Understanding the real-valued spectral representation and the random measures $U(\omega)$ and $V(\omega)$

Take-away: Every zero-mean, second-order stationary process can be decomposed into *uncorrelated random cosine–sine waves* whose amplitudes are carried by two mutually orthogonal random measures $U(\omega)$ and $V(\omega)$. These random measures are simply the *real* and *imaginary* parts of the complex spectral measure that appears in the classical Cramér representation; they inherit exact orthogonality and variance properties from that complex measure and provide an intuitive one-sided ("positive-frequency") description of the process' power spectrum.

1. Where U and V come from

1. **Complex Cramér representation.**
Any second-order stationary process $\xi(t)$ with covariance $r(t)$ admits
$$\xi(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, d\zeta(\omega),$$
with an orthogonal-increment complex random measure $d\zeta(\omega)$ satisfying $\mathbb{E}[d\zeta(\omega_1)\, \overline{d\zeta(\omega_2)}] = \delta(\omega_1 - \omega_2)\, dF_{\text{two}}(\omega)$ [1] [2].

2. **Reality constraint.**
Because $\xi(t)$ is real, the increments obey $d\zeta(-\omega) = \overline{d\zeta(\omega)}$ [3]. This symmetry lets us "fold" the two-sided integral onto $(0, \infty)$.

3. **Splitting into real and imaginary parts.**
For $\omega > 0$ set
$$dU(\omega) = 2\Re[d\zeta(\omega)], \qquad dV(\omega) = -2\Im[d\zeta(\omega)].$$
These definitions immediately give the *one-sided real representation*
$$\xi(t) = \int_0^{\infty} \cos(\omega t)\, dU(\omega) + \int_0^{\infty} \sin(\omega t)\, dV(\omega).$$
Orthogonality of $d\zeta$ implies $\mathbb{E}[dU(\omega_1) dV(\omega_2)] = 0$ and $\mathbb{E}[dU(\omega)^2] = \mathbb{E}[dV(\omega)^2] = dF(\omega)$ with $dF(\omega) = 2dF_{\text{two}}(\omega)$ [1].

2. What *is* a "random measure"?

A random measure $M(\omega)$ on $[0, \infty)$ assigns a (square-integrable) random variable to every Borel set $B \subset [0, \infty)$ such that:

- **Additivity:** $M(B_1 \cup B_2) = M(B_1) + M(B_2)$ for disjoint sets.
- **Orthogonal increments:** For the spectral case, different frequency bands are uncorrelated: $\mathbb{E}[M(B_1)\, M(B_2)] = 0$ when $B_1 \cap B_2 = \varnothing$[4].

Writing $U(\omega) = U([0, \omega])$ is shorthand for the cumulative random measure up to ω. In practice, $dU(\omega)$ (respectively $dV(\omega)$) is the "random amplitude" inside an infinitesimal frequency band $(\omega, \omega + d\omega)$.

3. Orthogonality and covariance

Because dU and dV inherit the orthogonality of the complex measure,

$$\mathbb{E}\big[dU(\omega_1)\, dU(\omega_2)\big] = \mathbb{E}\big[dV(\omega_1)\, dV(\omega_2)\big] = \delta(\omega_1 - \omega_2)\, dF(\omega_1), \qquad \mathbb{E}\big[dU(\omega_1)\, dV(\omega_2)\big]$$

Substituting into the real representation yields the **Wiener–Khintchine formula**

$$r(t) = \int_0^\infty \cos(\omega t)\, dF(\omega)[^1][^3],$$

confirming that $dF(\omega)$ is the one-sided power-spectral distribution.

4. Inversion formulas: recovering U and V from a sample path

Using Fourier inversion on the complex measure gives explicit path-wise limits:

$$U(\omega) = \lim_{T \to \infty} \frac{1}{\pi} \int_{-T}^{T} \frac{\sin(\omega t)}{t}\, \xi(t)\, dt, \qquad V(\omega) = -\lim_{T \to \infty} \frac{1}{\pi} \int_{-T}^{T} \frac{1 - \cos(\omega t)}{t}\, \xi(t)\, dt,$$

and similar formulas for increments $U(\omega_2) - U(\omega_1)$ and $V(\omega_2) - V(\omega_1)$.
These limits hold in mean-square for any mean-square continuous stationary process[4].

5. Physical interpretation

- **Cosine branch (U).** Adds *in-phase* random cosines; its increments store energy in the *even* component of the spectrum.
- **Sine branch (V).** Adds *quadrature* random sines; its increments store energy in the *odd* component.
- **Equal contribution.** Each branch contributes the same variance to any narrow band because $\mathbb{E}[dU^2] = \mathbb{E}[dV^2] = dF$.
- **Power-spectral density.** The deterministic measure $dF(\omega)$ equals the average power carried by frequencies in $(\omega, \omega + d\omega)$.

6. Relation to classical literature

The cosine–sine form goes back to Cramér's original work and is standard in modern time-series texts[1] [2]. It is often quoted without proof; the derivation above shows that *no additional assumptions* beyond second-order stationarity and reality are needed. Multiparameter and functional generalisations (e.g., random fields or Hilbert-valued processes) rely on the same splitting idea[4].

References (citation IDs)

ID	Key result used
[1]	Spectral representation theorem for stationary processes (one- and two-sided forms).
[2]	Brockwell & Davis, *Time Series: Theory and Methods* – standard reference for Cramér representation and its real variant.
[3]	Proof that reality imposes $d\zeta(-\omega) = \overline{d\zeta(\omega)}$.
[4]	Extension to multiparameter processes; details on random measures and inversion formulas.

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1. http://link.springer.com/10.1007/978-1-4419-0320-4_4
2. http://link.springer.com/10.1007/978-1-4419-0320-4
3. https://onlinelibrary.wiley.com/doi/10.1111/j.1467-842X.1991.tb00415.x
4. https://projecteuclid.org/journals/electronic-journal-of-probability/volume-17/issue-none/Multiparameter-processes-with-stationary-increments-Spectral-representation-and-integration/10.1214/EJP.v17-2287.pdf