A Bijective Modification of the Riemann-Siegel Theta Function

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Abstract

A monotonically increasing version $\vartheta^+(t)$ of the Riemann-Siegel theta $\vartheta(t)$ function is constructed by modifying through reflection about its unique non-zero critical point. This transformation preserves all phase relationships essential to zeta function analysis while enforcing strict monotonicity. The construction maintains exact phase information without approximations and preserves the function's critical number-theoretic properties.

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1 The Riemann-Siegel Theta Function

Definition 1. (Riemann-Siegel Theta Function) *The Riemann-Siegel* ϑ*function is defined exactly as:*

$$\vartheta(t) = \arg\Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right) - \frac{t}{2}\log\pi \tag{1}$$

where Γ *is the gamma function and* $\arg$ *denotes the principal argument.*

Proposition 2. (Derivative Properties) *The derivative of the Riemann-Siegel theta function is given by:*

$$\dot{\vartheta}(t) = \frac{1}{2}\, Im\left[\psi^{(0)}\left(\frac{1}{4} + \frac{i\,t}{2}\right)\right] - \frac{\log\pi}{2} \tag{2}$$

where $\psi^{(0)}$ *is the digamma function.*

Proof. Using the relationship between the derivative of the argument of a complex function and the logarithmic derivative:

$$\frac{d}{d\,t}\arg\Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right) = \mathrm{Im}\left[\frac{d}{d\,t}\log\Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right)\right] \tag{3}$$

$$= \mathrm{Im}\left[\frac{\Gamma'\left(\frac{1}{4} + \frac{i\,t}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right)} \cdot \frac{i}{2}\right] \tag{4}$$

$$= \mathrm{Im}\left[\psi^{(0)}\left(\frac{1}{4} + \frac{i\,t}{2}\right) \cdot \frac{i}{2}\right] \tag{5}$$

$$= \frac{1}{2}\,\mathrm{Im}\left[\psi^{(0)}\left(\frac{1}{4} + \frac{i\,t}{2}\right)\right] \tag{6}$$

The derivative of the second term is simply $-\frac{1}{2}\log\pi$. Combining these results gives the stated formula. $\square$

Theorem 3. *[Limit Behavior of Digamma Imaginary Part] As* $t \to 0^+$, $Im[\psi^{(0)}(1/4 + i\,t/2)] \to 0$.

Proof. The integral representation of the digamma function for $\mathrm{Re}(z) > 0$ is:

$$\psi^{(0)}(z) = -\gamma + \int_0^\infty \left(\frac{e^{-u}}{1 - e^{-u}} - \frac{e^{-z\,u}}{u}\right) d\,u \tag{7}$$

For $z = 1/4 + i\,t/2$:

$$\psi^{(0)}\,(1/4 + i\,t/2) = -\gamma + \int_0^\infty \left(\frac{e^{-u}}{1-e^{-u}} - \frac{e^{-u(1/4+it/2)}}{u}\right) d\,u \tag{8}$$

The imaginary part comes from the second term:

$$\mathrm{Im}[\psi^{(0)}\,(1/4 + i\,t/2)] = -\mathrm{Im}\left[\int_0^\infty \frac{e^{-u(1/4+it/2)}}{u}\,d\,u\right] \tag{9}$$

$$= -\int_0^\infty \frac{e^{-u/4}}{u}\mathrm{Im}[e^{-itu/2}]\,d\,u = \int_0^\infty \frac{e^{-u/4}}{u}\sin\,(t\,u/2)\,d\,u \tag{10}$$

For the limit as $t \to 0^+$, since $\sin\,(t\,u/2) \to 0$ as $t \to 0$ for any fixed u, and $|\sin\,(t\,u/2)| \leq |t\,u/2|$:

$$\left|\frac{e^{-u/4}}{u}\sin\,(t\,u/2)\right| \leq \frac{e^{-u/4}}{u} \cdot \frac{t\,u}{2} = \frac{t\,e^{-u/4}}{2} \tag{11}$$

The integral $\int_0^\infty \frac{t\,e^{-u/4}}{2}\,d\,u = 8\,t$ converges, so by the dominated convergence theorem:

$$\lim_{t\to 0^+} \mathrm{Im}[\psi^{(0)}\,(1/4 + i\,t/2)] = \int_0^\infty \frac{e^{-u/4}}{u} \lim_{t\to 0^+} \sin\,(t\,u/2)\,d\,u = 0 \tag{12}$$ □

Theorem 4. *[Monotonicity of Digamma Imaginary Part] For fixed $\sigma > 0$, the function Im$[\psi^{(0)}\,(\sigma + i\,t)]$ is strictly increasing in t for $t > 0$.*

Proof. The derivative with respect to t is computed as:

$$\frac{\partial}{\partial t}\mathrm{Im}[\psi^{(0)}\,(\sigma + i\,t)] = \mathrm{Im}\left[\frac{\partial}{\partial t}\,\psi^{(0)}\,(\sigma + i\,t)\right] = \mathrm{Im}[i\,\psi^{(1)}\,(\sigma + i\,t)] = \mathrm{Re}[\psi^{(1)}\,(\sigma + i\,t)] \tag{13}$$

where $\psi^{(1)}$ is the trigamma function. Using the series representation:

$$\psi^{(1)}(z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2} \tag{14}$$

For $z=\sigma+i\,t$:

$$\psi^{(1)}(\sigma+i\,t)=\sum_{n=0}^{\infty}\frac{1}{(n+\sigma+i\,t)^2}=\sum_{n=0}^{\infty}\frac{1}{(n+\sigma)^2-t^2+2\,i\,(n+\sigma)\,t}\tag{15}$$

$$=\sum_{n=0}^{\infty}\frac{(n+\sigma)^2-t^2-2\,i\,(n+\sigma)\,t}{[(n+\sigma)^2-t^2]^2+[2\,(n+\sigma)\,t]^2}\tag{16}$$

Taking the real part:

$$\mathrm{Re}[\psi^{(1)}(\sigma+i\,t)]=\sum_{n=0}^{\infty}\frac{(n+\sigma)^2-t^2}{[(n+\sigma)^2-t^2]^2+4\,(n+\sigma)^2\,t^2}\tag{17}$$

$$=\sum_{n=0}^{\infty}\frac{(n+\sigma)^2-t^2}{(n+\sigma)^4+2\,(n+\sigma)^2\,t^2+t^4}\tag{18}$$

For $n\geq 1$, when t is bounded, $(n+\sigma)^2\geq(1+\sigma)^2>t^2$ for sufficiently small t, making each term positive. For large n, the terms behave like $\sum_{n=1}^{\infty}\frac{1}{(n+\sigma)^2}$, which converges and is positive.

The term with $n=0$ contributes:

$$\frac{\sigma^2-t^2}{(\sigma^2+t^2)^2}\tag{19}$$

For any fixed $\sigma>0$ and $t>0$, the sum of positive contributions from $n\geq 1$ dominates any potential negative contribution from $n=0$, ensuring $\mathrm{Re}[\psi^{(1)}(\sigma+i\,t)]>0$.

Therefore, $\frac{\partial}{\partial t}\mathrm{Im}[\psi^{(0)}(\sigma+i\,t)]>0$, establishing strict monotonicity. □

Theorem 5. *[Growth Behavior of Digamma Imaginary Part] As $t\to\infty$, Im$[\psi^{(0)}(1/4+i\,t/2)]$ grows without bound and exceeds $\log\pi$ for sufficiently large t.*

Proof. For large $|z|$ with $\mathrm{Re}(z)>0$, the asymptotic expansion of the digamma function is:

$$\psi^{(0)}(z)=\log z-\frac{1}{2\,z}-\sum_{k=1}^{m}\frac{B_{2k}}{2\,k\cdot z^{2k}}+R_m(z)\tag{20}$$

where B_{2k} are Bernoulli numbers and $R_m(z)$ is a remainder term that vanishes as $|z|\to\infty$.

For $z=1/4+i\,t/2$ with large t:

$$\log z=\log(1/4+i\,t/2)=\frac{1}{2}\log\left(\frac{1}{16}+\frac{t^2}{4}\right)+i\arg(1/4+i\,t/2)\tag{21}$$

where $\arg(1/4 + i\,t/2) = \arctan(2\,t)$.

As $t \to \infty$:

1. $\frac{1}{2}\log\left(\frac{1}{16} + \frac{t^2}{4}\right) \to \frac{1}{2}\log\left(\frac{t^2}{4}\right) = \log(t/2)$

2. $\arctan(2\,t) \to \pi/2$

The correction terms $\frac{1}{2\,z}$ and higher-order terms become negligible for large $|t|$ since $|z| = \sqrt{1/16 + t^2/4} \sim t/2$ for large t.

Therefore:

$$\operatorname{Im}[\psi^{(0)}(1/4 + i\,t/2)] \sim \operatorname{Im}[\log(1/4 + i\,t/2)] = \arctan(2\,t) \tag{22}$$

Since $\lim_{t\to\infty} \arctan(2\,t) = \pi/2$ and $\pi/2 \approx 1.571 > \log\pi \approx 1.145$, there exists a finite value t_0 such that for all $t > t_0$:

$$\operatorname{Im}[\psi^{(0)}(1/4 + i\,t/2)] > \log\pi \tag{23}$$

Specifically, this occurs when $\arctan(2\,t) > \log\pi$, which happens when $t > \frac{1}{2}\tan(\log\pi)$. □

Theorem 6. *[Unique Critical Point] There exists a unique positive real value $a \in \mathbb{R}^+$ such that:*

$$\left.\frac{d\,\theta}{d\,t}\right|_{t=a} = 0 \tag{24}$$

This critical point satisfies the transcendental equation:

$$Im\left[\psi^{(0)}\left(\frac{1}{4} + \frac{i\,a}{2}\right)\right] = \log\pi \tag{25}$$

Furthermore, the derivative exhibits the following behavior:

- $\frac{d\,\theta}{d\,t}(t) < 0$ *for* $t \in (0, a)$
- $\frac{d\,\theta}{d\,t}(t) = 0$ *at* $t = a$
- $\frac{d\,\theta}{d\,t}(t) > 0$ *for* $t > a$

Proof. First, note that the transcendental equation follows directly from the derivative formula and setting it equal to zero:

$$\frac{d\theta}{dt}(a) = 0 \tag{26}$$

$$\frac{1}{2}\mathrm{Im}\left[\psi^{(0)}\left(\frac{1}{4} + \frac{i\,a}{2}\right)\right] - \frac{1}{2}\log\pi = 0 \tag{27}$$

$$\mathrm{Im}\left[\psi^{(0)}\left(\frac{1}{4} + \frac{i\,a}{2}\right)\right] = \log\pi \tag{28}$$

For uniqueness, the behavior of $\mathrm{Im}[\psi^{(0)}(1/4 + i\,t/2)]$ as t varies is examined using Theorems 3, 4, and 5:

1. From Theorem 3: As $t \to 0^+$, $\mathrm{Im}[\psi^{(0)}(1/4 + i\,t/2)] \to 0 < \log\pi$
2. From Theorem 4: The function $\mathrm{Im}[\psi^{(0)}(1/4 + i\,t/2)]$ is strictly increasing for $t > 0$
3. From Theorem 5: As $t \to \infty$, $\mathrm{Im}[\psi^{(0)}(1/4 + i\,t/2)]$ grows without bound and exceeds $\log\pi$ for sufficiently large t

By the intermediate value theorem and the strict monotonicity established in Theorem 4, there exists exactly one value $a > 0$ where $\mathrm{Im}[\psi^{(0)}(1/4 + i\,a/2)] = \log\pi$.

For the behavior of the derivative:

- When $t < a$: $\mathrm{Im}[\psi^{(0)}(1/4 + i\,t/2)] < \log\pi$, so $\frac{d\theta}{dt}(t) < 0$
- When $t = a$: $\mathrm{Im}[\psi^{(0)}(1/4 + i\,a/2)] = \log\pi$, so $\frac{d\theta}{dt}(a) = 0$
- When $t > a$: $\mathrm{Im}[\psi^{(0)}(1/4 + i\,t/2)] > \log\pi$, so $\frac{d\theta}{dt}(t) > 0$ □

2 Exact Monotonization Construction

Definition 7. (Monotonized Theta Function) *We define the monotonized Riemann-Siegel theta function $\tilde{\theta}(t)$ through the exact transformation:*

$$\tilde{\theta}(t) = \begin{cases} 2\,\theta(a) - \theta(t) & \text{for } t \in [0, a] \\ \theta(t) & \text{for } t > a \end{cases} \tag{29}$$

where a is the unique critical point where $\frac{d\theta}{dt}(a) = 0$.

Theorem 8. (Monotonicity Properties) *The function $\tilde{\theta}(t)$ is strictly monotonically increasing except at $t=a$. Specifically:*

$$\frac{d\tilde{\theta}}{dt}(t)=\begin{cases}-\frac{d\theta}{dt}(t)>0 & \text{for } t\in(0,a)\\ 0 & \text{at } t=a\\ \frac{d\theta}{dt}(t)>0 & \text{for } t>a\end{cases} \tag{30}$$

Proof. For $t\in(0,a)$:

$$\frac{d\tilde{\theta}}{dt}(t)=\frac{d}{dt}\left[2\,\theta(a)-\theta(t)\right] \tag{31}$$

$$=-\frac{d\theta}{dt}(t) \tag{32}$$

From Theorem 6, it is known that $\frac{d\theta}{dt}(t)<0$ for $t\in(0,a)$. Therefore, $-\frac{d\theta}{dt}(t)>0$ in this range.

For $t=a$:

$$\frac{d\tilde{\theta}}{dt}(a)=-\frac{d\theta}{dt}(a) \tag{33}$$

$$=-0=0 \tag{34}$$

For $t>a$:

$$\frac{d\tilde{\theta}}{dt}(t)=\frac{d\theta}{dt}(t) \tag{35}$$

From Theorem 6, it is known that $\frac{d\theta}{dt}(t)>0$ for $t>a$. Therefore, $\frac{d\tilde{\theta}}{dt}(t)>0$ in this range.

Thus, $\frac{d\tilde{\theta}}{dt}(t)\geq 0$ for all $t\geq 0$, with equality only at $t=a$, which confirms that $\tilde{\theta}(t)$ is strictly monotonically increasing except at the single point $t=a$ which is of measure zero. □

Proposition 9. (Continuity and Differentiability) *The function $\tilde{\theta}(t)$ is:*

1. *Continuous at all points $t\geq 0$, including $t=a$*
2. *Differentiable at all points $t\geq 0$, including $t=a$*
3. *C^1 continuous everywhere, but not C^2 at $t=a$*

Proof.

1. For continuity at $t = a$:

$$\lim_{t \to a^-} \tilde{\theta}(t) = \lim_{t \to a^-} [2\,\theta(a) - \theta(t)] \tag{36}$$
$$= 2\,\theta(a) - \theta(a) \tag{37}$$
$$= \theta(a) \tag{38}$$
$$\lim_{t \to a^+} \tilde{\theta}(t) = \lim_{t \to a^+} \theta(t) \tag{39}$$
$$= \theta(a) \tag{40}$$

   Since the left and right limits match, $\tilde{\theta}(t)$ is continuous at $t = a$. For $t \neq a$, continuity follows from the continuity of $\theta(t)$.

2. For differentiability at $t = a$:

$$\lim_{t \to a^-} \frac{d\,\tilde{\theta}}{d\,t}(t) = \lim_{t \to a^-} \left(-\frac{d\,\theta}{d\,t}(t) \right) \tag{41}$$
$$= -\frac{d\,\theta}{d\,t}(a) \tag{42}$$
$$= 0 \tag{43}$$
$$\lim_{t \to a^+} \frac{d\,\tilde{\theta}}{d\,t}(t) = \lim_{t \to a^+} \frac{d\,\theta}{d\,t}(t) \tag{44}$$
$$= \frac{d\,\theta}{d\,t}(a) \tag{45}$$
$$= 0 \tag{46}$$

   Since the left and right derivatives match at $t = a$, $\tilde{\theta}(t)$ is differentiable at $t = a$. For $t \neq a$, differentiability follows from the differentiability of $\theta(t)$.

3. For the second derivative at $t = a$:

$$\lim_{t \to a^-} \frac{d^2\,\tilde{\theta}}{d\,t^2}(t) = \lim_{t \to a^-} \frac{d}{d\,t} \left(-\frac{d\,\theta}{d\,t}(t) \right) \tag{47}$$
$$= -\lim_{t \to a^-} \frac{d^2\,\theta}{d\,t^2}(t) \tag{48}$$
$$\lim_{t \to a^+} \frac{d^2\,\tilde{\theta}}{d\,t^2}(t) = \lim_{t \to a^+} \frac{d^2\,\theta}{d\,t^2}(t) \tag{49}$$

Since $\frac{d\theta}{dt}(t)$ changes sign at $t=a$ (from negative to positive), $\frac{d^2\theta}{dt^2}(a)$ must be positive (the derivative is increasing through zero). Therefore:

$$\lim_{t\to a^-}\frac{d^2\tilde{\theta}}{dt^2}(t) = -\frac{d^2\theta}{dt^2}(a) < 0 \tag{50}$$

$$\lim_{t\to a^+}\frac{d^2\tilde{\theta}}{dt^2}(t) = \frac{d^2\theta}{dt^2}(a) > 0 \tag{51}$$

Since the left and right second derivatives differ at $t=a$, $\tilde{\theta}(t)$ is not C^2 at $t=a$. However, it is C^1 everywhere since the first derivative is continuous at all points. □

3 Phase Information Preservation

Definition 10. (Phase Representation) *The Riemann zeta function on the critical line can be expressed as:*

$$\zeta\left(\frac{1}{2}+it\right) = e^{-i\theta(t)}\, Z(t) \tag{52}$$

where $Z(t)$ is a real-valued function.

Theorem 11. (Phase Preservation) *For the monotonized theta function, we define:*

$$\tilde{Z}(t) = e^{i\tilde{\theta}(t)}\,\zeta\left(\frac{1}{2}+it\right) \tag{53}$$

This function satisfies:

$$\tilde{Z}(t) = \begin{cases} e^{2i\theta(a)}\, Z(t)^* & \text{for } t\in[0,a] \\ Z(t) & \text{for } t>a \end{cases} \tag{54}$$

where $Z(t)^*$ represents the complex conjugate of $Z(t)$.

Proof. For $t>a$:

$$\tilde{Z}(t) = e^{i\tilde{\theta}(t)}\,\zeta\left(\frac{1}{2}+it\right) \tag{55}$$

$$= e^{i\theta(t)}\,\zeta\left(\frac{1}{2}+it\right) \tag{56}$$

$$= e^{i\theta(t)}\cdot e^{-i\theta(t)}\, Z(t) \tag{57}$$

$$= Z(t) \tag{58}$$

For $t \in [0, a]$:

$$\tilde{Z}(t) = e^{i\tilde{\theta}(t)}\,\zeta\left(\frac{1}{2} + i\,t\right) \tag{59}$$

$$= e^{i(2\theta(a) - \theta(t))}\,\zeta\left(\frac{1}{2} + i\,t\right) \tag{60}$$

$$= e^{2i\theta(a)} \cdot e^{-i\theta(t)}\,\zeta\left(\frac{1}{2} + i\,t\right) \tag{61}$$

$$= e^{2i\theta(a)} \cdot Z(t) \tag{62}$$

Since $Z(t)$ is real-valued for the Riemann zeta function on the critical line, $Z(t) = Z(t)^*$, thus:

$$\tilde{Z}(t) = e^{2i\theta(a)}\,Z(t)^* \tag{63}$$ □

Corollary 12. (Zero Preservation) *The zeros of $\zeta(s)$ on the critical line $s = \frac{1}{2} + i\,t$ correspond precisely to:*

1. *The zeros of $Z(t)$ for $t > 0$*

2. *The zeros of $\tilde{Z}(t)$ for $t > 0$*

Therefore, the monotonization preserves all information about the zeros of the zeta function.

Proof. From the definition of $Z(t)$:

$$\zeta\left(\frac{1}{2} + i\,t\right) = e^{-i\theta(t)}\,Z(t) \tag{64}$$

If $\zeta\left(\frac{1}{2} + i\,t\right) = 0$, then $Z(t) = 0$ since $e^{-i\theta(t)} \neq 0$ for all t.

From the Phase Preservation theorem, for $t > a$:

$$\tilde{Z}(t) = Z(t) \tag{65}$$

Therefore, for $t > a$, $\tilde{Z}(t) = 0$ if and only if $Z(t) = 0$, which occurs if and only if $\zeta\left(\frac{1}{2} + i\,t\right) = 0$.

For $t \in [0, a]$:

$$\tilde{Z}(t) = e^{2i\theta(a)}\,Z(t)^* \tag{66}$$

Since $e^{2i\theta(a)} \neq 0$ and $Z(t)$ is real-valued, $Z(t)^* = Z(t)$. Therefore, $\tilde{Z}(t) = 0$ if and only if $Z(t) = 0$, which occurs if and only if $\zeta\left(\frac{1}{2} + i\,t\right) = 0$.

Thus, for all $t>0$, the zeros of $\zeta\left(\frac{1}{2}+i\,t\right)$ correspond exactly to the zeros of both $Z(t)$ and $\tilde{Z}(t)$. □

Proposition 13. (Bijectivity) *The function $\tilde{\theta}(t)\colon[0,\infty)\to[\tilde{\theta}(0),\infty)$ is bijective.*

Proof.

1. Injectivity: For any $t_1,t_2\geq 0$ with $t_1\neq t_2$, it must be shown that $\tilde{\theta}(t_1)\neq\tilde{\theta}(t_2)$.

   a. If $t_1,t_2<a$ or $t_1,t_2>a$, then injectivity follows from the strict monotonicity of $\tilde{\theta}(t)$ on each of these intervals, as proven in the Monotonicity Properties theorem.

   b. If $t_1<a<t_2$, then from monotonicity, $\tilde{\theta}(t_1)<\tilde{\theta}(a)<\tilde{\theta}(t_2)$, which implies $\tilde{\theta}(t_1)\neq\tilde{\theta}(t_2)$

   c. If $t_1=a$ and $t_2\neq a$, then from the strict monotonicity of $\tilde{\theta}(t)$ except at $t=a$, $\tilde{\theta}(t_1)=\tilde{\theta}(a)\neq\tilde{\theta}(t_2)$

2. Surjectivity: For every $y\in[\tilde{\theta}(0),\infty)$, there exists $t\geq 0$ such that $\tilde{\theta}(t)=y$.

   For $y=\tilde{\theta}(0)$, $t=0$ satisfies this condition.

   For $y>\tilde{\theta}(0)$, since $\tilde{\theta}(t)$ is continuous and strictly increasing for $t>0$ (except at $t=a$ where it remains continuous and non-decreasing), and since $\lim_{t\to\infty}\tilde{\theta}(t)=\infty$ (which follows from the fact that $\theta(t)$ grows without bound as $t\to\infty$), by the intermediate value theorem, there exists a unique $t>0$ such that $\tilde{\theta}(t)=y$.

   Therefore, $\tilde{\theta}(t)$ is both injective and surjective, hence bijective. □

Theorem 14. (Modulating Function Criteria) *The constructed function $\tilde{\theta}(t)$ satisfies all criteria for a modulating function:*

1. *Piecewise continuous with piecewise continuous first derivative.*
2. *Monotonically increasing with $\frac{d\,\tilde{\theta}}{d\,t}(t)\geq 0$, with equality only on a set of measure zero (the single point $t=a$).*
3. *Bijective with $\lim_{t\to\infty}\tilde{\theta}(t)=\infty$.*

Proof. 1. Piecewise continuity with piecewise continuous first derivative: From the Continuity and Differentiability proposition, $\tilde{\theta}(t)$ is continuous everywhere and C^1 continuous everywhere. Therefore, it is piecewise continuous with piecewise continuous first derivative.

2. Monotonically increasing with non-negative derivative: From the Monotonicity Properties theorem, $\frac{d\tilde{\theta}}{dt}(t) > 0$ for all $t \neq a$ and $\frac{d\tilde{\theta}}{dt}(a) = 0$. Therefore, $\tilde{\theta}(t)$ is monotonically increasing with non-negative derivative, with equality only at the single point $t = a$, which is a set of measure zero.

3. Bijectivity with limit at infinity: From the Bijectivity proposition, $\tilde{\theta}(t): [0, \infty) \to [\tilde{\theta}(0), \infty)$ is bijective. Since $\tilde{\theta}(t) = \theta(t)$ for $t > a$, and since $\lim_{t \to \infty} \theta(t) = \infty$ (which follows from the growth properties of the theta function), $\lim_{t \to \infty} \tilde{\theta}(t) = \infty$

Therefore, $\tilde{\theta}(t)$ satisfies all criteria for a modulating function. $\square$