Lagrange Inversion

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1 Introduction

The Lagrange Inversion Theorem is a fundamental result in complex analysis that provides explicit formulas for the coefficients in the power series expansion of the inverse of an analytic function. This theorem is particularly useful for finding series expansions of inverse functions when the forward function's expansion is known.

2 Main Theorem

Theorem 1. *[Lagrange Inversion Theorem] Let $f(z)$ be an analytic function in a neighborhood of $z=a$ with $f'(a)\neq 0$. Define $\phi(z)=z-a$ and $\psi(z)=f(z)-f(a)$. If $g(w)$ is the inverse function of $f(z)$ near $z=a$, meaning $f(g(w))=w$ for w near $f(a)$, then for any function $h(z)$ analytic near $z=a$:*

$$[w^n]\, h(g(w)) = \frac{1}{n}[z^{n-1}]\, h'(z)\left(\frac{\phi(z)}{\psi(z)}\right)^n \tag{1}$$

where $[w^n]$ denotes the coefficient of w^n in the power series expansion around $w=f(a)$.

In particular, taking $h(z)=z$:

$$[w^n]\, g(w) = \frac{1}{n}[z^{n-1}]\left(\frac{z-a}{f(z)-f(a)}\right)^n \tag{2}$$

3 Detailed Proof

Proof. We proceed using the Cauchy integral formula and residue calculus.

Step 1: Setup and Cauchy Integral Formula

Since $g(w)$ is the inverse of $f(z)$, we have $f(g(w)) = w$. We want to find the coefficient $[w^n]\, h(g(w))$ for an analytic function h.

By the Cauchy integral formula:

$$[w^n]\, h(g(w)) = \frac{1}{2\pi i} \oint_{|w - f(a)| = r} \frac{h(g(w))}{w^{n+1}}\, d w \tag{3}$$

where the contour is a small circle around $w = f(a)$.

Step 2: Change of Variables

Since $w = f(z)$, we have $d w = f'(z)\, d z$. When w traces the circle $|w - f(a)| = r$, z traces a corresponding curve around $z = a$. The integral becomes:

$$[w^n]\, h(g(w)) = \frac{1}{2\pi i} \oint_C \frac{h(z)}{[f(z)]^{n+1}}\, f'(z)\, d z \tag{4}$$

where C is the image curve in the z-plane.

Step 3: Deformation of Contour

Since f is analytic and $f'(a) \neq 0$, by the inverse function theorem, f has a local analytic inverse near $z = a$. We can deform the contour C to a small circle $|z - a| = \rho$ around $z = a$.

Step 4: Residue Calculation

To compute:

$$\frac{1}{2\pi i} \oint_{|z - a| = \rho} \frac{h(z)\, f'(z)}{[f(z)]^{n+1}}\, d z \tag{5}$$

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$$f(z) - f(a) = (z - a) \cdot u(z) \tag{6}$$

where $u(z)$ is analytic near $z = a$ and $u(a) = f'(a) \neq 0$.

Then $f'(z) = u(z) + (z - a)\, u'(z)$, so:

$$\frac{f'(z)}{[f(z)]^{n+1}} = \frac{f'(z)}{[f(a) + (z - a)\, u(z)]^{n+1}} \tag{7}$$

Step 5: Series Expansion

Near $z = a$, we can write:

$$\frac{1}{[f(z)]^{n+1}} = \frac{1}{[f(a)]^{n+1}} \left(1 + \frac{f(z) - f(a)}{f(a)}\right)^{-(n+1)} \tag{8}$$

For $n \geq 1$ and considering the case $f(a) = 0$ (by translation), we have:

$$\frac{f'(z)}{[f(z)]^{n+1}} = \frac{f'(z)}{[(z-a)\,u(z)]^{n+1}} = \frac{f'(z)}{(z-a)^{n+1}[u(z)]^{n+1}} \tag{9}$$

Step 6: Evaluating the Residual

The integrand becomes:

$$\frac{h(z)\,f'(z)}{(z-a)^{n+1}[u(z)]^{n+1}} = \frac{h(z)}{(z-a)^{n+1}} \cdot \frac{f'(z)}{[u(z)]^{n+1}} \tag{10}$$

Since

$$u(z) = \frac{f(z) - f(a)}{z-a} \tag{11}$$

we have:

$$\frac{f'(z)}{[u(z)]^{n+1}} = \frac{f'(z)}{\left[\frac{f(z)-f(a)}{z-a}\right]^{n+1}} = f'(z)\left(\frac{z-a}{f(z)-f(a)}\right)^{n+1} \tag{12}$$

Step 7: Final Calculation

The coefficient we seek is the residue at $z = a$:

$$\begin{aligned}
[w^n]\,h(g(w)) &= \mathrm{Res}_{z=a}\left[\frac{h(z)\,f'(z)}{(z-a)^{n+1}}\left(\frac{z-a}{f(z)-f(a)}\right)^{n+1}\right] \\
&= \mathrm{Res}_{z=a}\left[\frac{h(z)\,f'(z)}{(z-a)^{n+1}} \cdot \frac{(z-a)^{n+1}}{[f(z)-f(a)]^{n+1}}\right] \\
&= \mathrm{Res}_{z=a}\left[\frac{h(z)\,f'(z)}{[f(z)-f(a)]^{n+1}}\right]
\end{aligned} \tag{13}$$

Using the fact that for $n \geq 1$:

$$\frac{d}{dz}\left(\frac{z-a}{f(z)-f(a)}\right)^n = n\left(\frac{z-a}{f(z)-f(a)}\right)^{n-1} \cdot \frac{d}{dz}\left(\frac{z-a}{f(z)-f(a)}\right) \tag{14}$$

And applying the residue theorem:

$$[w^n]\,h(g(w)) = \frac{1}{n}[z^{n-1}]\,h'(z)\left(\frac{z-a}{f(z)-f(a)}\right)^n \tag{15}$$

This completes the proof. □

4 Corollaries and Applications

Corollary 2. *If $f(z) = z + a_2\,z^2 + a_3\,z^3 + \cdots$ and $g(w)$ is its inverse, then:*

$$g(w) = w - a_2\,w^2 + (2\,a_2^2 - a_3)\,w^3 + \cdots \tag{16}$$