Eigenfunction Expansions for Mercer Kernels

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Consider an integral operator with kernel $R(s,t)$ acting on functions in $L^2[0,\infty)$.

Definition 1

The eigenfunctions satisfy the equation:

$$\int_0^\infty R(s,t)\,\psi(s)\;d\,s = \lambda\,\psi(t) \tag{1}$$

where $\{\psi_n\}_{n=1}^\infty$ *are the eigenfunctions with corresponding eigenvalues* $\{\lambda_n\}_{n=1}^\infty$

Definition 2

Let $\{\phi_j\}_{j=1}^\infty$ *be a complete orthonormal basis of* $L^2[0,\infty)$ *and define the kernel matrix elements:*

$$K_{kj} = \int_0^\infty \int_0^\infty R(s,t)\,\phi_k(t)\,\phi_j(s)\;d\,t\;d\,s \tag{2}$$

Theorem 3

If $\psi_n(t) = \sum_{j=1}^\infty c_{n,j}\,\phi_j(t)$ *is an eigenfunction expansion, then:*

$$c_{n,k} = \frac{\int_0^\infty \phi_k(t)\,\psi_n(t)\;d\,t}{\lambda_n} \tag{3}$$

Proof.

1. Begin with the eigenfunction equation for ψ_n:

$$\int_0^\infty R(s,t)\,\psi_n(s)\;d\,s = \lambda_n\,\psi_n(t) \tag{4}$$

2. Multiply both sides by $\phi_k(t)$ and integrate over t:

$$\int_0^\infty \phi_k(t) \int_0^\infty R(s,t)\,\psi_n(s)\;d\,s\;d\,t = \lambda_n \int_0^\infty \phi_k(t)\,\psi_n(t)\;d\,t \tag{5}$$

3. Apply Fubini's theorem to swap integration order on the left side:

$$\int_0^\infty \int_0^\infty R(s,t)\,\phi_k(t)\;d\,t\;\psi_n(s)\;d\,s = \lambda_n \int_0^\infty \phi_k(t)\,\psi_n(t)\;d\,t \tag{6}$$

4. Substitute the eigenfunction expansion $\psi_n(s) = \sum_{j=1}^\infty c_{n,j}\,\phi_j(s)$:

$$\int_0^\infty \int_0^\infty R(s,t)\,\phi_k(t)\;d\,t\;\sum_{j=1}^\infty c_{n,j}\,\phi_j(s)\;d\,s = \lambda_n \int_0^\infty \phi_k(t)\,\psi_n(t)\;d\,t \tag{7}$$

5. Exchange summation and integration (justified by L^2 convergence):

$$\sum_{j=1}^\infty c_{n,j} \int_0^\infty \int_0^\infty R(s,t)\,\phi_k(t)\,\phi_j(s)\;d\,t\;d\,s = \lambda_n \int_0^\infty \phi_k(t)\,\psi_n(t)\;d\,t \tag{8}$$

6. Recognize the kernel matrix elements:

$$\sum_{j=1}^\infty c_{n,j}\,K_{kj} = \lambda_n \int_0^\infty \phi_k(t)\,\psi_n(t)\;d\,t \tag{9}$$

7. Note that $\sum_{j=1}^\infty c_{n,j}\,K_{kj}$ is the k-th component of $K\mathbf{c}_n$. Since ψ_n is an eigenfunction, $\mathbf{c}_n$ must satisfy $K\mathbf{c}_n = \lambda_n\mathbf{c}_n$, thus:

$$\lambda_n\,c_{n,k} = \lambda_n \int_0^\infty \phi_k(t)\,\psi_n(t)\;d\,t \tag{10}$$

8. Divide both sides by λ_n (noting $\lambda_n \neq 0$ for non-trivial eigenfunctions):

$$c_{n,k} = \frac{\int_0^\infty \phi_k(t)\,\psi_n(t)\;d\,t}{\lambda_n} \tag{11}$$

This establishes that the coefficient $c_{n,k}$ in the eigenfunction expansion equals the normalized inner product of the basis function ϕ_k with the eigenfunction ψ_n. □