Unitary Time Changes of Stationary Processes Yield Oscillatory Processes and a Functional Framework Toward a Hilbert–Pólya Construction

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are square-integrable over σ-compact sets. Applying U_θ to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\theta'(t)}\; e^{i\lambda\theta(t)}$ and evolutionary spectrum $d F_t(\lambda) = \theta'(t)\; d F(\lambda)$. It is proved that sample paths of any non-degenerate second-order stationary process almost surely lie in $L^2_{\sigma\text{-comp}}(\mathbb{R})$, making the operator applicable to typical realizations. A zero-localization measure $\mu(d t) = \delta(Z(t))\,|Z'(t)|\; d t$ induces a Hilbert space $L^2(\mu)$ on the zero set of an oscillatory process Z, and the multiplication operator $(L f)(t) = t\, f(t)$ has pure point, simple spectrum equal to the zero set of Z. This produces a concrete operator scaffold consistent with a Hilbert–Pólya-type viewpoint.

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1 Function Spaces and Unitary Time Change

1.1 σ-compact sets and local L^2

Definition 1. *[σ-compact sets] A subset $U \subseteq \mathbb{R}$ is σ-compact if*

$$U = \bigcup_{n=1}^{\infty} K_n \tag{1}$$

with each K_n compact.

Definition 2. *[Square-integrability on σ-compact sets] Define*

$$L^2_{\sigma\text{-}comp}(\mathbb{R}) := \left\{ f\colon \mathbb{R} \to \mathbb{C}\colon \int_U |f(t)|^2 \, dt < \infty \text{ for every } \sigma\text{-compact } U \subseteq \mathbb{R} \right\} \tag{2}$$

Remark 3. Every bounded measurable set in $\mathbb{R}$ is σ-compact; hence $L^2_{\sigma\text{-comp}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

1.2 Unitary time-change operator

Definition 4. *[Unitary time-change] Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with $\theta'(t) > 0$ almost everywhere and $\theta'(t) = 0$ only on sets of Lebesgue measure zero. The function θ maps σ-compact sets to σ-compact sets. Define, for f measurable,*

$$(U_\theta f)(t) := \sqrt{\theta'(t)}\; f(\theta(t)) \tag{3}$$

Proposition 5. *[Inverse map] The inverse map is given by*

$$(U_\theta^{-1}\, g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\theta'(\theta^{-1}(s))}} \tag{4}$$

which is well-defined almost everywhere on every σ-compact set.

Proof. Since $\theta'(t) = 0$ only on sets of measure zero, and θ^{-1} maps sets of measure zero to sets of measure zero (as absolutely continuous bijective functions preserve measure-zero sets), the denominator $\sqrt{\theta'(\theta^{-1}(s))}$ is positive almost everywhere. The expression is therefore well-defined almost everywhere on every σ-compact set, which suffices for defining an element of $L^2_{\sigma\text{-comp}}(\mathbb{R})$. □

Theorem 6. *[Local unitarity on σ-compact sets] For every σ-compact set $U \subseteq \mathbb{R}$ and $f \in L^2_{\sigma\text{-}comp}(\mathbb{R})$,*

$$\int_U |(U_\theta f)(t)|^2\, dt = \int_{\theta(U)} |f(s)|^2\, ds \tag{5}$$

Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\sigma\text{-}comp}(\mathbb{R})$.

Proof. Let $f \in L^2_{\sigma\text{-comp}}(\mathbb{R})$ and let U be any σ-compact set. The local L^2-norm of $U_\theta f$ over U is:

$$\int_U |(U_\theta f)(t)|^2\, dt = \int_U \left|\sqrt{\theta'(t)}\, f(\theta(t))\right|^2 dt \tag{6}$$

$$= \int_U \theta'(t) |f(\theta(t))|^2\, dt \tag{7}$$

Since θ is absolutely continuous and strictly increasing, applying the change of variables $s = \theta(t)$ gives $ds = \theta'(t)\, dt$ almost everywhere. Since θ maps σ-compact sets to σ-compact sets, as t ranges over U, $s = \theta(t)$ ranges over $\theta(U)$, which is σ-compact. Therefore:

$$\int_U \theta'(t)\, |f(\theta(t))|^2\, dt = \int_{\theta(U)} |f(s)|^2\, ds \tag{8}$$

To verify that U_θ^{-1} is indeed the inverse, we compute explicitly. For any $f \in L^2_{\sigma\text{-comp}}(\mathbb{R})$:

$$(U_\theta^{-1} U_\theta f)(s) = (U_\theta^{-1})\, [\sqrt{\theta'(\cdot)}\, f(\theta(\,\cdot\,))](s) \tag{9}$$

$$= \frac{[\sqrt{\theta'(\theta^{-1}(s))}\, f(\theta(\theta^{-1}(s)))])}{\sqrt{\theta'(\theta^{-1}(s))}} \tag{10}$$

$$= \frac{\sqrt{\theta'(\theta^{-1}(s))}\, f(s)}{\sqrt{\theta'(\theta^{-1}(s))}} \tag{11}$$

$$= f(s) \tag{12}$$

where $\theta(\theta^{-1}(s)) = s$. Similarly, for any $g \in L^2_{\sigma\text{-comp}}(\mathbb{R})$:

$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\theta'(t)}\, (U_\theta^{-1} g)(\theta(t)) \tag{13}$$

$$= \sqrt{\theta'(t)}\, \frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\theta'(\theta^{-1}(\theta(t)))}} \tag{14}$$

$$= \sqrt{\theta'(t)}\, \frac{g(t)}{\sqrt{\theta'(t)}} \tag{15}$$

$$= g(t) \tag{16}$$

where $\theta^{-1}(\theta(t)) = t$. Therefore

$$U_\theta U_\theta^{-1} = U_\theta^{-1} U_\theta = I \tag{17}$$

on $L^2_{\sigma\text{-comp}}(\mathbb{R})$. □

Theorem 7. *[Unitarity on $L^2(\mathbb{R})$] $U_\theta: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is unitary:*

$$\int_{\mathbb{R}} |(U_\theta f)(t)|^2\, dt = \int_{\mathbb{R}} |f(s)|^2\, ds \tag{18}$$

and U_θ^{-1} is its inverse.

Proof. For $f \in L^2(\mathbb{R})$, we have:

$$\int_{\mathbb{R}} |(U_\theta f)(t)|^2\, dt = \int_{\mathbb{R}} \theta'(t) |f(\theta(t))|^2\, dt \tag{19}$$

By the change of variables $s = \theta(t)$ with $ds = \theta'(t)\, dt$, and since $\theta: \mathbb{R} \to \mathbb{R}$ is bijective:

$$\int_{\mathbb{R}} \theta'(t)\, |f(\theta(t))|^2\, dt = \int_{\mathbb{R}} |f(s)|^2\, ds \tag{20}$$

The inverse relationship follows from the same computation as in Theorem 6, applied globally. $\square$

2 Oscillatory Processes (Priestley)

Definition 8. *[Oscillatory process] Let F be a finite nonnegative Borel measure on $\mathbb{R}$. For each $t \in \mathbb{R}$, let $A_t \in L^2(F)$ and set $\varphi_t(\lambda) := A_t(\lambda)\, e^{i\lambda t}$. An* oscillatory process *is a stochastic process*

$$Z(t) := \int_{\mathbb{R}} \varphi_t(\lambda)\ \Phi\,(d\,\lambda) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\ \Phi\,(d\,\lambda) \tag{21}$$

where Φ is a complex orthogonal random measure with spectral measure F, that is,

$$\mathbb{E}[\Phi\,(d\,\lambda)\,\overline{\Phi\,(d\,\mu)}] = \delta\,(\lambda-\mu)\ d\,F(\lambda) \tag{22}$$

Its covariance kernel is

$$R_Z(t,s) = \mathbb{E}[Z(t)\overline{Z(s)}] = \int_{\mathbb{R}} A_t(\lambda)\,\overline{A_s(\lambda)}\,e^{i\lambda(t-s)}\ d\,F(\lambda) \tag{23}$$

Remark 9. [Real-valuedness] Z is real-valued if and only if $A_t\,(-\lambda) = \overline{A_t(\lambda)}$ for F-a.e. λ, equivalently $\varphi_t\,(-\lambda) = \overline{\varphi_t(\lambda)}$ for F-a.e. λ.

Theorem 10. *[Existence] If F is finite and $(A_t)_{t\in\mathbb{R}}$ is measurable in t with*

$$\int_{\mathbb{R}} |A_t(\lambda)|^2\ d\,F(\lambda) < \infty \forall t \in \mathbb{R} \tag{24}$$

then there exists a complex orthogonal random measure Φ with spectral measure F such that

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda)\,e^{i\lambda t}\ \Phi\,(d\,\lambda) \tag{25}$$

is well-defined in $L^2(\Omega)$ and has covariance R_Z as in (23) above.

Proof. We construct the stochastic integral using the standard extension procedure. First, define the integral for simple functions of the form

$$g(\lambda)=\sum_{j=1}^{n} c_j\, 1_{E_j}(\lambda) \tag{26}$$

where $\{E_j\}$ are disjoint Borel sets with $F(E_j)<\infty$ and $c_j\in\mathbb{C}$:

$$\int_{\mathbb{R}} g(\lambda)\ \Phi\,(d\,\lambda):=\sum_{j=1}^{n} c_j\,\Phi(E_j) \tag{27}$$

For such simple functions, the isometry property holds:

$$\begin{aligned}
\mathbb{E}\left[\left|\int_{\mathbb{R}} g(\lambda)\ \Phi\,(d\,\lambda)\right|^2\right] &= \mathbb{E}\left[\left|\sum_{j=1}^{n} c_j\,\Phi(E_j)\right|^2\right] && (28)\\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} c_j\bar{c_k}\mathbb{E}[\Phi(E_j)\overline{\Phi(E_k)}] && (29)\\
&= \sum_{j=1}^{n} |c_j|^2\, F(E_j) && (30)\\
&= \int_{\mathbb{R}} |g(\lambda)|^2\ d\,F(\lambda) && (31)
\end{aligned}$$

Since simple functions are dense in $L^2(F)$, we extend by continuity to all $g\in L^2(F)$. For each t, since

$$\varphi_t(\lambda)=A_t(\lambda)\,e^{i\lambda t} \tag{32}$$

and $A_t\in L^2(F)$, we have $\varphi_t\in L^2(F)$. Therefore

$$Z(t)=\int_{\mathbb{R}} \varphi_t(\lambda)\ \Phi\,(d\,\lambda) \tag{33}$$

is well-defined in $L^2(\Omega)$. The covariance is computed as:

$$\begin{aligned}
R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] && (34)\\
&= \mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda)\ \Phi\,(d\,\lambda)\int_{\mathbb{R}} \overline{\varphi_s(\mu)}\ \overline{\Phi\,(d\,\mu)}\right] && (35)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\mu)}\mathbb{E}\,[\Phi\,(d\,\lambda)\overline{\Phi\,(d\,\mu)}] && (36)\\
&= \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)}\,d\,F(\lambda) && (37)\\
&= \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)}e^{i\lambda(t-s)}\ d\,F(\lambda) && (38)
\end{aligned}$$

□

3 Stationary Processes and Time Change

3.1 Stationary processes

Definition 11. *[Cramér representation] A zero-mean stationary process X with spectral measure F admits the sample path representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\ \Phi\,(d\,\lambda) \tag{39}$$

which has covariance

$$R_X\,(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)}\ d\,F(\lambda) \tag{40}$$

3.2 Stationary → oscillatory via U_θ

Theorem 12. *[Time change yields oscillatory process] Let X be zero-mean stationary as in Definition 11. For θ as in Definition 4, define*

$$\begin{aligned} Z(t) \ &= (U_\theta\, X)(t) \\ &= \sqrt{\theta'(t)}\ X(\theta(t)) \end{aligned} \tag{41}$$

Then Z is oscillatory with oscillatory function

$$\varphi_t(\lambda) = \sqrt{\theta'(t)}\ e^{i\lambda\theta(t)} \tag{42}$$

, gain function

$$A_t(\lambda) = \sqrt{\theta'(t)}\ e^{i\lambda(\theta(t)-t)} \tag{43}$$

, and covariance

$$R_Z(t,s) = \int_{\mathbb{R}} \sqrt{\theta'(t)\,\theta'(s)}\ e^{i\lambda(\theta(t)-\theta(s))}\ d\,F(\lambda) \tag{44}$$

Proof. Applying the unitary time change operator to the spectral representation of $X(t)$:

$$\begin{aligned} Z(t) \ &= (U_\theta\, X)(t) \\ &= \sqrt{\theta'(t)}\ X(\theta(t)) \\ &= \sqrt{\theta'(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)}\ \Phi\,(d\,\lambda) \\ &= \int_{\mathbb{R}} \sqrt{\theta'(t)}\, e^{i\lambda\theta(t)}\ \Phi\,(d\,\lambda) \\ &= \int_{\mathbb{R}} \varphi_t(\lambda)\ \Phi\,(d\,\lambda) \end{aligned} \tag{45}$$

where

$$\varphi_t(\lambda) = \sqrt{\theta'(t)}\, e^{i\lambda\theta(t)} \tag{46}$$

To verify this constitutes an oscillatory representation according to Definition 8, we must write $\varphi_t(\lambda)$ in the form $A_t(\lambda)\, e^{i\lambda t}$:

$$\begin{aligned} \varphi_t(\lambda) &= \sqrt{\theta'(t)}\, e^{i\lambda\theta(t)} && (47)\\ &= \sqrt{\theta'(t)}\, e^{i\lambda(\theta(t)-t)}\, e^{i\lambda t} && (48)\\ &= A_t(\lambda)\, e^{i\lambda t} && (49) \end{aligned}$$

where

$$A_t(\lambda) = \sqrt{\theta'(t)}\, e^{i\lambda(\theta(t)-t)} \tag{50}$$

Since $\theta'(t) \geq 0$ almost everywhere and $\theta'(t) = 0$ only on sets of measure zero, $A_t(\lambda)$ is well-defined almost everywhere. Moreover, $A_t \in L^2(F)$ for each t since:

$$\begin{aligned} \int_{\mathbb{R}} |A_t(\lambda)|^2\, dF(\lambda) &= \int_{\mathbb{R}} \left|\sqrt{\theta'(t)}\, e^{i\lambda(\theta(t)-t)}\right|^2\, dF(\lambda) && (51)\\ &= \int_{\mathbb{R}} \theta'(t) |e^{i\lambda(\theta(t)-t)}|^2\, dF(\lambda) && (52)\\ &= \theta'(t) \int_{\mathbb{R}} dF(\lambda) && (53)\\ &= \theta'(t)\, F(\mathbb{R}) < \infty && (54) \end{aligned}$$

where we used $|e^{i\alpha}| = 1$ for all real α. The covariance is computed as:

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}]\\ &= \mathbb{E}\left[\sqrt{\theta'(t)}\, X(\theta(t))\sqrt{\theta'(s)}\overline{X(\theta(s))}\right]\\ &= \sqrt{\theta'(t)\,\theta'(s)}\, \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}]\\ &= \sqrt{\theta'(t)\,\theta'(s)}\, R_X\,(\theta(t)-\theta(s))\\ &= \sqrt{\theta'(t)\,\theta'(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\, dF(\lambda) \end{aligned} \tag{55}$$

□

Corollary 13. *[Evolutionary spectrum] The evolutionary spectrum is*

$$\begin{aligned} dF_t(\lambda) &= |A_t(\lambda)|^2\, dF(\lambda)\\ &= \theta'(t)\, dF(\lambda) \end{aligned} \tag{56}$$

Proof. By definition of the evolutionary spectrum and using the gain function from Theorem 12:

$$\begin{aligned} d\,F_t(\lambda) \;&= |A_t(\lambda)|^2\; d\,F(\lambda) \\ &= \left|\sqrt{\theta'(t)}\, e^{i\lambda(\theta(t)-t)}\right|^2\; d\,F(\lambda) \\ &= \theta'(t)|e^{i\lambda(\theta(t)-t)}|^2\; d\,F(\lambda) \\ &= \theta'(t)\; d\,F(\lambda) \end{aligned} \tag{57}$$

since $|e^{i\alpha}| = 1$ for all real α. ☐

3.3 Covariance operator conjugation

Proposition 14. *[Operator conjugation] Let*

$$(T_K f)(t) := \int_{\mathbb{R}} K(|t-s|)\; f(s)\; d\,s \tag{58}$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h}\; d\,F(\lambda) \tag{59}$$

Define the transformed kernel

$$K_\theta(s,t) := \sqrt{\theta'(t)\,\theta'(s)}\;\; K(|\theta(t)-\theta(s)|) \tag{60}$$

and corresponding integral covariance operator

$$(T_{K_\theta} f)(t) := \int_{\mathbb{R}} K_\theta(s,t)\; f(s)\; d\,s \tag{61}$$

Then

$$T_{K_\theta} = U_\theta\; T_K\; U_\theta^{-1} \tag{62}$$

on $L^2_{\sigma\text{-}comp}(\mathbb{R})$.

Proof. For any $g \in L^2_{\sigma\text{-comp}}(\mathbb{R})$, we transform the integral operator from coordinates (r, w) to coordinates (t, s) by applying both coordinate transformations $r = \theta(t)$ and $w = \theta(s)$ simultaneously with Jacobians $d\,r = \theta'(t)\; d\,t$ and $d\,w = \theta'(s)\; d\,s$.

The operator T_K in (r, w) coordinates is:

$$(T_K f)(r) = \int_{\mathbb{R}} K(|r-w|)\; f(w)\; d\,w \tag{63}$$

Under the simultaneous transformation $r=\theta(t)$ and $w=\theta(s)$:

$$
\begin{aligned}
((U_\theta T_K U_\theta^{-1})\,g)(t) &= \sqrt{\theta'(t)\,\theta'(s)}\int_{\mathbb{R}} K(|\theta(t)-\theta(s)|)\,(U_\theta^{-1}\,g)(\theta(s))\frac{\theta'(s)}{\sqrt{\theta'(s)}}\;d\,s\\
&= \sqrt{\theta'(t)\,\theta'(s)}\int_{\mathbb{R}} K(|\theta(t)-\theta(s)|)\frac{g(s)}{\sqrt{\theta'(s)}}\,\sqrt{\theta'(s)}\;d\,s\\
&= \sqrt{\theta'(t)\,\theta'(s)}\int_{\mathbb{R}} K(|\theta(t)-\theta(s)|)\,g(s)\;d\,s\\
&= \int_{\mathbb{R}} \sqrt{\theta'(t)\,\theta'(s)}\,K(|\theta(t)-\theta(s)|)\,g(s)\;d\,s\\
&= \int_{\mathbb{R}} K_\theta(t,s)\,g(s)\;d\,s=(T_{K_\theta}\,g)(t)
\end{aligned}
\tag{64}
$$

where

$$
K_\theta(t,s)=\sqrt{\theta'(t)\,\theta'(s)}\,K(|\theta(t)-\theta(s)|)
\tag{65}
$$

Therefore $T_{K_\theta}=U_\theta T_K U_\theta^{-1}$. □

4 Sample Paths Live in $L^2_{\sigma\text{-comp}}$

Theorem 15. *[Sample paths in $L^2_{\sigma\text{-}comp}(\mathbb{R})$] Let $\{X(t)\}_{t\in\mathbb{R}}$ be a second-order stationary process with*

$$
\sigma^2:=\mathbb{E}[X(t)^2]<\infty
\tag{66}
$$

then, almost surely, every sample path $t\mapsto X(\omega,t)$ belongs to $L^2_{\sigma\text{-}comp}(\mathbb{R})$.

Proof. Fix any bounded interval $[a,b]$ and consider the random variable

$$
Y_{[a,b]}:=\int_a^b X(t)^2\;d\,t
\tag{67}
$$

By stationarity and Fubini's theorem:

$$
\mathbb{E}[Y_{[a,b]}]=\mathbb{E}\left[\int_a^b X(t)^2\;d\,t\right]=\int_a^b \mathbb{E}[X(t)^2]\;d\,t=\int_a^b \sigma^2\;d\,t=\sigma^2\,(b-a)<\infty
\tag{68}
$$

By Markov's inequality, for any $M>0$:

$$
P\,(Y_{[a,b]}>M)\le\frac{\mathbb{E}[Y_{[a,b]}]}{M}=\frac{\sigma^2\,(b-a)}{M}
\tag{69}
$$

Taking $M\to\infty$, we conclude

$$
P\,(Y_{[a,b]}<\infty)=1
\tag{70}
$$

, i.e., almost surely the sample path is square-integrable on $[a,b]$.

Since $\mathbb{R}$ is the countable union of bounded intervals:

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n] \tag{71}$$

by countable subadditivity of probability:

$$P\left(\bigcap_{n=1}^{\infty} \left\{\int_{-n}^{n} X(t)^2 \, dt < \infty\right\}\right) = 1 \tag{72}$$

Now let U be any σ-compact set. Then

$$U = \bigcup_{m=1}^{\infty} K_m \tag{73}$$

where each K_m is compact. Each compact set K_m is bounded, so

$$K_m \subseteq [-N_m, N_m] \tag{74}$$

for some N_m. Therefore:

$$\int_U X(t)^2 \, dt = \int_{\bigcup_{m=1}^{\infty} K_m} X(t)^2 \, dt \leq \sum_{m=1}^{\infty} \int_{K_m} X(t)^2 \, dt \leq \sum_{m=1}^{\infty} \int_{-N_m}^{N_m} X(t)^2 \, dt \tag{75}$$

Since each integral

$$\int_{-N_m}^{N_m} X(t)^2 \, dt < \infty \tag{76}$$

almost surely, and the sum of countably many finite terms is finite, we have

$$\int_U X(t)^2 \, dt < \infty \tag{77}$$

almost surely.

This holds for every σ-compact set U, so almost surely every sample path lies in $L^2_{\sigma\text{-comp}}(\mathbb{R})$. □

5 Zero Localization and Hilbert–Pólya Scaffold

5.1 Zero localization measure

Definition 16. *[Zero localization measure] Let Z be real-valued with $Z \in C^1(\mathbb{R})$ having only simple zeros*

$$Z(t_0) = 0 \Rightarrow Z'(t_0) \neq 0 \tag{78}$$

Define, for Borel $B \subset \mathbb{R}$,

$$\mu(B) := \int_{\mathbb{R}} 1_B(t)\ \delta(Z(t))\, |Z'(t)|\ d t \tag{79}$$

Theorem 17. *[Atomicity on the zero set] For every $\phi \in C_c^{\infty}(\mathbb{R})$,*

$$\int_{\mathbb{R}} \phi(t)\ \delta(Z(t))\, |Z'(t)|\ d t = \sum_{t_0 : Z(t_0) = 0} \phi(t_0) \tag{80}$$

hence

$$\mu = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0} \tag{81}$$

Proof. Since all zeros of Z are simple and $Z \in C^1(\mathbb{R})$, by the inverse function theorem each zero t_0 is isolated. Near each zero t_0, Z is locally monotonic, so we can apply the one-dimensional change of variables formula for the Dirac delta.

Specifically, near t_0 where $Z(t_0) = 0$ and $Z'(t_0) \neq 0$, we have locally

$$Z(t) = (t - t_0)\, Z'(t_0) + O((t - t_0)^2) \tag{82}$$

The distributional identity for the Dirac delta under smooth changes of variables gives:

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0) = 0} \frac{\delta\,(t - t_0)}{|Z'(t_0)|} \tag{83}$$

Therefore:

$$\int_{\mathbb{R}} \phi(t)\ \delta(Z(t))\, |Z'(t)|\ d t = \int_{\mathbb{R}} \phi(t)\, |Z'(t)| \sum_{t_0 : Z(t_0) = 0} \frac{\delta\,(t - t_0)}{|Z'(t_0)|}\ d t \tag{84}$$

$$= \sum_{t_0 : Z(t_0) = 0} \int_{\mathbb{R}} \phi(t) \frac{|Z'(t)|\, \delta\,(t - t_0)}{|Z'(t_0)|}\ d t \tag{85}$$

$$= \sum_{t_0 : Z(t_0) = 0} \frac{|Z'(t_0)|\, \phi(t_0)}{|Z'(t_0)|} \tag{86}$$

$$= \sum_{t_0: Z(t_0)=0} \phi(t_0) \tag{87}$$

This shows that μ is the discrete measure

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \tag{88}$$

assigning unit mass to each zero. □

5.2 Hilbert space on zeros and multiplication operator

Definition 18. *[Hilbert space on the zero set] Let $\mathcal{H} := L^2(\mu)$ with inner product $\langle f, g \rangle = \int f(t)\overline{g(t)}\, \mu(dt)$.*

Proposition 19. *[Atomic structure] With $\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0}$,*

$$\mathcal{H} \cong \left\{ f: \{t_0: Z(t_0) = 0\} \to \mathbb{C}: \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \tag{89}$$

with orthonormal basis $\{e_{t_0}\}_{t_0: Z(t_0)=0}$, where $e_{t_0}(t_1) = \delta_{t_0 t_1}$.

Proof. By the atomic form of μ, for any $f \in L^2(\mu)$:

$$\|f\|_{\mathcal{H}}^2 = \int |f(t)|^2 \, \mu(dt) \tag{90}$$

$$= \int |f(t)|^2 \sum_{t_0: Z(t_0)=0} \delta_{t_0}(dt) \tag{91}$$

$$= \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 \tag{92}$$

This shows the isomorphism with ℓ^2. The functions e_{t_0} defined by $e_{t_0}(t_1) = \delta_{t_0 t_1}$ satisfy:

$$\langle e_{t_0}, e_{t_1} \rangle = \int e_{t_0}(t)\overline{e_{t_1}(t)}\, \mu(dt) = \sum_{t: Z(t)=0} \delta_{t_0 t}\, \delta_{t_1 t} = \delta_{t_0 t_1} \tag{93}$$

so they form an orthonormal set. Any $f \in \mathcal{H}$ can be written as

$$f = \sum_{t_0: Z(t_0)=0} f(t_0)\, e_{t_0} \tag{94}$$

proving they form a basis. □

Definition 20. *[Multiplication operator] Define* $L: \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H}$ *by* $(L\,f)(t) = t\ f(t)$ *on* $\operatorname{supp}(\mu)$ *with domain*

$$\mathcal{D}(L) := \left\{ f \in \mathcal{H}: \int |t\ f(t)|^2\ \mu\,(d\,t) < \infty \right\} \tag{95}$$

Theorem 21. *[Self-adjointness and spectrum] L is self-adjoint on* $\mathcal{H}$ *and has pure point, simple spectrum*

$$\sigma(L) = \{t \in \mathbb{R}: Z(t) = 0\} \tag{96}$$

with eigenvalues $\lambda = t_0$ *and eigenvectors* e_{t_0}.

Proof. First, we verify self-adjointness. For $f, g \in \mathcal{D}(L)$:

$$\begin{aligned} \langle L\,f, g\rangle &= \int (L\,f)(t)\overline{g(t)}\ \mu\,(d\,t) && (97)\\ &= \int t\,f(t)\overline{g(t)}\ \mu\,(d\,t) && (98)\\ &= \int f(t)\overline{t\,g(t)}\ \mu\,(d\,t) && (99)\\ &= \int f(t)\overline{(L\,g)(t)}\ \mu\,(d\,t) && (100)\\ &= \langle f, L\,g\rangle && (101) \end{aligned}$$

Thus L is symmetric.

In the atomic representation, L acts as

$$(L\,f)(t_0) = t_0\,f(t_0) \tag{102}$$

for each t_0 where $Z(t_0) = 0$. This is unitarily equivalent to the diagonal operator on ℓ^2 with diagonal entries $\{t_0: Z(t_0) = 0\}$. Such diagonal operators are self-adjoint.

For the spectrum calculation: We have

$$L\,e_{t_0} = t_0\,e_{t_0} \tag{103}$$

so each t_0 where $Z(t_0) = 0$ is an eigenvalue of L with eigenvector e_{t_0}. Since $\{e_{t_0}\}$ forms an orthonormal basis, L has pure point spectrum.

To show there are no other spectral points, suppose $\lambda \notin \{t_0: Z(t_0) = 0\}$. Then for any $f \in \mathcal{D}(L)$, $(L - \lambda\,I)\,f$ has components

$$((L - \lambda\,I)\ f)(t_0) = (t_0 - \lambda)\ f(t_0) \tag{104}$$

Since $t_0 - \lambda \neq 0$ for all zeros t_0, we can solve

$$(L - \lambda I)\, f = g \tag{105}$$

uniquely for any $g \in \mathcal{H}$ by setting

$$f(t_0) = \frac{g(t_0)}{t_0 - \lambda} \tag{106}$$

This shows $L - \lambda I$ is invertible, so $\lambda \notin \sigma(L)$. Therefore

$$\sigma(L) = \{t_0 \colon Z(t_0) = 0\} \tag{107}$$

and the eigenvalues are simple. □

Remark 22. [Operator scaffold] The construction

$$\text{stationary } X \xrightarrow{U_\theta} \text{oscillatory } Z \xrightarrow{\mu = \delta(Z)|Z'|\, dt} L^2(\mu) \xrightarrow{L: t\cdot} (L, \sigma(L)) \tag{108}$$

produces a concrete self-adjoint operator whose spectrum equals the zero set of Z, determined by the choice of time-change θ and spectral measure F. This provides an explicit realization consistent with Hilbert–Pólya approaches to encoding arithmetic information in operator spectra.

6 Appendix: Regularity and Simple Zeros

Definition 23. *[Regularity and simplicity] Assume $Z \in C^1(\mathbb{R})$ and every zero is simple: $Z(t_0) = 0 \Rightarrow Z'(t_0) \neq 0$.*

Lemma 24. *[Local finiteness and delta decomposition] Under Definition 23, zeros are locally finite and*

$$\delta(Z(t)) = \sum_{t_0 \colon Z(t_0) = 0} \frac{\delta\,(t - t_0)}{|Z'(t_0)|} \tag{109}$$

whence $\mu = \sum_{t_0 \colon Z(t_0) = 0} \delta_{t_0}$.

Proof. Since $Z \in C^1(\mathbb{R})$ and $Z'(t_0) \neq 0$ at each zero t_0, the inverse function theorem implies that Z is locally invertible near each zero. Specifically, there exists a neighborhood U_{t_0} of t_0 such that $Z|_{U_{t_0}}$ is strictly monotonic and invertible.

This implies zeros are isolated: if $Z(t_0)=0$ and $Z'(t_0)\neq 0$, then there exists $\epsilon>0$ such that $Z(t)\neq 0$ for $0<|t-t_0|<\epsilon$. Therefore zeros are locally finite (finitely many in any bounded interval).

For the distributional identity, consider the one-dimensional change of variables formula for the Dirac delta. If $g\colon I\to\mathbb{R}$ is C^1 on interval I with $g'(x)\neq 0$ for all $x\in I$, then

$$\delta(g(x))=\sum_{x_0\colon g(x_0)=0}\frac{\delta(x-x_0)}{|g'(x_0)|} \tag{110}$$

Applying this locally around each zero t_0 of Z, and since zeros are isolated, we can patch together the local results to obtain the global identity:

$$\delta(Z(t))=\sum_{t_0\colon Z(t_0)=0}\frac{\delta(t-t_0)}{|Z'(t_0)|} \tag{111}$$

Consequently:

$$\mu(dt)=\delta(Z(t))|Z'(t)|\;dt=\sum_{t_0\colon Z(t_0)=0}\frac{|Z'(t)|}{|Z'(t_0)|}\,\delta(t-t_0)\;dt=\sum_{t_0\colon Z(t_0)=0}\delta_{t_0}(dt) \tag{112}$$

where the last equality uses the fact that $|Z'(t_0)|/|Z'(t_0)|=1$ when evaluating at $t=t_0$. $\square$