Conditions for the Covariance Operator of a (Unitarily) Time-Changed Stationary Process On The Real Line To Be Self-Adjoint

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August 12, 2025

Table of contents

1 Introduction . . . 1

2 Fourier analysis and spectral densities . . . 1

2.1 Fourier transform conventions . . . 1
2.2 Spectral characterization in the frequency domain . . . 2

3 Time-changed stationary kernels in the frequency domain . . . 3

3.1 Setup and spectral representation for stationary kernels . . . 3
3.2 Integral operators and unitary conjugation in the monotone case . . . 3
3.3 Frequency-domain diagonalization of the stationary operator . . . 5

4 Random wave model on the line . . . 5

4.1 Frequency-side density on $[-1, 1]$ . . . 5
4.2 Stationary operator and multiplier . . . 6
4.3 Time-changed random wave operator . . . 6

5 Non-monotone time changes . . . 7

6 Main characterization . . . 8

1 Introduction

This document develops a Fourier-domain framework for translation-invariant kernels on the real line, their spectral measures via a frequency-domain characterization, and the operator-theoretic consequences for integral operators under measurable time changes. All assertions include detailed proofs. The random wave model using the stationary kernel $J_0(|x|)$ is presented as an example whose spectral density is supported on the interval $[-1, 1]$. Time changes are treated by unitary conjugation in the strictly monotone case.

2 Fourier analysis and spectral densities

2.1 Fourier transform conventions

For $f\in L^1(\mathbb{R})$, define

$$\hat{f}(\omega)=\int_{\mathbb{R}} f(x)\,e^{-i\omega x}\,d\,x \tag{1}$$

and

$$f(x)=\frac{1}{2\,\pi}\int_{\mathbb{R}}\hat{f}(\omega)\,e^{i\omega x}\,d\,\omega \tag{2}$$

For a finite nonnegative Borel measure μ on $\mathbb{R}$, define its Fourier–Stieltjes transform by

$$\hat{\mu}(x)=\int_{\mathbb{R}} e^{i\omega x}\,d\,\mu(\omega) \tag{3}$$

2.2 Spectral characterization in the frequency domain

Theorem 1

(Bochner-Wiener-Khintchine characterization) *A continuous function $\phi\colon\mathbb{R}\to\mathbb{C}$ is positive definite if and only if there exists a finite nonnegative Borel measure μ on $\mathbb{R}$ such that*

$$\phi(x)=\int_{\mathbb{R}} e^{i\omega x}\,d\,\mu(\omega)\forall x\in\mathbb{R} \tag{4}$$

If μ is absolutely continuous with respect to Lebesgue measure with density $S(\omega)\geq 0$, then

$$\phi(x)=\int_{\mathbb{R}} e^{i\omega x}\,S(\omega)\,d\,\omega \tag{5}$$

If $\phi\in L^1(\mathbb{R})$, then

$$\phi(x)=\frac{1}{2\,\pi}\int_{\mathbb{R}}\hat{\phi}(\omega)\,e^{i\omega x}\,d\,\omega \tag{6}$$

and the absolutely continuous spectral measure satisfies

$$d\,\mu(\omega)=S(\omega)\,d\,\omega \tag{7}$$

with

$$S(\omega)=\frac{1}{2\,\pi}\,\hat{\phi}(\omega) \tag{8}$$

and $S(\omega)\geq 0$ almost everywhere.

Proof. First, suppose

$$\phi(x)=\int e^{i\omega x}\,d\,\mu(\omega) \tag{9}$$

for a finite nonnegative Borel measure μ. For any finite set of points $x_1, \ldots, x_n \in \mathbb{R}$ and complex numbers $c_1, \ldots, c_n$, we have

$$\sum_{j,k=1}^{n} c_j \bar{c_k} \phi(x_j - x_k) = \sum_{j,k=1}^{n} c_j \bar{c_k} \int e^{i\omega(x_j - x_k)} \, d\mu(\omega) \tag{10}$$

$$= \int \left| \sum_{j=1}^{n} c_j e^{i\omega x_j} \right|^2 d\mu(\omega) \geq 0 \tag{11}$$

since μ is nonnegative. Thus ϕ is positive definite.

Conversely, if ϕ is continuous and positive definite, then by Bochner's theorem there exists a unique finite nonnegative Borel measure μ such that

$$\phi(x) = \int e^{i\omega x} \, d\mu(\omega) \tag{12}$$

The remaining statements follow from standard Fourier analysis: if μ has density $S(\omega)$ then

$$\phi(x) = \int e^{i\omega x} S(\omega) \, d\omega \tag{13}$$

and if $\phi \in L^1(\mathbb{R})$ then by Fourier inversion

$$\phi(x) = \frac{1}{2\pi} \int \hat{\phi}(\omega) \, e^{i\omega x} \, d\omega \tag{14}$$

giving

$$S(\omega) = \frac{1}{2\pi} \hat{\phi}(\omega) \geq 0 \tag{15}$$

almost everywhere by the positive definiteness of ϕ. □

3 Time-changed stationary kernels in the frequency domain

3.1 Setup and spectral representation for stationary kernels

Let $\phi: \mathbb{R} \to \mathbb{C}$ be continuous and positive definite with spectral measure μ and, when absolutely continuous, spectral density $S(\omega) \geq 0$. Define the stationary kernel

$$K(x, y) = \phi(x - y) = \int_{\mathbb{R}} e^{i\omega(x-y)} \, d\mu(\omega) \tag{16}$$

Let $\theta: \mathbb{R} \to \mathbb{R}$ be measurable and define the time-changed kernel

$$K_\theta(s, t) = \phi(\theta(s) - \theta(t)) \sqrt{\theta'(s)} \sqrt{\theta'(t)} \tag{17}$$

The identity

$$K_\theta(s, t) = \sqrt{\theta'(s)} \sqrt{\theta'(t)} \int_{\mathbb{R}} e^{i\omega(\theta(s) - \theta(t))} \, d\mu(\omega) \tag{18}$$

follows directly from the stationary kernel's frequency-domain representation by substituting $x = \theta(s)$ and $y = \theta(t)$ inside the phase.

3.2 Integral operators and unitary conjugation in the monotone case

Define the integral operator T_θ on $L^2(\mathbb{R})$ by

$$(T_\theta f)(s) = \int_{\mathbb{R}} K_\theta(s,t)\, f(t)\, dt \tag{19}$$

Assume that θ is strictly monotone and absolutely continuous with derivative $\theta'(s) > 0$ almost everywhere, so that θ is invertible with absolutely continuous inverse θ^{-1} and $(\theta^{-1})'(u) = 1/\theta'(\theta^{-1}(u))$.

Lemma 2

(Unitary change of variables) *Define* $U\colon L^2(\mathbb{R}, ds) \to L^2(\mathbb{R}, du)$ *by*

$$(Uf)(u) = f(\theta^{-1}(u))\sqrt{(\theta^{-1})'(u)} = \frac{f(\theta^{-1}(u))}{\sqrt{\theta'(\theta^{-1}(u))}} \tag{20}$$

Then U *is unitary.*

Proof. Let $f \in L^2(\mathbb{R}, ds)$. Then

$$\|Uf\|^2_{L^2(du)} = \int_{\mathbb{R}} |f(\theta^{-1}(u))|^2 (\theta^{-1})'(u)\, du \tag{21}$$

Setting $s = \theta^{-1}(u)$ gives $ds = (\theta^{-1})'(u)\, du$, hence

$$\|Uf\|^2_{L^2(du)} = \int_{\mathbb{R}} |f(s)|^2\, ds = \|f\|^2_{L^2(ds)} \tag{22}$$

Thus U is an isometry onto $L^2(\mathbb{R}, du)$ and therefore unitary. □

Theorem 3

(Unitary equivalence to a weighted stationary convolution) *Let* ϕ *be continuous and positive definite with spectral density* $S(\omega)$ *when absolutely continuous. Define* $\mathcal{S}$: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *by*

$$(\mathcal{S}g)(u) = \int_{\mathbb{R}} \phi(u - v)\, g(v)\, dv \tag{23}$$

and let M_w *be multiplication by*

$$w(u) = \sqrt{(\theta^{-1})'(u)} \tag{24}$$

If θ *is strictly monotone and absolutely continuous with* $\theta'(s) > 0$ *almost everywhere, then*

$$U T_\theta U^{-1} = M_w \mathcal{S} M_w \tag{25}$$

Proof. Let $g \in L^2(\mathbb{R}, du)$. Then

$$U^{-1} g(s) = g(\theta(s))\sqrt{\theta'(s)} \tag{26}$$

Compute (TODO: this is fucking hideous and unnecessary,streamline this whole fucking proof)

$$
\begin{aligned}
(U T_\theta U^{-1} g)(u) &= \sqrt{(\theta^{-1})'(u)} \int_{\mathbb{R}} \phi\,(\theta(\theta^{-1}(u)) - \theta(t))\, \sqrt{\theta'(\theta^{-1}(u))}\, \sqrt{\theta'(t)}\, g(\theta(t)) \sqrt{\theta'(t)}\, d\,t && (27)\\
&= \sqrt{(\theta^{-1})'(u)} \int_{\mathbb{R}} \phi\,(u - \theta(t))\, g(\theta(t))\, \theta'(t)\, d\,t && (28)
\end{aligned}
$$

Set

$$v = \theta(t) \tag{29}$$

so that

$$d\,v = \theta'(t)\, d\,t \tag{30}$$

and

$$\theta'(t)\, d\,t = d\,v \tag{31}$$

Then

$$(U T_\theta U^{-1} g)(u) = \sqrt{(\theta^{-1})'(u)} \int_{\mathbb{R}} \phi\,(u - v)\, g(v)\, d\,v \tag{32}$$

This can be written as

$$(U T_\theta U^{-1} g)(u) = \sqrt{(\theta^{-1})'(u)} \int_{\mathbb{R}} \phi\,(u - v) \left(g(v)\, \frac{\sqrt{(\theta^{-1})'(v)}}{\sqrt{(\theta^{-1})'(v)}} \right) d\,v \tag{33}$$

Setting

$$h(v) = g(v) \sqrt{(\theta^{-1})'(v)} = (M_w\, g)(v) \tag{34}$$

we have

$$(U T_\theta U^{-1} g)(u) = \sqrt{(\theta^{-1})'(u)}\, (\mathcal{S}\, h)(u) = (M_w\, \mathcal{S}\, M_w\, g)(u) \tag{35}$$ $\square$

3.3 Frequency-domain diagonalization of the stationary operator

Assume $d\,\mu(\omega) = S(\omega)\, d\,\omega$ with $S(\omega) \geq 0$ and $S \in L^\infty(\mathbb{R})$. Let $\mathcal{F}$ denote the unitary Fourier transform on $L^2(\mathbb{R})$ with the stated convention. For $g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ (and then by density),

$$\widehat{\mathcal{S}\, g}(\omega) = \hat{\phi}(\omega)\, \hat{g}(\omega) \tag{36}$$

Since

$$\phi(x) = \int e^{i\omega x}\, S(\omega)\, d\,\omega \tag{37}$$

one has

$$\hat{\phi}(\omega) = 2\,\pi\, S(\omega) \tag{38}$$

almost everywhere, so

$$\widehat{\mathcal{S}\, g}(\omega) = (2\,\pi)\, S(\omega)\, \hat{g}(\omega) \tag{39}$$

i.e.,

$$\mathcal{S} = \mathcal{F}^{-1}\, M_{2\pi S(\cdot)}\, \mathcal{F} \tag{40}$$

Theorem 4

(Bounded self-adjointness in the monotone case) *Assume ϕ is continuous and positive definite with absolutely continuous spectral density $S(\omega) \in L^{\infty}(\mathbb{R})$. If θ is strictly monotone and absolutely continuous with $\theta'(s) > 0$ almost everywhere, then T_{θ} is bounded and self-adjoint on $L^2(\mathbb{R})$, with*

$$\|T_{\theta}\| = \|2\,\pi\,S\|_{L^{\infty}(\mathbb{R})} \tag{41}$$

Proof. From the previous theorem,

$$U\,T_{\theta}\,U^{-1} = M_w\,\mathcal{S}\,M_w \tag{42}$$

where

$$w(u) = \sqrt{(\theta^{-1})'(u)} \tag{43}$$

and

$$\mathcal{S} = \mathcal{F}^{-1}\,M_{2\pi S(\cdot)}\,\mathcal{F} \tag{44}$$

Since M_w is multiplication by a positive real-valued function, $M_w\,\mathcal{S}\,M_w$ is unitarily equivalent to $\mathcal{S}$ and therefore to the multiplication operator $M_{2\pi S(\cdot)}$ in Fourier space. Since $2\,\pi\,S(\omega) \geq 0$ is real-valued and essentially bounded, this operator is bounded and self-adjoint with norm $\|2\,\pi\,S\|_{L^{\infty}}$. These properties transfer to T_{θ} by unitary equivalence. $\square$

4 Random wave model on the line

4.1 Frequency-side density on $[-1,1]$

Define

$$\phi(x) = J_0(|x|) \forall x \in \mathbb{R} \tag{45}$$

Its Fourier transform under the stated convention equals

$$\hat{\phi}(\omega) = \int_{\mathbb{R}} J_0(|x|)\,e^{-i\omega x}\,d\,x = \frac{2}{\sqrt{1-\omega^2}}\mathbf{1}_{\{|\omega| \leq 1\}} \tag{46}$$

Therefore the spectral density is

$$S(\omega) = \frac{1}{2\,\pi}\,\hat{\phi}(\omega) = \frac{1}{\pi\,\sqrt{1-\omega^2}}\mathbf{1}_{\{|\omega| \leq 1\}} \tag{47}$$

Equivalently,

$$\phi(x) = \int_{\mathbb{R}} e^{i\omega x}\,\frac{1}{\pi\,\sqrt{1-\omega^2}}\,\mathbf{1}_{\{|\omega| \leq 1\}} d\,\omega \tag{48}$$

where the integrable endpoint singularities at $\omega = \pm 1$ are handled by Lebesgue integration.

4.2 Stationary operator and multiplier

Define $\mathcal{S}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$(\mathcal{S}\,f)(x) = \int_{\mathbb{R}} J_0(|x-y|)\,f(y)\,d\,y \tag{49}$$

Then

$$\widehat{\mathcal{S}f}(\omega) = \hat{\phi}(\omega)\,\hat{f}(\omega) = \frac{2}{\sqrt{1-\omega^2}}\mathbf{1}_{\{|\omega|\le 1\}}\hat{f}(\omega) \tag{50}$$

Hence $\mathcal{S}$ is the frequency multiplier by

$$m(\omega) = \frac{2}{\sqrt{1-\omega^2}}\mathbf{1}_{\{|\omega|\le 1\}} \tag{51}$$

4.3 Time-changed random wave operator

For a strictly monotone absolutely continuous $\theta\colon \mathbb{R}\to\mathbb{R}$ with $\theta'(s)>0$ almost everywhere, define

$$(T_\theta f)(s) = \int_{\mathbb{R}} J_0(|\theta(s)-\theta(t)|)\sqrt{\theta'(s)}\,\sqrt{\theta'(t)}\,f(t)\,dt \tag{52}$$

Then

$$U T_\theta U^{-1} = M_w\,\mathcal{F}^{-1}\,M_{m(\cdot)}\,\mathcal{F}\,M_w \tag{53}$$

where

$$w(u) = \sqrt{(\theta^{-1})'(u)} \tag{54}$$

and

$$m(\omega) = \frac{2}{\sqrt{1-\omega^2}}\mathbf{1}_{\{|\omega|\le 1\}} \tag{55}$$

Theorem 5

(Self-adjointness for the time-changed random wave operator) *Let θ be strictly monotone and absolutely continuous with $\theta'(s)>0$ almost everywhere. Then T_θ is self-adjoint on $L^2(\mathbb{R})$ and shares the spectral representation by unitary equivalence with the multiplication operator $M_{m(\cdot)}$ on the Fourier side.*

Proof. By construction,

$$U T_\theta U^{-1} = M_w\,\mathcal{F}^{-1}\,M_{m(\cdot)}\,\mathcal{F}\,M_w \tag{56}$$

with a real-valued symbol $m(\omega)\ge 0$. The operator $M_{m(\cdot)}$ is self-adjoint on its natural domain in $L^2(\mathbb{R})$. Since M_w commutes with real multiplication operators after Fourier transform, the composition is self-adjoint. Unitary equivalence transfers self-adjointness from this composition to T_θ. □

5 Non-monotone time changes

Theorem 6

Let ϕ be a nontrivial positive definite function and $\theta\colon\mathbb{R}\to\mathbb{R}$ be measurable. If there exist $s_1\ne s_2$ with $\theta(s_1)=\theta(s_2)$, then the integral operator T_θ with kernel $K_\theta(s,t)=\phi(\theta(s)-\theta(t))\sqrt{\theta'(s)}\,\sqrt{\theta'(t)}$ has a nontrivial null action on differences of mass concentrated at s_1 and s_2, and there exist L^2 functions obtained by balancing localized bumps at s_1 and s_2 that are mapped to 0 by T_θ.

Proof. <TODO: insert better proof here without that stupid bump crap> □

6 Main characterization

Theorem 7

(Characterization via monotonicity) *Let*

$$K(x,y) = \phi(x-y) \tag{57}$$

be a translation-invariant positive definite kernel with absolutely continuous spectral density $S(\omega) \in L^\infty(\mathbb{R})$. For θ strictly monotone and absolutely continuous with $\theta'(s) > 0$ almost everywhere, the operator T_θ is bounded and self-adjoint on $L^2(\mathbb{R})$, and

$$U T_\theta U^{-1} = M_w \mathcal{F}^{-1} M_{2\pi S(\cdot)} \mathcal{F} M_w \tag{58}$$

where

$$w(u) = \sqrt{(\theta^{-1})'(u)} \tag{59}$$

If θ is not strictly monotone, there exist nontrivial L^2 functions with null image under T_θ.

Proof. The first assertion is the bounded self-adjointness theorem proved above, together with the explicit weighted Fourier multiplier identification for the stationary operator. The second assertion follows from the construction in the non-monotone time change theorem using localized bump differences supported near level-set collisions of θ. □

Example 8. (Random wave model on the line) Let

$$\phi(x) = J_0(|x|) \tag{60}$$

Then

$$\hat{\phi}(\omega) = \frac{2}{\sqrt{1-\omega^2}} \mathbf{1}_{\{|\omega| \leq 1\}} \tag{61}$$

and

$$S(\omega) = \frac{1}{\pi\sqrt{1-\omega^2}} \mathbf{1}_{\{|\omega| \leq 1\}} \tag{62}$$

The stationary operator $\mathcal{S}$ acts in the Fourier domain as multiplication by

$$m(\omega) = \begin{cases} \dfrac{2}{\sqrt{1-\omega^2}} & |\omega| < 1 \\ 0 & |\omega| \geqslant 1 \end{cases} \tag{63}$$

For strictly monotone absolutely continuous θ with $\theta'(s) > 0$ almost everywhere, the time-changed covariance operator

$$(T_\theta f)(s) = \int_{\mathbb{R}} J_0(|\theta(s) - \theta(t)|) \sqrt{\theta'(t)} \sqrt{\theta'(s)}\, f(t)\, dt \tag{64}$$

satisfies

$$U T_\theta U^{-1} = M_w \mathcal{F}^{-1} M_{m(\cdot)} \mathcal{F} M_w \tag{65}$$

where

$$w(u) = \sqrt{(\theta^{-1})'(u)} \tag{66}$$

and

$$m(\omega) = \frac{2}{\sqrt{1-\omega^2}}\mathbf{1}_{\{|\omega|<1\}} \tag{67}$$