

Factorization of Stationary Gaussian Process Kernels

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1 Theorem: Spectral Representation

Theorem 1. *[Spectral Factorization Theorem]: Let $K: \mathbb{R} \rightarrow \mathbb{R}$ be a positive definite stationary kernel function.*

By Bochner's theorem, there exists a non-negative spectral density function $S: \mathbb{R} \rightarrow \mathbb{R}$ such that:

$$K(t-s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega(t-s)} d\omega \quad (1)$$

Let $h: \mathbb{R} \rightarrow \mathbb{C}$ be defined as:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega)} e^{i\omega t} d\omega \quad (2)$$

Then:

$$K(t-s) = \int_{-\infty}^{\infty} h(\tau) h(t-\tau-s) d\tau \quad (3)$$

2 Proof

1. Since K is positive definite and stationary, Bochner's theorem guarantees the existence of $S(\omega) \geq 0$ such that:

$$K(t-s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega(t-s)} d\omega \quad (4)$$

2. Since $S(\omega) \geq 0$ by Bochner's theorem:

$$K(t-s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega)} e^{i\omega t} \cdot \sqrt{S(\omega)} e^{-i\omega s} d\omega \quad (5)$$

3. Using the definition of $h(t)$ as the inverse Fourier transform of the square root of the spectral density $S(\omega)$:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega)} e^{i\omega t} d\omega \quad (6)$$

4. The Fourier transform of $h(t)$ gives:

$$\sqrt{S(\omega)} = \int_{-\infty}^{\infty} h(\tau) e^{-i\omega\tau} d\tau \quad (7)$$

5. Substituting this representation:

$$K(t-s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(\tau) e^{-i\omega\tau} d\tau \right) e^{i\omega t} \left(\int_{-\infty}^{\infty} h(\sigma) e^{-i\omega\sigma} d\sigma \right) e^{-i\omega s} d\omega \quad (8)$$

6. By Fubini's theorem (valid since K is PD):

$$K(t-s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau) h(\sigma) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-\tau-s+\sigma)} d\omega d\tau d\sigma \quad (9)$$

7. The inner integral yields the Dirac delta function:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-\tau-s+\sigma)} d\omega = \delta(t-\tau-s+\sigma) \quad (10)$$

8. Therefore:

$$K(t-s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau) h(\sigma) \delta(t-\tau-s+\sigma) d\tau d\sigma \quad (11)$$

9. Using the sifting property of δ :

$$\begin{aligned} K(t-s) &= \int_{-\infty}^{\infty} h(\tau) \int_{-\infty}^{\infty} h(\sigma) \delta(t-\tau-s+\sigma) d\sigma d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) h(t-\tau-s) d\tau \end{aligned} \quad (12)$$

3 Reverse Verification

Starting from the final result:

$$K(t-s) = \int_{-\infty}^{\infty} h(\tau) h(t-\tau-s) d\tau \quad (13)$$

Substituting the definition of $h(t)$:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega)} e^{i\omega t} d\omega \quad (14)$$

1) First substitution for $h(\tau)$:

$$K(t-s) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega_1)} e^{i\omega_1\tau} d\omega_1 \right] h(t-\tau-s) d\tau \quad (15)$$

2) Second substitution for $h(t-\tau-s)$:

$$K(t-s) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega_1)} e^{i\omega_1\tau} d\omega_1 \right] \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega_2)} e^{i\omega_2(t-\tau-s)} d\omega_2 \right] d\tau \quad (16)$$

3) Rearranging the integrals:

$$K(t-s) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{S(\omega_1)} \sqrt{S(\omega_2)} e^{i\omega_2 t} e^{-i\omega_2 s} e^{i(\omega_1-\omega_2)\tau} d\omega_1 d\omega_2 d\tau \quad (17)$$

4) The integral with respect to τ yields:

$$\int_{-\infty}^{\infty} e^{i(\omega_1-\omega_2)\tau} d\tau = 2\pi \delta(\omega_1 - \omega_2) \quad (18)$$

5) Applying this result:

$$K(t-s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega(t-s)} d\omega \quad (19)$$

This matches the original spectral representation from Bochner's theorem, confirming that:

- The substitutions were valid
- The use of Fubini's theorem was justified
- The manipulation of the Dirac delta function was correct
- The final result is consistent with the initial spectral representation