Unitary Time Changes of Stationary Processes Yield Oscillatory Processes

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are locally square-integrable (meaning over compact sets). Applying U_θ to the Cramér spectral representation of a stationary process $X(t)$ produces the transformed process $Z(t) = U_\theta X(t) = \sqrt{\dot{\theta}(t)}\ X(\theta(t)) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)}\ d\Phi(\lambda)$, which is an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\ e^{i\lambda\theta(t)}$, evolutionary spectrum $dF_t(\lambda) = \dot{\theta}(t)\ dF(\lambda)$, and covariance kernel $K_Z(t,s) = \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\ K_X(\theta(t), \theta(s))$ where K_X is the stationary covariance of $X(t) = \int_{\mathbb{R}} e^{i\lambda t}\ d\Phi(\lambda)$, and the expected zero-counting function $\mathbb{E}[N_{[a,b]}]$ of the oscillatory process paths equals $\sqrt{-\ddot{K}(0)}\ (\theta(a) - \theta(b))$. The sample paths of any non-degenerate second-order stationary process are locally square integrable, making the unitary time-change operator U_θ applicable to typical realizations. A zero-localization measure $d\mu(t) = \delta(Z(t))\,|\dot{Z}(t)|\ dt$ induces a Hilbert space $L^2(\mu)$ on the zero set of each oscillatory process realization $Z(t)$, and the multiplication operator $(L f)(t) = t f(t)$ has simple pure point spectrum equal to the zero crossing set of Z

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1 Gaussian Processes

1.1 Definition

Definition 1. ***(Gaussian process)*** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T a non-empty index set. A family $\{X_t : t \in T\}$ of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Gaussian process if for every finite subset $\{t_1, \ldots, t_n\} \subset T$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ is multivariate normal (possibly degenerate). Equivalently, every finite linear combination $\sum_{i=1}^n a_i X_{t_i}$ is either almost surely constant or Gaussian. The mean function is $m(t) := \mathbb{E}[X_t]$ and the covariance kernel is*

$$K(s,t) = \operatorname{Cov}(X_s, X_t) \tag{1}$$

For any finite $(t_i)_{i=1}^n \subset T$, the matrix $K_{ij} = K(t_i, t_j)$ is symmetric positive semidefinite, and a Gaussian process is completely determined in law by m and K

1.2 Stationary processes

Definition 2. ***[Cramér spectral representation]*** *[1] A zero-mean stationary process X with spectral measure F admits the sample path representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, d\,\Phi(\lambda) \tag{2}$$

which has covariance

$$R_X(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)}\, d\,F(\lambda) \tag{3}$$

1.2.1 Sample path realizations

Definition 3. ***[Locally square-integrable functions]*** *Define*

$$L^2_{\mathrm{loc}}(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} : \int_K |f(t)|^2\, d\,t < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \tag{4}$$

Remark 4. Every bounded measurable set in $\mathbb{R}$ is compact or contained in a compact set; hence $L^2_{\mathrm{loc}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity

Theorem 5. ***[Sample paths in $\mathbf{L^2_{loc}(\mathbb{R})}$]*** *Let $\{X(t)\}_{t \in \mathbb{R}}$ be a second-order stationary process with*

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \tag{5}$$

Then almost every sample path lies in $L^2_{\mathrm{loc}}(\mathbb{R})$

Proof. Fix an arbitrary bounded interval $[a, b] \subset \mathbb{R}$ with $a < b$. Define

$$Y_{[a,b]} := \int_a^b X(t)^2 \, dt \tag{6}$$

By Tonelli's theorem, since $X(t)^2 \geq 0$,

$$\mathbb{E}[Y_{[a,b]}] = \mathbb{E}\left[\int_a^b X(t)^2 \, dt\right] = \int_a^b \mathbb{E}[X(t)^2] \, dt \tag{7}$$

By stationarity, $\mathbb{E}[X(t)^2] = \sigma^2$ for all t, hence

$$\mathbb{E}[Y_{[a,b]}] = \sigma^2 (b - a) < \infty \tag{8}$$

Markov's inequality yields, for $M > 0$,

$$\mathbb{P}(Y_{[a,b]} > M) \leq \frac{\mathbb{E}[Y_{[a,b]}]}{M} = \frac{\sigma^2 (b - a)}{M} \tag{9}$$

and letting $M \to \infty$ gives $\mathbb{P}(Y_{[a,b]} < \infty) = 1$. Now let $K \subset \mathbb{R}$ be compact, so $K \subseteq [-N, N]$ for some $N > 0$. Then

$$\int_K X(t)^2 \, dt \leq \int_{-N}^N X(t)^2 \, dt < \infty \quad \text{a.s.} \tag{10}$$

hence almost every path satisfies $\int_K |X(t, \omega)|^2 \, dt < \infty$ for every compact K, i.e. $X(\cdot, \omega) \in L^2_{\text{loc}}(\mathbb{R})$ □

1.3 (Non-Stationary) Oscillatory Processes

Definition 6. ***[Oscillatory process]****[3] Let F be a finite nonnegative Borel measure on $\mathbb{R}$. Let*

$$A_t \in L^2(F) \quad \forall t \in \mathbb{R} \tag{11}$$

be the gain function and

$$\varphi_t(\lambda) = A_t(\lambda) \, e^{i\lambda t} \tag{12}$$

the corresponding oscillatory function. An oscillatory process is a stochastic process represented as

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda) \, d\Phi(\lambda) = \int_{\mathbb{R}} A_t(\lambda) \, e^{i\lambda t} \, d\Phi(\lambda) \tag{13}$$

where Φ is a complex orthogonal random measure with spectral measure F satisfying

$$d\,\mathbb{E}[\Phi(\lambda) \, \overline{\Phi(\mu)}] = \delta(\lambda - \mu) \, dF(\lambda) \tag{14}$$

and covariance

$$\begin{aligned} R_Z(t, s) = \mathbb{E}[Z(t) \, \overline{Z(s)}] &= \int_{\mathbb{R}} A_t(\lambda) \, \overline{A_s(\lambda)} \, e^{i\lambda(t-s)} \, dF(\lambda) \\ &= \int_{\mathbb{R}} \varphi_t(\lambda) \, \overline{\varphi_s(\lambda)} \, dF(\lambda \end{aligned} \tag{15}$$

Definition 7. ***[Evolutionary spectrum]*** *The evolutionary power spectral density of an oscillatory process is given by is*

$$\begin{aligned} d\,F_t(\lambda) &= S_t(\lambda)\mathrm{d}\lambda \\ &= |A_t(\lambda)|^2\; d\,F(\lambda) \\ &= |A_t(\lambda)|^2\; S(\lambda) d\lambda \end{aligned} \tag{16}$$

Definition 8. ***[Variance of evolutionary process]*** *The variance of an evolutionary process $Z(t)$ is given by integrating the evolutionary power spectral density $S_t(\lambda)$ over all frequencues*

$$\mathrm{var}(Z(t)) = \int_{-\infty}^{\infty} S_t(\lambda)\mathrm{d}\lambda = \int_{-\infty}^{\infty} \mathrm{d}F_t(\lambda) \tag{17}$$

Theorem 9. ***[Real-valuedness criterion for oscillatory processes]*** *Let Z be an oscillatory process with oscillatory function $\varphi_t(\lambda) = A_t(\lambda)\; e^{i\lambda t}$ and spectral measure F. Then Z is real-valued if and only if*

$$A_t\,(-\lambda) = \overline{A_t(\lambda)} \quad \textit{for } F\textit{-a.e. } \lambda \in \mathbb{R} \tag{18}$$

equivalently

$$\varphi_t\,(-\lambda) = \overline{\varphi_t(\lambda)} \quad \textit{for } F\textit{-a.e. } \lambda \in \mathbb{R} \tag{19}$$

Proof. If Z is real-valued, then $Z(t) = \overline{Z(t)}$ for all t. Taking conjugates in the representation $Z(t) = \int A_t(\lambda)\, e^{i\lambda t}\; d\,\Phi(\lambda)$ and using the symmetry relation for the orthogonal random measure appropriate for real-valued processes, a change of variable $\mu = -\lambda$ shows that the $L^2(F)$-integrands must agree F-a.e., i.e. $A_t(\lambda) = \overline{A_t\,(-\lambda)}$, which is equivalent to (18). Using $\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t}$ then gives (19). The converse follows by reversing the steps □

Theorem 10. ***[Existence of oscillatory processes with explicit L^2-limit construction]*** *Let F be an absolutely continuous spectral measure and the gain function $A_t(\lambda) \in L^2(F)$ for all $t \in \mathbb{R}$, measurable jointly in (t, λ). Define the time-dependent spectrum*

$$S_t := \int_{\mathbb{R}} |A_t(\lambda)|^2\; d\,F(\lambda) \;=\; \int_{\mathbb{R}} |A_t(\lambda)|^2\; S(\lambda)\; d\,\lambda \;<\; \infty \tag{20}$$

Then there exists a complex orthogonal random measure Φ with spectral measure F such that for each fixed t the stochastic integral

$$Z(t) \;=\; \int_{\mathbb{R}} A_t(\lambda)\; e^{i\lambda t}\; d\,\Phi(\lambda) \tag{21}$$

is well-defined as an $L^2(\Omega)$-limit and has covariance R_Z as in (15)

Proof. Step 1 (simple functions and isometry). Let S denote the set of simple functions

$$g(\lambda) \;=\; \sum_{j=1}^{n} c_j \; 1_{E_j}(\lambda) \tag{22}$$

with disjoint Borel E_j and $F(E_j) < \infty$, $c_j \in \mathbb{C}$. Define the stochastic integral on S by

$$\int_{\mathbb{R}} g(\lambda)\; d\,\Phi(\lambda) \;:=\; \sum_{j=1}^{n} c_j \; \Phi(E_j) \tag{23}$$

Using orthogonality of Φ,

$$\mathbb{E}\Bigg[\bigg|\int g\; d\,\Phi\bigg|^2\Bigg] = \sum_{j=1}^{n} |c_j|^2 \; F(E_j) = \int_{\mathbb{R}} |g(\lambda)|^2 \; d\,F(\lambda) \tag{24}$$

Thus the map $I\colon \mathsf{S} \to L^2(\Omega)$, $I(g) = \int g\; d\,\Phi$, is an isometry with respect to the $L^2(F)$-norm.

Step 2 (density and Cauchy property). Simple functions are dense in $L^2(F)$: for any $h \in L^2(F)$ there exists $g_n \in \mathsf{S}$ with $\|h - g_n\|_{L^2(F)} \to 0$. By (24),

$$\mathbb{E}\Bigg[\bigg|\int g_n\; d\,\Phi - \int g_m\; d\,\Phi\bigg|^2\Bigg] = \|g_n - g_m\|_{L^2(F)}^2 \xrightarrow{n,m\to\infty} 0 \tag{25}$$

so $\{\int g_n\; d\,\Phi\}$ is Cauchy in $L^2(\Omega)$.

Step 3 (definition by L^2-limit and independence of approximating sequence). Since $L^2(\Omega)$ is complete, the limit exists. Define, for $h \in L^2(F)$,

$$\int_{\mathbb{R}} h(\lambda)\; d\,\Phi(\lambda) \;:=\; \lim_{n\to\infty} \int_{\mathbb{R}} g_n(\lambda)\; d\,\Phi(\lambda) \tag{26}$$

where $g_n \in \mathsf{S}$ and $\|h - g_n\|_{L^2(F)} \to 0$. If g_n and $\tilde{g}_n$ are two such approximating sequences, then $\|g_n - \tilde{g}_n\|_{L^2(F)} \to 0$ and again by (24) the corresponding integrals differ by an $L^2(\Omega)$-null sequence, so the limit is independent of the sequence.

Step 4 (isometry and linearity extend). By continuity from (24) and (26),

$$\mathbb{E}\Bigg[\bigg|\int h\; d\,\Phi\bigg|^2\Bigg] = \int_{\mathbb{R}} |h(\lambda)|^2 \; d\,F(\lambda) \tag{27}$$

for $h \in L^2(F)$, and the map $h \mapsto \int h\; d\,\Phi$ is linear and isometric.

Step 5 (apply to φ_t). Since $|e^{i\lambda t}| = 1$, $\varphi_t(\lambda) = A_t(\lambda)\; e^{i\lambda t} \in L^2(F)$ and

$$\int_{\mathbb{R}} |\varphi_t(\lambda)|^2 \; d\,F(\lambda) = \int_{\mathbb{R}} |A_t(\lambda)|^2 \; d\,F(\lambda) = S_t < \infty \tag{28}$$

Hence $Z(t)$ in (21) is well-defined as the $L^2(\Omega)$-limit (26) with $h = \varphi_t$. Computing covariance via sesquilinearity together with (14) yields (15) $\square$

1.4 Operator Representations

[2]

2 Unitarily Time-Changed Stationary Processes

2.1 Unitary time-change operator $U_\theta f$

Theorem 11. ***[Unitary time-change and local isometry]*** *Let the time-scaling function $\theta: \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with*

$$\dot{\theta}(t) > 0 \tag{29}$$

almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero. For f measurable, define

$$(U_\theta f)(t) = \sqrt{\dot{\theta}(t)}\ f(\theta(t)) \tag{30}$$

Its inverse is given by

$$(U_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{31}$$

For every compact set $K \subseteq \mathbb{R}$ and $f \in L^2_{\text{loc}}(\mathbb{R})$,

$$\int_K |(U_\theta f)(t)|^2 \, d t = \int_{\theta(K)} |f(s)|^2 \, d s \tag{32}$$

Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\text{loc}}(\mathbb{R})$

Proof. By (30), $\int_K |(U_\theta f)(t)|^2 \, d t = \int_K \dot{\theta}(t) \, |f(\theta(t))|^2 \, d t$. With the change of variables $s = \theta(t)$ and $d s = \dot{\theta}(t) \, d t$, the domain maps to $\theta(K)$, giving (32). The two-sided inverse identities follow from direct substitution into (30) and (31) □

2.2 Time-Varying (Convolution) Filter Representations

Theorem 12. *TODO: insert time-varying filter representations (both forward and reverse)*

2.2.1 The Oscillatory Subclass $Z(t) = U_\theta X(t)$

Theorem 13. ***[Filter representations of unitarily time-changed stationary processes]*** *Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\theta'(t) > 0$ a.e. Let $X(u) = \int_{\mathbb{R}} e^{i\lambda t} \mathrm{d}\Phi(\lambda)$ be a realization of a stationary process , and set*

$$Z(t) = \sqrt{\dot{\theta}(t)}\ Y(\theta(t)) \tag{33}$$

Then:

1. *The forward filter kernel is*

$$h(t, u) = \sqrt{\dot{\theta}(t)}\ \delta(u - \theta(t)) \tag{34}$$

2. *The inverse filter kernel is*

$$g(t, s) = \frac{\delta(s - \theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \tag{35}$$

3. *The composition $(g \circ h)$ recovers the identity:*

$$Y(t) = \int_{\mathbb{R}} g(t, s)\ Z(s)\ d\, s = \frac{Z(\theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \tag{36}$$

Proof. Using the sifting property of the Dirac delta in (34) gives (33). Likewise, applying (35), then substituting (33) at $s = \theta^{-1}(t)$ and $\theta \circ \theta^{-1} = \mathrm{id}$ yields (36) □

2.3 Transformation of stationary to oscillatory processes via U_θ

Theorem 14. ***[Unitary time change produces oscillatory process]*** *Let X be zero-mean stationary as in Definition 2. For a scaling function θ as in Theorem 11, define*

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)}\ X(\theta(t)) \tag{37}$$

Then Z is a realization of an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\ e^{i\lambda\theta(t)} \tag{38}$$

gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\ e^{i\lambda(\theta(t) - t)} \tag{39}$$

and covariance

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t)\,\overline{Z(s)}] \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\ \mathbb{E}[X(\theta(t))\,\overline{X(\theta(s))}] \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\ R_X(\theta(t) - \theta(s)) \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t) - \theta(s))}\ d\, F(\lambda) \end{aligned} \tag{40}$$

Proof. From the Cramér representation (2), $X(\theta(t)) = \int e^{i\lambda\theta(t)}\, d\,\Phi(\lambda)$. Therefore

$$Z(t) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\,\Phi(\lambda) = \int_{\mathbb{R}} \left(\sqrt{\dot{\theta}(t)}\ e^{i\lambda\theta(t)}\right) d\,\Phi(\lambda) = \int \varphi_t(\lambda)\ d\,\Phi(\lambda)$$

which is of the oscillatory form with φ_t as in (38) and A_t as in (39). The covariance follows from stationarity via (3) □

Corollary 15. ***[Evolutionary spectrum of unitarily time-changed stationary process]*** *The evolutionary spectrum is*

$$d\,F_t(\lambda) = |A_t(\lambda)|^2\ d\,F(\lambda) = \dot{\theta}(t)\ d\,F(\lambda) \tag{41}$$

Proof. Since $|e^{i\alpha}| = 1$, $|A_t(\lambda)|^2 = \dot{\theta}(t)$, giving (41) □

2.4 Covariance operator conjugation

Proposition 16. ***[Operator conjugation]*** *Let*

$$(T_K\, f)(t) := \int_{\mathbb{R}} K(|t-s|)\ f(s)\ d\,s \tag{42}$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h}\ d\,F(\lambda) \tag{43}$$

Define the transformed kernel

$$K_\theta(s,t) := \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\ K(|\theta(t) - \theta(s)|) \tag{44}$$

Then for all $f \in L^2_{\mathrm{loc}}(\mathbb{R})$,

$$(T_{K_\theta} f)(t) = (U_\theta\ T_K\ U_\theta^{-1}\ f)(t) \tag{45}$$

Proof. Compute

$$(U_\theta\, T_K\, U_\theta^{-1}\ f)(t) = \sqrt{\dot{\theta}(t)}\ (T_K\, U_\theta^{-1}\ f)(\theta(t)) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} K(|\theta(t) - s|) \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}}\ d\,s$$

With $s = \theta(u)$, $d\,s = \dot{\theta}(u)\ d\,u$, obtain

$$\sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|)\, \sqrt{\dot{\theta}(u)}\ f(u)\ d\,u = \int_{\mathbb{R}} K_\theta(u,t)\ f(u)\ d\,u = (T_{K_\theta} f)(t)$$ □

3 Zero Localization

Definition 17. ***[Zero localization measure]*** *Let Z be real-valued with $Z \in C^1(\mathbb{R})$ having only simple zeros*

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \tag{46}$$

Define, for Borel $B \subset \mathbb{R}$,

$$\mu(B) := \int_B \delta(Z(t))\, |\dot{Z}(t)|\; dt \tag{47}$$

so that μ places unit mass at each simple zero of Z counted by the co-area/change-of-variables identity for C^1 functions. The induced space $L^2(\mu)$ consists of (equivalence classes of) functions supported on the zero set of Z, and the multiplication operator $(L\,f)(t) = t\,f(t)$ is essentially self-adjoint on C_c^∞ functions supported on the zero set with pure point spectrum equal to the zero-crossing set

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