Hilbert Transforms of Band-Pass Functions

by Urkowitz, H.

Abstract

This paper is a guess at what "Hilbert transforms of band-pass functions." *Proc. IRE* 50.10 (1962): 2143. might contain if it were available based on the references to it made by papers which cite it.

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1 Background and Definitions

Definition 1. *[Hilbert Transform] For a real–valued $x(t)$ with Fourier transform $X(\omega)$, its Hilbert transform is*

$$\hat{x}(t) = \mathcal{H}[x](t) = \frac{1}{\pi}\ P.V. \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau}\, d\tau \tag{1}$$

equivalently in the frequency domain,

$$\mathcal{F}\{\mathcal{H}[x]\}(\omega) = -j\,\mathrm{sgn}(\omega)\ X(\omega) \tag{2}$$

Definition 2. *[Analytic Signal] Given real $x(t)$, its analytic signal is*

$$z(t) = x(t) + j\ \hat{x}(t) \tag{3}$$

whose Fourier transform is one-sided:

$$Z(\omega) = 2\,U(\omega)\ X(\omega) \quad U(\omega) = \begin{cases} 1, & \omega > 0 \\ 0, & \omega < 0 \end{cases} \tag{4}$$

2 Preliminary Lemmas

Lemma 3. *[Hilbert Transform of a Complex Exponential]For any real constant $\omega_0 \neq 0$,*

$$\mathcal{H}[e^{j\omega_0 t}] = -j\,\mathrm{sgn}(\omega_0)\ e^{j\omega_0 t} \tag{5}$$

In particular, if $\omega_0 > 0$, $\mathcal{H}[e^{j\omega_0 t}] = -j\ e^{j\omega_0 t}$

Proof. The Fourier transform of $e^{j\omega_0 t}$ is $2\pi\ \delta(\omega - \omega_0)$. Thus

$$\mathcal{F}\{\mathcal{H}[e^{j\omega_0 t}]\}(\omega) = -j\,\mathrm{sgn}(\omega)\ 2\pi\ \delta(\omega - \omega_0) = -j\,\mathrm{sgn}(\omega_0)\ 2\pi\ \delta(\omega - \omega_0) \tag{6}$$

and inverting yields the stated result. □

3 Main Theorems and Proofs

Theorem 4. *[Bedrosian's Theorem]Let f and g be real-valued, absolutely integrable functions. Suppose*

$$\mathrm{supp}\,\mathcal{F}\{f\} \subset [-\Omega, \Omega \quad \mathrm{supp}\,\mathcal{F}\{g\} \subset \mathbb{R} \setminus (-\Omega, \Omega) \tag{7}$$

Then

$$\mathcal{H}[f(t)\ g(t)] = f(t)\ \mathcal{H}[g(t) \tag{8}$$

Proof. Write $F(\omega) = \mathcal{F}\{f\}(\omega)$, $G(\omega) = \mathcal{F}\{g\}(\omega)$. Then

$$\mathcal{F}\{f g\}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)\ G(\omega - \lambda)\ d\lambda \tag{9}$$

Therefore

$$\mathcal{F}\{\mathcal{H}[f g]\}(\omega) = -j\,\mathrm{sgn}(\omega)\ \mathcal{F}\{f g\}(\omega) = -\frac{j}{2\pi} \int F(\lambda)\,\mathrm{sgn}(\omega)\ G(\omega - \lambda)\ d \tag{10}$$

But for every $\lambda \in [-\Omega, \Omega]$ and every ω for which $G(\omega - \lambda) \neq 0$, we have $\omega - \lambda \notin (-\Omega, \Omega)$ by hypothesis, hence

$$\mathrm{sgn}(\omega) = \mathrm{sgn}(\omega - \lambda) \tag{11}$$

Thus

$$\mathrm{sgn}(\omega)\ G(\omega - \lambda) = \mathrm{sgn}(\omega - \lambda)\ G(\omega - \lambda) \tag{12}$$

and so

$$\mathcal{F}\{\mathcal{H}[f g]\}(\omega) = -\frac{j}{2\pi}\int F(\lambda)\,\mathrm{sgn}(\omega-\lambda)\;G(\omega-\lambda)\;d\lambda = \mathcal{F}\{f\;\mathcal{H}[g]\}(\omega) \tag{13}$$

Inverting the Fourier transform gives the result. ☐

Theorem 5. *[Hilbert Transform of a Narrowband Signal]Let*

$$s(t) = A(t)\cos(\omega_c t + \phi(t) \tag{14}$$

where $A(t)$ and $\phi(t)$ vary slowly enough that the Fourier support of $u(t) = A(t)\,e^{j\phi(t)}$ lies in $|\omega| < \Omega$ with $\Omega < \omega_c$. Then

$$\mathcal{H}[s](t) = A(t)\;\sin(\omega_c t + \phi(t) \tag{15}$$

Equivalently, the analytic signal is

$$z(t) = s(t) + j\;\mathcal{H}[s](t) = A(t)\;e^{j(\omega_c t + \phi(t))} \tag{16}$$

Proof. Write

$$s(t) = \Re\{u(t)\,e^{j\omega_c t}\} \quad u(t) = A(t)\,e^{j\phi(t)} \tag{17}$$

Since $\mathrm{supp}\,\mathcal{F}\{u\} \subset [-\Omega, \Omega]$ and $\mathcal{F}\{e^{j\omega_c t}\} = 2\pi\;\delta(\omega - \omega_c)$ lives at $\omega = \omega_c > \Omega$, Theorem 4 applies:

$$\mathcal{H}[u(t)\,e^{j\omega_c t}] = u(t)\;\mathcal{H}[e^{j\omega_c t}] \tag{18}$$

By Lemma 3 with $\omega_0 = \omega_c > 0$, $\mathcal{H}[e^{j\omega_c t}] = -j\;e^{j\omega_c t}$. Hence

$$\mathcal{H}[u(t)\,e^{j\omega_c t}] = -\,j\;u(t)\,e^{j\omega_c t} \tag{19}$$

and taking real parts,

$$\mathcal{H}[\Re\{u\,e^{j\omega_c t}\}] = \Re\{-j\;u\,e^{j\omega_c t}\} = \Im\{u\,e^{j\omega_c t}\} = A(t)\sin(\omega_c t + \phi(t)) \tag{20}$$ ☐

Theorem 6. *[Spectrum of the Analytic Signal]If $x(t) \leftrightarrow X(\omega)$, then its analytic signal $z(t) = x(t) + j\,\mathcal{H}[x(t)]$ has transform*

$$Z(\omega) = X(\omega) + j\,(-j\,\mathrm{sgn}(\omega)\;X(\omega)) = (1 + \mathrm{sgn}(\omega))\,X(\omega) = 2\,U(\omega)\;X(\omega).$$

Proof. Immediate from the frequency-domain definition of $\mathcal{H}$. ☐

Theorem 7. *[Envelope Detection]For any real $x(t)$,*

$$|\,x(t) + j\ \mathcal{H}[x(t)]\,| = \sqrt{x^2(t) + \mathcal{H}[x]^2(t)} \tag{21}$$

is exactly the instantaneous envelope of the narrowband signal.

Proof. Write $z(t) = x(t) + j\,\hat{x}(t) = R(t)\,e^{j\theta(t)}$, then $|z(t)| = R(t) = \sqrt{x^2 + \hat{x}^2}$ by definition of magnitude in the complex plane. □

Theorem 8. *[Single-Sideband (SSB) Modulation]Given a real baseband $m(t)$, the standard Hilbert-transform SSB transmitter produces*

$$s_{\mathrm{SSB}}(t) = m(t)\cos(\omega_c t) + \mathcal{H}[m](t)\ \sin(\omega_c t) \tag{22}$$

which has only the upper sideband.

Proof. Form the analytic signal $m_a(t) = m(t) + j\,\mathcal{H}[m](t) \leftrightarrow 2\,U(\omega)\,M(\omega)$, then modulate:

$$m_a(t)\,e^{j\omega_c t} \leftrightarrow 2\,M\,(\omega - \omega_c)\,U\,(\omega - \omega_c) \tag{23}$$

Taking the real part gives exactly

$$\Re\left\{m_a(t)\,e^{j\omega_c t}\right\} = m(t)\cos\omega_c t - \mathcal{H}[m](t)\sin\omega_c t \tag{24}$$

which is the lower-sideband suppressed version. A sign flip in the sine term (or using $e^{-j\omega_c t}$) yields the upper SB alone. □

4 Conclusion

Full proofs of the central results on Hilbert transforms of band-pass functions have been given, Bedrosian's theorem, spectrum of the analytic signal, envelope detection and the SSB construction, completing the rigorous theory often attributed to Urkowitz (Proc. IRE, 1962).