The Shift Group and Fourier Transform of Complex Exponentials

Definition 1. *[Shift Operator] For $t_0 \in \mathbb{R}$, the shift operator S_{t_0} is defined by*

$$(S_{t_0} f)(t) = f(t - t_0)$$

for any function f in the appropriate function space.

Definition 2. *[Fourier Transform] For $f \in L^1(\mathbb{R})$, the Fourier transform is defined by*

$$\hat{f}(\omega) = \mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

Theorem 3. *[Complex Exponentials as Eigenfunctions of Shift Operators] Let $\omega \in \mathbb{R}$ and consider the complex exponential function $e_\omega(t) = e^{i\omega t}$. Then for any $t_0 \in \mathbb{R}$,*

$$S_{t_0}[e_\omega] = e^{-i\omega t_0} \cdot e_\omega$$

That is, e_ω is an eigenfunction of S_{t_0} with eigenvalue $e^{-i\omega t_0}$.

Proof. By definition of the shift operator:

$$\begin{aligned}
(S_{t_0} e_\omega)(t) &= e_\omega(t - t_0) && (1)\\
&= e^{i\omega(t - t_0)} && (2)\\
&= e^{i\omega t - i\omega t_0} && (3)\\
&= e^{-i\omega t_0} \cdot e^{i\omega t} && (4)\\
&= e^{-i\omega t_0} \cdot e_\omega(t) && (5)
\end{aligned}$$

Therefore, $S_{t_0}[e_\omega] = e^{-i\omega t_0} \cdot e_\omega$. □

Theorem 4. *[Time Shift Property of Fourier Transform] Let $f \in L^1(\mathbb{R})$ and $t_0 \in \mathbb{R}$. Then*

$$\mathcal{F}[S_{t_0} f](\omega) = e^{-i\omega t_0}\, \hat{f}(\omega)$$

Proof. Let $g(t) = f(t - t_0) = (S_{t_0} f)(t)$. Then:

$$\begin{aligned}\mathcal{F}[g](\omega) &= \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t}\, d t &(6)\\ &= \int_{-\infty}^{\infty} f(t - t_0)\, e^{-i\omega t}\, d t &(7)\end{aligned}$$

Substituting $u = t - t_0$, so $t = u + t_0$ and $d t = d u$:

$$\begin{aligned}\mathcal{F}[g](\omega) &= \int_{-\infty}^{\infty} f(u)\, e^{-i\omega(u + t_0)}\, d u &(8)\\ &= e^{-i\omega t_0} \int_{-\infty}^{\infty} f(u)\, e^{-i\omega u}\, d u &(9)\\ &= e^{-i\omega t_0}\, \hat{f}(\omega) &(10)\end{aligned}$$

□

Theorem 5. *[Frequency Shift Property of Fourier Transform] Let $f \in L^1(\mathbb{R})$ and $\omega_0 \in \mathbb{R}$. Then*

$$\mathcal{F}\left[e^{i\omega_0 t} f(t)\right](\omega) = \hat{f}(\omega - \omega_0)$$

Proof.

$$\begin{aligned}\mathcal{F}\left[e^{i\omega_0 t} f(t)\right](\omega) &= \int_{-\infty}^{\infty} e^{i\omega_0 t} f(t)\, e^{-i\omega t}\, d t &(11)\\ &= \int_{-\infty}^{\infty} f(t)\, e^{-i(\omega - \omega_0)t}\, d t &(12)\\ &= \hat{f}(\omega - \omega_0) &(13)\end{aligned}$$

□

Theorem 6. *[Fourier Transform of Complex Exponential] Let $\omega_0 \in \mathbb{R}$. In the distributional sense,*

$$\mathcal{F}[e^{i\omega_0 t}](\omega) = 2\pi\,\delta(\omega - \omega_0)$$

where δ is the Dirac delta distribution.

Proof. We prove this by showing that for any test function $\phi \in \mathcal{S}(\mathbb{R})$ (Schwartz space):

$$\langle \mathcal{F}[e^{i\omega_0 t}], \phi \rangle = 2\pi\,\phi(\omega_0)$$

By definition of the Fourier transform of distributions:

$$\begin{aligned}
\langle \mathcal{F}[e^{i\omega_0 t}], \phi\rangle =& \langle e^{i\omega_0 t}, \mathcal{F}[\phi]\rangle && (14)\\
=& \int_{-\infty}^{\infty} e^{i\omega_0 t}\,\hat{\phi}(t)\,d t && (15)\\
=& \int_{-\infty}^{\infty} e^{i\omega_0 t} \int_{-\infty}^{\infty} \phi(\omega)\,e^{-i\omega t}\,d\omega\,d t && (16)
\end{aligned}$$

By Fubini's theorem (valid for $\phi \in \mathcal{S}(\mathbb{R})$):

$$\begin{aligned}
=& \int_{-\infty}^{\infty} \phi(\omega) \int_{-\infty}^{\infty} e^{i(\omega_0-\omega)t}\,d t\,d\omega && (17)\\
=& \int_{-\infty}^{\infty} \phi(\omega)\cdot 2\pi\,\delta\,(\omega_0-\omega)\,d\omega && (18)\\
=& 2\pi\,\phi(\omega_0) && (19)\\
=& \langle 2\pi\,\delta\,(\omega-\omega_0), \phi\rangle && (20)
\end{aligned}$$

Therefore, $\mathcal{F}[e^{i\omega_0 t}] = 2\pi\,\delta\,(\omega-\omega_0)$. □

Theorem 7. *[Diagonalization Property] Let T be a bounded linear operator on $L^2(\mathbb{R})$ that commutes with all shift operators, i.e., $T S_{t_0} = S_{t_0} T$ for all $t_0 \in \mathbb{R}$. Then T is diagonalized by the Fourier transform in the sense that there exists a function $m(\omega)$ such that*

$$\mathcal{F}\,[T f] = m\cdot\mathcal{F}[f]$$

for all f in the domain of T.

Proof. Since T commutes with all shift operators, by Theorem 1, the complex exponentials $e^{i\omega t}$ are eigenfunctions of T. Let $T e^{i\omega t} = m(\omega)\,e^{i\omega t}$ for some function $m(\omega)$.

For any $f \in L^2(\mathbb{R})$ with $\hat{f} \in L^2(\mathbb{R})$, we can write (by the inverse Fourier transform):

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\,e^{i\omega t}\,d\omega$$

Applying T and using linearity:

$$\begin{aligned}
T f(t) =& T\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\,e^{i\omega t}\,d\omega\right] && (21)\\
=& \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\,T[e^{i\omega t}]\,d\omega && (22)\\
=& \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\,m(\omega)\,e^{i\omega t}\,d\omega && (23)
\end{aligned}$$

Taking the Fourier transform:

$$\mathcal{F}[Tf](\omega) = m(\omega)\,\hat{f}(\omega) = m(\omega) \cdot \mathcal{F}[f](\omega) \qquad \square$$

Corollary 8. *The shift operators and multiplication by complex exponentials are the fundamental operations that generate all translation-invariant linear systems, and the Fourier transform provides the natural basis that simultaneously diagonalizes all such systems.*