The Hilbert-Schmidt Theory of Integral Equations With Symmetric Kernel

[1, 2.2 Sec 19-24]

1 Ascoli-Arzelà theorem

We first consider the integral equation

$$\psi(x) = \lambda \int_a^b G(x, \xi)\, \psi(\xi)\, d\xi \tag{1}$$

which is equivalent to a boundary value problem for $L\,\psi(y) = -\lambda\,\psi(y)$, as was shown in §1. Multiply both sides of (1) by $\sqrt{r(x)}$. (It is assumed that $r(x) > 0$, see Part 16.) Then, setting

$$z(x) = \sqrt{r(x)}\,\psi(x) \tag{2}$$

$$K(x, \xi) = \sqrt{r(x)}\,G(x, \xi)\sqrt{r(\xi)} \tag{3}$$

we obtain

$$z(x) = \lambda \int_a^b K(x, \xi)\, z(\xi)\, d\xi \tag{4}$$

Here $K(x, \xi)$ is a real-valued continuous function which has the symmetry property

$$K(x, \xi) = K(\xi, x) \tag{5}$$

just as $G(x, \xi)$.

If the integral equation (3) with the symmetric kernel $K(x, \xi)$ has a non-trivial continuous solution $z(x)$ for some value of parameter λ, then such z is called an eigenvalue and $z(x)$ an eigenfunction belonging to the eigenvalue λ. The Hilbert-Schmidt theory for this eigenvalue problem is based upon the following existence theorem:

Theorem 1

If $K(x, \xi)$ is a real-valued continuous symmetric kernel and does not vanish identically, then the integral equation (3) with the kernel $K(x, \xi)$ possesses at least one non-zero eigenvalue.

We shall prove, in this paragraph, the *Ascoli-Arzela theorem*, which will play a fundamental role in the proof of Theorem 1.

By means of the kernel $K\,(x,\xi)$, every complex-valued continuous function $f(x)$ defined on the interval $[a,b]$ is transformed into a continuous function

$$(K\,f)(x) = \int_a^b K\,(x,\xi)\; f(\xi)\, d\,\xi \tag{6}$$

on the same interval. This is so because

$$\left|\int_a^b (K\,(x_1,\xi) - K\,(x_2,\xi))\; f(\xi)\, d\,\xi\right| \leq \left\{\sup_{\xi\in[a,b]} |f(\xi)|\right\} \int_a^b |K\,(x_1,\xi) - K\,(x_2,\xi)|\, d\,\xi \tag{7}$$

tends to zero as $|x_1 - x_2| \to 0$ because of the uniform continuity of $K\,(x,\xi)$. We denote by K the transformation

$$(K\,f)(x) = \int_a^b K\,(x,\xi)\; f(\xi)\, d\,\xi \tag{8}$$

Obviously the transformation K satisfies

$$K\,(f_1 + f_2) = K\,f_1 + K\,f_2 \tag{9}$$

$$K\,(a\,f) = a\,(K\,f) \quad \text{(for any constant } a\text{)} \tag{10}$$

Accordingly, the transformation K is a *linear operator*. For each continuous function $f(x)$, we define its *norm* by

$$||\,f\,|| = \sqrt{\int_a^b |f(x)|^2\, d\,x} \tag{11}$$

Then we can prove that

$$|(K\,f)(x)| \leq ||\,f\,|| \sqrt{\int_a^b |K\,(x,\xi)|^2\, d\,\xi} \tag{12}$$

$$|(K\,f)(x_1) - (K\,f)(x_2)| \leq ||\,f\,|| \sqrt{\int_a^b |K\,(x_1,\xi) - K\,(x_2,\xi)|^2\, d\,\xi} \tag{13}$$

In fact, these are easily derived from the *Schwarz inequality*

$$\left|\int_a^b g(\xi)\, h(\xi)\, d\,\xi\right|^2 \leq \int_a^b |g(\xi)\,|^2 d\,\xi \int_a^b |\,h(\xi)|^2\, d\,\xi \tag{14}$$

which will be proved in Part 20. Accordingly, we obtain the following theorem.

Theorem 2

Let $\{f_n(x)\}$ be a sequence of continuous functions on $[a,b]$. Let $\{g_n(x)\}$ be $\{K f_n(x)\}$. Then $\{g_n(x)\}$ and $||\, f_n||$, $n=1,2,3,\ldots$ satisfies

$$\sup_{a\le\xi\le b} |g_n(\xi)| < \infty \tag{15}$$

$$\lim_{n\to\infty} \sup_{\substack{a\le\xi\le b\\ |x_1-x_2|<\delta}} |g_n(x_1)-g_n(x_2)| = 0 \tag{16}$$

Remark 3. A set of functions $\{g_n(x)\}$ is said to be *equibounded* on $[a,b]$ if it satisfies (19.8), and *equicontinuous* on $[a,b]$ if it satisfies (9). It should be noted that theorem 2 does not hold for all linear operators. For example, an operator T defined by

$$f(x) \to (T f)(x) = \alpha(x)\, f(x) \tag{17}$$

where $\alpha(x)$ is a continuous function does not satisfy (8) nor (9).

Owing to the properties (19.8') and (19.9') of the operator K, we can apply the Ascoli-Arzela theorem, which reads as follows:

Theorem 4

Let $\{g_n(x)\}$ be a sequence of continuous functions. If $\{g_n(x)\}$ satisfies the conditions (19.8') and (19.9'), then we can choose a subsequence $\{g_{n_k}(x)\}$ which converges uniformly on the interval $[a,b]$

Proof. Since the set of all rational numbers in the interval $[a,b]$ is denumerable, it may be arranged as $r_1, r_2, r_3, \ldots$

On account of (19.8'), the sequence of numbers

$$g_1(r_1), g_2(r_1), g_3(r_1), \ldots \tag{18}$$

is bounded; hence by the Bolzano-Weierstrass theorem there exists a convergent subsequence

$$g_{1'}(r_1), g_{2'}(r_1), g_{3'}(r_1), \ldots \tag{19}$$

Similarly, we can select from the sequence of numbers

$$g_{1'}(r_2), g_{2'}(r_2), g_{3'}(r_2), \ldots \tag{20}$$

a convergent subsequence

$$g_{1''}(r_2),\, g_{2''}(r_2),\, g_{3''}(r_2), \ldots \tag{21}$$

Repeating this procedure, we can select from each sequence of functions $n = 1, 2, \ldots$

$$g_{1^{(n-1)}}(r_n),\, g_{2^{(n-1)}}(r_n),\, g_{3^{(n-1)}}(r_n), \ldots, \quad (g_{k^{(0)}}(x) = g_k(x)) \tag{22}$$

a subsequence of functions

$$g_{1^{(n)}}(x),\, g_{2^{(n)}}(x),\, g_{3^{(n)}}(x), \ldots \quad (n = 1, 2, 3, \ldots) \tag{23}$$

which converges at the points $x = r_1, r_2, \ldots, r_n$. Accordingly, the subsequence

$$g_{1'}(x),\, g_{2'}(x),\, g_{3''}(x), \ldots, g_{n^n}(x), \ldots \tag{24}$$

of the original sequence $\{g_n(x)\}$ converges for every rational number $r = r_1, r_2, \ldots, r_n, \ldots$ (This method of selection (19.13) is an example of the so-called *diagonal method*.)

Next, we shall prove that the sequence (19.13) converges uniformly on the interval $[a, b]$. For the sake of simplicity, we shall denote the sequence (19.13) by $\{g_n(x)\}$. According to (19.9'), there exists, for any positive number $\varepsilon > 0$, a positive number $\delta = \delta(\varepsilon) > 0$ such that

$$|g_n(x_1) - g_n(x_2)| \leq \varepsilon \text{ for all } n$$

whenever $|x_1 - x_2| \leq \delta$.

On the other hand, the set of all rational numbers is dense in the interval $[a, b]$, hence for the above δ, there exists a number $N = N(\delta)$ such that

$$\min_{1 \leq k \leq N} |x - r_k| < \delta$$

for every number x in the interval. Further, since $\{g_n(x)\}$ converges at the points $x = r_1$, $r_2, \ldots, r_N$, there exists, for the $\varepsilon > 0$, a number $M = M(\varepsilon)$ such that

$$m, n \geq M \text{ implies } |g_n(r_k) - g_m(r_k)| < \varepsilon \quad (1 \leq k \leq N).$$

Therefore, for each x, there exists a rational number r_k $(1 \leq k \leq N)$ such that

$$|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(r_k)| + |g_n(r_k) - g_m(r_k)| + |g_m(r_k) - g_m(x)| \leq 3\,\varepsilon$$

wherever $m, n \geq M$. This means that $\{g_n(x)\}$ converges uniformly on the interval $[a, b]$, q.e.d.

Bibliography

[1] 吉田 耕作(Kōsaku Yosida). *Lectures on Differential and Integral Equations*, volume X of *Pure and Applied Mathematics*. Interscience Publishers/John Wiley Sons Inc. , New York/London/Sydney, 1960.