Multiplication Operators on ℓ^2 Space

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1 Introduction

This document presents a comprehensive analysis of multiplication operators on the Hilbert space ℓ^2 of square-summable sequences. The focus lies on establishing the fundamental properties of these operators through rigorous proofs.

2 The ℓ^2 Space

Definition 1. *[The ℓ^2 Space] The space ℓ^2 consists of all sequences $x = (x_0, x_1, x_2, \ldots)$ of complex numbers such that*

$$\sum_{n=0}^{\infty} |x_n|^2 < \infty.$$

Theorem 2. *[ℓ^2 is a Hilbert Space] The space ℓ^2 equipped with the inner product*

$$\langle x, y \rangle = \sum_{j=0}^{\infty} x_j \bar{y_j}$$

and induced norm $|x|_2 = \sqrt{\langle x, x \rangle}$ forms a complete Hilbert space.

Proof. The proof consists of several steps:

Step 1: Verify that the inner product is well-defined. For $x, y \in \ell^2$, the Cauchy-Schwarz inequality gives

$$\left|\sum_{j=0}^{n} x_j \bar{y_j}\right| \le \left(\sum_{j=0}^{n} |x_j|^2\right)^{1/2} \left(\sum_{j=0}^{n} |y_j|^2\right)^{1/2} \le |x|_2 |y|_2.$$

Since the right side is finite, the series $\sum_{j=0}^{\infty} x_j \bar{y_j}$ converges absolutely.

Step 2: Verify inner product axioms. The inner product satisfies linearity, conjugate symmetry, and positive definiteness by direct verification using properties of infinite series.

Step 3: Prove completeness. Let $(x^{(k)})_{k=1}^{\infty}$ be a Cauchy sequence in ℓ^2. For each fixed $n \in \mathbb{N}$, the sequence $(x_n^{(k)})_{k=1}^{\infty}$ is Cauchy in $\mathbb{C}$ and hence converges to some limit x_n.

One can show that $x = (x_0, x_1, x_2, \ldots) \in \ell^2$ and $x^{(k)} \to x$ in ℓ^2 norm, establishing completeness. □

3 Multiplication Operators

Definition 3. *[Multiplication Operator] Let $a = (a_0, a_1, a_2, \ldots) \in \ell^\infty$ be a bounded sequence. The multiplication operator $M_a \colon \ell^2 \to \ell^2$ is defined by*

$$M_a x = (a_0 x_0, a_1 x_1, a_2 x_2, \ldots)$$

for all $x = (x_0, x_1, x_2, \ldots) \in \ell^2$.

Lemma 4. *[Well-definedness of M_a] If $a \in \ell^\infty$ and $x \in \ell^2$, then $M_a x \in \ell^2$.*

Proof. For $x \in \ell^2$ and $a \in \ell^\infty$, one has

$$\sum_{j=0}^{\infty} |a_j x_j|^2 \le |a|_\infty^2 \sum_{j=0}^{\infty} |x_j|^2 = |a|_\infty^2 |x|_2^2 < \infty.$$

Therefore, $M_a x \in \ell^2$. □

Theorem 5. *[Operator Norm of Multiplication Operators] Let $a \in \ell^\infty$ and let $M_a \colon \ell^2 \to \ell^2$ be the corresponding multiplication operator. Then*

$$|M_a| = |a|_\infty = \sup_{n \ge 0} |a_n|.$$

Proof. The proof proceeds in two parts to establish both inequalities.

Step 1: Show $|M_a| \leq |a|_\infty$. For any $x \in \ell^2$ with $|x|_2 = 1$, one has

$$
\begin{aligned}
|M_a x|_2^2 &= \sum_{j=0}^{\infty} |a_j x_j|^2 && (1)\\
&\leq \sum_{j=0}^{\infty} |a_j|^2 |x_j|^2 && (2)\\
&\leq |a|_\infty^2 \sum_{j=0}^{\infty} |x_j|^2 && (3)\\
&= |a|_\infty^2 |x|_2^2 && (4)\\
&= |a|_\infty^2. && (5)
\end{aligned}
$$

Taking the supremum over all unit vectors x, this gives $|M_a| \leq |a|_\infty$.

Step 2: Show $|M_a| \geq |a|_\infty$. Let $\epsilon > 0$ be given. Since $|a|_\infty = \sup_{n \geq 0} |a_n|$, there exists an index n_0 such that $|a_{n_0}| > |a|_\infty - \epsilon$.

Consider the unit vector $e_{n_0} = (\delta_{n_0,0}, \delta_{n_0,1}, \delta_{n_0,2}, \dots)$ where δ is the Kronecker delta. Then

$$|M_a e_{n_0}|_2 = |(0, 0, \dots, a_{n_0}, 0, \dots)|_2 = |a_{n_0}| > |a|_\infty - \epsilon.$$

Since $|e_{n_0}|_2 = 1$, this shows $|M_a| \geq |a_{n_0}| > |a|_\infty - \epsilon$.

Since $\epsilon > 0$ was arbitrary, one obtains $|M_a| \geq |a|_\infty$.

Combining both inequalities yields $|M_a| = |a|_\infty$. □

Proposition 6. *[Action on Canonical Basis] Let $\{e_n\}_{n=0}^{\infty}$ denote the canonical orthonormal basis for ℓ^2. Then*

$$M_a e_n = a_n e_n$$

for all $n \geq 0$.

Proof. By definition, $e_n = (\delta_{n,0}, \delta_{n,1}, \delta_{n,2}, \dots)$. Therefore,

$$M_a e_n = (a_0 \delta_{n,0}, a_1 \delta_{n,1}, a_2 \delta_{n,2}, \dots) = a_n e_n.$$

This shows that multiplication operators are diagonal with respect to the canonical basis. □

4 Algebraic Properties

Theorem 7. *[Algebraic Structure of Multiplication Operators] The set of multiplication operators on ℓ^2 forms a commutative algebra under the operations:*

1. $M_a + M_b = M_{a+b}$
2. $\lambda M_a = M_{\lambda a}$ *for scalars* $\lambda \in \mathbb{C}$
3. $M_a M_b = M_{ab}$ *(componentwise product)*

Proof. Each property is verified by direct computation:

Property 1: For $x \in \ell^2$,

$$(M_a + M_b)\, x = M_a\, x + M_b\, x = (a_0\, x_0 + b_0\, x_0, a_1\, x_1 + b_1\, x_1, \dots) = M_{a+b}\, x.$$

Property 2: For $x \in \ell^2$ and $\lambda \in \mathbb{C}$,

$$(\lambda\, M_a)\, x = \lambda\, (M_a\, x) = (\lambda\, a_0\, x_0, \lambda\, a_1\, x_1, \dots) = M_{\lambda a}\, x.$$

Property 3: For $x \in \ell^2$,

$$(M_a\, M_b)\, x = M_a\, (M_b\, x) = M_a\, (b_0\, x_0, b_1\, x_1, \dots) \tag{6}$$
$$= (a_0\, b_0\, x_0, a_1\, b_1\, x_1, \dots) = M_{ab}\, x. \tag{7}$$

Commutativity follows since $M_a\, M_b = M_{ab} = M_{ba} = M_b\, M_a$. □

Theorem 8. *[Adjoint of Multiplication Operators] The adjoint of the multiplication operator M_a is given by*

$$M_a^* = M_{\bar{a}},$$

where $\bar{a} = (\bar{a_0}, \bar{a_1}, \bar{a_2}, \dots)$ is the componentwise complex conjugate.

Proof. For $x, y \in \ell^2$, one computes

$$\langle M_a\, x, y\, \rangle = \sum_{j=0}^{\infty} (a_j\, x_j)\bar{y_j} \tag{8}$$
$$= \sum_{j=0}^{\infty} x_j \bar{a_j} \bar{y_j} \tag{9}$$
$$= \sum_{j=0}^{\infty} x_j \overline{(\bar{a_j} y_j)} \tag{10}$$
$$= \langle x, M_{\bar{a}}\, y\, \rangle. \tag{11}$$

By the definition of the adjoint operator, this establishes $M_a^* = M_{\bar{a}}$. $\square$

5 Spectral Properties

Theorem 9. *[Invertibility of Multiplication Operators] The multiplication operator M_a is invertible if and only if*

$$\inf_{n \geq 0} |a_n| > 0.$$

When invertible, the inverse is given by $M_a^{-1} = M_{1/a}$ where $1/a = (1/a_0, 1/a_1, 1/a_2, \ldots)$.

Proof. Necessity: Suppose M_a is invertible. If $\inf_{n \geq 0} |a_n| = 0$, then there exists a subsequence (a_{n_k}) such that $|a_{n_k}| \to 0$ as $k \to \infty$.

For each k, consider the unit vector e_{n_k}. Then $M_a\, e_{n_k} = a_{n_k}\, e_{n_k}$, so $|M_a\, e_{n_k}|_2 = |a_{n_k}| \to 0$. This contradicts the existence of M_a^{-1} since M_a would not be bounded below.

Sufficiency: Suppose $\delta := \inf_{n \geq 0} |a_n| > 0$. Define the sequence $1/a = (1/a_0, 1/a_1, 1/a_2, \ldots)$. Since $|1/a_n| = 1/|a_n| \leq 1/\delta$ for all n, one has $1/a \in \ell^\infty$.

For any $x \in \ell^2$,

$$M_{1/a}\,(M_a\, x) = M_{1/a}\,(a_0\, x_0, a_1\, x_1, \ldots) = (x_0, x_1, x_2, \ldots) = x.$$

Similarly, $M_a\,(M_{1/a}\, x) = x$. Therefore, $M_a^{-1} = M_{1/a}$. $\square$

Theorem 10. *[Spectrum of Multiplication Operators] The spectrum of the multiplication operator M_a is given by*

$$\sigma(M_a) = \overline{\{a_n : n \geq 0\}},$$

the closure of the range of the sequence a.

Proof. Step 1: Show $\{a_n : n \geq 0\} \subseteq \sigma(M_a)$. For any $n \geq 0$, consider $\lambda = a_n$. Then $(M_a - \lambda\, I)\, e_n = a_n\, e_n - a_n\, e_n = 0$. Since $e_n \neq 0$, the operator $M_a - \lambda\, I$ is not injective, hence not invertible. Therefore, $a_n \in \sigma(M_a)$.

Step 2: Show $\sigma(M_a) \subseteq \overline{\{a_n : n \geq 0\}}$. Let $\lambda \notin \overline{\{a_n : n \geq 0\}}$. Then there exists $\epsilon > 0$ such that $|\lambda - a_n| \geq \epsilon$ for all $n \geq 0$. This means the sequence $b = ((\lambda - a_0)^{-1}, (\lambda - a_1)^{-1}, \ldots)$ is bounded, so M_b exists and satisfies $M_b\,(M_a - \lambda\, I) = (M_a - \lambda\, I)\, M_b = I$. Hence $\lambda \notin \sigma(M_a)$.

Since the spectrum is closed, one obtains $\sigma(M_a) = \overline{\{a_n : n \geq 0\}}$. $\square$

Remark 11. [Significance of ℓ^2 Space] The space ℓ^2 serves as the prototypical separable infinite-dimensional Hilbert space. Every separable Hilbert space is isometrically isomorphic to ℓ^2, making it the canonical model for quantum mechanical state spaces and numerous applications in functional analysis. The completeness of ℓ^2 enables the development of spectral theory, projection theory, and the rich geometric structure that characterizes Hilbert space analysis.