Shifted Jacobi Polynomial Integral Operational Matrix for Solving Fractional Riccati Equations

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1 Introduction

This article details the use of the shifted Jacobi polynomial integral operational matrix in solving fractional Riccati differential equations using a spectral method.

2 Shifted Jacobi Polynomials

The shifted Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$ on $[0,1]$ are defined by:

$$J_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+\alpha}{k} \binom{n+\beta}{n-k} x^k \tag{1}$$

where $\alpha, \beta > -1$ are parameters. These polynomials satisfy the orthogonality relation:

$$\int_0^1 x^\alpha (1-x)^\beta J_m^{(\alpha,\beta)}(x)\, J_n^{(\alpha,\beta)}(x)\, d\,x = h_n\, \delta_{mn} \tag{2}$$

where h_n is the normalization constant:

$$h_n = \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2\,n+\alpha+\beta+1)\,n!\Gamma(n+\alpha+\beta+1)} \tag{3}$$

3 Integral Operational Matrix

The integral operational matrix P for shifted Jacobi polynomials is defined as:

$$P_{ij} = \int_0^1 J_i^{(\alpha,\beta)}(x)\, J_j^{(\alpha,\beta)}(x)\, d\,x \tag{4}$$

For an N-term representation, P is an $N \times N$ matrix. This matrix allows us to express the integral of a function $f(x) = \sum_{i=0}^{\infty} c_i J_i^{(\alpha,\beta)}(x)$ as:

$$\int_0^x f(t)\, dt = \mathbf{X}^T P \mathbf{C} \tag{5}$$

where $\mathbf{X}$ is the vector of Jacobi polynomials and $\mathbf{C}$ is the vector of coefficients.

4 Application to Fractional Riccati Equations

Consider the fractional Riccati equation:

$$D^\alpha y(x) = p(x) + q(x)\, y(x) + r(x)\, y^2(x), \quad 0 < \alpha \leq 1 \tag{6}$$

with initial condition $y(0) = y_0$.

The spectral method involves the following steps:

1. Express $y(x)$ using shifted Jacobi polynomials:

$$y(x) = \sum_{i=0}^{\infty} c_i J_i^{(\alpha,\beta)}(x) = \mathbf{C}^T \mathbf{X}(x) \tag{7}$$

2. Express the fractional derivative using the Caputo definition:

$$D^\alpha y(x) = I^{1-\alpha} \frac{d}{d x} y(x) \tag{8}$$

   where $I^{1-\alpha}$ is the Riemann-Liouville fractional integral operator.

3. Use the operational matrix to represent the fractional integral:

$$I^{1-\alpha} y(x) = \mathbf{X}^T P^{1-\alpha} \mathbf{C} \tag{9}$$

4. Express the nonlinear term:

$$y^2(x) = (\mathbf{C}^T \mathbf{X}(x))^2 = \mathbf{C}^T \hat{\mathbf{X}} \mathbf{C} \tag{10}$$

   where $\hat{\mathbf{X}}$ is a matrix formed by the products of Jacobi polynomials.

5. Substitute these expressions into the original equation:

$$\mathbf{X}^T D\, P^{1-\alpha} \mathbf{C} = p(x) + q(x) \mathbf{C}^T \mathbf{X}(x) + r(x) (\mathbf{C}^T \hat{\mathbf{X}} \mathbf{C}) \tag{11}$$

where D is the operational matrix for differentiation.

6. Apply the Galerkin method by multiplying both sides by $\mathbf{X}(x)$ and integrating over $[0,1]$:

$$\int_0^1 \mathbf{X}(x)\mathbf{X}^T(x) \quad d \quad x \quad \cdot \quad D \quad P^{1-\alpha}\mathbf{C} = \int_0^1 \mathbf{X}(x) \quad [p(x) \quad + \quad q(x)\mathbf{C}^T\mathbf{X}(x) \quad + \quad r(x)(\mathbf{C}^T\hat{\mathbf{X}}\mathbf{C})]\, d\, x \tag{12}$$

7. This results in a system of nonlinear algebraic equations in the spectral domain:

$$F(\mathbf{C}) = 0 \tag{13}$$

8. Solve this system for the coefficients $\mathbf{C}$ using an iterative method like Newton-Raphson.

The solution is a sequence of spectral coefficients that define an exact representation of the solution in the basis of Jacobi polynomials. This spectral representation can be interpreted as a series of functions (which may be polynomial, rational, or more general, depending on the specific problem and basis functions used). For practical computation, this infinite series is typically truncated, introducing an approximation at that stage.