# Eigenfunctions of Stationary Gaussian processes

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**Definition 1.** *The Gram-Schmidt formula expresses the orthogonal complement $f_n^{\perp p}(x)$ of a function sequence $f_n(x)$ with respect to the orthogonality measure $p(x)$ by the recursive equation*

$$f_k^{\perp p}(x) = f_k(x) - \sum_{j=1}^{k-1} \frac{\langle f_k, f_j^{\perp p} \rangle_p}{\langle f_j^{\perp p}, f_j^{\perp p} \rangle} f_j^{\perp p}(x) \tag{1}$$

*where the inner product is defined as:*

$$\langle f, g \rangle_p = \int_0^\infty f(x)\, g(x)\, p(x)\, \mathrm{d}\, x \tag{2}$$

*where $\langle f, g \rangle = \langle f, g \rangle_1$ and the normalized functions are denoted with a wide bar*

$$\overline{f_k^{\perp}}(x) = \frac{f_k^{\perp}(x)}{\| f_k^{\perp} \|} = \frac{f_k^{\perp}(x)}{\sqrt{\langle f_k^{\perp}, f_k^{\perp} \rangle}} \tag{3}$$

**Definition 2.** *The Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are defined as:*

$$\mathcal{F}[f](\omega) = \int_0^\infty f(x)\, e^{-i\omega x}\, d\, x \tag{4}$$

$$\mathcal{F}^{-1}[g](x) = \frac{1}{2\,\pi} \int_0^\infty g(\omega)\, e^{i\omega x}\, d\,\omega \tag{5}$$

**Theorem 3.** *The eigenfunctions of a stationary gaussian process on the half-line are given by the orthogonal complement of the inverse Fourier transforms of the polynomials orthogonal to the square root of the spectral density.*

**Proof.** Let $S(\omega)$ be the spectral density of a stationary Gaussian process on $[0,\infty)$, $K(x-y) = (\mathcal{F}^{-1}[S(\omega)])(x-y)$ be its covariance kernel and $\{Q_k(\omega)\}$ be the sequence of polynomials orthogonal with respect to $\sqrt{S(\omega)}$

$$\int_0^\infty Q_k(\omega)\, Q_j(\omega) \sqrt{S(\omega)}\; d\,\omega = 0 \forall k \neq j \tag{6}$$

Let

$$\psi_k(x) = \mathcal{F}^{-1}[Q_k(\omega)](x) \tag{7}$$

be the inverse Fourier Transform of the polynomials orthogonal to the square root of the spectral density and

$$\psi_k^\perp(x) = \psi_k(x) - \sum_{j=1}^{k-1} \frac{\langle \psi_k, \psi_j^\perp \rangle}{\langle \psi_j^\perp, \psi_j^\perp \rangle}\, \psi_j^\perp(x) \tag{8}$$

denote its orthogonal complement. Then apply the covariance operator

$$T[f](y) = \int_0^\infty K(x-y)\, f(x)\, \mathrm{d}\, x \tag{9}$$

to $\psi_k^\perp(x)$ to get

$$\begin{aligned} T[\psi_k^\perp](x) \;\; &= \int_0^\infty K(x-y)\, \psi_k^\perp(y)\, \mathrm{d}\, y \\ &= \mathcal{F}^{-1}\left[S(\omega) \cdot \mathcal{F}[\psi_k^\perp](\omega)\right](x) \end{aligned} \tag{10}$$

where the equality is due to the convolution theorem on the half-line. By the linearity of the Fourier transform and the Gram-Schmidt construction in Equation (8):

$$\mathcal{F}[\psi_k^\perp](\omega) = Q_k(\omega) - \sum_{j=1}^{k-1} \frac{\langle \psi_k, \psi_j^\perp \rangle}{\langle \psi_j^\perp, \psi_j^\perp \rangle}\, \mathcal{F}[\psi_j^\perp](\omega) \tag{11}$$

Substituting this into Equation (10):

$$T[\psi_k^\perp](x) = \mathcal{F}^{-1}\left[S(\omega)\cdot\left(Q_k(\omega) - \sum_{j=1}^{k-1} c_j\,\mathcal{F}[\psi_j^\perp](\omega)\right)\right](x) \tag{12}$$

where

$$c_j = \frac{\langle \psi_k, \psi_j^\perp\rangle}{\langle \psi_j^\perp, \psi_j^\perp\rangle} \tag{13}$$

By the orthogonality of $Q_k(\omega)$ with respect to $\sqrt{S(\omega)}$, and the fact that $Q_k(\omega)$ are constructed as orthogonal polynomials with respect to the weight $\sqrt{S(\omega)}$, it follows that $Q_k(\omega)$ are eigenfunctions of the multiplication operator defined by $S(\omega)$. Specifically, since $S(\omega) = (\sqrt{S(\omega)})^2$, we have:

$$S(\omega)\,Q_k(\omega) = \lambda_k\,Q_k(\omega) \tag{14}$$

And its already known that:

$$S(\omega)\,\mathcal{F}[\psi_j^\perp](\omega) = \lambda_j\,\mathcal{F}[\psi_j^\perp](\omega) \forall j < k \tag{15}$$

Therefore:

$$T[\psi_k^\perp](x) = \lambda_k\,\psi_k(x) - \sum_{j=1}^{k-1} c_j\,\lambda_j\,\psi_j^\perp(x) \tag{16}$$

By the construction of $\psi_k^\perp(x)$, this equals:

$$T[\psi_k^\perp](x) = \lambda_k\,\psi_k^\perp(x) \tag{17}$$

Thus $\psi_k^\perp(x)$ is an eigenfunction of the kernel operator with eigenvalue $\lambda_k > 0$ on the half-line $[0,\infty)$. $\square$