Eigenfunction and Eigenvalue of the Sine Kernel

1 Introduction

The sine kernel is defined by:

$$K(x,y)=\frac{\sin(x-y)}{\pi(x-y)} \tag{1}$$

2 Eigenfunction and Eigenvalue

The eigenfunction $p_0(x)=\sin(x)$ satisfies:

$$\int_{-\infty}^{\infty}\frac{\sin(x-y)}{\pi(x-y)}\sin(y)\,d\,y=\sin(x) \tag{2}$$

with corresponding eigenvalue is 1.

3 Fourier Transform and Heaviside Functions

The Fourier transform of $p_0(y)$ is:

$$\int_{-\infty}^{\infty}e^{ixy}\frac{\sin(y)}{y\,\pi}\,d\,y=\theta(x+1)-\theta(x-1) \tag{3}$$

where $\theta(x)$ is the Heaviside step function

$$\theta(x)=\begin{cases}1 & x>0\\ \frac{1}{2}=\frac{\lim_{x\to 0^+}\theta(x)+\lim_{x\to 0^-}\theta(x)}{2} & x=0\\ 0 & x<0\end{cases} \tag{4}$$

4 Identity Validation

The convolution of the sine function under the sine kernel is:

$$\int_{-\infty}^{\infty}\frac{\sin(x-y)}{\pi(x-y)}\frac{\sin(y)}{\pi\,y}\,d\,y=\frac{\sin(x)}{x\,\pi} \tag{5}$$