# Mittag-Leffler Functions, Fourier Transforms, and Stochastic Processes

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November 22, 2024

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# 1 Mittag-Leffler Functions

## 1.1 One-Parameter Mittag-Leffler Function

The one-parameter Mittag-Leffler function $E_\alpha(z)$ is defined as:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \tag{1}$$

where $\alpha > 0$ and $z$ is a complex number.

## 1.2 Two-Parameter Mittag-Leffler Function

The two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined as:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{2}$$

where $\alpha > 0$, $\beta$ is a complex number, and $z$ is a complex number.

# 2 Fourier Transforms

## 2.1 One-Parameter Mittag-Leffler Function

For $0 < \alpha \leq 2$, the Fourier transform of $E_\alpha(-t^\alpha)$ is:

$$\mathcal{F}\{E_\alpha(-t^\alpha)\}(\omega) = \frac{1}{1 + |\omega|^\alpha e^{i\pi\alpha/2\mathrm{sgn}(\omega)}} \tag{3}$$

## 2.2 Two-Parameter Mittag-Leffler Function

For $0 < \alpha \leq 2$ and $\beta$ real, the Fourier transform of $t^{\beta-1} E_{\alpha,\beta}(-t^\alpha)$ is:

$$\mathcal{F}\{t^{\beta-1} E_{\alpha,\beta}(-t^\alpha)\}(\omega) = \frac{1}{|\omega|^\alpha} H^{2,1}_{1,2}\left[\frac{1}{|\omega|^\alpha}\middle| \begin{matrix} (1-\beta,\alpha) \\ (0,1),(1-\beta,1) \end{matrix}\right] \tag{4}$$

where $H^{2,1}_{1,2}$ is the H-function.

# 3 Special Cases and Connections

## 3.1 Cauchy Distribution

When $\alpha = 1$ in the one-parameter case, we get the Cauchy distribution:

$$\mathcal{F}\{E_1(-t)\}(\omega) = \frac{1}{1 + |\omega|} \tag{5}$$

This is related to the Ornstein-Uhlenbeck process.

## 3.2 Gaussian Distribution

When $\alpha = 2$ in the one-parameter case, we get a Gaussian distribution:

$$\mathcal{F}\{E_2(-t^2)\}(\omega) = \frac{1}{1 + \omega^2} \tag{6}$$

This is related to standard Brownian motion.

## 4 Orthogonal Polynomials

For the one-parameter case, we can define a weight function:

$$w(\omega) = \frac{1}{1 + |\omega|^{\alpha}} \tag{7}$$

And an inner product:

$$(f, g) = \int_{-\infty}^{\infty} f(\omega)\, g(\omega)\, w(\omega)\, d\omega \tag{8}$$

Orthogonal polynomials can be constructed using the Gram-Schmidt process on the monomials $1, \omega, \omega^2, \omega^3$,...

## 5 Implications for Stochastic Processes

The Mittag-Leffler function and its Fourier transform provide a framework for modeling a spectrum of processes:

- $\alpha = 1$: Ornstein-Uhlenbeck-like processes
- $1 < \alpha < 2$: Anomalous diffusion
- $\alpha = 2$: Brownian motion
- $0 < \alpha < 1$: Processes with long-range dependence

This framework is particularly useful in fractional calculus and for modeling phenomena that exhibit behavior intermediate between classical diffusion and pure randomness.