Envelopes and Pre-Envelopes of Real Waveforms

BY J. DUGUNDJI

University of Southern California, Los Angeles, Calif.

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Abstract

Rice's formula[rice1944] for the "envelope" of a given signal is very cumbersome; in any case where the signal is not a single sine wave, the analytical use and explicit calculation of the envelope is practically prohibitive. A different formula for the envelope is given herein which is much simpler and easier to handle analytically. We show precisely that if $\hat{u}(t)$ is the Hilbert transform of $u(t)$, then Rice's envelope of $u(t)$ is the absolute value of the complex-valued function $u(t) + i\,\hat{u}(t)$. The function $u + i\,\hat{u}$ is called the pre-envelope of u and is shown to be involved implicitly in some other usual engineering practices.

The Hilbert transform $\hat{u}$ is then studied; it is shown that $\hat{u}$ has the same power spectrum as u and is uncorrelated with u at the same time instant. Further, the autocorrelation of the pre-envelope of u is twice the pre-envelope of the autocorrelation of u.

By using the pre-envelope, the envelope of the output of a linear filter is easily calculated, and this is used to compute the first probability density for the envelope of the output of an arbitrary linear filter when the input is an arbitrary signal plus Gaussian noise. An application of pre-envelopes to the frequency modulation of an arbitrary waveform by another arbitrary waveform is also given.

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1 Introduction

We recall Rice's formulation[rice1944] of the envelope of a multichromatic waveform $u(t)$. He starts by writing $u(t)$ in the form

$$u(t) = \sum_n c_n \cos\left(\omega_n t + \varphi_n\right) \tag{1}$$

and then selects a frequency q called the "midband frequency." Using this selected frequency, he writes

$$\begin{aligned} u(t) =& \sum_n c_n \cos\left[(\omega_n - q)\, t + \varphi_n + q\, t\right] & (2) \\ =& I_c \cos q\, t - I_s \sin q\, t & (3) \end{aligned}$$

where

$$\begin{aligned} I_c =& \sum_n c_n \cos\left[(\omega_n - q)\, t + \varphi_n\right] & (4) \\ I_s =& \sum_n c_n \sin\left[(\omega_n - q)\, t + \varphi_n\right] & (5) \end{aligned}$$

The expression

$$R(t) = [I_c^2 + I_s^2]^{1/2} \tag{6}$$

is termed by Rice the "envelope of $u(t)$ referred to frequency q."

This formulation has several defects. First, to obtain the envelope, one must expand the given multichromatic waveform in the form above. Second, one must select a "midband frequency" q; it is not immediately evident whether or not a choice $q' \neq q$ leads to the same $R(t)$. Finally, the explicit calculation of $R(t)$ is a formidable task.

In this paper, we intend to give an equivalent, but more direct formulation of the concept "envelope." This is done by introducing the idea of the pre-envelope of a given real waveform, which is a complex-valued function whose absolute value turns out to be exactly the envelope of the given waveform. Applications of the pre-envelope are given: 1) to show that $R(t)$ depends only on the given waveform and not on the seemingly extraneous concept of "midband frequency," 2) to give direct methods for obtaining the envelope of the output of a linear filter when the input is an arbitrary waveform, 3) to calculate explicitly the first probability density for the envelope of the output of an arbitrary linear filter when the input is an arbitrary signal plus Gaussian noise, and 4) to give a definition of frequency modulation when both the waveform $u(t)$ and the modulating function $m(t)$ are arbitrary.

The paper is divided into six parts. Section 2 contains the essential facts about Hilbert transforms that are used; proofs will be found in Titchmarsh[titchmarsh1937]. In Section 3 the definitions of pre-envelope and envelope are given, and the equivalence of our envelope with that of Rice is established. Section 4 contains further properties of pre-envelopes and, in particular, shows how one is naturally led to their consideration. In Sections 5–7 the applications cited above are given.

We remark that Hilbert transforms have been used in electrical engineering before, notably by Gabor[gabor1946], Woodward[woodward1953], and Ville[ville1948]; the concept "pre-envelope" is in fact identical with the "signal analytique" of Ville, but our use and purpose differ from his.

2 Hilbert Transforms

Given a real-valued function $u(t)$ on $-\infty < t < +\infty$, its Hilbert transform $\hat{u}(t)$ is defined by

$$\hat{u}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(\xi)}{t - \xi} \, d\xi \tag{7}$$

where the principal value of the integral is always used. All functions considered here will be assumed to have Hilbert transforms.

Proposition 1. *The Hilbert transform of* $\cos(\omega t + \varphi)$ *is* $\sin(\omega t + \varphi)$.

It is proved in Titchmarsh[titchmarsh1937] that, under suitable conditions, $\hat{\hat{u}} = -u$.

Proposition 2. *The Hilbert transform of a Hilbert transform is the negative of the original function.*

Proposition 3. *If* $U(f) = \int_{-\infty}^{\infty} u(t) \exp(-2\pi i f t) \, dt$ *is the Fourier transform of* $u(t)$, *then the Fourier transform* $\hat{U}(f)$ *of* $\hat{u}(t)$ *is*

$$\hat{U}(f) = \begin{cases} -i\,U(f) & f > 0 \\ 0 & f = 0 \\ +i\,U(f) & f < 0 \end{cases} \tag{8}$$

Proposition 4. *The convolution*

$$w(t) = v(t) * u(t) = \int_{-\infty}^{\infty} v(\xi)\, u(t - \xi) \, d\xi \tag{9}$$

has Hilbert transform

$$\hat{w}(t) = v(t) * \hat{u}(t) \tag{10}$$

3 Definitions and Main Results

Definition 5. *Let $u(t)$ be a real waveform. The pre-envelope $z(t)$ of $u(t)$ is the complex-valued function*

$$z(t) = u(t) + i\,\hat{u}(t) \tag{11}$$

The envelope of $u(t)$ is the absolute value $|z(t)|$ of its pre-envelope.

Theorem 6. *This definition of envelope gives the same result as that of Rice, whenever Rice's definition is applicable.*

Proof. With the notations of Section ?, we start by writing $u(t)$ in the form of equation (1). Forming the pre-envelope $z(t)$ of $u(t)$ by using Proposition 1, one gets

$$z(t) = \sum_n c_n \cos\left(\omega_n t + \varphi_n\right) + i \sum_n c_n \sin\left(\omega_n t + \varphi_n\right) \tag{12}$$

Referring to the frequency q and noting that

$$\sum_n \sin\left[\left(\omega_n - q\right) t + \varphi_n + q\,t\right] = I_c \cos q\,t + I_s \sin q\,t \tag{13}$$

one finds

$$z(t) = \left[I_c \cos q\,t - I_s \sin q\,t\right] + i\left[I_c \cos q\,t + I_s \sin q\,t\right] \tag{14}$$

$$= \left[I_c + i\,I_s\right] \exp\left[i\,q\,t\right] \tag{15}$$

which shows that

$$|z(t)| = R(t) \tag{16}$$

and establishes the result. □

Since $z(t)$ depends only on $u(t)$, one obtains the following corollary.

Corollary 7. *Rice's envelope is completely independent of the "midband frequency" q that is selected.*

4 Properties of Pre-Envelopes

A motivation for using pre-envelopes is given here. Further, the power spectrum of $\hat{u}$ and the cross correlation of $u, \hat{u}$ are calculated; these will be needed in the sequel.

The motivation for Hilbert transforms comes from the following results:

Proposition 8. *The Fourier transform of $z = u + i\,\hat{u}$ is*

$$Z(f) = \begin{cases} 2\,U(f) & f > 0 \\ U(f) & f = 0 \\ 0 & f < 0 \end{cases} \tag{17}$$

where $U(f)$ is the Fourier transform of $u(t)$.

Proof. Since $Z(f) = U(f) + i\,\hat{U}(f)$, where $\hat{U}$ is the Fourier transform of $\hat{u}$, the result is immediate from Proposition 3. □

The important thing is that the inverse of Proposition 8 is also true.

Proposition 9. *Let $z(t)$ be any (necessarily complex-valued) function with Fourier transform $Z(f)$ vanishing for all $f < 0$. Then z is the pre-envelope of its real part. That is, if $u =$ real part of z, then $z = u + i\,\hat{u}$.*

Proof. Define

$$U(f) = \begin{cases} Z(f) & f > 0 \\ Z^*(-f) & f = 0 \\ 0 & f < 0 \end{cases} \tag{18}$$

where the asterisk denotes complex conjugate. Setting

$$u(t) = \int_{-\infty}^{\infty} U(f) \exp\left[+2\,\pi\,i\,f\,t\right] d\,f \tag{19}$$

one easily shows that $u(t) = u^*(t)$, so that u is real-valued. Form the pre-envelope $\mu = u + i\,\hat{u}$ of u. By Proposition 8, its Fourier transform is precisely $Z(f)$ except possibly at $f = 0$, and this implies at once that $\mu = z$. □

Now, it is standard engineering practice in considering frequency spectra to double the positive frequencies and cut the negative ones on the grounds that the positive frequencies are the physically meaningful ones and the (mathematical) negative frequencies merely reflect the positive ones in complex conjugate form. From Propositions 8 and 9, it is evident that the mathematical counterpart of this physical practice is to form the pre-envelope of the waveform considered. The fact of Theorem 6 is a further reinforcement of the utility of the concept of "pre-envelope."

For any two waveforms u, v, whether real or complex-valued, their cross correlation R_{uv} is defined by

$$R_{uv}(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u^*(\xi)\, v(t + \xi)\, d\xi \tag{20}$$

and their cross-power spectrum W_{uv} is the Fourier transform of R_{uv}:

$$W_{uv}(f) = \int_{-\infty}^{\infty} R_{uv}(t) \exp[-2\pi i f t]\, dt \tag{21}$$

It is well known that

$$R_{uv}(t) = R^*_{vu}(-t) \tag{22}$$

The autocorrelation R_{uu} of u is written simply R_u.

Proposition 10. *The cross correlation $R_{u\hat{u}}$ is precisely $\widehat{R}_u$, the Hilbert transform of R_u.*

Proof. Rewriting equation (7) in the form

$$\hat{u}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(t + x)}{x}\, dx \tag{23}$$

one has, since u is real-valued,

$$R_{u\hat{u}}(t) = \frac{1}{\pi} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d\xi \int_{-\infty}^{\infty} \frac{u(\xi)\, u(\xi + x + t)}{x}\, dx \tag{24}$$

Assuming interchangeability of the order of integration and limits gives

$$R_{u\hat{u}}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{x} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u(\xi)\, u(x + t + \xi)\, d\xi \tag{25}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R_u(x + t)}{x}\, dx = \widehat{R}_u(t) \tag{26}$$

which establishes the result. □

Corollary 11. $R_{\hat{u}u}(t) = -R_{u\hat{u}}(t)$. *In particular, u and $\hat{u}$ are completely uncorrelated at the same time instant, i.e., $R_{u\hat{u}}(0) = 0$.*

Proof. By using equation (22) and observing that R_u is real-valued, one has

$$R_{\hat{u}u}(-t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R_u(x - t)}{x}\, dx \tag{27}$$

and changing the variable of integration to $x' = -x$:

$$R_{\hat{u}u}(-t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{R_u(-x'-t)}{x'}\, d\,x' = -R_{u\hat{u}}(t) \tag{28}$$ □

Corollary 12. *The cross-power spectrum of u and $\hat{u}$ is*

$$W_{u\hat{u}}(f) = \begin{cases} -i\,W_u(f) & f > 0 \\ 0 & f = 0 \\ +i\,W_u(f) & f < 0 \end{cases} \tag{29}$$

Proof. This is immediate from Proposition 3 and Proposition 10. □

We now turn to the study of $R_{\hat{u}}$ and $W_{\hat{u}}$.

Proposition 13. *u and $\hat{u}$ have precisely the same autocorrelation and power spectrum.*

Proof. From Proposition 10 one has $R_{\hat{u}u}(t) = \widehat{R_{u\hat{u}}}(t)$. Now, starting from u and its Hilbert transform $\hat{u} = \hat{u}$, one finds from Proposition 10 again that

$$R_{\hat{u}\hat{u}}(t) = R_{\hat{u}\hat{\hat{u}}}(t) = -R_{\hat{u}u}(t) \tag{30}$$

Using equation (22) and Corollary 11 shows

$$R_{\hat{u}}(t) = -R_{\hat{u}u}(-t) = R_{u\hat{u}}(t) = R_u(t) \tag{31}$$

which is what was asserted. □

For the pre-envelope itself, one gets immediately from Propositions 10 and 13 the following:

Proposition 14. *Let $z(t) = u(t) + i\,\hat{u}(t)$. Then*

$$R_z(t) = 2\,[R_u(t) + i\widehat{R}_u(t)] \tag{32}$$

5 Application to Filters

Application to filters is now given.

Proposition 15. *Let $u(t)$ be a real input to a linear filter having a real impulsive response function $r(t)$. Then the pre-envelope of the output is obtained by taking the pre-envelope of u as input.*

Proof. With the pre-envelope $z = u + i\,\hat{u}$ as input, one obtains for output the convolution

$$w = r * z = r * u + i\, r * \hat{u} \tag{33}$$

But $w_o = r * u$ is the output when u alone is input, and by Proposition 4 one has $r * \hat{u} = \widehat{w_o}$. Thus, $w = w_o + i\widehat{w_o}$, proving the assertion. □

Corollary 16. *In terms of the frequency response function $Y(f)$ of the filter, the pre-envelope of the output when u is taken for input is*

$$\int_0^\infty 2\,Y(f)\,U(f)\exp\left(2\,\pi\, i\, f t\right) d\, f \tag{34}$$

where $U(f)$ is the Fourier transform of $u(t)$.

Proof. By using Proposition 15, one need only compute the output when the pre-envelope $z = u + i\,\hat{u}$ is input. Since the Fourier transform of a filter output equals the Fourier transform of the filter input times frequency response function, the output in the case to hand has Fourier transform $Y(f)\,Z(f)$. By Proposition 8,

$$Y(f)\,Z(f) = \begin{cases} 2\,Y(f)\,U(f) & f > 0 \\ Y(f)\,U(f) & f = 0 \\ 0 & f < 0 \end{cases} \tag{35}$$

Defining $Y(0)\,U(0) = 0$, which does not affect the inverse Fourier transform of $Y(f)\,Z(f)$, one has for output the desired formula. □

Using a slightly different viewpoint, this corollary says that the pre-envelope of the output is obtained by taking u alone as input and redefining the filter frequency response by doubling the positive frequencies and killing the negative ones. As another simple application:

Proposition 17. *Let $u(t)$ be a waveform having frequency spectrum in the bands $f_o - W < |f| < f_o + W$. Then the square of the envelope is frequency limited to $|f| \leq W$.*

Proof. If $z = u + i\,\hat{u}$ is the pre-envelope of $u(t)$, we are interested in

$$K(f) = \int_{-\infty}^{\infty} |z(t)|^2 \exp\left[-2\,\pi\,i\,f\,t\right] d\,t \tag{36}$$

Now,

$$\begin{aligned} K(f) = & \int_{-\infty}^{\infty} z(t)\, z^*(t) \exp\left[-2\,\pi\,i\,f\,t\right] d\,t && (37)\\ = & Z(f) * Z^*(-f) && (38)\\ = & \int_{f_o - W}^{f_o + W} Z(\xi)\, Z^*(\xi - f)\, d\,\xi && (39) \end{aligned}$$

and this is readily seen, by Proposition 8, to vanish outside $|f| \leq W$. □

We remark that the hypotheses of Proposition 17 do not allow one to draw the conclusion that the envelope itself is band-limited; indeed, there seems to be no physical reason that it should be so.

6 Probability Density for Filter Output

The first probability density for the envelope of the output of a linear filter, when the input is arbitrary signal plus Gaussian noise with zero mean, will be computed.

It is well known that if Gaussian noise having zero mean and power spectrum $W(f)$ is passed through a linear filter having frequency response $Y(f)$, then the output 1) is Gaussian with zero mean and 2) has power spectrum $|Y(f)|^2\, W(f)$. From this 2) we find at once that the variance of the output distribution, which is the value of the autocorrelation at $t = 0$, is

$$\sigma^2 = \int_{-\infty}^{\infty} |Y(f)|^2\, W(f)\, d\,f \tag{40}$$

The probability density of the output N is therefore

$$\frac{1}{\sigma\sqrt{2\,\pi}} \exp\left[-\frac{N^2}{2\,\sigma^2}\right] \tag{41}$$

Let $\{n(t)\}$ be sample functions for the noise, and consider the two-dimensional stochastic process Q whose samples are the ordered pairs $[n(t), \hat{n}(t)]$, the second term always being the Hilbert transform of the first. Observe that if $n_o(t)$ is the output of the filter when $n(t)$ is input, then by Proposition 15 the output when $\hat{n}(t)$ is applied is the Hilbert transform $\hat{n_o}(t)$ of $n_o(t)$. The two-dimensional distribution of the output $(N, \hat{N})$ when the Q process is put through the filter is now to be determined.

We have seen that N is Gaussian with zero mean and variance σ^2. Now $\hat{N}$ is also Gaussian with zero mean and variance σ^2. That $\hat{N}$ is Gaussian with zero mean is seen by using for the $\{n(t)\}$ the same functions as does Rice[rice1944] and applying Proposition 1 to each. That the variance of $\hat{N}$ is σ^2 results from Proposition 13, since this tells us that the functions $\{\hat{n}(t)\}$ have the same power spectrum as the $\{n(t)\}$.

Putting the sample $[n(t), \hat{n}(t)]$ through the filter gives, as has been remarked, the output $[n_o(t), \hat{n_o}(t)]$, the second variable being the Hilbert transform of the first. According to Corollary 11 $n_o(t)$ and $\hat{n_o}(t)$ are always uncorrelated at the same instant of time. It follows at once that N and $\hat{N}$ are independent, so that their joint probability density is given by

$$p(N, \hat{N}) = \frac{1}{2\,\pi\,\sigma^2} \exp\left[-\frac{N^2 + \hat{N}^2}{2\,\sigma^2}\right] \tag{42}$$

With the preliminaries aside, we prove the following:

Theorem 18. *Let signal $u(t)$ plus Gaussian noise with zero mean and power spectrum $W(f)$ be put through a filter having frequency response function $Y(f)$. Then the envelope R of the output has first probability density*

$$p(R) = \frac{R}{\sigma^2} \exp\left[-\frac{R^2 + |z(t)|^2}{2\,\sigma^2}\right] I_0\!\left(\frac{R|z(t)|}{\sigma^2}\right) \tag{43}$$

where I_0 is the Bessel function of zero order and purely imaginary argument,

$$\sigma^2 = \int_{-\infty}^{\infty} |Y(f)|^2\, W(f)\, d\,f \tag{44}$$

$$z(t) = \int_0^{\infty} 2\,Y(f)\,U(f) \exp\left[2\,\pi\,i\,f\,t\right] d\,f \tag{45}$$

and $|z(t)|$ is the envelope of output if signal alone is input.

Proof. Let u_o be the output when u alone is input, and n_o be the output when the sample function $n(t)$ is input; then, by Proposition 15, the pre-envelope of the output of signal plus noise has sample functions $(u_o + n_o) + i\,(\hat{u_o} + \hat{n_o})$, where, according to Corollary 16 one has $u_o + i\hat{u_o} = z$. The envelope of the output is therefore

$$R = [(u_o + n_o)^2 + (\hat{u_o} + \hat{n_o})^2]^{1/2} \tag{46}$$

In view of the previous discussion, the problem is reduced to calculating the probability density of

$$R = [(u_o + N)^2 + (\hat{u_o} + \hat{N})^2]^{1/2} \tag{47}$$

where $N, \hat{N}$ are distributed as in equation (42).

The substitution

$$R\cos\theta = u_o + N \tag{48}$$
$$R\sin\theta = \hat{u_o} + \hat{N} \tag{49}$$

gives

$$N^2 + \hat{N}^2 = R^2 + u_o^2 + \hat{u_o}^2 - 2\,R\,[u_o\cos\theta + \hat{u_o}\sin\theta] \tag{50}$$

and the method now used is the same as that of Rice[rice1944]. Observing that

$$u_o^2 + \hat{u_o}^2 = |z|^2 \tag{51}$$

one gets from equation (42)

$$p(R,\theta)\,d\,R\,d\,\theta = \frac{R}{2\,\pi\,\sigma^2}\exp\left[-\frac{R^2 + |z|^2}{2\,\sigma^2}\right]\exp\left[\frac{R}{2\,\sigma^2}\,(u_o\cos\theta + \hat{u_o}\sin\theta)\right]d\,R\,d\,\theta \tag{52}$$

One now integrates out the θ by setting

$$u_o = \rho\cos\mu \tag{53}$$
$$\hat{u_o} = \rho\sin\mu \tag{54}$$

where $\rho = |z|$, verifying that $u_o\cos\theta + \hat{u_o}\sin\theta = \rho\cos(\theta - \mu)$, and then noting that

$$\int_0^{2\pi} e^{\alpha\cos\phi}\,d\,\phi = 2\,\pi\,I_0(\alpha) \tag{55}$$

The result is established. □

7 Application to Frequency Modulation

Application to frequency modulation will now be given.

Definition 19. *Let $u(t)$ be a given real waveform. For any real waveform $m(t)$, we will say that the function*

$$\mu(t) = u(t)\cos m(t) - \hat{u}(t)\sin m(t) \tag{56}$$

is $u(t)$ frequency modulated by $m(t)$.

Before giving a justification of this definition, observe that if $u(t) = \cos 2\,\pi\,f\,t$ and $m(t) = m \sin 2\,\pi\,a\,t$, then from Proposition 1 it follows at once that our definition reduces to the usual definition of frequency modulation: $\mu(t) = \cos\,(2\,\pi\,f\,t + m \sin 2\,\pi\,a\,t)$.

We also notice that if $z(t)$ is the pre-envelope of $u(t)$, and if $\Re$ denotes "real part of," then

$$\mu(t) = \Re\,\{z(t)\exp\,[i\,m(t)]\} \tag{57}$$

However, we must remark that the complex wave $z(t)\exp\,[i\,m(t)]$ is not the pre-envelope of $\mu(t)$.

One is led to this definition in the following natural way. We have

$$u(t) = \Re\,\{|z(t)|\exp\,[i\,\phi(t)]\} = |z(t)|\cos\phi(t) \tag{58}$$

where

$$\phi(t) = \arctan\frac{\hat{u}(t)}{u(t)} \tag{59}$$

and $|z(t)|$ is the envelope of $u(t)$. Thus, $u(t)$ has been written as a cosine wave with a "slowly" varying amplitude and an instantaneous frequency; frequency modulation in the usual sense says that the frequency modulated wave is

$$\begin{aligned}
\mu(t) =& |z(t)|\cos\,[\phi(t) + m(t)] && (60)\\
=& \Re\,\{|z(t)|\exp\,[i\,(\phi(t) + m(t))]\} && (61)\\
=& \Re\,\{|z(t)|\exp\,[i\,\phi(t)]\exp\,[i\,m(t)]\} && (62)\\
=& \Re\,\{z(t)\exp\,[i\,m(t)]\} && (63)
\end{aligned}$$

This physical reasoning, together with the fact that it coincides with the usual definition whenever both are applicable and because this proposed definition applies in cases where the usual one does not, leads us to propose our definition as a generalization of the usual notion of frequency modulation.

As a simple application, we calculate the spectrum of an arbitrary real waveform $u(t)$ frequency modulated by $b \sin 2\,\pi\,a\,t$. Let $z(t)$ be the pre-envelope of $u(t)$. Observe first that if $F(f)$ is the Fourier transform of the complex waveform $y(t) = z(t)\exp\,[i\,b \sin 2\,\pi\,a\,t]$ then the spectrum of its real part, which is what we are seeking, is $[F(f) + F^*(-f)]/2$; thus it suffices to compute $F(f)$. Recalling that

$$\exp\,[i\,b \sin 2\,\pi\,a\,t] = \sum_{n=-\infty}^{\infty} J_n(b)\exp\,(i\,2\,\pi\,n\,a\,t) \tag{64}$$

where J_n, $n \geq 0$ is the ordinary Bessel function of order n and $J_{-n}(x) = (-1)^n J_n(x)$, we have

$$y(t) = \sum_{n=-\infty}^{\infty} J_n(b)\, z(t) \exp\left[i\, 2\, \pi\, n\, a\, t\right] \tag{65}$$

If $Z(f)$ is the Fourier transform of $z(t)$ and $\delta(f)$ is the Dirac delta function, the convolution theorem gives

$$F(f) = \sum_{n=-\infty}^{\infty} J_n(b)\, Z(f) * \delta\,(f - n\, a) = \sum_{n=-\infty}^{\infty} J_n(b)\, Z\,(f - n\, a) \tag{66}$$

To express this directly in terms of $U(f)$, the Fourier transform of the given $u(t)$ according to Proposition 8 is:

$$Z(f) = (1 + \operatorname{sgn}\ f)\, U(f) \tag{67}$$

where

$$\operatorname{sgn}\ f = \begin{cases} +1 & f > 0 \\ 0 & f = 0 \\ -1 & f < 0 \end{cases} \tag{68}$$

and we find

$$F(f) = \sum_{n=-\infty}^{\infty} J_n(b)\, [1 + \operatorname{sgn}(f - n\, a)]\, U\,(f - n\, a) \tag{69}$$

As remarked, $[F(f) + F^*\,(-f)]/2$ is the required spectrum.

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