Structure and Properties of Gaussian Kernels in Gaussian Processes Under Monotonic Modulation

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Definition 1

The standard Gaussian kernel $\exp(-(t-s)^2)$ *admits modulation through strictly monotonic functions. Let* $\mathcal{F}$ *denote the class of functions* $f\colon \mathbb{R}^+ \to \mathbb{R}$ *satisfying:*

1. $f \in C^2(\mathbb{R}^+)$
2. f *is strictly monotonically increasing*
3. $f(0) = 0$ *(since* $K(t,t) = \exp(-(f(t) - f(t))^2) = 1$ *for all* t*)*
4. f' *is locally bounded and measurable*
5. $\lim_{t\to\infty} f'(t)$ *exists (finite or infinite)*

Theorem 2

[Mean Zero-Crossing Rate] For any $f \in \mathcal{F}$*, the stationary Gaussian process with modulated kernel*

$$K(s,t) = \exp(-(f(t) - f(s))^2)$$

has mean zero-crossing rate:

$$\mathbb{E}[N([0,T])] = f(T)$$

where $N([0,T])$ *denotes the counting measure of zeros in* $[0,T]$*.*

Proof. By the Kac-Rice formula:

$$\mathbb{E}[N([0,T])] = \int_0^T \sqrt{-\lim_{s\to t} \frac{\partial^2}{\partial t\,\partial s} K(s,t)}\, dt$$

Computing the derivatives:

$$\frac{\partial}{\partial s} K(s,t) = 2\,(f(t) - f(s))\, f'(s) \exp(-(f(t) - f(s))^2)$$

$$\begin{aligned}\frac{\partial^2}{\partial t\,\partial s} K(s,t) &= -2\, f'(t)\, f'(s) \exp(-(f(t) - f(s))^2)\\ &\quad +2\,(f(t) - f(s))^2\, f'(t)\, f'(s) \exp(-(f(t) - f(s))^2)\end{aligned}$$

Taking the limit as $s \to t$:

$$\lim_{s \to t} \frac{\partial^2}{\partial t\,\partial s} K(s,t) = -(f'(t))^2$$

Therefore:

$$\mathbb{E}[N([0,T])] = \int_0^T |f'(t)|\, dt = f(T)$$ □

Theorem 3

[Eigenfunction Structure] For the modulated Gaussian kernel $K(s,t) = \exp(-(f(t) - f(s))^2)$ where $f \in \mathcal{F}$, the eigenfunctions take the form:

$$\phi_n(t) = c_n\, H_n(f(t)) \exp(-f(t)^2/2) \sqrt{f'(t)}$$

where H_n are the Hermite polynomials, c_n are normalization constants.

Proof. The eigenfunction equation for kernel K is:

$$\int_0^\infty \exp(-(f(t) - f(s))^2)\, \phi_n(s)\, ds = \lambda_n\, \phi_n(t)$$

Under the change of variables $u = f(s)$, $v = f(t)$:

$$\int_0^\infty \exp(-(v-u)^2)\, \phi_n(f^{-1}(u)) \frac{du}{f'(f^{-1}(u))} = \lambda_n\, \phi_n(f^{-1}(v))$$

Let $\psi_n(u) = \phi_n(f^{-1}(u)) \frac{1}{\sqrt{f'(f^{-1}(u))}}$. Then:

$$\int_0^\infty \exp(-(v-u)^2)\, \psi_n(u) \sqrt{f'(f^{-1}(u))}\, du = \lambda_n\, \psi_n(v) \sqrt{f'(f^{-1}(v))}$$

This reduces to the standard Gaussian kernel eigenfunction equation:

$$\int_0^\infty \exp\left(-(v-u)^2\right) \psi_n(u)\, d\,u = \lambda_n \psi_n(v)$$

Therefore $\psi_n(u) = H_n(u) \exp\left(-u^2/2\right)$, giving:

$$\phi_n(t) = c_n H_n(f(t)) \exp\left(-f(t)^2/2\right) \sqrt{f'(t)} \qquad \square$$

Corollary 4

[Eigenvalues and Normalization Constants] The eigenvalues of the modulated Gaussian kernel are:

$$\lambda_n = \sqrt{\pi}\, 2^{-n}$$

and the normalization constants c_n are given by:

$$c_n = \frac{1}{\sqrt{n! 2^n \sqrt{\pi}}}$$

Proof. The eigenvalues remain unchanged under our transformation since:

$$\int_0^\infty \exp\left(-(v-u)^2\right) \psi_n(u)\, d\,u = \lambda_n \psi_n(v)$$

is equivalent to the standard Gaussian kernel case where $\lambda_n = \sqrt{\pi}\, 2^{-n}$.

For the normalization constants, we require:

$$\int_0^\infty \phi_n(t)^2\, d\,t = 1$$

Substituting the form of ϕ_n:

$$\int_0^\infty c_n^2 H_n(f(t))^2 \exp\left(-f(t)^2\right) f'(t)\, d\,t = 1$$

Under the change of variables $u = f(t)$:

$$c_n^2 \int_0^\infty H_n(u)^2 \exp\left(-u^2\right) d\,u = 1$$

Using the known normalization of Hermite polynomials:

$$\int_{-\infty}^{\infty} H_n(u)^2 \exp(-u^2)\, d\, u = n! 2^n \sqrt{\pi}$$

Therefore:

$$c_n = \frac{1}{\sqrt{n! 2^n \sqrt{\pi}}} \qquad \square$$

Remark 5. The invariance of eigenvalues under modulation is a profound connection to linear algebra. Just as eigenvalues are invariant under similarity transformations $P^{-1}AP$, our modulation acts as a "continuous change of basis". The transformation operator

$$(T\phi)(t) = \sqrt{f'(t)}\, \phi(f(t))$$

makes this explicit, as our kernel transformation is essentially a conjugation:

$$K_f = T^{-1} K_{standard} T$$

The $\sqrt{f'(t)}$ terms act like the Jacobian of this transformation, perfectly balancing the change of measure to preserve spectral properties. While eigenfunctions transform (like eigenvectors under change of basis), the eigenvalues remain invariant - a beautiful infinite-dimensional analog of the basis-independence principle from linear algebra.