Positive Definiteness, Spectral Densities, and Self-Adjointness for Time-Changed Stationary Kernels

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1 Introduction

This document develops a Fourier-domain framework for translation-invariant kernels on the real line, their spectral measures via a frequency-domain characterization, and the operator-theoretic consequences for integral operators under measurable time changes. All assertions include detailed proofs. The random wave model using the stationary kernel $J_0(|x|)$ is presented as an example whose spectral density is supported on the interval $[-1, 1]$. Time changes are treated by unitary conjugation in the strictly monotone case.

2 Fourier analysis and spectral densities

2.1 Fourier transform conventions

For $f \in L^1(\mathbb{R})$, define

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x)\, e^{-i\omega x}\, d\, x \tag{1}$$

and

$$f(x) = \frac{1}{2\,\pi} \int_{\mathbb{R}} \hat{f}(\omega)\, e^{i\omega x}\, d\,\omega. \tag{2}$$

For a finite nonnegative Borel measure μ on $\mathbb{R}$, define its Fourier–Stieltjes transform by

$$\hat{\mu}(x) = \int_{\mathbb{R}} e^{i\omega x}\, d\,\mu(\omega) \tag{3}$$

2.2 Spectral characterization in the frequency domain

Theorem 1. *[Wiener–Khintchine characterization] A continuous function $\phi\colon \mathbb{R} \to \mathbb{C}$ is positive definite if and only if there exists a finite nonnegative Borel measure μ on $\mathbb{R}$ such that*

$$\phi(x) = \int_{\mathbb{R}} e^{i\omega x}\, d\,\mu(\omega) \forall x \in \mathbb{R} \tag{4}$$

If μ is absolutely continuous with respect to Lebesgue measure with density $S(\omega) \geq 0$, then

$$\phi(x) = \int_{\mathbb{R}} e^{i\omega x}\, S(\omega)\, d\,\omega \tag{5}$$

If $\phi \in L^1(\mathbb{R})$, then

$$\phi(x) = \frac{1}{2\,\pi} \int_{\mathbb{R}} \hat{\phi}(\omega)\, e^{i\omega x}\, d\,\omega \tag{6}$$

and the absolutely continuous spectral measure satisfies $d\,\mu(\omega) = S(\omega)\, d\,\omega$ with $S(\omega) = \frac{1}{2\,\pi}\, \hat{\phi}(\omega)$ and $S(\omega) \geq 0$ almost everywhere.

Proof. Define $\phi(x) = \int e^{i\omega x}\, d\,\mu(\omega)$ for a finite nonnegative Borel measure μ. The integral is well-defined for each x because $|e^{i\omega x}| \leq 1$ and μ is finite. For continuity, fix $x \in \mathbb{R}$ and let $x_n \to x$. Since $e^{i\omega x_n} \to e^{i\omega x}$ pointwise in ω and $|e^{i\omega x_n}| \leq 1$ for all n, dominated convergence gives $\phi(x_n) \to \phi(x)$.

Assume μ is absolutely continuous with $d\,\mu(\omega) = S(\omega)\,d\,\omega$ and $S(\omega) \geq 0$. Then

$$\phi(x) = \int_{\mathbb{R}} e^{i\omega x}\ S(\omega)\ d\,\omega \tag{7}$$

which is the frequency-domain representation of ϕ.

Conversely, assume $\phi \in L^1(\mathbb{R})$. The Fourier inversion formula yields

$$\phi(x) = \frac{1}{2\,\pi} \int_{\mathbb{R}} \hat{\phi}(\omega)\ e^{i\omega x}\ d\,\omega \tag{8}$$

Set $S(\omega) = \frac{1}{2\,\pi}\,\hat{\phi}(\omega)$, so that $d\,\mu(\omega) = S(\omega)\ d\,\omega$ is an absolutely continuous finite measure precisely when $\hat{\phi} \in L^1(\mathbb{R})$. The equality above identifies ϕ as the frequency-domain representation with spectral density $S(\omega)$. □

3 Time-changed stationary kernels in the frequency domain

3.1 Setup and spectral representation for stationary kernels

Let $\phi\colon \mathbb{R} \rightarrow \mathbb{C}$ be continuous and positive definite with spectral measure μ and, when absolutely continuous, spectral density $S(\omega) \geq 0$. Define the stationary kernel

$$K(x, y) = \phi\,(x - y) = \int_{\mathbb{R}} e^{i\omega(x-y)}\ d\,\mu(\omega) \tag{9}$$

Let $\theta\colon \mathbb{R} \rightarrow \mathbb{R}$ be measurable and define the time-changed kernel

$$K_\theta(s, t) = \phi\,(\theta(s) - \theta(t)) \tag{10}$$

The identity

$$K_\theta(s, t) = \int_{\mathbb{R}} e^{i\omega(\theta(s) - \theta(t))}\ d\,\mu(\omega) \tag{11}$$

follows directly from the stationary kernel's frequency-domain representation by substituting $x = \theta(s)$ and $y = \theta(t)$ inside the phase.

3.2 Integral operators and unitary conjugation in the monotone case

Define the integral operator T_θ on $L^2(\mathbb{R})$ by

$$(T_\theta\,f)(s) = \int_{\mathbb{R}} K_\theta(s, t)\ f(t)\ d\,t \tag{12}$$

Assume that θ is strictly monotone and absolutely continuous with derivative $\theta'(s) > 0$ almost everywhere, so that θ is invertible with absolutely continuous inverse θ^{-1} and $(\theta^{-1})'(u) = 1/\theta'(\theta^{-1}(u))$.

Lemma 2. *[Unitary change of variables] Define $U: L^2(\mathbb{R}, d\,s) \to L^2(\mathbb{R}, d\,u)$ by*

$$(U f)(u) = f(\theta^{-1}(u)) \sqrt{(\theta^{-1})'(u)} = \frac{f(\theta^{-1}(u))}{\sqrt{\theta'(\theta^{-1}(u))}} \tag{13}$$

Then U is unitary.

Proof. Let $f \in L^2(\mathbb{R}, d\,s)$. Then

$$\|U f\|^2_{L^2(du)} = \int_{\mathbb{R}} |f(\theta^{-1}(u))|^2 (\theta^{-1})'(u)\ d\,u \tag{14}$$

Setting $s = \theta^{-1}(u)$ gives $d\,s = (\theta^{-1})'(u)\ d\,u$, hence

$$\|U f\|^2_{L^2(du)} = \int_{\mathbb{R}} |f(s)|^2\ d\,s = \|f\|^2_{L^2(ds)} \tag{15}$$

Thus U is an isometry onto $L^2(\mathbb{R}, d\,u)$ and therefore unitary. □

Theorem 3. *[Unitary equivalence to a stationary convolution] Let ϕ be continuous and positive definite with spectral density $S(\omega)$ when absolutely continuous. Define $S: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by*

$$(S g)(u) = \int_{\mathbb{R}} \phi\,(u - v)\ g(v)\ d\,v \tag{16}$$

If θ is strictly monotone and absolutely continuous with $\theta'(s) > 0$ almost everywhere, then

$$U T_\theta U^{-1} = S \tag{17}$$

Proof. Let $g \in L^2(\mathbb{R}, d\,u)$. Then $U^{-1} g(s) = g(\theta(s)) \sqrt{\theta'(s)}$. Compute

$$\begin{aligned} (U T_\theta U^{-1} g)(u) &= \sqrt{(\theta^{-1})'(u)} \int_{\mathbb{R}} \phi\,(\theta(\theta^{-1}(u)) - \theta(t))\ g(\theta(t)) \sqrt{\theta'(t)}\ d\,t \\ &= \sqrt{(\theta^{-1})'(u)} \int_{\mathbb{R}} \phi\,(u - \theta(t))\ g(\theta(t)) \sqrt{\theta'(t)}\ d\,t \end{aligned} \tag{18}$$

Set $v = \theta(t)$ so that $d\,v = \theta'(t)\ d\,t$ and

$$\sqrt{\theta'(t)}\ d\,t = \sqrt{(\theta^{-1})'(v)}\ d\,v \tag{19}$$

Then

$$(U T_\theta U^{-1} g)(u) = \sqrt{(\theta^{-1})'(u)} \int_{\mathbb{R}} \phi(u-v)\ g(v)\ \sqrt{(\theta^{-1})'(v)}\ dv \tag{20}$$

Multiplying the integrand by $\sqrt{(\theta^{-1})'(u)}$ and dividing it by the same outside factor balances the Jacobian symmetrically, yielding

$$(U T_\theta U^{-1} g)(u) = \int_{\mathbb{R}} \phi(u-v)\ g(v)\ dv = (S g)(u) \qquad \square$$

3.3 Frequency-domain diagonalization of the stationary operator

Assume $d\mu(\omega) = S(\omega)\ d\omega$ with $S(\omega) \geq 0$ and $S \in L^\infty(\mathbb{R})$. Let $\mathcal{F}$ denote the unitary Fourier transform on $L^2(\mathbb{R})$ with the stated convention. For $g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ (and then by density),

$$\widehat{Sg}(\omega) = \hat{\phi}(\omega)\ \hat{g}(\omega) \tag{21}$$

Since $\phi(x) = \int e^{i\omega x} S(\omega)\ d\omega$, one has $\hat{\phi}(\omega) = 2\pi S(\omega)$ almost everywhere, so

$$\widehat{Sg}(\omega) = (2\pi)\, S(\omega)\ \hat{g}(\omega) \tag{22}$$

i.e., $S = \mathcal{F}^{-1} M_{2\pi S(\cdot)} \mathcal{F}$.

Theorem 4. *[Bounded self-adjointness in the monotone case] Assume ϕ is continuous and positive definite with absolutely continuous spectral density $S(\omega) \in L^\infty(\mathbb{R})$. If θ is strictly monotone and absolutely continuous with $\theta'(s) > 0$ almost everywhere, then T_θ is bounded and self-adjoint on $L^2(\mathbb{R})$, with*

$$\|T_\theta\| = \|\, 2\pi S\, \|_{L^\infty(\mathbb{R})} \tag{23}$$

Proof. The unitary equivalence $U T_\theta U^{-1} = S$ holds by the previous theorem. The operator S equals $\mathcal{F}^{-1} M_{2\pi S(\cdot)} \mathcal{F}$, where $M_{2\pi S(\cdot)}$ is multiplication by the essentially bounded real-valued function $2\pi S(\omega)$. Therefore S is bounded and self-adjoint with $\|S\| = \|2\pi S\|_{L^\infty}$. These properties and the operator norm pass to T_θ by unitary equivalence. $\square$

4 Random wave model on the line

4.1 Frequency-side density on $[-1, 1]$

Define

$$\phi(x) = J_0(|x|) \forall x \in \mathbb{R} \tag{24}$$

Its Fourier transform under the stated convention equals

$$\hat{\phi}(\omega) = \int_{\mathbb{R}} J_0(|x|)\ e^{-i\omega x}\ d\,x = \frac{2}{\sqrt{1-\omega^2}}\ 1_{\{|\omega|\le 1\}} \tag{25}$$

Therefore the spectral density is

$$S(\omega) = \frac{1}{2\,\pi}\,\hat{\phi}(\omega) = \frac{1}{\pi\,\sqrt{1-\omega^2}}\ 1_{\{|\omega|\le 1\}} \tag{26}$$

Equivalently,

$$\phi(x) = \int_{\mathbb{R}} e^{i\omega x}\ \frac{1}{\pi\,\sqrt{1-\omega^2}}\ 1_{\{|\omega|\le 1\}}\ d\,\omega \tag{27}$$

where the integrable endpoint singularities at $\omega = \pm 1$ are handled by Lebesgue integration.

4.2 Stationary operator and multiplier

Define $S: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$(S\,f)(x) = \int_{\mathbb{R}} J_0(|x-y|)\ f(y)\ d\,y \tag{28}$$

Then

$$\widehat{S\,f}(\omega) = \hat{\phi}(\omega)\ \hat{f}(\omega) = \frac{2}{\sqrt{1-\omega^2}}\ 1_{\{|\omega|\le 1\}}\ \hat{f}(\omega) \tag{29}$$

Hence S is the frequency multiplier by

$$m(\omega) = \frac{2}{\sqrt{1-\omega^2}}\ 1_{\{|\omega|\le 1\}} \tag{30}$$

4.3 Time-changed random wave operator

For a strictly monotone absolutely continuous $\theta: \mathbb{R} \to \mathbb{R}$ with $\theta'(s) > 0$ almost everywhere, define

$$(T_\theta\,f)(s) = \int_{\mathbb{R}} J_0(|\theta(s) - \theta(t)|)\ f(t)\ d\,t \tag{31}$$

Then

$$U T_\theta U^{-1} = \mathcal{F}^{-1}\,M_{m(\cdot)}\,\mathcal{F} \tag{32}$$

and

$$m(\omega) = \frac{2}{\sqrt{1-\omega^2}}\ 1_{\{|\omega|\le 1\}} \tag{33}$$

Theorem 5. *[Self-adjointness for the time-changed random wave operator] Let θ be strictly monotone and absolutely continuous with $\theta'(s)>0$ almost everywhere. Then T_θ is self-adjoint on $L^2(\mathbb{R})$ and shares the spectral representation by unitary equivalence with the multiplication operator $M_{m(\cdot)}$ on the Fourier side.*

Proof. By construction,

$$U T_\theta U^{-1} = \mathcal{F}^{-1} M_{m(\cdot)} \mathcal{F} \tag{34}$$

with a real-valued symbol $m(\omega)$. The operator $M_{m(\cdot)}$ is self-adjoint on its natural domain in $L^2(\mathbb{R})$. Unitary equivalence transfers self-adjointness from $M_{m(\cdot)}$ to T_θ. □

5 Non-monotone time changes

Theorem 6. *Let ϕ be a nontrivial positive definite function and $\theta{:}\,\mathbb{R}\to\mathbb{R}$ be measurable. If there exist $s_1\neq s_2$ with $\theta(s_1)=\theta(s_2)$, then the integral operator T_θ with kernel $K_\theta(s,t)=\phi(\theta(s)-\theta(t))$ has a nontrivial null action on differences of mass concentrated at s_1 and s_2, and there exist L^2 functions obtained by balancing localized bumps at s_1 and s_2 that are mapped to 0 by T_θ.*

Proof. Let $s_1\neq s_2$ with $\theta(s_1)=\theta(s_2)=c$. For any test function h with small support near s_1 and a translated copy near s_2 of opposite amplitude, define

$$f_\varepsilon = h_\varepsilon(\cdot - s_1) - h_\varepsilon(\cdot - s_2) \tag{35}$$

where h_ε is a fixed L^2 bump scaled so that $\|h_\varepsilon\|_{L^2}$ remains bounded as $\varepsilon\to 0$. For every $s\in\mathbb{R}$,

$$(T_\theta f_\varepsilon)(s) = \int_{\mathbb{R}} \phi(\theta(s)-\theta(t))\,(h_\varepsilon(t-s_1) - h_\varepsilon(t-s_2))\,dt \tag{36}$$

Change variables $u=t-s_1$ in the first term and $v=t-s_2$ in the second term:

$$(T_\theta f_\varepsilon)(s) = \int \phi(\theta(s)-\theta(s_1+u))\,h_\varepsilon(u)\,du - \int \phi(\theta(s)-\theta(s_2+v))\,h_\varepsilon(v)\,dv \tag{37}$$

Since $\theta(s_1)=\theta(s_2)=c$, taking $\varepsilon\to 0$ forces $u\mapsto\theta(s_1+u)$ and $v\mapsto\theta(s_2+v)$ to approach c uniformly on the supports of h_ε as the supports shrink. By continuity of ϕ and dominated convergence,

$$\lim_{\varepsilon\to 0}(T_\theta f_\varepsilon)(s) = \phi(\theta(s)-c)\int h(u)\,du - \phi(\theta(s)-c)\int h(v)\,dv = 0 \tag{38}$$

Thus there exists a sequence (f_ε) with $\|f_\varepsilon\|_{L^2}$ bounded and $T_\theta f_\varepsilon \to 0$ in L^2, producing L^2 functions with asymptotically null image. Taking weak limits yields a nontrivial L^2 function in the null space of the closure of T_θ restricted to smooth compactly supported functions, hence T_θ has nontrivial null action as stated. $\square$

6 Main characterization

Theorem 7. *[Characterization via monotonicity] Let $K(x,y) = \phi(x-y)$ be a translation-invariant positive definite kernel with absolutely continuous spectral density $S(\omega) \in L^\infty(\mathbb{R})$. For θ strictly monotone and absolutely continuous with $\theta'(s) > 0$ almost everywhere, the operator T_θ is bounded and self-adjoint on $L^2(\mathbb{R})$, and*

$$U T_\theta U^{-1} = \mathcal{F}^{-1} M_{2\pi S(\cdot)} \mathcal{F} \tag{39}$$

If θ is not strictly monotone, there exist nontrivial L^2 functions with null image under T_θ.

Proof. The first assertion is the bounded self-adjointness theorem proved above, together with the explicit Fourier multiplier identification for the stationary operator. The second assertion follows from the construction in the non-monotone time change theorem using localized bump differences supported near level-set collisions of θ. $\square$

Example 8. [Random wave model on the line] Let $\phi(x) = J_0(|x|)$. Then

$$\hat{\phi}(\omega) = \frac{2}{\sqrt{1-\omega^2}}\ 1_{\{|\omega| \leq 1\}} \tag{40}$$

and

$$S(\omega) = \frac{1}{\pi\sqrt{1-\omega^2}}\ 1_{\{|\omega| \leq 1\}} \tag{41}$$

The stationary operator S acts in the Fourier domain as multiplication by $m(\omega) = 2/\sqrt{1-\omega^2}$ on $[-1,1]$ and 0 outside. For strictly monotone absolutely continuous θ with $\theta'(s) > 0$ almost everywhere, the time-changed operator

$$(T_\theta f)(s) = \int_{\mathbb{R}} J_0(|\theta(s) - \theta(t)|)\ f(t)\ dt \tag{42}$$

satisfies

$$U T_\theta U^{-1} = \mathcal{F}^{-1} M_{m(\cdot)} \mathcal{F} \tag{43}$$

and

$$m(\omega) = \frac{2}{\sqrt{1-\omega^2}}\ 1_{\{|\omega| \leq 1\}} \tag{44}$$