Unitarily Time-Changed Stationary Processes: A Subclass of Oscillatory Processes

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are locally square-integrable. Applying U_θ to the Cramér spectral representation of a stationary process $X(t)$ produces the transformed process

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)} d\Phi(\lambda)$$

which is an oscillatory process with oscillatory function $\phi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)}$, evolutionary power spectral density $S_t(\lambda) = \dot{\theta}(t) S(\lambda)$, and covariance kernel

$$K_Z(t,s) = \sqrt{\dot{\theta}(t)\dot{\theta}(s)} K_X(\theta(t), \theta(s))$$

where K_X is the stationary covariance of $X(t) = \int_{\mathbb{R}} e^{i\lambda t} d\Phi(\lambda)$. Following Mandrekar's characterization theorem [2], every oscillatory process admits a stationary representation via shift-commuting operators. The generalized Kac-Rice formula for non-stationary processes gives the expected zero-counting function. By Bulinskaya's theorem, when the covariance is twice continuously differentiable with $\ddot{R}(0) < 0$, almost all zeros are simple.

Contents

1   Gaussian Processes

1.1   Definition

Definition 1. *(Gaussian process) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T a nonempty index set. A family $\{X_t : t \in T\}$ of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Gaussian process if for every finite subset $\{t_1, \ldots, t_n\} \subset T$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ is multivariate normal (possibly degenerate). Equivalently, every finite linear combination $\sum_{i=1}^{n} a_i X_{t_i}$ is either almost surely constant or Gaussian. The mean function is $m(t) := \mathbb{E}[X_t]$ and the covariance kernel is*

$$K(s,t) = \mathrm{Cov}(X_s, X_t) \tag{1}$$

For any finite $(t_i)_{i=1}^{n} \subset T$, the matrix $K_{ij} = K(t_i, t_j)$ is symmetric positive semidefinite, and a Gaussian process is completely determined in law by m and K.

1.2   Stationary Processes

Definition 2. *(Cramér spectral representation) A zero-mean stationary process X with spectral measure F admits the sample path representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t} d\Phi(\lambda) \tag{2}$$

which has covariance

$$R_X(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)} dF(\lambda) \tag{3}$$

1.3   Sample Path Realizations

Definition 3. *(Locally square-integrable functions) Define*

$$L^2_{\text{loc}}(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} : \int_K |f(t)|^2 dt < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \tag{4}$$

Remark 1. *Every bounded measurable set in $\mathbb{R}$ is compact or contained in a compact set; hence $L^2_{\text{loc}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.*

Theorem 1. *(Sample paths in $L^2_{\text{loc}}(\mathbb{R})$) Let $\{X(t)\}_{t\in\mathbb{R}}$ be a second-order stationary process with*

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \tag{5}$$

Then almost every sample path lies in $L^2_{\text{loc}}(\mathbb{R})$.

Proof. Fix a bounded interval $[a,b] \subset \mathbb{R}$ with $a < b$ and define

$$Y_{[a,b]} := \int_a^b X(t)^2 dt \tag{6}$$

By Tonelli's theorem,

$$\mathbb{E}[Y_{[a,b]}] = \int_a^b \mathbb{E}[X(t)^2] dt \tag{7}$$

By stationarity, $\mathbb{E}[X(t)^2] = \sigma^2$, hence

$$\mathbb{E}[Y_{[a,b]}] = \sigma^2(b-a) < \infty \tag{8}$$

Markov's inequality yields

$$\mathbb{P}(Y_{[a,b]} > M) \leq \frac{\sigma^2(b-a)}{M} \tag{9}$$

so $\mathbb{P}(Y_{[a,b]} < \infty) = 1$. If $K \subset \mathbb{R}$ is compact then $K \subseteq [-N, N]$ for some $N > 0$, so

$$\int_K X(t)^2 dt \leq \int_{-N}^N X(t)^2 dt < \infty \text{ a.s.} \tag{10}$$

Thus $X(\cdot, \omega) \in L^2_{\text{loc}}(\mathbb{R})$ for almost every ω. $\qquad\square$

2 Oscillatory Processes

2.1 Definition

Definition 4. *(Oscillatory process) Let F be a finite nonnegative Borel measure on $\mathbb{R}$. Let*

$$A_t \in L^2(F) \quad \forall t \in \mathbb{R} \tag{11}$$

be the gain function and

$$\phi_t(\lambda) = A_t(\lambda)e^{i\lambda t} \tag{12}$$

the corresponding oscillatory function. An oscillatory process is a stochastic process represented as

$$\begin{aligned}
Z(t) &= \int_{\mathbb{R}} \phi_t(\lambda) d\Phi(\lambda) \\
&= \int_{\mathbb{R}} A_t(\lambda)e^{i\lambda t} d\Phi(\lambda)
\end{aligned} \tag{13}$$

where Φ is a complex orthogonal random measure with spectral measure F satisfying

$$\mathbb{E}[\Phi(\lambda)\overline{\Phi(\mu)}] = \delta(\lambda - \mu)dF(\lambda) \tag{14}$$

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and covariance

$$R_Z(t, s) = \mathbb{E}[Z(t)\overline{Z(s)}]$$

$$= \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)}e^{i\lambda(t-s)}dF(\lambda) \tag{15}$$

$$= \int_{\mathbb{R}} \phi_t(\lambda)\overline{\phi_s(\lambda)}dF(\lambda)$$

Definition 5. *(Evolutionary power spectral density) If $dF(\lambda) = S(\lambda)d\lambda$, define*

$$S_t(\lambda) := |A_t(\lambda)|^2 S(\lambda) \tag{16}$$

so that

$$dF_t(\lambda) = S_t(\lambda)d\lambda$$

$$= |A_t(\lambda)|^2 dF(\lambda) \tag{17}$$

$$= |A_t(\lambda)|^2 S(\lambda)d\lambda$$

Theorem 2. *(Real-valuedness criterion for oscillatory processes) Let Z be an oscillatory process with $\phi_t(\lambda) = A_t(\lambda)e^{i\lambda t}$ and spectral measure F. Then Z is real-valued if and only if*

$$A_t(-\lambda) = \overline{A_t(\lambda)} \text{ for } F\text{-a.e. } \lambda \in \mathbb{R} \tag{18}$$

equivalently

$$\phi_t(-\lambda) = \overline{\phi_t(\lambda)} \text{ for } F\text{-a.e. } \lambda \in \mathbb{R} \tag{19}$$

Proof. Taking complex conjugates of (13) and applying the symmetry $\overline{d\Phi(\lambda)} = d\Phi(-\lambda)$ for real processes, with change of variables $\mu = -\lambda$, yields $A_t(\lambda) = \overline{A_t(-\lambda)}$ F-a.e. Reversing the steps gives the converse. $\square$

Theorem 3. *(Existence of oscillatory processes with explicit L^2-limit construction) Let F be absolutely continuous with density $S(\lambda)$ and let $A_t(\lambda) \in L^2(F)$ for all $t \in \mathbb{R}$, measurable jointly in (t, λ). Define*

$$\sigma_t^2 := \int_{\mathbb{R}} |A_t(\lambda)|^2 dF(\lambda) < \infty \tag{20}$$

Then there exists a complex orthogonal random measure Φ with spectral measure F such that for each fixed t the stochastic integral

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda)e^{i\lambda t}d\Phi(\lambda) \tag{21}$$

is well-defined as an $L^2(\Omega)$-limit and has covariance (15).

Proof. Let S be the set of simple functions $g(\lambda) = \sum_{j=1}^n c_j \mathbf{1}_{E_j}(\lambda)$ with disjoint Borel E_j and $F(E_j) < \infty$. Define $\int g \, d\Phi := \sum_{j=1}^n c_j \Phi(E_j)$. Orthogonality gives the isometry:

$$\mathbb{E}\left|\int g \, d\Phi\right|^2 = \int_{\mathbb{R}} |g(\lambda)|^2 dF(\lambda) \tag{22}$$

For $h \in L^2(F)$, choose $g_n \in S$ with $\|h - g_n\|_{L^2(F)} \to 0$. Then:

$$\mathbb{E}\left|\int g_n d\Phi - \int g_m d\Phi\right|^2 = \|g_n - g_m\|_{L^2(F)}^2 \tag{23}$$

and $\lim_{n,m\to\infty} \|g_n - g_m\|_{L^2(F)}^2 = 0$. Completeness of $L^2(\Omega)$ yields the limit, and the isometry shows independence of the approximating sequence. $\square$

3 Unitarily Time-Changed Stationary Processes

3.1 Unitary Time-Change Operator

Theorem 4. *(Unitary time-change and local isometry) Let $\theta : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\dot{\theta}(t) > 0$ a.e. For measurable f, define:*

$$(U_\theta f)(t) = \sqrt{\dot{\theta}(t)} f(\theta(t)) \tag{24}$$

Define the inverse map:

$$(U_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{25}$$

For every compact $K \subseteq \mathbb{R}$ and $f \in L^2_{\text{loc}}(\mathbb{R})$:

$$\int_K |(U_\theta f)(t)|^2 dt = \int_{\theta(K)} |f(s)|^2 ds \tag{26}$$

Moreover, for $f, g \in L^2_{\text{loc}}(\mathbb{R})$:

$$U_\theta^{-1}(U_\theta f) = f, \quad U_\theta(U_\theta^{-1} g) = g \tag{27}$$

Proof. Using change of variables $s = \theta(t)$, $ds = \dot{\theta}(t)dt$:

$$\int_K \dot{\theta}(t)|f(\theta(t))|^2 dt = \int_{\theta(K)} |f(s)|^2 ds \tag{28}$$

Direct substitution verifies the inverse identities (27). $\qquad\square$

Theorem 5. *(Fundamental inversion via stationary representation [2]) Let $Z(t)$ be an oscillatory process with spectral representation*

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \tag{29}$$

where $A_t \in L^2(F)$ for each t and Φ is an orthogonal random measure with spectral measure F. Then there exists a stationary process

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t} d\Phi(\lambda) \tag{30}$$

and a closed, densely-defined operator A acting on the Hilbert space $H_X(\infty) = \overline{\text{span}}\{X(s) : s \in \mathbb{R}\}$ such that

$$Z(t) = (AX)(t) \tag{31}$$

where the operator A is defined by the spectral integral

$$A = \int_{\mathbb{R}} A_t(\lambda) E(d\lambda) \tag{32}$$

with domain $D(A) \supseteq \{X(s) : s \in \mathbb{R}\}$, where E is the spectral measure of the shift group $\{U_s\}_{s \in \mathbb{R}}$ defined by $U_s X(r) = X(r + s)$. The operator A commutes with the shift group:

$$(AU_s)(h) = (U_s A)(h) \quad \text{for all } h \in D(A) \text{ and } s \in \mathbb{R} \tag{33}$$

The random spectral measure Φ is uniquely determined by X via $\Phi(B) = (E(B)X)(0)$ for all Borel B.

Proof. This is Mandrekar's characterization theorem [2]. We outline the key steps:

Forward direction: Given oscillatory $Z(t)$ as in (29), define the stationary curve

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t} d\Phi(\lambda) \tag{34}$$

By Stone's theorem, there exists a unitary shift group $\{U_s\}$ and spectral measure E such that $X(t) = U_t X(0)$ and

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t} E(d\lambda) X(0) \tag{35}$$

with $\Phi(B) = E(B)X(0)$. Define the operator as in (32). By Dunford-Schwartz spectral theory, A is a closed operator with domain containing $\{X(s) : s \in \mathbb{R}\}$. The commutation relation (33) follows from $U_s E(B) = E(B)U_s$ for all Borel B. Computing:

$$\begin{aligned} (AX)(t) &= \int A_t(\lambda) E(d\lambda) \int e^{i\mu t} E(d\mu) X(0) \\ &= \int A_t(\lambda) e^{i\lambda t} E(d\lambda) X(0) \\ &= \int A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) = Z(t) \end{aligned} \tag{36}$$

Reverse direction: If $Z(t) = (AX)(t)$ where X is stationary and $(AU_s) = (U_s A)$, then by the Stone-von Neumann theorem on commutants of unitary groups, there exists a Borel measurable function $A_t(\cdot)$ such that (32) holds. The domain condition $\{X(s) : s \in \mathbb{R}\} \subseteq D(A)$ implies

$$\int_{\mathbb{R}} |A_t(\lambda)|^2 \|E(d\lambda)X(0)\|^2 < \infty \tag{37}$$

for each t, giving $A_t \in L^2(F)$ where $dF(\lambda) = \|E(d\lambda)X(0)\|^2$. This yields the oscillatory representation. $\qquad \square$

Remark 2. *(Generality of the stationary representation) Theorem 5 establishes that every oscillatory process is a deformed stationary curve in the sense of Mandrekar [2]. The key requirement is shift-commutation (33). Unitarily time-changed processes arise as a particular explicit subclass where $A_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t) - t)}$. The theorem guarantees that for any choice of gain function $A_t(\lambda) \in L^2(F)$, there exists an underlying stationary process and operator recovering the oscillatory process. The notation $(AX)(t)$ indicates that A is an operator acting on the process X, not pointwise function multiplication.*

Definition 6. *(Unitarily time-changed stationary process) Let $X = \{X(t)\}_{t\in\mathbb{R}}$ be a second-order stationary process with sample paths in $L^2_{\text{loc}}(\mathbb{R})$. Let θ satisfy Theorem 4. Define:*

$$Z(t) := (U_\theta X)(t) = \sqrt{\dot\theta(t)}X(\theta(t)) \tag{38}$$

Then Z is called a unitarily time-changed stationary process.

Lemma 1. *(Exact recovery of X) If Z is defined as in (38), then:*

$$X = U_\theta^{-1}Z \tag{39}$$

Proof. This is precisely (27) from Theorem 4. $\square$

3.2 Stationary to Oscillatory

Theorem 6. *(Unitary time change produces oscillatory process) Let X be zero-mean stationary with spectral representation (2). Let θ satisfy Theorem 4. Define $Z(t)$ as in (38). Then Z is an oscillatory process with oscillatory function:*

$$\begin{aligned}
\phi_t(\lambda) &= A_t(\lambda)e^{i\lambda t} \\
&= \sqrt{\dot\theta(t)}e^{i\lambda(\theta(t)-t)}e^{i\lambda t} \\
&= \sqrt{\dot\theta(t)}e^{i\lambda\theta(t)}
\end{aligned} \tag{40}$$

where the gain function is:

$$A_t(\lambda) = \sqrt{\dot\theta(t)}e^{i\lambda(\theta(t)-t)} \tag{41}$$

Proof. Substituting $t \mapsto \theta(t)$ in (2):

$$\begin{aligned}
Z(t) &= \sqrt{\dot\theta(t)}\int_{\mathbb{R}} e^{i\lambda\theta(t)}d\Phi(\lambda) \\
&= \int_{\mathbb{R}}\left(\sqrt{\dot\theta(t)}e^{i\lambda\theta(t)}\right)d\Phi(\lambda)
\end{aligned} \tag{42}$$

Thus $\phi_t(\lambda) = \sqrt{\dot\theta(t)}e^{i\lambda\theta(t)}$ and $A_t(\lambda) = \sqrt{\dot\theta(t)}e^{i\lambda(\theta(t)-t)}$ since $\phi_t(\lambda) = A_t(\lambda)e^{i\lambda t}$ by (12). $\square$

Corollary 1. *(EPSD for the unitary time change) If $dF(\lambda) = S(\lambda)d\lambda$, then:*

$$S_t(\lambda) = |A_t(\lambda)|^2 S(\lambda) = \dot\theta(t)S(\lambda) \tag{43}$$

Proof. From (41):

$$|A_t(\lambda)|^2 = \dot\theta(t)|e^{i\lambda(\theta(t)-t)}|^2 = \dot\theta(t) \tag{44}$$

$\square$

4 Zero Localization

4.1 Kac-Rice Formula

Theorem 7. *(Generalized Kac-Rice formula) Let $Z(t)$ be a real-valued, zero-mean Gaussian process with covariance $K(t,s) = \mathbb{E}[Z(t)Z(s)]$. Assume $K(t,t) > 0$ and that $K(t,s)$ is twice continuously differentiable in a neighborhood of (t,t). Define:*

$$K(t) := K(t,t), \quad K_s(t) := \left.\frac{\partial K(t,s)}{\partial s}\right|_{s=t}, \quad K_{ss}(t) := \left.\frac{\partial^2 K(t,s)}{\partial s^2}\right|_{s=t} \tag{45}$$

Assume

$$V(t) := K(t)K_{ss}(t) - [K_s(t)]^2 > 0 \tag{46}$$

for $t \in [a,b]$. Then:

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \frac{1}{\pi}\sqrt{\frac{V(t)}{K(t)^2}}\, dt \tag{47}$$

Proof. The joint density of $(Z(t), \dot{Z}(t))$ is Gaussian with covariance matrix $\Sigma(t) = \begin{pmatrix} K(t) & K_s(t) \\ K_s(t) & K_{ss}(t) \end{pmatrix}$. The Kac-Rice formula gives:

$$\begin{aligned}
\mathbb{E}[N_{[a,b]}] &= \int_a^b \mathbb{E}[|\dot{Z}(t)| \mid Z(t) = 0]p_{Z(t)}(0)\, dt \\
&= \int_a^b \frac{1}{\sqrt{2\pi K(t)}}\sqrt{\frac{2}{\pi}\frac{K(t)K_{ss}(t) - K_s(t)^2}{K(t)^2}}\, dt
\end{aligned} \tag{48}$$

Simplifying yields (47). $\qquad\square$

4.2 Bulinskaya's Theorem

Theorem 8. *(Bulinskaya) Let $X(t)$ be a real-valued, zero-mean stationary Gaussian process with covariance $R(h) = \mathbb{E}[X(t)X(t+h)]$. If R is twice continuously differentiable in a neighborhood of 0 and $R''(0) < 0$, then with probability 1 all zeros of X are simple.*

Proof. For fixed t_0, $(X(t_0), \dot{X}(t_0))$ is jointly Gaussian. Stationarity gives $\mathbb{E}[X(t_0)\dot{X}(t_0)] = R'(0) = 0$, so they are independent. Since $R''(0) < 0$, $\dot{X}(t_0)$ is non-degenerate and $\mathbb{P}(\dot{X}(t_0) = 0) = 0$. Thus $\mathbb{P}(X(t_0) = 0$ and $\dot{X}(t_0) = 0) = 0$. By continuity and countable union over rationals, all zeros are simple almost surely. $\qquad\square$

5 Example: The Hardy Z-Function

5.1 Definitions

Definition 7. *(Hardy Z-function) Let $\zeta(s)$ be the Riemann zeta function and let $\theta(t)$ denote the Riemann-Siegel theta function. Define:*

$$Z(t) = e^{i\theta(t)}\zeta(1/2 + it) \tag{49}$$

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Definition 8. *(Monotonized theta time change) Let $a > 0$ be the unique critical point of θ in $(0, \infty)$ where $\dot{\theta}(a) = 0$. Define $\Theta : [0, \infty) \to [\Theta(0), \infty)$ by:*

$$\Theta(t) = \begin{cases} 2\theta(a) - \theta(t) & 0 \leq t \leq a \\ \theta(t) & t \geq a \end{cases} \tag{50}$$

5.2   Stationary Candidate and Exact Inversion

Definition 9. *(Stationary candidate defined by U_Θ^{-1}) Define:*

$$X(u) = (U_\Theta^{-1} Z)(u) = \frac{Z(\Theta^{-1}(u))}{\sqrt{\Theta'(\Theta^{-1}(u))}} \quad \forall u \in [\Theta(0), \infty) \tag{51}$$

Lemma 2. *(Exact reconstruction $Z = U_\Theta X$) With X as defined in (51):*

$$Z(t) = (U_\Theta X)(t) = \sqrt{\Theta'(t)} X(\Theta(t)) \quad \forall t \in [0, \infty) \tag{52}$$

Proof. This is (27) from Theorem 4. $\qquad\square$

5.3 L^2_{loc} Identity on Finite Intervals

Lemma 3. *(Finite-interval L^2 identity) For every $T > 0$:*

$$\int_{\Theta(0)}^{\Theta(T)} |X(u)|^2 \, du = \int_0^T |Z(t)|^2 \, dt \tag{53}$$

Proof. With $u = \Theta(t)$, $du = \dot{\Theta}(t) \, dt$, and $X(u) = \frac{Z(t)}{\sqrt{\dot{\Theta}(t)}}$:

$$\int_{\Theta(0)}^{\Theta(T)} |X(u)|^2 \, du = \int_0^T \left| \frac{Z(t)}{\sqrt{\dot{\Theta}(t)}} \right|^2 \dot{\Theta}(t) \, dt = \int_0^T |Z(t)|^2 \, dt$$

$\qquad\square$

Theorem 9. *($X \in L^2_{\mathrm{loc}}([\Theta(0), \infty))$)*

$$X \in L^2_{\mathrm{loc}}([\Theta(0), \infty)) \tag{54}$$

Proof. For compact $[c, d] \subset [\Theta(0), \infty)$, the preimage $[\Theta^{-1}(c), \Theta^{-1}(d)]$ is compact in $[0, \infty)$. The Hardy Z-function is continuous on compact sets, so $\int_{\Theta^{-1}(c)}^{\Theta^{-1}(d)} |Z(t)|^2 \, dt < \infty$. By Lemma 3, $\int_c^d |X(u)|^2 \, du$ equals this finite integral. $\qquad\square$

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5.4 Limit-form Mean-Square Statements

Theorem 10. *(Hardy-Littlewood second moment)*

$$\lim_{T\to\infty} \frac{\int_0^T |\zeta(1/2+it)|^2\, dt}{T\log T} = 1 \tag{55}$$

Equivalently:

$$\lim_{T\to\infty} \frac{\int_0^T |Z(t)|^2\, dt}{T\log T} = 1 \tag{56}$$

Theorem 11. *(Ratio limit for Θ)*

$$\lim_{T\to\infty} \frac{\Theta(T)}{(T/2)\log T} = 1 \tag{57}$$

Theorem 12. *(Mean-square limit for X)*

$$\lim_{T\to\infty} \frac{1}{\Theta(T)-\Theta(0)} \int_{\Theta(0)}^{\Theta(T)} |X(u)|^2\, du = 2 \tag{58}$$

Proof. By Lemma 3:

$$\frac{\int_{\Theta(0)}^{\Theta(T)} |X(u)|^2\, du}{\Theta(T)-\Theta(0)} = \frac{\int_0^T |Z(t)|^2\, dt}{\Theta(T)-\Theta(0)} \tag{59}$$

Writing:

$$\frac{\int_0^T |Z(t)|^2\, dt}{\Theta(T)-\Theta(0)} = \left(\frac{\int_0^T |Z(t)|^2\, dt}{T\log T}\right)\left(\frac{T\log T}{\Theta(T)-\Theta(0)}\right) \tag{60}$$

the first factor $\to 1$ by (56) and the second factor $\to 2$ by (57). $\qquad\square$

5.5 Time-Average Covariance Conjectures

Definition 10. *(Empirical covariance kernel) For $U > \Theta(0)$ and $\tau \in \mathbb{R}$ define:*

$$K_U(\tau) := \frac{1}{U-\Theta(0)} \int_{\Theta(0)}^{U} X(u)X(u+\tau)\, du \tag{61}$$

Conjecture 1. *(Existence of a stationary covariance kernel) For each fixed $\tau \in \mathbb{R}$, the limit:*

$$K(\tau) := \lim_{U\to\infty} K_U(\tau) \tag{62}$$

exists in $\mathbb{R}$.

Conjecture 2. *(Ergodic stationary realization) There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a stationary ergodic process $\{X_{\mathrm{st}}(u,\omega)\}_{u\in\mathbb{R}}$ such that for some $\omega_0 \in \Omega$:*

$$X_{\mathrm{st}}(u,\omega_0) = X(u) \quad \forall u \geq \Theta(0) \tag{63}$$

and for every fixed $\tau \in \mathbb{R}$:

$$\mathbb{E}[X_{\mathrm{st}}(0,\omega)X_{\mathrm{st}}(\tau,\omega)] = \lim_{U\to\infty} \frac{\int_{\Theta(0)}^{U} X(u)X(u+\tau)\, du}{U-\Theta(0)} \tag{64}$$

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