The Stationary J_0 Integral Covariance Operator

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Lemma 1. *The functions*

$$\psi_n(y) = \sqrt{\frac{4n+1}{y}}\,(-1)^n\,J_{2n+\frac{1}{2}}(y)$$

are orthonormal over the interval 0 *to* ∞*, i.e.,*

$$\int_0^\infty \psi_n(y)\,\psi_m(y)\,dy = \delta_{nm},$$

where δ_{nm} *is the Kronecker delta.*

Proof. Consider the integral

$$I = \int_0^\infty \psi_n(y)\,\psi_m(y)\,dy \tag{1}$$

Substituting $\psi_n(y)$ and $\psi_m(y)$, we get:

$$I = \int_0^\infty \sqrt{\frac{4n+1}{y}}\,(-1)^n\,J_{2n+\frac{1}{2}}(y)\sqrt{\frac{4m+1}{y}}\,(-1)^m\,J_{2m+\frac{1}{2}}(y)\,dy \tag{2}$$

This simplifies to:

$$I = \sqrt{(4n+1)(4m+1)}\,(-1)^{n+m}\int_0^\infty \frac{J_{2n+\frac{1}{2}}(y)\,J_{2m+\frac{1}{2}}(y)}{y}\,dy \tag{3}$$

Using the orthogonality relation for Bessel functions with $\nu = -\frac{1}{2}$:

$$\int_0^\infty \frac{J_{\nu+2n+1}(t)\,J_{\nu+2m+1}(t)}{t}\,dt = \frac{\delta_{nm}}{2(2n+\nu+1)} \tag{4}$$

for $\nu = -\frac{1}{2}$, we have:

$$\int_0^\infty \frac{J_{2n+\frac{1}{2}}(t)\, J_{2m+\frac{1}{2}}(t)}{t}\, dt = \frac{\delta_{nm}}{4n+1} \tag{5}$$

Substituting this result back into the integral, we have:

$$I = \sqrt{(4n+1)(4m+1)}\, (-1)^{n+m} \frac{\delta_{nm}}{4n+1} \tag{6}$$

For $n \neq m$, $\delta_{nm} = 0$, yielding $I = 0$. For $n = m$, $\delta_{nm} = 1$, giving:

$$I = \frac{\sqrt{(4n+1)^2}}{4n+1} = \frac{4n+1}{4n+1} = 1 \tag{7}$$

Hence, $\psi_n(y)$ and $\psi_m(y)$ are orthonormal. □

Theorem 2. *The eigenvalues of*

$$\int_0^\infty J_0(x-y) * \psi_n(x)\, dx = \lambda_n\, \psi_n(y) \tag{8}$$

are given by

$$\begin{aligned} \lambda_n &= \int_0^\infty J_0(x)\, \psi_n(x)\, dx \\ &= \int_0^\infty J_0(x) \sqrt{\frac{4n+1}{y}}\, (-1)^n J_{2n+\frac{1}{2}}(y)\, dx \\ &= \sqrt{\frac{4n+1}{\pi}} \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} \end{aligned} \tag{9}$$

where $\psi_n(x)$ is the orthonormal set defined in Lemma 1.

Proof. To show that $\psi_n(y)$ are eigenfunctions of the integral operator with kernel $J_0(x-y)$ and to find the corresponding eigenvalues, we start with the given equation:

$$\int_0^\infty J_0(x-y)\, \psi_n(x)\, dx = \lambda_n\, \psi_n(y)$$

where

$$\psi_n(y) = \sqrt{\frac{4n+1}{y}}\, (-1)^n J_{2n+\frac{1}{2}}(y)$$

and the eigenvalues are:

$$\lambda_n = \int_0^\infty J_0(x)\,\psi_n(x)\;d\,x = \sqrt{\frac{4\,n+1}{\pi}}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\,(n+1)^2}$$

To find λ_n, we compute:

$$\lambda_n = \int_0^\infty J_0(x)\,\psi_n(x)\;d\,x$$

Substituting $\psi_n(x)$:

$$\lambda_n = \int_0^\infty J_0(x)\sqrt{\frac{4\,n+1}{x}}\,(-1)^n\,J_{2n+\frac{1}{2}}(x)\;d\,x$$

Using the integral of products of Bessel functions:

$$\int_0^\infty x^{-\frac{1}{2}}\,J_0(x)\,J_{2n+\frac{1}{2}}(x)\;d\,x = \sqrt{\frac{4\,n+1}{\pi}}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\,(n+1)^2}$$

Thus:

$$\lambda_n = \sqrt{\frac{4\,n+1}{\pi}}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\,(n+1)^2}$$

This completes the proof that $\psi_n(y)$ are eigenfunctions of the integral operator with kernel $J_0\,(x-y)$ and the corresponding eigenvalues are as given. □

Definition 3. *Let $j_n(x)$ is the spherical Bessel function of the first kind,*

$$\begin{aligned} j_n(z) &= \sqrt{\frac{\pi}{2\,z}}\,J_{n+\frac{1}{2}}(x) \\ &= \frac{\sin\,(z)\,R_{n,\frac{1}{2}}(z) - \cos\,(z)\,R_{n-1,\frac{3}{2}}(z)}{\sqrt{z}} \end{aligned} \tag{10}$$

where $R_{n,v}(z)$ are the so-called Lommel polynomials [3]

$$R_{n,v}(z) = \frac{\Gamma\,(n+v)}{\Gamma(v)}\left(\frac{2}{z}\right)^n {}_2F_3\left(\left[-\frac{n}{2},\frac{1}{2}-\frac{n}{2}\right];[v,-n,1-v-n];-z^2\right) \tag{11}$$

where ${}_2F_3$ is a generalized hypergeometric function which are really rational functions which are said to be "polynomial in $\frac{1}{z}$".

Conjecture 4. *The series*

$$
\begin{aligned}
J_0(t) &= \sum_{k=0}^{\infty} \lambda_k \psi_k(t) \\
&= \sum_{k=0}^{\infty} \sqrt{\frac{4\,n+1}{\pi}} \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\,(n+1)^2} (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, j_{2n}(t) \\
&= \sum_{k=0}^{\infty} \frac{4\,n+1}{\pi} \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\,(n+1)^2} (-1)^n\, j_{2n}(t)
\end{aligned} \tag{12}
$$

converges uniformly for all complex t *except the origin where it has a regular singular point where* $\lim_{t\to 0} J_0(t) = 1$.

Conjecture 5. *The eigenfunctions of the stationary integral covariance operator*

$$
[T\psi_n](x) = \int_0^{\infty} J_0\,(x-y)\,\psi_n(x)\mathrm{d}x = \lambda_n \psi_n(x) \tag{13}
$$

are given by

$$
\psi_n(y) = (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, j_{2n}(y) \tag{14}
$$

and the eigenvalues are given by

$$
\begin{aligned}
\lambda_n &= \int_{-\infty}^{\infty} J_0(x)\,\psi_n(x)\,\mathrm{d}x \\
&= \sqrt{\frac{4\,n+1}{\pi}} \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\,(n+1)^2} \\
&= \sqrt{\frac{4\,n+1}{\pi}}\,(n+1)^2_{-\frac{1}{2}}
\end{aligned} \tag{15}
$$

where $(n+1)^2_{-\frac{1}{2}}$ *is the Pochhammer symbol(ascending/rising factorial).*

Definition 6. *The spectral density of a stationary process is the Fourier tranform of the covariance kernel due to Wiener-Khinchine theorem.*

Definition 7. *Let $S_n(x)$ be the orthogonal polynomials whose orthogonality measure is equal to the spectral density of the process. These polynomials shall be called the spectral polynomials corresponding to the process.*

Example 8. Let the kernel function be given by $K(t,s) = J_0(t-s)$ then identify the orthogonal polynomial sequence associated with the spectral density

$$S(\omega) = \int_{-\infty}^{\infty} J_0(x)\, e^{ix\omega}\, \mathrm{d}x = \begin{cases} \dfrac{2}{\sqrt{1-\omega^2}} & \omega \in (-1,1) \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

as being twice the orthogonality measure of the Type-I Chebyshev polynomials $T_n(x)$ so that the orthogonal polynomial sequence is identified as

$$S_n(x) = \sqrt{2} T_n(x) \tag{17}$$

so that

$$\int_{-1}^{1} S_n(\omega)\, S_m(\omega)\, S(\omega)\, \mathrm{d}\omega = \begin{cases} 0 & n \neq m \\ 2\pi & n = m = 0 \\ \pi & n = m \neq 0 \end{cases} \tag{18}$$

Remark 9. If the spectral density does not equal the orthogonality measure of a known set of orthogonal polynomials then such a set can always be generated by applying the Gram-Schmidt process to the monomials so that they are transformed into a set that is orthogonal with respect any given spectral density (of a stationary process).

Definition 10. *The sequence $\hat{S}_n(y)$ of Fourier transforms of the spectral polynomials $S_n(x)$ is given by*

$$\hat{S}_n(y) = \int_{-1}^{1} S_n(x) e^{ixy} \mathrm{d}x \tag{19}$$

Example 11. The Fourier transforms of the Chebyshev polynomials are just the usual infinite Fourier transforms with the integration restricted to the range $-1 \ldots 1$ since $T_n(x) = 0 \forall x \notin [-1,1]$). Equivalently, the spectral density function can be extended to take the value 0 outside the interval $[-1,1]$. The derivation of

$$\begin{aligned}\hat{T}_n(y) &= \int_{-\infty}^{\infty} e^{-ixy} T_n(x) d\,y = \int_{-1}^{1} e^{-ixy} T_n(x) d\,x \\ &= \int_{-\infty}^{\infty} e^{-ixy}\, {}_2F_1\left(\begin{matrix} n, & -n \\ & \frac{1}{2} \end{matrix}\middle| \frac{1}{2} - \frac{x}{2}\right) d\,x \\ &= \frac{i}{y}\left(e^{-iy} F_n^{+}(y) - e^{i(\pi n + y)} F_n^{-}(y)\right)\end{aligned} \tag{20}$$

where

$$F_n^{\pm}(y) = {}_3F_1\left(\begin{matrix} 1, & n, & -n \\ & & \frac{1}{2} \end{matrix}\middle| \frac{\pm i y}{2}\right) \tag{21}$$

can be found in [1].

Definition 12. *Let $Y_n(y)$ be the normalized spectral polynomials $S_n(x)$*

Example 13. When $K = J_0$ the spectral polynomials are given by

$$S_n(x) = \sqrt{2} T_n(x) \tag{22}$$

so that

$$\begin{aligned}Y_n(y) &= \frac{\hat{T}_n(y)}{|\hat{T}_n|} \\ &= \frac{i}{y}\left(\frac{e^{-iy} F_n^{+}(y) - e^{i(\pi n + y)} F_n^{-}(y)}{\sqrt{\frac{4(-1)^n \pi - (2n^2 - 1)}{4n^2 - 1}}}\right)\end{aligned} \tag{23}$$

where the L^2 norm of $\hat{T}_n(y)$ is given by

$$\begin{aligned}|\hat{T}_n| &= \sqrt{\int_{-\infty}^{\infty} \hat{T}_n(y)^2 \mathrm{d}y} \\ &= \sqrt{\frac{4(-1)^n \pi - (2n^2 - 1)}{4n^2 - 1}}\end{aligned} \tag{24}$$

Conjecture 14. *The eigenfunctions of the integral covariance operator (13) are given by the orthogonal complement of the normalized Fourier transforms $Y_n(y)$ of the spectral polynomials (via the Gram-Schmidt process)*

$$\psi_n(y) = Y_n^{\perp}(y) = Y_n(y) - \sum_{m=1}^{n-1} \frac{\langle Y_n(y), Y_m^{\perp}(y)\rangle}{\langle Y_m^{\perp}(y), Y_m^{\perp}(y)\rangle} Y_m^{\perp}(y) \tag{25}$$

can be equivalently expressed as

$$\begin{aligned} \psi_n(y) &= (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, j_{2n}(y) \\ &= (-1)^n \sqrt{\frac{4\,n+1}{\pi}} \sqrt{\frac{\pi}{2\,y}}\, J_{2n+\frac{1}{2}}(y) \\ &= (-1)^n \sqrt{\frac{4\,n+1}{2\,y}}\, J_{2n+\frac{1}{2}}(y) \\ &= \sqrt{\frac{4\,n+1}{4\,\pi}} \int_{-1}^{1} P_{2n}(x)\, e^{i x y}\, \mathrm{d}x \end{aligned} \tag{26}$$

Remark 15. Since T is compact due to its self-adjointness and convergence of the eigenvalues to 0 it converges uniformly since compactness implies uniform convergence of the eigenfunctions. TODO: cite/theorems from [4, 3. Reproducing Kernel Hilbert Space of a Gaussian Process]

1 Simplifying The Convolution

Apply the addition theorem

$$J_0\,(x-y) = \sum_{k=-\infty}^{\infty} J_k(x)\, J_k\,(y)$$

to the integral covariance operator

$$\begin{aligned} [T\psi_n](x) &= \int_0^{\infty} J_0\,(x-y)\, \psi_n(y)\; d\,y \\ &= \int_0^{\infty} \sum_{k=-\infty}^{\infty} J_k(x)\, J_k\,(y)\, \psi_n(y)\; d\,y \\ &= \sum_{k=-\infty}^{\infty} J_k(x) \int_0^{\infty} J_k\,(y)\, \psi_n(y)\; d\,y \\ &= \sum_{k=-\infty}^{\infty} J_k(x) \int_0^{\infty} J_k\,(y)\, (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, j_{2n}(y)\, d\,y \end{aligned}$$

Where $\psi_n(y)$ is:

$$\psi_n(y) = (-1)^n \sqrt{\frac{4n+1}{\pi}}\, j_{2n}(y) = (-1)^n \sqrt{\frac{4n+1}{\pi}}\, \sqrt{\frac{\pi}{2y}}\, J_{2n+\frac{1}{2}}(y)$$

Substituting

$$\begin{aligned}\int_0^\infty J_k(y)\, \psi_n(y)\, dy &= \int_0^\infty J_k(y)\, (-1)^n \sqrt{\frac{4n+1}{\pi}}\, j_{2n}(y)\, dy \\ &= \frac{\sqrt{4n+1}\, (-1)^n \sqrt{\pi}\, \Gamma\left(\frac{k}{2}+n+\frac{1}{2}\right)}{2\,\Gamma\left(-n+\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{k}{2}+n+1\right)\Gamma\left(n+1-\frac{k}{2}\right)}\end{aligned}$$

Now, putting it all back into the expansion for $[T\psi_n](x)$:

$$[T\psi_n](x) = \sum_{k=-\infty}^{\infty} J_k(x) \frac{\sqrt{4n+1}\, (-1)^n \sqrt{\pi}\, \Gamma\left(\frac{k}{2}+n+\frac{1}{2}\right)}{2\,\Gamma\left(-n+\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{k}{2}+n+1\right)\Gamma\left(n+1-\frac{k}{2}\right)}$$

Conjecture 16.

$$\sum_{k=0}^{\infty} \psi_k(x)^2 = \frac{1}{\pi} \tag{27}$$

1 Appendix

1.1 A Theorem On The Development of Symmetric Nuclei

Theorem 17. *[2, 11.70] Let $\phi_n(\alpha)$ be a complete set of orthogonal functions satisfying the homogeneous integral equation with symmetric nucleus*

$$\phi(\alpha) = \lambda \int_a^b K(\alpha, \xi)\, \phi(\xi)\, d\xi \tag{28}$$

the corresponding characteristic numbers being $\lambda_1, \lambda_2, \lambda_3, \ldots$ Now suppose that the series $\sum_{n=1}^{\infty} \frac{\phi_n(\alpha)\,\phi_n(\gamma)}{\lambda_n}$ is uniformly convergent when $0 \leqslant a \leq \alpha \leq b \leqslant \infty, 0 \leqslant a \leq \gamma \leq b \leqslant \infty$. Then it will be shown that

$$K(\alpha, \gamma) = \sum_{n=1}^{\infty} \frac{\phi_n(\alpha)\,\phi_n(\gamma)}{\lambda_n} \tag{29}$$

Proof. For consider the symmetric nucleus

$$H(\alpha,\gamma)=K(\alpha,\gamma)-\sum_{n=1}^{\infty}\frac{\phi_n(\alpha)\,\phi_n(\gamma)}{\lambda_n} \tag{30}$$

If this nucleus is not identically zero, it will possess at least one characteristic number μ. Let $\psi(\gamma)$ be any solution of the equation

$$\psi(\alpha)=\mu\int_a^b H(\alpha,\xi)\,\psi(\xi)\,d\,\xi \tag{31}$$

which does not vanish identically. Multiply by $\phi_m(\alpha)$ and integrate term-by-term (which we may do since the series converges uniformly by hypothesis proved by other means), and get

$$\int_a^b \psi(\alpha)\,\phi_m(\alpha)\,d\,\alpha=\mu\int_a^b\int_a^b\left[K(\alpha,\xi)-\sum_{n=1}^{\infty}\frac{\phi_n(\alpha)\,\phi_n(\xi)}{\lambda_n}\right]\psi(\xi)\,\phi_m(\alpha)\,d\,\xi\,d\,\alpha=0 \tag{32}$$

Therefore $\psi(\alpha)$ is orthogonal to $\phi_1(\alpha),\phi_2(\alpha),\ldots$; and so taking the equation

$$\psi(\alpha)=\mu\int_a^b\left[K(\alpha,\xi)-\sum_{n=1}^{\infty}\frac{\phi_n(\alpha)\,\phi_n(\xi)}{\lambda_n}\right]\psi(\xi)\,d\,\xi \tag{33}$$

we have

$$\psi(\alpha)=\mu\int_a^b K(\alpha,\xi)\,\psi(\xi)\,d\,\xi \tag{34}$$

Therefore μ is a characteristic number of $K(\alpha,\gamma)$, and so $\psi(\alpha)$ must be a linear combination of the functions $\phi_n(\alpha)$ corresponding to this number; let

$$\psi(\alpha)=\sum_m a_m\,\phi_m(\alpha) \tag{35}$$

Multiply by $\phi_m(\gamma)$ and integrate; then since $\psi(\alpha)$ is orthogonal to all the functions $\phi_m(\alpha)$, we see that $a_m=0$, so, contrary to hypothesis, $\psi(\alpha)=0$. The contradiction implies that the nucleus $H(\alpha,\gamma)$ must be identically zero; that is to say, $K(\alpha,\gamma)$ can be expanded in the given series, if it is uniformly convergent. □

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