The Operational Matrix of the Random Wave Process

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February 11, 2025

Abstract

An expression for the convolution of a pair of spherical Bessel functions is determined.

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1   Lemmas

Lemma 1

(Terminating Hypergeometric Series) *For any $p \in \mathbb{Z}_{\geq 0}$, the Gauss hypergeometric function terminates:*

$$_2F_1(-p, b; c; z) = \sum_{k=0}^{p} \frac{(-p)_k (b)_k}{(c)_k \, k!} z^k \tag{1}$$

where $(a)_k = \prod_{i=0}^{k-1} (a+i)$

Proof. By definition, the Gauss hypergeometric series is:

$$_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k \, k!} z^k \tag{2}$$

Setting $a = -p$ with $p \in \mathbb{Z}_{\geq 0}$, the Pochhammer symbol $(-p)_k$ becomes zero for all $k > p$. Explicitly:

$$(-p)_k = \prod_{i=0}^{k-1} (-p+i) = \begin{cases} (-p)(-p+1)\cdots(-p+k-1), & k \leq p \\ 0 & k > p \end{cases} \tag{3}$$

Thus, the series terminates at $k = p$, yielding:

$$_2F_1(-p, b; c; z) = \sum_{k=0}^{p} \frac{(-p)_k (b)_k}{(c)_k\, k!} z^k \tag{4} \quad \square$$

Lemma 2

(Integral with Incomplete Gamma Function) *For $j \geq 0$,*

$$\int_{-1}^{1} \left(\frac{1-x}{2}\right)^j e^{ixy}\, dx = \frac{e^{iy}}{2^j} \frac{\gamma(j+1, 2\,i\,y)}{(i\,y)^{j+1}} \tag{5}$$

where $\gamma(s, x)$ denotes the lower incomplete gamma function.

Proof. Substitute $t = \frac{1-x}{2} \implies x = 1 - 2t$, $d\,x = -2\,d\,t$, adjusting limits:

$$\int_{1}^{0} t^j e^{i(1-2t)y} (-2\,d\,t) = 2\, e^{iy} \int_{0}^{1} t^j e^{-2iyt}\, d\,t \tag{6}$$

Let $u = 2\,i\,y\,t \implies t = \frac{u}{2\,i\,y}$, $d\,t = \frac{d\,u}{2\,iy}$:

$$\frac{2\, e^{iy}}{(2\,i\,y)^{j+1}} \int_{0}^{2iy} u^j e^{-u}\, d\,u = \frac{e^{iy}}{2^j} \frac{\gamma(j+1, 2\,i\,y)}{(i\,y)^{j+1}} \qquad \square$$

Lemma 3

(Legendre Polynomial Representation) *The Legendre polynomials are hypergeometric functions*

$$P_m(x) =_2 F_1(-m, m+1; 1; \tfrac{1-x}{2}) \tag{7}$$

Proof. From the Rodrigues formula $P_m(x) = \frac{1}{2^m\, m!} \frac{d^m}{d\,x^m} (x^2 - 1)^m$ expand using the binomial theorem:

$$(x^2 - 1)^m = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} x^{2k} \tag{8}$$

Differentiating m times yields terms proportional to x^k, matching the hypergeometric series:

$$P_m(x) =_2 F_1\left(-m, m+1; 1; \tfrac{1-x}{2}\right) \qquad \square$$

2 Main Theorem

Theorem 4

(Fourier Transform of Legendre Polynomial Products) *The Fourier transform of the product of a pair of Legendre polynomials is the convolution of a pair of spherical Bessel functions of the first kind which can be expressed as*

$$
\begin{aligned}
I_{m,n}(y) &= \int_{-1}^{1} P_m(x)\, P_n(x)\, e^{ixy}\, dx \\
&= e^{iy} \sum_{j=0}^{m+n} \Psi_j(m,n) \frac{\gamma(j+1, 2\,i\,y)}{2^j\,(i\,y)^{j+1}}
\end{aligned}
\tag{9}
$$

where

$$
\begin{aligned}
\Psi_j(m,n) &= \frac{{}_4F_3\!\left(\begin{array}{c} -m,\ m+1,\ -n,\ n+1 \\ 1,\ 1,\ j+1 \end{array}; 1\right)}{j!} \\
&= \sum_{k=\max(0,\,j-n)}^{\min(j,m)} \frac{(-m)_k\,(m+1)_k\,(-n)_{j-k}\,(n+1)_{j-k}}{(1)_k (1)_{j-k}\, k!\,(j-k)!}
\end{aligned}
\tag{10}
$$

Proof. Expand both polynomials using their hypergeometric representations:

$$
P_m(x) = \sum_{k=0}^{m} \frac{(-m)_k\,(m+1)_k}{(1)_k\, k!}\left(\frac{1-x}{2}\right)^k
\tag{11}
$$

$$
P_n(x) = \sum_{\ell=0}^{n} \frac{(-n)_\ell\,(n+1)_\ell}{(1)_\ell\, \ell!}\left(\frac{1-x}{2}\right)^\ell
\tag{12}
$$

Their product becomes:

$$
\begin{aligned}
P_m(x)\, P_n(x) &= \sum_{k=0}^{m}\sum_{\ell=0}^{n} \frac{(-m)_k\,(m+1)_k\,(-n)_\ell\,(n+1)_\ell}{(1)_k(1)_\ell\, k!\,\ell!}\left(\frac{1-x}{2}\right)^{k+\ell} \\
&= \sum_{k=0}^{m}\sum_{\ell=0}^{n} \frac{(-m)_k\,(m+1)_k\,(-n)_\ell\,(n+1)_\ell}{k!^2\,\ell!^2}\left(\frac{1-x}{2}\right)^{k+\ell}
\end{aligned}
\tag{13}
$$

Reorganize using $j = k + \ell$, with explicit summation bounds:

$$
P_m(x)\, P_n(x) = \sum_{j=0}^{m+n} \underbrace{\sum_{k=\max(0,\,j-n)}^{\min(j,m)} \frac{(-m)_k\,(m+1)_k\,(-n)_{j-k}\,(n+1)_{j-k}}{(1)_k(1)_{j-k}\, k!\,(j-k)!}}_{\Psi_j(m,n)} \left(\frac{1-x}{2}\right)^{j}
\tag{14}
$$

3

The inner sum coefficients are:

$$\Psi_j(m,n) = \sum_{k=\max(0,j-n)}^{\min(j,m)} \frac{(-m)_k \, (m+1)_k \, (-n)_{j-k} \, (n+1)_{j-k}}{(1)_k (1)_{j-k} \, k! (j-k)!} \tag{15}$$

Apply the identities:

$$(-n)_{j-k} = (-1)^{j-k} \frac{\Gamma(n+1)}{\Gamma(n-j+k+1)} \tag{16}$$

$$(n+1)_{j-k} = \frac{\Gamma(n+j-k+1)}{\Gamma(n+1)} \tag{17}$$

$$(-a)_n (a+1)_n = (-1)^n \frac{\Gamma(a+1)}{\Gamma(a-n+1)} \tag{18}$$

so that the substitutions

$$(-n)_{j-k} (n+1)_{j-k} = \frac{(-1)^{j-k} \Gamma(n+j-k+1)}{\Gamma(n-j+k+1)} \tag{19}$$

and $h = j - k$ can be made so that $\Psi_j(m,n)$ can be expressed as

$$\Psi_j(m,n) = (-1)^j \sum_{h=0}^{\min(n,j) - \max(0,j-m)} \frac{(-m)_{j-h} \, (m+1)_{j-h} \, \Gamma(n+h+1)}{(j-h)!^2 \, \Gamma(n-j+h+1) \, h!} \tag{20}$$

to reveal the hypergeometric form by reversing the index transformation:

$$\begin{aligned}
\Psi_j(m,n) &= \frac{1}{j!} \sum_{k=0}^{\min(j,m,n)} \frac{(-m)_k \, (m+1)_k \, (-n)_k \, (n+1)_k}{(1)_k (1)_k (j+1)_k \, k!} \\
&= \frac{{}_4F_3\left(\begin{matrix} -m, m+1, -n, n+1 \\ 1,1,j+1 \end{matrix} ; 1 \right)}{j!}
\end{aligned} \tag{21}$$

To complete the proof, observe that the hypergeometric form has an extended summation range $k = 0$ to $\min(j,m,n)$ compared to the original bounds $k = \max(0, j-n)$ to $\min(j, m)$. However, terms outside the original bounds vanish identically. For $k > \min(j,m)$, the Pochhammer symbol $(-m)_k = 0$ by Lemma 1. For $k < \max(0, j-n)$, the term $(-n)_{j-k}$ in the original sum vanishes since $j - k > n$. Thus the extended summation yields the same result, establishing the equality of both representations of $\Psi_j(m,n)$. $\qquad\square$