The Riemann-Siegel Theta Function

by Stephen Crowley

February 19, 2025

Theorem 1. *For all real t,*

$$\theta(t) = \arg \Gamma \left(\frac{1}{4} + i \frac{t}{2} \right) - \frac{\log \pi}{2} t + \pi \left\lfloor \frac{t}{2} + \frac{1}{4} \right\rfloor \tag{1}$$

is continuous.

Proof. Let $z(t) = \frac{1}{4} + i \frac{t}{2}$. The reflection formula states:

$$\Gamma(z)\,\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)} \tag{2}$$

Taking arguments:

$$\arg \Gamma(z) + \arg \Gamma(1 - z) = -\arg \sin(\pi z) (\bmod 2\pi) \tag{3}$$

For t_0 where $\frac{t_0}{2} + \frac{1}{4} = n \in \mathbb{Z}$:

$$z(t_0 \pm \epsilon) = n \pm i \frac{\epsilon}{2} \tag{4}$$

The argument $\arg \Gamma(z(t))$ jumps by π at each such t_0:

$$\lim_{\epsilon \to 0^+} \arg \Gamma(z(t_0 + \epsilon)) = \lim_{\epsilon \to 0^-} \arg \Gamma(z(t_0 - \epsilon)) + \pi \tag{5}$$

At these same points t_0, the floor function term jumps:

$$\lim_{\epsilon \to 0^+} \pi \left\lfloor \frac{t_0 + \epsilon}{2} + \frac{1}{4} \right\rfloor = \pi n \tag{6}$$

$$\lim_{\epsilon \to 0^-} \pi \left\lfloor \frac{t_0 - \epsilon}{2} + \frac{1}{4} \right\rfloor = \pi (n - 1) \tag{7}$$

Therefore at each t_0:

$$\begin{aligned} \lim_{\epsilon \to 0^+} \theta(t_0 + \epsilon) &= [\arg \Gamma(z(t_0^+)) - \frac{\log \pi}{2} t_0 + \pi n] \\ &= [\arg \Gamma(z(t_0^-)) + \pi - \frac{\log \pi}{2} t_0 + \pi n] \\ &= [\arg \Gamma(z(t_0^-)) - \frac{\log \pi}{2} t_0 + \pi (n - 1)] \\ &= \lim_{\epsilon \to 0^-} \theta(t_0 - \epsilon) \end{aligned} \tag{8}$$

Between any two consecutive such points t_0, all terms in $\theta(t)$ are continuous. Thus $\theta(t)$ is continuous for all real t. □