Proof of Uniform Convergence of a Sequence of Orthogonal Functions to the Bessel function of the First Kind of Order 0

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Lemma 1

The functions

$$\psi_n(y) = \sqrt{\frac{4\,n+1}{y}}\,(-1)^n\,J_{2n+\frac{1}{2}}(y) \tag{1}$$

are orthonormal over the interval 0 to ∞, i.e.,

$$\int_0^\infty \psi_j(y)\,\psi_k(y)\;d\,y = \delta_{jk} \tag{2}$$

where δ_{jk} is the Kronecker delta.

Proof. *Consider the integral*

$$I = \int_0^\infty \psi_j(y)\,\psi_k(y)\;d\,y \tag{3}$$

which can be expressed as

$$I = \int_0^\infty \sqrt{\frac{4\,j+1}{y}}\,(-1)^j\,J_{2j+\frac{1}{2}}(y)\sqrt{\frac{4\,k+1}{y}}\,(-1)^k\,J_{2k+\frac{1}{2}}(y)\;d\,y \tag{4}$$

This simplifies to

$$I = \sqrt{(4\,j+1)\,(4\,k+1)}\,(-1)^{j+k}\int_0^\infty \frac{J_{2j+\frac{1}{2}}(y)\,J_{2k+\frac{1}{2}}(y)}{y}\;d\,y \tag{5}$$

Using the orthogonality relation for Bessel functions,

$$\int_0^\infty \frac{J_\nu(y)\,J_\mu(y)}{y}\;d\,y = \frac{\delta_{\nu\mu}}{2\,\nu} \tag{6}$$

where $\nu = 2j + \frac{1}{2}$ *and* $\mu = 2k + \frac{1}{2}$*, we find*

$$\int_0^\infty \frac{J_{2j+\frac{1}{2}}(y)\, J_{2k+\frac{1}{2}}(y)}{y}\, d y = \frac{\delta_{jk}}{4j+1} \tag{7}$$

Substituting this result back, we have

$$I = \sqrt{(4j+1)\,(4k+1)}\,(-1)^{j+k} \frac{\delta_{jk}}{4j+1} \tag{8}$$

For $j \neq k$*,* $\delta_{jk} = 0$*, yielding* $I = 0$*. For* $j = k$*,* $\delta_{jk} = 1$*, giving*

$$I = \frac{\sqrt{(4j+1)\,(4j+1)}}{4j+1} = 1 \tag{9}$$

Hence, $\psi_j(y)$ *and* $\psi_k(y)$ *are orthonormal.* □

Theorem 2

The partial sums $S_N(y)$ defined by

$$S_N(y) = \sqrt{\frac{2}{\pi}} \sum_{k=0}^{N} \frac{\psi_k(y)}{\left(\frac{1}{2}+k\right)_{\frac{1}{2}}^2} \tag{10}$$

uniformly converge to the Bessel function of the first kind of order zero, $J_0(y)$, as $N \to \infty$.

Proof. *From the orthonormality established in Lemma (1), consider the squared difference between* $J_0(y)$ *and* $S_N(y)$*:*

$$|J_0(y) - S_N(y)|^2 = \left| \sqrt{\frac{2}{\pi}} \sum_{k=N+1}^{\infty} \frac{\psi_k(y)}{\left(\frac{1}{2}+k\right)_{\frac{1}{2}}^2} \right|^2 \tag{11}$$

Using the Cauchy-Schwarz inequality and the orthonormality of $\psi_k(y)$*, this can be bounded as:*

$$|J_0(y) - S_N(y)|^2 \leq \frac{2}{\pi} \sum_{k=N+1}^{\infty} \frac{1}{\left(\frac{1}{2}+k\right)^4} \tag{12}$$

Using the trigamma function to express the tail sum, we have the limit

$$\lim_{N\to\infty} \frac{2}{\pi} \sum_{k=N+1}^{\infty} \frac{1}{\left(\frac{1}{2}+k\right)^4} = \lim_{N\to\infty} \frac{\Psi(3, \frac{3}{2}+N)}{3\pi} = 0 \tag{13}$$

as $N \to \infty$ *thus confirming uniform convergence.* □