Energetic Hilbert Subspaces

Introduction

In mathematics, more precisely in functional analysis, an energetic space is, intuitively, a subspace of a given real Hilbert space equipped with a new "energetic" inner product. The motivation for the name comes from physics, as in many physical problems the energy of a system can be expressed in terms of the energetic inner product.

Energetic Space

Formally, consider a real Hilbert space X with the inner product $(\cdot|\cdot)$ and the norm $\|\cdot\|$. Let Y be a linear subspace of X and $B\colon Y \to X$ be a strongly monotone symmetric linear operator, that is, a linear operator satisfying:

- $(B\,u|v) = (u|B\,v)\ \forall\ u, v \in Y$
- $(B\,u|u) \geq c\|u\|^2$ for some constant $c > 0$ and $\forall u \in Y$

The **energetic inner product** is defined as:

$$(u|v)_E = (B\,u|v) \tag{1}$$

for all $u, v \in Y$.

The **energetic norm** is:

$$\|u\|_E = \sqrt{(u|u)_E} \tag{2}$$

for all $u \in Y$.

The set Y together with the energetic inner product is a pre-Hilbert space. The **energetic space** X_E is defined as the completion of Y in the energetic norm. X_E can be considered a subset of the original Hilbert space X, since any Cauchy sequence in the energetic norm is also Cauchy in the norm of X (this follows from the strong monotonicity property of B).

The energetic inner product is extended from Y to X_E by:

$$(u|v)_E = \lim_{n\to\infty} (u_n|v_n)_E \tag{3}$$

where (u_n) and (v_n) are sequences in Y that converge to points in X_E in the energetic norm.

Energetic Extension

The operator B admits an **energetic extension** B_E:

$$B_E\colon X_E \to X_E^* \tag{4}$$

defined on X_E with values in the dual space X_E^* and given by the formula:

$$\langle B_E\,u|v\rangle_E = (u|v)_E \tag{5}$$

for all $u, v \in X_E$.

Here, $\langle\cdot|\cdot\rangle_E$ denotes the duality bracket between X_E^* and X_E, so $\langle B_E\,u|v\rangle_E$ actually denotes $(B_E\,u)(v)$.

If u and v are elements in the original subspace Y, then:

$$\langle B_E u|v\rangle_E = (u|v)_E = (B u|v) = \langle u|B|v\rangle$$

by the definition of the energetic inner product. If one views $B u$, which is an element in X, as an element in the dual X^* via the Riesz representation theorem, then $B u$ will also be in the dual X_E^* (by the strong monotonicity property of B). Via these identifications, it follows that $B_E u = B u$. In other words, the original operator $B\colon Y \to X$ can be viewed as an operator $B\colon Y \to X_E^*$, and then $B_E\colon X_E \to X_E^*$ is simply the function extension of B from Y to X_E.

An Example from Physics

Consider a string whose endpoints are fixed at two points $a < b$ on the real line. Let the vertical outer force density at each point x ($a \leq x \leq b$) on the string be $f(x)\mathbf{e}$, where $\mathbf{e}$ is a unit vector pointing vertically and $f\colon [a, b] \to \mathbb{R}$. Let $u(x)$ be the deflection of the string at the point x under the influence of the force. Assuming that the deflection is small, the elastic energy of the string is:

$$\frac{1}{2}\int_a^b u'(x)^2 \, d x \tag{6}$$

and the total potential energy of the string is:

$$F(u) = \frac{1}{2}\int_a^b u'(x)^2 \, d x - \int_a^b u(x) \, f(x) \, d x \tag{7}$$

The deflection $u(x)$ minimizing the potential energy will satisfy the differential equation:

$$-u'' = f \tag{8}$$

with boundary conditions:

$$u(a) = u(b) = 0 \tag{9}$$

To study this equation, consider the space $X = L^2(a, b)$, the L^2 space of all square-integrable functions $u\colon [a, b] \to \mathbb{R}$ with respect to the Lebesgue measure. This space is Hilbert with the inner product:

$$(u|v) = \int_a^b u(x) \, v(x) \, d x \tag{10}$$

and the norm given by:

$$\|u\| = \sqrt{(u|u)} \tag{11}$$

Let Y be the set of all twice continuously differentiable functions $u\colon [a, b] \to \mathbb{R}$ with the boundary conditions $u(a) = u(b) = 0$. Then Y is a linear subspace of X.

Consider the operator $B\colon Y \to X$ given by:

$$B u = -u'' \tag{12}$$

so the deflection satisfies the equation $B u = f$. Using integration by parts and the boundary conditions, one can see that:

$$(B u|v) = -\int_a^b u''(x) \, v(x) \, d x = \int_a^b u'(x) \, v'(x) \, d x = (u|B v) \tag{13}$$

for any u and $v \in Y$. Therefore, B is a symmetric linear operator.

B is also strongly monotone, since by Friedrichs's inequality:

$$\|u\|^2 = \int_a^b u^2(x)\ dx \leq C \int_a^b u'(x)^2\ dx = C\,(B\,u|u) \tag{14}$$

for some $C > 0$.

The energetic space with respect to the operator B is the Sobolev space $H_0^1(a,b)$. The elastic energy of the string which motivated this study is:

$$\frac{1}{2}\int_a^b u'(x)^2\ dx = \frac{1}{2}(u|u)_E \tag{15}$$

so it is half of the energetic inner product of u with itself.

To calculate the deflection u minimizing the total potential energy $F(u)$ of the string, one writes this problem in the form:

$$(u|v)_E = (f|v) \forall v \in X_E \tag{16}$$

Next, one usually approximates u by some u_h, a function in a finite-dimensional subspace of the true solution space. For example, one might let u_h be a continuous piecewise linear function in the energetic space, which gives the finite element method. The approximation u_h can be computed by solving a system of linear equations.

The energetic norm turns out to be the natural norm in which to measure the error between u and u_h, see Céa's lemma.

References

- Zeidler, Eberhard. *Applied functional analysis: applications to mathematical physics*. New York: Springer-Verlag, 1995. ISBN 0-387-94442-7.

- Johnson, Claes. *Numerical solution of partial differential equations by the finite element method*. Cambridge University Press, 1987. ISBN 0-521-34514-6.

Corollary 3. If $\{\varphi_n\} \subset D(P)$ and the sequence $\{P\varphi_n\}$ is complete in an initial Hilbert space H, then the sequence $\{\varphi_n\}$ is complete in the energetic space H_P.

2.3. Examples

The examples listed below present various ways of determining complete sequences with the aid of the theorems proved. Having a given sequence, it is sufficient to prove its completeness in the energetic space H_P, since every sequence complete in H_P is also complete in an initial Hilbert space H [?]. Sometimes, however, in order to prove the completeness of a sequence in H_P one should have shown its completeness in H from the start. Even when there is no need to do so, it is preferable to become acquainted with direct proofs of the completeness of the sequences in the initial space H with the aim of understanding how the theorems proved above are used.

In all examples, except one, H is assumed to be the space of quadratically integrable functions and P is always the operator $-\Delta$ with the homogeneous Dirichlet boundary condition. Therefore the domain $D(-\Delta)$ consists of those functions of the space H which are twice continuously differentiable in the considered area and vanish on its boundary.

Example 1. Consider on the interval $D=(0,a)$ the sequence $\{\varphi_n\}=\{x^n\,(a-x)\colon n=1,2,\ldots\}$.

Let $\mathcal{F}$ be the family consisting of φ_1 and φ_2. Raising the difference $a-x$ to mth power according to the binomial theorem and multiplying by $x^n\,(a-x)$ we ascertain that $x^n\,(a-x)^{m+1}$ is a linear combination of the terms of the sequence $\{\varphi_n\}$. Thus after addition of the constant $\varphi_0=1$ to the considered sequence, it is obvious that the sequence AS given by the formula (10) is a subset of the linear space $\mathrm{Lin}\{\varphi_n\colon n=0,1,2,\ldots\}$. Taking two points x' and x'' from the open interval $(0,a)$ such that $\varphi_1(x')=\varphi_1(x'')$ and $\varphi_2(x')=\varphi_2(x'')$, and dividing the second equation by the first, we obtain $x'=x''$. Therefore the family $\mathcal{F}$ separates the points of the interval $(0,a)$ and $R(\mathcal{F})$ is the two-point set $\{0,a\}$ being the boundary ∂D of the set D. Let AS_2=$\{\varphi_n\}$ and AS_1 will be the one-element sequence $\{\varphi_0\}$. Of course, AS_1 consisting only of a constant function fulfills the assumptions of the corollary of Theorem 2 (taking advantage of it one should assume $k=1$ and $S_1=\partial D$). Thus according to this corollary the sequence $\{\varphi_n\}$ is complete in $L_2(D)$.

To prove the completeness of the sequence $\{\varphi_n\}$ in $H_{-\Delta}$, let us observe that

$$1=-\Delta\,\frac{\varphi}{2} \tag{17}$$

$$x^{n-1}=\frac{[n\,(n-1)\,a\,x^{n-2}-2\,\Delta\,\varphi_n]}{(n^2+n)}\forall n=2,3,\ldots \tag{18}$$

, rom which it follows that every element of the sequence $\{\psi_n\}=\{x^n\colon n=0,1,2,\ldots\}$

Proof. Let $g\in C(D)$. By virtue of condition (a) there exist $f_n\in\mathrm{Lin}\{\varphi_n\}$ such that $\sup\{|f(u)\,P_n\,f(u)-f(u)|\colon u\in D\}\le 1/n$. Because $1/f\in L_2(D)$, the inverse image $f^{-1}[0]$ is of measure zero. Therefore $|P_n\,f-g|\to 0$ almost everywhere and $|P_n\,f-g|^2\le 1/|f|^2$. Consequently $\|P_n\,f-g\|\to 0$ by the Lebesgue dominated convergence theorem, which, together with the known fact that the set of continuous functions is dense in $L_2(D)$, proves that the sequence $\{P\varphi_n\}$ is complete in $L_2(D)$.

If condition (b) is fulfilled, then for any $g\in L_2(D)$ there exists a $g_n\in\mathrm{Lin}\{\varphi_n\}$ such that $\|f\,P_n-f\,g\|\le 1/n$. The inequality $|f|_{\inf}\cdot\|P_n-g\|\le\|f\,P_n-f\,g\|$ implies that $\|P_n-g\|\to 0$ since $|f|_{\inf}\ne 0$, from where it follows that the sequence $\{P\varphi_n\}$ is complete in $L_2(D)$ also for the case (b). From the two inequalities satisfied by every positive definite operator

$$\tag{19}$$

$$\begin{aligned}\|Ph\| &\ge\mu\|h\| \\ \|Ph\|\cdot\|h\| &\ge\|h\|^2\end{aligned}\qquad\forall h\in D(P) \tag{20}$$

where μ is a positive real number, and $\|h\|_P$ denotes the norm in H_P, we come to a conclusion that $\|h_n\|_P\to 0$ for any sequence $\{h_n\}\subset D(P)$ such that $\|P\,h_n\|\to 0$. Thus by the completeness of the sequence $\{P\varphi_n\}$ in $L_2(D)$, the sequence $\{\varphi_n\}$ is complete in H_P, which completes the proof. □

From the point (a) of the above theorem the following corollary results directly:

Corollary 2

If $\{\varphi_n\}$ is a sequence in $D(P)$, and there exists a sequence $\{\psi_n\}$ complete in $C(D)$ and a function $f\in C(D)$, $1/f\in L_2(D)$, such that each function ψ_n can be uniformly approximated by linear combinations of the terms of the sequence $\{f\,P\varphi_n\}$ (in particular when $\{\psi_n\}\subset Lin\{f\,P\varphi_n\}$, i.e., when each of the functions ψ_n is a linear combination of the terms of the sequence $\{f\,P\varphi_n\}$), then the sequence $\{\varphi_n\}$ is complete in the energetic space H_P.

Assuming that P is the identity operator $(P u = u)$, choosing the function $f = 1$ and taking advantage of the part of the proof of Theorem 6 concerning case (a) (where, among other things, we replace the limit $\|P_n f - g\|^2 \to 0$ by the limit $|f_n - g|^p \to 0$) we obtain a proof of the following corollary:

Corollary 3

Every sequence complete in $C(D)$ is also complete in $L_p(D)$.

Corollary 4

If $\{\varphi_n\} \subset D(P)$ and the sequence $\{P\varphi_n\}$ is complete in an initial Hilbert space H, then the sequence $\{\varphi_n\}$ is complete in the energetic space H_P.

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To prove the completeness of the sequence $\{\varphi_n\}$ in $H_{-\Delta}$, let us observe that $1 = -\Delta\, \varphi/2, x^{n-1} = [n\, (n - 1)\, a\, x^{n-2} - 2\, \Delta\, \varphi_n] / (n^2 + n), n = 2, 3, \ldots$, from which it follows that every element of the sequence $\{\psi_n\} = \{x^n\colon n = 0, 1, 2, \ldots\}$