Orthonormal Galerkin Method for Stationary Integral Covariance Operator Eigenfunction Expansions

BY STEPHEN CROWLEY

Table of contents

1 Given

1. $K(s,t) = K(t-s)$
2. $K(t-s) = \sum_{n=0}^{\infty} \psi_n(t-s)$ (uniformly convergent)
3. Eigenvalue equation: $\int_0^{\infty} K(t-s)\, \phi_k(t)\, d t = \lambda_k\, \phi_k(s)$
4. Eigenfunction expansion: $\phi_k(t) = \sum_{n=0}^{\infty} c_{n,k}\, \psi_n(t)$
5. The basis functions $\{\psi_n\}$ are orthonormal, i.e., $\int_0^{\infty} \psi_m(s)\, \psi_n(s)\, d s = \delta_{mn}$

2 Objective

Solve for the coefficient matrices $c_{n,k}$ for the eigenfunctions

$$T\phi_k(s) = \lambda_k\, \phi_k(s) \tag{1}$$

of the integral covariance operator

$$Tf(s) = \int_0^{\infty} K(t-s)\, f(t)\, d t \tag{2}$$

3 Proof

1. Substitute the eigenfunction expansion into the eigenvalue equation:

$$\int_0^\infty K(t-s) \sum_{n=0}^\infty c_{n,k} \psi_n(t)\ dt = \lambda_k \sum_{n=0}^\infty c_{n,k} \psi_n(s) \tag{3}$$

2. Use the uniform expansion of K:

$$\int_0^\infty \sum_{j=0}^\infty \psi_j(t-s) \sum_{n=0}^\infty c_{n,k} \psi_n(t)\ dt = \lambda_k \sum_{n=0}^\infty c_{n,k} \psi_n(s) \tag{4}$$

3. Apply Fubini's theorem (justified by uniform convergence):

$$\sum_{n=0}^\infty c_{n,k} \sum_{j=0}^\infty \int_0^\infty \psi_j(t-s)\, \psi_n(t)\ dt = \lambda_k \sum_{n=0}^\infty c_{n,k} \psi_n(s) \tag{5}$$

4. Define

$$G_{j,n}(s) = \int_0^\infty \psi_j(t-s)\, \psi_n(t)\ dt \tag{6}$$

   to express (5)

$$\begin{aligned} \lambda_k \sum_{n=0}^\infty c_{n,k} \psi_n(s) &= \sum_{j=0}^\infty \sum_{n=0}^\infty c_{n,k} \int_0^\infty \psi_j(t-s)\, \psi_n(t)\ dt \\ &= \sum_{n=0}^\infty c_{n,k} \sum_{j=0}^\infty G_{j,n}(s) \end{aligned} \tag{7}$$

5. Project onto the basis $\{\psi_m(s)\}$. Multiply both sides by $\psi_m(s)$ and integrate over s:

$$\int_0^\infty \sum_{n=0}^\infty c_{n,k} \sum_{j=0}^\infty G_{j,n}(s)\, \psi_m(s)\ ds = \lambda_k \int_0^\infty \sum_{n=0}^\infty c_{n,k} \psi_n(s)\, \psi_m(s)\ ds \tag{8}$$

6. Interchange summation and integration and utilize the orthonormality of $\{\psi_n\}$

$$\begin{aligned} \sum_{n=0}^\infty c_{n,k} \sum_{j=0}^\infty \int_0^\infty G_{j,n}(s)\, \psi_m(s)\ ds &= \lambda_k \sum_{n=0}^\infty c_{n,k} \int_0^\infty \psi_n(s)\, \psi_m(s)\ ds \\ &= \lambda_k \sum_{n=0}^\infty c_{n,k}\, \delta_{n,m} \\ &= \lambda_k\, c_{m,k} \end{aligned} \tag{9}$$

7. Define:

$$b_{m,n} = \sum_{j=0}^{\infty} \int_0^{\infty} G_{j,n}(s)\, \psi_m(s)\; d\,s \tag{10}$$

8. Our equation becomes:

$$\sum_{n=0}^{\infty} b_{m,n}\, c_{n,k} = \lambda_k\, c_{m,k} \tag{11}$$

9. This is a standard eigenvalue problem:

$$B\, \vec{c}_k = \lambda_k\, \vec{c}_k \tag{12}$$

where $B = (b_{m,n})$ and $\vec{c}_k = (c_{0,k}, c_{1,k}, \ldots)^T$

4 Verification that Solutions are Eigenfunctions

We will now prove that the solutions obtained are indeed eigenfunctions of the original integral equation.

- Let λ_k and $\vec{c}_k = (c_{0,k}, c_{1,k}, \ldots)^T$ be the eigenvalues and eigenvectors of the matrix equation:

$$B\, \vec{c}_k = \lambda_k\, \vec{c}_k \tag{13}$$

where $B = (b_{m,n})$ as derived above.

- construct the functions $\phi_k(t)$:

$$\phi_k(t) = \sum_{n=0}^{\infty} c_{n,k}\, \psi_n(t) \tag{14}$$

- Substitute this into the original integral equation:

$$T\phi_k(t) = \int_0^{\infty} K\,(t-s)\; \phi_k(t)\; d\,t = \int_0^{\infty} K\,(t-s) \left[\sum_{n=0}^{\infty} c_{n,k}\, \psi_n(t) \right] d\,t \tag{15}$$

- Use the expansion of $K\,(t-s)$ to interchanging summations:

$$= \int_0^{\infty} \left[\sum_{j=0}^{\infty} \psi_j\,(t-s) \right] \left[\sum_{n=0}^{\infty} c_{n,k}\, \psi_n(t) \right] d\,t \tag{16}$$

$$= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} c_{n,k} \int_0^{\infty} \psi_j\,(t-s)\; \psi_n(t)\; d\,t \tag{17}$$

- then rewrite the left-hand side of the integral equation:

$$\begin{aligned}
T\phi_k(s) &= \int_0^\infty K\,(t-s)\,\phi_k(t)\ d\,t \\
&= \int_0^\infty K\,(t-s)\left[\sum_{n=0}^\infty c_{n,k}\,\psi_n(t)\right] d\,t \\
&= \sum_{n=0}^\infty c_{n,k}\int_0^\infty K\,(t-s)\,\psi_n(t)\ d\,t \\
&= \sum_{n=0}^\infty c_{n,k}\left[\sum_{j=0}^\infty \int_0^\infty \psi_j\,(t-s)\,\psi_n(t)\ d\,t\right] \\
&= \sum_{n=0}^\infty c_{n,k}\left[\sum_{j=0}^\infty G_{j,n}(s)\right]
\end{aligned}$$

recalling that

$$\begin{aligned}
G_{j,n}(s) &= \int_0^\infty \psi_j\,(t-s)\,\psi_n(t)\ d\,t \\
b_{m,n} &= \sum_{j=0}^\infty \int_0^\infty G_{j,n}(s)\,\psi_m(s)\ d\,s
\end{aligned} \tag{18}$$

- finally, project $T\phi_k(s)$ onto $\psi_m(s)$ by multiplying it by $\psi_m(s)$ then integrating over s from 0 to ∞:

$$\begin{aligned}
\int_0^\infty \psi_m(s)\,T\phi_k(s)\ d\,s &= \int_0^\infty \psi_m(s)\left[\sum_{n=0}^\infty c_{n,k}\left[\sum_{j=0}^\infty G_{j,n}(s)\right]\right] d\,s \\
&= \sum_{n=0}^\infty c_{n,k}\left[\sum_{j=0}^\infty \int_0^\infty G_{j,n}(s)\,\psi_m(s)\ d\,s\right] \\
&= \sum_{n=0}^\infty c_{n,k}\,b_{m,n} \\
&= B\ \{\vec{c}_k\} \\
&= \lambda_k \vec{c}_k \\
&= \lambda_k\,c_{m,k}
\end{aligned} \tag{19}$$

- Since this holds for all m, and $\{\psi_m\}$ is a complete orthonormal basis, we conclude:

$$\int_0^\infty K\,(t-s)\,\phi_k(t)\,d\,t = \lambda_k\,\phi_k(s) \tag{20}$$

Therefore, the $\phi_k(s)$ constructed from the eigenvectors of B are indeed eigenfunctions of the original integral equation with eigenvalues λ_k.