1 The Discrete Fourier Transform

The Fourier transform establishes a bijective mapping between temporal and frequency domain representations of signals. For discrete-time signals of finite duration, the Discrete Fourier Transform (DFT) provides this mapping:

Definition 1. (Discrete Fourier Transform (DFT)) *Given a discrete-time signal $x(k)$ where $k = 0, 1, \ldots, N-1$, the DFT yields the same number of complex coefficients $X(k)$:*

$$X(k) = \sum_{n=0}^{N-1} x(n) \exp\left(-i \frac{2\pi k n}{N}\right) \tag{1}$$

The inverse DFT reconstructs the original signal:

$$x(k) = \frac{1}{N} \sum_{n=0}^{N-1} X(n) \exp\left(i \frac{2\pi k n}{N}\right) \tag{2}$$

The DFT presupposes periodic extension of the finite-length sequence $x(n)$, a fact that underlies many spectral estimation artifacts.

1.1 The Fast Fourier Transform

The Fast Fourier Transform denotes a family of algorithms that compute the DFT with $O(N \log N)$ complexity rather than the $O(N^2)$ complexity of direct matrix multiplication. Crucially, the FFT produces numerically identical results to the DFT; scaling relationships derived for the DFT apply without modification to FFT implementations.

1.2 Magnitude and Phase

The complex output $X(k)$ encodes both magnitude and phase information:

$$X(k) = |X(k)| \exp\left(i \arg\left(X(k)\right)\right) \tag{3}$$

where $|X(k)|$ represents the magnitude and $\arg\left(X(k)\right)$ the phase of the k-th frequency component.

Definition 2. (Argument (Phase)) *For a non-zero complex number $z = a + i\,b$ where $a, b \in \mathbb{R}$, the principal argument (phase) with range $(-\pi, \pi]$ is:*

$$\arg(z) = \begin{cases} 2\arctan\left(\dfrac{b}{\sqrt{a^2+b^2}+a}\right), & \text{if } b \neq 0 \text{ or } a > 0, \\ \pi, & \text{if } b = 0 \text{ and } a < 0. \end{cases} \tag{4}$$

where $\arg(0)$ *is undefined. This follows from the half-angle identity*

$$\lambda = 2\arctan\left(\frac{\sin(\lambda)}{1+\cos(\lambda)}\right), \quad \lambda \in (-\pi, \pi) \tag{5}$$

where

$$a(r, \lambda) = r\cos(\lambda) \tag{6}$$

and

$$b(r, \lambda) = r\sin(\lambda) \tag{7}$$

with

$$r = \sqrt{a^2 + b^2} \tag{8}$$

The conjugate relationship

$$\arg(\bar{z}) = \begin{cases} -\arg(z), & \text{if } z \notin (-\infty, 0], \\ \pi, & \text{if } z \in (-\infty, 0) \end{cases} \tag{9}$$

and the identity

$$b + i\,a = i\,\bar{z} \tag{10}$$

yield:

$$\arg(i\,\bar{z}) = \begin{cases} 2\arctan\left(\dfrac{a}{\sqrt{a^2+b^2}+b}\right), & \text{if } a \neq 0 \text{ or } b > 0, \\ \pi, & \text{if } a = 0 \text{ and } b < 0. \end{cases} \tag{11}$$

For real-valued input signals $x(n) \in \mathbb{R}$, the DFT exhibits conjugate symmetry:

$$X(k) = X^*(N-k) \tag{12}$$

Consequently, the magnitude spectrum is symmetric:

$$|X(k)| = |X(N-k)| \tag{13}$$

and the phase spectrum is antisymmetric modulo 2π:

$$\arg(X(k)) \equiv -\arg(X(N-k)) (\mathrm{mod}\, 2\pi) \tag{14}$$

with exact equality $\arg(X(k)) = -\arg(X(N-k))$ holding when $X(k)$ is not a negative real number. Thus in the real-valued (symmetric) case the coefficients contain redundant information.