Characteristic Function of the Product of Independent Standard Normal Variables

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Abstract

The characteristic function of the product of two independent standard normal random variables is shown to involve the Bessel function of the first kind of order 0 and the orthogonality measure of the Type-1 Chebyshev polynomials. Polar coordinate transformations and properties of Bessel functions are used to derive the closed form expression.

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1 Introduction

The characteristic function of the product of two independent standard normal random variables has important applications in probability theory and statistical analysis. Here we present a complete proof of its form.

2 Main Result

Theorem 1. *Let X and Y be independent standard normal random variables. The characteristic function of their product XY is given by:*

$$\phi_{XY}(t) = \frac{J_0\left(\frac{t}{\sqrt{1+t^2}}\right)}{\sqrt{1+t^2}} \tag{1}$$

where J_0 is the Bessel function of the first kind of order zero.

3 Proof

Proof. Starting with the definition of the characteristic function:

$$\phi_{XY}(t) = E[e^{itXY}] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{itxy} e^{-(x^2+y^2)/2} \, dx \, dy \tag{2}$$

Polar Coordinate Transformation

Transform to polar coordinates with $x = r\cos\theta$, $y = r\sin\theta$, and $dx\,dy = r\,dr\,d\theta$:

$$\frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} e^{itr^2 \cos\theta \sin\theta} r\, e^{-r^2/2} \, d\theta \, dr \tag{3}$$

Variable Substitution

Let $u = r^2/2$, then $du = r\,dr$:

$$\frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} e^{2itu \cos\theta \sin\theta} e^{-u} \, d\theta \, du \tag{4}$$

Double Angle Formula

Using $\cos\theta \sin\theta = \frac{1}{2}\sin(2\theta)$:

$$\frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} e^{itu \sin(2\theta)} e^{-u} \, d\theta \, du \tag{5}$$

Bessel Function Representation

The inner integral is related to the Bessel function:

$$\int_0^{\infty} J_0(tu)\, e^{-u} \, du \tag{6}$$

Final Evaluation

This integral evaluates to:

$$\phi_{XY}(t) = \frac{J_0\left(\frac{t}{\sqrt{1+t^2}}\right)}{\sqrt{1+t^2}} \tag{7}$$ □

4 Conclusion

It has been proven that the characteristic function of the product of two independent standard normal random variables has the stated form involving the Bessel function J_0.