Vitali and Fréchet Variations

1 Vitali Variation

1.1 Definition

Vitali variation is a fundamental concept in mathematical analysis, particularly in the study of functions of several variables. For a function $f: I \to \mathbb{R}$, where I is a rectangle in $\mathbb{R}^n$, the Vitali variation $V(f, I)$ is given by:

$$V(f, I) = \sup_P \sum_{J \in P} |f(a_J) - f(b_J)| \tag{1}$$

where P ranges over all partitions of I into subrectangles J, and a_J and b_J are opposite vertices of J.

Functions with finite Vitali variation possess several important properties:

1. They are bounded and continuous almost everywhere.
2. They can be expressed as the difference of two functions with nonnegative sums.
3. They have well-defined Riemann-Stieltjes integrals.

2 Fréchet Variation

2.1 Definition

A bimeasure F on a product space $\Omega_1 \times \Omega_2$ has finite Fréchet variation if:

$$V_F(\Omega_1, \Omega_2) = \sup_{\Pi_1, \Pi_2} \sum_{A \in \Pi_1} \sum_{B \in \Pi_2} |F(A, B)| \tag{2}$$

where Π_1 and Π_2 are finite partitions of Ω_1 and Ω_2 respectively.

3 Mathematical Implications

3.1 Strongly Harmonizable Processes

For strongly harmonizable processes, the correlation function is:

$$R(s, t) = \int_{\mathbb{R}} e^{i(s-t)\lambda} \, dF(\lambda) \tag{3}$$

where F is a complex-valued measure with finite Vitali variation.

3.2 Weakly Harmonizable Processes

For weakly harmonizable processes, the correlation function is:

$$R(s,t) = \int_{\mathbb{R}^2} e^{i(s\lambda_1 - t\lambda_2)}\, dF(\lambda_1, \lambda_2) \tag{4}$$

where F is a bimeasure with finite Fréchet variation.

3.3 Key Implications

1. Spectral measures: Strongly harmonizable processes have countably additive spectral measures.
2. Stochastic integration: More developed theory for strongly harmonizable processes.
3. Boundedness properties: Strongly harmonizable processes are bounded in probability.
4. Representation theory: Weakly harmonizable processes can be represented by positive definite contractive linear operators in a Hilbert space.
5. Continuity and differentiability: Strongly harmonizable processes have stronger continuity properties.