Unitary Time Changes of Stationary Processes Yield Oscillatory Processes

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are locally square-integrable. Applying U_θ to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}$, evolutionary spectrum $dF_t(\lambda) = \dot{\theta}(t)\, dF(\lambda)$, and expected zero-counting function $\mathbb{E}[N_{[0,T]}] = N_{\det}([0,T]) + \frac{[\theta(T) - \theta(0)]}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}}$, where $N_{\det}([0,T])$ counts deterministic zeros from critical points of the time-change. The sample paths of any non-degenerate second-order stationary process are locally square integrable, making the unitary time-change operator U_θ applicable to typical realizations. A zero-localization measure $d\mu(t) = \delta(Z(t))|\dot{Z}(t)|\, dt$ induces a Hilbert space $L^2(\mu)$ on the zero set of each oscillatory process realization $Z(t)$, and the multiplication operator $(Lf)(t) = t\, f(t)$ has simple pure point spectrum equal to the zero crossing set of Z.

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1 Gaussian Processes

Unless otherwise stated, all processes considered will be real-valued.

Theorem 1. *Let $X(u)$ be a real-valued process:*

$$X(u) \in \mathbb{R} \quad \forall u \in \mathbb{R} \tag{1}$$

Then its (complex-valued) random orthogonal spectral measure satisfies

$$d\,\bar{\Phi}(\lambda) = d\,\Phi\,(-\lambda) \tag{2}$$

and the corresponding covariance spectral measure F is even:

$$F\,(-A) = F(A) \tag{3}$$

Proof. 1. The spectral representation for $X(u)$ is

$$X(u) = \int_{-\infty}^{\infty} e^{i\lambda u}\; d\,\Phi(\lambda) \tag{4}$$

Since $X(u)$ is real-valued for each u,

$$\overline{X(u)} = X(u) \tag{5}$$

On the other hand,

$$\overline{X(u)} = \overline{\int_{-\infty}^{\infty} e^{i\lambda u}\; d\,\Phi(\lambda)} \tag{6}$$

$$= \int_{-\infty}^{\infty} \overline{e^{i\lambda u}}\, d\,\bar{\Phi}(\lambda) \tag{7}$$

$$= \int_{-\infty}^{\infty} e^{-i\lambda u}\; d\,\bar{\Phi}(\lambda) \tag{8}$$

By the substitution $\lambda \mapsto -\mu$,

$$\int_{-\infty}^{\infty} e^{-i\lambda u}\; d\,\bar{\Phi}(\lambda) = \int_{-\infty}^{\infty} e^{i\mu u}\; d\,\bar{\Phi}(-\mu) \tag{9}$$

So

$$X(u) = \int_{-\infty}^{\infty} e^{i\lambda u}\; d\,\Phi(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda u}\; d\,\bar{\Phi}(-\lambda) \tag{10}$$

By uniqueness of the spectral measure representation, it follows that

$$d\,\bar{\Phi}(\lambda) = d\,\Phi\,(-\lambda) \tag{11}$$

as (orthogonal) random measures.

2. The covariance function of X is

$$R(u) = \mathbb{E}\,(X(0)\,X(u)) = \int_{-\infty}^{\infty} e^{i\lambda u}\; d\,F(\lambda) \tag{12}$$

Since $X(u)$ is real-valued, $R(u)$ is real and $R\,(-u) = R(u)$. Thus,

$$R\,(-u) = \int_{-\infty}^{\infty} e^{-i\lambda u}\; d\,F(\lambda) = \int_{-\infty}^{\infty} e^{i\mu u}\; d\,F\,(-\mu) \tag{13}$$

Equating with $R(u)$,

$$\int_{-\infty}^{\infty} e^{i\lambda u}\, d\,F(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda u}\, d\,F\,(-\lambda) \tag{14}$$

for all u. By the uniqueness theorem for Fourier–Stieltjes transforms, this implies

$$d\,F(\lambda) = d\,F\,(-\lambda) \tag{15}$$

Thus for any Borel set A,

$$F\,(-A) = F(A) \tag{16}$$

establishing the evenness property. □

1.1 Definition

Definition 2. ***(Gaussian process)*** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T a nonempty index set. A family $\{X_t\colon t \in T\}$ of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Gaussian process if for every finite subset $\{t_1, \ldots, t_n\} \subset T$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ is multivariate normal (possibly degenerate). Equivalently, every finite linear combination $\sum_{i=1}^{n} a_i\, X_{t_i}$ is either almost surely constant or Gaussian. The mean function is $m(t) := \mathbb{E}[X_t]$ and the covariance kernel is*

$$K(s, t) = \operatorname{Cov}(X_s, X_t) \tag{17}$$

For any finite $(t_i)_{i=1}^{n} \subset T$, the matrix $K_{ij} = K(t_i, t_j)$ is symmetric positive semidefinite, and a Gaussian process is completely determined in law by m and K.

1.2 Stationary processes

Definition 3. ***[Cramér spectral representation]*** *[1] A zero-mean stationary process X with spectral measure F admits the sample path representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, d\,\Phi(\lambda) \tag{18}$$

which has covariance

$$R_X\,(t - s) = \int_{\mathbb{R}} e^{i\lambda(t-s)}\, d\,F(\lambda) \tag{19}$$

1.2.1 Fourier Transform Conventions

Definition 4. ***[Fourier transform conventions]*** *The forward and inverse Fourier transforms on $L^2(\mathbb{R})$ are defined by*

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, d\,u \tag{20}$$

and

$$f(u) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda)\, e^{i\lambda u}\, d\,\lambda \tag{21}$$

where the factor $\frac{1}{2\pi}$ is incorporated into the definition of the inverse transform.

1.2.2 Sample Path Realizations

Definition 5. ***[Locally square-integrable functions]*** *Define*

$$L^2_{\mathrm{loc}}(\mathbb{R}) := \left\{ f\colon \mathbb{R} \to \mathbb{C}\colon \int_K |f(t)|^2 \, dt < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \tag{22}$$

Remark 6. Every bounded measurable set in $\mathbb{R}$ is contained in a compact set; hence $L^2_{\mathrm{loc}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

Theorem 7. ***[Sample paths in $L^2_{\mathrm{loc}}(\mathbb{R})$]*** *Let $\{X(t)\}_{t\in\mathbb{R}}$ be a second-order stationary process with*

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \tag{23}$$

Then almost every sample path lies in $L^2_{\mathrm{loc}}(\mathbb{R})$.

Proof. Fix an arbitrary bounded interval $[a,b] \subset \mathbb{R}$ with $a < b$. Define the random variable

$$Y_{[a,b]} := \int_a^b X(t)^2 \, dt \tag{24}$$

By Fubini's theorem,

$$\mathbb{E}[Y_{[a,b]}] = \int_a^b \mathbb{E}[X(t)^2] \, dt = (b-a)\,\sigma^2 < \infty \tag{25}$$

By Markov's inequality, $\mathbb{P}\,(Y_{[a,b]} = \infty) = 0$. Thus $Y_{[a,b]} < \infty$ almost surely. Covering compacts by countably many dyadic intervals yields the result: for every compact $K \subset \mathbb{R}$, almost surely $\int_K X(t)^2 \, dt < \infty$. ☐

2 Oscillatory Processes

Definition 8. ***[Oscillatory process]*** *[2] Let F be a finite nonnegative Borel measure on $\mathbb{R}$. Let*

$$A_t \in L^2(F) \quad \forall t \in \mathbb{R} \tag{26}$$

be the gain function and

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{27}$$

be the corresponding oscillatory function; then an oscillatory process is a stochastic process which can be represented as

$$\begin{aligned} Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda)\, d\,\Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda) \end{aligned} \tag{28}$$

where Φ is a complex orthogonal random measure with spectral measure F which satisfies the relation

$$d\,\mathbb{E}[\Phi(\lambda)\overline{\Phi(\mu)}] = \delta\,(\lambda - \mu)\,d\,F(\lambda) \tag{29}$$

and has the corresponding covariance kernel

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)}e^{i\lambda(t-s)}\,d\,F(\lambda) \\ &= \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)}d\,F(\lambda) \end{aligned} \tag{30}$$

2.1 Triad of Gain Representations

Theorem 9. *[Triad of Gain Representations]Let $A_t(\omega)$ be a gain function as in (26) such that for each fixed $t \in \mathbb{R}$, the map $\omega \mapsto A_t(\omega)$ belongs to $L^2(\mathbb{R})$. Then the following relationships hold simultaneously:*

$$A_t(\omega) = \int_{-\infty}^{\infty} e^{i\lambda t}\,d\,H_\omega(\lambda) = \int_{-\infty}^{\infty} h_t(u)\,e^{i\omega u}\,d\,u \tag{31}$$

$$h_t(u) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} A_t(\omega)\,e^{-i\omega u}\,d\,\omega \tag{32}$$

$$H_\omega(\lambda) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} A_t(\omega)\,e^{-i\lambda t}\,d\,t \tag{33}$$

Proof. 1. Define $h(t,u)$ by (32):

$$h_t(u) := \frac{1}{2\,\pi}\int_{-\infty}^{\infty} A_t(\omega)\,e^{-i\omega u}\,d\,\omega \tag{34}$$

which is well-defined in $L^2(\mathbb{R})$ by Plancherel's theorem since $A_t \in L^2(\mathbb{R})$ by hypothesis. Substituting (34) into the right-hand side of (31) yields

$$\begin{aligned} \int_{-\infty}^{\infty} h_t(u)\,e^{i\omega u}\,d\,u &= \int_{-\infty}^{\infty}\left(\frac{1}{2\,\pi}\int_{-\infty}^{\infty} A_t(\nu)\,e^{-i\nu u}\,d\,\nu\right)e^{i\omega u}\,d\,u \\ &= \frac{1}{2\,\pi}\int_{-\infty}^{\infty} A_t(\nu)\left(\int_{-\infty}^{\infty} e^{i(\omega-\nu)u}\,d\,u\right)d\,\nu \end{aligned} \tag{35}$$

The inner integral is the delta function

$$\int_{-\infty}^{\infty} e^{i(\omega-\nu)u}\,d\,u = 2\,\pi\,\delta\,(\omega - \nu) \tag{36}$$

therefore,

$$\int_{-\infty}^{\infty} h_t(u)\,e^{i\omega u}\,d\,u = \int_{-\infty}^{\infty} A_t(\nu)\,\delta\,(\omega-\nu)\,d\,\nu = A_t(\omega) \tag{37}$$

which establishes (31).

2. To show that the gain $A_t(\lambda)$ and gain distributions $H_\omega(\lambda)$ are Fourier transform pairs fix $\omega \in \mathbb{R}$. By (31), the function $t \mapsto A_t(\omega)$ is the Fourier transform (in t) of the tempered distribution H_ω:

$$A_t(\omega) = \int_{-\infty}^{\infty} e^{i\lambda t}\, d\, H_\omega(\lambda) = \mathcal{F}_\lambda[H_\omega](t) \tag{38}$$

where $\mathcal{F}_\lambda$ denotes the Fourier transform with respect to λ. Consider the Fourier transform of $A_t(\omega)$ with respect to t:

$$\mathcal{F}_t[A_t(\omega)](\lambda) = \int_{-\infty}^{\infty} A_t(\omega)\, e^{-i\lambda t}\, d\, t \tag{39}$$

Substituting the representation of $A_t(\omega)$,

$$\begin{aligned}\int_{-\infty}^{\infty} A_t(\omega)\, e^{-i\lambda t}\, d\, t &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{i\lambda' t}\, d\, H_\omega(\lambda')\right) e^{-i\lambda t}\, d\, t\\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{i(\lambda'-\lambda)t}\, d\, t\right) d\, H_\omega(\lambda')\\ &= \int_{-\infty}^{\infty} 2\,\pi\,\delta\,(\lambda'-\lambda)\; d\, H_\omega(\lambda')\\ &= 2\,\pi\, H_\omega(\lambda)\end{aligned} \tag{40}$$

Thus,

$$\int_{-\infty}^{\infty} A_t(\omega)\, e^{-i\lambda t}\, d\, t = 2\,\pi\, H_\omega(\lambda) \tag{41}$$

and rearranging gives

$$H_\omega(\lambda) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} A_t(\omega)\, e^{-i\lambda t}\, d\, t \tag{42}$$

as an identity in $\mathcal{S}'(\mathbb{R})$ with respect to the λ-variable. This is precisely (31). □

Theorem 10. *[Duality of Gain Representations]Let $A_t(\omega)$ be a complex-valued function on $\mathbb{R}\times\mathbb{R}$ satisfying:*

1. *For each fixed $\omega \in \mathbb{R}$, the map $t \mapsto A_t(\omega)$ defines a tempered distribution in the time variable and admits a Fourier representation in the sense that there exists a tempered distribution $H_\omega \in \mathcal{S}'(\mathbb{R})$ with*

$$A_t(\omega) = \langle H_\omega, e^{i\lambda t}\rangle \quad \textit{for all } t \in \mathbb{R}.$$

2. *For each fixed $t \in \mathbb{R}$, the function $\omega \mapsto A_t(\omega)$ belongs to $L^2(\mathbb{R})$.*

Then there exist a family of tempered distributions $\{H_\omega\}_{\omega\in\mathbb{R}}$ on $\mathbb{R}$ and a function $h_t(u)$ on $\mathbb{R}\times\mathbb{R}$ such that the following relationships hold simultaneously:

1. *$A_t(\omega) = \langle H_\omega, e^{i\lambda t}\rangle$ for all $t \in \mathbb{R}$;*

2. *$A_t(\omega) = \int_{-\infty}^{\infty} h_t(u)\; e^{i\omega u}\; d\, u$ for all $\omega \in \mathbb{R}$, where the integral is taken in the sense of L^2-limits;*

3. *$h_t(u) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} A_t(\omega)\; e^{-i\omega u}\; d\,\omega$ for almost every $u \in \mathbb{R}$, with convergence in $L^2(\mathbb{R})$;*

4. *as an identity in $\mathcal{S}'(\mathbb{R})$ with respect to the variable λ,*

$$H_\omega = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} A_t(\omega)\; e^{-i\lambda t}\; d\, t,$$

i.e.

$$\langle H_\omega, \varphi(\lambda)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_t(\omega)\left(\int_{-\infty}^{\infty}\varphi(\lambda)\, e^{-i\lambda t}\, d\lambda\right) dt \quad \textit{for all } \varphi \in \mathcal{S}(\mathbb{R}).$$

Proof. 1. **Proof of (1).** By hypothesis 1, for each fixed ω there exists a tempered distribution $H_\omega \in \mathcal{S}'(\mathbb{R})$ such that

$$A_t(\omega) = \langle H_\omega, e^{i\lambda t}\rangle \quad \text{for all } t \in \mathbb{R}.$$

This is precisely the content of (1).

2. **Proof of (3).** Fix $t \in \mathbb{R}$. Hypothesis 2 gives $A_t(\cdot) \in L^2(\mathbb{R})$. By Plancherel's theorem, there exists a unique $h_t \in L^2(\mathbb{R})$ such that

$$h_t(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_t(\omega)\, e^{-i\omega u}\, d\omega$$

holds as an L^2-identity in u, with the integral understood as the L^2-limit over symmetric compact intervals in ω. This is exactly (3).

3. **Proof of (2).** For each fixed t, $A_t(\cdot) \in L^2(\mathbb{R})$ and h_t is its inverse Fourier transform in the L^2 sense. Applying the forward Fourier transform in the u-variable to h_t, one obtains

$$A_t(\omega) = \int_{-\infty}^{\infty} h_t(u)\, e^{i\omega u}\, du$$

as an L^2-identity in ω, again with the integral interpreted as the L^2-limit over symmetric compact intervals. This proves (2).

4. **Proof of (4).** For each fixed ω, hypothesis 1 and (1) say that $A_\cdot(\omega)$ is the Fourier transform (in t) of H_ω:

$$A_t(\omega) = \langle H_\omega, e^{i\lambda t}\rangle = \int_{-\infty}^{\infty} e^{i\lambda t}\, dH_\omega(\lambda)$$

in the distributional sense. Taking the Fourier transform in t of both sides and using the usual Fourier inversion formula in $\mathcal{S}'(\mathbb{R})$ yields

$$\int_{-\infty}^{\infty} A_t(\omega)\, e^{-i\lambda t}\, dt = 2\pi\, H_\omega$$

as an identity in $\mathcal{S}'(\mathbb{R})$ with respect to λ. Thus

$$H_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_t(\omega)\, e^{-i\lambda t}\, dt$$

in $\mathcal{S}'(\mathbb{R})$, which is the statement of (4). □

Theorem 11. ***[Real-valuedness criterion for oscillatory processes]*** *Let Z be an oscillatory process with oscillatory function*

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{43}$$

and spectral measure F. Then Z is real-valued if and only if

$$A_t(-\lambda) = \overline{A_t(\lambda)} \tag{44}$$

for F-almost every $\lambda \in \mathbb{R}$, equivalently

$$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \tag{45}$$

for F-almost every $\lambda \in \mathbb{R}$.

Proof.

1. Assume Z is real-valued. Then for all $t \in \mathbb{R}$,

$$Z(t) = \overline{Z(t)} \tag{46}$$

2. From the oscillatory representation (28),

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda) \tag{47}$$

3. Taking the complex conjugate of both sides of (47),

$$\overline{Z(t)} = \overline{\int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t}\, d\overline{\Phi(\lambda)} \tag{48}$$

4. For a real-valued process, the orthogonal random measure must satisfy the symmetry property from Theorem 1:

$$d\overline{\Phi(\lambda)} = d\,\Phi(-\lambda) \tag{49}$$

5. Substituting (49) into (48),

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t}\, d\,\Phi(-\lambda) \tag{50}$$

6. Apply the change of variables $\mu = -\lambda$, so $d\,\Phi(-\lambda) = d\,\Phi(\mu)$ and $e^{-i\lambda t} = e^{i\mu t}$:

$$\overline{Z(t)} \quad = \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t}\, d\,\Phi(\mu) \tag{51}$$

7. By (46), the right sides of (47) and (51) must be equal:

$$\int_{\mathbb{R}} A_t(\mu)\, e^{i\mu t}\, d\,\Phi(\mu) = \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t}\, d\,\Phi(\mu) \tag{52}$$

8. Since the stochastic integral representation is unique in $L^2(F)$, the integrands must be equal F-almost everywhere:

$$A_t(\lambda) = \overline{A_t(-\lambda)} \quad \text{for } F\text{-a.e. } \lambda \tag{53}$$

9. This is equivalent to (44). From (43),

$$\varphi_t(-\lambda) = A_t(-\lambda)\, e^{-i\lambda t} \tag{54}$$

10. Using (44),

$$\begin{aligned}\varphi_t(-\lambda) &= \overline{A_t(\lambda)} e^{-i\lambda t} \\ &= \overline{A_t(\lambda)\, e^{i\lambda t}} \\ &= \overline{\varphi_t(\lambda)}\end{aligned} \tag{55}$$

establishing (45).

11. Conversely, assume (44) holds. Reversing the steps from (51) to (46) shows that $\overline{Z(t)} = Z(t)$ for all t, so Z is real-valued. □

Theorem 12. ***[Existence of Oscillatory Processes]*** *Let F be an absolutely continuous spectral measure and the gain function*

$$A_t(\lambda) \in L^2(F) \quad \forall t \in \mathbb{R} \tag{56}$$

be measurable in both time and frequency; then the time-dependent spectral density is defined by

$$\begin{aligned}S_t(\lambda) &= \int_{\mathbb{R}} |A_t(\lambda)|^2 \, d\,F(\lambda) < \infty \\ &= \int_{\mathbb{R}} |A_t(\lambda)|^2 \, S(\lambda)\, d\,\lambda\end{aligned} \tag{57}$$

and there exists a complex orthogonal random measure Φ with spectral measure F such that for each sample path $\omega_0 \in \Omega$

$$Z(t, \omega_0) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda, \omega_0) \tag{58}$$

is well-defined in $L^2(\Omega)$ and has covariance R_Z as in (30).

Proof. 1. Define the space of simple functions on $\mathbb{R}$: for disjoint Borel sets $\{E_j\}_{j=1}^n$ with $F(E_j) < \infty$ and coefficients $\{c_j\}_{j=1}^n \subset \mathbb{C}$,

$$g(\lambda) = \sum_{j=1}^{n} c_j \mathbf{1}_{E_j}(\lambda) \tag{59}$$

2. For simple functions, define the stochastic integral

$$\int_{\mathbb{R}} g(\lambda)\, d\,\Phi(\lambda) := \sum_{j=1}^{n} c_j\, \Phi(E_j) \tag{60}$$

3. Compute the second moment:

$$\begin{aligned}\mathbb{E}\left[\left|\int_{\mathbb{R}} g(\lambda)\, d\,\Phi(\lambda)\right|^2\right] &= \mathbb{E}\left[\left|\sum_{j=1}^{n} c_j\, \Phi(E_j)\right|^2\right] \\ &= \mathbb{E}\left[\sum_{j=1}^{n}\sum_{k=1}^{n} c_j \bar{c_k} \Phi(E_j)\overline{\Phi(E_k)}\right]\end{aligned} \tag{61}$$

4. By linearity of expectation,

$$\mathbb{E}\left[\sum_{j=1}^{n}\sum_{k=1}^{n} c_j \bar{c_k} \Phi(E_j)\overline{\Phi(E_k)}\right] = \sum_{j=1}^{n}\sum_{k=1}^{n} c_j \bar{c_k} \mathbb{E}[\Phi(E_j)\overline{\Phi(E_k)}] \tag{62}$$

5. By the orthogonality relation (29), since $E_j \cap E_k = \emptyset$ for $j \neq k$,

$$\mathbb{E}[\Phi(E_j)\overline{\Phi(E_k)}] = \begin{cases} F(E_j) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \tag{63}$$

6. Substituting (63) into (62),

$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \bar{c_k} \mathbb{E}[\Phi(E_j)\overline{\Phi(E_k)}] = \sum_{j=1}^{n} |c_j|^2 F(E_j) \tag{64}$$

7. The right side of (64) equals

$$\sum_{j=1}^{n} |c_j|^2 F(E_j) = \int_{\mathbb{R}} |g(\lambda)|^2 \, d\, F(\lambda) \tag{65}$$

8. Therefore the isometry property holds for simple functions:

$$\mathbb{E}\left[\left|\int_{\mathbb{R}} g(\lambda) \, d\, \Phi(\lambda)\right|^2\right] = \int_{\mathbb{R}} |g(\lambda)|^2 \, d\, F(\lambda) \tag{66}$$

9. The space of simple functions is dense in $L^2(F)$. For any $h(\lambda) \in L^2(F)$ and $\epsilon > 0$, there exists a simple function $g(\lambda)$ such that

$$\int_{\mathbb{R}} |h(\lambda) - g(\lambda)|^2 \, d\, F(\lambda) < \epsilon \tag{67}$$

10. By the isometry (66) and completeness of $L^2(\Omega)$, the integral extends uniquely by continuity to all $h(\lambda) \in L^2(F)$.

11. Since $A_t \in L^2(F)$ by assumption (56), and $|e^{i\lambda t}| = 1$,

$$\int_{\mathbb{R}} |\varphi_t(\lambda)|^2 \, d\, F(\lambda) = \int_{\mathbb{R}} |A_t(\lambda)|^2 \, d\, F(\lambda) < \infty \tag{68}$$

so $\varphi_t \in L^2(F)$.

12. Therefore

$$\begin{aligned} Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda) \, d\, \Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda) \, e^{i\lambda t} \, d\, \Phi(\lambda) \end{aligned} \tag{69}$$

is well-defined in $L^2(\Omega)$.

13. To compute the covariance, use the sesquilinearity of the stochastic integral:

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda) \, d\, \Phi(\lambda) \overline{\int_{\mathbb{R}} \varphi_s(\mu) \, d\, \Phi(\mu)}\right] \end{aligned} \tag{70}$$

14. By Fubini's theorem for stochastic integrals,

$$\mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda) \, d\, \Phi(\lambda) \overline{\int_{\mathbb{R}} \varphi_s(\mu) \, d\, \Phi(\mu)}\right] = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\mu)} \mathbb{E}\left[d\, \Phi(\lambda) \overline{d\, \Phi(\mu)}\right] \tag{71}$$

15. Using the orthogonality relation (29),

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi_t(\lambda)\overline{\varphi_s(\mu)}\delta\,(\lambda-\mu)\,d\,F(\lambda)\,d\,F(\mu)=\int_{\mathbb{R}}\varphi_t(\lambda)\overline{\varphi_s(\lambda)}d\,F(\lambda) \tag{72}$$

16. Substituting the definition (27),

$$R_Z(t,s)=\int_{\mathbb{R}}A_t(\lambda)\overline{A_s(\lambda)}e^{i\lambda(t-s)}\,d\,F(\lambda) \tag{73}$$

as claimed in (30). □

Definition 13. ***[Forward impulse response]*** *For an oscillatory process $Z(t)=\int\varphi_t(\lambda)\,d\,\Phi(\lambda)$ with oscillatory function $\varphi_t(\lambda)$, define the forward impulse response function*

$$h(t,u)=\frac{1}{2\,\pi}\int_{-\infty}^{\infty}\varphi_t(\lambda)\,e^{-i\lambda u}\,d\,\lambda \tag{74}$$

That is, $h(t,\cdot)$ is the inverse Fourier transform (in the variable u) of the oscillatory function $\varphi_t(\cdot)$ at time t, with the $\frac{1}{2\,\pi}$ factor as in Definition 4.

Theorem 14. ***[Filter representation via impulse response]*** *Let X be a zero-mean stationary process with Cramér representation $X(u)=\int e^{i\lambda u}\,d\,\Phi(\lambda)$ and spectral measure F, and let Z be an oscillatory process with oscillatory function $\varphi_t(\lambda)$ and the same orthogonal random measure Φ. Then*

$$Z(t)=\int_{-\infty}^{\infty}h(t,u)\,X(u)\,d\,u \tag{75}$$

where $h(t,u)$ is the forward impulse response of Definition 13.

Proof. 1. Substitute the definitions of $h(t,u)$ and $X(u)$:

$$\begin{aligned}
\int_{-\infty}^{\infty}h(t,u)\,X(u)\,d\,u &= \int_{-\infty}^{\infty}\frac{1}{2\,\pi}\int_{-\infty}^{\infty}\varphi_t(\lambda)\,e^{-i\lambda u}\,d\,\lambda\int_{-\infty}^{\infty}e^{i\mu u}\,d\,\Phi(\mu)\,d\,u && \text{(substitute } h\text{)}\\
&= \frac{1}{2\,\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_t(\lambda)\int_{-\infty}^{\infty}e^{i(\mu-\lambda)u}\,d\,u\,d\,\lambda\;d\,\Phi(\mu) && \text{(apply Fubini)}\\
&= \frac{1}{2\,\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_t(\lambda)\cdot 2\,\pi\,\delta\,(\mu-\lambda)\;d\,\lambda\;d\,\Phi(\mu) && \text{(recognize } \delta\text{)}\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_t(\lambda)\,\delta\,(\mu-\lambda)\;d\,\lambda\;d\,\Phi(\mu) && \text{(simplify)}\\
&= \int_{-\infty}^{\infty}\varphi_t(\mu)\,d\,\Phi(\mu) && \text{(sift)}\\
&= Z(t) && \text{(recognize (28))}
\end{aligned} \tag{76}$$

□

3 Unitarily Time-Changed Stationary Processes

3.1 Unitary Time-Change Operator $U_\theta f$

Theorem 15. ***[Unitary time-change operator U_θ and its inverse U_θ^{-1}]*** *Let the time-change function $\theta\colon\mathbb{R}\to\mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with*

$$\dot{\theta}(t)>0 \tag{77}$$

almost everywhere and $\dot{\theta}(t)=0$ only on sets of Lebesgue measure zero. For f measurable, define

$$(U_\theta f)(t)=\sqrt{\dot{\theta}(t)}\, f(\theta(t)) \tag{78}$$

Its inverse is given by

$$(U_\theta^{-1} g)(s)=\frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{79}$$

For every compact set $K\subseteq\mathbb{R}$ and $f\in L^2_{\text{loc}}(\mathbb{R})$,

$$\int_K |(U_\theta f)(t)|^2 dt=\int_{\theta(K)} |f(s)|^2\, ds \tag{80}$$

Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\text{loc}}(\mathbb{R})$.

Proof. 1. Let $f\in L^2_{\text{loc}}(\mathbb{R})$ and let $K\subset\mathbb{R}$ be compact. From the definition (78),

$$\int_K |(U_\theta f)(t)|^2\, dt=\int_K \left|\sqrt{\dot{\theta}(t)}\, f(\theta(t))\right|^2 dt \tag{81}$$

2. Expanding the square,

$$\int_K \left|\sqrt{\dot{\theta}(t)}\, f(\theta(t))\right|^2 dt=\int_K \dot{\theta}(t)|f(\theta(t))|^2\, dt \tag{82}$$

3. Since θ is absolutely continuous and strictly increasing, $\theta'=\dot{\theta}$ exists almost everywhere and $\dot{\theta}(t)>0$ a.e.

4. Apply the change of variables $s=\theta(t)$. Then

$$ds=\dot{\theta}(t)\, dt \tag{83}$$

5. The inverse function $t=\theta^{-1}(s)$ exists since θ is strictly increasing and bijective.

6. As t ranges over K, the variable $s=\theta(t)$ ranges over $\theta(K)$.

7. Since θ is continuous and K is compact, $\theta(K)$ is compact.

8. Substituting (83) into (82),

$$\int_K \dot{\theta}(t)\, |f(\theta(t))|^2 dt=\int_{\theta(K)} |f(s)|^2\, ds \tag{84}$$

9. This establishes the local isometry (80).

10. To verify U_θ^{-1} is the inverse, compute:

$$(U_\theta^{-1} U_\theta f)(s)=U_\theta^{-1}(U_\theta f)(s) \tag{85}$$

11. By definition (79),

$$U_\theta^{-1}(U_\theta f)(s)=\frac{(U_\theta f)(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{86}$$

12. By definition (78),

$$(U_\theta f)(\theta^{-1}(s)) = \sqrt{\dot{\theta}(\theta^{-1}(s))}\, f(\theta(\theta^{-1}(s))) \tag{87}$$

13. Since $\theta \circ \theta^{-1} = \mathrm{id}$,

$$f(\theta(\theta^{-1}(s))) = f(s) \tag{88}$$

14. Substituting (87) and (88) into (86),

$$\frac{\sqrt{\dot{\theta}(\theta^{-1}(s))}\, f(s)}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} = f(s) \tag{89}$$

15. Therefore

$$U_\theta^{-1} U_\theta = \mathrm{id} \tag{90}$$

16. Similarly, compute:

$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)}\, (U_\theta^{-1} g)(\theta(t)) \tag{91}$$

17. By definition (79),

$$(U_\theta^{-1} g)(\theta(t)) = \frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} \tag{92}$$

18. Since $\theta^{-1} \circ \theta = \mathrm{id}$,

$$g(\theta^{-1}(\theta(t))) = g(t), \quad \theta^{-1}(\theta(t)) = t \tag{93}$$

19. Substituting (93) into (92),

$$\frac{g(t)}{\sqrt{\dot{\theta}(t)}} \tag{94}$$

20. Therefore from (91),

$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)} \cdot \frac{g(t)}{\sqrt{\dot{\theta}(t)}} = g(t) \tag{95}$$

21. Thus

$$U_\theta U_\theta^{-1} = \mathrm{id} \tag{96}$$

22. Combining (90) and (96), U_θ^{-1} is the two-sided inverse of U_θ on $L^2_{\mathrm{loc}}(\mathbb{R})$. □

3.2 Transformation of Stationary → Oscillatory Processes via U_θ

Theorem 16. ***[Unitary time changes of stationary processes produce oscillatory process]*** *Let X be zero-mean stationary as in Definition 3. For scaling function θ as in Theorem 15, define*

$$\begin{aligned} Z(t) &= (U_\theta X)(t) \\ &= \sqrt{\dot{\theta}(t)}\, X(\theta(t)) \end{aligned} \tag{97}$$

Then Z is a realization of an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \tag{98}$$

gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)} \tag{99}$$

and covariance kernel

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\Big[\sqrt{\dot{\theta}(t)}\, X(\theta(t))\overline{\sqrt{\dot{\theta}(s)}\, X(\theta(s))}\Big] \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\, \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\, R_X\,(\theta(t)-\theta(s)) \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\, \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\, dF(\lambda) \end{aligned} \tag{100}$$

Proof.

1. From the Cramér representation (18),

$$X(u) = \int_{\mathbb{R}} e^{i\lambda u}\, d\,\Phi(\lambda) \tag{101}$$

2. Substituting $u = \theta(t)$ into (101),

$$X(\theta(t)) = \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\,\Phi(\lambda) \tag{102}$$

3. From the definition (97),

$$Z(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) = \sqrt{\dot{\theta}(t)}\, \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\,\Phi(\lambda) \tag{103}$$

4. By linearity of the stochastic integral,

$$Z(t) = \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}\, d\,\Phi(\lambda) \tag{104}$$

5. Define

$$\varphi_t(\lambda) := \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \tag{105}$$

6. Then (104) becomes

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda)\, d\,\Phi(\lambda) \tag{106}$$

   which is the oscillatory representation (28).

7. To express this in terms of the standard oscillatory function form, define the gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)} \tag{107}$$

8. Then verify the oscillatory function form (27) factorizes

$$\begin{aligned}\varphi_t(\lambda) &= A_t(\lambda)\, e^{i\lambda t} \\ &= \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)}\, e^{i\lambda t} \\ &= \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t+t)} \\ &= \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}\end{aligned} \tag{108}$$

9. To compute the covariance, use (30):

$$R_Z(t,s) = \mathbb{E}[Z(t)\overline{Z(s)}] \tag{109}$$

10. Substituting (97),

$$R_Z(t,s) = \mathbb{E}\left[\sqrt{\dot{\theta}(t)}\, X(\theta(t))\overline{\sqrt{\dot{\theta}(s)}\, X(\theta(s))}\right] \tag{110}$$

11. Since $\dot{\theta}$ is deterministic,

$$R_Z(t,s) = \sqrt{\dot{\theta}(t)}\,\sqrt{\dot{\theta}(s)}\, \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \tag{111}$$

12. By stationarity of X, using (19),

$$\mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] = R_X(\theta(t)-\theta(s)) = \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\, dF(\lambda) \tag{112}$$

13. Substituting (112) into (111),

$$R_Z(t,s) = \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\, dF(\lambda) \tag{113}$$

establishing (100). □

3.2.1 Time-Varying Filter Representations

Theorem 17. *Let $\theta:\mathbb{R}\to\mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\dot{\theta}(t)>0$ almost everywhere. Let $X(u)$ be a stationary process, and define the oscillatory process obtained by the forward unitary time transformation U_θ*

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) = \int_{\mathbb{R}} h(t,u)\, X(u)\, du \tag{114}$$

where the forward impulse response function is given by

$$h(t,u) = \sqrt{\dot{\theta}(t)}\, \delta(u-\theta(t)) \tag{115}$$

For the unitary time-change case, this impulse response is a specialization of the general impulse response from Definition 13. Specifically, with $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}$, the Fourier-based impulse response

$$h(t,u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}\, e^{-i\lambda u}\, d\lambda = \sqrt{\dot{\theta}(t)}\, \delta(u-\theta(t)) \tag{116}$$

simplifies to the Dirac form via the inverse Fourier transform.

Then likewise the transformation can be reversed by expressing the stationary process as

$$X(u) = (U_\theta^{-1} Z)(u) = \frac{Z(\theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} = \int_{\mathbb{R}} g(u,t)\, Z(t)\, dt \tag{117}$$

where the inverse impulse response function is

$$g(u,t) = \frac{\delta\,(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \tag{118}$$

Proof. 1. Recall the forward unitary transformation from Theorem 15:

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) \tag{119}$$

2. To express this as a convolution integral, note that the Dirac delta function satisfies the sifting property: for any continuous function $f\colon \mathbb{R} \to \mathbb{R}$,

$$\int_{\mathbb{R}} f(u)\, \delta\,(u-a)\, du = f(a) \tag{120}$$

for any $a \in \mathbb{R}$.

3. Substituting $f(u) = X(u)$ and $a = \theta(t)$, which is well-defined since θ is bijective and continuous,

$$X(\theta(t)) = \int_{\mathbb{R}} X(u)\, \delta\,(u - \theta(t))\, du \tag{121}$$

4. Multiplying both sides by $\sqrt{\dot{\theta}(t)}$ and substituting into (119),

$$Z(t) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} X(u)\, \delta\,(u - \theta(t))\, du = \int_{\mathbb{R}} \Big[\sqrt{\dot{\theta}(t)}\, \delta\,(u - \theta(t))\Big] X(u)\, du \tag{122}$$

5. Thus, the forward impulse response function is

$$h(t,u) = \sqrt{\dot{\theta}(t)}\, \delta\,(u - \theta(t)) \tag{123}$$

establishing (114).

6. To verify this as a specialization of the Fourier-based impulse response from Definition 13, substitute $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}$:

$$h(t,u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}\, e^{-i\lambda u}\, d\lambda = \sqrt{\dot{\theta}(t)}\, \frac{1}{2\pi} \int e^{i\lambda(\theta(t)-u)}\, d\lambda = \sqrt{\dot{\theta}(t)}\, \delta\,(u - \theta(t)) \tag{124}$$

by the standard Fourier representation of the Dirac delta.

7. For the inverse transformation, recall from Theorem 15 that

$$X(u) = (U_\theta^{-1} Z)(u) = \frac{Z(\theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \tag{125}$$

8. Let $s=\theta^{-1}(u)$, so $u=\theta(s)$ and $Z(\theta^{-1}(u))=Z(s)$. The sifting property applied to $Z(t)$ with point $\theta^{-1}(u)$ gives

$$Z(\theta^{-1}(u))=\int_{\mathbb{R}} Z(t)\,\delta\,(t-\theta^{-1}(u))\,d\,t \tag{126}$$

9. Substituting into (125),

$$X(u)=\frac{1}{\sqrt{\dot{\theta}(\theta^{-1}(u))}}\int_{\mathbb{R}} Z(t)\,\delta\,(t-\theta^{-1}(u))\,d\,t=\int_{\mathbb{R}}\left[\frac{\delta\,(t-\theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}}\right] Z(t)\,d\,t \tag{127}$$

10. Thus, the inverse impulse response function is

$$g(u,t)=\frac{\delta\,(t-\theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \tag{128}$$

establishing (117) and (118).

11. To confirm invertibility, substitute (122) into (127). The integral becomes

$$X(u)=\int_{\mathbb{R}} g(u,t)\left[\int_{\mathbb{R}} h(t,v)\,X(v)\,d\,v\right]d\,t \tag{129}$$

12. By Fubini's theorem, since all measures are positive and the delta functions ensure finite support,

$$X(u)=\int_{\mathbb{R}}\int_{\mathbb{R}} g(u,t)\,h(t,v)\,X(v)\,d\,v\ \,d\,t \tag{130}$$

13. Integrating the impulse response composition

$$g(u,t)\,h(t,v)=\frac{\delta\,(t-\theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}}\cdot\sqrt{\dot{\theta}(t)}\,\delta\,(v-\theta(t)) \tag{131}$$

over t results in $t=\theta^{-1}(u)$, so

$$\sqrt{\dot{\theta}(t)}=\sqrt{\dot{\theta}(\theta^{-1}(u))} \tag{132}$$

and

$$\delta\,(v-\theta(t))=\delta\,(v-u) \tag{133}$$

yielding

$$\int_{\mathbb{R}} g(u,t)\,h(t,v)\,d\,t=\delta\,(v-u) \tag{134}$$

14. Thus, (130) simplifies to

$$\int_{\mathbb{R}} \delta\,(v-u)\,X(v)\,d\,v=X(u) \tag{135}$$

confirming the transformations are inverses. □

Corollary 18. *The evolutionary spectrum is*

$$d\,F_t(\lambda) = \dot{\theta}(t)\,d\,F(\lambda) \tag{136}$$

Proof. 1. The evolutionary spectrum is defined by

$$d\,F_t(\lambda) = |A_t(\lambda)|^2\,d\,F(\lambda) \tag{137}$$

2. From (99),

$$|A_t(\lambda)|^2 = \left|\sqrt{\dot{\theta}(t)}\,e^{i\lambda(\theta(t)-t)}\right|^2 \tag{138}$$

3. Since $|e^{i\alpha}| = 1$ for all real α,

$$|e^{i\lambda(\theta(t)-t)}|^2 = 1 \tag{139}$$

4. Therefore

$$|A_t(\lambda)|^2 = \left(\sqrt{\dot{\theta}(t)}\right)^2 \cdot 1 = \dot{\theta}(t) \tag{140}$$

5. Substituting (140) into (137),

$$d\,F_t(\lambda) = \dot{\theta}(t)\,d\,F(\lambda) \tag{141}$$ □

3.3 Covariance operator conjugation

Proposition 19. *Let*

$$(T_K\,f)(t) := \int_{\mathbb{R}} K(|t-s|)\,f(s)\,d\,s \tag{142}$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h}\,d\,F(\lambda) \tag{143}$$

Define the transformed kernel

$$K_\theta(s,t) := \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\,K(|\theta(t)-\theta(s)|) \tag{144}$$

then the corresponding integral covariance operator is conjugated for all $f \in L^2_{\mathrm{loc}}(\mathbb{R})$ by

$$(T_{K_\theta}\,f)(t) = (U_\theta\,T_K\,U_\theta^{-1}\,f)(t) \tag{145}$$

Proof. 1. From (145), expand the right side:

$$(U_\theta\,T_K\,U_\theta^{-1}\,f)(t) = \sqrt{\dot{\theta}(t)}\,(T_K\,U_\theta^{-1}\,f)(\theta(t)) \tag{146}$$

2. By definition (142),

$$(T_K\,U_\theta^{-1}\,f)(\theta(t)) = \int_{\mathbb{R}} K(|\theta(t)-s|)\,(U_\theta^{-1}\,f)(s)\,d\,s \tag{147}$$

3. By definition (79),

$$(U_\theta^{-1}\,f)(s) = \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{148}$$

4. Substituting (148) into (147),

$$\int_{\mathbb{R}} K(|\theta(t)-s|)\frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}}\,d\,s \tag{149}$$

5. Apply the change of variables $s=\theta(u)$, so $d\,s=\dot{\theta}(u)\,d\,u$ and $\theta^{-1}(s)=u$:

$$\int_{\mathbb{R}} K(|\theta(t)-\theta(u)|)\frac{f(u)}{\sqrt{\dot{\theta}(u)}}\dot{\theta}(u)\,d\,u \tag{150}$$

6. Simplify:

$$\int_{\mathbb{R}} K(|\theta(t)-\theta(u)|)\frac{\dot{\theta}(u)}{\sqrt{\dot{\theta}(u)}}\,f(u)\,d\,u=\int_{\mathbb{R}} K(|\theta(t)-\theta(u)|)\sqrt{\dot{\theta}(u)}\,f(u)\,d\,u \tag{151}$$

7. Substituting (151) into (146),

$$\sqrt{\dot{\theta}(t)}\int_{\mathbb{R}} K(|\theta(t)-\theta(u)|)\sqrt{\dot{\theta}(u)}\,f(u)\,d\,u \tag{152}$$

8. Bring the constant inside the integral:

$$\int_{\mathbb{R}} \sqrt{\dot{\theta}(t)}\,\sqrt{\dot{\theta}(u)}\,K(|\theta(t)-\theta(u)|)\,f(u)\,d\,u \tag{153}$$

9. By definition (144),

$$\sqrt{\dot{\theta}(t)}\,\sqrt{\dot{\theta}(u)}\,K(|\theta(t)-\theta(u)|)=K_\theta(u,t) \tag{154}$$

10. Therefore

$$\int_{\mathbb{R}} K_\theta(u,t)\,f(u)\,d\,u=(T_{K_\theta}f)(t) \tag{155}$$

establishing (145). □

4 Zero Localization

Definition 20. *Let Z be real-valued with $Z\in C^1(\mathbb{R})$ having only simple zeros*

$$Z(t_0)=0\Rightarrow\dot{Z}(t_0)\neq 0 \tag{156}$$

Define, for Borel $B\subset\mathbb{R}$,

$$\mu(B)=\int_{\mathbb{R}} \mathbf{1}_B(t)\,\delta(Z(t))|\dot{Z}(t)|\,d\,t \tag{157}$$

Theorem 21. *Under the assumptions of Definition 20, zeros are locally finite and one has*

$$\delta(Z(t))=\sum_{t_0:Z(t_0)=0}\frac{\delta\,(t-t_0)}{|\dot{Z}(t_0)|} \tag{158}$$

whence

$$\mu = \sum_{t_0 : Z(t_0)=0} \delta_{t_0} \tag{159}$$

Proof. 1. For any smooth test function ϕ with compact support, apply the standard change of variables formula for the delta function. Let $\{t_0^{(1)}, t_0^{(2)}, \ldots\}$ denote the zeros of Z.

2. By the change of variables formula for distributions,

$$\int_{\mathbb{R}} \phi(t)\, \delta(Z(t))\, d\, t = \sum_{t_0 : Z(t_0)=0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} \tag{160}$$

3. The right side of (160) equals

$$\sum_{t_0 : Z(t_0)=0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} = \sum_{t_0 : Z(t_0)=0} \int_{\mathbb{R}} \phi(t) \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|}\, d\, t \tag{161}$$

4. By Fubini's theorem (justified since the sum has locally finite terms due to C^1 regularity and simple zeros),

$$\sum_{t_0 : Z(t_0)=0} \int_{\mathbb{R}} \phi(t) \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|}\, d\, t = \int_{\mathbb{R}} \phi(t) \sum_{t_0 : Z(t_0)=0} \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|}\, d\, t \tag{162}$$

5. Comparing (160) and (162),

$$\int_{\mathbb{R}} \phi(t)\, \delta(Z(t))\, d\, t = \int_{\mathbb{R}} \phi(t) \sum_{t_0 : Z(t_0)=0} \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|}\, d\, t \tag{163}$$

6. Since ϕ is arbitrary,

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0)=0} \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|} \tag{164}$$

establishing (158).

7. Substituting (164) into the definition (157),

$$\mu(B) = \int_{\mathbb{R}} \mathbf{1}_B(t) \sum_{t_0 : Z(t_0)=0} \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|} |\dot{Z}(t)|\, d\, t \tag{165}$$

8. By the sifting property of the delta function, $|\dot{Z}(t)|$ evaluated at $t = t_0$ gives $|\dot{Z}(t_0)|$:

$$\int_{\mathbb{R}} \mathbf{1}_B(t) \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|} |\dot{Z}(t)|\, d\, t = \frac{\mathbf{1}_B(t_0) |\dot{Z}(t_0)|}{|\dot{Z}(t_0)|} = \mathbf{1}_B(t_0) \tag{166}$$

9. Summing over all zeros,

$$\mu(B) = \sum_{t_0 : Z(t_0)=0} \mathbf{1}_B(t_0) = \sum_{t_0 \in B : Z(t_0)=0} 1 \tag{167}$$

10. This is precisely the atomic measure

$$\mu = \sum_{t_0 : Z(t_0)=0} \delta_{t_0} \tag{168}$$

establishing (159). □

Definition 22. *Let $\mathcal{H}=L^2(\mu)$ be the Hilbert space with inner product*

$$\langle f,g\rangle=\int_{-\infty}^{\infty} f(t)\overline{g(t)}d\mu(t) \tag{169}$$

Proposition 23. ***[Atomic structure]*** *Let*

$$\mu=\sum_{t_0:Z(t_0)=0}\delta_{t_0} \tag{170}$$

then

$$\mathcal{H}\cong\left\{f:\{t_0:Z(t_0)=0\}\to\mathbb{C}:\sum_{t_0:Z(t_0)=0}|f(t_0)|^2<\infty\right\}\cong\ell^2 \tag{171}$$

with orthonormal basis $\{e_{t_0}\}_{t_0:Z(t_0)=0}$ where

$$e_{t_0}(t_1)=\delta_{t_0,t_1} \tag{172}$$

Proof.

1. By (170), μ is a purely atomic measure with atoms at the zero set.

2. For any $f\in L^2(\mu)$, the L^2 norm is

$$\|f\|^2_{L^2(\mu)}=\int_{\mathbb{R}}|f(t)|^2\,d\mu(t) \tag{173}$$

3. Substituting (170),

$$\int_{\mathbb{R}}|f(t)|^2\,d\mu(t)=\int_{\mathbb{R}}|f(t)|^2\sum_{t_0:Z(t_0)=0}\delta_{t_0}(dt)=\sum_{t_0:Z(t_0)=0}|f(t_0)|^2 \tag{174}$$

4. Therefore

$$\|f\|^2_{L^2(\mu)}=\sum_{t_0:Z(t_0)=0}|f(t_0)|^2 \tag{175}$$

5. This is precisely the ℓ^2 norm on the zero set.

6. Define the map $\Psi: L^2(\mu)\to\ell^2$ by

$$\Psi(f)=(f(t_0))_{t_0:Z(t_0)=0} \tag{176}$$

7. From (175), Ψ is an isometry:

$$\|\Psi(f)\|^2_{\ell^2}=\sum_{t_0:Z(t_0)=0}|f(t_0)|^2=\|f\|^2_{L^2(\mu)} \tag{177}$$

8. Ψ is surjective: for any sequence $(c_{t_0})\in\ell^2$, define $f(t)=\sum_{t_0}c_{t_0}\delta_{t_0}(t)$, which is in $L^2(\mu)$.

9. Therefore Ψ is a Hilbert space isomorphism, establishing (171).

10. For the orthonormal basis, define e_{t_0} by (172).

11. Then

$$\langle e_{t_0}, e_{t_1}\rangle = \int_{\mathbb{R}} e_{t_0}(t)\overline{e_{t_1}(t)} d\,\mu(t) = \sum_{s:Z(s)=0} \delta_{t_0,s}\,\delta_{t_1,s} = \delta_{t_0,t_1} \tag{178}$$

12. Therefore $\{e_{t_0}\}$ is an orthonormal set.

13. Since every $f \in L^2(\mu)$ can be written as

$$f = \sum_{t_0:Z(t_0)=0} f(t_0)\, e_{t_0} \tag{179}$$

the set $\{e_{t_0}\}$ is complete, hence an orthonormal basis. □

Definition 24. ***[Multiplication operator]*** *Define the linear operator*

$$L: \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H} \tag{180}$$

by

$$(L\,f)(t) = t\,f(t) \tag{181}$$

on the support of μ with domain

$$\mathcal{D}(L) := \left\{ f \in \mathcal{H}: \int |t\,f(t)|^2\, d\,\mu(t) < \infty \right\} \tag{182}$$

Theorem 25. ***[Self-adjointness and spectrum]*** *L is self-adjoint on $\mathcal{H}$ and has pure point, simple spectrum*

$$\sigma(L) = \overline{\{t \in \mathbb{R}: Z(t) = 0\}} \tag{183}$$

with eigenvalues $\lambda = t_0$ for each zero t_0 and corresponding eigenvectors e_{t_0}.

Proof.

1. For $f, g \in \mathcal{D}(L)$, compute the inner product:

$$\langle L\,f, g\rangle = \int_{\mathbb{R}} (L\,f)(t)\overline{g(t)} d\,\mu(t) \tag{184}$$

2. By definition (181),

$$\int_{\mathbb{R}} t\,f(t)\overline{g(t)} d\,\mu(t) \tag{185}$$

3. Since t is real-valued, $\bar{t} = t$, so

$$\int_{\mathbb{R}} t\,f(t)\overline{g(t)} d\,\mu(t) = \int_{\mathbb{R}} f(t)\overline{t\,g(t)} d\,\mu(t) \tag{186}$$

4. The right side of (186) is

$$\int_{\mathbb{R}} f(t)\overline{(L\,g)(t)} d\,\mu(t) = \langle f, L\,g\rangle \tag{187}$$

5. Therefore

$$\langle L\,f, g\rangle = \langle f, L\,g\rangle \tag{188}$$

for all $f, g \in \mathcal{D}(L)$, establishing that L is symmetric.

6. Since L is a multiplication operator on $L^2(\mu)$, it is self-adjoint (by standard functional analysis).

7. To determine the spectrum, compute the action on basis vectors. From (181) and (172),

$$(L\, e_{t_0})(t) = t\, e_{t_0}(t) = t\, \delta_{t_0}(t) \tag{189}$$

8. By the sifting property,

$$t\, \delta_{t_0}(t) = t_0\, \delta_{t_0}(t) = t_0\, e_{t_0}(t) \tag{190}$$

9. Therefore

$$L\, e_{t_0} = t_0\, e_{t_0} \tag{191}$$

10. This shows that each t_0 is an eigenvalue with eigenvector e_{t_0}.

11. Since the $\{e_{t_0}\}$ form a complete orthonormal basis (Proposition 23), the spectrum is pure point.

12. Each eigenspace is one-dimensional (spanned by e_{t_0}), so the spectrum is simple and given by the closure of the zero set

$$\sigma(L) = \{t_0 \colon Z(t_0) = 0\} = \overline{\{t \in \mathbb{R} \colon Z(t) = 0\}} \tag{192}$$ □

4.1 Simplicity of Zeros and Their Expected Counting Function

Theorem 26. *[Bulinskaya]Let $X(t)$ be a centered stationary Gaussian process with covariance function $K(h) = \mathbb{E}[X(t)\, X(t+h)]$ that is twice differentiable at $h = 0$ with $K(0) > 0$ and $\ddot{K}(0) < 0$. Then the zero set of X has no accumulation points almost surely. In particular, on any compact interval $[a, b]$, the number of zeros is almost surely finite.*

Proof. 1. The twice-differentiability of K at $h = 0$ implies that the spectral measure F has finite second moment:

$$\lambda_2 = \int_{\mathbb{R}} \omega^2\, d\, F(\omega) = -\ddot{K}(0) < \infty. \tag{193}$$

2. This finite second moment implies that X has mean-square continuous derivative $\dot{X}(t)$, with

$$\mathbb{E}[\dot{X}(t)^2] = -\ddot{K}(0) = \lambda_2 > 0. \tag{194}$$

3. Since $\dot{X}(t)$ is a non-degenerate centered Gaussian process, it is continuous almost surely and does not vanish identically on any interval.

4. For any zero t_0 of X (i.e., $X(t_0) = 0$), if $\dot{X}(t_0) \neq 0$, then X crosses transversally through zero at t_0, making t_0 an isolated zero.

5. The joint distribution of $(X(t_0), \dot{X}(t_0))$ is bivariate Gaussian with covariance matrix

$$\begin{pmatrix} K(0) & K'(0) \\ K'(0) & -\ddot{K}(0) \end{pmatrix} = \begin{pmatrix} K(0) & 0 \\ 0 & -\ddot{K}(0) \end{pmatrix}, \tag{195}$$

where $K'(0) = 0$ by evenness of K. Since X and $\dot{X}$ are uncorrelated Gaussians, they are independent.

6. At any zero t_0 of X, the derivative $\dot{X}(t_0)$ is Gaussian with mean zero and variance $-\ddot{K}(0)>0$, hence

$$\mathbb{P}\,[\dot{X}(t_0)=0|\,X(t_0)=0]=\mathbb{P}\,[\dot{X}(t_0)=0]=0. \tag{196}$$

   Thus almost surely $\dot{X}(t_0)\neq 0$, making t_0 an isolated zero.

7. Since every zero of X is isolated almost surely, the zero set can have no accumulation points almost surely.

8. On a compact interval $[a,b]$, a set with no accumulation points is finite, completing the proof. ☐

Theorem 27. *[Expected Zero-Counting Function with Deterministic Atoms]Let $\theta\colon\mathbb{R}\to\mathbb{R}$ be C^1 (continuously differentiable), strictly increasing, and bijective with $\dot{\theta}(t)\geq 0$ for all t and $\dot{\theta}(t)>0$ almost everywhere. Define the zero-derivative set*

$$T_0:=\{t\in\mathbb{R}\colon\dot{\theta}(t)=0\} \tag{197}$$

and assume that T_0 is at most countable with no accumulation points on any compact interval. Let X be a centered stationary Gaussian process with spectral measure F and covariance function

$$K(h)=\int_{\mathbb{R}}e^{i\omega h}\,d\,F(\omega) \tag{198}$$

twice differentiable at $h=0$ with $\ddot{K}(0)<0$ and $K(0)>0$. Define the unitarily time-changed process

$$Z(t)=\sqrt{\dot{\theta}(t)}\;X(\theta(t)) \tag{199}$$

Then Z is a centered Gaussian process with covariance

$$K_Z(t,s)=\sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\;K\left(\theta(t)-\theta(s)\right) \tag{200}$$

For any compact interval $[0,T]$, define

$$N_{\det}([0,T]):=\#(T_0\cap[0,T]) \tag{201}$$

By the assumption on T_0, $N_{\det}([0,T])$ is finite. The expected number of zeros of Z in $[0,T]$ decomposes as

$$\mathbb{E}[N_{[0,T]}(Z)]=N_{\det}([0,T])+\frac{\theta(T)-\theta(0)}{\pi}\sqrt{-\frac{\ddot{K}(0)}{K(0)}} \tag{202}$$

Proof.

1. Define

$$T_0:=\{t\in\mathbb{R}\colon\dot{\theta}(t)=0\},\qquad I_T:=[0,T]\setminus T_0 \tag{203}$$

   For any $t_0\in T_0$,

$$Z(t_0)=\sqrt{\dot{\theta}(t_0)}\;X(\theta(t_0))=0\cdot X(\theta(t_0))=0 \tag{204}$$

   so each $t_0\in T_0$ is a deterministic zero of Z on every sample path.

2. By the assumption that T_0 has no accumulation points on compact intervals, $T_0\cap[0,T]$ is finite. Thus

$$N_{\det}([0,T])=\#(T_0\cap[0,T])<\infty. \tag{205}$$

3. On the complement $I_T = [0,T] \setminus T_0$, the derivative satisfies $\dot\theta(t) > 0$, hence

$$Z(t) = 0 \quad \Longleftrightarrow \quad X(\theta(t)) = 0, \qquad t \in I_T \tag{206}$$

Define

$$Y(t) := X(\theta(t)). \tag{207}$$

The random zero set of Z on $[0,T]$ coincides with the zero set of Y on I_T, and the total zero count decomposes as

$$N_{[0,T]}(Z) = N_{\mathrm{det}}([0,T]) + N_{\mathrm{rand}}([0,T]) \tag{208}$$

where $N_{\mathrm{rand}}([0,T])$ counts zeros of Y in I_T.

4. For $t, s \in \mathbb{R}$,

$$K_Y(t,s) = \mathbb{E}\left[X(\theta(t))\, X(\theta(s))\right] = K\left(\theta(t) - \theta(s)\right) \tag{209}$$

Differentiate with respect to s:

$$\frac{\partial}{\partial s} K_Y(s,t) = -\dot\theta(s)\ \dot K\left(\theta(t) - \theta(s)\right) \tag{210}$$

Since $K(h)$ is even, $\dot K(0) = 0$. Taking $s \to t$,

$$\lim_{s \to t} \frac{\partial}{\partial s} K_Y(s,t) = 0 \tag{211}$$

5. The mixed partial derivative is

$$\frac{\partial^2}{\partial s\, \partial t} K_Y(t,s) = -\dot\theta(t)\, \dot\theta(s)\ \ddot K\left(\theta(t) - \theta(s)\right) \tag{212}$$

Taking $s \to t$,

$$\lim_{s \to t} \frac{\partial^2}{\partial s\, \partial t} K_Y(t,s) = -\dot\theta(t)^2\, \ddot K(0) \tag{213}$$

Also $K_Y(t,t) = K(0)$.

6. The Kac–Rice zero intensity for Y on I_T is

$$\rho_Y(t) = \frac{1}{\pi} \sqrt{\frac{K_Y(t,t) \cdot \lim_{s \to t} \frac{\partial^2}{\partial s\, \partial t} K_Y(t,s) - \left(\lim_{s \to t} \frac{\partial}{\partial s} K_Y(s,t)\right)^2}{K_Y(t,t)^2}} \tag{214}$$

Substituting,

$$K_Y(t,t) \cdot \lim_{s \to t} \frac{\partial^2}{\partial s\, \partial t} K_Y(t,s) - 0^2 = K(0) \cdot (-\dot\theta(t)^2\, \ddot K(0)) = K(0)\, \dot\theta(t)^2\, (-\ddot K(0)) \tag{215}$$

Therefore,

$$\rho_Y(t) = \frac{1}{\pi} \sqrt{\frac{\dot\theta(t)^2\, (-\ddot K(0))}{K(0)}} = \frac{\dot\theta(t)}{\pi} \sqrt{-\frac{\ddot K(0)}{K(0)}} \tag{216}$$

7. The expected random zero count is

$$\mathbb{E}[N_{\mathrm{rand}}([0,T])] = \int_{I_T} \rho_Y(t)\ d\,t \tag{217}$$

Since T_0 is countable (hence Lebesgue measure zero),

$$\mathbb{E}[N_{\mathrm{rand}}([0,T])] = \int_0^T \frac{\dot{\theta}(t)}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}}\ d\,t = \frac{\theta(T)-\theta(0)}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}} \tag{218}$$

8. The total zero count factors as

$$N_{[0,T]}(Z) = N_{\mathrm{det}}([0,T]) + N_{\mathrm{rand}}([0,T]) \tag{219}$$

so

$$\mathbb{E}[N_{[0,T]}(Z)] = N_{\mathrm{det}}([0,T]) + \frac{\theta(T)-\theta(0)}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}} \tag{220}$$ □

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