Quantum Field Theory

by Stanley P. Gooder (remixed by Steve)

7.1. Canonical Commutation Relations

Let us begin with a mathematical description of a single spinless, nonrelativistic particle with one degree of freedom. Such a system is traditionally described by the Hilbert space $L^2(\mathbb{R})$. The position operator Q and the momentum operator P play fundamental roles. The position operator Q is the self-adjoint operator

$$(Q\,\phi)(x) = x\,\phi(x) \tag{1}$$

with domain

$$D(Q) = \{\phi \in L^2(\mathbb{R}) \colon x\,\phi(x) \in L^2(\mathbb{R})\} \tag{2}$$

and the momentum operator P is the self-adjoint operator

$$P = -i\frac{\mathrm{d}}{\mathrm{d}x} \tag{3}$$

(we assume $\hbar = 1$) whose domain is the set of all absolutely continuous functions ϕ on $\mathbb{R}$ such that $\phi' \in L^2(\mathbb{R})$

$$D(P) = \{\phi \in L^2(\mathbb{R}) \colon \phi \text{ is absolutely continuous and } \phi' \in L^2(\mathbb{R})\} \tag{4}$$

Notice that the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R})$ satisfies $\mathcal{S} \subseteq D(Q) \cap D(P)$ and that $\mathcal{S}$ is a dense subspace of $L^2(\mathbb{R})$ which is left invariant by Q and P. Moreover, it is clear that

$$Q\,P - PQ = i\,I \tag{5}$$

on $\mathcal{S}$. We call (5) the *Heisenberg form of the canonical commutation relations (CCR)*.

There are other forms of the CCR. Since P is self-adjoint, the operator

$$U(a) = e^{-iaP} \forall a \in \mathbb{R} \tag{6}$$

is unitary. In fact, it is well known that for any self-adjoint operator P, the set of operators $\{U(a) \colon a \in \mathbb{R}\}$ forms a one-parameter strongly continuous unitary group. That is,

1. $U(a_1)\,U(a_2) = U\,(a_1 + a_2)\ \forall a_1, a_2 \in \mathbb{R}$

2. $\lim_{a\to a_0} U(a)\,\phi = U(a_0)\,\phi\ \forall \phi \in L^2(\mathbb{R})$

For this reason, we call $a \mapsto U(a)$ a unitary representation of the additive group $\mathbb{R}$. Using Taylor's theorem we see that for all $\phi \in \mathcal{S}$

$$\begin{aligned} e^{-iaP}\phi(x) &= \sum \frac{(-i\,a\,P)^n}{n!}\,\phi(x) \\ &= \sum \frac{(-a)^n}{n!}\,\phi^{(n)}(x) \\ &= \phi\,(x-a) \end{aligned} \tag{7}$$

Extending by continuity we conclude that

$$U(a)\,\phi(x) = \phi\,(x-a) \tag{8}$$

for all $\phi \in L^2(\mathbb{R})$. Now $D(Q)$ is invariant under $U(a)$ for every $a \in \mathbb{R}$, and for every $\phi \in D(Q)$ we have

$$\begin{aligned} U(a)\,Q\,U\,(-a)\,\phi(x) &= Q\,U\,(-a)\,\phi\,(x-a) \\ &= (x-a)\,U\,(-a)\,\phi\,(x-a) = (x-a)\,\phi(x) \end{aligned} \tag{9}$$

Hence, on $D(Q)$ we have

$$U(a)\,Q\,U\,(-a) = Q - a\,I \tag{10}$$

We call (10) the Schrödinger form of the CCR. Since Q is self-adjoint it generates a strongly continuous one-parameter unitary group $V(b)$ given by

$$V(b)\,\phi(x) = e^{-ibQ_\phi}(x) = e^{-ibx}\,\phi(x) \tag{11}$$

If $\phi \in \mathcal{S}, a, b \in \mathbb{R}$ we obtain

$$\begin{aligned} U(a)\,V(b)\,\phi(x) &= V(b)\,\phi\,(x-a) = e^{-ib(x-a)}\,\phi\,(x-a) \\ &= e^{iab}\,e^{-ibx}\,\phi\,(x-a) = e^{iab}\,V(b)\,U(a)\,\phi(x) \end{aligned} \tag{12}$$

Extending this equation to $L^2(\mathbb{R})$ by continuity we conclude that

$$U(a)\,V(b) = e^{iab}\,V(b)\,U(a) \tag{13}$$

for every $a, b \in \mathbb{R}$. Equation (13) is called the Weyl form of the CCR.

Let us now approach the subject from an axiomatic point of view as Heisenberg did when he introduced matrix mechanics. For a quantum system with one degree of freedom we assume that there are two fundamental observables, These observables are represented by two self-adjoint operators Q and P which leave invariant a dense subspace $D \subseteq L^2(\mathbb{R})$ and which satisfy the Heisenberg form (5) of the CCR on D. Unfortunately, these conditions do not determine Q and P uniquely to within a unitary equivalence. To see this, we have already observed that $Q\,\phi(x) = x\,\phi(x)$ and $P\phi(x) = -i\,\phi'(x)$ satisfy (5). Now let $H_1 = L^2([0,1])$ and define the self-adjoint operators Q_1 and P_1 as follows:

$$\begin{array}{ll} D(Q_1) = \{\phi \in \mathscr{H}_1 \colon x\,\phi(x) \in \mathscr{H}_1\} & Q_1\,\phi(x) = x\,\phi(x) \\ D(P_1) = \{\phi \in H_1 \colon \phi(0) = \phi(1),\, \phi' \in \mathscr{F}_1\} & P_1\,\phi(x) = -i\,\phi'(x) \end{array} \tag{14}$$

Then it is easy to see that

$$Q_1\,P_1 - P_1\,Q_1 = i\,I \tag{15}$$

on the dense subspace of $\mathscr{K}_1$ consisting of the infinitely differentiable functions ϕ satisfying $\phi(0) = \phi(1)$. But the pair (Q, P) is not unitarily equivalent to the pair (Q_1, P_1) since Q and P are unbounded with purely continuous spectrum, while Q_1 is bounded and P_1 has discrete point spectrum. We thus see that there are inequivalent ways of representing the CCR in the Heisenberg form (5).

The situation is quite different for the Weyl form (13) of the CCR. In this case the von Neumann uniqueness theorem, which we shall consider in detail later, takes effect. This theorem states that all irreducible [i.e., $U(a)\,M \subseteq M, V(b)\,M \subseteq M$, $a, b \in \mathbb{R}$, for a closed subspace M implies $M = \{0\}$ or $\mathscr{H}$] representations of the Weyl form of the CCR for one degree of freedom are unitarily equivalent. Since any representation is the direct sum of irreducible representations it follows that if $U_1(a)$ and $V_1(b)$ satisfy (13), then $U_1(a)$ and $V_1(b)$ must be the direct sum of copies of operators of the form

$$U(a)\,\phi(x) = \phi\,(x - a) \tag{16}$$

$$V(b)\,\phi(x) = e^{-ibx}\,\phi(x) \tag{17}$$

It follows from the above that the Heisenberg and Weyl forms of the CCR are not logically equivalent. It can be shown, however, that the Schrödinger and the Weyl forms are equivalent. The Weyl form (13) thus enjoys at least two advantages. It is phrased in terms of bounded operators and it uniquely determines the operators $U(a)$ and $V(b)$ to within an equivalence (in the irreducible case).

We have noted that the Heisenberg form does not have the uniqueness property of the Weyl form. We now show that it cannot have the boundedness property.

Lemma 1. *If $Q\,P - PQ = i\,I$, then at least one of the operators Q or P must be unbounded.*

Proof. Suppose, on the contrary, that there exist two bounded operators Q and P acting on a Hilbert space and satisfying (5). We can assume without loss of generality that P is invertible, for if not we could replace P by $P-\lambda I$, where $\lambda > \|P\|$, without changing the commutation relation. Since

$$PQ-\alpha I = P(QP-\alpha I)P^{-1} \tag{18}$$

for all $\alpha \in C$, we conclude that the spectrum $\sigma(PQ)=\sigma(QP)$. Since $PQ = QP - iI$, we have

$$PQ - i(\alpha-1)I = QP - i\alpha I \tag{19}$$

for every $\alpha \in C$. Hence $i\alpha \in \sigma(QP)$ if and only if $i(\alpha-1)\in\sigma(PQ)$. Since $\sigma(PQ)\neq\varnothing$ it follows that there exists an $\alpha\in C$ such that

$$\{i(\alpha+n): n=0,1,2,\dots\}\subseteq\sigma(PQ) \tag{20}$$

Thus $\sigma(PQ)$ is unbounded. This contradicts the fact that PQ is a bounded operator. □

All that we have said so far can be easily generalized to systems with a finite number n of degrees of freedom. In this case we have $2n$ operators $Q_1,\dots,Q_n$, $P_1,\dots,P_n$ on $L^2(\mathbb{R}^n)$ satisfying the Heisenberg form of the CCR

$$Q_\kappa P_j - P_j Q_\kappa = i\,\delta_{\kappa j} I \tag{21}$$

For $a=(a_1,\dots,a_n), b=(b_1,\dots,b_n)\in\mathbb{R}^n$ we form the unitary operators $U(a)\,\phi(x) = \phi(x-a)$, $V(b)\,\phi(x)=e^{-i\langle b,x\rangle}\,\phi(x)$, $\phi\in L^2(\mathbb{R}^n)$. As before, $U(a)$ and $V(b)$ are unitary representations of the additive group $\mathbb{R}^n$ i.e., satisfy (i) and (ii)] and, moreover, the Weyl form of the CCR holds:

$$U(a)\,V(b) = e^{i\langle a,b\rangle}\,V(b)\,U(a) \tag{22}$$

for every $a,b\in\mathbb{R}^n$. Again, the von Neumann uniqueness theorem states that (22) uniquely determines $U(a)$ and $V(b)$ to within an equivalence (in the irreducible case).

7.2. Quantum Fields

We now come to some of the mathematical problems of quantum field theory. Roughly speaking, a quantum field is a system with infinitely many degrees of freedom. We shall need an infinite-dimensional analogue of the Weyl form of the CCR given in (22). In order to state what this analogue is we need to introduce some more structure.

Let $\mathscr{H}$ be a separable real Hilbert space. For $f \in \mathscr{H}$ we define the unitary operators $U(f)$ and $V(f)$ on a complex Hilbert space $\mathscr{K}$ satisfying

$$U(f)\,V(g) = e^{i\langle f, g\rangle}\,V(g)\,U(f) \tag{23}$$

for all $f, g \in \mathscr{H}$. As before, $U(f)$ and $V(g)$ are strongly continuous unitary representations of the additive group $\mathscr{H}$. That is,

1. $U(f_1)\,U(f_2) = U\,(f_1 + f_2)$
   and
   $V(g_1)\,V(g_2) = V\,(g_1 + g_2)$
   $\forall f_1, f_2, g_1, g_2 \in \mathscr{H}$

2. $\lim_{n\to\infty} U(f_n)\,\phi = U(f)\,\phi$
   and
   $\lim_{n\to\infty} V(g_n)\,\phi = V(g)\,\phi$
   $\forall \phi \in \mathscr{K}$ whenever $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$ in $\mathscr{H}$

We now describe three standard examples of quantum fields. For a boson field we have $\mathscr{H} = L^2(\mathbb{R}^3)$, and $\mathscr{K}$ is the boson Fock space which we shall now define. Let $\mathscr{K}_0 = \mathbb{C}$ be the 1-dimensional Hilbert space consisting of all complex numbers, and let $\mathscr{K}_n = (L^2(\mathbb{R}^3) \otimes \cdots \otimes L^2(\mathbb{R}^3))^S$ be the n-fold symmetric tensor product of $L^2(\mathbb{R}^3)$ with itself. The boson Fock space is then defined as

$$\mathscr{K} = \mathscr{K}_0 \oplus \mathscr{K}_1 \oplus \cdots \oplus \mathscr{K}_n \oplus \cdots \tag{24}$$

Thus a vector $\phi \in \mathscr{K}$ is a sequence $\phi = (\phi_0, \phi_1, \ldots, \phi_n, \ldots)$, where $\phi_0 \in \mathbb{C}$, $\phi_n \in \mathscr{K}_n$, and $\sum_0^\infty |\phi_n|^2 < \infty$. The inner product of $\phi, \psi \in \mathscr{K}$ is given by

$$\langle \phi, \psi \rangle = \sum_0^\infty \langle \phi_n, \psi_n \rangle \tag{25}$$

The unit vector $\Omega = (1, 0, 0, \ldots)$ is called the vacuum state, and any vector in $\mathscr{K}$ can be approximated arbitrarily closely by (finite) linear combinations of multiple applications of the creation operators to the vacuum state. (The creation operators will be defined below.)

For $f \in \mathscr{H}$, the Segal field operator is defined by

$$\Phi(f) = (Q(f) + P(f))/\sqrt{2} \tag{26}$$

where $Q(f) = a^*(f) + a(f)$ (position operator) and $P(f) = i\,(a^*(f) - a(f))$ (momentum operator). Here, $a^*(f)$ is the creation operator and $a(f)$ is the annihilation operator. For $f_1, \ldots, f_n \in L^2(\mathbb{R}^3)$, the creation operator $a^*(f)$ is defined by

$$\begin{aligned} a^*(f)\,\Omega &= (0, f, 0, \ldots) \\ a^*(f)(0, \ldots, \phi_n, 0, \ldots) &= (0, \ldots, 0, S\,(f \otimes \phi_n), 0, \ldots) \end{aligned} \tag{27}$$

where S is the operator that symmetrizes the tensor product. The annihilation operator $a(f)$ is defined as the adjoint of $a^*(f)$. These operators have the property that $a(f)\,\Omega = 0$ for all f. We also have the following commutation relations:

$$\begin{aligned} &[a(f), a^*(g)] = \langle f, g\rangle\, I \\ &[a(f), a(g)] = [a^*(f), a^*(g)] = 0 \end{aligned} \tag{28}$$

for all $f, g \in L^2(\mathbb{R}^3)$.

Returning to the Segal field operators $\Phi(f)$, let us define the unitary operators

$$\begin{aligned} U(f) &= e^{-iP(f)} \\ V(f) &= e^{-iQ(f)} \end{aligned} \tag{29}$$

for $f \in L^2(\mathbb{R}^3)$.

It can be shown that (23) holds for these operators. A fermion field is similar to a boson field with a few changes. The position operator is

$$Q(f) = b^*(f) + b(f) \tag{30}$$

and the momentum operator is

$$P(f) = i\,(b^*(f) - b(f)) \tag{31}$$

for $f \in \mathscr{H} = L^2(\mathbb{R}^3)$. Here $b^*(f)$ is the fermion creation operator and $b(f)$ is the fermion annihilation operator. As before, the Segal field operator is given by (26). The fermion space $\mathscr{K}$ is similar to the boson space defined in (24) except that $\mathscr{K}_n = (L^2(\mathbb{R}^3) \otimes \cdots \otimes L^2(\mathbb{R}^3))^A$ is the n-fold antisymmetric tensor product. The relations for the creation and annihilation operators are now

$$\begin{aligned} &[b(f), b^*(g)]_+ = \langle f, g\rangle\, I \\ &[b(f), b(g)]_+ = [b^*(f), b^*(g)]_+ = 0 \end{aligned} \tag{32}$$

where $[A, B]_+ = A\,B + B\,A$ is the anticommutator. In particular, we have that $b(f)^2 = b^*(f)^2 = 0$. This result is consistent with the Pauli exclusion principle and tells us that a fermion cannot be in the same state twice.

Our third example is that of a free scalar field. Let $\mathcal{H}$ be the Hilbert space consisting of the set of all real-valued solutions f of the Klein-Gordon equation

$$\Box\, f + m^2\, f = 0 \tag{33}$$

where $\Box = \partial^2/\partial t^2 - \nabla^2$ is the d'Alembert operator, with the inner product

$$\langle f, g\rangle = \int \left[f(x,0)\frac{\partial g}{\partial t}(x,0) - \frac{\partial f}{\partial t}(x,0)\, g\,(x,0) \right] d\,x \tag{34}$$

This inner product is time-independent. That is, if we replace 0 by t in (34), then the inner product remains the same. This is essentially Green's identity of ordinary differential equations. The quantum space $\mathscr{K}$ is the boson Fock space. We can construct the field operator as follows. For simplicity we consider the case where $m=0$. In this case solutions of (33) are of the form $u(t,x)=v\,(x+t)+w\,(x-t)$. We define

$$a(k)=\frac{1}{(2\,\pi)^{3/2}}\int e^{-ik\cdot x}\left[\omega(k)\,u(0,x)+i\,\frac{\partial u}{\partial t}\,(0,x)\right]d\,x \tag{35}$$

and

$$a^{*}(k)=\frac{1}{(2\,\pi)^{3/2}}\int e^{ik\cdot x}\left[\omega(k)\,u(0,x)-i\,\frac{\partial u}{\partial t}\,(0,x)\right]d\,x \tag{36}$$

where $\omega(k)=|k|$. These are the standard annihilation and creation operators. This last field appears to be different from the previous two, but this is in fact not the case. It has been proved that all three fields are unitarily equivalent.

We now summarize some of the general theory. Let $U(f)$ and $V(g)$, $f,g\in\mathscr{H}$, be operators on $\mathscr{K}$ satisfying (23). Assume for simplicity that these operators are irreducible; i.e., the only closed subspaces of $\mathscr{K}$ that are simultaneously invariant under all the operators $U(f)$ and $V(g)$ are $\{0\}$ and $\mathscr{K}$. According to the Stone-von Neumann-Mackey theorem, the representation is unitarily equivalent to the one described in our first example. That is, we can find a unitary operator W: $\mathscr{K}\rightarrow\mathscr{K}_B$, where $\mathscr{K}_B$ is the boson Fock space, such that

$$\begin{aligned} W U(f)\,W^{-1} &= U_B(f)\\ W V(g)\,W^{-1} &= V_B(g)\end{aligned} \tag{37}$$

for all $f,g\in\mathscr{H}$. Here $U_B(f)$ and $V_B(g)$ are the unitary operators for the boson field. In this sense, all quantum fields (at least of bosonic type) are unitarily equivalent.

However, the situation changes drastically if we consider dynamics. Let H be the Hamiltonian (energy operator) for a quantum field. Let $U_t=e^{itH}$ be the corresponding time-evolution operator. Then $U_t\,K=K$ if and only if K commutes with H. It can be shown that for many interacting-field Hamiltonians, the operators $U_t\,U(f)\,U_{-t}$ and $U_t\,V(g)\,U_{-t}$ do not satisfy the Weyl form of the CCR. In this case we say that the fields are not unitarily equivalent to free fields. In fact, it has been proved that all known physically relevant interacting fields are not unitarily equivalent to free fields. This is one of the central problems in quantum field theory.

Another problem in quantum field theory is the following. Let us first consider the Klein-Gordon equation

$$\Box\,\phi+m^2\,\phi=0 \tag{38}$$

for a free scalar field. A general solution is of the form

$$\phi(t,x) = \int [a(k)\, e^{i(k\cdot x - \omega(k)t)} + a^*(k)\, e^{-i(k\cdot x - \omega(k)t)}]\, d\,k \tag{39}$$

where $\omega(k) = \sqrt{|k|^2 + m^2}$. The energy of this field is given by

$$H = \frac{1}{2} \int [\Pi^2(x) + (\nabla\, \phi(x))^2 + m^2\, \phi^2(x)]\, d\,x \tag{40}$$

where $\Pi(x) = \partial\phi/\partial t$ is the field momentum.

Now let us consider the case when a self-interaction term $g\,\phi^4(x)$ is added to the energy integral. Then the energy becomes

$$H = \frac{1}{2} \int [\Pi^2(x) + (\nabla\, \phi(x))^2 + m^2\, \phi^2(x) + g\, \phi^4(x)]\, d\,x \tag{41}$$

For this interaction, new and even more formidable problems arise. It is not clear that the solutions of the interacting field equation are operator-valued. They may produce a distribution-valued field rather than operator-valued. Infinities may arise in perturbation expansions. All these problems are areas of active research in quantum field theory.

In the remainder of this section we describe a formulation of the dynamics of quantum fields. Let the Hamiltonian H for the system be a self-adjoint operator. If ψ_t denotes the state of the system at time t and ψ_0 the initial state, then we have

$$\psi_t = e^{-itH}\, \psi_0 \tag{42}$$

This is the Schrödinger picture.

Another approach is the Heisenberg picture. In this approach it is the observable rather than the state that varies with time. If A is an observable, then its value at time t is

$$A(t) = e^{itH}\, A\, e^{-itH} \tag{43}$$

Presumably, the expectation value $\langle \psi_t, A\, \psi_t \rangle$ of A at time t in the Schrödinger picture should equal the expectation value $\langle \psi_0, A(t)\, \psi_0 \rangle$ of $A(t)$ at time t, relative to the initial state, in the Heisenberg picture:

$$\langle \psi_t, A\, \psi_t \rangle = \langle \psi_0, A(t)\, \psi_0 \rangle \tag{44}$$

This is indeed the case since

$$\begin{aligned} \langle \psi_t, A\, \psi_t \rangle &= \langle e^{-itH}\, \psi_0, A\, e^{-itH}\, \psi_0 \rangle \\ &= \langle \psi_0, e^{itH}\, A\, e^{-itH}\, \psi_0 \rangle \\ &= \langle \psi_0, A(t)\, \psi_0 \rangle \end{aligned} \tag{45}$$

It is quite useful to have the operator satisfy a differential equation. Differentiating (43) by t we obtain

$$\begin{aligned}\frac{d\,A(t)}{d\,t} &= i\,e^{itH}\,H\,A\,e^{-itH} - i\,e^{itH}\,A\,H\,e^{-itH} \\ &= i\,[H, A(t)]\end{aligned} \tag{46}$$

where we have used the fact that H is self-adjoint.

Let us apply this to the canonical field variables

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\omega(k)}}\,[a(k)\,e^{ik\cdot x} + a^*(k)\,e^{-ik\cdot x}]\,d\,k \\ \pi(x) &= \frac{i}{\sqrt{2}}\int \sqrt{\omega(k)}\,[a^*(k)\,e^{-ik\cdot x} - a(k)\,e^{ik\cdot x}]\,d\,k\end{aligned} \tag{47}$$

The usual canonical commutation relations for the field operators are

$$\begin{aligned}&[\phi(x), \pi(y)] = i\,\delta\,(x-y) \\ &[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0\end{aligned} \tag{48}$$

An example of a Hamiltonian for a free field is

$$H = \frac{1}{2}\int [\pi^2(x) + (\nabla\,\phi(x))^2 + m^2\,\phi^2(x)]\,d\,x \tag{49}$$

For this Hamiltonian we have

$$\begin{aligned}&[H, \phi(x)] = i\,\pi(x) \\ &[H, \pi(x)] = -i\,[-\nabla^2 + m^2]\,\phi(x)\end{aligned}$$

(50)

It follows from the Heisenberg equation (46) that

$$\frac{d\,\phi(x,t)}{d\,t} = \pi(x,t) \tag{51}$$

and

$$\frac{d\,\pi(x,t)}{d\,t} = [\nabla^2 - m^2]\,\phi(x,t) \tag{52}$$

Eliminating $\pi(x,t)$ we obtain

$$\frac{d^2\,\phi(x,t)}{d\,t^2} = [\nabla^2 - m^2]\,\phi(x,t) \tag{53}$$

or

$$\Box\,\phi(x,t)+m^2\,\phi(x,t)=0 \tag{54}$$

We therefore end up with the Klein-Gordon equation (38) again. For the more general Hamiltonian (41) defining the ϕ^4 interaction, the evolution equations are more difficult to solve.

In the next section we introduce the Euclidean approach to quantum field theory. We shall see the advantages of this approach for the study of the various problems that appear in quantum field theory, especially interacting field theory.