Theorem 1

(Integral Operator Coverring Number Upper Bounds) *Let T_K be the compact self-adjoint integral operator on $L^2[0,\infty)$ defined by kernel K:*

$$(T_K f)(z) = \int_0^\infty K(z,w)\, f(w)\, dw \tag{1}$$

where

$$K(z,w) = \sum_{n=1}^{\infty} \lambda_n\, \phi_n(z)\, \phi_n(w) \tag{2}$$

with $\{\phi_n\}_{n=1}^{\infty}$ an orthonormal sequence in $L^2[0,\infty)$ and $\{\lambda_n\}_{n=1}^{\infty}$ the corresponding eigenvalues ordered such that

$$|\lambda_n| \geq |\lambda_{n+1}| \forall n \tag{3}$$

Let T_{K_N} be the truncated operator with kernel

$$K_N(z,w) = \sum_{n=1}^{N} \lambda_n\, \phi_n(z)\, \phi_n(w) \tag{4}$$

then:

$$\|T_K - T_{K_N}\| \leq |\lambda_{N+1}| \tag{5}$$

Proof. Let $E_N = T_K - T_{K_N}$ be the difference of integral operators. For any unit vector $f \in L^2[0,\infty)$:

$$\begin{aligned} \|E_N f\|^2 &= \langle E_N f, E_N f\rangle \\ &= \langle E_N^* E_N f, f\rangle \end{aligned} \tag{6}$$

Since E_N is self-adjoint (as difference of self-adjoint operators T_K and T_{K_N}), we have:

$$\|E_N\| = \sup_{\|f\|=1} |\langle E_N f, f\rangle| \tag{7}$$

Let $f = g + h$ where g is in $\text{span}\{\phi_k\}_{k \leq N}$ and h is in $\text{span}\{\phi_k\}_{k>N}$. Then:

$$\langle E_N f, f\rangle = \langle E_N g, g\rangle + 2\,\Re\langle E_N g, h\rangle + \langle E_N h, h\rangle \tag{8}$$

By construction of E_N, for any g in $\text{span}\{\phi_k\}_{k\leq N}$:

$$E_N g = 0 \Longrightarrow \langle E_N g, g\rangle = 0 \text{ and } \langle E_N g, h\rangle = 0 \tag{9}$$

For h in $\text{span}\{\phi_k\}_{k>N}$, $E_N h = \lambda_k h$ where $|\lambda_k| \leq |\lambda_{N+1}|$, thus:

$$|\langle E_N h, h\rangle| \leq |\lambda_{N+1}| \|h\|^2 \leq |\lambda_{N+1}| \|f\|^2 \tag{10}$$

Therefore:

$$\|E_N\| \leq |\lambda_{N+1}| \tag{11} \quad \square$$

Remark 2. The appearance of $\Re$ in the expansion $\langle E_N f, f\rangle = \langle E_N g, g\rangle + 2\,\Re\langle E_N g, h\rangle + \langle E_N h, h\rangle$ is due to the properties of inner products in complex Hilbert spaces. When expanding $\langle E_N f, f\rangle$ with $f = g + h$, the cross terms $\langle E_N g, h\rangle$ and $\langle E_N h, g\rangle$ are complex conjugates. Since $\langle E_N h, g\rangle = \overline{\langle E_N g, h\rangle}$, their sum equals $2\,\Re\langle E_N g, h\rangle$. While this term ultimately vanishes in our proof due to orthogonality, this expansion technique is standard when dealing with sesquilinear forms.