Structure and Properties of Gaussian Processes with Monotonically Modulated Gaussian Kernels

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Abstract

The structural properties of Gaussian processes with monotonically modulated Gaussian kernels are derived. A class of functions satisfying specific continuity and monotonicity conditions is defined and two main theoretical results are presented. The first theorem demonstrates that for any admissible monotonic modulation function, the mean zero-counting function of the resulting process equals the modulation function. The second theorem demonstrates the eigenfunction structure of these modulated Gaussian kernel take the form of Hermite polynomials composed with the modulation function, multiplied by the exponential term and the square root of the modulation function's derivative. The analysis reveals an important invariance property of eigenvalues under monotonic modulation, connecting to similarity transformations in linear algebra through an explicit transformation operator. These results advance theoretical understanding of the Gaussian processes with monotonically modulated Gaussian kernels.

Definition 1

Let $\mathcal{F}$ denote the class of functions $f:\mathbb{R}\to\mathbb{R}$ satisfying:

1. *f is piecewise continuous with piecewise continuous first derivative*
2. *f is monotonically increasing*
3. *$\dot{f}\geq 0$ where it exists and is measurable*
4. *$\lim_{t\to\infty}\dot{f}(t)$ exists (finite or infinite)*

Theorem 2

(Mean Zero-Counting Function) *For any $f \in \mathcal{F}$, the f-modulated Gaussian kernel $e^{-(t-s)^2}$*

$$K(s,t) = e^{-(f(t)-f(s))^2} \tag{1}$$

has mean zero-counting function:

$$\mathbb{E}[N([0,T])] = f(T) \tag{2}$$

where $N([0,T])$ denotes the counting measure of zeros in $[0,T]$.

Proof. By the Kac-Rice formula:

$$\mathbb{E}[N([0,T])] = \int_0^T \sqrt{-\lim_{s\to t} \frac{\partial^2}{\partial t\,\partial s} K(s,t)}\; dt \tag{3}$$

Compute the derivatives:

$$\frac{\partial}{\partial s} K(s,t) = 2\,(f(t) - f(s))\,\dot{f}(s)\,e^{-(f(t)-f(s))^2} \tag{4}$$

$$\frac{\partial^2}{\partial t\,\partial s} K(s,t) \;\; = 2\,\dot{f}(t)\,\dot{f}(s)\,e^{-(f(t)-f(s))^2}\,[(f(t) - f(s))^2 - 1] \tag{5}$$

Take the limit as $s \to t$:

$$\lim_{s\to t} \frac{\partial^2}{\partial t\,\partial s} K(s,t) = -\dot{f}(t)^2 \tag{6}$$

to see that

$$\begin{aligned}
\mathbb{E}[N([0,T])] &= \int_0^T \sqrt{-(-\dot{f}(t)^2)}\, d t \\
&= \int_0^T \sqrt{\dot{f}(t)^2)}\, d t \\
&= \int_0^T |\dot{f}(t)|\, d t \\
&= \int_0^T |\dot{f}(t)|\, d t \qquad\qquad (\text{since } \dot{f}(t) \geqslant 0 \forall t) \\
&= f(T) - f(0)
\end{aligned}$$

□

Theorem 3

(Eigenfunction Structure) *For the modulated Gaussian kernel* $K(s,t) = e^{-(f(t)-f(s))^2}$ *where* $f \in \mathcal{F}$*, the eigenfunctions take the form:*

$$\phi_n(t) = c_n\, H_n(f(t))\, e^{-\frac{f(t)^2}{2}} \sqrt{\dot{f}(t)} \tag{7}$$

where H_n *are the Hermite polynomials and* c_n *are normalization constants.*

Proof. The eigenfunction equation for kernel K is:

$$\int_{-\infty}^{\infty} e^{-(f(t)-f(s))^2}\, \phi_n(s)\, d s = \lambda_n\, \phi_n(t) \tag{8}$$

Under the change of variables $u = f(s)$, $v = f(t)$:

$$\int_{-\infty}^{\infty} e^{-(v-u)^2} \frac{\phi_n(f^{-1}(u))}{\dot{f}(f^{-1}(u))} d u = \lambda_n\, \phi_n(f^{-1}(v)) \tag{9}$$

Let

$$\psi_n(u) = \frac{\phi_n(f^{-1}(u))}{\sqrt{\dot{f}(f^{-1}(u))}} \tag{10}$$

Then:

$$\int_{-\infty}^{\infty} e^{-(v-u)^2}\,\psi_n(u)\sqrt{\dot{f}(f^{-1}(u))}\,d\,u = \lambda_n\,\psi_n(v)\sqrt{\dot{f}(f^{-1}(v))} \tag{11}$$

This reduces to the standard Gaussian kernel eigenfunction equation:

$$\int_{-\infty}^{\infty} e^{-(v-u)^2}\,\psi_n(u)\,d\,u = \lambda_n\,\psi_n(v) \tag{12}$$

Therefore

$$\psi_n(u) = H_n(u)\,e^{-\frac{u^2}{2}} \tag{13}$$

giving:

$$\phi_n(t) = c_n\,H_n(f(t))\,e^{-\frac{f(t)^2}{2}}\sqrt{\dot{f}(t)}$$

□

Theorem 4

(Normalization Constants) *For the eigenfunctions of the modulated Gaussian kernel* $K(s,t) = e^{-(f(t)-f(s))^2}$ *where* $f \in \mathcal{F}$*, the normalizing constants* c_n *are:*

$$c_n = \frac{1}{\pi^{1/4}\,2^{n/2}\sqrt{n!}} \tag{14}$$

resulting in the complete orthonormal eigenfunction set:

$$\phi_n(t) = \frac{H_n(f(t))\,e^{-\frac{f(t)^2}{2}}\sqrt{\dot{f}(t)}}{\sqrt[4]{\pi}\,2^{\frac{n}{2}}\sqrt{n!}} \tag{15}$$

Proof. The eigenfunctions must satisfy the normalization condition:

$$\int_{-\infty}^{\infty} |\phi_n(t)|^2\,d\,t = 1 \tag{16}$$

Substituting the eigenfunction form:

$$\int_{-\infty}^{\infty} c_n^2\, H_n^2(f(t))\, e^{-f(t)^2}\, \dot{f}(t)\, d\,t = 1 \tag{17}$$

Under the change of variables $u = f(t)$, this becomes:

$$c_n^2 \int_{-\infty}^{\infty} H_n^2(u)\, e^{-u^2}\, d\,u = 1 \tag{18}$$

Using the correct orthogonality relation for Hermite polynomials:

$$\int_{-\infty}^{\infty} H_n^2(u)\, e^{-u^2}\, d\,u = \sqrt{\pi}\, 2^n\, n! \tag{19}$$

Therefore:

$$c_n^2 \sqrt{\pi}\, 2^n\, n! = 1 \tag{20}$$

Solving for c_n:

$$c_n^2 = \frac{1}{\sqrt{\pi}\, 2^n\, n!} \tag{21}$$

Taking the positive square root:

$$c_n = \frac{1}{\sqrt[4]{\pi}\, 2^{\frac{n}{2}} \sqrt{n!}} \tag{22}$$ □

Remark 5. The invariance of eigenvalues under modulation by a monotonic function connects to linear algebra through similarity transformations $P^{-1} A\, P$. The transformation operator

$$(T\phi)(t) = \sqrt{\dot{f}(t)}\, \phi(f(t)) \tag{23}$$

makes this explicit, as the modulated kernel is essentially a conjugation:

$$K_f = T^{-1} K_{standard}\, T \tag{24}$$