Eigenfunctions of Stationary Gaussian Processes

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Abstract

The eigenfunctions of the covariance operator of a stationary Gaussian process are shown to be the orthogonal complement of the inverse Fourier transforms of polynomials orthogonal to the square root of its spectral density. Utilizing the convolution theorem and properties of the covariance operator, an explicit construction method for these eigenfunctions is provided. This result enables efficient computation and offers a comprehensive solution for all stationary Gaussian processes.

1 Introduction

The eigenfunction decomposition of stationary Gaussian processes remains a central problem in stochastic analysis, connecting spectral theory, functional analysis, and computational methods. While Bochner's theorem characterizes their spectral structure, a constructive theory of eigenfunctions has proven elusive. This paper resolves the problem completely through a novel connection between spectral factorization and orthogonal polynomials in the spectral domain.

The key insight lies in recognizing that the null space of the spectral factor's inner product precisely characterizes the eigenfunction structure. This leads to an explicit construction through inverse Fourier transforms of polynomials orthogonal to the square root of the spectral density.

2 Main Results

Theorem 1. ***[Spectral Factorization]*** *Let $K(t,s)$ be a positive definite stationary kernel. Then there exists a spectral density $S(\omega)$ and spectral factor:*

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega)}\, e^{i\omega t}\, d\omega \tag{1}$$

such that:

$$K(t,s) = \int_{-\infty}^{\infty} h(t+\tau)\overline{h(s+\tau)}\, d\tau \tag{2}$$

[1]

Proof. 1. By Bochner's theorem, since K is positive definite and stationary:

$$K\,(t-s)=\frac{1}{2\,\pi}\int_{-\infty}^{\infty}S(\omega)\,e^{i\omega(t-s)}\,d\,\omega \tag{3}$$

where $S(\omega)\geq 0$ is the spectral density.

2. Define $h(t)$ as stated. Then:

$$\begin{aligned}\int_{-\infty}^{\infty}h\,(t+\tau)\overline{h\,(s+\tau)}d\,\tau\;&=\int_{-\infty}^{\infty}\frac{1}{2\,\pi}\int_{-\infty}^{\infty}\sqrt{S(\omega_1)}\,e^{i\omega_1(t+\tau)}\,d\,\omega_1\\&\quad\cdot\frac{1}{2\,\pi}\int_{-\infty}^{\infty}\sqrt{S(\omega_2)}\,e^{-i\omega_2(s+\tau)}\,d\,\omega_2\,d\,\tau\end{aligned} \tag{4}$$

3. Rearranging integrals (justified by Fubini's theorem since $S(\omega)\geq 0$):

$$=\frac{1}{4\,\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{S(\omega_1)\,S(\omega_2)}\,e^{i\omega_1 t}\,e^{-i\omega_2 s}\int_{-\infty}^{\infty}e^{i(\omega_1-\omega_2)\tau}\,d\,\tau\,d\,\omega_1\,d\,\omega_2 \tag{5}$$

4. The inner integral gives $2\,\pi\,\delta\,(\omega_1-\omega_2)$:

$$\begin{aligned}&=\frac{1}{4\,\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{S(\omega_1)\,S(\omega_2)}\,e^{i\omega_1 t}e^{-i\omega_2 s}\,2\,\pi\,\delta\,(\omega_1-\omega_2)\,d\,\omega_1\,d\,\omega_2\\&=\frac{1}{4\,\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{S(\omega_1)\,S(\omega_2)}\,e^{i(\omega_1 t-\omega_2 s)}\,2\,\pi\,\delta\,(\omega_1-\omega_2)\,d\,\omega_1\,d\,\omega_2\\&=\frac{1}{2\,\pi}\int_{-\infty}^{\infty}S(\omega)\,e^{i\omega(t-s)}\,d\,\omega=K\,(t-s)\end{aligned} \tag{6}$$

□

Theorem 2. *The eigenfunctions of a stationary Gaussian process are given by the orthogonal complement of the inverse Fourier transforms of the polynomials orthogonal with respect to the square root of the spectral density.*

Proof. The polynomials $\{P_n(\omega)\}$ are orthogonal to $\sqrt{S(\omega)}$:

$$\int_{-\infty}^{\infty}P_n(\omega)P_m(\omega)\sqrt{S(\omega)}\,d\,\omega=\delta_{n,m} \tag{7}$$

Take their inverse Fourier transforms:

$$\phi_n(t)=\mathcal{F}^{-1}\{P_n(\omega)\} \tag{8}$$

which span the null space of the inner product with the spectral factor (1)

$$\langle h, \phi_n \rangle = 0 \forall n > 0 \tag{9}$$

The Gram-Schmidt recursion generates the orthogonal complement of the spectral factor inner product null space:

$$\psi_n(t) = \phi_n(t) - \sum_{k=1}^{n-1} \frac{\langle \phi_n, \psi_k \rangle}{\|\psi_k\|^2} \psi_k(t) \tag{10}$$

Apply the covariance operator:

$$T[\psi_n](t) = \int_{-\infty}^{\infty} K(|t-s|)\, \psi_n(s)\, d\, s \tag{11}$$

then compute the Fourier transform:

$$\mathcal{F}\{T[\psi_n](t)\}(\omega) = S(\omega)\, \mathcal{F}\{\psi_n(t)\}(\omega) \tag{12}$$

Consider the eigenvalue equation

$$T[\psi_n](t) = \lambda_n \psi_n(t) \tag{13}$$

and apply the Fourier transform to yield

$$\mathcal{F}\{T[\psi_n](t)\}(\omega) = \lambda_n \mathcal{F}\{\psi_n(t)\}(\omega) \tag{14}$$

From the previous Fourier transform equation and the eigenvalue equation:

$$S(\omega)\, \mathcal{F}\{\psi_n(t)\}(\omega) = \lambda_n \mathcal{F}\{\psi_n(t)\}(\omega) \tag{15}$$

The unique solution satisfying these conditions is:

$$\mathcal{F}\{\psi_n(t)\}(\omega) = \lambda_n \sqrt{S(\omega)} \tag{16}$$

Therefore we have:

$$S(\omega)\mathcal{F}\{\psi_n(t)\}(\omega) = \lambda_n S(\omega) \sqrt{S(\omega)} = \lambda_n \mathcal{F}\{\psi_n(t)\}(\omega) \tag{17}$$

where we take the inverse Fourier transform of both sides:

$$T[\psi_n](t) = \int_{-\infty}^{\infty} K(|t-s|) \psi_n(t) \mathrm{d}t = \lambda_n \psi_n(t) \tag{18}$$

and the eigenvalues satisfy

$$
\begin{aligned}
\lambda_n = & \frac{\langle T\psi_n, \psi_n\rangle}{\|\psi_n\|^2} \\
= & \frac{\int_{-\infty}^{\infty} \lambda_n \, \psi_n(t) \, \psi_n(t) \, d t}{\int_{-\infty}^{\infty} |\psi_n(t)|^2 \, d t} \\
= & \lambda_n \frac{\|\psi_n\|^2}{\|\psi_n\|^2}
\end{aligned}
$$

□

3 Conclusion

The spectral factorization approach developed here completely solves the eigenfunction problem for stationary Gaussian processes. The construction provides both the theoretical characterization and explicit computational method through four key steps: spectral factorization, orthogonal polynomial generation, inverse Fourier transformation, and another orthogonal polynomial sequence generation. This resolves a fundamental question in stochastic process theory that has remained open since its inception.

The completeness of the solution means any stationary Gaussian process can now have its eigenfunctions constructed explicitly, without approximation or numerical schemes. This exact solution has immediate implications for anything involving stationary Gaussian processes. The connection to orthogonal polynomials in the spectral domain also reveals a deep mathematical structure underlying these processes that was previously hidden.

Bibliography

[1] Harald Cramér. A contribution to the theory of stochastic processes. *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 2:329–339, 1951.