

The Lamperti Transformation and Its Mathematical Foundations in Stochastic Processes

The Lamperti transformation represents a cornerstone in the analysis of stochastic differential equations (SDEs) and their associated processes, enabling simplifications through strategic variable changes. This exposition synthesizes the core formulas and theoretical underpinnings of the Lamperti transformation, its representation theorems, and connections to branching processes, Lévy dynamics, and self-similar Markov processes.

Lamperti Transformation for Stochastic Differential Equations

Definition and Core Formula

Given an Itô process X_t governed by the SDE:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t,$$

the **Lamperti transformation** maps X_t to a new process Z_t with constant diffusion coefficient. The transformation is defined as:

$$Z_t = \phi(X_t, t) = \int^{X_t} \frac{1}{\sigma(x, t)} dx.$$

This integral effectively removes state-dependent volatility, yielding a transformed SDE:

$$dZ_t = \left(\frac{\mu(X_t, t)}{\sigma(X_t, t)} - \frac{1}{2} \frac{\partial \sigma(X_t, t)}{\partial x} \right) dt + dW_t,$$

where the diffusion term becomes unity [1][3][13].

Key Properties

- Unit Diffusion:** The transformed process Z_t has $\sigma_Z = 1$, simplifying analysis and simulation [1][7].
- Separable Volatility:** When $\sigma(X_t, t) = \sigma_1(X_t)\sigma_2(t)$, the transformation decouples spatial and temporal dependencies, aiding analytical solutions [1].
- Geometric Brownian Motion Example:** For $dX_t = \mu X_t dt + \sigma X_t dW_t$, the Lamperti transform gives $Z_t = \frac{1}{\sigma} \ln X_t$, reducing to $dZ_t = \left(\frac{\mu}{\sigma} - \frac{\sigma}{2} \right) dt + dW_t$ [13].

Lamperti Representation for Continuous-State Branching Processes (CSBPs)

Connection to Lévy Processes

The **Lamperti representation** establishes a duality between CSBPs and spectrally positive Lévy processes (no negative jumps). Let X be a Lévy process with Laplace exponent $\psi(\lambda)$, and define the time-change:

$$A_t = \int_0^t \frac{1}{X_s} ds, \quad \theta(t) = \inf\{s > 0 : A_s > t\}.$$

The time-changed process $Y_t = X_{\theta(t)}$ is a CSBP with branching mechanism ψ . Conversely, every CSBP arises from such a time-change [4][9][10][14].

Branching Property and Laplace Exponent

A CSBP Y_t satisfies the branching property:

$$\mathbb{E}_x [e^{-\lambda Y_t}] = e^{-xu_t(\lambda)},$$

where $u_t(\lambda)$ solves:

$$\int_{\lambda}^{u_t(\lambda)} \frac{1}{\psi(u)} du = t.$$

Here, ψ is the Laplace exponent of the associated Lévy process [9][10].

Extinction Criteria

A CSBP Y_t almost surely reaches extinction (i.e., $Y_t \rightarrow 0$) if and only if:

$$\int_0^{\infty} \frac{1}{\psi(u)} du < \infty \quad \text{and} \quad \psi'(0+) \geq 0.$$

This links the long-term behavior of Y_t to the drift of its Lévy counterpart [9][14].

Self-Similar Markov Processes and the Lamperti Transform

Self-Similarity and Time Scaling

A process X_t is H -self-similar if $X_{at} \stackrel{d}{=} a^H X_t$ for all $a > 0$. The **Lamperti transform** relates such processes to stationary ones via:

$$Y(t) = e^{-Ht} X(e^t), \quad X(t) = t^H Y(\ln t).$$

This bijection preserves Markov properties and facilitates spectral analysis [6][15].

Killing at Zero

For real-valued self-similar Markov processes (RSSMPs) killed at hitting zero, the Lamperti representation generalizes to:

$$X_t = \exp(\xi_{\tau(t)}), \quad \tau(t) = \inf \left\{ s : \int_0^s \exp(\alpha \xi_u) du > t \right\},$$

where ξ is a Lévy process and α governs self-similarity [6].

Inverse Lamperti Transformation and Multidimensional Extensions

Inverting the Transformation

While the Lamperti transform maps state-dependent diffusions to constant volatility, the inverse problem—converting a constant-drift SDE to a driftless one—requires stringent conditions. For $dX_t = f(X_t)dt + \sigma dW_t$, an inverse transform exists only if $f(X_t)$ derives from a potential function, ensuring the driftless SDE $dY_t = \sigma(Y_t)dW_t$ maintains the Markov property [7].

Multidimensional Challenges

Extending the Lamperti transform to multidimensional SDEs necessitates that the drift vector $\mathbf{f}(\mathbf{X}_t)$ satisfies integrability conditions akin to gradient fields. This ensures the existence of a scalar potential $\Phi(\mathbf{X}_t)$ such that $\mathbf{f} = \nabla \Phi$, permitting coordinate transformations to remove drift terms [7][12].

Mathematical Underpinnings and Itô Calculus

Itô's Lemma and Transformation Derivation

The Lamperti transform is a direct application of Itô's lemma. For $Z_t = \phi(X_t, t)$, the lemma gives:

$$dZ_t = \left(\frac{\partial \phi}{\partial t} + \mu \frac{\partial \phi}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 \phi}{\partial x^2} \right) dt + \sigma \frac{\partial \phi}{\partial x} dW_t.$$

Choosing ϕ to satisfy $\frac{\partial \phi}{\partial x} = \frac{1}{\sigma}$ eliminates state-dependent diffusion, yielding the transformed drift [3][13].

Fokker-Planck Equation

The transformation also simplifies the Fokker-Planck equation. For the original process X_t , the probability density $p(x, t)$ satisfies:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x} [\mu p] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [\sigma^2 p].$$

Under \$ Z_t \$, this reduces to:

$$\frac{\partial p_Z}{\partial t} = -\frac{\partial}{\partial z} \left[\left(\frac{\mu}{\sigma} - \frac{1}{2} \frac{\partial \sigma}{\partial x} \right) p_Z \right] + \frac{1}{2} \frac{\partial^2 p_Z}{\partial z^2},$$

highlighting the role of the adjusted drift ^[1][3].

Conclusion

The Lamperti transformation and its associated representations provide a unified framework for analyzing stochastic processes with state-dependent dynamics. By converting complex SDEs into tractable forms, linking branching processes to Lévy flights, and encoding self-similarity through time changes, these tools remain indispensable in stochastic calculus. Future research directions include extending the formalism to non-Markovian settings and refining inverse transformations for multidimensional systems.

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