The Integral $\int_{-1}^{\omega} \frac{1}{\sqrt[4]{1-\lambda^2}} d\lambda$

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Theorem 1. *[Integral of $(1-y^2)^{-1/4}$] For $\lambda \in [-1,1]$, the definite integral*

$$\int_{-1}^{\lambda} \sqrt{\frac{1}{\sqrt{1-y^2}}}\, d\,y \tag{1}$$

can be expressed in terms of incomplete elliptic integrals of the second kind as:

$$\int_{-1}^{\lambda} \sqrt{\frac{1}{\sqrt{1-y^2}}}\, d\,y = 2\left[\mathrm{EllipticE}\left(\frac{\arcsin \lambda}{2}\middle|2\right) - \mathrm{EllipticE}\left(-\frac{\pi}{4}\middle|2\right)\right] \tag{2}$$

where $\mathrm{EllipticE}(\phi|m)$ *denotes the incomplete elliptic integral of the second kind.*

Proof. We proceed in several steps.

Step 1: Simplify the integrand.

$$\sqrt{\frac{1}{\sqrt{1-y^2}}} = \left(\frac{1}{\sqrt{1-y^2}}\right)^{1/2} = (1-y^2)^{-1/4} \tag{3}$$

Therefore, our integral becomes:

$$I = \int_{-1}^{\lambda} (1-y^2)^{-1/4}\, d\,y \tag{4}$$

Step 2: Apply the trigonometric substitution $y = \sin\theta$.

Under this substitution:

$$d\,y = \cos\theta\, d\,\theta \tag{5}$$
$$1-y^2 = 1-\sin^2\theta = \cos^2\theta \tag{6}$$
$$(1-y^2)^{-1/4} = (\cos^2\theta)^{-1/4} = |\cos\theta|^{-1/2} \tag{7}$$

For $\theta \in [-\pi/2, \pi/2]$, we have $\cos\theta \geq 0$, so $|\cos\theta| = \cos\theta$.

The limits of integration transform as:

$$y = -1 \Rightarrow \theta = \arcsin(-1) = -\frac{\pi}{2} \tag{8}$$

$$y = \lambda \Rightarrow \theta = \arcsin(\lambda) \tag{9}$$

Step 3: Transform the integral.

$$I = \int_{-\pi/2}^{\arcsin\lambda} (\cos\theta)^{-1/2} \cos\theta \, d\theta = \int_{-\pi/2}^{\arcsin\lambda} (\cos\theta)^{1/2} \, d\theta \tag{10}$$

Step 4: Express in terms of elliptic integrals.

It is a known result that:

$$\int \sqrt{\cos\theta} \, d\theta = 2 \, \text{EllipticE}\left(\frac{\theta}{2} \middle| 2\right) + C \tag{11}$$

where EllipticE($\phi|m$) is the incomplete elliptic integral of the second kind.

Step 5: Evaluate the definite integral.

$$I = \left[2 \, \text{EllipticE}\left(\frac{\theta}{2} \middle| 2\right) \right]_{-\pi/2}^{\arcsin\lambda} \tag{12}$$

$$= 2 \, \text{EllipticE}\left(\frac{\arcsin\lambda}{2} \middle| 2\right) - 2 \, \text{EllipticE}\left(\frac{-\pi/2}{2} \middle| 2\right) \tag{13}$$

$$= 2 \left[\text{EllipticE}\left(\frac{\arcsin\lambda}{2} \middle| 2\right) - \text{EllipticE}\left(-\frac{\pi}{4} \middle| 2\right) \right] \tag{14}$$

This completes the proof. $\square$