Unitary Time Changes of Stationary Processes Yield Oscillatory Processes

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are locally square-integrable (meaning over compact sets). Applying U_θ to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} \, e^{i\lambda\theta(t)}$, evolutionary spectrum $d\,F_t(\lambda) = \dot{\theta}(t)\,d\,F(\lambda)$, and expected zero-counting function $\mathbb{E}[N_{[0,T]}] = \frac{[\theta(T) - \theta(0)]}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}}$. The sample paths of any non-degenerate second-order stationary process are locally square integrable, making the unitary time-change operator U_θ applicable to typical realizations. A zero-localization measure $d\,\mu(t) = \delta(Z(t))|\dot{Z}(t)|\,d\,t$ induces a Hilbert space $L^2(\mu)$ on the zero set of each oscillatory process realization $Z(t)$, and the multiplication operator $(L\,f)(t) = t\,f(t)$ has simple pure point spectrum equal to the zero crossing set of Z.

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1 Gaussian Processes and Simplicity of Zeros

Theorem 1. *[Bulinskaya] Let $X(t)$ be a centered stationary Gaussian process with covariance function $K(h) = \mathbb{E}\left[X(t)\, X\,(t+h)\right]$ twice differentiable at $h = 0$ with $K(0) > 0$ and $\ddot{K}(0) < 0$. Then all zeros of X are simple: for every t_0 such that $X(t_0) = 0$, it follows that $\dot{X}(t_0) \neq 0$.*

Proof. The twice-differentiability of K at $h = 0$ implies that the spectral measure F possesses a finite second moment. Define

$$\sigma_1^2 := \int_{\mathbb{R}} \lambda^2 \, d\, F(\lambda) \tag{1}$$

By properties of stationary Gaussian processes,

$$\sigma_1^2 = -\ddot{K}(0) < \infty \tag{2}$$

The finiteness of σ_1^2 guarantees the existence of a mean-square continuous derivative $\dot{X}(t)$ with

$$\mathbb{E}[\dot{X}(t)^2] = -\ddot{K}(0) = \sigma_1^2 > 0 \tag{3}$$

For any fixed t, the joint distribution of $(X(t), \dot{X}(t))$ is bivariate Gaussian. The covariance kernel $K(h)$ is even by stationarity, so $\dot{K}(0) = 0$. Therefore, the covariance matrix is

$$\Sigma(t) = \begin{pmatrix} K(0) & \dot{K}(0) \\ \dot{K}(0) & -\ddot{K}(0) \end{pmatrix} = \begin{pmatrix} K(0) & 0 \\ 0 & -\ddot{K}(0) \end{pmatrix} \tag{4}$$

The determinant of $\Sigma(t)$ is

$$\det\left(\Sigma(t)\right) = K(0) \cdot (-\ddot{K}(0)) > 0 \tag{5}$$

so $\Sigma(t)$ is positive definite.

The inverse matrix is

$$\Sigma(t)^{-1} = \begin{pmatrix} \frac{1}{K(0)} & 0 \\ 0 & \frac{1}{-\ddot{K}(0)} \end{pmatrix} \tag{6}$$

The joint density of $(X(t), \dot{X}(t))$ is

$$p(x, y; t) = \frac{1}{2\,\pi\,\sqrt{\det\left(\Sigma(t)\right)}} \exp\left(-\frac{1}{2}\,(x, y)\,\Sigma(t)^{-1}(x, y)^T\right) \tag{7}$$

Substituting the explicit forms:

$$p(x, y; t) = \frac{1}{2\pi \sqrt{K(0) \cdot (-\ddot{K}(0))}} \exp\left(-\frac{1}{2}\left[\frac{x^2}{K(0)} + \frac{y^2}{-\ddot{K}(0)}\right]\right) \tag{8}$$

The marginal density of $X(t)$ is obtained by integrating over y:

$$p_X(x; t) = \int_{-\infty}^{\infty} p(x, y; t)\, dy \tag{9}$$

Factor out the x-dependent part:

$$p_X(x; t) = \frac{1}{2\pi \sqrt{K(0) \cdot (-\ddot{K}(0))}} \exp\left(-\frac{x^2}{2\,K(0)}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2\,(-\ddot{K}(0))}\right) dy \tag{10}$$

The integral over y evaluates to $\sqrt{2\pi\,(-\ddot{K}(0))}$:

$$\int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2\,(-\ddot{K}(0))}\right) dy = \sqrt{2\pi\,(-\ddot{K}(0))} \tag{11}$$

Therefore:

$$p_X(x; t) = \frac{1}{\sqrt{2\pi\,K(0)}} \exp\left(-\frac{x^2}{2\,K(0)}\right) \tag{12}$$

At $x = 0$:

$$p_X(0; t) = \frac{1}{\sqrt{2\pi\,K(0)}} > 0 \tag{13}$$

The joint density evaluated at $x = 0$ is:

$$p(0, y; t) = \frac{1}{2\pi \sqrt{K(0) \cdot (-\ddot{K}(0))}} \exp\left(-\frac{y^2}{2\,(-\ddot{K}(0))}\right) \tag{14}$$

The conditional density is

$$p\left(\dot{X}(t)\,\middle|\, X(t) = 0\right) = \frac{p(0, \dot{X}(t); t)}{p_X(0; t)} \tag{15}$$

Substituting:

$$p\left(\dot{X}(t)\,\middle|\, X(t) = 0\right) = \frac{1}{\sqrt{2\pi\,(-\ddot{K}(0))}} \exp\left(-\frac{y^2}{2\,(-\ddot{K}(0))}\right) \tag{16}$$

This is the density of a Gaussian random variable with mean zero and variance $-\ddot{K}(0) > 0$. For any $c \in \mathbb{R}$,

$$\mathbb{P}\left[\dot{X}(t) = c\,\middle|\, X(t) = 0\right] = \int_c^c p\left(\dot{X}(t)\,\middle|\, X(t) = 0\right) dy = 0 \tag{17}$$

In particular,

$$\mathbb{P}\left[\dot{X}(t_0)=0 \,\middle|\, X(t_0)=0\right]=0 \tag{18}$$

so

$$\mathbb{P}\left[\dot{X}(t_0)\neq 0 \,\middle|\, X(t_0)=0\right]=1 \tag{19}$$

for every t_0 with $X(t_0)=0$. All zeros are simple almost surely. $\square$

Theorem 2. *[Simplicity under time change] Let $X(u)$ satisfy the hypotheses of Theorem 1. Let $\theta\colon \mathbb{R}\to\mathbb{R}$ be strictly increasing and absolutely continuous with $\dot{\theta}(t)>0$ almost everywhere. Define*

$$Z(t)=\sqrt{\dot{\theta}(t)}\,X(\theta(t)) \tag{20}$$

Then all zeros of Z are simple: for every t_0 such that $Z(t_0)=0$, it follows that $\dot{Z}(t_0)\neq 0$.

Proof. Let $u(t)=\sqrt{\dot{\theta}(t)}$ and $v(t)=X(\theta(t))$, so $Z(t)=u(t)\,v(t)$.

The derivative of $u(t)$ is

$$\dot{u}(t)=\frac{d}{dt}\sqrt{\dot{\theta}(t)}=\frac{\ddot{\theta}(t)}{2\sqrt{\dot{\theta}(t)}} \tag{21}$$

The derivative of $v(t)$ by the chain rule is

$$\dot{v}(t)=\dot{X}(\theta(t))\,\dot{\theta}(t) \tag{22}$$

By the product rule,

$$\dot{Z}(t)=\dot{u}(t)\,v(t)+u(t)\dot{v}(t) \tag{23}$$

Substitute:

$$\dot{Z}(t)=\frac{\ddot{\theta}(t)}{2\sqrt{\dot{\theta}(t)}}\,X(\theta(t))+\sqrt{\dot{\theta}(t)}\,\dot{X}(\theta(t))\,\dot{\theta}(t) \tag{24}$$

Let t_0 satisfy $Z(t_0)=0$ and $\dot{\theta}(t_0)>0$. Then $\sqrt{\dot{\theta}(t_0)}>0$ and

$$0=Z(t_0)=\sqrt{\dot{\theta}(t_0)}\,X(\theta(t_0)) \tag{25}$$

so

$$X(\theta(t_0))=0 \tag{26}$$

By Theorem 1, $\dot{X}(\theta(t_0))\neq 0$ almost surely.

Evaluate $\dot{Z}(t_0)$:

$$\dot{Z}(t_0)=\frac{\ddot{\theta}(t_0)}{2\sqrt{\dot{\theta}(t_0)}}\cdot 0+\sqrt{\dot{\theta}(t_0)}\,\dot{X}(\theta(t_0))\,\dot{\theta}(t_0) \tag{27}$$

So

$$\dot{Z}(t_0) = [\dot{\theta}(t_0)]^{3/2} \dot{X}(\theta(t_0)) \tag{28}$$

Since $\dot{\theta}(t_0) > 0$ and $\dot{X}(\theta(t_0)) \neq 0$, one has $\dot{Z}(t_0) \neq 0$. The set where $\dot{\theta}(t) = 0$ has Lebesgue measure zero, so all zeros of Z are simple almost surely. $\square$

All processes in the remainder of the document are real-valued unless otherwise stated.

Theorem 3. *Let $X(u)$ be a real-valued process:*

$$X(u) \in \mathbb{R} \quad \forall u \in \mathbb{R} \tag{29}$$

Then its (complex-valued) random orthogonal spectral measure satisfies

$$d\,\bar{\Phi}(\lambda) = d\,\Phi(-\lambda) \tag{30}$$

and the corresponding covariance spectral measure F is even:

$$F(-A) = F(A) \tag{31}$$

Proof. 1. The spectral representation for $X(u)$ is

$$X(u) = \int_{-\infty}^{\infty} e^{i\lambda u} \, d\,\Phi(\lambda) \tag{32}$$

Since $X(u)$ is real-valued for each u,

$$\overline{X(u)} = X(u) \tag{33}$$

The complex conjugate of the integral representation is

$$\overline{X(u)} = \overline{\int_{-\infty}^{\infty} e^{i\lambda u} \, d\,\Phi(\lambda)} \tag{34}$$

$$= \int_{-\infty}^{\infty} \overline{e^{i\lambda u}} d\,\bar{\Phi}(\lambda) \tag{35}$$

$$= \int_{-\infty}^{\infty} e^{-i\lambda u} \, d\,\bar{\Phi}(\lambda) \tag{36}$$

Apply the substitution $\lambda \mapsto -\mu$:

$$\int_{-\infty}^{\infty} e^{-i\lambda u} \, d\,\bar{\Phi}(\lambda) = \int_{-\infty}^{\infty} e^{i\mu u} \, d\,\bar{\Phi}(-\mu) \tag{37}$$

Equating the two expressions for $\overline{X(u)}$:

$$X(u) = \int_{-\infty}^{\infty} e^{i\lambda u} \, d\,\Phi(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda u} \, d\,\bar{\Phi}(-\lambda) \tag{38}$$

By uniqueness of the spectral representation,

$$d\,\Phi(\lambda) = d\,\bar{\Phi}(-\lambda) \tag{39}$$

Taking complex conjugates,

$$d\,\bar{\Phi}(\lambda) = d\,\Phi(-\lambda) \tag{40}$$

2. The covariance function of X is

$$R(u) = \mathbb{E}\left(X(0)\,X(u)\right) = \int_{-\infty}^{\infty} e^{i\lambda u}\,d\,F(\lambda) \tag{41}$$

Since X is real-valued, $R(u)$ is real and stationary implies $R(-u) = R(u)$. Compute

$$R(-u) = \int_{-\infty}^{\infty} e^{-i\lambda u}\,d\,F(\lambda) \tag{42}$$

Substitute $\mu = -\lambda$:

$$R(-u) = \int_{-\infty}^{\infty} e^{i\mu u}\,d\,F(-\mu) \tag{43}$$

Equating $R(u)$ and $R(-u)$ for all u gives

$$\int_{-\infty}^{\infty} e^{i\lambda u}\,d\,F(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda u}\,d\,F(-\lambda) \tag{44}$$

By uniqueness of Fourier–Stieltjes transforms,

$$d\,F(\lambda) = d\,F(-\lambda) \tag{45}$$

For any Borel set A,

$$F(-A) = \int_{-A} d\,F(\lambda) = \int_{A} d\,F(-\lambda) = \int_{A} d\,F(\lambda) = F(A) \tag{46} \quad \square$$

1.1 Definition

Definition 4. *(Gaussian process) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T a nonempty index set. A family $\{X_t \colon t \in T\}$ of real-valued random variables is called a Gaussian process if for every finite subset $\{t_1, \ldots, t_n\} \subset T$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ is multivariate normal. The mean function is*

$$m(t) := \mathbb{E}[X_t] \tag{47}$$

and the covariance kernel is

$$K(s,t) = \mathrm{Cov}(X_s, X_t) = \mathbb{E}\left[\left(X_s - m(s)\right)\left(X_t - m(t)\right)\right] \tag{48}$$

1.2 Stationary processes

Definition 5. *(Cramér spectral representation) A zero-mean stationary process X with spectral measure F admits the representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t} \, d\,\Phi(\lambda) \tag{49}$$

where Φ is a complex orthogonal random measure. The covariance is

$$R_X(t - s) = \mathbb{E}[X(t)\overline{X(s)}] = \int_{\mathbb{R}} e^{i\lambda(t-s)} \, d\,F(\lambda) \tag{50}$$

where F is the spectral measure defined by $d\,F(\lambda) = \mathbb{E}\left[d\,\Phi(\lambda)\overline{d\,\Phi(\lambda)}\right]$.

1.2.1 Fourier Transform Conventions

Definition 6. *(Fourier transform conventions) The forward and inverse Fourier transforms on $L^2(\mathbb{R})$ are*

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(u) \, e^{-i\lambda u} \, d\,u \tag{51}$$

and

$$f(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda) \, e^{i\lambda u} \, d\,\lambda \tag{52}$$

1.2.2 Sample Path Realizations

Definition 7. *(Locally square-integrable functions)*

$$L_{\mathrm{loc}}^2(\mathbb{R}) := \left\{ f \colon \mathbb{R} \to \mathbb{C} \colon \int_K |f(t)|^2 \, d\,t < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \tag{53}$$

Remark 8. The space $L_{\mathrm{loc}}^2(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

Theorem 9. *(Sample paths in $L_{\mathrm{loc}}^2(\mathbb{R})$) Let $\{X(t)\}_{t \in \mathbb{R}}$ be a second-order stationary process with*

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \tag{54}$$

Then almost every sample path lies in $L_{\mathrm{loc}}^2(\mathbb{R})$.

Proof. Let $[a, b] \subset \mathbb{R}$ be bounded and define

$$Y_{[a,b]} := \int_a^b X(t)^2 \, d\,t \tag{55}$$

By Fubini,

$$\mathbb{E}[Y_{[a,b]}] = \int_a^b \mathbb{E}[X(t)^2] \, d\,t = \int_a^b \sigma^2 \, d\,t = \sigma^2 \, (b - a) < \infty \tag{56}$$

Thus $\mathbb{P}\left(Y_{[a,b]} < \infty\right) = 1$.

Cover $\mathbb{R}$ by $I_n = [n, n+1]$, $n \in \mathbb{Z}$. For each n,

$$\mathbb{P}\left(\int_{I_n} X(t)^2 \, dt < \infty\right) = 1 \tag{57}$$

The countable intersection has probability one, so for almost every ω_0 and every compact K,

$$\int_K |X(t, \omega_0)|^2 \, dt < \infty \tag{58}$$

Hence the sample path lies in $L^2_{\mathrm{loc}}(\mathbb{R})$. $\square$

2 Oscillatory Processes

Definition 10. *(Oscillatory process) Let F be a finite nonnegative Borel measure on $\mathbb{R}$. Let*

$$A_t \in L^2(F) \quad \forall t \in \mathbb{R} \tag{59}$$

be the gain function and

$$\varphi_t(\lambda) = A_t(\lambda) \, e^{i\lambda t} \tag{60}$$

be the oscillatory function. An oscillatory process is

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda) \, d\,\Phi(\lambda) = \int_{\mathbb{R}} A_t(\lambda) \, e^{i\lambda t} \, d\,\Phi(\lambda) \tag{61}$$

where Φ is a complex orthogonal random measure with spectral measure F satisfying

$$d\,\mathbb{E}[\Phi(\lambda)\overline{\Phi(\mu)}] = \delta\left(\lambda - \mu\right) d\,F(\lambda) \tag{62}$$

Theorem 11. *(Real-valuedness criterion) Let Z be an oscillatory process with oscillatory function $\varphi_t(\lambda) = A_t(\lambda) \, e^{i\lambda t}$ and spectral measure F. Then Z is real-valued if and only if*

$$A_t\left(-\lambda\right) = \overline{A_t(\lambda)} \tag{63}$$

for F-almost every λ, equivalently

$$\varphi_t\left(-\lambda\right) = \overline{\varphi_t(\lambda)} \tag{64}$$

for F-almost every λ.

Proof. Assume Z real-valued. Then

$$Z(t) = \overline{Z(t)} \tag{65}$$

and

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda) \, e^{i\lambda t} \, d\,\Phi(\lambda) \tag{66}$$

Taking conjugates,

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t} \, d\,\bar{\Phi}(\lambda) \tag{67}$$

By Theorem 3, $d\,\bar{\Phi}(\lambda) = d\,\Phi(-\lambda)$, so

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t}\, d\,\Phi(-\lambda) \tag{68}$$

Set $\mu = -\lambda$:

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t}\, d\,\Phi(\mu) \tag{69}$$

Equality $Z(t) = \overline{Z(t)}$ gives

$$\int_{\mathbb{R}} A_t(\mu)\, e^{i\mu t}\, d\,\Phi(\mu) = \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t}\, d\,\Phi(\mu) \tag{70}$$

so

$$A_t(\lambda) = \overline{A_t(-\lambda)} \quad F\text{-a.e.} \tag{71}$$

Then

$$\varphi_t(-\lambda) = A_t(-\lambda)\, e^{-i\lambda t} = \overline{A_t(\lambda)} e^{-i\lambda t} = \overline{A_t(\lambda)\, e^{i\lambda t}} = \overline{\varphi_t(\lambda)} \tag{72}$$

The converse follows by reversing the argument. $\qquad\square$

Theorem 12. *(Existence of Oscillatory Processes) Let F be absolutely continuous with gain function $A_t(\lambda) \in L^2(F)$ for all t, measurable in both time and frequency. Then there exists a complex orthogonal random measure Φ with spectral measure F such that*

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda) \tag{73}$$

is well-defined in $L^2(\Omega)$, and

$$R_Z(t,s) = \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)} e^{i\lambda(t-s)}\, d\,F(\lambda). \tag{74}$$

Proof. Define the stochastic integral first for simple $g(\lambda) = \sum c_j\, 1_{E_j}(\lambda)$ by

$$\int g(\lambda)\, d\,\Phi(\lambda) = \sum c_j\, \Phi(E_j) \tag{75}$$

Using orthogonality,

$$\mathbb{E}\left| \int g\, d\Phi \right|^2 = \int |g(\lambda)|^2\, d\,F(\lambda) \tag{76}$$

Simple functions are dense in $L^2(F)$, so the integral extends uniquely to all $L^2(F)$.

For each t, $A_t(\lambda)\, e^{i\lambda t} \in L^2(F)$, so $Z(t)$ is defined. Covariance follows from bilinearity and orthogonality exactly as in standard spectral theory:

$$R_Z(t,s) = \int A_t(\lambda)\overline{A_s(\lambda)} e^{i\lambda(t-s)}\, d\,F(\lambda) \tag{77} \;\square$$

Definition 13. *(Forward impulse response) For $Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda)\, d\,\Phi(\lambda)$, define*

$$h(t,u) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_t(\lambda)\, e^{-i\lambda u}\, d\,\lambda \tag{78}$$

Theorem 14. *(Filter representation via impulse response) Let X be a zero-mean sta-tionary process with Cramér representation $X(u) = \int_{\mathbb{R}} e^{i\lambda u} d\Phi(\lambda)$ and spectral measure F. Let Z be an oscillatory process with oscillatory function $\varphi_t(\lambda)$ using the same orthogonal random measure Φ. Then*

$$Z(t) = \int_{-\infty}^{\infty} h(t, u) X(u) \, du \tag{79}$$

with h from Definition 13.

Proof. Substitute:

$$\int h(t, u) X(u) \, du = \int \frac{1}{2\pi} \int \varphi_t(\lambda) e^{-i\lambda u} d\lambda \int e^{i\lambda' u} d\Phi(\lambda') \, du \tag{80}$$

Interchange integrals:

$$= \frac{1}{2\pi} \iint \varphi_t(\lambda) \left[\int e^{i(\lambda' - \lambda) u} \, du \right] d\lambda' \, d\lambda \tag{81}$$

Use

$$\int_{-\infty}^{\infty} e^{i(\lambda' - \lambda) u} \, du = 2\pi \delta(\lambda' - \lambda) \tag{82}$$

to get

$$= \int \varphi_t(\lambda') \, d\Phi(\lambda') = Z(t) \tag{83} \quad \square$$

3   Unitarily Time-Changed Stationary Processes

3.1   Unitary Time-Change Operator

Theorem 15. *(Unitary time-change operator) Let $\theta: \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with $\dot{\theta}(t) > 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero. For f measurable, define*

$$(U_\theta f)(t) := \sqrt{\dot{\theta}(t)} \, f(\theta(t)) \tag{84}$$

Its inverse is

$$(U_\theta^{-1} g)(s) := \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{85}$$

For every compact $K \subseteq \mathbb{R}$ and $f \in L^2_{\mathrm{loc}}(\mathbb{R})$,

$$\int_K |(U_\theta f)(t)|^2 dt = \int_{\theta(K)} |f(s)|^2 ds \tag{86}$$

Moreover, U_θ^{-1} is the two-sided inverse of U_θ on $L^2_{\mathrm{loc}}(\mathbb{R})$.

Proof. For $f \in L^2_{\mathrm{loc}}$,

$$\int_K |U_\theta f(t)|^2 \, dt = \int_K \dot{\theta}(t) |f(\theta(t))|^2 \, dt \tag{87}$$

Set $s = \theta(t)$ so $d\, s = \dot{\theta}(t)\, d\, t$ and $s \in \theta(K)$:

$$= \int_{\theta(K)} |f(s)|^2\, d\, s \tag{88}$$

For the inverse, compute

$$(U_\theta^{-1} U_\theta f)(s) = \frac{(U_\theta f)(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} = \frac{\sqrt{\dot{\theta}(\theta^{-1}(s))}\, f(s)}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} = f(s) \tag{89}$$

and similarly $(U_\theta U_\theta^{-1} g)(t) = g(t)$. □

3.2 Transformation of Stationary to Oscillatory Processes

Theorem 16. *(Time changes produce oscillatory processes) Let X be zero-mean stationary as in Definition 5. For θ as in Theorem 15, define*

$$Z(t) := \sqrt{\dot{\theta}(t)}\, X(\theta(t)) \tag{90}$$

Then Z is an oscillatory process with oscillatory function

$$\varphi_t(\lambda) := \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \tag{91}$$

gain function

$$A_t(\lambda) := \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)} \tag{92}$$

and covariance kernel

$$R_Z(t,s) := \sqrt{\dot{\theta}(t)\, \dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\, d\, F(\lambda) \tag{93}$$

Proof. From Cramér,

$$X(u) = \int e^{i\lambda u}\, d\, \Phi(\lambda) \tag{94}$$

so

$$X(\theta(t)) = \int e^{i\lambda\theta(t)}\, d\, \Phi(\lambda) \tag{95}$$

Then

$$Z(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) = \int \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}\, d\, \Phi(\lambda) \tag{96}$$

Set

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \tag{97}$$

and

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)} \tag{98}$$

so $\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t}$.

For covariance,

$$R_Z(t,s) = \mathbb{E}[Z(t)\overline{Z(s)}] = \sqrt{\dot{\theta}(t)\, \dot{\theta}(s)}\, \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \tag{99}$$

and stationarity gives

$$\mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] = \int e^{i\lambda(\theta(t)-\theta(s))} \, dF(\lambda) \tag{100}$$

so

$$R_Z(t,s) = \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \int e^{i\lambda(\theta(t)-\theta(s))} \, dF(\lambda) \tag{101} \ \square$$

Corollary 17. *The evolutionary spectrum is*

$$dF_t(\lambda) := \dot{\theta}(t)\, dF(\lambda) \tag{102}$$

Proof. By definition $dF_t(\lambda) = |A_t(\lambda)|^2 \, dF(\lambda)$ and

$$|A_t(\lambda)|^2 = \left|\sqrt{\dot{\theta}(t)}\right|^2 |e^{i\lambda(\theta(t)-t)}|^2 = \dot{\theta}(t) \tag{103}$$

so $dF_t(\lambda) = \dot{\theta}(t)\, dF(\lambda)$. $\square$

3.2.1 Time-Varying Filter Representations

Theorem 18. *Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\dot{\theta}(t) > 0$ almost everywhere. Let $X(u)$ be a stationary process. The oscillatory process obtained by U_θ is*

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) \tag{104}$$

with forward kernel

$$h(t,u) := \sqrt{\dot{\theta}(t)}\, \delta\,(u - \theta(t)) \tag{105}$$

and inverse

$$X(u) = (U_\theta^{-1} Z)(u) = \frac{Z(\theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \tag{106}$$

with inverse kernel

$$g\,(u,t) := \frac{\delta\,(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \tag{107}$$

Proof. Using δ,

$$X(\theta(t)) = \int X(u)\, \delta\,(u - \theta(t))\, du \tag{108}$$

hence

$$Z(t) = \sqrt{\dot{\theta}(t)} \int X(u)\, \delta\,(u - \theta(t))\, du = \int \sqrt{\dot{\theta}(t)}\, \delta\,(u - \theta(t))\, X(u)\, du \tag{109}$$

so $h(t,u) = \sqrt{\dot{\theta}(t)}\, \delta\,(u - \theta(t))$.

For the inverse, from Theorem 15,

$$X(u) = \frac{Z(\theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} = \int Z(t) \frac{\delta\,(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}}\, dt \tag{110}$$

so $g(u,t)$ as claimed. Composition of kernels yields the identity via standard delta sifting. $\square$

3.3 Covariance Operator Conjugation

Proposition 19. *Let*
$$(T_K f)(t) := \int_{\mathbb{R}} K(|t-s|)\, f(s)\, d\,s \tag{111}$$
with
$$K(h) := \int_{\mathbb{R}} e^{i\lambda h}\, d\, F(\lambda) \tag{112}$$
Define
$$K_\theta(s,t) := \sqrt{\dot{\theta}(t)\, \dot{\theta}(s)}\, K(|\theta(t)-\theta(s)|) \tag{113}$$
Then
$$(T_{K_\theta} f)(t) = (U_\theta\, T_K\, U_\theta^{-1}\, f)(t) \tag{114}$$

Proof. Compute
$$(U_\theta\, T_K\, U_\theta^{-1}\, f)(t) = \sqrt{\dot{\theta}(t)} \int K(|\theta(t)-s|)\, U_\theta^{-1}\, f(s)\, d\,s \tag{115}$$
Substitute U_θ^{-1} and $s = \theta(r)$:
$$= \sqrt{\dot{\theta}(t)} \int K(|\theta(t)-\theta(r)|)\sqrt{\dot{\theta}(r)}\, f(r)\, d\,r \tag{116}$$
$$= \int \sqrt{\dot{\theta}(t)\, \dot{\theta}(r)}\, K(|\theta(t)-\theta(r)|)\, f(r)\, d\,r \tag{117}$$
which is $(T_{K_\theta} f)(t)$. $\square$

4 Zero Localization

Definition 20. *Let* $Z: \mathbb{R} \to \mathbb{R}$ *be real-valued with* $Z \in C^1(\mathbb{R})$. *By Theorem 2, all zeros of Z satisfy*
$$Z(t_0) = 0 \implies \dot{Z}(t_0) \neq 0 \tag{118}$$
Define, for Borel $B \subseteq \mathbb{R}$:
$$\mu(B) := \int_{\mathbb{R}} 1_B(t)\, \delta(Z(t))|\dot{Z}(t)|\, d\,t \tag{119}$$

Theorem 21. *The zeros are locally finite and*
$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta\,(t-t_0)}{|\dot{Z}(t_0)|} \tag{120}$$
and
$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \tag{121}$$

Proof. For smooth test ϕ,

$$\int \phi(t)\,\delta(Z(t))\,dt = \sum_{t_0:\,Z(t_0)=0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} \tag{122}$$

Zeros are isolated (simplicity and C^1), so the sum is finite on compacts. Thus

$$\delta(Z(t)) = \sum_{t_0} \frac{\delta\,(t-t_0)}{|\dot{Z}(t_0)|} \tag{123}$$

Substitute into μ:

$$\mu(B) = \sum_{t_0} \int 1_B(t)\frac{\delta\,(t-t_0)}{|\dot{Z}(t_0)|}|\dot{Z}(t)|\,dt = \sum_{t_0} 1_B(t_0) \tag{124}$$

so $\mu = \sum_{t_0} \delta_{t_0}$. □

Definition 22. *Define*

$$H := L^2(\mu) \tag{125}$$

with

$$\langle f, g \rangle := \int f(t)\,g(t)\,d\mu(t) \tag{126}$$

Proposition 23. *[Atomic structure] Let* $\mu = \sum_{t_0:\,Z(t_0)=0}\delta_{t_0}$. *Then*

$$H = \left\{ f\colon \{t_0\colon Z(t_0)=0\} \to \mathbb{C}\colon \sum_{t_0:\,Z(t_0)=0} |f(t_0)|^2 < \infty \right\} \tag{127}$$

with orthonormal basis $\{e_{t_0}\}_{t_0:\,Z(t_0)=0}$ *where*

$$e_{t_0}(t_1) := \delta_{t_0,t_1} \tag{128}$$

Proof. For $f \in L^2(\mu)$,

$$\|f\|_{L^2(\mu)}^2 = \sum_{t_0} |f(t_0)|^2 \tag{129}$$

Define $\Phi(f) = (f(t_0))_{t_0}$; this is an isometric isomorphism onto ℓ^2.

For orthonormality,

$$\langle e_{t_0}, e_{t_1} \rangle = \sum_s e_{t_0}(s)\,e_{t_1}(s) = \delta_{t_0,t_1} \tag{130}$$

Any f has expansion $f = \sum f(t_0)\,e_{t_0}$, so $\{e_{t_0}\}$ is an orthonormal basis. □

Definition 24. *Define*

$$L\colon D(L) \subseteq H \to H, \quad (L\,f)(t) := t\,f(t) \tag{131}$$

with domain

$$D(L) := \left\{ f \in H\colon \int |t\,f(t)|^2\,d\mu(t) < \infty \right\} \tag{132}$$

Theorem 25. *The operator L is self-adjoint on H and has pure point, simple spectrum*

$$\sigma(L) = \{t \in \mathbb{R} : Z(t) = 0\} \tag{133}$$

with eigenvalues t_0 and eigenvectors e_{t_0}.

Proof. For $f, g \in D(L)$,

$$\langle L f, g \rangle = \int t\, f(t)\, g(t)\, d\mu(t) = \int f(t)\, t\, g(t)\, d\mu(t) = \langle f, L g \rangle \tag{134}$$

so L is symmetric and as a multiplication operator is self-adjoint.

On e_{t_0},

$$(L\, e_{t_0})(t) = t\, e_{t_0}(t) = t_0\, e_{t_0}(t) \tag{135}$$

so e_{t_0} is an eigenvector with eigenvalue t_0. The basis is complete, so the spectrum is pure point and simple. $\qquad\square$

4.1 Expected Zero-Counting Function

Theorem 26. *[Expected Zero-Counting Function with Deterministic Atoms] Let $\theta : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\dot{\theta}(t) \geq 0$ for all t and $\dot{\theta}(t) > 0$ almost everywhere. Define*

$$T_0 := \{t \in \mathbb{R} : \dot{\theta}(t) = 0\} \tag{136}$$

Let X be a centered stationary Gaussian process with spectral measure F and covariance

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h}\, dF(\lambda) \tag{137}$$

twice differentiable at $h = 0$ with $\ddot{K}(0) < 0$ and $K(0) > 0$. Define

$$Z(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) \tag{138}$$

Then for any $T > 0$,

$$\mathbb{E}[N_{[0,T]}(Z)] = N_{\det}([0,T]) + \frac{\theta(T) - \theta(0)}{\pi} \sqrt{\frac{-\ddot{K}(0)}{K(0)}} \tag{139}$$

where $N_{\det}([0,T]) := \#(T_0 \cap [0,T])$.

Proof. If $t_0 \in T_0$ then $\dot{\theta}(t_0) = 0$ and

$$Z(t_0) = 0 \cdot X(\theta(t_0)) = 0 \tag{140}$$

so these are deterministic zeros. By Theorem 1 and monotonicity of θ, there are finitely many such points in $[0, T]$.

On $I_T = [0, T] \setminus T_0$, $\dot\theta(t) > 0$ and

$$Z(t) = 0 \Longleftrightarrow X(\theta(t)) = 0 \tag{141}$$

Define $Y(t) = X(\theta(t))$. Then

$$K_Y(t, s) = \mathbb{E}\left[Y(t)\,Y(s)\right] = K\left(\theta(t) - \theta(s)\right) \tag{142}$$

Hence

$$\frac{\partial}{\partial s}\,K_Y(t, s) = -\dot\theta(s)\,\dot K\left(\theta(t) - \theta(s)\right) \tag{143}$$

and since K is even, $\dot K(0) = 0$ and

$$\lim_{s \to t} \frac{\partial}{\partial s}\,K_Y(t, s) = 0 \tag{144}$$

Similarly,

$$\frac{\partial^2}{\partial s\,\partial t}\,K_Y(t, s) = -\dot\theta(t)\,\dot\theta(s)\,\ddot K\left(\theta(t) - \theta(s)\right) \tag{145}$$

so

$$\lim_{s \to t} \frac{\partial^2}{\partial s\,\partial t}\,K_Y(t, s) = -\dot\theta(t)^2\,\ddot K(0) \tag{146}$$

and $K_Y(t, t) = K(0)$.

Kac–Rice gives the zero intensity

$$\lambda_Y(t) = \frac{1}{\pi}\sqrt{\frac{K(0)\,(-\dot\theta(t)^2\,\ddot K(0))}{K(0)^2}} = \frac{\dot\theta(t)}{\pi}\sqrt{\frac{-\ddot K(0)}{K(0)}} \tag{147}$$

for $t \in I_T$. Then

$$\mathbb{E}[N_{\mathrm{rand}}([0, T])] = \int_{I_T} \lambda_Y(t)\,dt = \frac{1}{\pi}\sqrt{\frac{-\ddot K(0)}{K(0)}}\int_0^T \dot\theta(t)\,dt \tag{148}$$

and

$$\int_0^T \dot\theta(t)\,dt = \theta(T) - \theta(0) \tag{149}$$

so

$$\mathbb{E}[N_{\mathrm{rand}}([0, T])] = \frac{\theta(T) - \theta(0)}{\pi}\sqrt{\frac{-\ddot K(0)}{K(0)}} \tag{150}$$

The total is

$$\mathbb{E}[N_{[0, T]}(Z)] = N_{\mathrm{det}}([0, T]) + \mathbb{E}[N_{\mathrm{rand}}([0, T])] \tag{151}$$

as claimed. $\square$

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