An Orthogonal Basis for the Bessel Functions of the First Kind of Orders 0 and 1

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Abstract

The even-indexed orthonormalized Fourier transforms of the Chebyshev polynomials of the first kind form a basis in a reproducing-kernel Hilbert space for the Bessel function of the first kind J_0 and likewise for the odd-indexed functions which form a basis that reproduces $\dot{J}_0 = -J_1$. Suprisingly, such a basis for these functions was not known to exist before this.

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1 The Type-I Chebyshev Polynomials $T_n(x)$

Let T_n be the Chebyshev polynomials of the first kind, also said to be of Type-I, defined by

$$\begin{aligned}
T_n(x) &= {}_2F_1\left(\begin{matrix} n, & -n \\ & \frac{1}{2} \end{matrix} \middle| \frac{1}{2} - \frac{x}{2} \right) \\
&= \int_{-\infty}^{\infty} e^{ixy} \hat{T}_n(y) d y \\
&= \int_{-\infty}^{\infty} e^{ixy} \frac{i}{y} (\mathrm{e}^{-\mathrm{i}y} F_n^+(y) - \mathrm{e}^{\mathrm{i}y} (-1)^n F_n^-(y)) d y \\
&= \int_{-\infty}^{\infty} e^{ixy} \int_{-\infty}^{\infty} e^{-iyz} T_n(z) d z d y
\end{aligned} \tag{1}$$

where ${}_2F_1$ is the (Gauss) hypergeometric function. [1, (13.140)]

1.1 The Fourier Transforms $\hat{T}_n(y)$ of $T_n(x)$

The functions $\hat{T}_n(y)$ are Fourier transforms of $T_n(x)$ defined by

$$\begin{aligned}\hat{T}_n(y) &= \int_{-\infty}^{\infty} e^{-ixy} T_n(x) d\,y = \int_{-1}^{1} e^{-ixy} T_n(x) d\,x \\ &= \int_{-\infty}^{\infty} e^{-ixy} {}_2F_1\left(\begin{array}{cc} n, & -n \\ & \frac{1}{2} \end{array}\middle| \frac{1}{2} - \frac{x}{2}\right) d\,x \\ &= \frac{i}{y}\,(e^{-iy} F_n^{+}(y) - e^{i(\pi n + y)} F_n^{-}(y))\end{aligned} \tag{2}$$

where

$$F_n^{\pm}(y) = {}_3F_1\left(\begin{array}{ccc} 1, & n, & -n \\ & & \frac{1}{2} \end{array}\middle| \frac{\pm i y}{2}\right) \tag{3}$$

The L^2 norm of $\hat{T}_n(y)$ is

$$|\hat{T}_n| = \sqrt{\int_{-\infty}^{\infty} \hat{T}_n(y)^2 \mathrm{d}y} = \sqrt{\frac{4(-1)^n \pi - (2n^2 - 1)}{4n^2 - 1}} \tag{4}$$

then define the normalized Fourier transforms $Y_n(y)$ of $T_n(x)$ by

$$\begin{aligned}Y_n(y) &= \frac{\hat{T}_n(y)}{|\hat{T}_n|} \\ &= \frac{i}{y}\left(\frac{e^{-iy} F_n^{+}(y) - e^{i(\pi n + y)} F_n^{-}(y)}{\sqrt{\frac{4(-1)^n \pi - (2n^2 - 1)}{4n^2 - 1}}}\right)\end{aligned} \tag{5}$$

For a proof see [2]. It just so happens to be that $Y_n(y)$ enumerates the elements of the kernel of the integral covariance operator, aka its null space, defined by

$$\int_{-\infty}^{\infty} J_0(y) Y_n(y) \mathrm{d}y = \delta_{n,0} = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{6}$$

where $\delta_{n,0}$ is the Kronecker delta function which takes the value 1 when its arguments are equal and 0 when they are not.

1.2 Orthogonalizing $Y_n(y)$ Via The Gram-Schmidt Process

Apply the Gram-Schmidt process to the normalized Fourier transforms of the Type I Chebyshev polynomials $Y_n(y)$ to get $Y_n^{\perp}(y)$

$$Y_n^{\perp}(y) = Y_n(y) - \sum_{m=1}^{n-1} \frac{\langle Y_n(y), Y_m^{\perp}(y)\rangle}{\langle Y_m^{\perp}(y), Y_m^{\perp}(y)\rangle} Y_m^{\perp}(y) \tag{7}$$

then the limits of $Y_n^{\perp}(y)$ at y=0 are equal to

$$\lim_{y\to 0} Y_n^{\perp}(y) = \begin{cases} \frac{1}{\sqrt{\pi}} & n=0 \\ 0 & n\neq 0 \end{cases} \tag{8}$$

Let

$$A_{k,n} = -(-1)^{n+\binom{k}{2}}(k-2n+1)!2^{2n-1-k}\binom{k+1}{k-2n+1}\binom{2k+2-2n}{k+1} \tag{9}$$

and

$$B_{k,n} = \frac{(-1)^{n+\binom{k}{2}} 2^{k-2n}(k-n)!\binom{\frac{1}{2}-n+k}{k-2n}}{n!} \tag{10}$$

then defined the associated functions

$$\Psi_n^{\sin}(y) = \frac{\sin(y)\sqrt{2n-1}}{x^n\sqrt{\pi}}\sum_{k=0}^{n-2} x^{2k}A_{k,n-2} \tag{11}$$

and

$$\Psi_n^{\cos}(y) = \frac{\cos(y)\sqrt{2n-1}}{x^n\sqrt{\pi}}\sum_{k=0}^{n-2} x^{2k+1}B_{k,n-2} \tag{12}$$

then $Y_n^{\perp}(y)$ can be expressed as

$$Y_n^{\perp} = \begin{cases} \frac{\sin(y)}{y\sqrt{\pi}} & n=1 \\ \Psi_n^{\sin}(y) + \Psi_n^{\cos}(y) & n>1 \end{cases} \tag{13}$$

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 15 & -6 & 0 & 0 & 0 & 0 & 0 & \dots \\ 105 & -45 & 1 & 0 & 0 & 0 & 0 & \dots \\ -945 & 420 & -15 & 0 & 0 & 0 & 0 & \dots \\ -10395 & 4725 & -210 & 1 & 0 & 0 & 0 & \dots \\ 135135 & -62370 & 3150 & -28 & 0 & 0 & 0 & \dots \\ 2027025 & -945945 & 51975 & -630 & 1 & 0 & 0 & \dots \\ -34459425 & 16216200 & -945945 & 13860 & -45 & 0 & 0 & \dots \\ -654729075 & 310134825 & -18918900 & 315315 & -1485 & 1 & 0 & \dots \\ 13749310575 & -6547290750 & 413513100 & -7567560 & 45045 & -66 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

Table 1. The first 10 row-vectors of $A_{k,n}$ matrix

Bibliography

[1] G. Arfken and H. Weber. *Mathematical Methods for Physicists*. Elsevier AP, Boston, 6th edition, 2005.

[2] Atul Dixit, Lin Jiu, Victor H Moll, and Christophe Vignat. The finite fourier transform of classical polynomials. *Journal of the Australian Mathematical Society*, 98:145–160, 2015.