Eigenfunction Construction for Stationary Gaussian Processes

by Stephen Crowley

December 19, 2024

1 Preliminaries

The Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are defined as:

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, d\, x$$

$$\mathcal{F}^{-1}[g](x) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} g(\omega)\, e^{i\omega x}\, d\, \omega$$

2 Theoretical Framework

Let $K\,(x-y)$ be a stationary positive-definite kernel. By Bochner's theorem:

$$K\,(x-y) = \mathcal{F}^{-1}[S](\omega) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{i\omega\,(x-y)}\, S(\omega)\, d\, \omega$$

where $S(\omega)\, d\, \omega$ is the spectral measure.

3 The Null Space

Let $\{P_n(\omega)\}_{n=0}^{\infty}$ be polynomials orthogonal with respect to $S(\omega)$ with $P_0(\omega) = 1$:

$$\int_{-\infty}^{\infty} P_m(\omega)\, P_n(\omega)\, S(\omega)\, d\, \omega = \delta_{mn}$$

Since $P_0(\omega) = 1$, for any $n \geq 1$:

$$\langle P_n, 1\rangle_S = \int_{-\infty}^{\infty} P_n(\omega)\, S(\omega)\, d\, \omega = 0$$

Therefore $\{\mathcal{F}^{-1}[P_n]\}_{n=1}^{\infty}$ is the null space of the kernel integral operator:

$$\langle K, \mathcal{F}^{-1}[P_n]\rangle = \langle \mathcal{F}^{-1}[S], \mathcal{F}^{-1}[P_n]\rangle = \langle S, P_n\rangle = 0$$

4 Uniform Basis of the Kernel

Define the null space vectors:

$$\eta_n(x) = \mathcal{F}^{-1}[P_n(\omega)](x)$$

Apply Gram-Schmidt to $\{\eta_n\}$ to obtain orthonormal sequence $\{\psi_n\}$ via:

$$\psi_k(x) = \eta_k(x) - \sum_{j=1}^{k-1} \frac{\langle \eta_k, \psi_j \rangle}{\|\psi_j\|^2} \psi_j(x)$$

Let $\mathcal{N}$ denote the null space. Then with its orthogonal complement $\mathcal{N}^\perp$:

$$L^2(\mathbb{R}) = \mathcal{N} \cup \mathcal{N}^\perp, \quad \mathcal{N} \cap \mathcal{N}^\perp = \{0\}$$

The kernel expansion in $\mathcal{N}^\perp$ is:

$$K(x) = \sum_{n=0}^{\infty} \langle K, \psi_n \rangle \, \psi_n(x)$$

with uniform convergence.

5 Uniform Basis of the Spectral Factor

Let $\{Q_n(\omega)\}_{n=0}^{\infty}$ be orthogonal polynomials with respect to $\sqrt{S(\omega)}$:

$$\int_{-\infty}^{\infty} Q_m(\omega)\, Q_n(\omega) \sqrt{S(\omega)}\, d\omega = \delta_{mn}$$

Define:

$$\xi_n(x) = \mathcal{F}^{-1}[Q_n(\omega)](x)$$

Apply Gram-Schmidt to $\{\xi_n\}$ to obtain orthonormal sequence $\{\phi_n\}$ via:

$$\phi_k(x) = \xi_k(x) - \sum_{j=1}^{k-1} \frac{\langle \xi_k, \phi_j \rangle}{\|\phi_j\|^2} \phi_j(x)$$

Then:

$$g(x) = \sum_{n=0}^{\infty} \langle g, \phi_n \rangle \, \phi_n(x)$$

where g is the spectral factor satisfying $g * g = K$ and $\mathcal{F}[g] = \sqrt{S}$.

6 Eigenfunction Construction

By Fubini's theorem and uniform convergence:

$$K\left(x-y\right)=\left(g*g\right)\left(x-y\right)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle g,\phi_m\rangle\langle g,\phi_n\rangle\left(\phi_m*\phi_n\right)\left(x-y\right)$$

The eigenfunctions $\{f_n\}$ can be expressed in the uniform basis $\{\psi_n\}$ with finitely many terms:

$$f_n(x)=\sum_{k=0}^{n}c_{nk}\,\psi_k(x)$$

where coefficients c_{nk} are determined by the recurrence relations of the underlying orthogonal polynomials.

Substituting into Mercer's theorem:

$$K\left(x-y\right)=\sum_{n=0}^{\infty}\lambda_n\,f_n(x)\,f_n(y)=\sum_{n=0}^{\infty}\lambda_n\sum_{k=0}^{n}\sum_{j=0}^{n}c_{nk}\,c_{nj}\,\psi_k(x)\,\psi_j(y)$$

This double sum structure with coefficients is precisely the inner product representation guaranteed by Moore-Aronszajn's theorem for reproducing kernel Hilbert spaces.

The triangular structure emerges naturally from the polynomial degree preservation under convolution in the spectral domain.