Compact Self-Adjoint Integral Covariance Operators on $L^2[0,\infty)$

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Abstract

The spectral theory of compact self-adjoint integral operators on $L^2[0,\infty)$ is applied to extend Mercer's theorem, which is relegated to bounded(compact) intervals to the non-compact (unbounded) semi-infinite interval $[0,\infty)$, focusing on eigenfunction expansions and norm bounds. For operators with kernels $K(z,w)$ represented as Mercer expansions, which are infinite series of eigenfunction products, the main result establishes the operator norm bound $\|T_K - T_{K_N}\| \leq |\lambda_{N+1}|$ for the integral covariance operator $(Tf)(y) = \int_0^\infty K(x,y) f(x) \mathrm{d}x$ derived using orthogonal projection methods. An extension of Mercer's Theorem to the semi-infinite interval $[0,\infty)$ demonstrates that it is the the compactness of the operator relative to the induced canonical metric(square root of the variance structure function which is in one-to-one correspondence with the covariance kernel function K in the case of real-valued processes), rather than the domain, underpins these results. Furthermore, the completeness of the eigenfunction system is proven through the spectral properties of compact self-adjoint operators. These findings provide a refined understanding of integral covariance operators operators on $L^2[0,\infty)$ and their finite-dimensional subspaces.

Theorem 1

Let T_K be a compact self-adjoint integral covariance operator on $L^2[0,\infty)$

$$(T_K f)(z) = \int_0^\infty K(z,w)\, f(w)\, d\, w \tag{1}$$

defined by kernel K:

$$K(z,w) = \sum_{k=0}^{\infty} \lambda_k \phi_k(z) \phi_k(w)$$

where $\{\phi_n\}_{n=0}^{\infty}$ is a sequence of orthonormal eigenfunctions in $L^2[0,\infty)$ and the corresponding eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ which are ordered by decreasing magnitude

$$|\lambda_{n+1}| < |\lambda_n| \forall n \tag{2}$$

satisfy the eigenfunction equations

$$\begin{aligned}
(T_K\, \phi_n)(z) &= \int_0^\infty K(z,w)\, \phi_n(w)\, d\, w \\
&= \int_0^\infty \left(\sum_{k=0}^{\infty} \lambda_k \phi_k(z) \phi_k(w) \right) \phi_n(w)\, d\, w \\
&= \int_0^\infty \phi_n(w) \left(\sum_{k=0}^{\infty} \lambda_k \phi_k(z) \phi_k(w) \right) d\, w \\
&= \int_0^\infty \phi_n(w) \sum_{k=0}^{\infty} \lambda_k \phi_k(z) \phi_k(w)\, d\, w \\
&= \sum_{k=0}^{\infty} \phi_n(w) \int_0^\infty \lambda_k \phi_k(z) \phi_k(w)\, d\, w \\
&= \sum_{k=0}^{\infty} \phi_k(z) \lambda_k \int_0^\infty \phi_n(w) \phi_k(w)\, d\, w \\
&= \sum_{k=0}^{\infty} \phi_k(z) \lambda_k \delta_{n,k} \\
&= \phi_n(z)\, \lambda_n
\end{aligned} \tag{3}$$

Let T_{K_N} be the truncated operator with kernel

$$K_N(z,w) = \sum_{n=0}^{N} \lambda_n\, \phi_n(z)\, \phi_n(w) \tag{4}$$

then:

$$\|T_K - T_{K_N}\| \le |\lambda_{N+1}| \tag{5}$$

Proof. Let

$$(E_N f)(z) = (T_K f)(z) - (T_{K_N} f)(z) \tag{6}$$

be the difference $T_K - T_{K_N}$ then let

$$f = g + h \forall f \in L^2[0,\infty) \tag{7}$$

where

$$g(x) = \sum_{k=0}^{N} \langle f, \phi_k \rangle \, \phi_k(x) \forall g \in \operatorname{span}\{\phi_k\}_{k \leq N} \tag{8}$$

and

$$h(x) = \sum_{k=N+1}^{\infty} \langle f, \phi_k \rangle \, \phi_k(x) \forall h \in \operatorname{span}\{\phi_k\}_{k > N} \tag{9}$$

where by orthogonality of g and h (which follows from the orthogonality of eigenfunctions φ_k)

$$\begin{aligned} \langle g, h \rangle &= \left\langle \sum_{k=0}^{N} \langle f, \phi_k \rangle \, \phi_k, \sum_{j=N+1}^{\infty} \langle f, \phi_j \rangle \, \phi_j \right\rangle \\ &= \sum_{k=0}^{N} \sum_{j=N+1}^{\infty} \langle f, \phi_k \rangle \langle f, \phi_j \rangle \langle \phi_k, \phi_j \rangle \\ &= \sum_{k=0}^{N} \sum_{j=N+1}^{\infty} \langle f, \phi_k \rangle \langle f, \phi_j \rangle \, \delta_{k,j} \\ &= 0 \end{aligned} \tag{10}$$

we have

$$\|E_N f\|^2 = \langle E_N f, E_N f \rangle = \langle E_N h, h \rangle \tag{11}$$

since

$$\begin{aligned} (E_N g)(x) &= \left(E_N \sum_{k=0}^{N} \langle f, \phi_k \rangle \, \phi_k \right)(x) \\ &= (T_K g)(x) - (T_{K_N} g)(x) \\ &= \sum_{k=0}^{N} \lambda_k \langle f, \phi_k \rangle \, \phi_k(z) - \sum_{k=0}^{N} \lambda_k \langle f, \phi_k \rangle \, \phi_k(z) \\ &= 0 \end{aligned} \tag{12}$$

by construction and since h is orthogonal to the first N eigenfunctions and along with the fact that the eigenvalues are ordered by decreasing magnitude

$$|\lambda_k| \leq |\lambda_{N+1}| \forall k > N \tag{13}$$

we have

$$\begin{aligned} |\langle E_N h, h \rangle| &\leq |\lambda_{N+1}| \|h\|^2 \\ &\leq |\lambda_{N+1}| \|f\|^2 \end{aligned} \tag{14}$$

Therefore:

$$\|E_N\| \leq |\lambda_{N+1}| \tag{15}$$ □

Remark 2. This extension of Mercer's Theorem to $[0, \infty)$ reveals a deeper truth about integral operators that is obscured in most presentations. The key insight is that compactness of the interval plays no essential role - what matters is the compactness of the operator itself.

The traditional presentation of Mercers theorem on compact $[a, b]$ emphasize properties that are merely convenient rather than fundamental:

- Compactness of $[a, b]$ provides easy continuity arguments
- Finite measure simplifies certain technical steps
- Historical development focused on these cases first

However, the proof above shows that the essential structure depends only on:

1. The spectral properties of compact self-adjoint operators
2. The precise operator norm bound $\|E_N\| \leq |\lambda_{N+1}|$
3. The fact that $\{\lambda_n\}_{n=1}^{\infty}$ converges to zero

This reveals that Mercer's Theorem is fundamentally about the behavior of integral operators themselves, not about properties of their domains. The extension to $[0, \infty)$ is not just a generalization - it's a clearer view of the true mathematical structure.

Theorem 3

(Completeness) *Let T_K be a compact self-adjoint integral operator on $L^2[0, \infty)$ with kernel $K(z, w)$. Then the eigenfunctions $\{\phi_n\}_{n=0}^{\infty}$ form a complete orthonormal system in $L^2[0, \infty)$.*

Proof. Suppose there exists $f \in L^2[0, \infty)$ orthogonal to all ϕ_n. Then:

$$\langle f, \phi_n \rangle = 0 \quad \forall n \tag{16}$$

Therefore:

$$T_K f = \sum_{n=0}^{\infty} \lambda_n \langle f, \phi_n \rangle \, \phi_n = 0 \tag{17}$$

Since T_K is compact and self-adjoint, $\ker(T_K)^{\perp} = \overline{\mathrm{range}(T_K)}$ contains all eigenvectors. Thus f must be zero, proving completeness. $\square$