Testing for Harmonizability

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Abstract

Let $R(s,t)$ be a covariance function having the representation

$$R(s,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(i\,s\,x-i\,t\,y\right)\,d^2\,G(x,y) \tag{1}$$

where $G(x,y)$ is continuous to the right in both variables and is of bounded variation in the plane; then $R(s,t)$ is harmonizable in that $G(x,y)$ is also a covariance. Examples are given in which this result is used to determine the harmonizability of new processes and covariances that are formed by operations on old processes and covariances. Specifically, if $X(t)$ is a real Gaussian harmonizable process, then $X^n(t)$ is harmonizable. If $X(t)$ is harmonizable, $G(x,y)$ has bounded support and $g(t)$ is a Fourier-Stieltjes transform, then $X(g(t))$ and $X\,(t+g(t))$ are harmonizable. If

$$X(t)=\int_{-\infty}^{\infty}f(t,u)\;d\,Z(u) \tag{2}$$

where $f(t,u)=f\,(t-u)$ is a Fourier-Stieltjes transform and $G(u,v)=\mathbb{E}\{Z(u)\,Z^*(v)\}$ has finite total variation, then $X(t)$ is harmonizable. A sufficient condition for Priestley's oscillatory processes to be harmonizable is also obtained. The Bochner-Eberlein characterization of Fourier-Stieltjes transforms is particularly convenient for determining harmonizability in these cases.

1 Introduction

Let $\{X(t,\omega),-\infty<t<\infty,\omega\in\Omega\}$ be a second-order continuous-parameter stochastic process defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$. The process $X(t,\omega)$ is said to be harmonizable [loeve1955probability, p. 474] if it has the quadratic mean representation

$$X(t,\omega)=\int_{-\infty}^{\infty}\exp\left(i\,t\,x\right)\;d\,Z(x,\omega) \tag{3}$$

where $\{Z(x,\omega),-\infty<x<\infty\}$ is a process whose covariance is of bounded variation (BV) in the plane. Harmonizable processes are of engineering interest because decomposition relative to $\exp\left(i\,t\,x\right)$ admits the usual frequency interpretation of linear filtering. If $H(x)$ is the frequency response of a stable, linear time-invariant system, then the system output process $Y(t,\omega)$ is given by the quadratic mean integral

$$Y(t,\omega)=\int_{-\infty}^{\infty}\exp\left(i\,t\,x\right)H(x)\;d\,Z(x,\omega) \tag{4}$$

A detailed account may be found in [blanc-lapierre1968random, Ch. 8]. For recent results on harmonizable processes in engineering, see [cambanis1970harmonizable, donati1971spectra, ogura1971spectral].

The covariance functions for $X(t)$ and $Y(t)$ are

$$R(s,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(i\,s\,x-i\,t\,y)\ d^2G(x,y) \tag{5}$$

$$R_Y(s,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(i\,s\,x-i\,t\,y)\,H(x)\,H^*(y)\ d^2G(x,y) \tag{6}$$

where $d^2G(x,y)=\mathbb{E}\{d\,Z(x)\,d\,Z^*(y)\}$. Any covariance with representation (5), with $G(x, y)$ a covariance of bounded variation, is called harmonizable; harmonizable processes have harmonizable covariances. Conversely, processes with harmonizable covariances are themselves harmonizable [loeve1955probability, p. 476]. For brevity, we call those corresponding to G with finite total variation simply "harmonizable."

This paper addresses the determination of harmonizability for new processes or covariances constructed from old ones. The main results are as follows:

- If $X(t)$ is a real Gaussian harmonizable process, then $X^n(t)$ is harmonizable.

- If $X(t)$ is harmonizable with spectral decomposition of bounded support and $g(t)$ is a Fourier-Stieltjes transform, then $X(t+g(t))$ and $X(g(t))$ are harmonizable.

- If R_1 and R_2 are harmonizable covariances, then for T of finite Lebesgue measure,

$$R_3(s,t)=\int_T R_1(s,u)\,R_2(u,t)\ d\,u \tag{7}$$

  is harmonizable.

- If $X(t)$ is a moving average as in (2) with $f(t,u)=f(t-u)$ a Fourier-Stieltjes transform and $G(u,v)$ of bounded variation, then $X(t)$ is harmonizable.

- If $X(t)$ is as above, $Z(u)$ has orthogonal increments with $d\,F(u)=\mathbb{E}\,|d\,Z(u)|^2$ and

$$f(t,u)=\exp(i\,u\,t)\int_{-\infty}^{\infty}\exp(i\,t\,x)\ d\,H_u(x) \tag{8}$$

  with $H_u(x)$ of bounded variation for every u, the resulting processes include Priestley's oscillatory processes [priestley1965evolutionary], which are harmonizable under suitable conditions.

The method is to use the following result.

Theorem 1. *If $R(s,t)$ is simultaneously a covariance and a Fourier-Stieltjes (FS) transform with respect to some $G(x,y)$ of bounded variation, then $R(s,t)$ is harmonizable in that G is necessarily a covariance.*

Proof. Sufficiency is immediate: one can find a process $Z(x,\omega)$ whose covariance is G and whose FS transform $X(t,\omega)$ as in (3) has covariance $R(s,t)$. Conversely, suppose R is both a covariance and an FS transform with respect to some G that is BV. Define

$$G_a(x,y) = G(x,y) - G(a,y) - G(x,a) + G(a,a) \tag{9}$$

For any n and sequence $\{x_j \geq a,\ j = 1, \ldots, n\}$ and complex $\{c_j\}$,

$$\sum_{j,k=1}^{n} c_j\, c_k^*\, G_a(x_j, x_k) \geq 0$$

This follows by constructing $g_a(s) = \sum_{j=1}^{n} c_j\, [1 - \exp(-i\, s\, x_j)]\, \exp(-i\, s\, a)$ and applying the inversion theorem [loeve1955probability, p. 475]. Letting $a \to -\infty$, $G_a(x,y) \to G(x,y)$. Thus $G(x,y)$ is non-negative definite. □

Thus, any characterization of FS transforms, such as the Bochner-Eberlein theorem, also provides a characterization for harmonizable covariances [bochner1934fst, eberlein1955fst, rudin1962groups]. Cramér [cramer1939representation] and Dominguez [dominguez1940fst] provide alternative characterizations.

2 Mathematical Preliminaries

Assume $G(x,y)$ is normalized, e.g.,

$$G(x,y) = \frac{1}{4}\,[G(x+0, y+0) + G(x+0, y-0) + G(x-0, y+0) + G(x-0, y-0)] \tag{10}$$

and satisfies

$$\lim_{x \to -\infty} G(x,y) = \lim_{y \to -\infty} G(x,y) = 0.$$

We now state the key characterizations.

Theorem 2. *[Bochner]A necessary and sufficient condition that $f(t)$, $-\infty < t < \infty$, has the representation*

$$f(t) = \int_{-\infty}^{\infty} \exp(i\,x\,t)\; d\,G(x) \tag{11}$$

for a complex measure G of bounded variation is that, for any n, any sequence $\{t_j\}_{j=1}^{n}$ and any complex $\{c_j\}_{j=1}^{n}$,

$$\left|\sum_{j=1}^{n} c_j\, f(t_j)\right| \le M\left[\sum_{j=1}^{n}\sum_{k=1}^{n} c_j\, c_k^{*} \exp\left(i\,x\,(t_j - t_k)\right)\right]^{1/2} \tag{12}$$

for some $M > 0$.

Theorem 3. *[Bochner-Eberlein]A necessary and sufficient condition for a function $R(s, t)$ to have representation (1) with*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |d^2\,G(x, y)| \le M \tag{13}$$

is that for any n, any sequence of pairs $\{(s_j, t_j)\}_{j=1}^{n}$ and any complex $\{c_j\}_{j=1}^{n}$,

$$\left|\sum_{j=1}^{n} c_j\, R(s_j, t_j)\right| \le M\left[\sum_{j=1}^{n}\sum_{k=1}^{n} c_j\, c_k^{*} \exp\left(i\,(x\,s_j - y\,t_j - x\,s_k + y\,t_k)\right)\right]^{1/2} \tag{14}$$

for some $M > 0$.

3 Examples and Application

Example 4. Let $X(t)$ be a zero-mean real Gaussian harmonizable process with covariance $R(s,t)$. Then, for any $n \ge 1$, $X^n(t)$ is harmonizable.

Proof. An exercise with characteristic functions shows that

$$\mathbb{E}\,[X^n(s)\, X^n(t)] = \sum_{p,q,r \ge 0} c_z\,(p,q,r,n)\, R^p(s,s)\, R^q(s,t)\, R^r(t,s)\, R^p(t,t) \tag{15}$$

where the sum is over all $p, q, r \ge 0$ with $n = 2\,p + q + r$ and $c_z\,(p,q,r,n)$ are combinatorial coefficients.

Since both FS transforms [bochner1934fst, p.151] and covariances [loeve1955probability, p.468] are closed under products, $R^q(s,t)\, R^r(t,s) = R^{q+r}(s,t)$ is an FS transform and a covariance. The product $R^p(s,s)\, R^p(t,t)$ is also an FS transform, as for $f(s) = R(s,s)$

$$f^p(s)\, f^p(t)$$

is NND and an FS transform by Theorem 2. The sum in (15) is harmonizable since FS transforms and covariances are closed under positive sums. □

Example 5. Suppose $X(t)$ is harmonizable with spectral decomposition supported by a bounded set A and $g(t)$ is the FS transform of some $G(x)$ of finite variation. Then $X\,(t + g(t))$ and $X(g(t))$ are harmonizable.

Proof. Set $Y(t) = X\,(t + g(t))$, so

$$R_Y(s,t) = R_X(s + g(s), t + g(t)) = \langle\mathrm{iint}\rangle_A \exp\,(i\,x\,(s + g(s)) - i\,y\,(t + g(t)))\, d^2\, G(x,y).$$

Let M_A denote the variation over $A \times A$ of $G(x,y)$. For any complex $\{c_j\}$ and parameter pairs,

$$Q = \left|\sum_{j=1}^{n} c_j\, R_Y(s_j, t_j)\right| \le M_A \left|\sum_{j=1}^{n} c_j \exp\,(i\,x'\,s_j + i\,x'\,g(s_j) - i\,y'\,t_j - i\,y'\,g(t_j))\right|$$

where x', y' in closure of A. The mappings $t \mapsto \exp\,[i\,x'\,g(t)]$ and $t \mapsto \exp\,[-i\,y'\,g(t)]$ are FS transforms, so by repeated application of Bochner's condition this is bounded, and $R_Y(s, t)$ is an FS transform. □

Example 6. Suppose R_1 and R_2 are harmonizable covariances, and for T of finite Lebesgue measure define $R_3(s,t)$ as in (7). Then R_3 is harmonizable.

Proof. By the Bochner-Eberlein condition, for any $\{c_j\}$,

$$Q = \left|\sum_{j=1}^{n} c_j \int_T R_1(s_j, u)\, R_2(u, t_j)\ d\,u\right| \le M_1 \sup_{x,y} \left|\sum_{j=1}^{n} c_j \int_T R_2(u, t_j) \exp\,(i\,s_j\,x - i\,u\,y)\, d\,u\right|.$$

With M_2 the variation bound for R_2,

$$Q \le M_1\, M_2\, m(T) \sup_{x,y} \left|\sum_{j=1}^{n} c_j \exp\,(i\,s_j\,x - i\,t_j\,y)\right|$$

where $m(T)$ is Lebesgue measure. Thus R_3 is an FS transform. □

Example 7. Suppose $X(t)$ is a moving average as in (2) with $f(t,u)=f(t-u)$ a Fourier-Stieltjes transform and $G(u,v)$ of bounded variation. Then $X(t)$ is harmonizable.

Proof. From (2) and the bounded variation of $H(x)$ and $G(u,v)$,

$$R(s,t)=\langle\text{iiiint}\rangle\exp\left[i\,x\,(s-u)-i\,y\,(t-v)\right]\;d^2\,G(u,v)\;d\,H(x)\;d\,H^*(y).$$

This is an FS transform with variation bound $M_G\,M_H^2$, where M_G bounds $G(u,v)$ and M_H bounds $H(x)$. □

Example 8. Suppose $X(t)$ is as above, $Z(u)$ has orthogonal increments with $d\,F(u)=\mathbb{E}[|d\,Z(u)|^2]$ and $f(t,u)$ as in (8). Then $X(t)$ is a Priestley oscillatory process, and is harmonizable provided

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|d\,H_u(x)|\,|\,d\,H_u^*(y)|\;d\,F(u)<\infty. \tag{16}$$

Proof. The covariance is

$$R(s,t)=\int_{-\infty}^{\infty}\exp\left[i\,u\,(s-t)\right]\,f(u,s)\;f^*(u,t)\,d\,F(u).$$

By repeated application of Bochner's and Fubini's theorems, provided (16) holds, $R(s,t)$ is an FS transform and thus harmonizable. □

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