Harmonizable Representation and Evolutionary Spectrum of Monotonically Modulated Stationary Gaussian Processes

Definition 1. *[Harmonizable Process] A stochastic process $\{X_t, t \in \mathbb{R}\}$ is harmonizable if it admits the representation:*

$$X_t = \int_{\mathbb{R}} e^{i\lambda t} \, d\, Z(\lambda) \tag{1}$$

where $d\, Z$ is a complex-valued random measure with bounded variation, not necessarily having orthogonal increments. The correlation structure is given by:

$$\mathbb{E}\, [d\, Z(\lambda)\, d\overline{Z(\mu)}] = F(d\,\lambda, d\,\mu) \tag{2}$$

where F is a measure on $\mathbb{R}^2$ of bounded variation.

Definition 2. *[Projection Operator for Time-Modulated Processes] Let $\{Y_{(t,\tau)}\}$ be a stochastic process defined on $\mathbb{R}^2$ and $\theta\colon \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function. The projection operator P_θ is defined as:*

$$(P_\theta Y)_t = Y_{(t,\theta(t))} \tag{3}$$

for all $t \in \mathbb{R}$. This operator projects from the space of processes on $\mathbb{R}^2$ to the space of processes on $\mathbb{R}$ by restricting to the curve $\{(t, \theta(t))\colon t \in \mathbb{R}\}$.

The projection operator P_θ satisfies:

1. *$P_\theta^2 = P_\theta$ (idempotent):*

$$\begin{aligned} (P_\theta^2 Y)_t &= (P_\theta\, (P_\theta Y))_t \\ &= P_\theta (Y_{(\cdot,\theta(\cdot))})_t \\ &= Y_{(t,\theta(t))} \\ &= (P_\theta Y)_t \end{aligned} \tag{4}$$

2. *$P_\theta^* = P_\theta$ (self-adjoint): If $\langle \cdot, \cdot \rangle$ denotes the inner product in the appropriate Hilbert space, then*

$$\langle P_\theta Y, Z \rangle = \langle Y, P_\theta Z \rangle \tag{5}$$

Definition 3. *[Evolutionary Spectrum] A non-stationary process $\{X_t, t\in\mathbb{R}\}$ has an evolutionary spectral representation if:*

$$X_t = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\, Z(\lambda) \tag{6}$$

where:

- $d\, Z(\lambda)$ *is an orthogonal increment process with* $\mathbb{E}\,|d\, Z(\lambda)|^2 = d\,\lambda$
- $A_t(\lambda)$ *is a time-varying amplitude function*
- *The evolutionary spectral density is* $h_t(\lambda) = |A_t(\lambda)|^2$

Definition 4. *[Monotonically Modulated Process] Let $X_0(t)$ be a stationary process with kernel $K_0\,(t-s)$. A monotonically modulated process is defined as:*

$$X_t = X_0(\theta(t)) \tag{7}$$

where $\theta\colon \mathbb{R}\to\mathbb{R}$ is a monotonically increasing function, yielding the kernel:

$$K(t,s) = K_0\,(\theta(t) - \theta(s)) \tag{8}$$

Theorem 5. *[Harmonizable Structure of Modulated Processes] The monotonically modulated process $X_t = X_0(\theta(t))$ is a harmonizable process with spectral representation:*

$$X_t = \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\, Z_0(\lambda) \tag{9}$$

where $d\, Z_0$ is the spectral measure of the original stationary process X_0.

Proof. Step 1: By Cramér's representation theorem, the stationary process $X_0(t)$ has representation:

$$X_0(t) = \int_{\mathbb{R}} e^{i\lambda t}\, d\, Z_0(\lambda) \tag{10}$$

where $d\, Z_0$ has orthogonal increments with $\mathbb{E}\,[d\, Z_0(\lambda)\, d\overline{Z_0(\mu)}] = \delta\,(\lambda-\mu)\, f_0(\lambda)\, d\,\lambda\, d\,\mu$.

Step 2: For any fixed time point $u\in\mathbb{R}$, we have:

$$X_0(u) = \int_{\mathbb{R}} e^{i\lambda u}\, d\, Z_0(\lambda) \tag{11}$$

Step 3: Setting $u = \theta(t)$ specifically, we get:

$$X_0(\theta(t)) = \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\, Z_0(\lambda) \tag{12}$$

Step 4: By definition of the modulated process $X_t = X_0(\theta(t))$, we have:

$$X_t = \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\, Z_0(\lambda) \tag{13}$$

Step 5: The covariance function is directly calculated:

$$\begin{aligned}
K(t,s) &= \mathbb{E}[X_t \overline{X_s}] \\
&= \mathbb{E}\left[\int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\, Z_0(\lambda)\, \overline{\int_{\mathbb{R}} e^{i\mu\theta(s)}\, d\, Z_0(\mu)}\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda\theta(t)}\, e^{-i\mu\theta(s)}\, d\, Z_0(\lambda)\, d\overline{Z_0(\mu)}\right] \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda\theta(t)}\, e^{-i\mu\theta(s)}\, \mathbb{E}\left[d\, Z_0(\lambda)\, d\overline{Z_0(\mu)}\right] \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda\theta(t)}\, e^{-i\mu\theta(s)}\, \delta\,(\lambda-\mu)\; f_0(\lambda)\, d\,\lambda\, d\,\mu \\
&= \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, e^{-i\lambda\theta(s)}\; f_0(\lambda)\, d\,\lambda \\
&= \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\; f_0(\lambda)\, d\,\lambda \\
&= K_0\,(\theta(t)-\theta(s))
\end{aligned} \tag{14}$$

Thus, X_t is harmonizable with the specified covariance structure. ☐

Theorem 6. *[Evolutionary Spectral Representation] The harmonizable process $X_t = X_0(\theta(t))$ has an exact evolutionary spectral representation:*

$$X_t = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\, Z_0(\lambda) \tag{15}$$

where $A_t(\lambda) = e^{i\lambda(\theta(t)-t)}$ is the time-varying amplitude function.

Proof. Step 1: Starting from the harmonizable representation:

$$X_t = \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\, Z_0(\lambda) \tag{16}$$

Step 2: We perform exact algebraic manipulation of the complex exponential term:

$$\begin{aligned} e^{i\lambda\theta(t)} &= e^{i\lambda\theta(t)} \cdot \frac{e^{i\lambda t}}{e^{i\lambda t}} \\ &= e^{i\lambda t} \cdot e^{i\lambda\theta(t) - i\lambda t} \\ &= e^{i\lambda t} \cdot e^{i\lambda(\theta(t) - t)} \end{aligned} \tag{17}$$

Step 3: Substituting this factorization back:

$$\begin{aligned} X_t &= \int_{\mathbb{R}} e^{i\lambda\theta(t)} \, d\, Z_0(\lambda) \\ &= \int_{\mathbb{R}} e^{i\lambda t} \cdot e^{i\lambda(\theta(t) - t)} \, d\, Z_0(\lambda) \end{aligned} \tag{18}$$

Step 4: Define the time-varying amplitude function:

$$A_t(\lambda) = e^{i\lambda(\theta(t) - t)} \tag{19}$$

Step 5: This gives us the evolutionary spectral representation:

$$X_t = \int_{\mathbb{R}} A_t(\lambda) \, e^{i\lambda t} \, d\, Z_0(\lambda) \tag{20}$$

Step 6: The evolutionary spectral density is:

$$\begin{aligned} h_t(\lambda) &= |A_t(\lambda)|^2 \cdot f_0(\lambda) \\ &= |e^{i\lambda(\theta(t) - t)}|^2 \cdot f_0(\lambda) \\ &= 1 \cdot f_0(\lambda) \\ &= f_0(\lambda) \end{aligned} \tag{21}$$

where we used the fact that $|e^{ix}|^2 = 1$ for any real x.

□

Theorem 7. *[Stationary Dilation via Naimark's Theorem] The harmonizable process $X_t = X_0(\theta(t))$ admits a stationary dilation $Y_{(t,\tau)}$ in an expanded space:*

$$Y_{(t,\tau)} = \int_{\mathbb{R}} e^{i\lambda\tau} \, d\, Z_0(\lambda) \tag{22}$$

The original harmonizable process is recovered via the projection operator P_θ:

$$X_t = (P_\theta Y)_t = Y_{(t,\theta(t))} \tag{23}$$

Proof. **Step 1:** We construct the stationary dilation:

$$Y_{(t,\tau)} = \int_{\mathbb{R}} e^{i\lambda\tau} \, d\, Z_0(\lambda) \tag{24}$$

Step 2: This process is stationary in the parameter τ as shown by its covariance:

$$\begin{aligned}
\tilde{K}((t,\tau),(s,\sigma)) &= \mathbb{E}[Y_{(t,\tau)}\overline{Y_{(s,\sigma)}}] \\
&= \mathbb{E}\left[\int_{\mathbb{R}} e^{i\lambda\tau} \, d\, Z_0(\lambda) \overline{\int_{\mathbb{R}} e^{i\mu\sigma} \, d\, Z_0(\mu)}\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda\tau} e^{-i\mu\sigma} \, d\, Z_0(\lambda) \, d\overline{Z_0(\mu)}\right] \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda\tau} e^{-i\mu\sigma} \, \mathbb{E}\left[d\, Z_0(\lambda) \, d\overline{Z_0(\mu)}\right] \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda\tau} e^{-i\mu\sigma} \, \delta\,(\lambda-\mu) \, f_0(\lambda) \, d\,\lambda \, d\,\mu \\
&= \int_{\mathbb{R}} e^{i\lambda\tau} e^{-i\lambda\sigma} \, f_0(\lambda) \, d\,\lambda \\
&= \int_{\mathbb{R}} e^{i\lambda(\tau-\sigma)} \, f_0(\lambda) \, d\,\lambda \\
&= K_0\,(\tau-\sigma)
\end{aligned} \tag{25}$$

The covariance depends only on $\tau - \sigma$, confirming stationarity.

Step 3: Apply the projection operator P_θ defined earlier:

$$\begin{aligned}
(P_\theta Y)_t &= Y_{(t,\theta(t))} \\
&= \int_{\mathbb{R}} e^{i\lambda\theta(t)} \, d\, Z_0(\lambda) \\
&= X_t
\end{aligned} \tag{26}$$

Step 4: Verify that P_θ is idempotent (already established in the definition):

$$\begin{aligned}
(P_\theta^2 Y)_t &= (P_\theta\,(P_\theta Y))_t \\
&= P_\theta(Y_{(\cdot,\theta(\cdot))})_t \\
&= Y_{(t,\theta(t))} \\
&= (P_\theta Y)_t
\end{aligned} \tag{27}$$

Step 5: This confirms that $Y_{(t,\tau)}$ is the stationary dilation of X_t, and the original process is precisely the projection of this stationary process via the projection operator P_θ. □

Corollary 8. *[Complete Characterization] For a monotonically modulated process* $X_t = X_0(\theta(t))$*:*

1. *It is harmonizable with representation*

$$X_t = \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\, Z_0(\lambda) \tag{28}$$

2. *It has evolutionary spectral representation*

$$X_t = \int_{\mathbb{R}} e^{i\lambda(\theta(t)-t)}\, e^{i\lambda t}\, d\, Z_0(\lambda) \tag{29}$$

3. *It is the projection of a stationary process*

$$Y_{(t,\tau)} = \int_{\mathbb{R}} e^{i\lambda\tau}\, d\, Z_0(\lambda) \tag{30}$$

via

$$X_t = (P_\theta Y)_t = Y_{(t,\theta(t))} \tag{31}$$

4. *Its kernel*

$$K(t,s) = K_0\,(\theta(t) - \theta(s)) \tag{32}$$

maintains positive definiteness from the original process