Theorem 1

Let T_K be a compact self-adjoint integral covariance operator on $L^2[0,\infty)$

$$(T_K f)(z) = \int_0^\infty K(z,w)\, f(w)\, d\,w \tag{1}$$

defined by kernel K:

$$K(z,w) = \sum_{k=0}^{\infty} \lambda_k \phi_k(z) \phi_k(w)$$

where $\{\phi_n\}_{n=0}^{\infty}$ is a sequence of orthonormal eigenfunctions in $L^2[0,\infty)$ and $\{\lambda_n\}_{n=0}^{\infty}$ the corresponding eigenvalues ordered by decreasing magnitude

$$|\lambda_{n+1}| < |\lambda_n| \forall n \tag{2}$$

satisfy the eigenfunction equations

$$\begin{aligned}
(T_K \phi_n)(z) &= \int_0^\infty K(z,w)\, \phi_n(w)\, d\,w \\
&= \int_0^\infty \left(\sum_{k=0}^{\infty} \lambda_k \phi_k(z) \phi_k(w) \right) \phi_n(w)\, d\,w \\
&= \int_0^\infty \phi_n(w) \left(\sum_{k=0}^{\infty} \lambda_k \phi_k(z) \phi_k(w) \right) d\,w \\
&= \int_0^\infty \sum_{k=0}^{\infty} \phi_n(w) \lambda_k \phi_k(z) \phi_k(w)\, d\,w \\
&= \sum_{k=0}^{\infty} \int_0^\infty \phi_n(w) \lambda_k \phi_k(z) \phi_k(w)\, d\,w \\
&= \sum_{k=0}^{\infty} \phi_k(z) \lambda_k \int_0^\infty \phi_n(w) \phi_k(w)\, d\,w \\
&= \sum_{k=0}^{\infty} \phi_k(z) \lambda_k \delta_{n,k} \\
&= \phi_n(z)\, \lambda_n
\end{aligned} \tag{3}$$

Let T_{K_N} be the truncated operator with kernel

$$K_N(z,w) = \sum_{n=0}^{N} \lambda_n\, \phi_n(z)\, \phi_n(w) \tag{4}$$

then:

$$\|T_K - T_{K_N}\| \le |\lambda_{N+1}| \tag{5}$$

Proof. Let E_N be the difference $T_K - T_{K_N}$. For any $f \in L^2[0,\infty)$:

Let $f = g + h$ where $g \in \mathrm{span}\{\phi_k\}_{k \leq N}$ and $h \in \mathrm{span}\{\phi_k\}_{k > N}$ so that

$$g(x) = \sum_{k=0}^{N} \langle f, \phi_k \rangle \, \phi_k(x) \tag{6}$$

and

$$h(x) = \sum_{k=N+1}^{\infty} \langle f, \phi_k \rangle \, \phi_k(x) \tag{7}$$

where by orthogonality of g and h

$$\langle g, h \rangle = \int_0^\infty g(x) h(x) \mathrm{d}x = 0 \tag{8}$$

we have

$$\|E_N f\|^2 = \langle E_N f, E_N f \rangle = \langle E_N h, h \rangle \tag{9}$$

because $E_N g = 0$ by construction and since h is orthogonal to the first N eigenfunctions and

$$|\lambda_k| \leq |\lambda_{N+1}| \forall k > N \tag{10}$$

we have

$$\begin{aligned} |\langle E_N h, h \rangle| &\leq |\lambda_{N+1}| \|h\|^2 \\ &\leq |\lambda_{N+1}| \|f\|^2 \end{aligned} \tag{11}$$

Therefore:

$$\|E_N\| \leq |\lambda_{N+1}| \tag{12}$$

□

Remark 2. This extension of Mercer's Theorem to $[0, \infty)$ reveals a deeper truth about integral operators that is obscured in most presentations. The key insight is that compactness of the interval plays no essential role - what matters is the compactness of the operator itself.

The traditional presentation of Mercers theorem on compact $[a, b]$ emphasize properties that are merely convenient rather than fundamental:

- Compactness of $[a, b]$ provides easy continuity arguments
- Finite measure simplifies certain technical steps
- Historical development focused on these cases first

However, the proof above shows that the essential structure depends only on:

1. The spectral properties of compact self-adjoint operators

2. The precise operator norm bound $\|E_N\| \leq |\lambda_{N+1}|$
3. The fact that $\{\lambda_n\}_{n=1}^{\infty}$ converges to zero

This reveals that Mercer's Theorem is fundamentally about the behavior of integral operators themselves, not about properties of their domains. The extension to $[0, \infty)$ is not just a generalization - it's a clearer view of the true mathematical structure.

Theorem 3

(Completeness) *Let T_K be a compact self-adjoint integral operator on $L^2[0, \infty)$ with kernel $K(z, w)$. Then the eigenfunctions $\{\phi_n\}_{n=0}^{\infty}$ form a complete orthonormal system in $L^2[0, \infty)$.*

Proof. Suppose there exists $f \in L^2[0, \infty)$ orthogonal to all ϕ_n. Then:

$$\langle f, \phi_n \rangle = 0 \quad \forall n \tag{13}$$

Therefore:

$$T_K f = \sum_{n=0}^{\infty} \lambda_n \langle f, \phi_n \rangle \, \phi_n = 0 \tag{14}$$

Since T_K is compact and self-adjoint, $\ker(T_K)^{\perp} = \overline{\mathrm{range}(T_K)}$ contains all eigenvectors. Thus f must be zero, proving completeness. $\square$