Injective Measure-Preserving Time-Changes of Stationary Processes are Oscillatory

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Oscillatory Processes and Normalized Injective Time-Changes

Definition 1. *[Oscillatory Process] A complex-valued second-order stochastic process $\{X_t\}_{t\in I}$ is said to be oscillatory if there exists a family of functions $\phi_t(\omega)$ and a complex orthogonal increment process $Z(\omega)$ with $E\,|d\,Z(\omega)|^2 = d\,\mu(\omega)$ such that*

$$X_t = \int_{-\infty}^{\infty} \phi_t(\omega)\ d\,Z(\omega) \tag{1}$$

where $\phi_t(\omega) = A_t(\omega)\,e^{i\omega t}$ and $A_t(\omega)$ is quadratically integrable with respect to $d\,\mu$.

Definition 2. *[Stationary Process] A second-order process $\{S_t\}_{t\in J}$ is stationary if it admits the spectral representation*

$$S_t = \int_{-\infty}^{\infty} e^{i\omega t}\ d\,Z(\omega) \tag{2}$$

for some orthogonal increment process $Z(\omega)$ with $E\,|d\,Z(\omega)|^2 = d\,\mu(\omega)$.

Theorem 3. *[Time-Varying Filter for Injective Time-Change] Let S_t be a stationary process and $\theta\colon \mathbb{R}\to\mathbb{R}$ be smooth and strictly increasing with $\theta'(t) > 0$. To achieve the transformation*

$$X_t = \sqrt{\theta'(t)}\ S_{\theta(t)} \tag{3}$$

via convolution $X_t = \int_{-\infty}^{\infty} S_{t-u}\,h_t(u)\,d\,u$, the time-varying impulse response must be

$$h_t(u) = \sqrt{\theta'(t)}\ \delta\left(u - (t - \theta(t))\right) \tag{4}$$

Proof. For the convolution to yield $X_t = \sqrt{\theta'(t)}\ S_{\theta(t)}$, the argument of S in the integrand must equal $\theta(t)$ when the delta function is activated. This requires:

$$t - u = \theta(t)$$

Solving for u:

$$u = t - \theta(t) \tag{5}$$

Therefore:

$$h_t(u) = \sqrt{\theta'(t)}\, \delta\left(u - (t - \theta(t))\right) \tag{6}$$

Verification by direct computation:

$$\begin{aligned}
X_t &= \int_{-\infty}^{\infty} S_{t-u}\, h_t(u)\, d\,u && (7)\\
&= \int_{-\infty}^{\infty} S_{t-u}\, \sqrt{\theta'(t)}\, \delta\left(u - (t - \theta(t))\right) d\,u && (8)\\
&= \sqrt{\theta'(t)}\, S_{t-(t-\theta(t))} \quad \text{(by sifting property)} && (9)\\
&= \sqrt{\theta'(t)}\, S_{\theta(t)} && (10)
\end{aligned}$$

□

Theorem 4. *[Spectral Envelope for Injective Time-Change] For the time-varying filter* $h_t(u) = \sqrt{\theta'(t)}\, \delta\left(u - (t - \theta(t))\right)$*, the spectral envelope is*

$$A_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega(t-\theta(t))}$$

Proof. The spectral envelope is the Fourier transform of $h_t(u)$:

$$\begin{aligned}
A_t(\omega) &= \int_{-\infty}^{\infty} e^{i\omega u}\, h_t(u)\, d\,u && (11)\\
&= \int_{-\infty}^{\infty} e^{i\omega u}\, \sqrt{\theta'(t)}\, \delta\left(u - (t - \theta(t))\right) d\,u && (12)\\
&= \sqrt{\theta'(t)}\, e^{i\omega(t-\theta(t))} \quad \text{(by sifting property)} && (13)
\end{aligned}$$

□

Theorem 5. *[Oscillatory Representation of Injective Time-Change] The process* $X_t = \sqrt{\theta'(t)}\, S_{\theta(t)}$ *admits the oscillatory representation*

$$X_t = \int_{-\infty}^{\infty} \phi_t(\omega)\, d\,Z(\omega)$$

where

$$\phi_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega\theta(t)}$$

Proof. Starting from the spectral representation of S_t:

$$X_t = \sqrt{\theta'(t)}\, S_{\theta(t)} \tag{14}$$

$$= \sqrt{\theta'(t)} \int_{-\infty}^{\infty} e^{i\omega\theta(t)}\, d\, Z(\omega) \tag{15}$$

$$= \int_{-\infty}^{\infty} \sqrt{\theta'(t)}\, e^{i\omega\theta(t)}\, d\, Z(\omega) \tag{16}$$

Thus $\phi_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega\theta(t)}$. □

Corollary 6. *[Envelope in Standard Form] The oscillatory functions can be written in the standard form* $\phi_t(\omega) = A_t(\omega)\, e^{i\omega t}$ *where*

$$A_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega(\theta(t)-t)}$$

Proof. Factor the exponential:

$$\phi_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega\theta(t)} \tag{17}$$

$$= \sqrt{\theta'(t)}\, e^{i\omega(\theta(t)-t)}\, e^{i\omega t} \tag{18}$$

$$= A_t(\omega)\, e^{i\omega t} \tag{19}$$

where $A_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega(\theta(t)-t)}$. □

Theorem 7. *[Verification of Filter-Envelope Consistency] The spectral envelope*

$$A_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega(\theta(t)-t)} \tag{20}$$

is the Fourier transform of the time-varying impulse response

$$h_t(u) = \sqrt{\theta'(t)}\, \delta\,(u - (t - \theta(t))) \tag{21}$$

Proof. Compute the Fourier transform:

$$\mathcal{F}[h_t](\omega) = \int_{-\infty}^{\infty} e^{i\omega u}\, \sqrt{\theta'(t)}\, \delta\,(u - (t - \theta(t)))\, d\, u \tag{22}$$

$$= \sqrt{\theta'(t)}\, e^{i\omega(t-\theta(t))} \tag{23}$$

$$= A_t(\omega) \tag{24}$$

Compute the inverse Fourier transform:

$$
\begin{aligned}
\mathcal{F}^{-1}[A_t](u) =& \frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{-i\omega u} \, \sqrt{\theta'(t)} \, e^{i\omega\,(\theta(t)-t)} \, d\,\omega && (25)\\
=& \frac{\sqrt{\theta'(t)}}{2\,\pi} \int_{-\infty}^{\infty} e^{i\omega\,(\theta(t)-t-u)} \, d\,\omega && (26)\\
=& \sqrt{\theta'(t)} \, \delta\,(\theta(t) - t - u) && (27)\\
=& \sqrt{\theta'(t)} \, \delta\,(u - (t - \theta(t))) && (28)\\
=& h_t(u) && (29)
\end{aligned}
$$

$\square$