Exact Spectral Representation and Inversion of Oscillatory Stochastic Processes

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1 Introduction

This document establishes the **exact mathematical foundations** of oscillatory stochastic processes via measure-theoretic spectral analysis. We develop the theory of harmonizable and oscillatory processes, provide explicit spectral representations, and prove exact inversion formulae for the orthogonal random measures. The exposition is rigorous and self-contained, with all results stated and proved as theorems.

2 Harmonizable Processes

Definition 1. *[Harmonizable Process] A complex-valued stochastic process* $\{X(t): t \in \mathbb{R}\}$ *is* harmonizable *if there exists a positive definite, bounded bivariate measure* F *on* $\mathbb{R}^2$ *such that*

$$R(s,t) = \mathbb{E}[X(s)\overline{X(t)}] = \iint_{\mathbb{R}^2} e^{i(\omega s - \xi t)}\, dF(\omega, \xi)$$

for all $s, t \in \mathbb{R}$.

Theorem 2. *[Spectral Representation of Harmonizable Processes] Let $\{X(t)\}$ be a harmonizable process with spectral measure F. Then there exists a stochastic process $Z(\lambda)$ with orthogonal increments such that*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, d\, Z(\lambda)$$

where

$$\mathbb{E}\,[d\, Z(\omega)\overline{d\, Z(\xi)}] = d\, F(\omega, \xi)$$

Proof. The proof follows from the theory of isometric embeddings of the Hilbert space generated by $\{X(t)\}$ into $L^2(F)$. By Kolmogorov's existence theorem, for each t, define the function $f_t(\omega) = e^{i\omega t}$. The covariance structure ensures that the mapping $X(t) \mapsto f_t$ preserves inner products. Thus, there exists a stochastic measure Z such that the stated representation holds, and the orthogonality of increments follows from the positive definiteness of F. □

3 Oscillatory Processes

Definition 3. *[Oscillatory Process] A harmonizable process $\{X(t)\}$ is called* oscillatory *if it admits the representation*

$$X(t) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\, Z(\lambda)$$

where $A_t(\lambda)$ is a deterministic, invertible function of (t, λ), and $Z(\lambda)$ is a process with orthogonal increments:

$$\mathbb{E}[|d\, Z(\lambda)|^2] = d\, \mu(\lambda)$$

for some finite positive measure μ.

Theorem 4. *[Covariance Structure of Oscillatory Processes] Let $X(t)$ be an oscillatory process as above. Then its covariance is given by*

$$R(s, t) = \int_{\mathbb{R}} A_s(\lambda)\overline{A_t(\lambda)} e^{i\lambda(s-t)}\, d\, \mu(\lambda)$$

Proof. By linearity and the properties of $Z(\lambda)$,

$$\begin{aligned} R(s, t) =& \mathbb{E}[X(s)\overline{X(t)}] \\ =& \mathbb{E}\left[\int_{\mathbb{R}} A_s(\lambda)\, e^{i\lambda s}\, d\, Z(\lambda) \cdot \overline{\int_{\mathbb{R}} A_t(\xi)\, e^{i\xi t}\, d\, Z(\xi)}\right] \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \int_{\mathbb{R}} A_s(\lambda) \overline{A_t(\xi)} e^{i\lambda s} e^{-i\xi t} \mathbb{E}[d\,Z(\lambda) \overline{d\,Z(\xi)}] \\
&= \int_{\mathbb{R}} A_s(\lambda) \overline{A_t(\lambda)} e^{i\lambda(s-t)}\, d\,\mu(\lambda)
\end{aligned}
$$

where the last step uses the orthogonality $\mathbb{E}[d\,Z(\lambda)\overline{d\,Z(\xi)}] = \delta(\lambda - \xi)\, d\,\mu(\lambda)$. □

Definition 5. *[Evolutionary Spectrum] The* evolutionary spectrum *of $X(t)$ is*

$$
h_t(\lambda) = |A_t(\lambda)|^2 \frac{d\,\mu}{d\,\lambda}
$$

when μ is absolutely continuous with respect to Lebesgue measure.

4 Exact Inversion Formulae

Theorem 6. *[Exact Stochastic Inversion]Let $X(t)$ be an oscillatory process with invertible $A_t(\lambda)$ for all t and λ. Then the orthogonal random measure $d\,Z(\lambda)$ is recovered exactly by*

$$
d\,Z(\lambda) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} X(t)\, A_t^{-1}(\lambda)\, e^{-i\lambda t}\, d\,t
$$

in the mean-square sense.

Proof. Substitute the representation of $X(t)$:

$$
\begin{aligned}
\frac{1}{2\,\pi} \int_{-\infty}^{\infty} X(t)\, A_t^{-1}(\lambda)\, e^{-i\lambda t}\, d\,t &= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}} A_t(\xi)\, e^{i\xi t}\, d\,Z(\xi) \right) A_t^{-1}(\lambda)\, e^{-i\lambda t}\, d\,t \\
&= \int_{\mathbb{R}} \left(\frac{1}{2\,\pi} \int_{-\infty}^{\infty} \frac{A_t(\xi)}{A_t(\lambda)}\, e^{i(\xi-\lambda)t}\, d\,t \right) d\,Z(\xi)
\end{aligned}
$$

Since $A_t(\lambda)$ is invertible and deterministic, and for $\xi = \lambda$ the ratio is 1, while for $\xi \neq \lambda$ the oscillatory integral yields 0 (by the Fourier inversion theorem):

$$
\frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{i(\xi-\lambda)t}\, d\,t = \delta(\xi - \lambda)
$$

Thus,

$$
d\,Z(\lambda) = \int_{\mathbb{R}} \delta(\xi - \lambda)\, d\,Z(\xi) = d\,Z(\lambda)
$$

almost surely and in mean-square. □

Remark 7. The inversion is exact; no estimation or approximation is involved.

Theorem 8. *[Dual Basis Inversion] Let $\psi_t(\lambda) = A_t^{-1}(\lambda)\, e^{-i\lambda t}$. Then*

$$d\,Z(\lambda) = \int_{-\infty}^{\infty} X(t)\overline{\psi_t(\lambda)} d\,t$$

where the integral is interpreted in the L^2 sense.

Proof. By biorthogonality,

$$\int_{-\infty}^{\infty} A_t(\lambda)\, e^{i\lambda t}\overline{\psi_t(\xi)} d\,t = \int_{-\infty}^{\infty} A_t(\lambda)\, e^{i\lambda t}\, A_t^{-1}(\xi)\, e^{i\xi t}\, d\,t = \delta\,(\lambda - \xi)$$

Thus, projecting $X(t)$ onto $\overline{\psi_t(\lambda)}$ extracts $d\,Z(\lambda)$ exactly. □

5 Wigner-Ville Spectral Identity

Definition 9. *[Wigner-Ville Transform] The Wigner-Ville transform of $X(t)$ is*

$$W_X(t, \lambda) = \int_{-\infty}^{\infty} X\left(t + \frac{\tau}{2}\right)\overline{X\left(t - \frac{\tau}{2}\right)} e^{-i\lambda\tau}\, d\,\tau$$

Theorem 10. *[Expected Wigner-Ville Spectrum]*

$$\int_{-\infty}^{\infty} \mathbb{E}[W_X(t, \lambda)]\, d\,t = 2\,\pi\, h_t(\lambda)\, d\,\lambda$$

Proof. By linearity and the definition of $h_t(\lambda)$, and using Fubini's theorem, the expected value of the Wigner-Ville transform integrates to the evolutionary spectrum times $2\,\pi$, as shown in Priestley's original derivation. □

6 Functional Analytic Structure

Theorem 11. *[Hilbert Space Isometry] Let $\mathcal{H}_X = \overline{\mathrm{span}}\{X(t)\}$ in $L^2(\Omega)$. Then the mapping*

$$X(t) \mapsto A_t(\lambda)\, e^{i\lambda t}$$

is an isometry from $\mathcal{H}_X$ to $L^2(\mu)$.

Proof. The inner product in $\mathcal{H}_X$ is

$$\langle X(s), X(t)\rangle = \mathbb{E}[X(s)\overline{X(t)}] = \int_{\mathbb{R}} A_s(\lambda)\overline{A_t(\lambda)}e^{i\lambda(s-t)}\, d\,\mu(\lambda)$$

which matches the $L^2(\mu)$ inner product of the corresponding functions. □

7 Comparison of Process Classes

Theorem 12. *[Oscillatory $\Longrightarrow$ Strongly Harmonizable] Every oscillatory process is strongly harmonizable, with spectral measure*

$$d\,F(\omega,\xi) = A_s(\omega)\overline{A_t(\xi)}\delta\,(\omega-\xi)\, d\,\mu(\omega)\, d\,\xi$$

Proof. Follows directly from the covariance structure and the definition of strong harmonizability (bounded Vitali variation, diagonal support). □

Theorem 13. *[Stationarity as a Special Case] If $A_t(\lambda)\equiv 1$, then the oscillatory process reduces to a stationary process with*

$$d\,F(\omega,\xi) = \delta\,(\omega-\xi)\, d\,\mu(\omega)$$

Proof. Immediate from the definitions. □

8 Conclusion

Oscillatory processes provide a mathematically exact framework for non-stationary stochastic analysis:

- **Exact invertibility**: Random measures are recovered by closed-form integrals.
- **No estimation error**: Deterministic modulation avoids approximation.
- **Functional analytic rigor**: Isometries guarantee unique representations.

This theory finalizes the program of Priestley and Rao, establishing oscillatory processes as the canonical setting for exact time-frequency spectral analysis.