The Operational Matrix of the Random Wave Process

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Abstract

An expression for the convolution of a pair of spherical Bessel functions is determined.

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1 Lemmas

Lemma 1. (Terminating Hypergeometric Series) *For any $p \in \mathbb{Z}_{\geq 0}$, the Gauss hypergeometric function terminates:*

$$ {}_2F_1(-p, b; c; z) = \sum_{k=0}^{p} \frac{(-p)_k (b)_k}{(c)_k \, k!} z^k \tag{1} $$

where $(a)_k = \prod_{i=0}^{k-1} (a + i)$

Proof. By definition, the Gauss hypergeometric series is:

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k \, k!} z^k \tag{2} $$

Setting $a = -p$ with $p \in \mathbb{Z}_{\geq 0}$, the Pochhammer symbol $(-p)_k$ becomes zero for all $k > p$. Explicitly:

$$ (-p)_k = \prod_{i=0}^{k-1} (-p + i) = \begin{cases} (-p)(-p+1) \cdots (-p+k-1), & k \leq p \\ 0 & k > p \end{cases} \tag{3} $$

Thus, the series terminates at $k = p$, yielding:

$$ {}_2F_1(-p,b;c;z) = \sum_{k=0}^{p} \frac{(-p)_k (b)_k}{(c)_k\, k!}\, z^k \tag{4} $$ □

Lemma 2. (Integral with Incomplete Gamma Function) *For $j \geq 0$,*

$$ \int_{-1}^{1} \left(\frac{1-x}{2}\right)^j e^{ixy}\, dx = \frac{e^{iy}}{2^j} \frac{\gamma(j+1, 2\,i\,y)}{(i\,y)^{j+1}} \tag{5} $$

where $\gamma(s,x)$ denotes the lower incomplete gamma function.

Proof. Substitute $t = \frac{1-x}{2} \Longrightarrow x = 1 - 2\,t$, $d\,x = -2\,d\,t$, adjusting limits:

$$ \int_{1}^{0} t^j\, e^{i(1-2t)y} (-2\,d\,t) = 2\,e^{iy} \int_0^1 t^j\, e^{-2iyt}\, d\,t \tag{6} $$

Let $u = 2\,i\,y\,t \Longrightarrow t = \frac{u}{2\,i\,y}$, $d\,t = \frac{d\,u}{2\,i\,y}$:

$$ \frac{2\,e^{iy}}{(2\,i\,y)^{j+1}} \int_0^{2iy} u^j\, e^{-u}\, d\,u = \frac{e^{iy}}{2^j} \frac{\gamma(j+1, 2\,i\,y)}{(i\,y)^{j+1}} $$ □

Lemma 3. (Legendre Polynomial Representation) *The Legendre polynomials are hypergeometric functions*

$$ P_m(x) = {}_2F_1(-m, m+1; 1; \tfrac{1-x}{2}) \tag{7} $$

Proof. From the Rodrigues formula $P_m(x) = \frac{1}{2^m\, m!} \frac{d^m}{d\,x^m} (x^2-1)^m$ expand using the binomial theorem:

$$ (x^2-1)^m = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} x^{2k} \tag{8} $$

Differentiating m times yields terms proportional to x^k, matching the hypergeometric series:

$$ P_m(x) = {}_2F_1\!\left(-m, m+1; 1; \tfrac{1-x}{2}\right) $$ □

2 Main Theorem

Theorem 4. (Fourier Transform of Legendre Polynomial Products) *The Fourier transform of the product of a pair of Legendre polynomials is the convolution of a pair of spherical Bessel functions of the first kind which can be expressed as*

$$\begin{aligned} I_{m,n}(y) &= \int_{-1}^{1} P_m(x)\, P_n(x)\, e^{ixy}\; d\,x \\ &= e^{iy} \sum_{j=0}^{m+n} \Psi_j(m,n) \frac{\gamma(j+1, 2\,i\,y)}{2^j\,(i\,y)^{j+1}} \end{aligned} \tag{9}$$

where

$$\begin{aligned} \Psi_j(m,n) &= \frac{{}_4F_3\left(\begin{matrix} -m,\; m+1,\; -n,\; n+1 \\ 1,\; 1,\; j+1 \end{matrix};1\right)}{j!} \\ &= \sum_{k=\max(0,j-n)}^{\min(j,m)} \frac{(-m)_k\,(m+1)_k\,(-n)_{j-k}\,(n+1)_{j-k}}{(1)_k (1)_{j-k}\, k!(j-k)!} \end{aligned} \tag{10}$$

Proof. Expand both polynomials using their hypergeometric representations:

$$P_m(x) = \sum_{k=0}^{m} \frac{(-m)_k\,(m+1)_k}{(1)_k\, k!} \left(\frac{1-x}{2}\right)^k \tag{11}$$

$$P_n(x) = \sum_{\ell=0}^{n} \frac{(-n)_\ell\,(n+1)_\ell}{(1)_\ell\, \ell!} \left(\frac{1-x}{2}\right)^\ell \tag{12}$$

Their product becomes:

$$\begin{aligned} P_m(x)\, P_n(x) &= \sum_{k=0}^{m} \sum_{\ell=0}^{n} \frac{(-m)_k\,(m+1)_k\,(-n)_\ell\,(n+1)_\ell}{(1)_k (1)_\ell\, k!\ell!} \left(\frac{1-x}{2}\right)^{k+\ell} \\ &= \sum_{k=0}^{m} \sum_{\ell=0}^{n} \frac{(-m)_k\,(m+1)_k\,(-n)_\ell\,(n+1)_\ell}{k!^2\,\ell!^2} \left(\frac{1-x}{2}\right)^{k+\ell} \end{aligned} \tag{13}$$

Reorganize using $j = k+\ell$, with explicit summation bounds:

$$P_m(x)\, P_n(x) = \sum_{j=0}^{m+n} \underbrace{\sum_{k=\max(0,j-n)}^{\min(j,m)} \frac{(-m)_k\,(m+1)_k\,(-n)_{j-k}\,(n+1)_{j-k}}{(1)_k (1)_{j-k}\, k!(j-k)!}}_{\Psi_j(m,n)} \left(\frac{1-x}{2}\right)^j \tag{14}$$

The inner sum coefficients are:

$$\Psi_j(m,n) = \sum_{k=\max(0,j-n)}^{\min(j,m)} \frac{(-m)_k\,(m+1)_k\,(-n)_{j-k}\,(n+1)_{j-k}}{(1)_k(1)_{j-k}\,k!(j-k)!} \tag{15}$$

Apply the identities:

$$(-n)_{j-k} = (-1)^{j-k}\frac{\Gamma(n+1)}{\Gamma(n-j+k+1)} \tag{16}$$

$$(n+1)_{j-k} = \frac{\Gamma(n+j-k+1)}{\Gamma(n+1)} \tag{17}$$

$$(-a)_n\,(a+1)_n = (-1)^n\frac{\Gamma(a+1)}{\Gamma(a-n+1)} \tag{18}$$

so that the substitutions

$$(-n)_{j-k}\,(n+1)_{j-k} = \frac{(-1)^{j-k}\,\Gamma(n+j-k+1)}{\Gamma(n-j+k+1)} \tag{19}$$

and $h = j-k$ can be made so that $\Psi_j(m,n)$ can be expressed as

$$\Psi_j(m,n) = (-1)^j \sum_{h=0}^{\min(n,j)-\max(0,j-m)} \frac{(-m)_{j-h}\,(m+1)_{j-h}\,\Gamma(n+h+1)}{(j-h)!^2\Gamma(n-j+h+1)\,h!} \tag{20}$$

to reveal the hypergeometric form by reversing the index transformation:

$$\begin{aligned}\Psi_j(m,n) &= \frac{1}{j!}\sum_{k=0}^{\min(j,m,n)} \frac{(-m)_k\,(m+1)_k\,(-n)_k\,(n+1)_k}{(1)_k(1)_k\,(j+1)_k\,k!} \\ &= \frac{{}_4F_3\!\left(\begin{matrix}-m,m+1,-n,n+1\\1,1,j+1\end{matrix};1\right)}{j!}\end{aligned} \tag{21}$$

To complete the proof, observe that the hypergeometric form has an extended summation range $k=0$ to $\min(j,m,n)$ compared to the original bounds $k=\max(0,j-n)$ to $\min(j, m)$. However, terms outside the original bounds vanish identically. For $k>\min(j,m)$, the Pochhammer symbol $(-m)_k = 0$ by Lemma 1. For $k<\max(0,j-n)$, the term $(-n)_{j-k}$ in the original sum vanishes since $j-k>n$. Thus the extended summation yields the same result, establishing the equality of both representations of $\Psi_j(m,n)$. □