Approximate Riemann-Siegel Theta Function Via Stirling's Gamma Function Approximation

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May 17, 2025

The Riemann-Siegel theta function $\theta(t)$ is defined as:

$$\theta(t) = \arg\Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right) - \frac{t}{2}\log\pi$$

Theorem. (Stirling Approximation of $\boldsymbol{\theta(t)}$) *The approximation of the Riemann-Siegel theta function is:*

$$\theta(t) = \frac{t}{2}\log\left(\frac{t}{2\,\pi}\right) - \frac{t}{2} - \frac{\pi}{8} + O\left(\frac{1}{t}\right)$$

Theorem 1. (Inverse Formula) *The inverse theta function θ^{-1} is:*

$$\theta^{-1}(x) = 2\,\pi\exp\left(W\left(\frac{x}{\pi\,e}\right)\right) + O\left(\frac{\log x}{x}\right)$$

where W is the Lambert W function.

Proof. The definition of the theta function gives $\theta(t) = \arg\Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right) - \frac{t}{2}\log\pi$. Stirling's formula for the gamma function states:

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\,\pi) + \frac{1}{12\,z} + O\left(\frac{1}{z^3}\right)$$

Substituting $z = \frac{1}{4} + \frac{i\,t}{2}$:

$$\log\Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right) = \left(-\frac{1}{4} + \frac{i\,t}{2}\right)\log\left(\frac{1}{4} + \frac{i\,t}{2}\right) - \frac{1}{4} - \frac{i\,t}{2} + \frac{1}{2}\log(2\,\pi) + \frac{1}{12\left(\frac{1}{4} + \frac{i\,t}{2}\right)} + O\left(\frac{1}{t^3}\right)$$

For the complex number $\frac{1}{4} + \frac{i\,t}{2}$, the modulus equals $\frac{1}{2}\sqrt{\frac{1}{4} + t^2}$ and the argument equals $\arctan(2\,t)$.

The logarithm in polar form equals:

$$\log\left(\frac{1}{4}+\frac{i\,t}{2}\right)=\log\left(\frac{1}{2}\sqrt{\frac{1}{4}+t^2}\right)+i\arctan(2\,t)$$

Taking the imaginary part and subtracting $\frac{t}{2}\log\pi$ gives:

$$\theta(t)=\frac{t}{2}\log\left(\frac{t}{2\,\pi}\right)-\frac{t}{2}-\frac{\pi}{8}+O\!\left(\frac{1}{t}\right)$$

For the inverse theta function, given $x=\theta(t)$:

$$x=\frac{t}{2}\log\left(\frac{t}{2\,\pi}\right)-\frac{t}{2}-\frac{\pi}{8}+O\!\left(\frac{1}{t}\right)$$

Rearranging terms:

$$x+\frac{\pi}{8}=\frac{t}{2}\log\left(\frac{t}{2\,\pi}\right)-\frac{t}{2}+O\!\left(\frac{1}{t}\right)$$

Substituting $u=\frac{t}{2\,\pi}$:

$$x+\frac{\pi}{8}=\pi\,u\log(u)-\pi\,u+O\!\left(\frac{1}{u}\right)$$

This equation has the form $\pi\,u\log(u)-\pi\,u=x+\frac{\pi}{8}$. Dividing by π:

$$u\log(u)-u=\frac{x+\frac{\pi}{8}}{\pi}=\frac{x}{\pi}+\frac{1}{8}$$

The Lambert W function directly gives:

$$u=\exp\left(W\!\left(\frac{x}{\pi\,e}\right)\right)$$

Therefore, the inverse theta function is:

$$\theta^{-1}(x)=2\,\pi\exp\left(W\!\left(\frac{x}{\pi\,e}\right)\right)+O\!\left(\frac{\log x}{x}\right) \qquad \square$$