The Eigenfunctions of Stationary Integral Covariance Operators

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Abstract

The null spaces of the kernel inner product operators of Gaussian process are shown to be the Fourier transforms of the polynomials orthogonal with respect to the spectral densities of the processes and it is furthermore shown that the orthogonal complements of the null spaces are given by the Gram-Schmidt recursions which enumerate the products $\Psi_k(t-s) = \sqrt{\frac{c_p}{c_q}} \frac{\prod_{i=1}^{n_k} (t-s-\alpha_{k,i})}{\prod_{j=1}^{m_k} (t-s-\beta_{k,j})} = \psi_k(t)\,\psi_k(s)$ of the eigenfunctions $\psi_k(t)$ of the corresponding integral covariance operators.

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1 Introduction

Let $C(x)$ be the covariance function of a stationary Gaussian process on $[0, \infty)$. Define the integral covariance operator T by:

$$(Tf)(x) = \int_0^\infty C\,(x-y)\,f(y)\,\mathrm{d}y \tag{1}$$

Let $S(\omega)$ be the spectral density related to $C(x)$ by the Wiener-Khinchin theorem:

$$C(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{i\omega x}\, S(\omega)\, \mathrm{d}\omega \tag{2}$$

$$S(\omega) = \int_{0}^{\infty} C(x)\, e^{-ix\omega}\, \mathrm{d}x \tag{3}$$

1.1 Null Space of the Inner Product Operator

Consider polynomials $\{p_n(\omega)\}$ orthogonal with respect to $S(\omega)$:

$$\int_{-\infty}^{\infty} p_n(\omega)\, p_m(\omega)\, S(\omega)\, \mathrm{d}\omega = \delta_{nm} \tag{4}$$

Define $r_n(x)$ as the inverse Fourier transforms of $p_n(\omega)$:

$$r_n(x) = \int_{-\infty}^{\infty} p_n(\omega)\, e^{i\omega x}\, \mathrm{d}\omega \tag{5}$$

Lemma 1. *The functions $r_n(x)$ form the null space of the kernel inner product:*

$$\int_{0}^{\infty} C(x)\, r_n(x)\, \mathrm{d}x = 0 \tag{6}$$

Proof. Let $C(x)$ and $r_n(x)$ be defined as:

$$C(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{iax}\, S(a)\, \mathrm{d}a \tag{7}$$

$$r_n(x) = \int_{-\infty}^{\infty} p_n(b)\, e^{ibx}\, \mathrm{d}b \tag{8}$$

where $S(a)$ is the spectral density and $p_n(b)$ are orthogonal polynomials with respect to $S(a)$. Note that $C(x)$ and $r_n(x)$ are even functions, as they depend on the difference between two variables.

Substitute the definitions of $C(x)$ and $r_n(x)$, and apply Fubini's theorem:

$$\begin{aligned}\int_{0}^{\infty} C(x)\, r_n(x)\, \mathrm{d}x &= \int_{0}^{\infty} \frac{1}{\pi}\int_{-\infty}^{\infty} e^{iax}\, S(a)\, \mathrm{d}a \int_{-\infty}^{\infty} p_n(b)\, e^{ibx}\, \mathrm{d}b\, \mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_n(b)\, S(a) \int_{0}^{\infty} e^{i(a+b)x}\, \mathrm{d}x\, \mathrm{d}b\, \mathrm{d}a\end{aligned}$$

Since $C(x)$ and $r_n(x)$ are even functions, we can write:

$$\int_0^\infty C(x)\, r_n(x)\, \mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} C(x)\, r_n(x)\, \mathrm{d}x \tag{9}$$

Now we have:

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{\infty} C(x)\, r_n(x)\, \mathrm{d}x &= \frac{1}{2\,\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_n(b)\, S(a) \int_{-\infty}^{\infty} e^{i(a+b)x}\, \mathrm{d}x\, \mathrm{d}b\, \mathrm{d}a \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_n(b)\, S(a)\, \delta\,(a+b)\, \mathrm{d}b\, \mathrm{d}a \\ &= \frac{1}{2}\int_{-\infty}^{\infty} p_n\,(-a)\, S(a)\, \mathrm{d}a\end{aligned}$$

By the orthogonality of $p_n(a)$ with respect to $S(a)$, we conclude:

$$\frac{1}{2}\int_{-\infty}^{\infty} p_n\,(-a)\, S(a)\, \mathrm{d}a = 0 \tag{10}$$

Thus, $\int_0^\infty C(x)\, r_n(x)\, \mathrm{d}x = 0$, which completes the proof. □

2 Eigenfunctions Are The Facorizations of the Orthogonal Complement of the Null Space

By orthogonalizing the null space $\{r_n(x)\}$, we obtain the product of the eigenfunctions $\{\psi_n(x)\psi_n(y) = \Psi_n(x-y)\}$ of the covariance operator T. The orthogonalization process gives:

$$r_n^\perp(y) = \psi_n(x) = \sum_{k=0}^{n} a_{nk}\, r_k(x) = r_n(y) - \sum_{m=0}^{n-1} \frac{\langle r_n(y), r_m^\perp(y)\rangle}{\langle r_m^\perp(y), r_m^\perp(y)\rangle}\, r_m^\perp(y)$$

where the coefficients a_{nk} are given by:

$$a_{nk} = \begin{cases} 1 & \text{if } k = n \\ -\displaystyle\sum_{j=k}^{n-1} a_{nj}\, \langle r_n, \psi_j\rangle & \text{if } k\, n \\ 0 & \text{if } k > n \end{cases} \tag{11}$$

Theorem 2. *The eigenvalues of the integral covariance operator (1) are given by*

$$\lambda_n = \int_0^\infty C(z)\, \Psi_n(z)\, \mathrm{d}z \tag{12}$$

Proof. TOOD: Insert the way to show this, its standard □

Theorem 3. *Let* $\Psi_n(y)$ *be the normalized Fourier transforms of the spectral polynomials* $\Psi_n(y) = \frac{\hat{S}_n(y)}{|\hat{S}_n|}$ *where the sequence* $\hat{S}_n(y)$ *of inverse Fourier transforms of the spectral polynomials* $S_n(x)$ *is given by*

$$\hat{S}_n(y) = \int_{-1}^{1} S_n(x)\, e^{ixy}\, \mathrm{d}x \tag{13}$$

The product of the eigenfunctions of the integral covariance operator (1) are given by

$$\psi_n(x)\, \psi_n(y) = \Psi_n^{\perp}(x-y) \tag{14}$$

of the elements of orthogonal complement of the normalized Fourier transforms $\Psi_n(y)$ *of the spectral polynomials (via the Gram-Schmidt process)*

$$\begin{aligned} \psi_n(x)\, \psi_n(y) &= \Psi_n^{\perp}(x-y) \\ &= \Psi_n(x-y) - \sum_{m=0}^{n-1} \frac{\langle \Psi_m(x-y), \Psi_m^{\perp}(x-y) \rangle}{\langle \Psi_m^{\perp}(x-y), \Psi_m^{\perp}(x-y) \rangle} \Psi_m^{\perp}(x-y) \end{aligned} \tag{15}$$

Proof. ...TODO □

Example 4. Let the kernel function be given by $K(t,s) = J_0(t-s)$ then identify the orthogonal polynomial sequence associated with the spectral density

$$S(\omega) = \int_0^{\infty} J_0(x)\, e^{ix\omega}\, \mathrm{d}x = \begin{cases} \frac{1}{\sqrt{1-\omega^2}} & \omega \in (-1,1) \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

as being the orthogonality measure of the Type-I Chebyshev polynomials $T_n(x)$ so that the orthogonal polynomial sequence is identified as

$$S_n(x) = T_n(x) \tag{17}$$

so that

$$\int_{-1}^{1} S_n(\omega)\, S_m(\omega)\, S(\omega)\, \mathrm{d}\omega = \begin{cases} 0 & n \neq m \\ 2\,\pi & n = m = 0 \\ \pi & n = m \neq 0 \end{cases} \tag{18}$$

The finite Fourier transforms of the Chebyshev polynomials[1] are just the usual infinite Fourier transforms with the integration restricted to the range $-1\ldots 1$ since $T_n(x) = 0 \forall x \notin [-1,1]$). Equivalently, the spectral density function can be extended to take the value 0 outside the interval $[-1,1]$.

$$\begin{aligned} \hat{T}_n(y) &= \int_{-\infty}^{\infty} e^{-i x y}\, T_n(x)\, \mathrm{d}y = \int_{-1}^{1} e^{-i x y}\, T_n(x)\, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} e_2^{-i x y}\, F_1\left(\begin{array}{cc} n, & -n \\ & \frac{1}{2} \end{array} | \frac{1}{2} - \frac{x}{2} \right) \mathrm{d}x \\ &= \frac{i}{y}\, (e^{-i y}\, F_n^{+}(y) - e^{i(\pi n + y)}\, F_n^{-}(y)) \end{aligned} \tag{19}$$

where

$$F_n^{\pm}(y) = {}_3F_1\left(\begin{array}{ccc} 1, & n, & -n \\ & & \frac{1}{2} \end{array} | \frac{\pm i\, y}{2} \right) \tag{20}$$

the spectral polynomials S_n are given by the Type-I Chebyshev polynomials

$$S_n(x) = T_n(x) \tag{21}$$

and their normalization is

$$\begin{aligned} Y_n(y) &= \frac{\hat{T}_n(y)}{|\hat{T}_n|} \\ &= \frac{i}{y} \left(\frac{e^{-i y}\, F_n^{+}(y) - e^{i(\pi n + y)}\, F_n^{-}(y)}{\sqrt{\frac{4\,(-1)^n\, \pi - (2\, n^2 - 1)}{4\, n^2 - 1}}} \right) \end{aligned} \tag{22}$$

where the L^2 norm of $\hat{T}_n(y)$ is given by

$$\begin{aligned} |\hat{T}_n| &= \sqrt{\int_{-\infty}^{\infty} \hat{T}_n(y)^2\, \mathrm{d}y} \\ &= \sqrt{\frac{4\,(-1)^n\, \pi - (2\, n^2 - 1)}{4\, n^2 - 1}} \end{aligned} \tag{23}$$

Definition 5. *Let $j_n(x)$ is the spherical Bessel function of the first kind,*

$$\begin{aligned} j_n(z) &= \sqrt{\frac{\pi}{2\,z}}\,J_{n+\frac{1}{2}}(x) \\ &= \frac{1}{\sqrt{z}}\left(\sin(z)\,R_{n,\frac{1}{2}}(z) - \cos(z)\,R_{n-1,\frac{3}{2}}(z)\right) \end{aligned} \tag{24}$$

where $R_{n,v}(z)$ are the (misnamed) Lommel polynomials

$$R_{n,v}(z) = \frac{\Gamma(n+v)}{\Gamma(v)}\left(\frac{2}{z}\right)^n {}_2F_3\left(\left[-\frac{n}{2}, \frac{1}{2} - \frac{n}{2}\right]; [v, -n, 1-v-n]; -z^2\right) \tag{25}$$

where ${}_2F_3$ is a generalized hypergeometric function. The "Lommel polynomials" are actually rational functions of z, not polynomial; but rather "polynomial in $\frac{1}{z}$".

Theorem 6. *The eigenfunctions of the stationary integral covariance operator*

$$[T\psi_n](x) = \int_0^\infty J_0(x-y)\,\psi_n(x)\,\mathrm{d}x = \lambda_n\,\psi_n(x) \tag{26}$$

are given by

$$Y_n^\perp(x-y) = \psi_n(x)\,\psi_n(y) = (-1)^n\sqrt{\frac{8\,n+2}{\pi}}\,j_{2n}(x-y) \tag{27}$$

and the eigenvalues are given by

$$\begin{aligned} \lambda_n &= \int_0^\infty J_0(x)\,\psi_n(x)\,\mathrm{d}x \\ &= \sqrt{\frac{2\,n+\frac{1}{2}}{\pi}}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} \\ &= \sqrt{\frac{2\,n+\frac{1}{2}}{\pi}}\,(n+1)^2_{-\frac{1}{2}} \end{aligned} \tag{28}$$

where $(n+1)_{-\frac{1}{2}}$ is the Pochhammer symbol (ascending/rising factorial).

3 Appendix

3.1 Proofs & Lemmas

3.1.1 Unique Uniform Convergence of Eigenfunctions to Kernel

Theorem 7. *Let $K(\alpha, \gamma)$ be a symmetric kernel defined on $[0, \infty) \times [0, \infty)$, and let $\{\phi_n(\alpha)\}_{n=0}^{\infty}$ be the set of orthonormal eigenfunctions of the integral equation*

$$\phi(\alpha) = \lambda \int_0^{\infty} K(\alpha, \xi)\, \phi(\xi)\, \mathrm{d}\xi \tag{29}$$

with corresponding eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$. The eigenfunctions satisfy the orthonormality condition:

$$\int_0^{\infty} \phi_n(\alpha)\, \phi_m(\alpha)\, \mathrm{d}\alpha = \delta_{nm} \tag{30}$$

where δ_{nm} is the Kronecker delta. Then, if the series

$$\sum_{n=0}^{\infty} \frac{\phi_n(\alpha)\, \phi_n(\gamma)}{\lambda_n} \tag{31}$$

is uniformly convergent for $0 \leq \alpha, \gamma < \infty$, we have

$$K(\alpha, \gamma) = \sum_{n=0}^{\infty} \frac{\phi_n(\alpha)\, \phi_n(\gamma)}{\lambda_n} \tag{32}$$

Proof. Let

$$S(\alpha, \gamma) = \sum_{n=0}^{\infty} \frac{\phi_n(\alpha)\, \phi_n(\gamma)}{\lambda_n} \tag{33}$$

Consider the action of $S(\alpha, \gamma)$ on an eigenfunction $\phi_m(\gamma)$:

$$\begin{aligned}
\int_0^{\infty} S(\alpha, \gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma &= \int_0^{\infty} \sum_{n=0}^{\infty} \frac{\phi_n(\alpha)\, \phi_n(\gamma)}{\lambda_n}\, \phi_m(\gamma)\, \mathrm{d}\gamma && (34)\\
&= \sum_{n=0}^{\infty} \frac{\phi_n(\alpha)}{\lambda_n} \int_0^{\infty} \phi_n(\gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma && (35)\\
&= \sum_{n=0}^{\infty} \frac{\phi_n(\alpha)}{\lambda_n}\, \delta_{nm} && (36)\\
&= \frac{\phi_m(\alpha)}{\lambda_m} && (37)
\end{aligned}$$

The interchange of summation and integration is justified by the uniform convergence of the series. For the eigenfunction $\phi_m(\alpha)$:

$$\phi_m(\alpha) = \lambda_m \int_0^\infty K(\alpha, \gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma \tag{38}$$

Comparing this with our result for $S(\alpha, \gamma)$, we see that

$$\int_0^\infty S(\alpha, \gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma = \int_0^\infty K(\alpha, \gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma \tag{39}$$

for all eigenfunctions $\phi_m(\alpha)$. For any square-integrable function $f(\alpha)$:

$$f(\alpha) = \sum_{m=0}^\infty c_m\, \phi_m(\alpha) \tag{40}$$

where $c_m = \int_0^\infty f(\gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma$. Then:

$$\begin{aligned}
\int_0^\infty S(\alpha, \gamma)\, f(\gamma)\, \mathrm{d}\gamma &= \int_0^\infty S(\alpha, \gamma) \sum_{m=0}^\infty c_m\, \phi_m(\gamma)\, \mathrm{d}\gamma && (41)\\
&= \sum_{m=0}^\infty c_m \int_0^\infty S(\alpha, \gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma && (42)\\
&= \sum_{m=0}^\infty c_m \int_0^\infty K(\alpha, \gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma && (43)\\
&= \int_0^\infty K(\alpha, \gamma) \sum_{m=0}^\infty c_m\, \phi_m(\gamma)\, \mathrm{d}\gamma && (44)\\
&= \int_0^\infty K(\alpha, \gamma)\, f(\gamma)\, \mathrm{d}\gamma && (45)
\end{aligned}$$

Since this equality holds for all square-integrable functions $f(\alpha)$, we conclude that

$$S(\alpha, \gamma) = K(\alpha, \gamma) \tag{46}$$

To prove uniqueness, suppose there exists another expansion

$$K(\alpha, \gamma) = \sum_{n=0}^\infty \frac{\phi_n(\alpha)\, \phi_n(\gamma)}{\lambda_n} + H(\alpha, \gamma) \tag{47}$$

where $H(\alpha, \gamma)$ is a non-zero symmetric function. Then for any eigenfunction $\phi_m(\alpha)$:

$$\int_0^\infty H(\alpha, \gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma = 0 \tag{48}$$

This implies $H(\alpha, \gamma)$ must be identically zero, contradicting our assumption. Therefore, the expansion is unique. [2, ..?]

For a specific eigenfunction $\phi_k(\alpha)$:

$$\phi_k(\alpha) = \lambda_k \frac{\phi_k(\alpha)}{\lambda_k} \int_0^\infty [\phi_k(\xi)]^2 \, \mathrm{d}\xi \tag{49}$$

This leads to the normalization condition: $\int_0^\infty [\phi_k(\xi)]^2 \, \mathrm{d}\xi = 1$. □

3.1.2 Factorization of Rational $f(t-s) = g(t)g(s)$

Lemma 8. *For any rational function*

$$f(t-s) = \frac{P(t-s)}{Q(t-s)} \tag{50}$$

, where P and Q are polynomials, there exist rational functions $g(t)$ and $g(s)$ such that

$$f(t-s) = g(t)\, g(s) \tag{51}$$

Proof. Let

$$P(t-s) = c_p \prod_{i=1}^{n} (t - s - \alpha_i) \tag{52}$$

and

$$Q(t-s) = c_q \prod_{j=1}^{m} (t - s - \beta_j) \tag{53}$$

then define

$$g(t) = \sqrt{\frac{c_p}{c_q}} \frac{\prod_{i=1}^{n} (t - \alpha_i)}{\prod_{j=1}^{m} (t - \beta_j)} \tag{54}$$

such that

$$\begin{aligned} g(t)\, g(s) &= \frac{c_p}{c_q} \frac{\prod_{i=1}^{n} (t - \alpha_i)}{\prod_{j=1}^{m} (t - \beta_j)} \frac{\prod_{i=1}^{n} (s - \alpha_i)}{\prod_{j=1}^{m} (s - \beta_j)} \\ &= \frac{c_p}{c_q} \frac{\prod_{i=1}^{n} (t - \alpha_i)(s - \alpha_i)}{\prod_{j=1}^{m} (t - \beta_j)(s - \beta_j)} \\ &= \frac{c_p \prod_{i=1}^{n} ((t-s) - \alpha_i)}{c_q \prod_{j=1}^{m} ((t-s) - \beta_j)} \\ &= f(t-s) \end{aligned} \tag{55}$$

For complex roots, we pair each α_i or β_j with its complex conjugate in the factorization of $g(t)$. This ensures that the product $(t - \alpha_i)(t - \bar{\alpha_i})$ results in a quadratic polynomial with real coefficients, making $g(t)$ a real-valued function. □

Bibliography

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[2] George Neville Watson Edmund Taylor Whittaker. *A course of modern analysis: an introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions*. University Press, 1920.