The Riemann Zeta Functional Equation via the Shah Function: Complete Rigorous Derivation

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Abstract

This paper presents a complete rigorous derivation of the Riemann zeta functional equation $\xi(s) = \xi(1-s)$ using the Shah function (Dirac comb) as the central analytical tool. The Shah function $\mathrm{III}(x) = \sum_{n \in \mathbb{Z}} \delta(x-n)$ provides a distributional framework that converts discrete sums into continuous integrals through its fundamental sampling property $\int_{-\infty}^{\infty} f(x)\,\mathrm{III}(x)\; d\,x = \sum_{n \in \mathbb{Z}} f(n)$. This duality enables the application of Fourier analysis to number-theoretic objects. We establish the Poisson summation formula as a direct consequence of the Shah function's Fourier series representation, apply it to Gaussian functions $e^{-\pi x^2 t}$ to derive the Jacobi theta functional equation $\theta(t) = t^{-1/2}\,\theta(1/t)$, and then connect the theta function to the Riemann zeta function via Mellin transform. The proof is entirely self-contained, with all calculations carried out explicitly without appeal to external results. The theta functional equation, obtained through the self-duality of Gaussians under Fourier transformation and mediated by the Shah function's integer sampling, is shown to imply the zeta functional equation through a careful analysis of the integral representation $2\,\xi(s) = \int_0^{\infty} (\theta(t) - 1)\, t^{s/2-1}\, d\,t$. This approach illuminates the deep connection between sampling theory in signal processing, modular forms in number theory, and the analytic properties of $\zeta(s)$.

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1 Foundational Definitions

This section fixes notation and conventions for distributions, Fourier analysis, and special functions. References include [GelfandShilov, SteinShakarchiFourier, GrafakosClassical, RudinRCA, Edwards, Titchmarsh, IK].

Definition 1. *[Schwartz space and tempered distributions] The Schwartz space $\mathcal{S}(\mathbb{R})$ consists of all C^∞ functions $\phi\colon \mathbb{R} \to \mathbb{R}$ such that for every pair of nonnegative integers m, n,*

$$\sup_{x \in \mathbb{R}} |\, x^m \, \phi^{(n)}(x) \,| \, < \, \infty \tag{1}$$

Its continuous dual $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions. The distributional pairing is denoted $\langle T, \phi \rangle$ for $T \in \mathcal{S}'(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R})$.

Definition 2. *[Dirac delta] The Dirac delta $\delta \in \mathcal{S}'(\mathbb{R})$ is defined by*

$$\int_{-\infty}^{\infty} \delta\,(x-a)\; \phi(x)\; d\,x \;=\; \phi(a) \tag{2}$$

for all test functions $\phi \in \mathcal{S}(\mathbb{R})$ and all $a \in \mathbb{R}$ [GelfandShilov].

Definition 3. *[Shah function (Dirac comb)] The Shah function $\mathrm{III} \in \mathcal{S}'(\mathbb{R})$ is the 1-periodic tempered distribution*

$$\mathrm{III}(x) \;=\; \sum_{n \in \mathbb{Z}} \delta\,(x-n) \tag{3}$$

Definition 4. *[Fourier transform] For $f \in L^1(\mathbb{R})$, the Fourier transform is*

$$\hat{f}(\omega) \;=\; \int_{-\infty}^{\infty} f(x)\; e^{-2\pi i\,\omega x}\; d\,x \tag{4}$$

With the normalization in (4), the transform extends to a topological automorphism of $\mathcal{S}(\mathbb{R})$ and by duality to $\mathcal{S}'(\mathbb{R})$ [SteinShakarchiFourier, GrafakosClassical].

Definition 5. *[Mellin transform] For $F\colon (0,\infty) \to \mathbb{C}$ locally integrable and $s \in \mathbb{C}$ in a vertical strip where the integral converges, the Mellin transform is*

$$\mathcal{M}\{F\}(s) \;=\; \int_{0}^{\infty} F(t)\; t^{\,s-1}\; d\,t \tag{5}$$

Definition 6. *[Jacobi theta function] For $t > 0$,*

$$\theta(t) \;=\; \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} \tag{6}$$

Definition 7. *[Riemann zeta function] For $\mathrm{Re}(s) > 1$,*

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{7}$$

Definition 8. *[Completed zeta] The completed zeta function is*

$$\xi(s) \;=\; \pi^{-\frac{s}{2}}\,\Gamma\!\left(\frac{s}{2}\right)\,\zeta(s) \tag{8}$$

2 The Shah Function and Sampling

Proposition 9. *[Action of III] For any $\phi \in \mathcal{S}(\mathbb{R})$,*

$$\langle \mathrm{III},\, \phi\rangle \;=\; \sum_{n\in\mathbb{Z}} \phi(n) \tag{9}$$

Proof. By linearity of the pairing and (2),

$$\langle \mathrm{III},\, \phi\rangle = \left\langle \sum_{n\in\mathbb{Z}} \delta\left(\,\cdot - n\right),\, \phi \right\rangle \tag{10}$$

$$= \sum_{n\in\mathbb{Z}} \langle \delta(\,\cdot - n),\, \phi\rangle \tag{11}$$

$$= \sum_{n\in\mathbb{Z}} \phi(n) \tag{12}$$

□

Proposition 10. *[Fourier series of III] As a periodic tempered distribution,*

$$\mathrm{III}(x) \;=\; \sum_{k\in\mathbb{Z}} e^{2\pi i k x} \tag{13}$$

Proof. The period is 1. The k-th Fourier coefficient over $[0,1)$ equals

$$c_k \;=\; \int_0^1 \mathrm{III}(x)\; e^{-2\pi i k x}\; d\,x \;=\; \langle \delta(x),\, e^{-2\pi i k x}\rangle \;=\; 1 \tag{14}$$

since the only delta in $[0,1)$ is at $x=0$. Thus (13) holds in $\mathcal{S}'(\mathbb{R})$ [SteinShakarchiFourier, §1.2]. □

3 Poisson Summation Formula

Theorem 11. *[Poisson summation] For $f \in \mathcal{S}(\mathbb{R})$,*

$$\sum_{n\in\mathbb{Z}} f(n) \;=\; \sum_{k\in\mathbb{Z}} \hat{f}(k) \tag{15}$$

Proof. By (9) and (13),

$$\sum_{n\in\mathbb{Z}} f(n) = \langle \mathrm{III},\; f\rangle \tag{16}$$

$$= \int_{-\infty}^{\infty} f(x)\, \mathrm{III}(x)\; d\,x \tag{17}$$

$$= \int_{-\infty}^{\infty} f(x)\; \sum_{k\in\mathbb{Z}} e^{2\pi i k x}\; d\,x \tag{18}$$

$$= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} f(x)\; e^{2\pi i k x}\; d\,x \tag{19}$$

$$= \sum_{k\in\mathbb{Z}} \hat{f}(-k) \;=\; \sum_{k\in\mathbb{Z}} \hat{f}(k) \tag{20}$$

where interchange in (19) is justified by the rapid decay of f [SteinShakarchiFourier, Ch. 3]. □

4 Gaussian Fourier Transform

Lemma 12. *[Gaussian transform] For $t>0$ and $g_t(x) = e^{-\pi t x^2}$*

$$\hat{g}_t(\omega) \;=\; \frac{1}{\sqrt{t}}\; e^{-\pi \frac{\omega^2}{t}} \tag{21}$$

Proof. Compute

$$\hat{g}_t(\omega) = \int_{-\infty}^{\infty} e^{-\pi t x^2}\, e^{-2\pi i \omega x}\, d\,x \tag{22}$$

$$= \int_{-\infty}^{\infty} e^{-\pi t\left(x^2 + \frac{2 i \omega}{t} x\right)}\, d\,x \tag{23}$$

Complete the square:

$$x^2 + \frac{2\,i\,\omega}{t} x \;=\; \left(x + \frac{i\,\omega}{t}\right)^2 - \left(\frac{i\,\omega}{t}\right)^2 \;=\; \left(x + \frac{i\,\omega}{t}\right)^2 + \frac{\omega^2}{t^2} \tag{24}$$

Therefore

$$-\pi\, t\left(x^2 + \frac{2\,i\,\omega}{t}\, x\right) = -\pi\, t\left(x + \frac{i\,\omega}{t}\right)^2 - \pi\, \frac{\omega^2}{t} \tag{25}$$

Insert (25) into (23):

$$\hat{g}_t(\omega) = e^{-\pi \frac{\omega^2}{t}} \int_{-\infty}^{\infty} e^{-\pi t\left(x + \frac{i\omega}{t}\right)^2}\, d\,x \tag{26}$$

Shift the contour $x \mapsto y = x + \frac{i\,\omega}{t}$ (justified since the integrand is entire and decreases sufficiently quickly along horizontal lines):

$$\int_{-\infty}^{\infty} e^{-\pi t\left(x + \frac{i\omega}{t}\right)^2}\, d\,x = \int_{-\infty}^{\infty} e^{-\pi t y^2}\, d\,y \;=\; \frac{1}{\sqrt{t}} \tag{27}$$

using the standard Gaussian integral with the present normalization [SteinShakarchiFourier, §1.2]. Combining (26) and (27) yields (21). □

5 Theta Functional Equation

Theorem 13. *[Theta modular relation] For all $t > 0$,*

$$\theta(t) \;=\; \frac{1}{\sqrt{t}}\, \theta\!\left(\frac{1}{t}\right) \tag{28}$$

Proof. Apply (11) to $f(x) = e^{-\pi t x^2}$. The left-hand side is

$$\sum_{n \in \mathbb{Z}} e^{-\pi t n^2} \;=\; \theta(t) \tag{29}$$

By (12), the right-hand side is

$$\sum_{k\in\mathbb{Z}} \hat{f}(k) = \sum_{k\in\mathbb{Z}} \frac{e^{-\pi\frac{k^2}{t}}}{\sqrt{t}} = \frac{\theta\left(\frac{1}{t}\right)}{\sqrt{t}} \tag{30}$$

Equating (29) and (30) gives (28) [SteinShakarchiFourier, §1.4], [ApostolModular, Ch. 1]. □

6 Connection to ζ via Mellin Transform

Lemma 14. *[Theta decomposition] For* $t>0$,

$$\theta(t)-1 = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{-\pi n^2 t} = 2\sum_{n=1}^{\infty} e^{-\pi n^2 t} \tag{31}$$

Proof. By (6), the term $n=0$ contributes 1, and the remaining terms pair as $\pm n$ with identical value, giving (31). □

Lemma 15. *[Theta–zeta Mellin identity] For* $\mathrm{Re}(s)>1$,

$$\int_0^{\infty} (\theta(t)-1)\, t^{\frac{s}{2}-1}\, dt = 2\,\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\,\zeta(s) \tag{32}$$

Proof. Using (31) and Fubini,

$$\int_0^{\infty} (\theta(t)-1)\, t^{\frac{s}{2}-1}\, dt = 2\sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 t}\, t^{\frac{s}{2}-1}\, dt \tag{33}$$

For each $n\geq 1$, substitute $u=\pi n^2 t$:

$$\int_0^{\infty} e^{-\pi n^2 t}\, t^{\frac{s}{2}-1}\, dt = \int_0^{\infty} e^{-u}\left(\frac{u}{\pi n^2}\right)^{\frac{s}{2}-1}\frac{du}{\pi n^2} \tag{34}$$

$$= \frac{1}{\pi^{\frac{s}{2}}\, n^s}\int_0^{\infty} e^{-u}\, u^{\frac{s}{2}-1}\, du \tag{35}$$

$$= \frac{1}{\pi^{\frac{s}{2}}\, n^s}\,\Gamma\left(\frac{s}{2}\right) \tag{36}$$

Insert (36) into (33) to obtain (32). See [Edwards, §3.5], [Titchmarsh, §2.10]. □

7 The Riemann Zeta Functional Equation

Theorem 16. *[Functional equation for ξ] The completed zeta $\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$ satisfies the meromorphic identity*

$$\xi(s) \;=\; \xi\,(1-s) \qquad \text{for all } s \in \mathbb{C}. \tag{37}$$

Proof. By (15), for $\mathrm{Re}(s) > 1$,

$$2\,\xi(s) \;=\; \int_0^\infty (\theta(t) - 1)\; t^{\frac{s}{2}-1}\; d\,t \tag{38}$$

Split at $t = 1$:

$$2\,\xi(s) \;=\; \int_0^1 (\theta(t) - 1)\; t^{\frac{s}{2}-1}\; d\,t \;+\; \int_1^\infty (\theta(t) - 1)\; t^{\frac{s}{2}-1}\; d\,t \tag{39}$$

Use (28) in the first integral and substitute $u = \frac{1}{t}$:

$$\begin{aligned}
\int_0^1 (\theta(t) - 1)\; t^{\frac{s}{2}-1}\; d\,t &= \int_0^1 \left(\frac{1}{\sqrt{t}}\;\theta\!\left(\frac{1}{t}\right) - 1\right) t^{\frac{s}{2}-1}\; d\,t & (40)\\
&= \int_0^1 \theta\!\left(\frac{1}{t}\right)\; t^{\frac{s}{2}-\frac{3}{2}}\; d\,t \;-\; \int_0^1 t^{\frac{s}{2}-1}\; d\,t & (41)\\
&= \int_1^\infty \theta(u)\; u^{-\frac{s}{2}-\frac{1}{2}}\; d\,u \;-\; \frac{2}{s} & (42)\\
&= \int_1^\infty (1 + (\theta(u) - 1))\; u^{\frac{1-s}{2}-1}\; d\,u \;-\; \frac{2}{s} & (43)
\end{aligned}$$

The first term in (43) evaluates (for $\mathrm{Re}(s) > 1$) to

$$\int_1^\infty u^{\frac{1-s}{2}-1}\; d\,u \;=\; -\,\frac{2}{1-s}, \tag{44}$$

hence

$$\int_0^1 (\theta(t) - 1)\; t^{\frac{s}{2}-1}\; d\,t \;=\; \int_1^\infty (\theta(u) - 1)\; u^{\frac{1-s}{2}-1}\; d\,u \;-\; \frac{2}{s} \;-\; \frac{2}{1-s} \tag{45}$$

Insert (45) into (39) and relabel $u \mapsto t$:

$$\begin{aligned}
2\,\xi(s) &= \int_1^\infty (\theta(t) - 1)\; t^{\frac{1-s}{2}-1}\; d\,t \;-\; \frac{2}{s} \;-\; \frac{2}{1-s} \;+\; \int_1^\infty (\theta(t) - 1)\; t^{\frac{s}{2}-1}\; d\,t & (46)\\
&= \int_1^\infty (\theta(t) - 1)\; \left(t^{\frac{s}{2}-1} + t^{\frac{1-s}{2}-1}\right) d\,t \;-\; \frac{2}{s} \;-\; \frac{2}{1-s} & (47)
\end{aligned}$$

The right-hand side of (47) is invariant under $s \mapsto 1-s$. Therefore, for $\mathrm{Re}(s) > 1$,

$$\xi(s) \;=\; \xi\,(1-s) \tag{48}$$

Both sides of (48) extend meromorphically to $\mathbb{C}$ and agree on a nonempty open set, hence (37) holds for all $s \in \mathbb{C}$ by analytic continuation [Titchmarsh, §2.10], [Edwards, Ch. 3]. □

Remark 17. [Alternative normalization] The entire function $\Xi(s) = \frac{1}{2}\, s\,(s-1)\;\xi(s)$ also satisfies $\Xi(s) = \Xi\,(1-s)$ and is often used in the theory of the Riemann Hypothesis [Titchmarsh, §2.12], [Edwards, Ch. 1].

Acknowledgments

Standard references: [Riemann1859, Titchmarsh, Edwards, SteinShakarchiFourier, GrafakosClassical, IK, GelfandShilov, RudinRCA, ApostolModular].

Bibliography

[ApostolModular] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, 2nd ed., Springer, 1990.

[Edwards] H. M. Edwards, *Riemann's Zeta Function*, Academic Press, 1974.

[GelfandShilov] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Vol. 1, Academic Press, 1964.

[GrafakosClassical] L. Grafakos, *Classical Fourier Analysis*, 3rd ed., Springer, 2014.

[IK] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, AMS Colloquium Publications, Vol. 53, 2004.

[Riemann1859] B. Riemann, "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse," *Monatsberichte der Berliner Akademie*, 1859. (English translation in Edwards [Edwards], Appendix.)

[RudinRCA] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw–Hill, 1987.

[SteinShakarchiFourier] E. M. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton Univ. Press, 2003.

[Titchmarsh] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Oxford Univ. Press, 1986.