1 Translation-Invariant (Stationary) Gaussian Processes

Proof. Let $R(s,t)$ be a positive definite kernel on $[0,\infty)$ and $\{\phi_k(t)\}_{k=1}^{\infty}$ be an orthonormal basis for $L^2[0,\infty)$. Define the matrix K with elements:

$$K_{ij} = \int_0^{\infty} \int_0^{\infty} R(s,t)\, \phi_i(s)\, \phi_j(t)\ d\,s\ d\,t \tag{1}$$

Consider the eigenvalue problem:

$$K\mathbf{c}_n = \lambda_n \mathbf{c}_n \tag{2}$$

where $\mathbf{c}_n = (c_{n,1}, c_{n,2}, \ldots)^T$ is the eigenvector corresponding to eigenvalue λ_n. Now, let

$$\psi_n(t) = \sum_{k=1}^{\infty} c_{n,k}\, \phi_k(t) \tag{3}$$

We will show that $\psi_n(t)$ is an eigenfunction of the integral operator with kernel $R(s,t)$.

$$\begin{aligned}
\int_0^{\infty} R(s,t)\, \psi_n(s)\ d\,s\ &= \int_0^{\infty} R(s,t) \sum_{k=1}^{\infty} c_{n,k}\, \phi_k(s)\ d\,s \\
&= \sum_{k=1}^{\infty} c_{n,k} \int_0^{\infty} R(s,t)\ \phi_k(s)\ d\,s \\
&= \sum_{k=1}^{\infty} c_{n,k} \sum_{j=1}^{\infty} \left(\int_0^{\infty} R(s,t)\ \phi_k(s)\ \phi_j(t)\ d\,s \right) \phi_j(t) \\
&= \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} K_{jk}\, c_{n,k} \right) \phi_j(t) \\
&= \sum_{j=1}^{\infty} \lambda_n\, c_{n,j}\, \phi_j(t) \\
&= \lambda_n\, \psi_n(t)
\end{aligned} \tag{4}$$

Thus, $\psi_n(t)$ is indeed an eigenfunction of the integral operator with eigenvalue λ_n, and the coefficients $c_{n,k}$ satisfy:

$$c_{n,k} = \frac{\sum_{j=1}^{\infty} K_{kj}\, c_{n,j}}{\lambda_n} \tag{5}$$

This proves that the expansion

$$\psi_n(t) = \sum_{k=1}^{\infty} c_{n,k}\, \phi_k(t) \tag{6}$$

holds for any positive definite kernel $R(s,t)$ and any orthonormal basis $\{\phi_k(t)\}_{k=1}^{\infty}$ on $[0, \infty)$, with the coefficients $c_{n,k}$ determined by the eigenvalue problem of the matrix K. □

Proof. We start with the eigenfunction equation:

$$\int_0^{\infty} R(s,t)\,\psi_n(s)\;ds = \lambda_n\,\psi_n(t) \tag{7}$$

Multiply both sides by $\phi_k(t)$ and integrate over t:

$$\int_0^{\infty} \phi_k(t) \int_0^{\infty} R(s,t)\,\psi_n(s)\;ds\;dt = \lambda_n \int_0^{\infty} \phi_k(t)\,\psi_n(t)\;dt \tag{8}$$

By Fubini's theorem, we can swap the order of integration on the left side:

$$\int_0^{\infty} \int_0^{\infty} R(s,t)\,\phi_k(t)\;dt\;\psi_n(s)\;ds = \lambda_n \int_0^{\infty} \phi_k(t)\,\psi_n(t)\;dt \tag{9}$$

Recall the definition of $\psi_n(t)$:

$$\psi_n(t) = \sum_{j=1}^{\infty} c_{n,j}\,\phi_j(t) \tag{10}$$

Substitute this into the left side of the equation from step 3:

$$\int_0^{\infty} \int_0^{\infty} R(s,t)\,\phi_k(t)\;dt\;\sum_{j=1}^{\infty} c_{n,j}\,\phi_j(s)\;ds = \lambda_n \int_0^{\infty} \phi_k(t)\,\psi_n(t)\;dt \tag{11}$$

Expand the left side:

$$\sum_{j=1}^{\infty} c_{n,j} \int_0^{\infty} \int_0^{\infty} R(s,t)\,\phi_k(t)\,\phi_j(s)\;dt\;ds = \lambda_n \int_0^{\infty} \phi_k(t)\,\psi_n(t)\;dt \tag{12}$$

Recognize the definition of K_{kj} from the original proof:

$$\sum_{j=1}^{\infty} c_{n,j}\,K_{kj} = \lambda_n \int_0^{\infty} \phi_k(t)\,\psi_n(t)\;dt \tag{13}$$

The left side is exactly $\lambda_n\,c_{n,k}$ from the eigenvalue equation $K\mathbf{c}_n = \lambda_n\mathbf{c}_n$:

$$\lambda_n\,c_{n,k} = \lambda_n \int_0^{\infty} \phi_k(t)\,\psi_n(t)\;dt \tag{14}$$

Divide both sides by λ_n:

$$c_{n,k} = \frac{\int_0^\infty \phi_k(t)\, \psi_n(t)\; d\,t}{\lambda_n} \tag{15}$$

This completes the proof. $\square$

Proof. We start with the eigenvalue equation for the matrix K:

$$K\mathbf{c}_n = \lambda_n \mathbf{c}_n \tag{16}$$

Multiply both sides by $\mathbf{c}_n^T$ from the left:

$$\mathbf{c}_n^T K\mathbf{c}_n = \lambda_n \mathbf{c}_n^T \mathbf{c}_n \tag{17}$$

Divide both sides by $\mathbf{c}_n^T \mathbf{c}_n$:

$$\lambda_n = \frac{\mathbf{c}_n^T K\mathbf{c}_n}{\mathbf{c}_n^T \mathbf{c}_n} \tag{18}$$

Now, let's expand the numerator using the definition of K:

$$\begin{aligned} \mathbf{c}_n^T K\mathbf{c}_n &= \sum_{i,j} c_{n,i}\, K_{ij}\, c_{n,j} \\ &= \sum_{i,j} c_{n,i}\, c_{n,j} \int_0^\infty \int_0^\infty R(s,t)\, \phi_i(s)\, \phi_j(t)\; d\,s\; d\,t \end{aligned} \tag{19}$$

Recall that $\psi_n(t) = \sum_{k=1}^\infty c_{n,k}\, \phi_k(t)$. Using this, we can rewrite the above as:

$$\begin{aligned} \mathbf{c}_n^T K\mathbf{c}_n &= \int_0^\infty \int_0^\infty R(s,t) \left(\sum_i c_{n,i}\, \phi_i(s) \right) \left(\sum_j c_{n,j}\, \phi_j(t) \right) d\,s\; d\,t \\ &= \int_0^\infty \int_0^\infty R(s,t)\, \psi_n(s)\, \psi_n(t)\; d\,s\; d\,t \end{aligned} \tag{20}$$

For the denominator, note that $\mathbf{c}_n^T \mathbf{c}_n = \|\psi_n\|^2 = 1$ due to normalization of eigenfunctions. Therefore, we conclude:

$$\lambda_n = \int_0^\infty \int_0^\infty R(s,t)\, \psi_n(s)\, \psi_n(t)\; d\,s\; d\,t \tag{21}$$

This completes the proof. $\square$

Therefore, (15) becomes

$$c_{n,k} = \frac{\int_0^\infty \phi_k(t)\, \psi_n(t)\; d\,t}{\int_0^\infty \int_0^\infty R(s,t)\, \psi_n(s)\, \psi_n(t)\; d\,s\; d\,t} \tag{22}$$