Notes on Measures Equivalent to Wiener Measure[1]

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Recently, Shepp [shepp1966] has given, among several other results, a simple condition for the equivalence of a Gaussian measure to Wiener measure. His proof is ingenious but long. The purpose of this note is to record a somewhat simpler proof of this result which was obtained independently. The proof uses a reproducing kernel Hilbert space (RKHS) criterion, due to Oodaira [oodaira1963], for the equivalence of Gaussian measures.

We first state Shepp's theorem (in a terminology more familiar to engineers):

Theorem 1. *[Shepp's Theorem] Discrimination between a zero-mean Gaussian process with covariance function $R(t,s)$ and a Wiener process with covariance function $\min(t,s)$, $0 \le t, s \le T$, will be nonsingular if and only if there exists a unique, symmetric, square-integrable function K on $[0,T] \times [0,T]$,*

$$K(t,s) = K(s,t), \qquad \iint_{[0,T]^2} K^2(t,s)\, dt\, ds < \infty \tag{1}$$

such that

$$R(t,s) - \min(t,s) = \iint_{[0,T]^2} K(u,v)\, du\, dv \tag{2}$$

and

$$1 \text{ is not an eigenvalue of } K(u,v). \tag{3}$$

Remark 2. That is, the measures corresponding to these two processes will be equivalent.

As mentioned above, this result will be obtained by direct application of the following theorem of Oodaira [oodaira1963]:

Theorem 3. *[Oodaira] Discrimination between zero-mean Gaussian processes with covariance functions $R_1(t,s)$, $R_2(t,s)$, $t, s \in [0,T]$, will be nonsingular if and only if*

1. *$R_1 - R_2$ belongs to the direct product space $H(R_2) \otimes H(R_2)$, where $H(R_2)$ is the reproducing kernel Hilbert space (RKHS) of R_2;*
2. *There exist positive constants $c_1, c_2 > 0$ such that $c_1 R_2 \le R_1 \le c_2 R_2$.*

We apply this theorem to $R_1(t,s)=R(t,s)$ and $R_2(t,s)=\min(t,s)$, $0\leq t,s\leq T$. The calculation of $H(R_2)\otimes H(R_2)$ is straightforward and actually available in the literature on RKHS (see, e.g., Parzen [parzen1963], p. 164):

A function $g(t,s)$, $0\leq t,s\leq T$, will belong to $H(R_2)\otimes H(R_2)$, where $R_2(t,s)=\min(t,s)$, if and only if

$$\iint_{[0,T]^2}\left|\frac{\partial^2 g(t,s)}{\partial t\,\partial s}\right|^2 d t\, d s<\infty \tag{4}$$

It follows that conditions (1) and (2) are equivalent to condition (i) of Theorem 3. To show the equivalence of (3) and condition (ii), proceed as follows.

The inner product in $H(R_2)$ is

$$(f,g)_{H(R_2)}=\int_0^T f'(t)\,g'(t)\,d t \tag{5}$$

where the primes denote differentiation with respect to t.

Let the (orthonormalized) eigenfunctions and eigenvalues of the (Hilbert-Schmidt) kernel $K(t,s)$ be given by

$$\int_0^T K(t,s)\,\psi_j(s)\,d s=\lambda_j\,\psi_j(t),\qquad j=1,2,\ldots \tag{6}$$

and define

$$\Phi_j(t)=\int_0^T\psi_j(s)\,d s,\qquad j=1,2,\ldots \tag{7}$$

It is easy to verify that

$$(\min(t,s),\Phi_j(s))_{H(R_2)}=\int_0^T\psi_j(s)\,d s=\Phi_j(t) \tag{8}$$

$$\left(\iint_{[0,T]^2}K(u,v)\;d u\;d v,\Phi_j(s)\right)_{H(R_2)}=\lambda_j\,\Phi_j(t) \tag{9}$$

Taking inner products with ψ_j in $H(R_2)$, the condition

$$c_1\min(t,s)\leq\min(t,s)-\iint_{[0,T]^2}K(u,v)\;d u\;d v\leq c_2\min(t,s) \tag{10}$$

becomes

$$c_1\,\Phi_j(t)\leq(1-\lambda_j)\,\Phi_j(t)\leq c_2\,\Phi_j(t),\qquad j=1,2,\ldots \tag{11}$$

Taking inner products with ψ_j again gives

$$c_1 \leq 1 - \lambda_j \leq c_2 \tag{12}$$

i.e.,

$$1 - c_2 \leq \lambda_j \leq 1 - c_1 < 1 \tag{13}$$

which is equivalent to condition (3).

The above arguments establish the equivalence of the RKHS conditions to the requirements on K and the eigenvalues, thus proving the theorem.

Additional Remarks

If the process with covariance R has a nonzero mean $m(t)$, $0 \leq t \leq T$, then for nonsingular detection $m(t)$ must belong to $H(R_2)$, which in turn will only be true if $\frac{d\,m(t)}{d\,t}$ is square integrable on $(0, T)$ (see, e.g., Parzen [parzen1963], p. 159).

This condition, due to Parzen [parzen1961], has also been obtained in a different way by Shepp [shepp1966], Eq. 1.3. In his paper, Shepp treats several questions related to Wiener measure. Some of these results (Eqs. 5.6, 5.14, 5.15, 6.8–6.10) can again be obtained by RKHS methods. The expansion theorems in Shepp [shepp1966], Theorems 2 and 3, can also be generalized by such methods. Details will be given in a later paper.

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