Stationary Isotropic Correlation Functions

Abstract

Examples of Stationary Isotropic Correlation Functions in 1, 2 and 3 dimensions are recalled from Yaglom'87[1]

Example 1. [1, Example 6]Consider the stationary correlation function (2.101) given by the equation

$$B(\tau) = C \exp(-\alpha\,\tau)\cos(\omega_0\,\tau) \tag{1}$$

where $\alpha > 0$, $\omega_0 > 0$, $\tau > 0$.

It is clear that if α is small enough, this function does not satisfy even the first inequality (4.1.13), i.e. it does not belong to the class D_0 of isotropic correlation function in the space $\mathbb{R}^n$. By evaluating, with the aid of (4.1.18) or (2.102) and (4.1.32), the two-dimensional spectral density $f(k)$, which corresponds to the correlation function $B(\tau)$, and by studying the conditions for the nonnegativity of this $f(k)$, one can show that the function (2.101) is a possible correlation function of the isotropic field in the space $\mathbb{R}^n$ if, and only if, $\alpha > \sqrt{\omega_0^2 + \lambda_n^2}$. It is still easier to obtain conditions guaranteeing that $B(\tau)$ belongs to D_n. Indeed, it was noted on pp. 124-125 that for a (one-dimensional) spectral density corresponding to the correlation function (2.101) is monotonically nonincreasing on the positive half-axis if, and only if, $\alpha \geq \sqrt{3}\,\omega_0$. It follows that the function (2.101) is a three-dimensional isotropic correlation function in the space $\mathbb{R}^3$ if, and only if, $\alpha \geq \sqrt{3}\,\omega_0$.

Example 2. [1, Example 7]Formula (4.105) shows that the function

$$B(\tau) = \frac{J_{\frac{n-2}{2}}(\alpha\,\tau)}{C\,(\alpha\,\tau)^{\frac{n-2}{2}}} \tag{2}$$

where $C > 0$, $\alpha > 0$, is an n-dimensional isotropic correlation function. The spectral distribution function $\Phi(k)$ corresponding to this correlation function is clearly of the form

$$\Phi(k) = \begin{cases} 0, & \text{for } k < \alpha \\ \frac{\frac{C}{2}\left(\frac{\alpha}{2}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)k^{n+1}} & \text{for } k > \alpha \end{cases}$$

i.e., the n-dimensional spectral density is given here by the relation

$$f(k) = \Phi'(k)\left(\frac{\alpha}{2}\right)^{\frac{n}{2}} 2\,\pi^{-\frac{n}{2}}\,k^{n-1} = \frac{\frac{C}{(2\,\pi)^{\frac{n}{2}}}\,\alpha^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}\,k^{n-1}\,\delta\,(k-\alpha) \tag{3}$$

where $\delta(x)$ is the Dirac δ-function. Using (4.120), we easily obtain an equation of the form (2.138), with $\nu = \frac{n-2}{2}$, for the corresponding one-dimensional spectral density $f_1(k)$. Replacing $\frac{n-2}{2}$ in the right-hand side of (4.1.39) by an arbitrary real number $\nu > -\frac{1}{2}$, we arrive at a more general stationary correlation function (2.137). The inequality (4.112) clearly implies that the function (2.137) does not belong to the class D_n (i.e. cannot be an isotropic correlation function in the n-dimensional space $\mathbb{R}^n$) if $\nu < \frac{n-2}{2}$. On the other hand, it is not hard to show that if $\nu > \frac{n-2}{2}$ the function (2.137) belongs to the class D_{n} . In fact, it is well known that for $\nu > \nu_* = -1$ the following identity holds:

$$\int_0^\infty k^\mu\,J_\mu\,(k\,x)\,J_\mu\,(k\,y)\,k\;d\,k = \frac{2^{\mu-1}\,\Gamma\,(\nu+\mu)}{x^\mu\,\Gamma(\mu)\,y^{\nu+1}}\,\delta\,(x-y) \quad \forall \mu > -\frac{1}{2} \tag{4}$$

(see, e.g., Erdelyi et al., 1953, Vol. 2, Eq. 7.14.2(34) or Gradshteyn and Ryzhik, 1980, Eq. 6.575(1)). According to (4.114) it follows from this that, for $\nu > (n-2)/2$, the n-dimensional spectral density $f(k)$, which corresponds to the correlation function (2.137), has the form

$$f(k) = \begin{cases} C\,(\alpha^2 - k^2)\nu - \frac{n}{2} & \text{for } k < \alpha \\ 0 & \text{for } k > \alpha \end{cases} \tag{5}$$

where C is a normalization constant. The function (4.139) is everywhere nonnegative; hence the function (2.137), where $\nu > \frac{n-2}{2}$, belongs to D_n.

Some more examples of isotropic correlation functions can also be found in the available literature, but we will not pursue this topic any further here.

Bibliography

[1] A.M. Yaglom. *Correlation Theory of Stationary and Related Random Functions: Volume I: Basic Results*. Applied Probability. Springer New York, 1987.