Factorization of Stationary Gaussian Process Kernels

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1 Theorem: Spectral Representation

Theorem 1

[Spectral Factorization Theorem]*[2] : Let $K:\mathbb{R}\to\mathbb{R}$ be a positive definite stationary kernel function.*

By Bochner's theorem, there exists a non-negative spectral density function $S:\mathbb{R}\to\mathbb{R}$ such that:

$$K(t-s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(\omega)\,e^{i\omega(t-s)}\,d\omega \tag{1}$$

Let $h:\mathbb{R}\to\mathbb{C}$ be defined as:

$$h(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{S(\omega)}\,e^{i\omega t}\,d\omega \tag{2}$$

Then:

$$K(t-s)=\int_{-\infty}^{\infty}h(t+\tau)\,\overline{h(s+\tau)}\,d\tau \tag{3}$$

2 Proof

1. Since K is positive definite and stationary, Bochner's theorem guarantees the existence of $S(\omega)\geq 0$ such that:

$$K(t-s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(\omega)\,e^{i\omega(t-s)}\,d\omega \tag{4}$$

2. Since $S(\omega)\geq 0$ by Bochner's theorem:

$$K(t-s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{S(\omega)}\,e^{i\omega t}\cdot\sqrt{S(\omega)}\,e^{-i\omega s}\,d\omega \tag{5}$$

3. Using the definition of $h(t)$ as the inverse Fourier transform of the square root of the spectral density $S(\omega)$:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega)}\, e^{i\omega t}\, d\omega \tag{6}$$

4. The Fourier transform of $h(t)$ gives:

$$\sqrt{S(\omega)} = \int_{-\infty}^{\infty} h(\tau)\, e^{-i\omega\tau}\, d\tau \tag{7}$$

5. Substituting this representation:

$$K(t-s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(\tau)\, e^{-i\omega\tau}\, d\tau \right) e^{i\omega t} \left(\int_{-\infty}^{\infty} h(\sigma)\, e^{-i\omega\sigma}\, d\sigma \right) e^{-i\omega s}\, d\omega \tag{8}$$

6. By Fubini's theorem (valid since K is PD):

$$K(t-s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau)\, h(\sigma) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-\tau-s+\sigma)}\, d\omega\, d\tau\, d\sigma \tag{9}$$

7. The inner integral yields the Dirac delta function:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-\tau-s+\sigma)}\, d\omega = \delta(t-\tau-s+\sigma) \tag{10}$$

8. Therefore:

$$K(t-s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau)\, h(\sigma)\, \delta(t-\tau-s+\sigma)\, d\tau\, d\sigma \tag{11}$$

9. Using the sifting property of δ:

$$\begin{aligned} K(t-s) &= \int_{-\infty}^{\infty} h(\tau) \int_{-\infty}^{\infty} h(\sigma)\, \delta(t-\tau-s+\sigma)\, d\sigma\, d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)\, h(t-\tau-s)\, d\tau \end{aligned} \tag{12}$$

3 Reverse Verification

Starting from the final result:

$$K(t-s) = \int_{-\infty}^{\infty} h(\tau)\, h(t-\tau-s)\, d\tau \tag{13}$$

Substituting the definition of $h(t)$:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega)}\, e^{i\omega t}\, d\omega \tag{14}$$

1) First substitution for $h(\tau)$:

$$K(t-s) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega_1)}\, e^{i\omega_1 \tau}\, d\omega_1 \right] h(t-\tau-s)\, d\tau \tag{15}$$

2) Second substitution for $h(t-\tau-s)$:

$$K(t-s) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega_1)}\, e^{i\omega_1 \tau}\, d\omega_1 \right] \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega_2)}\, e^{i\omega_2 (t-\tau-s)}\, d\omega_2 \right] d\tau \tag{16}$$

3) Rearranging the integrals:

$$K(t-s) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{S(\omega_1)}\, \sqrt{S(\omega_2)}\, e^{i\omega_2 t}\, e^{-i\omega_2 s}\, e^{i(\omega_1-\omega_2)\tau}\, d\omega_1\, d\omega_2\, d\tau \tag{17}$$

4) The integral with respect to τ yields:

$$\int_{-\infty}^{\infty} e^{i(\omega_1-\omega_2)\tau}\, d\tau = 2\pi\, \delta(\omega_1-\omega_2) \tag{18}$$

5) Applying this result:

$$K(t-s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{i\omega(t-s)}\, d\omega \tag{19}$$

This matches the original spectral representation from Bochner's theorem, confirming that:

- The substitutions were valid
- The use of Fubini's theorem was justified
- The manipulation of the Dirac delta function was correct
- The final result is consistent with the initial spectral representation

[2][1]

Bibliography

[1] Derek K. Chang and M.M. Rao. Special representations of weakly harmonizable processes. *Stochastic Analysis and Applications*, 6(2):169–189, 1988.

[2] Harald Cramér. A contribution to the theory of stochastic processes. *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 2:329–339, 1951.