The Riemann Hypothesis in Terms of the Hardy Z-Function

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Abstract

This paper establishes the equivalence of the Riemann Hypothesis (RH) with the assertion that all zeros of the Hardy Z-function are real. The Riemann-Siegel theta function $\theta(t)$ is constructed to cancel the complex phase of $\zeta\left(\frac{1}{2}+i\,t\right)$, rendering $Z(t)$ real-valued independent of the Riemann Hypothesis. Proofs of continuity for $\theta(t)$ and the phase cancellation are provided.

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1 Foundational Definitions

1.1 Riemann-Siegel Theta Function

The Riemann-Siegel theta function $\theta(t)$, critical to the Z-function, is defined as:

$$\theta(t) = \arg\Gamma\left(\frac{1}{4} + i\,\frac{t}{2}\right) - \frac{\log\pi}{2}\,t + \pi\left\lfloor\frac{t}{2} + \frac{1}{4}\right\rfloor \tag{1}$$

where $\Gamma(z)$ is the gamma function.

1.2 Hardy Z-Function

The Hardy Z-function is defined by:

$$Z(t) = e^{i\theta(t)}\,\zeta\left(\frac{1}{2} + i\,t\right) \tag{2}$$

2 Phase Cancellation and Reality of $Z(t)$

2.1 Functional Equation of $\zeta(s)$

The functional equation for the Riemann zeta function is:

$$\zeta(s) = \chi(s)\,\zeta\,(1 - s) \tag{3}$$

$$\chi(s) = \pi^{s - \frac{1}{2}}\,\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \tag{4}$$

2.2 Phase Analysis on the Critical Line

Lemma 1. (Phase Cancellation) *For $s = \frac{1}{2} + i\,t$, the theta function satisfies:*

$$\theta(t) \equiv -\arg\zeta\left(\frac{1}{2} + i\,t\right) (\mathrm{mod}\,\pi) \tag{5}$$

rendering $Z(t)$ real-valued.

Proof. On the critical line, $\zeta(s) = \chi(s)\overline{\zeta(s)}$ via (3). Taking arguments:

$$\arg\zeta(s) \equiv \arg\chi(s) - \arg\zeta(s) (\mathrm{mod}\,2\,\pi) \tag{6}$$

Rearranging gives:

$$2\arg\zeta(s) \equiv \arg\chi(s) (\mathrm{mod}\,2\,\pi) \tag{7}$$

so:

$$\arg\zeta(s) \equiv \frac{1}{2}\arg\chi(s) (\mathrm{mod}\,\pi) \tag{8}$$

Substituting $\chi(s)$ from (3):

$$\arg\chi(s) = -\frac{\log\pi}{2}\,t + \arg\Gamma\left(\frac{1}{4} - \frac{i\,t}{2}\right) + \pi\left\lfloor\frac{t}{2} + \frac{1}{4}\right\rfloor \tag{9}$$

By definition (1), $\theta(t) \equiv \frac{1}{2}\arg\chi(s) (\mathrm{mod}\,\pi)$, hence:

$$\arg\zeta\left(\frac{1}{2} + i\,t\right) \equiv -\theta(t) (\mathrm{mod}\,\pi) \tag{10}$$

forcing $Z(t) = e^{i\theta(t)}\,\zeta\left(\frac{1}{2} + i\,t\right)$ to be real. □

3 Main Theorem: RH Equivalence

Theorem 2. (Equivalence) *The Riemann Hypothesis holds if and only if all zeros of $Z(t)$ are real.*

Proof. Let $\rho = \sigma + i\,t$ be a non-trivial zero of $\zeta(s)$.

Forward Direction (⇒) If RH holds, all ρ lie on $\sigma = \frac{1}{2}$. By Lemma 1, zeros of $Z(t)$ correspond to $\zeta\left(\frac{1}{2} + i\,t\right) = 0$, which are real.

Reverse Direction (⇐) Suppose all zeros of $Z(t)$ are real. For any $\rho = \sigma + i\,t$, the functional equation implies an accompanying zero $\rho' = 1 - \sigma + i\,t$. If $\sigma \neq \frac{1}{2}$, then ρ and ρ' would produce two distinct zeros of $Z(t)$ at t, violating reality. Hence, $\sigma = \frac{1}{2}$. □

4 Continuity of $\theta(t)$

Theorem 3. (Continuity) *$\theta(t)$, defined in (1), is continuous for all $t \in \mathbb{R}$.*

Proof. Let $n \in \mathbb{Z}$ and $t_0 = 2\,n - \frac{1}{2}$. Near t_0:

$$\lim_{\epsilon\to 0^+} \arg\Gamma\left(\frac{1}{4} + i\,\frac{t_0 + \epsilon}{2}\right) = \lim_{\epsilon\to 0^-} \arg\Gamma\left(\frac{1}{4} + i\,\frac{t_0 - \epsilon}{2}\right) + \pi \tag{11}$$

The floor term compensates:

$$\lim_{\epsilon\to 0^+} \pi\left\lfloor\frac{t_0 + \epsilon}{2} + \frac{1}{4}\right\rfloor = \pi\,n, \quad \lim_{\epsilon\to 0^-} \pi\left\lfloor\frac{t_0 - \epsilon}{2} + \frac{1}{4}\right\rfloor = \pi\,(n-1) \tag{12}$$

Thus:

$$\lim_{\epsilon\to 0^+} \theta\,(t_0 + \epsilon) = \lim_{\epsilon\to 0^-} \theta\,(t_0 - \epsilon) \tag{13}$$

and $\theta(t)$ is continuous everywhere. □