

The Operational Matrix of the Random Wave Process

BY STEPHEN CROWLEY

Department of Intuition
The Royal Citadel of Pure Being

February 11, 2025

Abstract

An expression for the convolution of a pair of spherical Bessel functions is determined.

1 Lemmas

Lemma 1. (Terminating Hypergeometric Series) *For any $p \in \mathbb{Z}_{\geq 0}$, the Gauss hypergeometric function terminates:*

$${}_2F_1(-p, b; c; z) = \sum_{k=0}^p \frac{(-p)_k (b)_k}{(c)_k k!} z^k \quad (1)$$

where $(a)_k = \prod_{i=0}^{k-1} (a + i)$

Proof. By definition, the Gauss hypergeometric series is:

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k \quad (2)$$

Setting $a = -p$ with $p \in \mathbb{Z}_{\geq 0}$, the Pochhammer symbol $(-p)_k$ becomes zero for all $k > p$. Explicitly:

$$(-p)_k = \prod_{i=0}^{k-1} (-p + i) = \begin{cases} (-p)(-p+1) \cdots (-p+k-1), & k \leq p \\ 0 & k > p \end{cases} \quad (3)$$

Thus, the series terminates at $k = p$, yielding:

$${}_2F_1(-p, b; c; z) = \sum_{k=0}^p \frac{(-p)_k (b)_k}{(c)_k k!} z^k \quad (4) \quad \square$$

Lemma 2. (Integral with Incomplete Gamma Function) For $j \geq 0$,

$$\int_{-1}^1 \left(\frac{1-x}{2} \right)^j e^{ixy} dx = \frac{e^{iy}}{2^j} \frac{\gamma(j+1, 2iy)}{(iy)^{j+1}} \quad (5)$$

where $\gamma(s, x)$ denotes the lower incomplete gamma function.

Proof. Substitute $t = \frac{1-x}{2} \implies x = 1 - 2t$, $dx = -2dt$, adjusting limits:

$$\int_1^0 t^j e^{i(1-2t)y} (-2dt) = 2e^{iy} \int_0^1 t^j e^{-2iyt} dt \quad (6)$$

Let $u = 2iyt \implies t = \frac{u}{2iy}$, $dt = \frac{du}{2iy}$:

$$\frac{2e^{iy}}{(2iy)^{j+1}} \int_0^{2iy} u^j e^{-u} du = \frac{e^{iy}}{2^j} \frac{\gamma(j+1, 2iy)}{(iy)^{j+1}} \quad \square$$

Lemma 3. (Legendre Polynomial Representation) The hypergeometric function equals the Legendre polynomial $P_m(x)$. The Legendre polynomials are hypergeometric functions

$$P_m(x) = {}_2F_1\left(-m, m+1; 1; \frac{1-x}{2}\right) \quad (7)$$

Proof. From the Rodrigues formula $P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m} (x^2 - 1)^m$ expand using the binomial theorem:

$$(x^2 - 1)^m = \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} x^{2k} \quad (8)$$

Differentiating m times yields terms proportional to x^k , matching the hypergeometric series:

$$P_m(x) = {}_2F_1\left(-m, m+1; 1; \frac{1-x}{2}\right) \quad \square$$

2 Main Theorem

Theorem 4. (Fourier Transform of Legendre Polynomial Products)

$$\begin{aligned} I_{m,n}(y) &= \int_{-1}^1 P_m(x) P_n(x) e^{ixy} dx \\ &= e^{iy} \sum_{j=0}^{m+n} \frac{\Psi_j(m,n)}{2^j} \left[\frac{\gamma(j+1, 2iy)}{(iy)^{j+1}} \right] \end{aligned} \quad (9)$$

where

$$\Psi_j(m,n) = \frac{{}_4F_3\left(\begin{matrix} -m, m+1, -n, n+1 \\ 1, 1, j+1 \end{matrix}; 1\right)}{j!} \quad (10)$$

Proof. Part 1: Integral Reduction

Expand $P_m(x) P_n(x)$ using Lemma 1:

$$P_m(x) P_n(x) = \sum_{k=0}^m \sum_{\ell=0}^n \frac{(-m)_k (m+1)_k (-n)_\ell (n+1)_\ell}{(1)_k (1)_\ell k! \ell!} \left(\frac{1-x}{2}\right)^{k+\ell} \quad (11)$$

Let $j = k + \ell$, valid for $0 \leq k \leq m$, $0 \leq \ell \leq n$. Then:

$$I_{m,n}(y) = \sum_{j=0}^{m+n} \underbrace{\sum_{k=\max(0, j-n)}^{\min(j, m)} \frac{(-m)_k (m+1)_k (-n)_{j-k} (n+1)_{j-k}}{(1)_k (1)_{j-k} k! (j-k)!}}_{\Psi_j(m,n)} \int_{-1}^1 \left(\frac{1-x}{2}\right)^j e^{ixy} dx \quad (12)$$

Apply Lemma 2 to obtain the result.

Part 2: $\Psi_j(m,n)$ as a ${}_4F_3$ Function

Expand the ${}_4F_3$ series:

$${}_4F_3\left(\begin{matrix} -m, m+1, -n, n+1 \\ 1, 1, j+1 \end{matrix}; 1\right) = \sum_{k=0}^j \frac{(-m)_k (m+1)_k (-n)_k (n+1)_k}{(1)_k (1)_k (j+1)_k k!} \quad (13)$$

The main result follows. □