An Operator Characterization of Oscillatory Harmonizable Processes

RANDALL J. SWIFT Department of Mathematics,
Western Kentucky University, Bowling Green, Kentucky

Dedicated to Professor M.M. Rao, advisor and friend, on the occasion of his 65th birthday.

1 INTRODUCTION

A class of nonstationary stochastic processes which are encountered in some applications is the class of modulated stationary processes $X(t)$. These processes are obtained when a stationary process $X_0(t)$ is multiplied by some nonrandom modulating function $A(t)$:

$$X(t) = A(t)X_0(t).$$

This class of processes has been investigated by Joyeux (1987) and Priestley (1981). The book by Yaglom (1987) provides a nice treatment of these processes. In particular, if $A(t)$ admits a generalized Fourier transform, the class of oscillatory processes, studied by Priestley (1981) is obtained. In some physical situations, the assumption of stationarity for the process $X_0(t)$ is unrealistic Rao (1982). If this condition is relaxed, and $X_0(t)$ is assumed to be harmonizable and if $A(t)$ admits a generalized Fourier transform, the process $X(t)$ is not oscillatory, but is oscillatory harmonizable.

This paper investigates the properties of oscillatory harmonizable processes. Section 2 recalls the basic theory of harmonizable processes required for the subsequent analysis. Section 3 introduces and develops the class of oscillatory harmonizable processes. In this section, the spectral representation of oscillatory harmonizable processes is obtained. This representation is used to deduce relationships between the oscillatory harmonizable processes and other classes of nonstationary processes. Section 4 obtains an important and useful operator characterization for oscillatory harmonizable processes.

2 PRELIMINARIES

In the following work, there is always an underlying probability space, (Ω, Σ, P), whether this is explicitly stated or not.

DEFINITION 2.1 For $p \geq 1$, define $L_0^p(P)$ to be the set of all complex valued $f \in L^p(\Omega, \Sigma, P)$ such that $E(f) = 0$, where $E(f) = \int_\Omega f(\omega)dP(\omega)$ is the expectation.

In this paper, we will consider second order stochastic processes. More specifically, mappings $X : I\!R \to L_0^2(P)$.

DEFINITION 2.2 A stochastic process $X : I\!R \to L_0^2(P)$ is *stationary* (stationary in the wide or Khintchine sense) if its covariance $r(s,t) = E(X(s)\overline{X(t)})$ is continuous and is a function of the difference of its arguments, so that

$$r(s,t) = \tilde{r}(s-t).$$

An equivalent definition of a stationary process is one whose covariance function can be represented as

$$\tilde{r}(\tau) = \int_{I\!R} e^{i\lambda\tau} dF(\lambda), \tag{1}$$

for a unique non-negative bounded Borel measure $F(\cdot)$. This alternate definition is a consequence of a classical theorem of Bochner's (Gihman and Skorohod, 1974), and motivates the following definition.

DEFINITION 2.3 A stochastic process $X : I\!R \to L_0^2(P)$ is *weakly harmonizable* if its covariance $r(\cdot,\cdot)$ is expressible as

$$r(s,t) = \int_{I\!R}\int_{I\!R} e^{i\lambda s - i\lambda' t} dF(\lambda, \lambda') \tag{2}$$

where $F : I\!R \times I\!R \to \mathbb{C}$ is a positive semi-definite bimeasure, hence of finite Fréchet variation.

The integrals in (2) are strict Morse-Transue, (Chang and Rao, 1986). A stochastic process, $X(\cdot)$, is *strongly harmonizable* if the bimeasure $F(\cdot,\cdot)$ in (2) extends to a complex measure and hence is of bounded Vitali variation. In either case, $F(\cdot,\cdot)$ is termed the *spectral bi-measure* (or *spectral measure*) of the harmonizable process.

Comparison of equation (2) with equation (1) shows that when $F(\cdot,\cdot)$ concentrates on the diagonal $\lambda = \lambda'$, both the weak and strong harmonizability concepts reduce to the stationary concept. Harmonizable processes retain the powerful Fourier analytic methods inherent with stationary processes, as seen in Bochner's theorem, (1); but they relax the requirement of stationarity.

The structure and properties of harmonizable processes has been investigated and developed extensively by M.M. Rao and others. The following sources are listed here to provide a partial summary of the literature. The papers by Rao (1978, 1982, 1989, 1991, 1994) provide a basis for the theory. Chang and Rao (1986) develop the necessary bi-measure theory. A study of sample path behavior for harmonizable processes is considered by Swift (1996b). Some results on moving average representations were obtained by Mehlman (1992). The structure of harmonizable isotropic random fields and some applications has been considered by Swift (1994, 1995, 1996a). Second order processes with harmonizable increments has been investigated also by Swift (1996c). The forthcoming book by Kakihara gives a general treatment of multidimensional second order processes which include the harmonizable class.

3 OSCILLATORY HARMONIZABLE PROCESSES

M.B. Priestley (1981), introduced and studied a generalization of the class of stationary processes. This generalization is given by:

DEFINITION 3.1 A stochastic process $X : \mathbb{R} \to L_0^2(P)$ is *oscillatory* if it has representation

$$X(t) = \int_R A(t, \lambda)e^{i\lambda t} dZ(\lambda)$$

where $Z(\cdot)$ is a stochastic measure with orthogonal increments and

$$A(t, \lambda) = \int_R e^{itx} H(\lambda, dx)$$

with $H(\cdot, B)$ a Borel function on $\mathbb{R}$, $H(\lambda, \cdot)$ a signed measure and $A(t, \lambda)$ having an absolute maximum at $\lambda = 0$ independent of t.

Using this representation the covariance of an oscillatory process is

$$r(s, t) = \int_{\mathbb{R}} A(s, \lambda)\overline{A(t, \lambda)} e^{i\lambda(s-t)} d\Phi(\lambda).$$

The idea of definition 2.3 provides the motivation for the following definition:

DEFINITION 3.2 A stochastic process $X : \mathbb{R} \to L_0^2(P)$ is *oscillatory weakly harmonizable*, if its covariance has representation

$$r(s, t) = \int_{\mathbb{R}} \int_{\mathbb{R}} A(s, \lambda)\overline{A(t, \lambda')} e^{i\lambda s - i\lambda' t} dF(\lambda, \lambda')$$

where $F(\cdot, \cdot)$ is a function of bounded Fréchet variation, and

$$A(t, \lambda) = \int_{\mathbb{R}} e^{itx} H(\lambda, dx)$$

with $H(\cdot, B)$ a Borel function on $\mathbb{R}, H(\lambda, \cdot)$ a signed measure and $A(t, \lambda)$ having an absolute maximum at $\lambda = 0$ independent of t.

Note that if $A(t, \lambda) = 1$, this class coincides with the weakly harmonizable processes. As Priestley's definition provides an extension to the class of stationary processes, definition 3.2 provides an extension to the class of weakly harmonizable processes.

Observe, further, that in this definition, for $F(\cdot, \cdot)$ concentrating on the diagonal, $\lambda = \lambda'$, the oscillatory processes are obtained. Thus the oscillatory harmonizable processes also provide an extension to the class introduced by Priestley, which we will now term *oscillatory stationary* .

Using this definition, it is possible to obtain the spectral representation of an oscillatory harmonizable process $X(\cdot)$.

THEOREM 3.1 *The spectral representation of an oscillatory weakly harmonizable stochastic process is:*

$$X(t) = \int_{\mathbb{R}} A(t,\lambda) e^{i\lambda t} dZ(\lambda)$$

where $Z(\cdot)$ is a stochastic measure satisfying

$$E(Z(B_1)\overline{Z(B_2)}) = F(B_1, B_2)$$

with $F(\cdot,\cdot)$ a function of bounded Fréchet variation.

Proof: Let $X(\cdot)$ be an oscillatory weakly harmonizable process. Then, the covariance $r(\cdot,\cdot)$ has representation

$$r(s,t) = \int_{\mathbb{R}} \int_{\mathbb{R}} A(s,\lambda)\overline{A(t,\lambda')} e^{i\lambda s - i\lambda' t} dF(\lambda,\lambda').$$

Applying a form of Karhunen's theorem, (Yaglom, 1987, volume 2, pages 33 - 41) gives the spectral representation of $X(\cdot)$ as

$$X(t) = \int_{\mathbb{R}} A(t,\lambda) e^{i\lambda t} dZ(\lambda),$$

which is the desired result. □

The following condition on the signed measure H, for oscillatory strongly harmonizable processes show these processes are actually a subclass of the strongly harmonizable processes. A similar result was obtained by R. Joyeux (1987), for the oscillatory stationary processes.

THEOREM 3.2 *If $X(\cdot)$ is an oscillatory strongly harmonizable process with*

$$\int_{\mathbb{R}} \mid H(\lambda, dx) \mid < \infty$$

uniformly in $\lambda \in \mathbb{R}$, then $X(\cdot)$ is strongly harmonizable.

Proof: Let

$$\tilde{Z}(A) = \int_{\mathbb{R}} H(\lambda, A-\lambda) dZ(\lambda)$$

where A is a Borel set of $\mathbb{R}$ and

$$A - \lambda = \{x - \lambda : x \in A\}.$$

$\tilde{Z}(\cdot)$ is a stochastic measure since $H(\lambda,\cdot)$ is a signed measure, and uniformly bounded by K.

Now set

$$\tilde{X}(t) = \int_{\mathbb{R}} e^{i\lambda t} d\tilde{Z}(\lambda).$$

Claim: $\tilde{X}(\cdot)$ is a strongly harmonizable process.

If one lets $\tilde{Z}(A,B) = E(\tilde{Z}(A)\overline{\tilde{Z}(B)})$ A, B Borel sets of $\mathbb{R}$, it must be shown that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \mid \tilde{F}(d\omega, d\omega') \mid < \infty.$$

Now

$$\begin{aligned} E(\tilde{Z}(d\omega)\overline{\tilde{Z}(d\omega')}) &= \int_{\mathbb{R}} \int_{\mathbb{R}} H(\lambda, d(\omega-\lambda)) \overline{H(\lambda', d(\omega'-\lambda))} E(Z(d\lambda)\overline{Z(d\lambda)}) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} H(\lambda, d(\omega-\lambda)) \overline{H(\lambda', d(\omega'-\lambda))} F(d\lambda, d\lambda') \end{aligned}$$

where $F(A,B) = E(Z(A)\overline{Z(B)})$ is of finite Vitali variation since $X(t)$ is strongly harmonizable.
Thus,

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} | \tilde{F}(d\omega, d\omega') | &= \int_{\mathbb{R}}\int_{\mathbb{R}} \left| \int_{\mathbb{R}}\int_{\mathbb{R}} H(\lambda, d(\omega-\lambda))\overline{H(\lambda', d(\omega'-\lambda'))}F(d\lambda, d\lambda') \right| \\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} | H(\lambda, d(\omega-\lambda))\overline{H(\lambda', d(\omega'-\lambda'))}F(d\lambda, d\lambda') | \\
&< \infty
\end{aligned}$$

since $| H | (\lambda, \mathbb{R})$ is bounded, proving the claim.
Now

$$\begin{aligned}
\tilde{X}(t) &= \int_{\mathbb{R}} e^{i\omega t}\tilde{Z}(d\omega) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\omega t} H(\lambda, d(\omega-\lambda))Z(d\lambda) \\
&= \int_{\mathbb{R}} e^{i\lambda t}\int_{\mathbb{R}} e^{i(\omega-\lambda)t} H(\lambda, d(\omega-\lambda))Z(d\lambda) \\
&= \int_{\mathbb{R}} e^{i\lambda t} A(t,\lambda)Z(d\lambda) \\
&= X(t).
\end{aligned}$$

So $X(t)$ is strongly harmonizable, which completes the proof of the theorem. □
An additional class of processes related to the oscillatory processes is given by:
DEFINITION 3.3 An oscillatory weakly harmonizable stochastic process $X : \mathbb{R} \to L_0^2(P)$ is *ε-slowly changing weakly harmonizable* if

$$B(\lambda) = \int_{\mathbb{R}} | x | \, | H | (\lambda, dx) \leq \epsilon, \ \forall \ \lambda \in \mathbb{R}.$$

Slowly changing stationary processes where first considered by Priestley (1981) and are of interest not only in engineering but also in economics. Priestley showed that it is possible to define a spectral measure for these processes. The class of slowly changing harmonizable processes introduced above extend the class of slowly changing stationary processes. The following corollary shows that it is possible to consider a similar concept for the slowly changing harmonizable class.
COROLLARY 3.1 *Slowly changing strongly harmonizable processes form a subclass of strongly harmonizable processes.*
Proof: The assumption is

$$\int_{\mathbb{R}} | x | \, | H | (\lambda, dx) \leq \varepsilon \ \forall \lambda \in \mathbb{R}.$$

Claim:

$$\int_{\mathbb{R}} | H | (\lambda, dx) < \infty.$$

In fact,

$$\begin{aligned} \mid H \mid (\lambda, I\!R) &= \int_{I\!R} \mid H \mid (\lambda, dx) \\ &= \int_{|x|<1} \mid H \mid (\lambda, dx) + \int_{|x|\geq 1} \mid H \mid (\lambda, dx) \\ &\leq K + \int_{|x|\geq 1} \mid H \mid (\lambda, dx) \leq K + \int_{|x|\geq 1} \mid x \mid\mid H \mid (\lambda, dx) \\ &\leq K + \int_{I\!R} \mid x \mid\mid H \mid (\lambda, dx) \leq K + \epsilon < \infty \end{aligned}$$

which is the claim. Now since K is finite, by the theorem $X(t)$ is strongly harmonizable, proving the corollary. □

4 AN OPERATOR CHARACTERIZATION

Using oscillatory harmonizable processes, it is possible to obtain a representation of a broader class of processes on $I\!R$.

DEFINITION 4.1 Let $\boldsymbol{S}$ be a locally compact space with $\mathcal{B}_0$ as the σ-ring generated by the bounded Borel sets of $\boldsymbol{S}$. If T is any index set, $\{X(t), t \in T\} \subset L_0^2(P)$ a second order process, r its covariance and $\tilde{\beta} : \mathcal{B}_0 \times \mathcal{B}_0 \to \mathbb{C}$, a bimeasure having locally bounded Fréchet variation then $X(\cdot)$ is said to be *(locally) weakly of class (C)* when $\tilde{\beta}$ is positive definite and

$$r(s,t) = \int_{\boldsymbol{S}} \int_{\boldsymbol{S}} g_s(\lambda)\overline{g_t(\lambda')}\beta(d\lambda, \lambda') \ \ (s,t) \in T \times T \text{ (strict MT-integral)}$$

where $g_t : \boldsymbol{S} \to \mathbb{C}, t \in T$ a family of Borel functions for which the integral exists. If $\tilde{\beta}$ has locally finite Vitali variation, then the process is termed of *class (C)* relative to $\{g_s, S \in T\}$ and $\tilde{\beta}$.

Weak class (C) processes are considered extensively in Chang and Rao (1986). Oscillatory harmonizable processes affords this broad class of processes to have a simple representation on $I\!R$ as seen in the following

PROPOSITION 4.1 *The class of oscillatory weakly harmonizable processes $\{X(t), t \in I\!R\} \subset L_0^2(P)$ coincides with the class of weak class (C) processes indexed on $I\!R$.*

Proof: This follows by setting $g_s(\lambda) = e^{is\lambda}A_s(\lambda)$ in the definition of weak class (c) processes, since $F(\cdot,\cdot)$ always has finite Fréchet variation. □

Using this simple identification, an operator representation of weak class (C) processes indexed on $I\!R$ is possible. This result is an extension of that given in Chang and Rao (1988) for the oscillatory stationary class.

THEOREM 4.1 *$X(\cdot)$ is an oscillatory weakly harmonizable process iff it is representable as*

$$X(t) = a(t)T(t)Y(0), \quad t \in I\!R,$$

where

$$Y_0 = Y(0)$$

is some point in

$$H(X) = \overline{sp}\{X(t), t \in \mathbb{R}\}$$

with $a(t)$ a densely defined closed operator in $H(X)$ for each $t \in \mathbb{R}$ and

$$\{T(s), s \in \mathbb{R}\}$$

a weakly continuous family of positive definite contractive operators in $H(X)$ which commutes with each $a(t)$, $t \in \mathbb{R}$.

Proof: Suppose $X(t)$ is oscillatory weakly harmonizable, then

$$X(t) = \int_{\mathbb{R}} A(t, \lambda) e^{i\lambda t} dZ(\lambda)$$

where $Z(\cdot)$ is a stochastic measure satisfying

$$E(Z(B_1)\overline{Z(B_2)}) = F(B_1, B_2)$$

with $F(\cdot, \cdot)$ of bounded Fréchet variation.

Let

$$Y(t) = \int_{\mathbb{R}} e^{i\lambda t} dZ(\lambda)$$

then $Y(\cdot)$ is weakly harmonizable.

Now by a theorem of Rao (1982) there is a weakly continuous family of positive definite contractive operators

$$\{T(t), t \in \mathbb{R}\}$$

on

$$H(X) = \overline{sp}\{X(t), t \in \mathbb{R}\}$$

so that

$$Y(t) = T(t)Y_0.$$

Using the spectral theorem for this family of operators, (cf. Rao, 1982)

$$T(t) = \int_{\mathbb{R}} e^{i\lambda t} \tilde{E}(d\lambda), \quad t \in \mathbb{R}$$

where $\{\tilde{E}(\cdot), \boldsymbol{B}\}$ is the resolution of the identity of $\{T(t), t \in \mathbb{R}\}$ with $\boldsymbol{B}$ as the Borel σ-algebra of $\mathbb{R}$. So

$$Z(A) = \tilde{E}(A)Y_0, \quad A \in \boldsymbol{B}.$$

Now define

$$a(t) = \int_{\mathbb{R}} A(t, \lambda) \tilde{E}(d\lambda) \quad t \in \mathbb{R}.$$

It follows that $a(t)$ is closed and densely defined on $H(X)$ with its domain containing

$$\{Y(s), \ s \in \mathbb{R}\}.$$

Now since $T(t)$ and $\tilde{E}(D)$ commute for all t and D, then $a(t)$ and $\{\tilde{E}(D), \ D \in \boldsymbol{B}\}$ commute, so that $a(t)$ and $\{T(s), \ s \in \mathbb{R}\}$ commute for each t.

Thus

$$\begin{aligned} a(t)T(t)Y_0 &= a(t)\left(\int_{\mathbb{R}} e^{i\omega t}\tilde{E}(d\omega)Y_0\right) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} A(t,\lambda)\tilde{E}(d\lambda)\left(\int_{\mathbb{R}} e^{i\omega t}E(d\omega)Y_0\right) \\ &= \int_{\mathbb{R}} A(t,\lambda)e^{i\lambda t}\tilde{E}(d\lambda)Y_0 \\ &= \int_{\mathbb{R}} A(t,\lambda)e^{i\lambda t}dZ(\lambda) \\ &= X(t) \end{aligned}$$

where (3) follows since

$$\tilde{E}(D)\int_{\mathbb{R}} e^{i\omega t}Y_0 = \int_D e^{i\omega t}\tilde{E}(d\omega)Y_0, \quad D \in \boldsymbol{B}.$$

Thus if $X(t)$ is oscillatory weakly harmonizable, then

$$X(t) = a(t)T(t)Y(0)$$

where

$$Y_0 = Y(0)$$

is some point in

$$H(X) = \overline{sp}\{X(t), t \in \mathbb{R}\}$$

$a(t)$ is a densely defined closed operator in $H(X)$ for each $t \in \mathbb{R}$ and

$$\{T(s), s \in \mathbb{R}\}$$

is a weakly continuous family of positive definite contractive operators in $H(X)$ which commutes with each $a(t)$, $t \in \mathbb{R}$.

Now suppose $X(t)$ can be represented as

$$X(t) = a(t)T(t)Y(0)$$

with $a(t)$, $T(t)$, and $Y(0)$ as stated in the theorem.

Then, using a classical result of von Neumann and F. Riesz (1990), $a(t)$ is a function $g(t)$ of $T(t)$ and further

$$a(t) = g(t)T(t) = \int_{\mathbb{R}} g(t,\lambda)\tilde{E}(d\lambda).$$

Thus

$$\begin{aligned} X(t) &= a(t)T(t)Y(0) \\ &= \int_{\mathbb{R}} g(t,\lambda)\tilde{E}(d\lambda)\int_{\mathbb{R}} e^{i\omega t}\tilde{E}(d\omega)Y_0 \end{aligned}$$

$$= \int_{\mathbb{R}} e^{i\lambda t} g(t,\lambda)\tilde{E}(d\lambda)Y_0$$

$$= \int_{\mathbb{R}} e^{i\lambda t} g(t,\lambda)Z(d\lambda)$$

but this is the representation of a oscillatory weakly harmonizable process. □

ACKNOWLEDGEMENTS

The author expresses his thanks to Professor M.M. Rao for his advice and encouragement during the work of this project. The author also expresses his gratitude to the Mathematics department at Western Kentucky University for release time during the Spring 1995 semester, during which this work was completed.

REFERENCES

1. D. K. Chang and M. M. Rao. (1986). Bimeasures and Nonstationary Processes. Real and Stochastic Analysis John Wiley and Sons, New York, p. 7.
2. D. K. Chang and M. M. Rao. (1988). Special Representations of Weakly Harmonizable Processes. Stoc. Anal. and Appl., 6(2):169.
3. I. I. Gihman and A. V. Skorohod. (1974). The Theory of Stochastic Processes I. Springer-Verlag, New York.
4. R. Joyeux. (1987). Slowly Changing Processes and Harmonizability. J. Time Series Anal.. 8, No.4.
5. Y. Kakihara. Multidimensional Second Order Stochastic Processes. World Scientific, In preparation.
6. M. H. Mehlman. (1992). Prediction and Fundamental Moving Averages for Discrete Multidimensional Harmonizable Processes. J. Multiv. Anal., 43, No.1.
7. M. B. Priestley. (1981). Spectral Analysis and Time Series. Vol. 1 and 2, Academic Press, London.
8. M. M. Rao. (1978). Covariance Analysis of Non Stationary Time Series, Developments in Statistics. 1, p. 171.
9. M. M. Rao. (1982). Harmonizable Processes: Structure Theory. L'Enseign Math, 28, p. 295.
10. M. M. Rao. (1989). Harmonizable Signal Extraction, Filtering and Sampling. Topics in Non-Gaussian Signal Processing. (E.J. Wegman, S.C. Schwartz, J.B. Thomas, eds.), Springer-Verlag, New York.
11. M. M. Rao. (1991). Sampling and Prediction for Harmonizable Isotropic Random Fields. J. Comb., Info. and Sys. Sci.. 16 No. 2- 3 p. 207.
12. M. M. Rao. (1994). Harmonizable processes and inference: unbiased prediction for stochastic flows. J. Stat. Plan. and Infer.. 39 p. 187.
13. F. Riesz & B. Sz-Nagy. (1990). Functional Analysis. Dover, New York.
14. R. Swift. (1994). The Structure of Harmonizable Isotropic Random Fields. Stoch. Anal. and Appl., 12, No. 5, p. 583.

15. R. Swift. (1995). Representation and Prediction for Locally Harmonizable Isotropic Random Fields. J. Appl. Math. and Stoch. Anal.. VIII, p. 101.
16. R. Swift. (1996a). A Class of Harmonizable Isotropic Random Fields. J. Comb., Info. and Sys. Sci., (to appear).
17. R. Swift. (1996b). Almost Periodic Harmonizable Processes. Georgian Math. J., (to appear).
18. R. Swift. (1996c). Stochastic Processes with Harmonizable Increments. J. Comb., Info. and Sys. Sci., (to appear).
19. A.M. Yaglom. (1987). Correlation Theory of Stationary and Related Random Functions. Vol. 1 and 2, Springer-Verlag, New York.