a slightly modified treatment which will enable us to carry this generalization to its logical conclusion. To do this it will be necessary to use extended definitions of correlation functions and power spectra, definitions which remain valid even for nonstationary time series.

2   Analysis

Let us consider a pair of nonstationary time series $i_1(x)$ and $i_2(x)$ which are defined in the range $-\infty < x \le t$ and which are zero for $x > t$.

We note that in the case of autocorrelation of a stationary time series this definition is equivalent to the temporal one usually given in the literature (see Rice,[2] p. 312) (see Wiener,[3] p. 38), namely,

$$\psi(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} i(t)i(t + \tau)\, dt. \tag{1}$$

We now define the (cross)correlation function $\psi_{12}(t; \tau)$ of the i_1 and i_2 in this order by

$$\psi_{12}(t; \tau) = \langle i_1(t)i_2(t - \tau)\rangle_{\text{av}}, \tag{2}$$

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where the angular brackets denote that the ensemble average of the product is to be taken. It is clear that in this work concerning nonstationary time series, ensemble rather than time averaging must be employed.

If we denote their Fourier transforms by $S_1(t; f)$ and $S_2(t; f)$, respectively, we have

$$S_1(t; f) = \int_{-\infty}^{t} i_1(t_1)e^{-i2\pi f t_1}\, dt_1, \tag{3}$$

$$S_2^*(t; f) = \int_{-\infty}^{t} i_2(t_2)e^{+i2\pi f t_2}\, dt_2, \tag{4}$$

where the star notation for the complex conjugate has been used.

Now the "joint energy" of the component of frequency f of the i_1 and i_2 in this order is

$$\begin{aligned} E_{12}(t; f) &= S_1(t; f)S_2^*(t; f) \\ &= \int_{-\infty}^{t} \int_{-\infty}^{t} i_1(t_1)i_2(t_2)e^{i2\pi f(t_2-t_1)}\, dt_1\, dt_2. \end{aligned} \tag{5}$$

Let us now take the ensemble average of both sides of Eq. (5) and make use of Eq. (2). We obtain

$$\langle E_{12}(t; f)\rangle_{\mathrm{av}} = \int_{-\infty}^{t} \int_{-\infty}^{t} \psi_{12}(t_1; t_1 - t_2)e^{i2\pi f(t_2-t_1)}\, dt_1\, dt_2. \tag{6}$$

If we make the substitution $\tau = t_2 - t_1$ we may write symbolically

$$\int_{-\infty}^{t} \int_{-\infty}^{t} dt_1\, dt_2 = \int_{-\infty}^{t} \int_{0}^{\infty} dt_2\, d\tau + \int_{-\infty}^{t} \int_{-\infty}^{0} dt_1\, d\tau. \tag{7}$$

But by Eq. (2) we have

$$\begin{aligned} \psi_{12}(t_1 - \tau; -\tau) &= \langle i_1(t_1 - \tau)i_2(t_1)\rangle_{\mathrm{av}} \\ &= \psi_{21}(t_1; \tau), \end{aligned} \tag{8}$$

so that Eq. (6) becomes

$$\langle E_{12}(t; f)\rangle_{\mathrm{av}} = \int_{-\infty}^{t} \int_{0}^{\infty} \{\psi_{12}(t_1; \tau)e^{-i2\pi f\tau} + \psi_{21}(t_1; \tau)e^{+i2\pi f\tau}\}\, dt_1\, d\tau. \tag{9}$$

We now define[1] the (cross)power spectrum $w_{12}(t; f)$ by the relation

$$w_{12}(t; f) = 2\frac{\partial}{\partial t}\{\langle E_{12}(t; f)\rangle_{\mathrm{av}}\}, \tag{10}$$

where the factor 2 has been introduced to make the integral of $w_{12}(t; f)$ over all positive real frequencies equal the total (cross)power.

If we carry out the indicated differentiation we have immediately

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[1]This definition of power spectrum is essentially equivalent to that of C. H. Page,[4] The author is grateful to Professor Zadeh of Columbia University for drawing his attention, in a private communication, to Page's important paper and pointing out this equivalence.

Theorem 1 (Generalized Wiener-Khintchine Theorem). *The cross-power spectrum for non-stationary processes is given by*

$$w_{12}(t;f) = 2 \int_0^\infty \{\psi_{12}(t;\tau)e^{-i2\pi f\tau} + \psi_{21}(t;\tau)e^{+i2\pi f\tau}\} \, d\tau, \tag{11}$$

which is the main result of this paper.

Proof. The result follows directly from differentiating Eq. (9) with respect to t and applying the definition in Eq. (10). □

If $i_1 = i_2$, we have, dropping the subscripts,

$$w(t;f) = 4 \int_0^\infty \psi(t;\tau) \cos 2\pi f\tau \, d\tau, \tag{12}$$

and this equation clearly reduces to the Wiener-Khintchine result if the time series is stationary.

Finally, let us find the inversion of Eq. (11). We have

$$\int_0^\infty w_{12}(t;f)e^{i2\pi f\rho} \, df = 2 \int_0^\infty \int_0^\infty \{\psi_{12}(t;\tau)e^{i2\pi f(\rho-\tau)} + \psi_{21}(t;\tau)e^{i2\pi f(\rho+\tau)}\} \, d\tau \, df \tag{13}$$

and

$$\int_0^\infty w_{21}(t;f)e^{-i2\pi f\rho} \, df = 2 \int_0^\infty \int_0^\infty \{\psi_{21}(t;\tau)e^{-i2\pi f(\rho+\tau)} + \psi_{12}(t;\tau)e^{-i2\pi f(\rho-\tau)}\} \, d\tau \, df. \tag{14}$$

Adding these gives

$$\int_0^\infty \{w_{12}(t;f)e^{i2\pi f\rho} + w_{21}(t;f)e^{-i2\pi f\rho}\} \, df = 4 \int_0^\infty \int_0^\infty \{\psi_{12}(t;\tau) \cos 2\pi f(\rho-\tau) \\ + \psi_{21}(t;\tau) \cos 2\pi f(\rho+\tau)\} \, d\tau \, df. \tag{15}$$

We now carry out the integration with respect to f on the right-hand side, getting

$$\int_0^\infty \{w_{12}(t;f)e^{i2\pi f\rho} + w_{21}(t;f)e^{-i2\pi f\rho}\} \, df = 2 \int_0^\infty \{\psi_{12}(t;\tau)\delta(\rho-\tau) \\ + \psi_{21}(t;\tau)\delta(\rho+\tau)\} \, d\tau, \tag{16}$$

where the interpretation (see Rice,[2] p. 314)

$$\int_0^\infty \cos 2\pi fx \, df = \frac{1}{2}\delta(x) \tag{17}$$

has been used. Here $\delta(x)$ is the even unit impulse function (Dirac δ-function).

Performing the remaining integration gives at once if $\rho \geq 0$,

Theorem 2 (Inversion Formula). *The correlation function can be recovered from the power spectrum via*

$$\psi_{12}(t;\rho) = \frac{1}{2}\int_0^\infty \{w_{12}(t;f)e^{i2\pi f\rho} + w_{21}(t;f)e^{-i2\pi f\rho}\}\, df, \tag{18}$$

which is the required inversion of Eq. (11).

Proof. The result follows from Eq. (16) and the properties of the Dirac delta function, noting that $\delta(\rho + \tau) = 0$ for $\rho, \tau \geq 0$. $\qquad\square$

Again if $i_1 = i_2$, we obtain

$$\psi(t;\rho) = \int_0^\infty w(t;f)\cos 2\pi f\rho\, df, \tag{19}$$

which also reduces to the Wiener-Khintchine result in the stationary case. We note finally the work of Fano,[5] in which another aspect of this problem is discussed.

Acknowledgment

References

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