Spectral Analysis of the Ornstein-Uhlenbeck Process on $\mathbb{R}$

1 Covariance Function and Spectral Density

The covariance function of the Ornstein-Uhlenbeck (OU) process is given by:

$$C(x) = \sigma^2 e^{-\alpha |x|} \tag{1}$$

where σ^2 is the variance and $\alpha > 0$ is the mean reversion rate.

The spectral density $S(\omega)$ is the Fourier transform of $C(x)$:

$$\begin{aligned}
S(\omega) &= \int_{-\infty}^{\infty} C(x)\, e^{-i\omega x}\, d x \\
&= \sigma^2 \int_{-\infty}^{\infty} e^{-\alpha|x|}\, e^{-i\omega x}\, d x \\
&= \sigma^2 \left(\int_0^{\infty} e^{-(\alpha + i\omega)x}\, d x + \int_0^{\infty} e^{-(\alpha - i\omega)x}\, d x \right) \\
&= \sigma^2 \left[\frac{1}{\alpha + i\,\omega} + \frac{1}{\alpha - i\,\omega} \right] \\
&= \frac{2\,\sigma^2\,\alpha}{\alpha^2 + \omega^2}
\end{aligned} \tag{2}$$

2 Orthogonal Polynomials

The polynomials orthogonal with respect to the spectral density $S(\omega)$ are related to the Routh-Romanovski polynomials. Let $x = \omega / \alpha$, then the weight function becomes:

$$w(x) = \frac{1}{1 + x^2} \tag{3}$$

The Routh-Romanovski polynomials $R_n(x)$ are defined by the recurrence relation:

$$\begin{aligned}
R_0(x) &= 1 \\
R_1(x) &= x \\
R_{n+1}(x) &= x\, R_n(x) - k_n\, R_{n-1}(x) \quad \text{for } n \geq 1
\end{aligned} \tag{4}$$

where k_n is a coefficient that can vary depending on the specific normalization used. In some sources, $k_n = n^2$, but this may differ.

These polynomials satisfy the orthogonality relation:

$$\int_{-\infty}^{\infty} R_m(x)\, R_n(x)\, w(x)\, d\,x = h_n\, \delta_{mn} \tag{5}$$

where h_n is a normalization constant and δ_{mn} is the Kronecker delta.

The first few polynomials are:

$$\begin{aligned} R_0(x) &= 1 \\ R_1(x) &= x \\ R_2(x) &= x^2 - 1 \\ R_3(x) &= x^3 - 3\,x \\ R_4(x) &= x^4 - 6\,x^2 + 3 \end{aligned} \tag{6}$$

The polynomials orthogonal with respect to $S(\omega)$ are:

$$P_n(\omega) = R_n\,(\omega / \alpha) \tag{7}$$

3 Fourier Transforms of Orthogonal Polynomials

The Fourier transforms of the first few Routh-Romanovski polynomials are:

$$\begin{aligned} r_0(t) &= \sqrt{2\,\pi}\, \delta(t) \\ r_1(t) &= i\, \sqrt{2\,\pi / \alpha}\, \frac{d}{d\,t} [e^{-\alpha |t|}] \\ r_2(t) &= -\sqrt{2\,\pi / \alpha^2}\, \frac{d^2}{d\,t^2} [e^{-\alpha |t|}] - \sqrt{2\,\pi}\, \delta(t) \end{aligned} \tag{8}$$

where $\delta(t)$ is the Dirac delta function.

4 Entropy Integral and Non-Compactness

To show that the covariance operator of the OU process is not compact on $L^2(\mathbb{R})$, we analyze the ϵ-entropy integral.

The ϵ-covering number $N(\epsilon)$ is related to the spectral density:

$$N(\epsilon) \approx \int_{-\infty}^{\infty} \max\left(1, \sqrt{\frac{S(\omega)}{\epsilon^2}}\right) d\,\omega \tag{9}$$

For large ω, $S(\omega) \sim 2\,\sigma^2\,\alpha\,/\,\omega^2$, so:

$$N(\epsilon) \approx 2\int_0^\infty \max\left(1, \frac{\sqrt{2\,\sigma^2\,\alpha}}{\epsilon\,\omega}\right) d\,\omega \tag{10}$$

Let $\omega_\epsilon = \sqrt{2\,\sigma^2\,\alpha}\,/\,\epsilon$. Then:

$$\begin{aligned} N(\epsilon) &\approx 2\left[\omega_\epsilon + \int_{\omega_\epsilon}^\infty \frac{\sqrt{2\,\sigma^2\,\alpha}}{\epsilon\,\omega}\, d\,\omega\right] \\ &= 2\left[\frac{\sqrt{2\,\sigma^2\,\alpha}}{\epsilon} + \frac{\sqrt{2\,\sigma^2\,\alpha}}{\epsilon}\log\left(\frac{\infty}{\omega_\epsilon}\right)\right] \\ &\approx \frac{C}{\epsilon}\log\left(\frac{1}{\epsilon}\right) \end{aligned} \tag{11}$$

where C is a constant depending on σ and α.

The ϵ-entropy $H(\epsilon)$ is defined as $\log(N(\epsilon))$, so:

$$H(\epsilon) \approx \log\left(\frac{C}{\epsilon}\right) + \log\left(\log\left(\frac{1}{\epsilon}\right)\right) \tag{12}$$

The entropy integral is:

$$\int_0^1 H(\epsilon)\, d\,\epsilon \approx \int_0^1 \left[\log\left(\frac{C}{\epsilon}\right) + \log\left(\log\left(\frac{1}{\epsilon}\right)\right)\right] d\,\epsilon \tag{13}$$

The second term causes the divergence. Using the change of variable $u = \log(1/\epsilon)$:

$$\int_0^1 \log\left(\log\left(\frac{1}{\epsilon}\right)\right) d\,\epsilon = \int_0^\infty \log(u)\, e^{-u}\, d\,u = \infty \tag{14}$$

Therefore:

$$\int_0^1 H(\epsilon)\, d\,\epsilon = \infty \tag{15}$$

This divergence of the entropy integral demonstrates that the covariance operator of the OU process is not compact on $L^2(\mathbb{R})$.