perplexity

Discrete Approximation and Convergence

Setup: Frequency Discretization

Let $ \Delta\lambda $ be the frequency bin size. Discretize the frequency axis as:

$$\omega_k = k \cdot \Delta\lambda, \quad k = \ldots, -2, -1, 0, 1, 2, \ldots$$

For a finite frequency range $ [-W, W] $, the number of frequency bins is:

$$N = \frac{2W}{\Delta\lambda}$$

Therefore:

$$\boxed{\Delta\lambda = \frac{2W}{N}}$$

confirming that $ \Delta\lambda $ and $ N $ are inversely related.

Discrete Approximation of the Forward Transform

The continuous representation:

$$X(t) = \int_{-\infty}^{\infty} \phi_t(\omega) F(\omega)\, d\omega$$

is approximated by the Riemann sum:

$$X(t) \approx \sum_{k=-\infty}^{\infty} \phi_t(\omega_k) F(\omega_k) \Delta\lambda$$

For finite bandwidth $ [-W, W] $:

$$\boxed{X(t) \approx \sum_{k=-N/2}^{N/2-1} \phi_t(k\Delta\lambda) F(k\Delta\lambda) \Delta\lambda}$$

Discrete Approximation of the Inverse Transform

The continuous inversion:

$$F(\omega) = \int_{-\infty}^{\infty} X(t) \overline{\phi_t(\omega)}\, dt$$

becomes:

$$\boxed{F(\omega_k) = \int_{-\infty}^{\infty} X(t)\overline{\phi_t(\omega_k)}\, dt}$$

Note: The inverse is still an integral over time, not a sum, because we're inverting from the continuous-time process $ X(t) $.

Convergence as $ \Delta\lambda \to 0 $

Theorem: If $ F(\omega) $ is continuous and integrable, then as $ \Delta\lambda \to 0 $ (equivalently, $ N \to \infty $):

$$\lim_{\Delta\lambda \to 0} \sum_{k=-\infty}^{\infty} \phi_t(k\Delta\lambda)F(k\Delta\lambda)\Delta\lambda = \int_{-\infty}^{\infty} \phi_t(\omega)F(\omega)\, d\omega$$

Proof: This is the standard convergence of Riemann sums to the Riemann integral. For any fixed $ t $:

1. The function $ g(\omega) = \phi_t(\omega) F(\omega) $ is integrable by assumption.
2. The Riemann sum is:
$$S_{\Delta\lambda} = \sum_k g(k\Delta\lambda)\Delta\lambda$$
3. By the Riemann integral theorem:
$$\lim_{\Delta\lambda \to 0} S_{\Delta\lambda} = \int_{-\infty}^{\infty} g(\omega)\, d\omega = \int_{-\infty}^{\infty} \phi_t(\omega)F(\omega)\, d\omega$$

Summary Table: Continuous vs. Discrete

Representation	Continuous	Discrete Approximation
Forward	$ X(t) = \int \phi_t(\omega) F(\omega) d\omega $	$ X(t) \approx \sum_k \phi_t(k\Delta\lambda) F(k\Delta\lambda) \Delta\lambda $
Inverse	$ F(\omega) = \int X(t) \overline{\phi_t(\omega)} dt $	$ F(k\Delta\lambda) = \int X(t) \overline{\phi_t(k\Delta\lambda)} dt $
Relationship	$ \Delta\lambda \to 0 $	$ N = \frac{2W}{\Delta\lambda} \to \infty $

Key Point:

As $ \Delta\lambda \to 0 $, we get $ N \to \infty $, and the discrete approximation converges pointwise to the continuous integral, provided $ F(\omega) $ has sufficient regularity (continuity and integrability).