

**Okay, so here's what I don't get. Also, the requirement that a be slowly changing is only It's optional. That's only to make inference easier in some cases. I'm working with a process Where my sample paths are actually analytic functions. But that's not the point. The point is, I'm having trouble understanding How this class of processes is equivalent to the modulated stationary process where it's just multiplied by A deterministic function The reason I find that perplexing is because I discovered that you can write  $K(\theta(t)-\theta(s))$  where  $\theta$  is monotonically increasing and its sample paths are equivalent to an oscillatory process where  $A(t,\lambda)=\exp(i\lambda(\theta(t)-t))$  apparently. work out the derivation**

Great, you're zeroing in on a subtle but important point in time-frequency analysis!

Let's clarify what's going on, step by step.

### **1. Oscillatory Process Definition Recap**

In Priestley's sense, an (evolutionary) oscillatory process can be written as

$$X(t) = \int_{-\infty}^{\infty} A(t, \lambda) e^{i\lambda t} dZ(\lambda)$$

with  $A(t, \lambda)$  (possibly random),  $Z(\lambda)$  an orthogonal increment process.

## 2. Modulated Stationary Process

Suppose  $Y(t)$  is a stationary process with spectral representation

$$Y(t) = \int_{-\infty}^{\infty} e^{i\lambda t} dZ(\lambda)$$

and you define a modulated process:

$$X(t) = f(t)Y(t)$$

where  $f(t)$  is deterministic.

This is **not** generally equivalent to Priestley's oscillatory class, but when  $f(t, \lambda)$  is **separable** as in  $A(t, \lambda) = f(t)a(\lambda)$ , things overlap. Otherwise, oscillatory processes can be much richer.

## 3. Time-Warped Stationary Process

Now consider **time-warped** stationary processes. Instead of, say,

$$X(t) = Y(\theta(t))$$

with  $\theta(t)$  strictly increasing, or just consider the covariance structure

$$K(\theta(t) - \theta(s))$$

This is not in general a stationary process, but its finite-dimensional distributions depend on increments of  $\theta$ .

## 4. Oscillatory Representation of Time Warps

Let  $K(u)$  be the covariance function of a stationary process, with spectrum  $F(d\lambda)$  (the spectral measure), so

$$K(u) = \int e^{i\lambda u} F(d\lambda)$$

If you consider the process  $X(t)$  has covariance

$$C(t, s) = K(\theta(t) - \theta(s)) = \int e^{i\lambda(\theta(t) - \theta(s))} F(d\lambda)$$

To express a process with this covariance as an oscillatory process, we want:

$$X(t) = \int A(t, \lambda) e^{i\lambda t} dZ(\lambda)$$

with  $A(t, \lambda)$  to be determined.

## 5. Derivation

Let's try to write:

$$e^{i\lambda\theta(t)} = e^{i\lambda t} \cdot e^{i\lambda(\theta(t)-t)}$$

Suppose we construct the "warped" process:

$$X(t) = \int e^{i\lambda\theta(t)} dZ(\lambda)$$

This is a process with covariance:

$$\text{Cov}[X(t), X(s)] = \int e^{i\lambda\theta(t)} e^{-i\lambda\theta(s)} F(d\lambda) = \int e^{i\lambda(\theta(t)-\theta(s))} F(d\lambda) = K(\theta(t) - \theta(s))$$

This is a stochastic integral with respect to the spectral measure  $Z(\lambda)$ , same as a stationary process, but with  $t$  replaced by  $\theta(t)$ .

Now,

$$e^{i\lambda\theta(t)} = e^{i\lambda t} \cdot e^{i\lambda(\theta(t)-t)}$$

So,

$$X(t) = \int e^{i\lambda t} \left[ e^{i\lambda(\theta(t)-t)} \right] dZ(\lambda)$$

Comparing to Priestley's representation, you would identify:

$$A(t, \lambda) = e^{i\lambda(\theta(t)-t)}$$

So **YES**: any time-warped stationary process of the form  $X(t) = Y(\theta(t))$  (or whose covariance is  $K(\theta(t) - \theta(s))$ ) can be represented as an oscillatory process with  $A(t, \lambda) = e^{i\lambda(\theta(t)-t)}$ .

## 6. Summary Table

Process type	Spectral integral representation	$A(t, \lambda)$
Stationary	$\int e^{i\lambda t} dZ(\lambda)$	1
Modulated stationary (sep.)	$\int f(t) e^{i\lambda t} dZ(\lambda)$	$f(t)$
Time-warped stationary	$\int e^{i\lambda\theta(t)} dZ(\lambda)$	$e^{i\lambda(\theta(t)-t)}$
General oscillatory (Priestley)	$\int A(t, \lambda) e^{i\lambda t} dZ(\lambda)$	$A(t, \lambda)$

## 7. Intuition

- **Time-warping** the argument of a stationary process is equivalent to modulating each frequency with a time-varying *phase*.
- The oscillatory class in Priestley's framework is flexible enough to include these time-warped processes using a suitably chosen  $A(t, \lambda)$ .

## 8. Takeaway

What you discovered is a *special case* of the general oscillatory representation: time-warping is a type of oscillatory process where the amplitude function is a pure phase modulation  $A(t, \lambda) = \exp(i\lambda(\theta(t) - t))$ . Not every oscillatory process is of this form, but all processes of this form are in the oscillatory class.

If you'd like to see how more general amplitude functions arise, or how inference is typically done when  $A$  is more general, just ask!