Parseval Frames

BY $S\tau\Sigma v\varepsilon$

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Parseval frames are a specialized type of frame in Hilbert spaces that extend the concept of orthonormal bases. They maintain the preservation of vector norms while introducing redundancy into the system. This redundancy allows for greater flexibility in representation and analysis, distinguishing Parseval frames as a valuable mathematical tool in the study of Hilbert spaces.

1 Definition of Parseval Frames

A Parseval frame is a specific type of frame in linear algebra and functional analysis that generalizes the concept of orthonormal bases while maintaining certain desirable properties. Formally, a sequence of vectors $\{f_j\}$ in a Hilbert space H is called a Parseval frame if it satisfies the following condition for all vectors x in H:

$$\sum_j |\langle x, f_j \rangle|^2 = \|x\|^2 \tag{1}$$

This equation, known as the frame condition, is a generalization of Parseval's identity for orthonormal bases. It ensures that the norm of any vector x is preserved when expressed in terms of its inner products with the frame elements.

Parseval frames can be characterized by their frame operator S, defined as:

$$S x = \sum_j \langle x, f_j \rangle f_j \tag{2}$$

For a Parseval frame, the frame operator is equal to the identity operator, i.e., $S = I$. This property distinguishes Parseval frames from general frames and tight frames.

An equivalent definition of a Parseval frame can be given in terms of the analysis operator T and its adjoint T^*:

$$T^* T = I$$

where T is the operator that maps a vector x to its sequence of frame coefficients $\{\langle x, f_j \rangle\}$.

In finite-dimensional spaces, Parseval frames have an additional characterization: a set of vectors $\{f_j\}$ forms a Parseval frame if and only if the matrix F whose columns are the frame vectors satisfies $F F^* = I$, where F^* is the conjugate transpose of F.

2 Key Properties of Parseval Frames

1. Norm Equivalence: One of the fundamental properties of Parseval frames is their ability to maintain norm equivalence between a vector in the Hilbert space and its sequence of coefficients.

2. Redundancy: Unlike orthonormal bases, Parseval frames can have more vectors than the dimension of the space they span.

3. Tight Frame Property: Parseval frames are a special case of tight frames, where the frame bounds A and B are equal to 1.

4. Reconstruction Formula: For any vector x in the Hilbert space, a Parseval frame $\{f_j\}$ satisfies the reconstruction formula:

$$x = \sum_j \langle x, f_j \rangle f_j \tag{3}$$

5. Parseval's Identity: Parseval frames satisfy a generalized version of Parseval's identity:

$$\|x\|^2 = \sum_j |\langle x, f_j \rangle|^2 \tag{4}$$

6. Duality: Every Parseval frame is self-dual, meaning that the frame itself serves as its own dual frame.

7. Invariance Under Unitary Transformations: If $\{f_j\}$ is a Parseval frame and U is a unitary operator, then $\{U f_j\}$ is also a Parseval frame.

8. Finite-Dimensional Characterization: In finite-dimensional Hilbert spaces, a set of vectors forms a Parseval frame if and only if the matrix whose columns are the frame vectors has orthonormal rows.

3 Construction via Orthogonal Projection

Let H be infinite-dimensional Hilbert space
Let $W \subseteq H$ be finite-dimensional subspace
Let $\{e_1, e_2, ..., e_n\}$ be orthonormal basis for W
Let $\{f_k\}_{k=1}^{\infty}$ be orthonormal basis for H

Orthogonal projection P_W onto W:
$P_W x = \sum \langle x, e_i \rangle e_i$

Construction of Parseval frame:

$$\tilde{f}_k = \frac{P_W f_k}{\sqrt{\sum |\langle f_k, e_i \rangle|^2}} \tag{5}$$

Verification:

1. For any $x \in W$:

$$\langle x, \tilde{f}_k \rangle = \frac{\langle x, P_W f_k \rangle}{\sqrt{\sum |\langle f_k, e_i \rangle|^2}} = \frac{\langle P_W x, f_k \rangle}{\sqrt{\sum |\langle f_k, e_i \rangle|^2}} = \frac{\langle x, f_k \rangle}{\sqrt{\sum |\langle f_k, e_i \rangle|^2}} \tag{6}$$

2. Parseval frame condition:

$$\sum |\langle x, \tilde{f}_k \rangle|^2 = \sum \frac{|\langle x, f_k \rangle|^2}{\sum |\langle f_k, e_i \rangle|^2} \tag{7}$$

3. Interchanging sums:

$$\sum_i \sum_k \frac{|\langle x, f_k \rangle|^2 |\langle f_k, e_i \rangle|^2}{(\sum |\langle f_k, e_j \rangle|^2)^2} \tag{8}$$

4. Using orthonormal basis property:

$$\sum |\langle x, e_i \rangle|^2 = \|P_W x\|^2 = \|x\|^2 \tag{9}$$