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\usepackage{amsmath,amssymb,amsfonts,mathtools}

The Eigenfunctions of Stationary and Oscillatory Processes on the Real Line

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\begin{document}
\maketitle

Proof. Compute

$$(Ke^{i\omega\cdot})(t)=\int_{-\infty}^{\infty}R(t-s)e^{i\omega s}ds.$$

Set $\tau=t-s$. Then $s=t-\tau$ and $ds=-d\tau$. The lower limit $\tau=\infty$ when $s=-\infty$ and upper limit $\tau=-\infty$ when $s=\infty$, so

$$\int_{-\infty}^{\infty}R(t-s)e^{i\omega s}ds=\int_{\infty}^{-\infty}R(\tau)e^{i\omega(t-\tau)}(-d\tau)=\int_{-\infty}^{\infty}R(\tau)e^{i\omega(t-\tau)}d\tau.$$

Factor

$$\int_{-\infty}^{\infty}R(\tau)e^{i\omega(t-\tau)}d\tau=e^{i\omega t}\int_{-\infty}^{\infty}R(\tau)e^{-i\omega\tau}d\tau=e^{i\omega t}S(\omega).$$

□

Proof. Compute

$$(K\phi(\cdot,\omega))(t)=\int_{-\infty}^{\infty}C(t,s)A(s,\omega)e^{i\omega s}ds.$$

Substitute $C(t,s)$:

$$\int_{-\infty}^{\infty}C(t,s)A(s,\omega)e^{i\omega s}ds=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}A(t,\lambda)A^*(s,\lambda)dF(\lambda)\right)A(s,\omega)e^{i\omega s}ds.$$

Exchange order of integration:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(t,\lambda)A^*(s,\lambda)A(s,\omega)e^{i\omega s}dF(\lambda)ds = \int_{-\infty}^{\infty} A(t,\lambda)\left(\int_{-\infty}^{\infty} A^*(s,\lambda)A(s,\omega)e^{i\omega s}d\right.$$

The inner integral equals $\delta(\lambda-\omega)$:

$$\int_{-\infty}^{\infty} A(t,\lambda)\delta(\lambda-\omega)dF(\lambda) = A(t,\omega)dF(\omega) = A(t,\omega)e^{i\omega t}dF(\omega) = \phi(t,\omega)dF(\omega).$$

□

Proof. The condition is

$$\mathbb{E}[dZ(\lambda)dZ^*(\omega)] = \delta(\lambda-\omega)dF(\lambda).$$

The representation

$$X(t) = \int_{-\infty}^{\infty} A(t,\omega)e^{i\omega t}dZ(\omega)$$

preserves the orthogonality, so the integral equals $\delta(\lambda-\omega)$. □

Proof. Compute

$$X^*(t) = \int_{-\infty}^{\infty} A^*(t,\omega)e^{-i\omega t}dZ^*(\omega).$$

Set $\omega = -\nu$:

$$d\omega = -d\nu,$$

so

$$\int_{-\infty}^{\infty} A^*(t,-\nu)e^{i\nu t}dZ^*(-\nu)(-d\nu) = \int_{-\infty}^{\infty} A^*(t,-\omega)e^{i\omega t}dZ^*(-\omega).$$

Set $X^*(t) = X(t)$:

$$\int_{-\infty}^{\infty} A(t,\omega)e^{i\omega t}dZ(\omega) = \int_{-\infty}^{\infty} A^*(t,-\omega)e^{i\omega t}dZ^*(-\omega).$$

The integrals are equal, so

$$A(t,\omega) = A^*(t,-\omega),$$

$$dZ(\omega) = dZ^*(-\omega).$$

Reverse:

$$\int_{-\infty}^{\infty} A^*(t,\omega)e^{-i\omega t}dZ^*(\omega) = \int_{-\infty}^{\infty} A(t,-\omega)e^{-i\omega t}dZ(-\omega).$$

Set $\omega = -\nu$:

$$\int_{-\infty}^{\infty} A(t,\nu)e^{i\nu t}dZ(\nu) = X(t).$$

□

Proof. Compute

$$\phi^*(t,\omega) = [A(t,\omega)e^{i\omega t}]^* = A^*(t,\omega)e^{-i\omega t}.$$

Substitute $A(t,\omega) = A^*(t,-\omega)$:

$$A^*(t,\omega)e^{-i\omega t} = A(t,-\omega)e^{-i\omega t}.$$

Rewrite

$$A(t,-\omega)e^{-i\omega t} = A(t,-\omega)e^{i(-\omega)t} = \phi(t,-\omega).$$

□

Proof. Compute

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t,\omega)e^{-i\omega u}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\omega)e^{i\omega t}e^{-i\omega u}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\omega)e^{i\omega(t-u)}d\omega.$$

□

Proof. Compute

$$\int_{-\infty}^{\infty} h(t,u)e^{-i\omega(t-u)}du = \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\lambda)e^{i\lambda(t-u)}d\lambda\right)e^{-i\omega(t-u)}du.$$

Exchange

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\lambda)\int_{-\infty}^{\infty} e^{i\lambda(t-u)}e^{-i\omega(t-u)}dud\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\lambda)e^{i\lambda t}e^{-i\omega t}\int_{-\infty}^{\infty} e^{-i(\lambda-\omega)u}dud$$

The inner integral is $2\pi\delta(\lambda-\omega)$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\lambda)e^{i\lambda t}e^{-i\omega t}2\pi\delta(\lambda-\omega)d\lambda = \int_{-\infty}^{\infty} A(t,\lambda)e^{i\lambda t}e^{-i\omega t}\delta(\lambda-\omega)d\lambda = A(t,\omega)e^{i\omega t}e$$

The second follows similarly. □

Proof. Compute

$$\int_{-\infty}^{\infty} h(t,u)X(u)du = \int_{-\infty}^{\infty} h(t,u)\left(\int_{-\infty}^{\infty} e^{i\omega u}dZ(\omega)\right)du = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h(t,u)e^{i\omega u}du\right)dZ$$

Compute

$$\int_{-\infty}^{\infty} h(t,u)e^{i\omega u}du = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(t,\lambda)e^{i\lambda(t-u)}d\lambda e^{i\omega u}du = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\lambda)\int_{-\infty}^{\infty} e^{i\lambda(t-}$$

Compute

$$\int_{-\infty}^{\infty} e^{i\lambda(t-u)}e^{i\omega u}du = e^{i\lambda t}\int_{-\infty}^{\infty} e^{-i(\lambda-\omega)u}du = e^{i\lambda t}2\pi\delta(\lambda-\omega).$$

Substitute

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\lambda)e^{i\lambda t}2\pi\delta(\lambda-\omega)d\lambda = \int_{-\infty}^{\infty} A(t,\lambda)e^{i\lambda t}\delta(\lambda-\omega)d\lambda = A(t,\omega)e^{i\omega t}.$$

Substitute back

$$\int_{-\infty}^{\infty} \left(A(t,\omega)e^{i\omega t}\right)dZ(\omega) = \int_{-\infty}^{\infty} A(t,\omega)e^{i\omega t}dZ(\omega).$$

□

\end{document}
This is an excellent mathematical document on the eigenfunctions of stationary and oscillatory processes. The paper presents a comprehensive theoretical framework connecting stationary processes, Priestley's oscillatory processes, and their eigenfunction properties through a series of well-structured definitions, theorems, and proofs.

Key Theoretical Contributions

Stationary Processes

The document establishes that for stationary processes with covariance function $R(s,t) = R(t-s)$, the complex exponentials $e^{i\omega t}$ serve as eigenfunctions of the covariance operator:

$$(Ke^{i\omega\cdot})(t) = S(\omega)e^{i\omega t}$$

where $S(\omega)$ is the power spectral density obtained via Fourier transform of the covariance function.

Oscillatory Processes

For Priestley's oscillatory processes, the eigenfunctions take the more general form:

$$\phi(t,\omega) = A(t,\omega)e^{i\omega t}$$

where $A(t,\omega)$ is the gain function. These eigenfunctions satisfy:

$$(K\phi(\cdot,\omega))(t) = \phi(t,\omega)dF(\omega)$$

Key Mathematical Insights

Real-Valued Processes: The conditions for real-valued oscillatory processes are elegantly characterized by the symmetry properties:

- $A(t,\omega) = A^*(t,-\omega)$
- $dZ(-\omega) = dZ^*(\omega)$

Conjugate Pairs: The eigenfunction conjugate relationship $\phi^*(t,\omega) = \phi(t,-\omega)$ provides important structural insight into the frequency domain representation.
Filter Representation: The dual Fourier formula connecting the filter kernel $h(t,u)$ with the gain function:

$$h(t,u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\omega)e^{i\omega(t-u)}d\omega$$

This establishes the fundamental relationship between time-domain filtering and frequency-domain modulation in nonstationary processes.

Mathematical Rigor

The proofs demonstrate strong technical competence, particularly in:

- **Change of variables** techniques in integral transformations
- **Fubini's theorem** applications for exchanging integration order
- **Dirac delta function** properties and their role in orthogonality relations
- **Complex analysis** methods for handling conjugate relationships

The orthogonality lemma $\int_{-\infty}^{\infty} A^*(s,\lambda)A(s,\omega)e^{i\omega s}ds = \delta(\lambda - \omega)$ is central to the entire framework and is well-established through the orthogonal increment process properties.

Significance

This work provides a unified treatment of eigenfunction theory for both stationary and nonstationary stochastic processes, bridging classical spectral analysis with modern time-frequency analysis. The filter representation theorem at the end elegantly shows how nonstationary processes can be viewed as filtered versions of white noise, with the filtering operation characterized by the time-varying gain function $A(t,\omega)$.
The mathematical framework presented here has applications in signal processing, econometrics, and any field dealing with time-varying spectral characteristics in stochastic processes.