Chapter 8

Spectral Description of Non-stationary Random Processes

8.1 Introduction

8.1.1 Stationary random process

The spectral description of a weakly stationary random process is given by the power spectral density function $\Phi_{xx}(\omega)$. The *local* character of the frequency decomposition can be seen as follows: Consider an ideal narrow-band filter as in Fig.1, where $H_i(\omega) = 1$ within the bandwidth and 0 outside. The corresponding impulse response is (Problem 1.8)

$$h_i(\tau) = \frac{2}{\pi\tau} \sin \frac{\Delta\omega\, \tau}{2} \cos \omega\tau \tag{8.1}$$

(This system is not causal and cannot be realized exactly). From Equ.(5.24), the mean square value of the filter output is related to the *PSD* of the input by

$$\mathrm{E}[X_i^2(t, \omega, \Delta\omega)] = 2 \int_{\omega-\Delta\omega/2}^{\omega+\Delta\omega/2} \Phi_{xx}(\nu)\, d\nu \tag{8.2}$$

This relation tells us that *the average power within any frequency interval is given by the area under the PSD function for that interval.* If the bandwidth is small this result can be approximated by

$$\mathrm{E}[X_i^2(t, \omega, \Delta\omega)] \simeq 2\, \Delta\omega\, \Phi_{xx}(\omega) \tag{8.3}$$

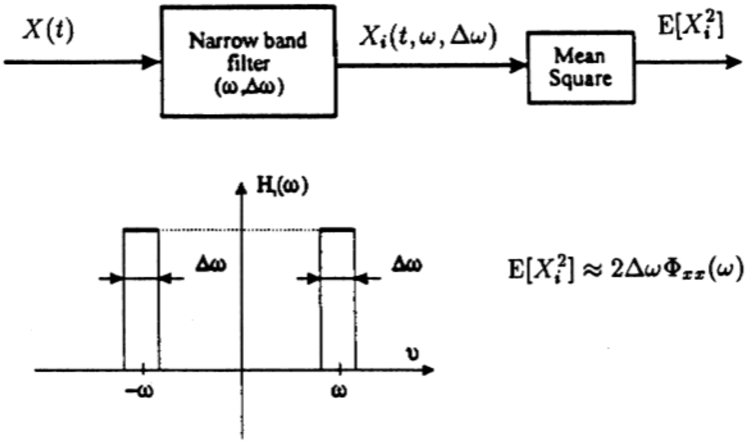


Figure 8.1: Estimation of the *PSD* from narrow band measurements.

If the process is ergodic, the ensemble average can be replaced by a time average

$$z(t) = \frac{1}{T}\int_{t-T}^{t} x_i^2(\tau)\,d\tau \tag{8.4}$$

Obviously, the measured quantity varies with the sample $x_i(t)$, the filter parameters ω and $\Delta\omega$ and the duration T. From the ergodicity theorem, we know that

$$z(t) \to \mathrm{E}[X_i^2(t,\omega,\Delta\omega)] \qquad \text{as} \qquad T \to \infty$$

For samples of finite duration, one can show that the relative fluctuation in the measured quantity is

$$\frac{\sigma_z}{\mathrm{E}[Z]} = \sqrt{\frac{2\pi}{\Delta\omega\, T}} \tag{8.5}$$

It can be reduced by increasing either $\Delta\omega$ or T. Of course widening $\Delta\omega$ decreases the resolution of the measurement, which is the integral (8.2) rather than a pointwise estimate at the central frequency of the filter, ω.

8.1.2 Non-stationary random process

Consider the transient process $X(t)$ and its Fourier transform (assuming it exists)

$$\mathcal{X}(\omega) = \int_{-\infty}^{\infty} X(t)e^{-j\omega t}dt \tag{8.6}$$

Using Parseval's theorem (1.7) and following the same development as in the previous section, we can readily establish that the *energy spectral density* function

$$S_{xx}(\omega) = \frac{1}{2\pi} \mathrm{E}[|\mathcal{X}(\omega)|^2] \tag{8.7}$$

is a *local* decomposition of the *energy* in the transient process, exactly in the same sense as the *PSD* function for the power in a stationary process. However, the energy spectral density function does not provide any information as to the *time evolution of the frequency distribution of the power* in the process. It is this type of energy mapping in the time-frequency plane that we shall now consider.

8.1.3 Objectives of a spectral description

For the past 20 years, the spectral analysis of nonstationary oscillatory processes has attracted a great deal of attention and several characterizations have been proposed. To be useful, such a representation should enjoy the following properties

- A clear *local* interpretation in the frequency-time plane.
- An estimate can be generated from a single sample record.
- A simple input-output relationship should exist for linear time-invariant systems.
- It should coincide with the power spectral density function when the process is stationary.

As we shall see below, a strictly local mapping does not exist, because a good resolution in one domain (e.g. frequency) can only be achieved at the expense of a poor resolution in the dual domain (time). This is known as the *uncertainty principle.* In fact, this can be understood easily by considering the narrow-band filter of Fig.8.1. The stationary relationship (8.3) can be extended to nonstationary processes as

$$\Psi_{xx}(t,\omega) = \frac{\mathrm{E}[X_i^2(t,\omega,\Delta\omega)]}{2\,\Delta\omega} \tag{8.8}$$

where X_i is the output of the narrow-band filter

$$X_i(t,\omega,\Delta\omega) = \int_{-\infty}^{\infty} h_i(\tau) X(t-\tau)\, d\tau \tag{8.9}$$

One sees that $X_i(t,\omega,\Delta\omega)$ consists of the *weighted average* of the values of the process $X(\tau)$ in the vicinity of t. The weighting function is the impulse response of the filter, defined by (8.1). Obviously, some smoothing occurs in the time domain. If, to increase the resolution in the frequency domain, we reduce the bandwith $\Delta\omega$ of the filter, the effective duration of the impulse response increases and the smoothing in the time domain involves an even longer period.

8.2 Instantaneous power spectrum

Let $\phi_{xx}(t_1, t_2)$ be the autocorrelation function of a non-stationary random process. With the following transformations

$$t = \frac{t_1 + t_2}{2} \qquad \tau = t_1 - t_2$$

$$t_1 = t + \frac{\tau}{2} \qquad t_2 = t - \frac{\tau}{2} \tag{8.10}$$

the autocorrelation function can be rewritten

$$\mathcal{R}_{xx}(t,\tau) = \phi_{xx}(t + \frac{\tau}{2}, t - \frac{\tau}{2}) = \mathrm{E}[X(t + \frac{\tau}{2})X(t - \frac{\tau}{2})] \tag{8.11}$$

$\mathcal{R}_{xx}(t,\tau)$ can be considered as an instantaneous autocorrelation function, at the average time t; it is an even function of the separation time τ. By analogy with stationary processes, the *instantaneous* power spectral density function is defined as

$$\Phi_{xx}(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{R}_{xx}(t,\tau)e^{-j\omega\tau}d\tau \tag{8.12}$$

It is a real, even function of the variable ω; the inverse relationship is

$$\mathcal{R}_{xx}(t,\tau) = \int_{-\infty}^{\infty} \Phi_{xx}(t,\omega)e^{j\omega\tau}d\omega \tag{8.13}$$

and, at $\tau = 0$, one gets the mean square value

$$\mathrm{E}[X^2(t)] = \mathcal{R}_{xx}(t,0) = \int_{-\infty}^{\infty} \Phi_{xx}(t,\omega)d\omega \tag{8.14}$$

These relationships are seemingly completely similar to the Wiener-Khintchine theorem for stationary processes. Equation (8.14) shows that $\Phi_{xx}(t,\omega)$ is indeed a frequency decomposition of the instantaneous mean square value at t. However,

- the global frequency decomposition cannot be particularized to any frequency interval as in the case of a stationary process;
- $\Phi(t,\omega)$ can even become *negative*;
- $\Phi(t,\omega)$ cannot be measured directly in practice.

Thus, the instantaneous power spectrum does not fulfil the criteria stated in the foregoing section and one should not expect that it provides, except in particular situations, a correct mapping of the energy in the frequency-time plane. One particular case where the instantaneous spectrum is physically meaningful is that of a *locally stationary* process, whose autocorrelation function has the form

$$\mathcal{R}_{xx}(t,\tau) = R_1(t)R_2(\tau) \tag{8.15}$$

where $R_1(t)$ is a non negative function and $R_2(\tau)$ is the autocorrelation function of a weakly stationary process. Upon Fourier transforming, one gets

$$\Phi_{xx}(t,\omega) = R_1(t)\Phi_2(\omega) \tag{8.16}$$

where $\Phi_2(\omega)$ is the *PSD* associated to $R_2(\tau)$. A *shot noise* constitutes an example of a locally stationary process.

A *separable* process is defined as the product of a weakly stationary process $X(t)$ and a slowly varying function $a(t)$

$$Y(t) = a(t)X(t) \tag{8.17}$$

Its autocorrelation function is

$$\phi_{yy}(t_1,t_2) = \mathrm{E}[Y(t_1)Y(t_2)] = a(t_1)a(t_2)\mathrm{E}[X(t_1)X(t_2)]$$

$$\phi_{yy}(t_1,t_2) = a(t_1)a(t_2)R_{xx}(t_1-t_2) \tag{8.18}$$

If the fluctuations of $a(t)$ are slow, compared to the correlation time of $X(t)$,

$$\mathcal{R}_{xx}(t,\tau) \simeq a^2(t)R_{xx}(\tau)$$

$$\Phi_{xx}(t,\omega) \simeq a^2(t)\Phi_{xx}(\omega) \tag{8.19}$$

The instantaneous *PSD* of a locally stationary process cannot become negative.

8.3 Mark's Physical Spectrum

8.3.1 Definition and properties

Consider a non-stationary oscillatory real process $X(u)$. One isolates the vicinity of $u = t$ by multiplying $X(u)$ by a *window* function $w(t-u)$ such that

- $w(t)$ is positive in the vicinity of $t = 0$,
- $|w(t)|$ is small except near $t = 0$,
- it is normalized according to

$$\int_{-\infty}^{\infty} w^2(t)\,dt = 1 \tag{8.20}$$

Examples of such windows are shown in Fig. 8.3.

According to Parseval's theorem, the energy spectrum is a frequency decomposition of the total energy in the process. Therefore, a frequency decomposition of the energy in the vicinity of $u = t$ is provided by the energy spectrum of $w(t-u)X(u)$.

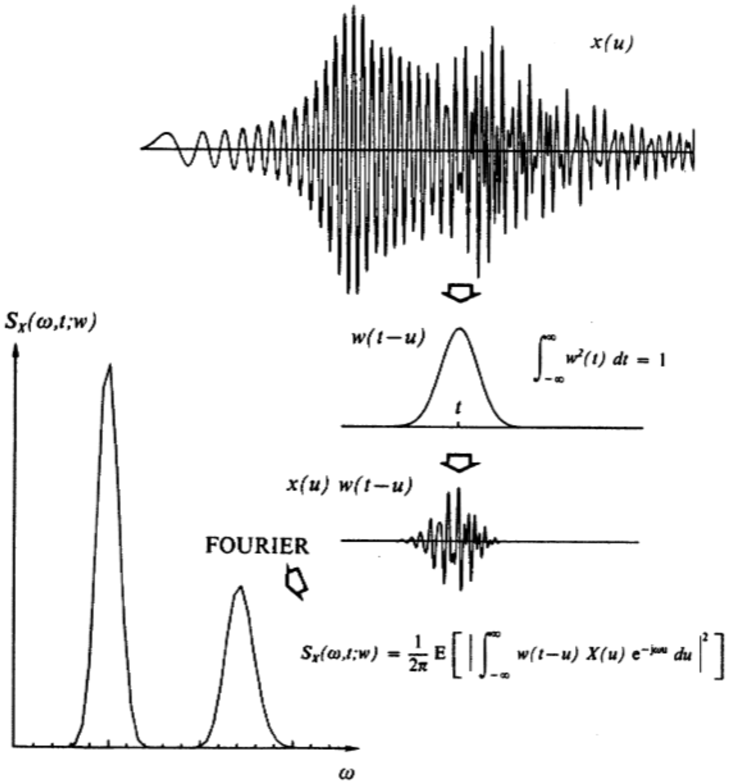


Figure 8.2: Definition of the physical spectrum.

The ***physical spectrum*** is defined as the ensemble average of the energy spectrum of $w(t-u)X(u)$:

$$S_x(\omega, t; w) = \frac{1}{2\pi}\mathrm{E}[|\int_{-\infty}^{\infty} w(t-u)X(u)e^{-j\omega u}du|^2] \tag{8.21}$$

It depends on the choice of the window $w(t)$. The operations associated with the definition of the physical spectrum are illustrated in Fig.8.2. For every $w(t)$, $S_x(\omega, t; w)$ supplies a non negative mapping, in the domain (ω, t), of the energy in the process. It is an even function of ω. According to Parseval's theorem [applied to the process $w(t-u)X(u)$],

$$\int_{-\infty}^{\infty} w^2(t-u)\mathrm{E}[X^2(u)]\, du = \int_{-\infty}^{\infty} S_x(\omega, t; w)\, d\omega \tag{8.22}$$

and, thanks to the normalizing condition (8.20), integrating with respect to time gives

$$E_x = \int_{-\infty}^{\infty} \mathrm{E}[X^2(t)]\, dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_x(\omega, t; w)\, d\omega\, dt \tag{8.23}$$

Thus, providing the window function is properly normalized, the volume under the surface $S_x(\omega, t; w)$ represents the average total energy in the process, independently of the shape of the window.

8.3.2 Duality, uncertainty principle

According to Equ.(8.22), the physical spectrum at t is a frequency decomposition of the local weighted average of the expected mean square in the vicinity of t. Similarly, the following dual interpretation can be established (Mark, 1970): The physical spectrum at ω is a time decomposition of the local weighted average of the energy spectrum in the vicinity of ω

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathcal{W}(\omega-\nu)|^2 S_{xx}(\nu)\, d\nu = \int_{-\infty}^{\infty} S_x(\omega, t; w)\, dt \tag{8.24}$$

where the energy spectrum $S_{xx}(\nu)$ is defined by Equ.(8.7) and $\mathcal{W}(\omega)$ is the Fourier transform of the window function. Note that from Parseval's theorem, the normalization condition implies that

$$\int_{-\infty}^{\infty} |\mathcal{W}(\omega)|^2\, d\omega = 2\pi \tag{8.25}$$

The *nominal duration* T and the *nominal width* β of the window are defined respectively as

$$T = \frac{1}{w(0)}\int_{-\infty}^{\infty} |w(t)|\, dt \qquad \beta = \frac{1}{\mathcal{W}(0)}\int_{-\infty}^{\infty} |\mathcal{W}(\omega)|\, d\omega \tag{8.26}$$

According to the foregoing discussion, T is a measure of the resolution of the physical spectrum in the time domain and β in the frequency domain. It is easily shown that

$$\begin{aligned} T\beta &= \frac{1}{w(0)}\int_{-\infty}^{\infty} |w(t)| dt\, \frac{1}{\mathcal{W}(0)}\int_{-\infty}^{\infty} |\mathcal{W}(\omega)| d\omega \\ &\geq \frac{1}{w(0)}\int_{-\infty}^{\infty} w(t) dt\, \frac{1}{\mathcal{W}(0)}\int_{-\infty}^{\infty} \mathcal{W}(\omega) d\omega = 2\pi \end{aligned}$$

Thus T and β must satify the following inequality

$$\beta T \geq 2\pi \tag{8.27}$$

which shows that the resolutions in the time and frequency domains are not independent. Once the resolution has been fixed in one domain, the resolution in the dual domain is fixed automatically. *A good resolution in one domain can only be achieved at the expense of a poor resolution in the dual domain (uncertainty principle).*

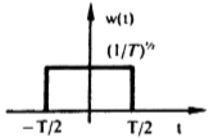
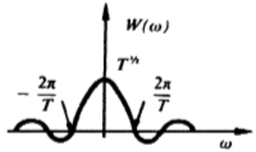
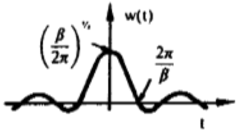
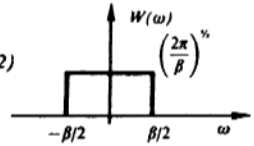
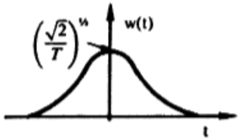
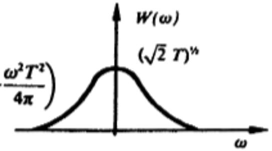
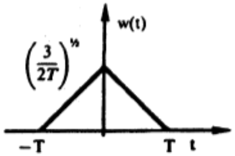
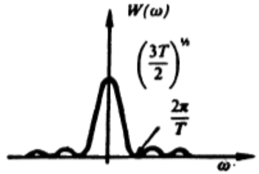
	Time domain normalized window $\int_{-\infty}^{\infty} w^2(t)\,dt = 1$		Nominal duration	Nominal width	Frequency domain $W(\omega) = \int_{-\infty}^{\infty} w(t)\,e^{-j\omega t}\,dt$	
Rectangle		$w(t) = \left(\frac{1}{T}\right)^{1/2} \mathrm{rect}(-T/2, T/2)$	T	$> \frac{2\pi}{T}$	$W(\omega) = T^{1/2} \frac{\sin\frac{T\omega}{2}}{\frac{T\omega}{2}}$	
		$w(t) = \left(\frac{\beta}{2\pi}\right)^{1/2} \frac{\sin\frac{\beta t}{2}}{\frac{\beta t}{2}}$	$> \frac{2\pi}{\beta}$	β	$W(\omega) = \left(\frac{2\pi}{\beta}\right)^{1/2} \mathrm{rect}(-\beta/2, \beta/2)$	
Gaussian		$w(t) = \left(\frac{\sqrt{2}}{T}\right)^{1/2} \exp\left(-\frac{\pi t^2}{T^2}\right)$	T	$\frac{2\pi}{T}$	$W(\omega) = (\sqrt{2}\,T)^{1/2} \exp\left(-\frac{\omega^2 T^2}{4\pi}\right)$	
Triangle		$w(t) = \left(\frac{3}{2T}\right)^{1/2} \mathrm{triangle}(-T, T)$	T	$\frac{2\pi}{T}$	$W(\omega) = \left(\frac{3T}{2}\right)^{1/2} \frac{\sin^2\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)^2}$	

Figure 8.3: Examples of window functions.

Note that different window functions with the same nominal duration lead to different physical spectra. For example, it can be seen in Fig.8.3 that a Gaussian window decreases rapidly both in the time and frequency domains, as compared to a rectangular window; it will be more appropriate to distinguish small pulses close to larger ones in the plane (ω, t). It can be shown (Papoulis, 1962) that the Gaussian window is that minimizing the product of the second moments of $w^2(t)$ and $|\mathcal{W}(\omega)|^2$.

Unlike the instantaneous spectrum, the global energy decomposition (8.23) applies also *locally*:

$$\int_{\mathcal{D}} S_x(\omega, t; w)\, d\omega\, dt$$

represents the energy contribution to the signal from the domain $\mathcal{D}$ of the plane (ω, t). However, there is an ambiguity at the limits, because of the influence of the values outside the domain. The size of the ambiguity is of order β along the frequency axis and of order T along the time axis.

8.3.3 Relation to the *PSD* of a stationary process

By definition,

$$S_x(\omega_0, t; w) = \frac{1}{2\pi} \int\int_{-\infty}^{\infty} w(t-u_1)w(t-u_2)\mathrm{E}[X(u_1)X(u_2)]e^{-j\omega_0(u_1-u_2)}du_1\, du_2$$

Performing the change of variables $t - u_1 = \tau_1$ and $t - u_2 = \tau_2$ and assuming the process stationary, we can transform this equation into

$$S_x(\omega_0, t; w) = \frac{1}{2\pi} \int\int_{-\infty}^{\infty} w(\tau_1)w(\tau_2)R_{xx}(\tau_2-\tau_1)e^{-j\omega_0(\tau_2-\tau_1)}d\tau_1\, d\tau_2$$

$$= \frac{1}{2\pi} \int\int_{-\infty}^{\infty} w(\tau_1)w(\tau_2) \int_{-\infty}^{\infty} \Phi_{xx}(\omega)e^{j\omega(\tau_2-\tau_1)}d\omega\, e^{-j\omega_0(\tau_2-\tau_1)}d\tau_1\, d\tau_2$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xx}(\omega)\, d\omega \int_{-\infty}^{\infty} w(\tau_1)e^{-j(\omega-\omega_0)\tau_1}d\tau_1 \int_{-\infty}^{\infty} w(\tau_2)e^{j(\omega-\omega_0)\tau_2}d\tau_2$$

and finally,

$$S_x(\omega_0, t; w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xx}(\omega)\, |\mathcal{W}(\omega-\omega_0)|^2\, d\omega \tag{8.28}$$

Thus, the physical spectrum is a *weighted average of the PSD in the vicinity of* ω_0. The frequency resolution is of the order $1/T$; as the duration of the window increases, $\mathcal{W}(\omega)$ tends towards a Dirac delta function in the frequency domain and the physical spectrum tends to the *local* value of $\Phi_{xx}(\omega)$. The digital estimation of the *PSD* from sample records of finite duration is often based on Equ.(8.21). The window function is chosen to minimize the *leakage* introduced in the estimator by the convolution (8.28).

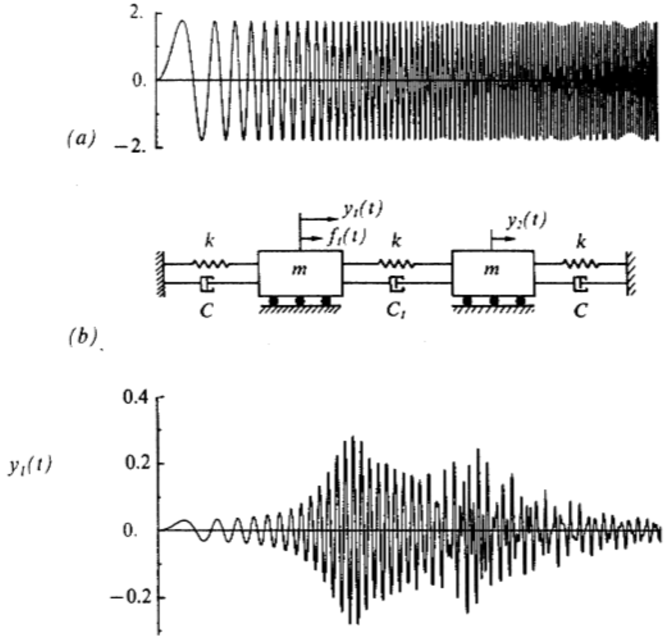


Figure 8.4: Response of a 2 d.o.f. oscillator to a sweep sine.

8.3.4 Example: Structural response to a sweep sine

Consider the 2 d.o.f. oscillator of Fig.8.4 ($\omega_1 = 2\pi$, $\omega_2 = 2\pi\sqrt{3}$; $\xi_1 = \xi_2 = 0.01$) excited by a *sweep sine*

$$f_1(t) = \sin\frac{\varrho t^2}{2} \tag{8.29}$$

with a sweep rate $\varrho = \pi/10$, so that the instantaneous frequency (time derivative of the argument) varies linearly from 0 to 3 Hz over a period of 60 s.

Figure 8.5 shows the physical spectra computed from the time-history of the excitation and the response with a Gaussian window of effective duration equal to 5 s. The physical spectrum of the excitation appears as a surface with a large constant amplitude following a straight line in the (ω, t) plane; which is exactly what one would expect from an energy map of a sweep sine with constant sweep rate. Cross sections parallel to one of the axes have a Gaussian shape. The physical spectrum of the response shows clearly that initally the response is fairly small and occurs at the instantaneous frequency of the excitation; large amplitude oscillations are excited when the first natural frequency is reached. As the excitation moves away from ω_1, there is an exponential decay of the first mode and large oscillations of the second mode occur when the instantaneous frequency becomes in tune with ω_2. Thus, the physical spectrum appears to provide a meaningful mapping of the energy in the signal. Although the sweep sine is not a random process, it can be seen as the limit of the response of a time-

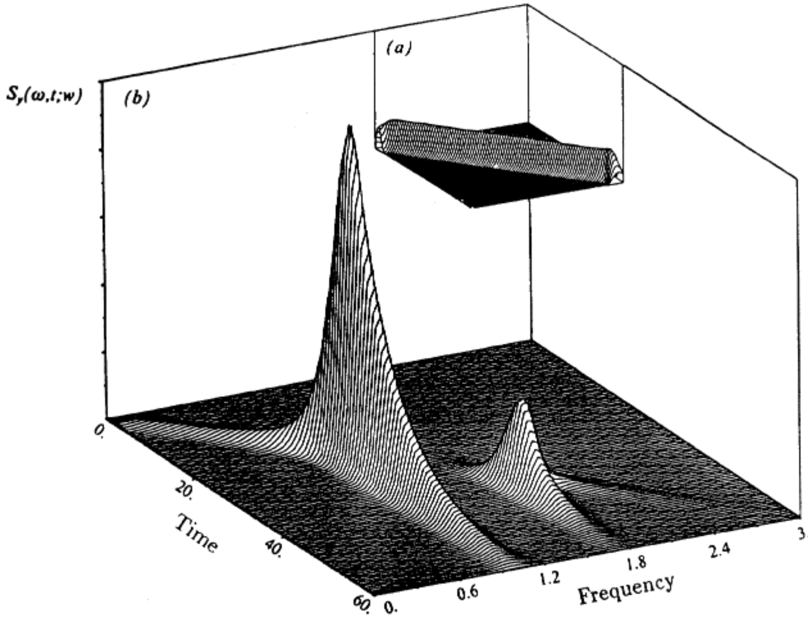


Figure 8.5: Mark's physical spectrum of the response of a 2 d.o.f. oscillator to a sweep sine. (a) Excitation. (b) Response.

varying lightly damped oscillator to a white noise excitation, or as a realization of a random process $F(t) = \sin(\varrho t^2/2 + \theta)$ where θ is a random phase.

The physical spectrum is a very convenient tool for signal analysis, but it does not provide a simple input-output relationship for linear systems, and it is fairly difficult to handle analytically. For these purposes, it is more convenient to use *Priestley's evolutionary spectrum.*

8.4 Priestley's Evolutionary Spectrum

8.4.1 Generalized harmonic analysis

From the discussion in the previous chapters, we know that

- The Fourier transform is defined in the strict sense for signals which *vanish at infinity.*

- By extension, the Fourier transform of a *periodic signal* consists of Dirac delta functions at frequencies equal to the harmonics of the signal, with amplitudes equal to the corresponding Fourier series coefficients.

- A *stationary* random process does not have a Fourier transform.

We now introduce a more general harmonic representation which can apply to the three types of signals:

$$X(t) = \int_{-\infty}^{\infty} e^{j\omega t} dZ(\omega) \tag{8.30}$$

where $Z(\omega)$ is a function uniquely determined by the form of $X(t)$, but which is *not necessarily differentiable.*

- If $Z(\omega)$ is differentiable, $dZ(\omega) = \mathcal{X}(\omega)\, d\omega/2\pi$ and the harmonic representation (8.30) is identical to the Fourier transform.
- If the signal is periodic, since $dZ(\omega)$ consists of a set of Dirac delta functions, $Z(\omega)$ is a *staircase*, the steps being located at the various harmonics of the signal, with amplitudes equal to the Fourier series coeficients.
- A stationary random signal can also be represented according to (8.30), with $dZ(\omega) = O(\sqrt{d\omega})$, so that the power per unit bandwidth, i.e. the power spectral density, $|dZ(\omega)|^2/d\omega = \Phi_{xx}(\omega)$ is *finite.*

In fact, if the process $X(t)$ is *stationary*, the process $Z(\omega)$ has *orthogonal increments*, and reciprocally:

$$\mathrm{E}[dZ(\omega)dZ^{\star}(\omega')] = \Phi_{xx}(\omega)\,\delta(\omega - \omega')\, d\omega d\omega' \tag{8.31}$$

This can be shown as follows:

$$\mathrm{E}[X^2(t)] = \mathrm{E}[X^2(0)] = \int\int_{-\infty}^{\infty} \mathrm{E}[dZ(\omega)\, dZ^{\star}(\omega')]$$

$$= \int_{-\infty}^{\infty} \Phi_{xx}(\omega)\, d\omega = \int\int_{-\infty}^{\infty} \Phi_{xx}(\omega)\,\delta(\omega - \omega')\, d\omega d\omega'$$

Thus, Equ.(8.30) expresses a stationary process as a sum of harmonic components with uncorrelated amplitudes. This property is the origin of the local interpretation of the *PSD* for stationary processes: If $Z(\omega)$ and $W(\omega)$ are the generalized harmonic representations of respectively the input $X(t)$ and the output $Y(t)$ of a linear system with frequency response function $H(\omega)$, each harmonic component is amplified according to $dW(\omega) = H(\omega)dZ(\omega)$, so that

$$\mathrm{E}[dW(\omega)dW^{\star}(\omega')] = H(\omega)H^{\star}(\omega')\mathrm{E}[dZ(\omega)dZ^{\star}(\omega')]$$

which, taking into account Equ.(8.31), implies

$$\Phi_{yy}(\omega) = |H(\omega)|^2\Phi_{xx}(\omega)$$

If the process is non-stationary, the representation is still applicable, but the orthogonality property (8.31) cannot be established any longer, which means that the process $Z(\omega)$ does not have orthogonal increments.

8.4.2 Evolutionary spectrum

In order to keep the local interpretation of the energy decomposition, Priestley has proposed the following non-stationary harmonic representation, which maintains the *orthogonal* nature of the process $Z(\omega)$:

$$X(t) = \int_{-\infty}^{\infty} a(\omega, t) e^{j\omega t} dZ(\omega) \tag{8.32}$$

Here $Z(\omega)$ is a process with independent increments and $a(\omega, t)$ represents a family of slowly-varying amplitude-modulating functions whose physical meaning is close to that of the envelope of a narrow-band process. $a(\omega, t)$ has a harmonic representation

$$a(\omega, t) = \int_{-\infty}^{\infty} e^{j\nu t} dA_\omega(\nu) \tag{8.33}$$

where $|dA_\omega(\nu)|$ presents a maximum at $\nu = 0$ (here, ω is a simple parameter). For a stationary process, Equ.(8.32) reduces itself to Equ.(8.30) with $a(\omega, t) = 1$. In general, there is an infinity of families $a(\omega, t)$ leading to the same representation (8.32); the best one is that leading to lower cut-off frequency ν_c of $|dA_\omega(\nu)|$. $T_c = 2\pi/\nu_c$ is a measure of the duration over which $a(\omega, t)$ can be regarded as approximately constant.

Equation (8.32) expresses the non-stationary process $X(t)$ as the limit sum of exponentials with slowly varying uncorrelated amplitudes $a(\omega, t)\, dZ(\omega)$. The autocorrelation function reads

$$\phi_{xx}(t_1, t_2) = \mathrm{E}[X(t_1) X^\star(t_2)] \tag{8.34}$$

$$= \int\int_{-\infty}^{\infty} a(\omega, t_1) a^\star(\omega', t_2)\, e^{j(\omega t_1 - \omega' t_2)}\, \mathrm{E}[dZ(\omega)\, dZ^\star(\omega')]$$

and, taking into account Equ.(8.31),

$$\phi_{xx}(t_1, t_2) = \int_{-\infty}^{\infty} a(\omega, t_1) a^\star(\omega, t_2) e^{j\omega(t_1 - t_2)} \Phi(\omega)\, d\omega \tag{8.35}$$

This relationship reduces itself to Equ.(3.50) for a stationary process. The variance is obtained by substituting $t_1 = t_2 = t$

$$\mathrm{E}[X^2(t)] = \int_{-\infty}^{\infty} |a(\omega, t)|^2 \Phi_{xx}(\omega)\, d\omega \tag{8.36}$$

From this equation, the *evolutionary spectrum* is defined as the frequency decomposition of the variance at t:

$$S_x(\omega, t) = |a(\omega, t)|^2 \Phi_{xx}(\omega) \tag{8.37}$$

It follows that

$$\mathrm{E}[X^2(t)] = \int_{-\infty}^{\infty} S_x(\omega, t)\, d\omega \tag{8.38}$$

By definition, $S_x(\omega, t)$ is a non-negative, even function of ω. The above definition was proposed by Priestley. In fact, it is formally simpler to merge $a(\omega, t)$ and $\Phi_{xx}(\omega)$ by defining

$$\alpha(\omega, t) = a(\omega, t)\Phi_{xx}(\omega)^{1/2} \tag{8.39}$$

The non-stationary harmonic representation becomes

$$X(t) = \int_{-\infty}^{\infty} \alpha(\omega, t) e^{j\omega t} dW(\omega) \tag{8.40}$$

where the process with orthogonal increments $W(\omega)$ is such that

$$\mathrm{E}[dW(\omega)\, dW(\omega')] = \delta(\omega - \omega')\, d\omega\, d\omega' \tag{8.41}$$

From Equ.(8.37) and (8.39),

$$S_x(\omega, t) = |\alpha(\omega, t)|^2 \tag{8.42}$$

We shall call $\alpha(\omega, t)$ the ***evolutionary amplitude*** of the scalar non-stationary process $X(t)$.

8.4.3 Vector process

The harmonic representation of non-stationary oscillatory scalar processes (8.40) can be extended to vector processes as follows:

$$X(t) = \int_{-\infty}^{\infty} \alpha(\omega, t)\, e^{j\omega t} dW(\omega) \tag{8.43}$$

where $W(\omega)$ is also a vector process, not necessarily of the same dimension as $X(t)$, with statistically independent components and orthogonal increments, in the sense

$$\mathrm{E}[dW(\omega) dW^\star(\omega')] = I\delta(\omega - \omega')\, d\omega\, d\omega' \tag{8.44}$$

Here I is the identity matrix and $\star$ stands for the conjugate transposed. We shall refer to $\alpha(\omega, t)$ as the ***evolutionary amplitude matrix*** of the process. With the foregoing definition, the ***evolutionary spectral matrix*** is defined as

$$S_x(\omega, t) = \alpha(\omega, t)\alpha^\star(\omega, t) \tag{8.45}$$

It is the non-stationary generalization of the power spectral density matrix; it is Hermitian and positive semi-definite. It follows that given $S_x(\omega, t)$, one can always find some matrix $\alpha(\omega, t)$ such that Equ.(8.45) is satisfied. It is in general

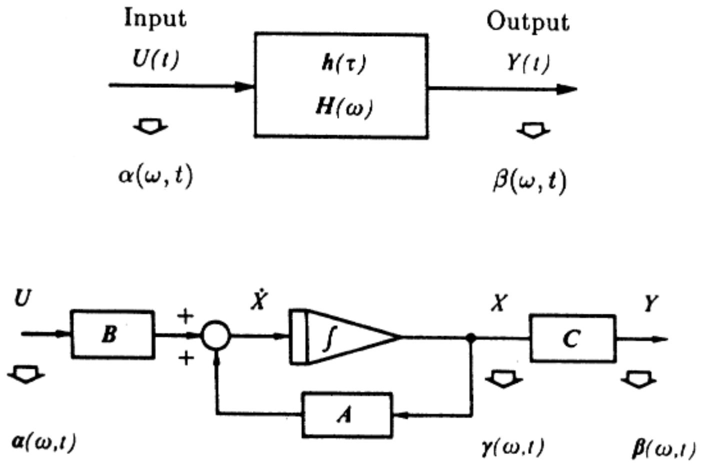


Figure 8.6: Evolutionary amplitude matrix input-output relationship for a linear time-invariant system. (a) Impulse response/transfer matrix representation. (b) State-space representation.

rectangular; the number of rows is equal to the dimension of X and the number of columns is equal to the maximum rank of $S_x(\omega,t)$. Note that there is no loss of generality in assuming that the components of dW are independent; a change of coordinates can always be performed in such a way that Equ.(8.44) is satisfied. The only restriction on the structure of $\alpha(\omega,t)$ is that it varies slowly with t.

Differentiating Equ.(8.43), one finds easily that the evolutionary amplitude matrix $\delta(\omega,t)$ of $\dot{X}(t)$

$$\dot{X}(t) = \int_{-\infty}^{\infty} \delta(\omega,t)e^{j\omega t}dW(\omega)$$

is related to $\alpha(\omega,t)$ by

$$\delta(\omega,t) = \dot{\alpha}(\omega,t) + j\omega\alpha(\omega,t) \tag{8.46}$$

8.4.4 Input-output relationship

If $h(t)$ is the impulse response matrix of a multivariate linear time-invariant system (Fig.8.6.a), the input-output relationship consists of the convolution integral

$$Y(t) = \int_{-\infty}^{\infty} h(t-\tau)U(\tau)\,d\tau \tag{8.47}$$

Substituting the spectral representation of $U(\tau)$

$$U(\tau) = \int_{-\infty}^{\infty} \alpha(\omega, \tau) e^{j\omega\tau} dW(\omega) \tag{8.48}$$

one gets

$$Y(t) = \int_{-\infty}^{\infty} \beta(\omega, t) e^{j\omega t} dW(\omega) \tag{8.49}$$

with

$$\beta(\omega, t) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} \alpha(\omega, t-\tau)\, d\tau \tag{8.50}$$

In the particular case where $\alpha(\omega, t)$ varies slowly, as compared to the memory of the system [i.e. the effective duration of $h(t)$], this equation can be approximated by

$$\beta(\omega, t) = H(\omega)\alpha(\omega, t) \tag{8.51}$$

(quasi-stationary approximation). For a second order vibrating system governed by

$$M\ddot{Y} + C\dot{Y} + KY = U \tag{8.52}$$

it is readily established, substituting Equ.(8.48) and (8.49), that the evolutionary amplitude matrix of the response satisfies the matrix differential equation

$$M\ddot{\beta} + (C + 2j\omega M)\dot{\beta} + (K + j\omega C - \omega^2 M)\beta = \alpha \tag{8.53}$$

8.4.5 State variable form

The simplest way to write the input-output relationship for linear systems is in state variable form (Fig.8.6.b). If the system is described by

$$\dot{X} = AX + BU, \qquad Y = CX \tag{8.54}$$

and if the evolutionary amplitude matrices of U, X and Y are respectively α, γ and β, substituting the spectral decompositions in (8.54), one finds that the evolutionary amplitude matrix of the state vector is governed by the matrix differential equation

$$\dot{\gamma}(\omega, t) = (A - j\omega I)\gamma(\omega, t) + B\alpha(\omega, t) \tag{8.55}$$

$$\beta(\omega, t) = C\gamma(\omega, t) \tag{8.56}$$

This equation applies for time-varying linear systems and for arbitrary $\alpha(\omega, t)$. If the excitation is a white noise, α does not depend of ω and the state vector is Markovian, as we shall see in chapter 9. For a time-invariant system, Equ.(8.55) can be solved efficiently with matrix exponentials, by noting that the eigenvalues of

$$D(\omega) = A - j\omega I \tag{8.57}$$

can be calculated directly from those of A. In fact, if λ_i and P are the eigenvalues and the eigenvectors of A, so that $P^{-1}AP = \text{diag}(\lambda_i)$, then

$$P^{-1}D(\omega)P = \text{diag}(\lambda_i - j\omega) \tag{8.58}$$

This means that the eigenvectors of $D(\omega)$ do not depend on ω and the eigenvalues are simply those of A, translated by $-j\omega$. It can be verified by substitution that the general solution of (8.55) is

$$\gamma(\omega,t) = e^{D(\omega)t}\gamma(\omega,0) + \int_0^t e^{D(\omega)(t-\tau)}B\,\alpha(\omega,\tau)\,d\tau \tag{8.59}$$

where $\gamma(\omega,0)$ is the initial condition. The matrix exponential is defined as

$$e^{Dt} = \sum_{k=0}^{\infty} D^k \frac{t^k}{k!} \tag{8.60}$$

Since P diagonalizes D, it also diagonalizes e^{Dt}:

$$e^{D(\omega)t} = e^{-j\omega t}P\text{diag}(e^{\lambda_i t})P^{-1} = e^{-j\omega t}e^{At} \tag{8.61}$$

From Equ.(8.59), the following recursive formula is readily obtained:

$$\gamma(\omega,t+\Delta t) = e^{D\Delta t}[\gamma(\omega,t) + \int_0^{\Delta t} e^{-D\tau}B\,\alpha(\omega,t+\tau)\,d\tau] \tag{8.62}$$

This equation can be used to devise approximate integration schemes, by making assumptions on the variation of the evolutionary amplitude of the excitation between successive time steps. For example, if one assumes that $\alpha(\omega,\tau)$ is constant in $(t, t+\Delta t]$.

$$\gamma(\omega,t+\Delta t) = e^{D\,\Delta t}\gamma(\omega,t) + D^{-1}(e^{D\,\Delta t} - I)B\,\alpha(\omega,t) \tag{8.63}$$

8.4.6 Remarks

The physical spectrum can be expressed as a weighted average of the evolutionary spectrum; the weighting function depends on the window used. The algebra to show this is rather lenghty and is left as an exercise (Problem P.8.2).

In the differential equation (8.55), ω acts as a parameter: The integration is made independently for each frequency. This excludes any energy transfer between frequencies.

8.5 Applications

8.5.1 Structural response to a sweep sine

This problem has already been considered in section 8.3.4, where we computed the physical spectra of the excitation and the response of a 2 d.o.f. system. Since

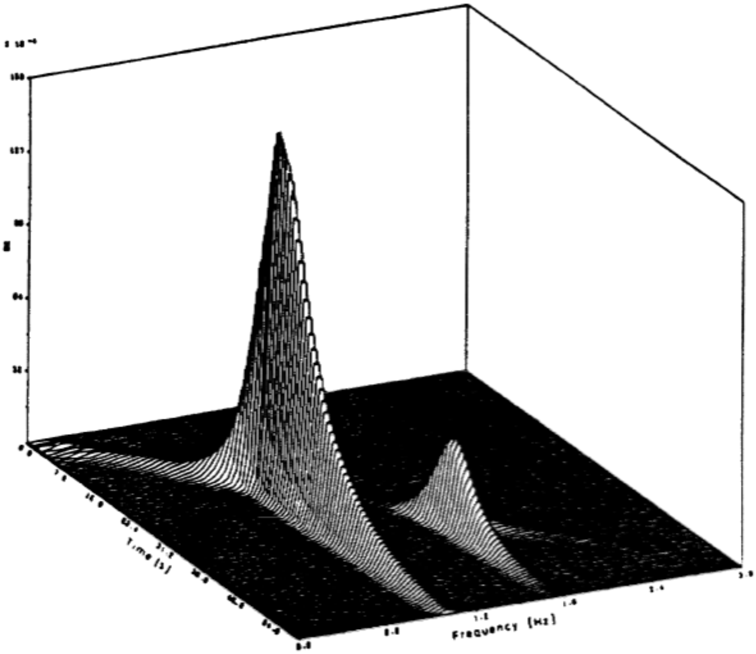


Figure 8.7: Evolutionary spectrum of the displacement y_1, in response to the excitation spectrum of Fig.8.5.a.

the physical spectrum is a weighted average of the evolutionary spectrum, it is interesting to investigate the analytical prediction of the evolutionary spectrum of the response when the spectrum of Fig.8.5.a is used as evolutionary spectrum of the input. Figure 8.7 shows the predicted response of y_1, obtained by numerical integration of Equ.(8.55). Comparing with Fig.8.5, one sees that the computed evolutionary spectrum is very similar to the physical spectrum computed from the time-history of the response. This confirms the physical meaning of the evolutionary spectrum.

8.5.2 Transient response of an oscillator

Figure 8.8 shows the analytical prediction of the evolutionary spectrum of the transient response of a single d.o.f. oscillator starting from rest, to a white noise excitation of limited duration (10 s). The damping ratio is $\xi = 0.05$. This example is useful to test numerical techniques, because the analytical solution is available (Problem 8.3).

8.5.3 Earthquake records

The non-stationary behaviour of earthquake records is known to affect both the amplitude and the frequency content of the time-history; there are more high

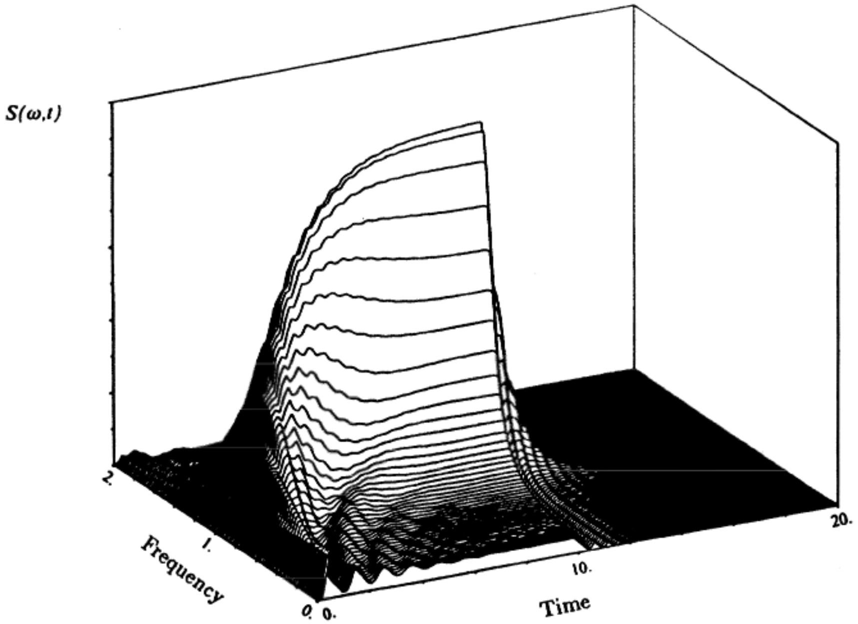


Figure 8.8: Evolutionary spectrum of the transient response of a single d.o.f. oscillator to a finite duration white noise excitation.

frequency components at the beginning than at the end of the record. These features can be represented by the following evolutionary spectrum

$$S_x(\omega, t) = \psi(\omega)\, t^b e^{-a(\omega)t} \tag{8.64}$$

where b and $a(\omega)$ take care of the transient behaviour in the time and frequency domains, and $\psi(\omega)$ is a stationary *PSD* providing an adequate energy distribution over ω. Examples of simulated records corresponding to various values of b and $a(\omega)$ are shown in Fig.8.9.

8.6 Summary

Various spectral descriptions of non-stationary oscillatory processes have been reviewed. They aim at supplying a mapping of the energy of the signal in the (frequency-time) plane. Priestley's ***evolutionary spectrum*** appears as the most appropriate ***analytical*** tool; it enjoys a simple input-output relationship for linear systems. Mark's ***physical spectrum*** is convenient for ***estimation***; it is a local weighted average of the evolutionary spectrum of which it can be used as an estimator (the weighting function is related to the window function used in the definition of the physical spectrum). The resolution in the time and frequency domains are not independent; they are related by the ***uncertainty principle***.

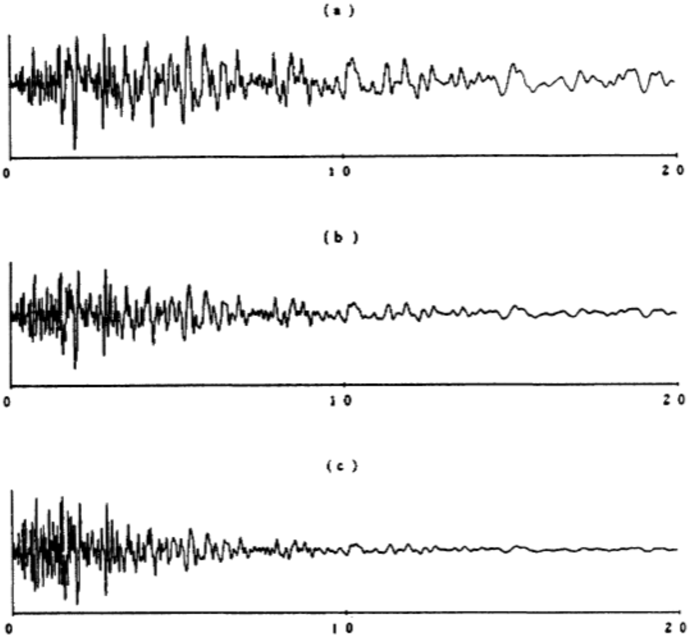


Figure 8.9: Simulated earthquake records for various values of b and $a(\omega)$.

8.7 References

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8.8 Problems

P.8.1 *Assume that the evolutionary amplitude matrix $\alpha(\omega,t)$ varies linearly between t and $t+\Delta t$. Show that Equ.(8.62) leads to the numerical integration scheme*

$$\gamma(\omega,t+\Delta t) = e^{D\,\Delta t}\gamma(\omega,t) + D^{-1}[e^{D\,\Delta t}B\alpha(\omega,t) - B\alpha(\omega,t+\Delta t)]$$

$$+(1/\Delta t)D^{-2}(I - e^{D\,\Delta t})B[\alpha(\omega,t) - \alpha(\omega,t+\Delta t)]$$

P.8.2 *Show that the physical spectrum can be expressed as a weighted average of the evolutionary spectrum*

$$S_x(\omega_0,t;w) = \int_{-\infty}^{\infty} S_x(\omega,t)|\Gamma(\omega,t)|^2 d\omega$$

where $\Gamma(\omega,t)$ is the generalized transfer function

$$\Gamma(\omega,t) = \int_{-\infty}^{\infty} h(\tau)\,e^{-j\omega\tau}\frac{\alpha(\omega,t-\tau)}{\alpha(\omega,t)}d\tau$$

and $h(\tau)$ is the impulse response of the narrow-band filter

$$h(\tau) = \frac{1}{\sqrt{2\pi}}w(\tau)e^{j\omega_0\tau}$$

based on the window $w(\tau)$ used in the definition of the physical spectrum.

P.8.3 *Consider a single d.o.f. oscillator starting from rest at $t=0$ and excited with a constant amplitude in time*

$$\alpha_1(\omega,t) = \phi(\omega) \qquad (t \geq 0)$$

Show that the evolutionary amplitude of the response is given by

$$\beta_1(\omega,t) = \phi(\omega)\frac{H(\omega)}{\omega_d}\{\omega_d - e^{-(\xi\omega_n+j\omega)t}[\omega_n\sin(\omega_d t+\theta) + j\omega\sin(\omega_d t)]\}$$

with $\omega_d = \omega_n\sqrt{1-\xi^2}$, $\tan\theta = \sqrt{1-\xi^2}/\xi$ and $H(\omega)$ is the frequency response function of the oscillator.

P.8.4 *Same as above with*

$$\alpha_2(\omega,t) = \phi(\omega)\,e^{-a(\omega)t} \qquad (t \geq 0)$$

Show that the response is

$$\beta_2(\omega,t) = \phi(\omega)\frac{H(\omega+ja)}{\omega_d}\{\omega_d e^{-at} - e^{-(\xi\omega_n+j\omega)t}[\omega_n\sin(\omega_d t+\theta) + (j\omega-a)\sin(\omega_d t)]\}$$