Journal of Computational and Applied Mathematics 49 (1993) 51–57
North-Holland

CAM 1415

Real orthogonalizing weights for Bessel polynomials

W.D. Evans
School of Mathematics, University of Wales, Cardiff, United Kingdom

W.N. Everitt *
Department of Mathematics, The University of Birmingham, United Kingdom

K.H. Kwon *,**
Department of Mathematics, Korea Advanced Institute of Science and Technology, Taejon, South Korea

L.L. Littlejohn *
Department of Mathematics, Utah State University, Logan, UT, United States

Received 12 February 1992
Revised 2 April 1992

Abstract

Evans, W.D., W.N. Everitt, K.H. Kwon and L.L. Littlejohn, Real orthogonalizing weights for Bessel polynomials, Journal of Computational and Applied Mathematics 49 (1993) 51–57.

We construct real orthogonalizing weights of bounded variation for the generalized Bessel polynomials.

Keywords: Generalized Bessel polynomials; real orthogonalizing weights

1. Introduction

It is well known that there are essentially four distinct orthogonal polynomial sets (OPSs in short), i.e., three classical OPSs (Jacobi, Laguerre, Hermite polynomials) and Bessel polynomials, which arise as polynomial solutions of

$$\sigma(x)y_n''(x)+\tau(x)y_n'(x)-\lambda_n y_n(x)=0,\quad n=0,1,2,\ldots, \tag{1.1}$$

Correspondence to: Prof. L.L. Littlejohn, Department of Mathematics & Statistics, Utah State University, Logan, UT 84322-3900, United States. e-mail: lance@sunfs.math.usu.edu.
* These authors express their thanks to the first author for his hospitality during their visit to the University of Wales.
** This author is partially supported by KOSEF and GARC.

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SSDI 0377-0427(93)E0042-K

where $\sigma(x)=ax^2+bx+c$ and $\tau(x)=dx+e$ are real polynomials and $\lambda_n=n(an+d-a)$. Here, we are mainly concerned with the orthogonality of the Bessel polynomials which were first studied systematically in [12]. We refer to [5] for other details on Bessel polynomials.

The (generalized) Bessel polynomials are polynomial solutions of the equation

$$x^2y_n''(x)+(dx+e)y_n'(x)-n(n+d-1)y_n(x)=0, \quad n=0, 1, 2, \ldots, \tag{1.2}$$

where $d \neq 0, -1, -2, \ldots$ and $e \neq 0$. After replacing x by $\frac{1}{2}ex$ and d by $2+\alpha$, the equation (1.2) becomes

$$x^2y_n''(x)+[(2+\alpha)x+2]y_n'(x)-n(n+\alpha+1)y_n(x)=0, \quad n=0, 1, 2, \ldots, \tag{1.3}$$

where $\alpha \neq -2, -3, \ldots$, of which the monic polynomial solutions [5,12] are

$$B_n^\alpha(x)=2^n\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(n+\alpha+k+1)}{\Gamma(2n+\alpha+1)}\left(\tfrac{1}{2}x\right)^k, \quad n=0, 1, 2, \ldots . \tag{1.4}$$

They satisfy a three-term recurrence relation:

$$B_{n+1}^\alpha(x)=(x-\beta_n)B_n^\alpha(x)-\gamma_nB_{n-1}^\alpha(x), \quad (B_{-1}^\alpha(x)\equiv 0) \tag{1.5}$$

$$\begin{aligned}\beta_n&=\frac{-2\alpha}{(2n+\alpha)(2n+\alpha+2)},\\ \gamma_n&=\frac{-4n(n+\alpha)}{(2n+\alpha+1)(2n+\alpha)^2(2n+\alpha-1)}, \quad n=0, 1, 2, \ldots .\end{aligned} \tag{1.6}$$

Since $\gamma_n \neq 0$ for $n \geqslant 1$, the Bessel polynomials form an OPS by Favard's theorem [4]. Krall and Frink [12] established the complex orthogonality of Bessel polynomials and raised the problem of finding a real Stieltjes measure $d\mu(x)$ with respect to which the Bessel polynomials are orthogonal. Since any OPS is orthogonal with respect to any linear functional on polynomials which generates the moments of the OPS [9], it is a problem of finding a real-valued function $\mu(x)$ of bounded variation such that

$$\int_{-\infty}^{\infty} x^n \,d\mu(x)=\mu_n^\alpha, \quad n=0, 1, 2, \ldots, \tag{1.7}$$

where $\{\mu_n^\alpha\}_0^\infty$ are the moments of the Bessel polynomials $\{B_n^\alpha(x)\}_0^\infty$. Due to the classical theorem of Boas [1] on the Stieltjes moment problem, there must be such a function $\mu(x)$ which we may even take to be such that $\mu'(x)$ is in S^+, the Schwartz space on $(0, \infty)$ by the beautiful theorem of Duran [2].

After a long search for $\mu(x)$ [7,10,11,14,15], several such functions were found independently by Duran [3] and Kwon et al. [13] using different methods. Duran [3] approached the problem from a viewpoint of the pure moment problem and obtained $\mu(x)$ in the series form, while in [13] it was obtained in the closed form by solving a certain first-order nonhomogeneous differential equation associated to (1.3):

$$x^2w'(x)-(ax+2)w(x)=g(x), \tag{1.8}$$

where $g(x)$ is any "polynomial killer".

In Section 2, we explain the complex orthogonality of Bessel polynomials and the reason why some of the previous methods in [9,14,15] fail to give a real orthogonalizing weight.

In Section 3, we first characterize the polynomial killers $g(x)$ that work and construct two examples of real orthogonalizing weights for the Bessel polynomials extending the work in [13].

2. Complex orthogonality

The differential equation (1.1) becomes symmetric when it is multiplied by any nontrivial classical solution of the so-called symmetry equation

$$[\sigma(x)w(x)]' - \tau(x)w(x) = 0. \tag{2.1}$$

Moreover, the orthogonalizing weights for all three classical OPSs satisfy (2.1) classically or distributionally [14]. The symmetry equation for Bessel polynomials is

$$x^2 w'(x) - (ax + 2)w(x) = 0, \tag{2.2}$$

for which the only one nontrivial distributional (and even classical) solution is

$$w_+(x) = \begin{cases} x^{\alpha} \exp\left(-\dfrac{2}{x}\right), & x > 0 \\ 0, & x \leqslant 0, \end{cases} \tag{2.3}$$

which cannot be a weight since $\lim_{x\to\infty} x^n w_+(x) = \infty$ for $n + \alpha > 0$. Solving (2.1) in distribution space gives weights for the classical OPSs, but not for Bessel polynomials, since (2.1) has at most regular singular points for the classical OPSs, but has an irregular singular point at $x = 0$ for the Bessel polynomials.

By Komatsu's index theorem (cf. [8, Theorem 3.3]), equation (2.2) has exactly three linearly independent hyperfunction solutions: $w_+(x)$, $w_-(x) = [-(x + \mathrm{i}0)]^{\alpha} \exp(-2/(x + \mathrm{i}0))$ and $w_0(x)$ which must have its support at $\{0\}$. The hyperfunction $w_-(x)$ cannot be a weight either since $w_-(x)|_{(-\infty,0)} = (-x)^{\alpha} \exp(-2/x)$ and so $\lim_{x\to-\infty} x^{2n} w_-(x) = \infty$ for $2n + \alpha > 0$. As a hyperfunction with support at $\{0\}$, $w_0(x)$ must be of the form [6]

$$w_0(x) = \sum_0^{\infty} c_n \delta^{(n)}(x). \tag{2.4}$$

Substituting expression (2.4) into (2.2), we obtain

$$(n + 1)(n + \alpha + 2)c_{n+1} - 2c_n = 0, \quad n = 0, 1, 2, \dots . \tag{2.5}$$

Taking $c_0 = -2$, we obtain

$$w_0(x) = -\sum_0^{\infty} \frac{2^{n+1}\Gamma(\alpha + 2)}{n!\, \Gamma(n + \alpha + 2)} \delta^{(n)}(x). \tag{2.6}$$

Note that when $\alpha = 0$, it is exactly the formal δ-series used in [15]. Since $w_0(x)$ has its support at $\{0\}$, it can act on polynomials and we have

$$\mu_n^{\alpha} \equiv \langle w_0(x), x^n \rangle = \frac{(-2)^{n+1}\Gamma(\alpha + 2)}{\Gamma(n + \alpha + 2)}, \quad n = 0, 1, 2, \dots, \tag{2.7}$$

which are exactly the moments of Bessel polynomials in [12, formula (42)]. The Cauchy representation (also called the Stieltjes transform) of $w_0(x)$ is

$$W_0(z) = \tfrac{1}{2}\pi i\left\langle w_0(x), \frac{1}{x-z}\right\rangle = \frac{-1}{2\pi i}\sum_0^\infty \frac{\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)}\left(\frac{-2}{z}\right)^{n+1}, \tag{2.8}$$

which is analytic for all $z \neq 0$ and coincides essentially with the function $\rho(x)$ in [12, formula (36)]. In terms of hyperfunction theory, the function $W_0(z)$ is called the *standard defining function* of $w_0(x)$.

From (2.7) we have the following theorem.

Theorem 1. *The Bessel polynomials $\{B_n^\alpha(x)\}$ form an OPS with respect to the hyperfunctional weight $w_0(x)$, that is, for any numbers a and b with $a < 0 < b$,*

$$\begin{aligned}\int_a^b B_m^\alpha(x)B_n^\alpha(x)w_0(x)\,\mathrm{d}x &= \int_C B_m^\alpha(z)B_n^\alpha(z)W_0(z)\,\mathrm{d}z \\ &= K_n\delta_{mn}, \quad m \text{ and } n = 0, 1, 2, \ldots,\end{aligned} \tag{2.9}$$

where $K_n \neq 0$, δ_{mn} is the Kronecker delta and C is any closed path through a and b which encircles the point $z = 0$ once in the clockwise direction.

Formula (2.9) is exactly that for the complex orthogonality of Bessel polynomials in [12]. We refer to [5] for the value of K_n.

3. Real orthogonality

In Section 2, we have seen that solving the homogeneous symmetry equation (2.2) even in the hyperfunction space cannot yield a real weight for Bessel polynomials. In order to find a real weight, we first need the following fact.

Lemma 2 (Kwon, Kim and Han [13], Littlejohn [14]). *Assume that* (1.1) *has an OPS $\{P_n(x)\}_0^\infty$ as solutions. Then $\{P_n(x)\}_0^\infty$ is an OPS with respect to a distribution $w(x)$ if and only if*

(i) *$w(x)$ decays very rapidly at infinity so that it can act on polynomials*;

(ii) $\langle w(x), 1\rangle \neq 0$;

(iii) *$w(x)$ satisfies a nonhomogeneous differential equation*

$$[\sigma(x)w(x)]' - \tau(x)w(x) = g(x), \tag{3.1}$$

where $g(x)$ is a distribution having 0-moments, i.e.,

$$\langle g(x), x^n\rangle = 0, \quad n = 0, 1, 2, \ldots .$$

Moreover, there are infinitely many such $w(x)$ even in S^+. It is enough to take $w(x)$ in S^+ which generates the moments of the OPS $\{P_n(x)\}_0^\infty$ (cf. [2, Theorem 3.1]). We call the equation (3.1) the weight equation; in case of Bessel polynomials it is

$$x^2w'(x) - (ax+2)w(x) = g(x). \tag{3.2}$$

From now on, we assume that $g(x)$ is a nontrivial real-valued continuous function on $(-\infty, \infty)$ with $g(x) \equiv 0$ on $(-\infty, 0]$, and that

$$\int_0^\infty x^n g(x)\, dx = 0, \quad n = 0, 1, 2, \dots . \tag{3.3}$$

We further assume that

$$\lim_{x \to \infty} x^n g(x) = 0, \quad n = 0, 1, 2, \dots . \tag{3.4}$$

Define a function $w_\alpha(x)$ by

$$w_\alpha(x) = \begin{cases} 0, & x \leqslant 0, \\ -x^\alpha \exp\left(-\dfrac{2}{x}\right) \displaystyle\int_x^\infty t^{-\alpha-2} g(t) \exp\left(\dfrac{2}{t}\right) \mathrm{d}t, & x > 0. \end{cases} \tag{3.5}$$

Then $w_\alpha(x)$ is C^1 for $x \neq 0$ and satisfies (3.2) for $x \neq 0$. Furthermore, we have the following lemma.

Lemma 3. *Let $g(x)$ and $w_\alpha(x)$ be the same as above. Then, with $\mathbb{R}^+ = [0, \infty)$, we have*

(i) $x^n g(x) \in L^1(\mathbb{R}) \cap L^1(\mathbb{R}^+)$, $n = 0, 1, 2, \dots$;

(ii) $\lim_{x \to 0+} w_\alpha(x) = 0$, *so that $w_\alpha(x)$ is continuous on $\mathbb{R}$*;

(iii) $x^n w_\alpha(x) \in L^1(\mathbb{R}) \cap L^1(\mathbb{R}^+)$, $n = 0, 1, 2, \dots$;

(iv) *if we set $\mu_n^\alpha = \int_0^\infty x^n w_\alpha(x)\, \mathrm{d}x$, $n = 0, 1, 2, \dots$, then they satisfy the Bessel moments recurrence relation*

$$(n + \alpha + 2)\mu_{n+1}^\alpha + 2\mu_n^\alpha = 0, \quad n = 0, 1, 2, \dots; \tag{3.6}$$

(v) *$w_\alpha(x)$ satisfies (3.2) distributionally on $\mathbb{R}$.*

Proof. We shall prove only (iv) and (v) assuming (i), (ii), and (iii). (iv):

$$\begin{aligned} \mu_{n+1}^\alpha &= \int_0^\infty x^{n+1} w_\alpha(x)\, \mathrm{d}x = \lim_{\epsilon \to 0+} \int_\epsilon^\infty w_\alpha(x)\, \mathrm{d}\left(\frac{x^{n+2}}{n+2}\right) \\ &= \frac{-1}{n+2} \lim_{\epsilon \to 0+} \int_\epsilon^\infty x^n [(\alpha x + 2) w_\alpha(x) + g(x)]\, \mathrm{d}x \\ &= \frac{-1}{n+2} \int_0^\infty (\alpha x^{n+1} + 2x^n) w_\alpha(x)\, \mathrm{d}x = \frac{-1}{n+2}(\alpha \mu_{n+1}^\alpha + 2\mu_n^\alpha). \end{aligned}$$

(v): for any test function $\phi(x)$, by integration by parts as in (iv), we have

$$\int_0^\infty w_\alpha(x) x^2 \phi'(x)\, \mathrm{d}x = -\int_0^\infty \phi(x)[(\alpha x + 2) w_\alpha(x) + g(x) + 2x w_\alpha(x)]\, \mathrm{d}x.$$

Hence,

$$\begin{aligned} &\langle x^2 w_\alpha'(x) - (\alpha x + 2) w_\alpha(x), \phi(x) \rangle \\ &\quad = -\int_0^\infty w_\alpha(x)[x^2 \phi'(x) + ((2 + \alpha)x + 2)\phi(x)]\, \mathrm{d}x \\ &\quad = \int_0^\infty g(x)\phi(x)\, \mathrm{d}x = \langle g(x), \phi(x) \rangle. \end{aligned}$$

Finally, define a function $\mu_\alpha(x)$ by

$$\mu_\alpha(x) = \int_0^x w_\alpha(t)\,\mathrm{d}t. \tag{3.7}$$

Then $\mu_\alpha(x)$ is a C^1-function of bounded variation with support in $\mathbb{R}^+$ since $w_\alpha(x)$ is in $L^1(\mathbb{R}^+)$. □

In summary, we have the following theorem.

Theorem 4. *The set of Bessel polynomials* $\{B_n^\alpha(x)\}_0^\infty$ *is an OPS with respect to the Stieltjes measure* $w_\alpha(x)\,\mathrm{d}x = \mathrm{d}\mu_\alpha(x)$ *on* $[0, \infty)$, *provided that*

$$\int_0^\infty w_\alpha(x)\,\mathrm{d}x \neq 0. \tag{3.8}$$

Proof. It suffices to see that the recurrence relation (3.6) yields the Bessel moments in (2.7) modulo a nonzero constant multiple under the condition (3.8). □

As for the differentiability of $w_\alpha(x)$, we have the following lemma.

Lemma 5. *Assume further that* $g(x)$ *is in* $C^\infty(\mathbb{R})$ (*respectively* S^+). *Then the corresponding* $w_\alpha(x)$ *is in* $C^\infty(\mathbb{R})$ (*respectively* S^+), *provided that*

$$\lim_{x\to 0+} \frac{g^{(m)}(x)}{x^n} = 0, \quad m \text{ and } n = 0, 1, 2, \dots . \tag{3.9}$$

Furthermore, the function $w_\alpha(x)$ *satisfies* (3.1) *classically.*

There are at least two known polynomial killers both of which are due to Stieltjes:

$$g_1(x) = \begin{cases} 0, & \text{for } x \leqslant 0, \\ \exp(-x^{1/4}) \sin x^{1/4}, & \text{for } x > 0, \end{cases} \tag{3.10}$$

$$g_2(x) = \begin{cases} 0, & \text{for } x \leqslant 0, \\ \exp(-\ln^2 x) \sin(2\pi \ln x), & \text{for } x > 0. \end{cases} \tag{3.11}$$

Thus $g_1(x)$ is continuous on $\mathbb{R}$ but not differentiable at $x = 0$ and $g_2(x)$ is in S^+. Moreover, both $g_1(x)$ and $g_2(x)$ satisfy condition (3.4). Let $w_{\alpha_j}(x)$ be the functions $w_\alpha(x)$ in (3.5) corresponding to $g(x) = g_j(x)$, $j = 1, 2$. Then $w_{\alpha_2}(x)$ is in S^+ since $g_2(x)$ satisfies condition (3.9). They provide two real orthogonalizing weights for Bessel polynomials at least for $\alpha = -1$, 0, 1, for which condition (3.8) can be checked numerically (cf. [13]).

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