Theorem 1. *For the phase function $\theta(t)$ used in Hardy's Z-function and the gamma factor $\chi(s)$ in the functional equation of the Riemann zeta function:*

$$e^{i\theta(t)} = \frac{1}{\sqrt{\chi(1/2+i\,t)}}$$

for all real t.

Proof. 1. By definition, Hardy's Z-function is real-valued and given by:

$$Z(t) = e^{i\theta(t)}\,\zeta(1/2+i\,t)$$

2. By the functional equation:

$$\zeta(1/2+i\,t) = \chi(1/2+i\,t)\,\zeta(1/2-i\,t)$$

3. For real t:

$$\zeta(1/2-i\,t) = \overline{\zeta(1/2+i\,t)}$$

4. Therefore:

$$Z(t) = e^{i\theta(t)}\,\chi(1/2+i\,t)\overline{\zeta(1/2+i\,t)}$$

5. Since $Z(t)$ is real:

$$Z(t) = \overline{Z(t)}$$

$$e^{i\theta(t)}\,\chi(1/2+i\,t)\overline{\zeta(1/2+i\,t)} = \overline{e^{i\theta(t)}}\;\overline{\chi(1/2+i\,t)}\zeta(1/2+i\,t)$$

6. For the gamma factor $\chi(s)$, we know:

$$\chi(s)\overline{\chi(s)} = 1$$

when $s = 1/2+i\,t$ for real t

7. The equation in step 5 must hold for all values of $\zeta(1/2+i\,t)$. This is only possible if:

$$e^{i\theta(t)}\,\chi(1/2+i\,t) = \frac{1}{e^{i\theta(t)}}$$

8. Therefore:

$$e^{2i\theta(t)}\,\chi(1/2+i\,t) = 1$$

$$e^{i\theta(t)} = \frac{1}{\sqrt{\chi(1/2+i\,t)}}$$

□

Corollary 2. *For the Hardy Z-function and the Riemann zeta function:*

$$Z(t) = \frac{\zeta(1/2+i\,t)}{\sqrt{\chi(1/2+i\,t)}}$$

for all real t.

Proof. From the definition of Hardy's Z-function:

$$Z(t) = e^{i\theta(t)}\,\zeta(1/2+i\,t)$$

Using the theorem that:

$$e^{i\theta(t)} = \frac{1}{\sqrt{\chi(1/2+i\,t)}}$$

Substituting this expression for $e^{i\theta(t)}$:

$$Z(t) = \frac{1}{\sqrt{\chi(1/2+i\,t)}}\,\zeta(1/2+i\,t) = \frac{\zeta(1/2+i\,t)}{\sqrt{\chi(1/2+i\,t)}}$$

□