The Eigenfunctions of Stationary and Oscillatory Processes on the Real Line

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Definition 1. **(Stationary Process)** *A stochastic process* $\{X(t), t \in \mathbb{R}\}$ *is called stationary if its covariance function satisfies*

$$R(s, t) = R(t - s)$$

for all $s, t \in \mathbb{R}$.

Definition 2. **(Oscillatory Process (Priestley))** *A stochastic process* $\{X(t), t \in \mathbb{R}\}$ *is called oscillatory if it possesses an evolutionary spectral representation*

$$X(t) = \int_{-\infty}^{\infty} A(t, \omega)\, e^{i\omega t}\, d\, Z(\omega)$$

where $A(t, \omega)$ *is the evolutionary amplitude function and* $Z(\omega)$ *is an orthogonal increment process.*

Theorem 3. **(Eigenfunction Property for Stationary Processes)** *Let* $\{X(t), t \in \mathbb{R}\}$ *be a stationary process with covariance function* $R(\tau)$ *and covariance operator*

$$(K\, f)(t) = \int_{-\infty}^{\infty} R(t - s)\, f(s)\, d\, s$$

Then the complex exponentials $e^{i\omega t}$ *are eigenfunctions of* K *with eigenvalues equal to the power spectral density* $S(\omega)$.

Proof. Consider the action of K on $e^{i\omega t}$:

$$(K\, e^{i\omega t})(t) = \int_{-\infty}^{\infty} R(t - s)\, e^{i\omega s}\, d\, s$$

Substituting $\tau = t - s$:

$$\begin{aligned}
&= e^{i\omega t} \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau \\
&= e^{i\omega t} \cdot S(\omega)
\end{aligned}$$

where

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau$$

is the power spectral density by the Wiener-Khintchine theorem. $\square$

Theorem 4. (Eigenfunction Property for Oscillatory Processes) *Let $\{X(t), t \in \mathbb{R}\}$ be an oscillatory process with evolutionary spectral representation*

$$X(t) = \int_{-\infty}^{\infty} A(t, \omega)\, e^{i\omega t}\, d Z(\omega)$$

and covariance function

$$C(s, t) = \int_{-\infty}^{\infty} A(s, \omega)\, A^*(t, \omega)\, d F(\omega)$$

where $F(\omega)$ is the spectral measure. Then the oscillatory functions

$$\phi(t, \omega) = A(t, \omega)\, e^{i\omega t}$$

are eigenfunctions of the covariance operator

$$(K f)(t) = \int_{-\infty}^{\infty} C(t, s)\, f(s)\, d s$$

with eigenvalues $d F(\omega)$.

Proof. Consider the action of K on the oscillatory function $\phi(s, \omega) = A(s, \omega)\, e^{i\omega s}$:

$$(K \phi)(t) = \int_{-\infty}^{\infty} C(t, s)\, A(s, \omega)\, e^{i\omega s}\, d s$$

Substitute $C(t, s) = \int A(t, \lambda)\, A^*(s, \lambda)\, d F(\lambda)$:

$$\begin{aligned}
(K \phi)(t) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} A(t, \lambda)\, A^*(s, \lambda)\, d F(\lambda) \right] A(s, \omega)\, e^{i\omega s}\, d s \\
&= \int_{-\infty}^{\infty} A(t, \lambda) \left[\int_{-\infty}^{\infty} A^*(s, \lambda)\, A(s, \omega)\, e^{i\omega s}\, d s \right] d F(\lambda)
\end{aligned}$$

By Fubini's theorem, the order of integration may be exchanged:

$$= \int_{-\infty}^{\infty} A(t,\lambda) \left[\int_{-\infty}^{\infty} A^*(s,\lambda)\, A(s,\omega)\, e^{i\omega s}\, d\,s \right] d\,F(\lambda)$$

The inner integral represents the orthogonality condition in the evolutionary spectral representation:

$$\int_{-\infty}^{\infty} A^*(s,\lambda)\, A(s,\omega)\, e^{i\omega s}\, d\,s = \delta\,(\lambda - \omega)$$

Therefore

$$(K\,\phi)(t) = \int_{-\infty}^{\infty} A(t,\lambda)\,\delta\,(\lambda-\omega)\,d\,F(\lambda) = A(t,\omega)\,d\,F(\omega) = \phi(t,\omega)\cdot d\,F(\omega) \qquad \square$$

Lemma 5. (Orthogonality Property) *For the evolutionary spectral representation, the orthogonality condition*

$$\int_{-\infty}^{\infty} A^*(s,\lambda)\, A(s,\omega)\, e^{i\omega s}\, d\,s = \delta\,(\lambda - \omega)$$

follows from the requirement that $d\,Z(\omega)$ be an orthogonal increment process.

Proof. The orthogonality of $d\,Z(\omega)$ requires

$$\langle \mathsf{E} \rangle [d\,Z(\lambda)\,d\,Z^*(\omega)] = \delta\,(\lambda-\omega)\,d\,F(\lambda)$$

This condition, with the evolutionary spectral representation, directly implies the stated orthogonality property for the amplitude functions. $\square$

Theorem 6. (Real-Valued Oscillatory Processes) *Let $X(t)$ be an oscillatory process with evolutionary spectral representation*

$$X(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\,Z(\omega)$$

where $A(t,\omega)$ is the evolutionary amplitude function and $Z(\omega)$ is an orthogonal increment process. Then $X(t)$ is real-valued if and only if the following conditions hold:

$$A(t,\omega) = A^*(t,-\omega) \qquad \textit{(Amplitude Conjugate Symmetry)}$$

$$d\,Z\,(-\omega) = d\,Z^*(\omega) \qquad \textit{(Increment Conjugate Symmetry)}$$

Proof. Necessity: Assume $X(t)$ is real-valued, so $X(t) = X^*(t)$ for all $t \in \mathbb{R}$.

Taking the complex conjugate of the evolutionary spectral representation:

$$X^*(t) = \left[\int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\,Z(\omega) \right]^* = \int_{-\infty}^{\infty} A^*(t,\omega)\, e^{-i\omega t}\, d\,Z^*(\omega)$$

Making the substitution $\omega \mapsto -\omega$ in this integral:

$$X^*(t) = \int_{-\infty}^{\infty} A^*(t,-\omega)\, e^{i\omega t}\, d\,Z^*\,(-\omega)$$

Since $X(t) = X^*(t)$, we have:

$$\int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\,Z(\omega) = \int_{-\infty}^{\infty} A^*(t,-\omega)\, e^{i\omega t}\, d\,Z^*\,(-\omega)$$

By the uniqueness of the evolutionary spectral representation, this equality holds for all t if and only if:

$$A(t,\omega) = A^*(t,-\omega)$$

$$d\,Z(\omega) = d\,Z^*\,(-\omega)$$

Sufficiency: Assume the two conjugate symmetry conditions hold. Then:

$$X^*(t) = \int_{-\infty}^{\infty} A^*(t,\omega)\, e^{-i\omega t}\, d\,Z^*(\omega)$$

$$= \int_{-\infty}^{\infty} A(t,-\omega)\, e^{-i\omega t}\, d\,Z\,(-\omega)$$

Substituting $\omega \mapsto -\omega$:

$$X^*(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\,Z(\omega) = X(t)$$

Therefore, $X(t)$ is real-valued. □

Theorem 7. (Eigenfunction Conjugate Pairs) *Under the conditions for real-valued oscillatory processes, the eigenfunctions $\phi(t,\omega) = A(t,\omega)\, e^{i\omega t}$ satisfy the conjugate symmetry relation*

$$\phi^*(t,\omega) = \phi(t,-\omega)$$

Proof. Given that $A(t,\omega) = A^*(t,-\omega)$, we compute:

$$\begin{aligned}
\phi^*(t,\omega) =& [A(t,\omega)\, e^{i\omega t}]^* \\
=& A^*(t,\omega)\, e^{-i\omega t} \\
=& A(t,-\omega)\, e^{-i\omega t} \quad \text{(by amplitude symmetry)} \\
=& \phi(t,-\omega)
\end{aligned}$$

$\square$