Generation of Oscillatory Processes via Unitarily Time-Changed Stationary Processes

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Definition 1. *[Time Change Function] Let $T(t)$ be a function that is strictly increasing except possibly on a set of Lebesgue measure zero.*

Definition 2. *[Stationary Gaussian Process] Let $X(t)$ be a stationary Gaussian process with spectral representation:*

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\, d\,\Phi(\lambda) \tag{1}$$

where $d\,\Phi(\lambda)$ is an orthogonal increment process and $S(\lambda)$ is the spectral density of $X(t)$.

Definition 3. *[Unitarity-Preserving Time Change] The unitarily time-changed process is defined as:*

$$Z(t) = \sqrt{T'(t)} \cdot X(T(t)) \tag{2}$$

where $T'(t)$ is the derivative of $T(t)$.

Theorem 4. *[Oscillatory Function Representation] The oscillatory function for the unitarily time-changed process is:*

$$\phi_t(\lambda) = \sqrt{T'(t)}\, e^{i\lambda T(t)} \tag{3}$$

Proof. Substituting the spectral representation of $X(t)$ into the definition of $Z(t)$:

$$\begin{aligned} Z(t) &= \sqrt{T'(t)} \cdot X(T(t)) && (4)\\ &= \sqrt{T'(t)} \int_{-\infty}^{\infty} e^{i\lambda T(t)}\, d\,\Phi(\lambda) && (5)\\ &= \int_{-\infty}^{\infty} \sqrt{T'(t)}\, e^{i\lambda T(t)}\, d\,\Phi(\lambda) && (6)\\ &= \int_{-\infty}^{\infty} \phi_t(\lambda)\, d\,\Phi(\lambda) && (7) \end{aligned}$$

□

Theorem 5. *[Priestley Gain Function] The gain function in Priestley's oscillatory process representation is:*

$$A(t,\lambda) = \sqrt{T'(t)}\, e^{i\lambda(T(t)-t)} \tag{8}$$

Proof. The Priestley representation requires:

$$Z(t) = \int_{-\infty}^{\infty} A(t,\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda) \tag{9}$$

Comparing with the oscillatory function representation:

$$\int_{-\infty}^{\infty} \phi_t(\lambda)\, d\,\Phi(\lambda) = \int_{-\infty}^{\infty} A(t,\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda) \tag{10}$$

Therefore:

$$\phi_t(\lambda) = A(t,\lambda)\, e^{i\lambda t} \tag{11}$$

Solving for $A(t,\lambda)$:

$$\begin{aligned} A(t,\lambda) &= \phi_t(\lambda)\, e^{-i\lambda t} && (12)\\ &= \sqrt{T'(t)}\, e^{i\lambda T(t)}\, e^{-i\lambda t} && (13)\\ &= \sqrt{T'(t)}\, e^{i\lambda (T(t)-t)} && (14) \end{aligned}$$

□

Theorem 6. *[Covariance Kernel] The covariance kernel $R(s,t)$ of the unitarily time-changed process $Z(t)$ is:*

$$R(s,t) = \sqrt{T'(s)\, T'(t)} \int_{-\infty}^{\infty} e^{i\lambda (T(s)-T(t))}\, S(\lambda)\, d\,\lambda$$

Proof. The covariance kernel is defined as:

$$R(s,t) = \mathrm{Cov}[Z(s), Z(t)] = E[Z(s)\overline{Z(t)}] \tag{15}$$

Using the oscillatory function representation:

$$\begin{aligned} R(s,t) &= E\left[\int_{-\infty}^{\infty} \phi_s(\lambda)\, d\,\Phi(\lambda) \int_{-\infty}^{\infty} \overline{\phi_t(\mu)}\, \overline{d\,\Phi(\mu)}\right] && (16)\\ &= \int_{-\infty}^{\infty} \phi_s(\lambda)\overline{\phi_t(\lambda)} S(\lambda)\, d\,\lambda && (17)\\ &= \int_{-\infty}^{\infty} \sqrt{T'(s)}\, e^{i\lambda T(s)}\, \sqrt{T'(t)}\, e^{-i\lambda T(t)}\, S(\lambda)\, d\,\lambda && (18)\\ &= \sqrt{T'(s)\, T'(t)} \int_{-\infty}^{\infty} e^{i\lambda (T(s)-T(t))}\, S(\lambda)\, d\,\lambda && (19) \end{aligned}$$

□

Theorem 7. *[Variance Function] The variance function of the unitarily time-changed process is:*

$$\sigma_Z^2(t) = Var[Z(t)] = T'(t) \cdot \sigma_X^2$$

where $\sigma_X^2 = \int_{-\infty}^{\infty} S(\lambda)\, d\,\lambda$.

Proof. Setting $s = t$ in the covariance kernel:

$$\begin{aligned}
\sigma_Z^2(t) &= R(t,t) &(20)\\
&= \sqrt{T'(t)\,T'(t)} \int_{-\infty}^{\infty} e^{i\lambda(T(t)-T(t))}\, S(\lambda)\ d\lambda &(21)\\
&= T'(t) \int_{-\infty}^{\infty} e^{i\lambda\cdot 0}\, S(\lambda)\ d\lambda &(22)\\
&= T'(t) \int_{-\infty}^{\infty} S(\lambda)\ d\lambda &(23)\\
&= T'(t)\cdot\sigma_X^2 &(24)
\end{aligned}$$

□

Theorem 8. *[Time-Dependent Spectral Density] The time-dependent spectral density (evolutionary spectral density) is:*

$$S_Z(t,\lambda) = T'(t)\cdot S(\lambda)$$

Proof. The time-dependent spectral density is defined as:

$$S_Z(t,\lambda) = |A(t,\lambda)|^2\cdot S(\lambda) \tag{25}$$

Computing the modulus squared of the gain function:

$$\begin{aligned}
|A(t,\lambda)|^2 &= \left|\sqrt{T'(t)}\, e^{i\lambda(T(t)-t)}\right|^2 &(26)\\
&= \left|\sqrt{T'(t)}\right|^2 \left|e^{i\lambda(T(t)-t)}\right|^2 &(27)\\
&= T'(t)\cdot 1 &(28)\\
&= T'(t) &(29)
\end{aligned}$$

Therefore:

$$S_Z(t,\lambda) = T'(t)\cdot S(\lambda)$$ □

Theorem 9. *[Expected Zero Count via Kac-Rice Formula] The expected zero count of the unitarily time-changed process $Z(t)$ in interval $[a,b]$ is:*

$$E[N_Z[a,b]] = \rho_X\cdot(T(b)-T(a))$$

where $\rho_X = \frac{1}{\pi}\sqrt{\frac{-R_X''(0)}{\sigma_X^2}}$ is the constant zero-crossing rate of the stationary process $X(t)$.

Proof. TODO: merge with other paper where I stated this most excellently □