

# Oscillatory Processes with Monotonic Phase Functions

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**Definition 1.** *[Oscillatory Process] Let  $\{X_t\}_{t \in \mathbb{R}}$  be a complex second-order process.  $\{X_t\}$  is called oscillatory if*

$$X_t = \int_{-\infty}^{\infty} \phi_t(\omega) dZ(\omega) \quad (1)$$

where  $Z(\omega)$  has orthogonal increments where

$$E |dZ(\omega)|^2 = d\mu(\omega) \quad (2)$$

and

$$\phi_t(\omega) = A_t(\omega) e^{i\omega t} \quad (3)$$

**Theorem 2.** *Let  $\theta: \mathbb{R} \rightarrow \mathbb{R}$  be smooth and strictly monotonically increasing. Define*

$$\phi_t(\omega) = e^{i\omega\theta(t)} \quad (4)$$

Set

$$A_t(\omega) = e^{i\omega(\theta(t)-t)} \quad (5)$$

Then the time-varying impulse response  $h_t(u)$  defined by

$$A_t(\omega) = \int_{-\infty}^{\infty} e^{i\omega u} h_t(u) du \quad (6)$$

is

$$h_t(u) = \delta(u - [\theta(t) - t]) \quad (7)$$

**Proof.** The inverse Fourier transform yields

$$\begin{aligned}
h_t(u) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega u} A_t(\omega) d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega u} e^{i\omega(\theta(t)-t)} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(\theta(t)-t-u)} d\omega \\
&= \delta(\theta(t) - t - u) \\
&= \delta(u - [\theta(t) - t])
\end{aligned} \tag{8}$$

□

**Corollary 3.** Let  $S_t = \int_{-\infty}^{\infty} e^{i\omega t} dZ(\omega)$ . Then

$$X_t = \int_{-\infty}^{\infty} S_{t-u} h_t(u) du = S_{2t-\theta(t)} \tag{9}$$

and

$$X_t = \int_{-\infty}^{\infty} e^{i\omega\theta(t)} dZ(\omega) \tag{10}$$

**Proof.** By the sifting property of the Dirac delta,

$$\begin{aligned}
X_t &= \int_{-\infty}^{\infty} S_{t-u} \delta(u - [\theta(t) - t]) du \\
&= S_{t-[\theta(t)-t]} \\
&= S_{2t-\theta(t)}
\end{aligned} \tag{11}$$

By direct substitution, the oscillatory representation holds by definition. □