# Eigenfunction Construction for Stationary Gaussian Processes

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## 1 Preliminaries

The Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are defined as:

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, d\, x \tag{1}$$

$$\mathcal{F}^{-1}[g](x) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} g(\omega)\, e^{i\omega x}\, d\,\omega \tag{2}$$

## 2 Theoretical Framework

Let $K\,(x-y)$ be a stationary positive-definite kernel. By Bochner's theorem:

$$\begin{aligned} K\,(x-y) &= \mathcal{F}^{-1}[S](\omega) \\ &= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{i\omega(x-y)}\, S(\omega)\, d\,\omega \end{aligned} \tag{3}$$

where $S(\omega)\, d\,\omega$ is the spectral measure.

## 3 The Null Space

**Definition 1**

***[Spectral Polynomials]*** *Let* $\{P_n(\omega)\}_{n=0}^{\infty}$ *be polynomials orthogonal with respect to* $S(\omega)$ *normalized so that* $P_0(\omega) = 1$*:*

$$\int_{-\infty}^{\infty} P_m(\omega)\, P_n(\omega)\, S(\omega)\, d\,\omega = \delta_{mn} \tag{4}$$

**Theorem 2**

***[Null Space Theorem]*** *Let the inverse Fourier transforms of* $P_n(\omega)$ *be defined by*

$$f_n(x) = \mathcal{F}^{-1}[P_n(\omega)](x) \tag{5}$$

*then*

$$\begin{aligned} \mathcal{N} &= \{f_n(x)\}_{n=1}^{\infty} \\ &= \{\mathcal{F}^{-1}[P_n(\omega)](x)\}_{n=1}^{\infty} \end{aligned} \tag{6}$$

*so that they constitute the null-space of the kernel inner-product*

$$\mathcal{N} = \{f_n : \langle K, f_n \rangle = 0\}_{n=1}^{\infty} \tag{7}$$

*which is evinced by an application of Parseval's theorem*

$$\begin{aligned} \langle K, \mathcal{F}^{-1}[P_n] \rangle &= \langle K, f_n \rangle \\ &= \int_0^{\infty} K(x) f_n(x) \mathrm{d}x \\ &= \langle \mathcal{F}^{-1}[S], \mathcal{F}^{-1}[P_n] \rangle \\ &= \langle S, P_n \rangle \\ &= 0 \end{aligned} \quad \forall n \geqslant 1 \tag{8}$$

**Remark 3.** The null-space of an operator is also called a kernel, but to avoid confusion with the kernel referring to the autocovariance kernel function of the Gaussian process integral covariance operator, the null-space terminology is preferred

# 4 Uniform Basis of the Kernel

Define the null space vectors:

$$\eta_n(x) = \mathcal{F}^{-1}[P_n(\omega)](x) \tag{9}$$

Apply Gram-Schmidt to $\{\eta_n\}$ to obtain orthonormal sequence $\{\psi_n\}$ via:

$$\psi_k(x) = \eta_k(x) - \sum_{j=1}^{k-1} \frac{\langle \eta_k, \psi_j \rangle}{\|\psi_j\|^2} \psi_j(x) \tag{10}$$

Then with its orthogonal complement $\mathcal{N}^{\perp}$:

$$L^2(\mathbb{R}) = \mathcal{N} \cup \mathcal{N}^{\perp} \tag{11}$$

$$\mathcal{N} \cap \mathcal{N}^{\perp} = \{0\} \tag{12}$$

The kernel expansion in $\mathcal{N}^{\perp}$ is:

$$K(x) = \sum_{n=0}^{\infty} \langle K, \psi_n \rangle \psi_n(x) \tag{13}$$

with uniform convergence.

# 5 Uniform Basis of the Spectral Factor

Let $\{Q_n(\omega)\}_{n=0}^{\infty}$ be orthogonal polynomials with respect to $\sqrt{S(\omega)}$:

$$\langle Q_m, Q_n \rangle_{\sqrt{S}} = \int_{-\infty}^{\infty} Q_m(\omega)\, Q_n(\omega) \sqrt{S(\omega)}\, d\omega = \delta_{mn} \tag{14}$$

Define:

$$\xi_n(x) = \mathcal{F}^{-1}[Q_n(\omega)](x) \tag{15}$$

Apply Gram-Schmidt to $\{\xi_n\}$ to obtain orthonormal sequence $\{\phi_n\}$ via:

$$\phi_k(x) = \xi_k(x) - \sum_{j=1}^{k-1} \frac{\langle \xi_k, \phi_j \rangle}{\| \phi_j \|^2} \phi_j(x) \tag{16}$$

Then:

$$\begin{aligned} g(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \sqrt{S(\omega)}\, d\omega \\ &= \sum_{n=0}^{\infty} \langle g, \phi_n \rangle\, \phi_n(x) \end{aligned} \tag{17}$$

where $g$ is the spectral factor with $\mathcal{F}[g] = \sqrt{S}$ satisfying

$$\begin{aligned} g(x) * g(y) &= K(x-y) \\ &= \int_{-\infty}^{\infty} g(x+z)\overline{g(y-z)} \mathrm{d}z \end{aligned} \tag{18}$$

# 6 Eigenfunction Construction

By Fubini's theorem and uniform convergence:

$$\begin{aligned} K\,(x-y) &= (g * g)\,(x-y) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle g, \phi_m \rangle \langle g, \phi_n \rangle\, (\phi_m * \phi_n)\,(x-y) \end{aligned} \tag{19}$$

The eigenfunctions $\{f_n\}$ can be expressed in the uniform basis $\{\psi_n\}$ with finitely many terms:

$$f_n(x) = \sum_{k=0}^{n} c_{nk}\, \psi_k(x) \tag{20}$$

where coefficients $c_{nk}$ are determined by the recurrence relations of the underlying orthogonal polynomials.

Substituting into Mercer's theorem:

$$\begin{aligned} K\,(x-y) \; &= \sum_{n=0}^{\infty} \lambda_n\, f_n(x)\, f_n(y) \\ &= \sum_{n=0}^{\infty} \lambda_n \sum_{k=0}^{n} \sum_{j=0}^{n} c_{nk}\, c_{nj}\, \psi_k(x)\, \psi_j(y) \end{aligned} \tag{21}$$

This double sum structure with coefficients is precisely the inner product representation guaranteed by Moore-Aronszajn's theorem for reproducing kernel Hilbert spaces.

The triangular structure emerges naturally from the polynomial degree preservation under convolution in the spectral domain.