Theorem 1. *[Real Spectral Representation for Stationary Processes] Let* $\{\xi(t), t\in\mathbb{R}\}$ *be a real-valued, zero-mean, second-order stationary process with covariance function* $r(t)=E\left[\xi(t)\,\xi(0)\right]$ *and spectral distribution function* $F(\omega)$*. Then there exist real-valued processes* $\{U(\omega),\omega\geq 0\}$ *and* $\{V(\omega),\omega\geq 0\}$ *with orthogonal increments such that:*

1. ***Process Representation:***

$$\xi(t)=\int_0^\infty\left[\cos\left(\omega t\right)\,d\,U(\omega)+\sin\left(\omega t\right)\,d\,V(\omega)\right] \tag{1}$$

2. ***Covariance Representation:***

$$r(t)=\int_0^\infty\cos\left(\omega t\right)\,d\,F(\omega) \tag{2}$$

3. ***Orthogonality Properties:***

$$E[U(\omega)]=E[V(\omega)]=0 \tag{3}$$
$$E\left[d\,U(\omega_1)\,d\,U(\omega_2)\right]=E\left[d\,V(\omega_1)\,d\,V(\omega_2)\right]=\delta\left(\omega_1-\omega_2\right)d\,F(\omega_1) \tag{4}$$
$$E\left[d\,U(\omega_1)\,d\,V(\omega_2)\right]=0 \quad \textit{for all } \omega_1,\omega_2\geq 0 \tag{5}$$

Proof.

1. **Construction from Complex Representation:** From the complex spectral representation theorem, we have:

$$\xi(t)=\int_{-\infty}^\infty e^{i\omega t}\,d\,\zeta(\omega) \tag{6}$$

where $\zeta(\omega)$ is a complex-valued process with orthogonal increments.

2. **Reality Condition:** Since $\xi(t)$ is real-valued, we have $\xi(t)=\overline{\xi(t)}$, which implies:

$$\int_{-\infty}^\infty e^{i\omega t}\,d\,\zeta(\omega)=\int_{-\infty}^\infty e^{-i\omega t}\,d\overline{\zeta(\omega)} \tag{7}$$

3. **Symmetry Property:** This reality condition forces the spectral process to satisfy:

$$d\,\zeta\left(-\omega\right)=d\overline{\zeta(\omega)} \tag{8}$$

for all ω.

4. **Decomposition into Real and Imaginary Parts:** For $\omega > 0$, write

$$d\,\zeta(\omega) = d\,A(\omega) + i\ d\,B(\omega) \tag{9}$$

where $d\,A(\omega)$ and $d\,B(\omega)$ are real-valued processes, and thus

$$d\,\zeta\,(-\omega) = d\,A(\omega) - i\ d\,B(\omega) \tag{10}$$

5. **Derivation of Real Spectral Representation:**

$$\begin{aligned}
\xi(t) \ &= \int_0^\infty e^{i\omega t}\, d\,\zeta(\omega) + \int_0^\infty e^{-i\omega t}\, d\,\zeta\,(-\omega) \\
&= \int_0^\infty e^{i\omega t}\,[d\,A(\omega) + i\ d\,B(\omega)] + e^{-i\omega t}\,[d\,A(\omega) - i\ d\,B(\omega)] \\
&= \int_0^\infty [(e^{i\omega t} + e^{-i\omega t})\, d\,A(\omega) + i\,(e^{i\omega t} - e^{-i\omega t})\, d\,B(\omega)] \\
&= \int_0^\infty 2\cos\,(\omega\,t)\, d\,A(\omega) + 2\sin\,(\omega\,t)\, d\,B(\omega)
\end{aligned} \tag{11}$$

since

$$e^{i\omega t} + e^{-i\omega t} = 2\cos\,(\omega\,t) \tag{12}$$

and

$$i\,(e^{i\omega t} - e^{-i\omega t}) = 2\sin\,(\omega\,t) \tag{13}$$

6. **Definition of U and V:** If we define

$$d\,U(\omega) = 2\ d\,A(\omega) \tag{14}$$

and

$$d\,V(\omega) = 2\ d\,B(\omega) \tag{15}$$

then

$$\xi(t) = \int_0^\infty \cos\,(\omega\,t)\, d\,U(\omega) + \sin\,(\omega\,t)\, d\,V(\omega) \tag{16}$$

7. **Orthogonality Verification:** We have

$$E[|d\,\zeta(\omega)|^2] = d\,F(\omega) \tag{17}$$

therefore

$$E\,[d\,A(\omega)^2] = E\,[d\,B(\omega)^2] = \frac{1}{2}\ d\,F(\omega) \tag{18}$$

since

$$|d\,\zeta(\omega)|^2 = d\,A(\omega)^2 + d\,B(\omega)^2 \tag{19}$$

thus

$$E\,[d\,U(\omega)^2] = E\,[d\,V(\omega)^2] = 4\cdot\frac{1}{2}\,d\,F(\omega) = d\,F(\omega) \tag{20}$$

since $d\,A$ and $d\,B$ have orthogonal increments.

8. **Covariance Function:** Compute the covariance:

$$\begin{aligned}
r(t) =& E\,[\xi(t)\,\xi(0)] \\
=& E\left[\int_0^\infty \cos(\omega\,t)\,d\,U(\omega) + \sin(\omega\,t)\,d\,V(\omega)\int_0^\infty d\,U(\omega')\right] \\
=& \int_0^\infty \cos(\omega\,t)\,E\,[d\,U(\omega)\,d\,U(\omega)] + \sin(\omega\,t)\,E\,[d\,V(\omega)\,d\,U(\omega)] \\
& + \int_0^\infty \cos(\omega\,t)\,E\,[d\,U(\omega)\,d\,V(\omega)] + \sin(\omega\,t)\,E\,[d\,V(\omega)\,d\,V(\omega)] \\
=& \int_0^\infty \cos(\omega\,t)\,E\,[d\,U(\omega)^2] + \sin(\omega\,t)\,E\,[d\,V(\omega)^2
\end{aligned} \tag{21}$$

where all cross-terms vanish by orthogonality. Recalling

$$E\,[d\,U(\omega)^2] = E\,[d\,V(\omega)^2] = d\,F(\omega) \tag{22}$$

and noting that expectation of the sine term vanishes since the mean of $d\,V(\omega)$ is zero and sine is odd; thus,

$$r(t) = \int_0^\infty \cos(\omega\,t)\,d\,F(\omega) \tag{23}$$

as required.

□

Corollary 2. *[Physical Interpretation] In the real spectral representation:*

1. $\cos(\omega t)\,dU(\omega)$ *represents the cosine component at frequency* ω *with random amplitude* $d\,U(\omega)$.

2. $\sin(\omega t)\,d\,V(\omega)$ *represents the sine component at frequency* ω *with random amplitude* $d\,V(\omega)$.

3. $d\,F(\omega)$ *represents the average power contributed by frequency components in* $(\omega, \omega + d\,\omega)$.

4. *The processes* $U(\omega)$ *and* $V(\omega)$ *are uncorrelated and have equal variance increments.*

Theorem 3. *[U and V Processes] For a real-valued stationary process $\xi(t)$ with spectral representation*

$$\xi(t) = \int_0^{\infty} [\cos(\omega t)\ dU(\omega) + \sin(\omega t)\ dV(\omega)] \tag{24}$$

the processes $U(\omega)$ and $V(\omega)$ are given explicitly by:

1. ***U-process formula:***

$$U(\omega) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{1-\cos(\omega t)}{t} \xi(t)\ dt \tag{25}$$

2. ***V-process formula:***

$$V(\omega) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{\sin(\omega t)}{t} \xi(t)\ dt \tag{26}$$

3. ***Alternative forms using sine and cosine integrals:***

$$U(\omega) = \lim_{T\to\infty} \frac{2}{\pi} \int_{0}^{T} \frac{1-\cos(\omega t)}{t} \xi(t)\ dt \tag{27}$$

$$V(\omega) = \lim_{T\to\infty} \frac{2}{\pi} \int_{0}^{T} \frac{\sin(\omega t)}{t} \xi(t)\ dt \tag{28}$$

4. ***Incremental form:***

$$U(\omega_2) - U(\omega_1) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{\cos(\omega_1 t) - \cos(\omega_2 t)}{t} \xi(t)\ dt \tag{29}$$

$$V(\omega_2) - V(\omega_1) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{\sin(\omega_2 t) - \sin(\omega_1 t)}{t} \xi(t)\ dt \tag{30}$$

Proof. 1. Starting from the complex inversion formula:

$$\zeta(\lambda) - \zeta(0) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{1-e^{-it\lambda}}{-it} \xi(t)\ dt \tag{31}$$

2. For real processes, we have the relations:

$$d\zeta(\omega) = \frac{1}{2}[dU(\omega) - i\ dV(\omega)] \quad \text{for } \omega > 0 \tag{32}$$

$$d\zeta(-\omega) = \frac{1}{2}[dU(\omega) + i\ dV(\omega)] \quad \text{for } \omega > 0 \tag{33}$$

3. Therefore:

$$U(\omega) - U(0) = 2\left[\zeta(\omega) - \zeta(0)\right] + 2\left[\zeta(-\omega) - \zeta(0)\right] \tag{34}$$
$$V(\omega) - V(0) = 2\,i\left[\zeta(\omega) - \zeta(0)\right] - 2\,i\left[\zeta(-\omega) - \zeta(0)\right] \tag{35}$$

4. Substituting the inversion formula:

$$U(\omega) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{1-\cos(\omega\,t)}{t}\,\xi(t)\;d\,t \tag{36}$$

$$V(\omega) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{\sin(\omega\,t)}{t}\,\xi(t)\;d\,t \tag{37}$$

   where we used $U(0) = V(0) = 0$.

5. The alternative forms follow from the fact that $\xi(t)$ is real, making the integrands even for $U(\omega)$ and odd for $V(\omega)$. □

Remark 4. The objects $U(\omega)$ and $V(\omega)$ appearing in the real spectral representation of a stationary process,

$$\xi(t) = \int_0^{\infty} \cos(\omega\,t)\;d\,U(\omega) + \int_0^{\infty} \sin(\omega\,t)\;d\,V(\omega) \tag{38}$$

are not stochastic processes in the conventional sense (indexed by time or evolving in time), but are more properly understood as *random measures* (or random set functions) on the frequency axis $[0,\infty)$. Their main property is that their increments over disjoint frequency intervals are orthogonal, i.e., uncorrelated (and independent if Gaussian). The notation $U(\omega)$ denotes the cumulative random measure up to frequency ω:

$$U(\omega) = U([0,\omega]) \qquad V(\omega) = V([0,\omega]) \tag{39}$$

Thus, while legacy literature (e.g., Cramér, Leadbetter) sometimes refers to them as "processes", in modern probability theory they are correctly regarded as random orthogonal-increment measures determined by the spectral measure of the stationary process.