Stationary Processes: Spectral Representation

by Cramer & Leadbetter

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1 Stationary Processes

In particular, this gives for $t = 0$

$$\int_{-\infty}^{\infty} g(\lambda, A)\, d\,\lambda = r(0) \tag{1}$$

Thus, if we think of the nonnegative function $g(\lambda, A) / r(0)$ as a probability density, the corresponding characteristic function will be $u\,(t\,A)\,r(t) / r(0)$. Now as $A \to \infty$, this tends for every t to $r(t) / r(0)$, which is a continuous function of t. It then follows from the continuity theorem for c.f.'s (see Section 2.7) that $r(t) / r(0)$ is a c.f., so that $r(t)$ is a c.f. multiplied with the constant $r(0)$. This immediately gives relation (3), and so Bochner's theorem is proved.

Thus, every covariance function of a stationary process $\xi(t)$ can be represented in the form (3), which may be regarded as a spectral representation of $r(t)$ (see Section 6.4). The never-decreasing function $F(\lambda)$ is called the spectral distribution function of the $\xi(t)$ process. Clearly, $F(\lambda)$ is only defined up to an additive constant, and we can always assume

$$F\,(-\infty) = 0, \quad F\,(+\infty) = r(0). \tag{2}$$

For the spectral representation (3), it evidently does not matter how we determine the value of $F(\lambda)$ in a discontinuity point, as long as the relation $F\,(\lambda - 0) \leq F(\lambda) \leq F\,(\lambda + 0)$ is satisfied for all λ. We shall usually take $F(\lambda) = F\,(\lambda + 0)$, so that $F(\lambda)$ is everywhere continuous to the right.

If $F(\lambda)$ is absolutely continuous, the derivative $f(\lambda) = F'(\lambda)$ will be called the spectral density of the process.

In the following section we shall obtain an analogous spectral representation for the stationary process $\xi(t)$ itself. The spectral representations for $r(t)$ and $\xi(t)$ will then be compared and used for various applications.

2 The Spectral Representation

As before, we consider a stationary process $\xi(t)$ with zero mean, and continuous in q.m. Let $F(\lambda)$ be the spectral distribution function defined in the preceding section, continuous to the right, and satisfying (2). The following theorem is fundamental for the theory of stationary processes.

Theorem 1. *[Spectral Representation Theorem]To every stationary $\xi(t)$ there can be assigned a process $\zeta(\lambda)$ with orthogonal increments, such that we have for each fixed t the spectral representation*

$$\xi(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, d\,\zeta(\lambda) \tag{3}$$

the stochastic integral being defined as a q.m. integral. $\zeta(\lambda)$ is defined up to an additive random variable. If this is fixed by taking $\zeta(-\infty) = 0$, we have

$$E\,\zeta(\lambda) = 0 \tag{4}$$

$$E\,|\zeta(\lambda)|^2 = F(\lambda) \tag{5}$$

$$E\,|d\,\zeta(\lambda)|^2 = d\,F(\lambda) \tag{6}$$

As already pointed out in Section 6.4, the spectral representation (3) shows how the $\xi(t)$ process is additively built up by elementary and mutually orthogonal harmonic oscillations

$$e^{it\lambda}\, d\,\zeta(\lambda) \tag{7}$$

each of which has an angular frequency λ, while the random amplitude and the random phase are determined by the elementary increment $d\,\zeta(\lambda)$. The $\zeta(\lambda)$ process will be called the spectral process associated with $\xi(t)$.

It has also been pointed out in Section 6.4 that an equality between random variables such as (3) should always be understood in the sense that the two members are equivalent random variables.

Proof. The spectral representation theorem may be proved by various methods. We shall first give a complete proof using Hilbert space theory, and then briefly indicate another proof by means of trigonometric integrals. From an abstract point of view, the theorem is equivalent to a theorem on groups of unitary transformations in Hilbert space due to Stone [Stone1], while the above probabilistic version was given by Cramér [Cramer3, Cramer4], and independently by Loève [Loeve1]. The proof given here will be framed in terms of random variables, and will only rest on the elements of Hilbert space theory as developed in Sections 5.6 and 5.7.

We shall consider two different realizations of Hilbert space, and establish a correspondence between them in such a way that corresponding pairs of elements have the same inner products.

Let $H(\xi)$ be the Hilbert space of the $\xi(t)$ process as defined in Section 5.7. This is the space spanned by the random variables $\xi(t)$ for all real t. The elements of $H(\xi)$ are random variables, namely, all finite linear combinations

$$\alpha_1 \xi(t_1) + \cdots + \alpha_n \xi(t_n) \tag{8}$$

and all limits in q.m. of sequences of such combinations. The inner product of any two elements η_1 and η_2 of $H(\xi)$ is

$$(\eta_1, \eta_2) = E \eta_1 \bar{\eta}_2 \tag{9}$$

Two elements are regarded as identical if their distance, as defined by this inner product (see Section 5.6), is zero.

On the other hand, we denote as usual by $L_2(F)$ the set of all non-random complex-valued functions $g(\lambda)$ such that the Lebesgue-Stieltjes integral

$$\int_{-\infty}^{\infty} |g(\lambda)|^2 \, dF(\lambda) \tag{10}$$

exists and is finite. The set $L_2(F)$ forms a Hilbert space, say $H(F)$, if addition and scalar multiplication are defined in the ordinary way, while the inner product of g_1 and g_2 is

$$(g_1, g_2) = \int_{-\infty}^{\infty} g_1(\lambda) \overline{g_2(\lambda)} \, dF(\lambda) \tag{11}$$

Two elements g_1 and g_2 are regarded as identical if their distance, as defined by this inner product, is zero, that is, if we have

$$\int_{-\infty}^{\infty} |g_1(\lambda) - g_2(\lambda)|^2 \, dF(\lambda) = 0 \tag{12}$$

We now establish a correspondence between $H(\xi)$ and $H(F)$ by successive steps in the following way: For every real t, let $\xi(t) \in H(\xi)$ and $e^{it\lambda} \in H(F)$ be corresponding elements. By Bochner's theorem we have

$$E \xi(t) \overline{\xi(u)} = \int_{-\infty}^{\infty} e^{it\lambda} e^{-iu\lambda} \, dF(\lambda) \tag{13}$$

so that inner products are preserved.

The correspondence is now extended to finite linear combinations of the $\xi(t)$ and the $e^{it\lambda}$, by letting

$$\eta = \sum_{k=1}^{n} \alpha_k \, \xi(t_k) \tag{14}$$

$$g(\lambda) = \sum_{k=1}^{n} \alpha_k \, e^{it_k \lambda} \tag{15}$$

be corresponding elements. Clearly, Bochner's theorem shows that inner products are still preserved, so that we have for any corresponding pairs η_1, η_2 and g_1, g_2,

$$(\eta_1, \eta_2) = (g_1, g_2) \tag{16}$$

and also for the corresponding squares of distances

$$E\,|\eta_1 - \eta_2|^2 = \int_{-\infty}^{\infty} |g_1(\lambda) - g_2(\lambda)|^2 \, d\,F(\lambda) \tag{17}$$

The last relation shows that the correspondence is one-one, since η_1 and η_2 cannot be identical without g_1 and g_2 also being identical, and conversely.

If $\eta_1, \eta_2, \ldots$ is a sequence of random variables of the form (14) converging in q.m. to a random variable η, we have for the corresponding $g_1, g_2, \ldots,$

$$E\,|\eta_m - \eta_n|^2 = \int_{-\infty}^{\infty} |g_m(\lambda) - g_n(\lambda)|^2 \, d\,F(\lambda) \tag{18}$$

It follows that the nonrandom functions $g_m(\lambda)$ converge in q.m. relative to the spectral d.f. $F(\lambda)$. If $g(\lambda)$ is their limit, we finally extend the correspondence by letting η and $g(\lambda)$ be corresponding elements, and conversely, when starting from a convergent $g_m(\lambda)$ sequence.

Now any element of $H(\xi)$ is, by definition, the limit in q.m. of a sequence of elements of the form (14). Similarly, any element of $H(F)$ is the limit in q.m. of a sequence of elements of the form (15). Thus, we have now extended the correspondence to the whole spaces $H(\xi)$ and $H(F)$. It follows from the properties of convergence in q.m. that the relations (16) and (17) will still hold under the extended conditions, so that inner products and distances are still preserved, and the correspondence is one-one.

For every λ_0, the function $g\,(\lambda - \lambda_0)$, where

$$g(\lambda) = \begin{cases} 1 & \text{for } \lambda \leq 0 \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

is an element of $H(F)$. If the corresponding element of $H(\xi)$ is denoted by $\zeta(\lambda_0)$, the increment $\zeta(\lambda_1)-\zeta(\lambda_0)$ corresponds to $g(\lambda-\lambda_1)-g(\lambda-\lambda_0)$. For $\lambda_0<\lambda_1$ this is the function taking the value 1 for $\lambda_0<\lambda\le\lambda_1$, and zero otherwise. If (λ_0,λ_1) and (λ_2,λ_3) are disjoint intervals, it follows from the preservation of inner products that

$$E\{[\zeta(\lambda_1)-\zeta(\lambda_0)]\cdot\overline{[\zeta(\lambda_3)-\zeta(\lambda_2)]}\}=\int_{-\infty}^{\infty}[g(\lambda-\lambda_1)-g(\lambda-\lambda_0)]\cdot\overline{[g(\lambda-\lambda_3)-g(\lambda-\lambda_2)]}\,dF(\lambda)=0 \tag{20}$$

so that $\zeta(\lambda)$ is a process with orthogonal increments. Taking $\lambda_1=\lambda_0$, and $\lambda_2=\lambda_0$, we obtain, since $F(\lambda)$ is continuous to the right,

$$E|[\zeta(\lambda_1)-\zeta(\lambda_0)]|^2=F(\lambda_1)-F(\lambda_0) \tag{21}$$

$$E|\zeta(\lambda_0)|^2=F(\lambda_0) \tag{22}$$

This process satisfies (6), the relation $E\zeta(\lambda)=0$ being evident.

In a discontinuity point λ of F, the limits in q.m. $\zeta(\lambda\pm0)$ both exist by Section 6.1, and we have $\zeta(\lambda)=\zeta(\lambda+0)$, while $\Delta\,\zeta(\lambda)=\zeta(\lambda)-\zeta(\lambda-0)$ is a random variable such that $E\,|\Delta\,\zeta(\lambda)|^2=\Delta\,F(\lambda)$. Further, let $-A=\lambda_1<\lambda_2<\cdots<\lambda_{n+1}=A$ be a partition of the interval $(-A,A)$. The random variable

$$\eta=\sum_{j=1}^{n}e^{it\lambda_j}[\zeta(\lambda_{j+1})-\zeta(\lambda_j)] \tag{23}$$

will then correspond to the λ function

$$g(\lambda)=\begin{cases}e^{it\lambda_j} & \text{for } \lambda_j<\lambda\le\lambda_{j+1} \quad (j=1,\ldots,n)\\ 0 & \text{otherwise}\end{cases} \tag{24}$$

As A tends to infinity, while at the same time the maximum distance between consecutive λ_j tends to zero, $g(\lambda)$ will converge in q.m. to $e^{it\lambda}\in H(F)$, while η converges to the q.m. integral (see Sections 5.3 and 6.3)

$$\int_{-\infty}^{\infty}e^{it\lambda}\,d\,\zeta(\lambda)\in H \tag{25}$$

so that these limits are corresponding elements. However, we already know that $e^{it\lambda}$ corresponds to $\xi(t)$, and since the correspondence is one-one, it now follows that we have

$$\xi(t)=\int_{-\infty}^{\infty}e^{it\lambda}\,d\,\zeta(\lambda) \tag{26}$$

and the proof of the theorem is completed. □

Incidentally, we remark that, between any η of the form (14) and the corresponding $g(\lambda)$ given by (15) we have the relation

$$\eta=\int_{-\infty}^{\infty}g(\lambda)\,d\,\zeta(\lambda) \tag{27}$$

This relation will subsist for any pair η and $g(\lambda)$ such that the integral (10) exists as a Riemann-Stieltjes integral. In fact, the stochastic integral in the second member of (27) is then by Section 6.3 defined as a q.m. integral. For a random variable $\eta \in H(\xi)$ which corresponds to a $g(\lambda) \in L_2(F)$ such that the integral in (27) is not defined by Section 6.3, we regard (27) as a definition of this integral. It will then be seen that the set of all random variables η representable in the form (27) with a $g(\lambda) \in L_2(F)$ is identical with the Hilbert space $H(\xi)$ spanned by the random variables $\xi(t)$ for all t.

If we compare the spectral representations of $r(t)$ and $\xi(t)$

$$r(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, dF(\lambda) \tag{28}$$

$$\xi(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, d\zeta(\lambda) \tag{29}$$

it will be seen that the elementary harmonic oscillations are respectively $e^{it\lambda}\, dF(\lambda)$ and $e^{it\lambda}\, d\zeta(\lambda)$.

In both cases the angular frequency is λ. The squared random amplitude appearing in the latter case has a mean value equal to the nonrandom amplitude in the former case

$$E\, |d\zeta(\lambda)|^2 = dF(\lambda) \tag{30}$$

On the other hand, the random phases arising from the factor

$$d\zeta(\lambda) = |d\zeta(\lambda)|\, e^{i \arg d\zeta(\lambda)} \tag{31}$$

are entirely obliterated in the nonrandom elementary oscillations, where all the phases are reduced to zero.

If we think of $\xi(t)$ as representing the temporal development of some concrete physical system, for example, a fluctuating electric voltage, the spectral representation (29) gives the decomposition of the total fluctuation in its elementary harmonic components. The relation (30) then shows that $dF(\lambda)$ is the average power dissipated across a unit resistance by the component with frequency in the element $(\lambda, \lambda + d\lambda)$. The spectral d.f. $F(\lambda)$ thus determines the distribution of the total average power in the $\xi(t)$-fluctuation over the range of angular frequency λ. The average power assigned to the frequency interval $\lambda_1 < \lambda < \lambda_2$ is $F(\lambda_2) - F(\lambda_1)$, which for the whole infinite λ range becomes

$$E|\xi(t)|^2 = r(0) = F(+\infty) - F(-\infty) \tag{32}$$

Thus, $F(\lambda)$ determines the power spectrum of the $\xi(t)$ process. We may think of this as a distribution of a spectral mass of total amount $r(0)$ over the λ axis. We shall say that a point λ belongs to the power spectrum, whenever the interval $(\lambda - h, \lambda + h)$ carries a positive mass for every $h > 0$.

The spectral moments

$$\mu_{2k} = \int_{-\infty}^{\infty} \lambda^{2k}\, d\, F(\lambda) \tag{33}$$

may or may not be finite. The moment μ_{2k} is finite if and only if $r(t)$ has a derivative of order $2\,k$ at $t=0$, and there will then be an expansion of $r(t)$ for small t up to the power t^{2k}, just as for a characteristic function (see Section 2.7.4).

Since $F(\lambda)$ only differs by a multiplicative constant from an ordinary d.f. as used in probability theory, there is a unique representation (see Section 2.5) of the form $F = F_1 + F_2 + F_3$, where each F_i is real, never-decreasing, and bounded. F_1 is a step function which contains all the jumps of F, while F_2 is absolutely continuous, and F_3 is singular (see Section 2.5). In the applications, the singular component F_3 as a rule does not occur, and we shall not consider it in the sequel.

The discontinuity points of F, which all enter in F_1, form the point spectrum of $\xi(t)$. If F has a jump of magnitude $\Delta\, F$ at the point λ, this will introduce a discrete harmonic term into the spectral representations for $r(t)$ and $\xi(t)$, respectively, of the form

$$\Delta\, F e^{it\lambda} \quad \text{and} \quad \Delta\, \zeta\, e^{it\lambda} \tag{34}$$

where $\Delta\,\zeta$ is a random variable with zero mean and variance $E\,|\Delta\,\zeta|^2 = \Delta\, F$. In the extreme case when $F = F_1$ is a pure step function, we have

$$r(t) = \sum \Delta\, F e^{it\lambda} \tag{35}$$

$$\xi(t) = \sum \Delta\, \zeta\, e^{it\lambda} \tag{36}$$

Both sums are extended over all jumps of $F = F_1$, and the first sum is absolutely convergent, while the second converges in q.m.

For a great number of applications, the most important case is when $F = F_2$, so that we have an absolutely continuous spectrum, with a spectral density $f(\lambda) = F'(\lambda)$. Then, (28) takes the form

$$r(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, f(\lambda)\, d\,\lambda \tag{37}$$

The spectral representations (28) and (29) are formally identical with the relation defining a characteristic function (see Section 2.7.1), and there are also corresponding reciprocal relations expressing $F(\lambda)$ and $\zeta(\lambda)$ in terms of $r(t)$ and $\xi(t)$. For any two continuity points λ_1 and λ_2 of $F(\lambda)$ we have, in fact,

$$F(\lambda_2) - F(\lambda_1) = \frac{1}{2\,\pi} \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{-it\lambda_1} - e^{-it\lambda_2}}{-i\,t}\, r(t)\, d\,t \tag{38}$$

$$\zeta(\lambda_2) - \zeta(\lambda_1) = \lim_{T\to\infty} \frac{1}{2\,\pi} \int_{-T}^{T} \frac{e^{-it\lambda_1} - e^{-it\lambda_2}}{-i\,t}\, \xi(t)\, d\,t \tag{39}$$

The first of these relations differs only by a multiplicative constant from the corresponding relation for a characteristic function given in Section 2.7.2. The second relation contains a q.m. integral, which, by means of the criterion (3.5.7), is found to converge in q.m. as $T \to \infty$. Using the well-known properties of trigonometric integrals, it is then proved without difficulty that the $\zeta(\lambda)$ process defined by the second member of (39) is a process with orthogonal increments, which satisfies the spectral relation (29). Starting from the expression in the second member of (39), with a given stationary $\xi(t)$, we thus obtain an independent proof of the spectral representation (29).

We finally note that the inversion formulae (38) and (39) will hold even for discontinuity points λ_1 and λ_2 if in the first members we replace $F(\lambda_1)$ by $\frac{1}{2}[F(\lambda_1)+F(\lambda_1-0)]$ and $\zeta(\lambda_1)$ by $\frac{1}{2}[\zeta(\lambda_1)+\zeta(\lambda_1-0)]$ and similarly for $F(\lambda_2)$ and $\zeta(\lambda_2)$. This remark will be used in the following section.

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