perplexity

ok then mayne i specified the filter incorrectly, because A_t(lambda) should equal exp(i*lambda*(theta(t)-t)) so that phi_t(lambda)=exp(I*lambda*theta(t)) . Determine the h_t(u) corresponding to that

Great clarification!
You want the family of functions

$$\phi_t(\lambda) = e^{i\lambda\theta(t)}$$

and you want to express this as

$$A_t(\lambda) = e^{i\lambda(\theta(t)-t)}$$

in the form

$$X_t = \int A_t(\lambda) e^{i\lambda t} dZ(\lambda)$$

and to identify the time-varying filter $ h_t(u) $ corresponding to this $ A_t(\lambda) $.

Step 1: Express $ A_t(\lambda) $ in Terms of $ h_t(u) $

Recall from Priestley:

$$A_t(\lambda) = \int e^{i\lambda u} h_t(u) du$$

with the inverse (for nice enough $ h_t $):

$$h_t(u) = \frac{1}{2\pi} \int e^{-i\lambda u} A_t(\lambda)\, d\lambda$$

Given

$$A_t(\lambda) = e^{i\lambda(\theta(t)-t)}$$

so

$$h_t(u) = \frac{1}{2\pi} \int e^{-i\lambda u} e^{i\lambda(\theta(t)-t)} d\lambda = \frac{1}{2\pi} \int e^{-i\lambda[u-(\theta(t)-t)]} d\lambda$$

Step 2: Evaluate the Integral

$$
h_t(u) = \frac{1}{2\pi} \int e^{-i\lambda[u-(\theta(t)-t)]}\, d\lambda
$$

But

$$
\frac{1}{2\pi} \int e^{-i\lambda a} d\lambda = \delta(a)
$$

Therefore,

$$
h_t(u) = \delta\big(u - [\theta(t) - t]\big)
$$

Step 3: Write Final Answer

$$
\boxed{h_t(u) = \delta\Big(u - [\theta(t) - t]\Big)}
$$

In words:

- The filter at time $ t $ is a pure delay of $ \theta(t) - t $.
- The corresponding family is $ \phi_t(\lambda) = e^{i \lambda \theta(t)} $.

Check:
Plugging back into the formula:

$$
A_t(\lambda) = \int e^{i\lambda u}\, \delta(u - [\theta(t) - t])\, du = e^{i\lambda(\theta(t)-t)}
$$

which matches your requirement.

Summary Table

Quantity	Formula
$ \phi_t(\lambda) $	$ e^{i\lambda \theta(t)} $
$ A_t(\lambda) $	$ e^{i\lambda (\theta(t)-t)} $
Corresponding filter $ h_t(u) $	$ \delta(u - [\theta(t)-t]) $

If you want to know what sort of process $ X_t $ results when a stationary process is filtered through this $ h_t(u) $, let me know!