Correlation Functions of Eigenvalues of Random Matrices

by Stephen Crowley

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The study of correlation functions of eigenvalues of random matrices is a significant area in mathematical physics, especially in quantum theory and statistical mechanics. These correlation functions are statistical measures that describe the interrelationships among the eigenvalues of random matrices.

The joint probability density of the eigenvalues of $n \times n$ random Hermitian matrices $M \in \mathbf{H}^{n \times n}$ with partition functions of the form

$$Z_{n,V} = \int_{M \in \mathbf{H}^{n \times n}} e^{\operatorname{tr}(V(M))} \, d\mu_0(M) \tag{1}$$

where

$$V(x) := \sum_{j=1}^{\infty} v_j x^j \tag{2}$$

and $d\mu_0(M)$ is the standard Lebesgue measure on the space $\mathbf{H}^{n \times n}$ of Hermitian $n \times n$ matrices, is given by

$$p_{n,V}(x_1, \ldots, x_n) = \left(\frac{e^{-\sum_{i=1}^{n} V(x_i)}}{Z_{n,V}} \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (x_i - x_j)^2 \right) \tag{3}$$

The k-point correlation functions (or marginal distributions) are defined as

$$R_{n,V}^{(k)}(x_1, \ldots, x_k) = \frac{n!}{(n-k)!} \int_{\mathbf{R}} p_{n,V}(x_1, x_2, \ldots, x_n) d x_{k+1} \cdots \int_{\mathbf{R}} d x_n \tag{4}$$

which are skew symmetric functions of their variables. In particular, the one-point correlation function, or density of states, is

$$R_{n,V}^{(1)}(x_1) = n \int_{\mathbf{R}} p_{n,V}(x_1, x_2, \ldots, x_n) d x_2 \cdots \int_{\mathbf{R}} d x_n \tag{5}$$

Its integral over a Borel set $B \subset \mathbf{R}$ gives the expected number of eigenvalues contained in B:

$$\int_B R_{n,V}^{(1)}(x) \, dx = \mathbf{E}(\#\{\text{eigenvalues in } B\}) \tag{6}$$

Theorem 1

The Dyson-Mehta Theorem

For any k, $1 \leq k \leq n$ *the* k*-point correlation function* $R^{(k)}_{n,V}$ *can be written as a determinant*

$$R^{(k)}_{n,V}(x_1, x_2, \ldots, x_k) = \det_{1 \leq i,j \leq k}(K_{n,V}(x_i, x_j)) \tag{7}$$

where $K_{n,V}(x, y)$ *is the* n*-th Christoffel-Darboux kernel*

$$K_{n,V}(x, y) := \sum_{k=0}^{n-1} \psi_k(x)\, \psi_k(y), \tag{8}$$

associated to V*, written in terms of the quasipolynomials*

$$\psi_k(x) = \frac{p_k(z)\ e^{-\frac{V(z)}{2}}}{\sqrt{h_k}} \tag{9}$$

where $\{p_k(x)\}_{k \in \mathbf{N}}$ *is a complete sequence of monic polynomials, of the degrees indicated, satisfying the orthogonality conditions*

$$\int_{\mathbf{R}} \psi_j(x)\, \psi_k(x)\ d\,x = \delta_{jk} \tag{10}$$