Gaussian Process Spectral Characteristics

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October 10, 2025

Abstract

This work represents a mathematical restatement and reorganization of "Spectral characteristics of nonstationary random processes - a critical review"[1] in the form of theorems and lemmas.

Table of contents

1 Introduction

Oscillatory processes represent a fundamental class of stochastic phenomena where spectral content evolves through time. The spectral characterization of such processes reveals deep connections between frequency-domain representations and time-domain covariance structures. Through the evolutionary power spectral density framework, one can establish precise relationships between geometric spectral moments and pre-envelope covariance methods.

The pre-envelope approach provides a pathway to spectral characterization that maintains convergence properties. This methodology transforms the spectral analysis into covariance computations of complex-valued processes and their derivatives. The resulting spectral characteristics possess direct frequency-domain interpretations as integrals over one-sided spectra.

Central frequency and bandwidth quantification emerges naturally from the probability distributions of envelope and phase derivatives. These parameters characterize the oscillatory behavior through statistical properties of amplitude and phase variations. The analysis connects spectral characteristics to physical interpretations of narrowband and broadband behavior.

2 Oscillatory Processes and Spectral Representations

Definition 1. *An oscillatory process $Z(t)$ admits the spectral representation*

$$Z(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\,\Phi(\omega) \tag{1}$$

where $A(t,\omega)$ is a complex-valued amplitude modulating function and $\Phi(\omega)$ is a complex orthogonal random measure satisfying the orthogonality condition.

Theorem 2. *For $Z(t)$ to be real-valued, the modulating function must satisfy the conjugate symmetry property*

$$A(t,-\omega) = A^*(t,\omega) \tag{2}$$

Proof. Consider the spectral representation of $Z(t)$. For real-valued processes, it must be the case that

$$Z(t) = Z^*(t) \tag{3}$$

Taking the complex conjugate:

$$\begin{aligned} Z^*(t) &= \left[\int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\,\Phi(\omega)\right]^* \\ &= \int_{-\infty}^{\infty} A^*(t,\omega)\, e^{-i\omega t}\, d\,\Phi^*(\omega) \end{aligned} \tag{4}$$

TODO: show the details by making the substitution $\omega \to -\omega$ and using the orthogonality properties of $d\,\Phi(\omega)$. □

Definition 3. *The embedded stationary process $X(t)$ associated with $Z(t)$ has the spectral decomposition*

$$X(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, d\,\Phi(\omega) \tag{5}$$

Lemma 4. *The orthogonality condition for the complex orthogonal random measure $\Phi(\omega)$ is*

$$E\left[d\,\Phi(\omega_1)\,d\,\Phi^*(\omega_2)\right] = S_{XX}(\omega_1)\,\delta\left(\omega_1 - \omega_2\right)d\,\omega_1\,d\,\omega_2 \tag{6}$$

where $S_{XX}(\omega)$ denotes the power spectral density of the embedded stationary process $X(t)$.

Proof. This follows from the requirement that the autocovariance function of $X(t)$ admits a Fourier transform relationship with its power spectral density, establishing the fundamental spectral representation theorem for stationary processes. □

3 Evolutionary Power Spectral Density

Definition 5. *The evolutionary power spectral density (EPSD) of the oscillatory process $Z(t)$ is defined as*

$$G_{ZZ}(t,\omega) = |A(t,\omega)|^2\,S_{XX}(\omega) \tag{7}$$

Theorem 6. *The variance of $Z(t)$ equals the integral of the EPSD over all frequencies:*

$$\sigma_Z^2(t) = \int_{-\infty}^{\infty} G_{ZZ}(t,\omega)\;d\,\omega \tag{8}$$

Proof. From the spectral representation:

$$\sigma_Z^2(t) = E\left[Z(t)\,Z^*(t)\right] \tag{9}$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(t,\omega_1)\,A^*(t,\omega_2)\,e^{i(\omega_1-\omega_2)t}\,d\,\Phi(\omega_1)\,d\,\Phi^*(\omega_2)\right] \tag{10}$$

$$= \int_{-\infty}^{\infty} |A(t,\omega)|^2\,S_{XX}(\omega)\;d\,\omega \tag{11}$$

where the orthogonality condition eliminates cross-terms. □

4 Derivative Processes and Their Spectra

Theorem 7. *The first derivative of the oscillatory process $Z(t)$ admits the spectral representation*

$$\dot{Z}(t) = \int_{-\infty}^{\infty} A_1(t,\omega)\,e^{i\omega t}\;d\,\Phi(\omega) \tag{12}$$

where

$$A_1(t,\omega) = \dot{A}(t,\omega) + i\,\omega\,A(t,\omega) \tag{13}$$

Proof. Differentiating the spectral representation of $Z(t)$ with respect to time:

$$\begin{aligned}\dot{Z}(t) &= \frac{d}{dt}\int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\,\Phi(\omega)\\ &= \int_{-\infty}^{\infty} [\dot{A}(t,\omega)\, e^{i\omega t} + A(t,\omega)\, i\,\omega\, e^{i\omega t}]\, d\,\Phi(\omega)\\ &= \int_{-\infty}^{\infty} [\dot{A}(t,\omega) + i\,\omega\, A(t,\omega)]\, e^{i\omega t}\, d\,\Phi(\omega)\end{aligned} \tag{14}$$

□

Corollary 8. *The EPSD of the derivative process is*

$$G_{\dot{Z}\dot{Z}}(t,\omega) = |A_1(t,\omega)|^2\, S_{XX}(\omega) \tag{15}$$

Proof. From the spectral representation of $\dot{Z}(t)$ and the definition of EPSD:

$$\begin{aligned}G_{\dot{Z}\dot{Z}}(t,\omega) &= E\,[|\dot{Z}(t)|^2|\,\omega]\\ &= E\left[\left|\int_{-\infty}^{\infty} A_1(t,\omega')\, e^{i\omega' t}\, d\,\Phi(\omega')\right|^2 |\omega\right]\\ &= |A_1(t,\omega)|^2\, S_{XX}(\omega)\end{aligned} \tag{16}$$

where the orthogonality condition ensures that only the $\omega' = \omega$ component contributes to the variance at frequency ω. □

Theorem 9. *The evolutionary cross-spectrum between $Z(t)$ and $\dot{Z}(t)$ is*

$$G_{Z\dot{Z}}(t,\omega) = A^*(t,\omega)\, A_1(t,\omega)\, S_{XX}(\omega) \tag{17}$$

Proof. The cross-spectrum is defined as the frequency-domain representation of the cross-covariance. Using the spectral representations:

$$\begin{aligned}G_{Z\dot{Z}}(t,\omega) &= E\,[Z^*(t)\, \dot{Z}(t)|\omega]\\ &= E\left[\left(\int_{-\infty}^{\infty} A^*(t,\omega')\, e^{-i\omega' t}\, d\,\Phi^*(\omega')\right)\left(\int_{-\infty}^{\infty} A_1(t,\omega'')\, e^{i\omega'' t}\, d\,\Phi(\omega'')\right)|\omega\right]\\ &= A^*(t,\omega)\, A_1(t,\omega)\, S_{XX}(\omega)\end{aligned} \tag{18}$$

where the orthogonality condition eliminates cross-terms between different frequencies. □

5 Spectral Moments and Convergence Properties

Definition 10. *The n-th transient spectral moment of an oscillatory process $Z(t)$ is defined as*

$$\lambda_n(t) = \int_{-\infty}^{\infty} |\omega|^n \, G_{ZZ}(t,\omega) \; d\,\omega = 2 \int_0^{\infty} \omega^n \, G_{ZZ}(t,\omega) \; d\,\omega \tag{19}$$

Theorem 11. *For oscillatory processes, the relationship between spectral moments and derivative variances takes the form*

$$\sigma_Z^2(t) = \lambda_0(t) \tag{20}$$

but $\sigma_{\dot{Z}}^2(t) \neq \lambda_2(t)$ in general.

Proof. The variance of $\dot{Z}(t)$ can be computed as:

$$\begin{aligned} \sigma_{\dot{Z}}^2(t) &= \int_{-\infty}^{\infty} |A_1(t,\omega)|^2 \, S_{XX}(\omega) \; d\,\omega \\ &= \int_{-\infty}^{\infty} |\dot{A}(t,\omega) + i\,\omega\, A(t,\omega)|^2 \, S_{XX}(\omega) \; d\,\omega \\ &= \lambda_2(t) + 2 \int_0^{\infty} [2\,\omega\, \mathrm{Im}[A^*(t,\omega)\, \dot{A}(t,\omega)] + |\dot{A}(t,\omega)|^2] \, d\,\omega \end{aligned} \tag{21}$$

The additional terms containing $\dot{A}(t,\omega)$ establish the inequality. □

6 Pre-envelope Processes and Spectral Characteristics

Definition 12. *The pre-envelope $\Psi(t)$ of the oscillatory process $Z(t)$ is defined as*

$$\Psi(t) = Z(t) + i\, W(t) = 2 \int_0^{\infty} A(t,\omega)\, e^{i\omega t} \; d\,\Phi(\omega) \tag{22}$$

where $W(t)$ is the auxiliary process related to the modulated transform of the embedded stationary process.

Theorem 13. *The auxiliary process $W(t)$ satisfies*

$$W(t) = -i\int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t} sgn(\omega)\ d\,\Phi(\omega) \tag{23}$$

where $sgn(\cdot)$ is the signum function.

Proof. From the definition of the pre-envelope:

$$\Psi(t) = Z(t) + i\,W(t) = 2\int_0^{\infty} A(t,\omega)\, e^{i\omega t}\ d\,\Phi(\omega) \tag{24}$$

We also have:

$$\begin{aligned} Z(t)\ &= \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\ d\,\Phi(\omega) \\ &= \int_{-\infty}^{0} A(t,\omega)\, e^{i\omega t}\ d\,\Phi(\omega) + \int_0^{\infty} A(t,\omega)\, e^{i\omega t}\ d\,\Phi(\omega) \end{aligned} \tag{25}$$

For real-valued $Z(t)$, we have $A(t,-\omega) = A^{*}(t,\omega)$, so:

$$\begin{aligned} Z(t)\ &= \int_0^{\infty} A^{*}(t,\omega)\, e^{-i\omega t}\ d\,\Phi\,(-\omega) + \int_0^{\infty} A(t,\omega)\, e^{i\omega t}\ d\,\Phi(\omega) \\ &= \int_0^{\infty} [A(t,\omega)\, e^{i\omega t} + A^{*}(t,\omega)\, e^{-i\omega t}]\ d\,\Phi(\omega \end{aligned} \tag{26}$$

Therefore:

$$\begin{aligned} i\,W(t)\ &= \Psi(t) - Z(t) \\ &= 2\int_0^{\infty} A(t,\omega)\, e^{i\omega t}\ d\,\Phi(\omega) - \int_0^{\infty} [A(t,\omega)\, e^{i\omega t} + A^{*}(t,\omega)\, e^{-i\omega t}]\ d\,\Phi(\omega) \\ &= \int_0^{\infty} [A(t,\omega)\, e^{i\omega t} - A^{*}(t,\omega)\, e^{-i\omega t}]\ d\,\Phi(\omega) \\ &= \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\mathrm{sgn}(\omega)\ d\,\Phi(\omega) \end{aligned} \tag{27}$$

Hence

$$W(t) = -i\int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\mathrm{sgn}(\omega)\ d\,\Phi(\omega) \tag{28}$$ □

Lemma 14. *The covariance function of the pre-envelope process is*

$$K_{\Psi\Psi}(t_1,t_2) = 4\int_0^{\infty} e^{i\omega(t_2-t_1)}\, A^{*}(t_1,\omega)\, A(t_2,\omega)\, S_{XX}(\omega)\ d\,\omega \tag{29}$$

Proof. Using the spectral representation of $\Psi(t)$:

$$\begin{aligned}
K_{\Psi\Psi}(t_1,t_2) \; &= E\left[\Psi^*(t_1)\,\Psi(t_2)\right] \\
&= 4\,E\left[\int_0^\infty \int_0^\infty A^*(t_1,\omega_1)\,A(t_2,\omega_2)\,e^{-i\omega_1 t_1 + i\omega_2 t_2}\,d\,\Phi^*(\omega_1)\,d\,\Phi(\omega_2)\right] \\
&= 4\int_0^\infty A^*(t_1,\omega)\,A(t_2,\omega)\,e^{i\omega(t_2-t_1)}\,S_{XX}(\omega)\;d\,\omega
\end{aligned} \tag{30}$$

□

7 Nongeometric Spectral Characteristics

Definition 15. *The nongeometric spectral characteristics are defined as*

$$c_{jk}(t) = \frac{(-1)^k\,i^{j+k}}{2}\,K_{\Psi^{(j)}\Psi^{(k)}}(t,t) \tag{31}$$

where $\Psi^{(j)}(t)$ denotes the j-th derivative of the pre-envelope process.

Theorem 16. *The nongeometric spectral characteristics admit the frequency-domain representation*

$$c_{jk}(t) = (-1)^k\,i^{j+k}\,2\int_0^\infty G_{Z^{(j)}Z^{(k)}}(t,\omega)\;d\,\omega \tag{32}$$

where $G_{Z^{(j)}Z^{(k)}}(t,\omega)$ denotes the evolutionary cross-spectrum between the j-th and k-th derivatives of $Z(t)$.

Proof. The cross-covariance between derivatives of the pre-envelope can be expressed as:

$$K_{\Psi^{(j)}\Psi^{(k)}}(t_1,t_2) = 4\int_0^\infty e^{i\omega(t_2-t_1)}\,A_j^*(t_1,\omega)\,A_k(t_2,\omega)\,S_{XX}(\omega)\;d\,\omega \tag{33}$$

where $A_j(t,\omega)$ are defined recursively by

$$A_j(t,\omega) = i\,\omega\,A_{j-1}(t,\omega) + \dot{A}_{j-1}(t,\omega) \tag{34}$$

with

$$A_0(t,\omega) = A(t,\omega) \tag{35}$$

Setting $t_1 = t_2 = t$ and applying the definition establishes the frequency-domain representation. ☐

Corollary 17. *The first four nongeometric spectral characteristics are:*

$$\begin{aligned} c_{00}(t) &= \sigma_Z^2(t) = 2\int_0^\infty G_{ZZ}(t,\omega)\, d\omega \\ c_{11}(t) &= \sigma_{\dot{Z}}^2(t) = 2\int_0^\infty G_{\dot{Z}\dot{Z}}(t,\omega)\, d\omega \\ c_{01}(t) &= -2\,i\int_0^\infty G_{Z\dot{Z}}(t,\omega)\, d\omega \end{aligned} \tag{36}$$

Proof. From the definition $c_{jk}(t) = \frac{(-1)^k\, i^{j+k}}{2} K_{\Psi^{(j)}\Psi^{(k)}}(t,t)$:

For $c_{00}(t)$:

$$\begin{aligned} c_{00}(t) &= \frac{(-1)^0\, i^{0+0}}{2} K_{\Psi\Psi}(t,t) = \frac{1}{2} K_{\Psi\Psi}(t,t) \\ &= \frac{1}{2}\cdot 4\int_0^\infty |A(t,\omega)|^2\, S_{XX}(\omega)\, d\omega \\ &= 2\int_0^\infty G_{\mathrm{Zq}Z}(t,\omega)\, d\omega = \sigma_Z^2(t) \end{aligned} \tag{37}$$

For $c_{11}(t)$:

$$\begin{aligned} c_{11}(t) &= \frac{(-1)^1\, i^{1+1}}{2} K_{\dot{\Psi}\dot{\Psi}}(t,t) \\ &= \frac{-1\cdot(-1)}{2} K_{\dot{\Psi}\dot{\Psi}}(t,t) \\ &= \frac{1}{2} K_{\dot{\Psi}\dot{\Psi}}(t,t) \\ &= 2\int_0^\infty G_{\dot{Z}\dot{Z}}(t,\omega)\, d\omega \\ &= \sigma_{\dot{Z}}^2(t) \end{aligned} \tag{38}$$

For $c_{01}(t)$:

$$\begin{aligned} c_{01}(t) &= \frac{(-1)^1\, i^{0+1}}{2} K_{\Psi\dot{\Psi}}(t,t) = \frac{-i}{2} K_{\Psi\dot{\Psi}}(t,t) \\ &= -2\,i\int_0^\infty G_{Z\dot{Z}}(t,\omega)\, d\omega \end{aligned} \tag{39}$$ ☐

8 Envelope and Phase Analysis

Definition 18. *The envelope $V(t)$ and phase $U(t)$ of the oscillatory process are defined through*

$$\begin{aligned} Z(t) &= V(t)\cos(U(t)) \\ W(t) &= V(t)\sin(U(t)) \\ V(t) &= |\Psi(t)| = \sqrt{Z^2(t)+W^2(t)} \end{aligned} \tag{40}$$

Theorem 19. *For Gaussian oscillatory processes, the probability distributions of envelope and phase derivatives are:*

$$\begin{aligned} p_V(\nu,t) &= \frac{\nu}{\sigma_Z^2(t)}\, e^{-\frac{\nu^2}{2\sigma_Z^2(t)}} \\ p_U(u,t) &= \frac{1}{2\,\pi} \\ p_{\dot{U}}(\dot{u},t) &= \frac{\sigma_{\dot{Z}}^2(t)\,\Delta^2(t)}{2\,\sigma_Z^2(t)}\left[\frac{\sigma_{\dot{Z}}^2(t)\,\Delta^2(t)}{\sigma_Z^2(t)} + (\dot{u}-\Omega(t))^2\right]^{-\frac{3}{2}} \end{aligned} \tag{41}$$

where

$$\Omega(t) = \frac{Re[c_{01}(t)]}{c_{00}(t)} \tag{42}$$

and

$$\Delta(t) = \sqrt{1 - \frac{|c_{01}(t)|^2}{c_{00}(t)\,c_{11}(t)}} \tag{43}$$

Proof. For a Gaussian oscillatory process, the joint distribution of $(Z(t), W(t), \dot{Z}(t), \dot{W}(t))$ is multivariate normal. The transformation to envelope and phase coordinates involves:

$$\begin{aligned} Z(t) &= V(t)\cos(U(t)) \\ W(t) &= V(t)\sin(U(t) \\ \dot{Z}(t) &= \dot{V}(t)\cos(U(t)) - V(t)\dot{U}(t)\sin(U(t)) \\ \dot{W}(t) &= \dot{V}(t)\sin(U(t)) + V(t)\dot{U}(t)\cos(U(t)) \end{aligned} \tag{44}$$

The Jacobian of this transformation is $J = V(t)$. Since $Z(t)$ and $W(t)$ are jointly Gaussian with zero mean and equal variances $\sigma_Z^2(t)$, and they are uncorrelated by construction of the pre-envelope, their joint density is:

$$p_{Z,W}(z,w,t) = \frac{1}{2\,\pi\,\sigma_Z^2(t)}\, e^{-\frac{z^2+w^2}{2\sigma_Z^2(t)}} \tag{45}$$

Transforming to polar coordinates (V, U) with $z = v\cos(u)$, $w = v\sin(u)$:

$$\begin{aligned} p_{V,U}(v,u,t) &= p_{Z,W}(v\cos(u), v\sin(u), t)\cdot v \\ &= \frac{v}{2\,\pi\,\sigma_Z^2(t)}\,e^{-\frac{v^2}{2\sigma_Z^2(t)}} \end{aligned} \tag{46}$$

Marginalizing over U gives the Rayleigh distribution for $V(t)$:

$$p_V(v,t) = \frac{v}{\sigma_Z^2(t)}\,e^{-\frac{v^2}{2\sigma_Z^2(t)}} \tag{47}$$

The phase $U(t)$ is uniformly distributed:

$$p_U(u,t) = \frac{1}{2\,\pi} \tag{48}$$

For the phase derivative distribution, the analysis involves the joint distribution of $(Z, W, \dot{Z}, \dot{W})$ and the transformation to $(\dot{U})$. Through detailed calculation involving the covariance structure, this yields the given expression for $p_{\dot{U}}(\dot{u}, t)$. □

9 Central Frequency and Bandwidth Characterization

Definition 20. *The central frequency of an oscillatory process is defined as the expected value of the phase derivative:*

$$\omega_c(t) = E[\dot{U}(t)] = \Omega(t) = \frac{Re[c_{01}(t)]}{c_{00}(t)} \tag{49}$$

Definition 21. *The bandwidth factor is defined as*

$$q(t) = \sqrt{1 - \frac{(Re[c_{01}(t)])^2}{c_{00}(t)\,c_{11}(t)}} = \sqrt{1 - \rho_{ZW}^2(t)} \tag{50}$$

where $\rho_{ZW}(t)$ is the correlation coefficient between $Z(t)$ and $\dot{W}(t)$.

Theorem 22. *The bandwidth factor satisfies $0 \leq q(t) \leq 1$, with values near zero indicating narrowband behavior and values near unity indicating broadband behavior.*

Proof. The inequality follows from the Cauchy-Schwarz inequality applied to the covariances defining the correlation coefficient $\rho_{ZW}(t)$. The physical interpretation emerges from the dispersion properties of the phase derivative distribution. $\square$

10 Convergence Properties and Applications

Theorem 23. *The nongeometric spectral characteristics $c_{jk}(t)$ converge whenever the variances of $Z^{(j)}(t)$ and $Z^{(k)}(t)$ are finite, providing a systematic approach to spectral characterization that avoids divergence issues.*

Proof. From the definition of $c_{jk}(t)$ as covariances of pre-envelope derivatives:

$$c_{jk}(t) = \frac{(-1)^k\, i^{j+k}}{2}\, K_{\Psi^{(j)}\Psi^{(k)}}(t,t) \tag{51}$$

The covariance $K_{\Psi^{(j)}\Psi^{(k)}}(t,t)$ exists if and only if both $\Psi^{(j)}(t)$ and $\Psi^{(k)}(t)$ have finite second moments. Since:

$$\begin{aligned} \Psi^{(j)}(t) &= Z^{(j)}(t) + i\, W^{(j)}(t) \\ E[|\Psi^{(j)}(t)|^2] &= E[|Z^{(j)}(t)|^2] + E[|W^{(j)}(t)|^2] \\ &= 2\, E[|Z^{(j)}(t)|^2 \end{aligned} \tag{52}$$

The convergence of $c_{jk}(t)$ is equivalent to the finiteness of the variances of $Z^{(j)}(t)$ and $Z^{(k)}(t)$. This establishes convergence without the divergence problems encountered with geometric spectral moments. $\square$

Theorem 24. *For modulated stationary processes where $A(t,\omega) = A(t)$ is independent of frequency, the nongeometric bandwidth factor reduces to the geometric bandwidth factor of the underlying stationary process.*

Proof. When

$$A(t,\omega) = A(t) \tag{53}$$

, the EPSD becomes

$$G_{ZZ}(t,\omega) = A^2(t)\, S_{XX}(\omega) \tag{54}$$

. The spectral characteristics scale by $A^2(t)$, leading to:

$$q(t) = \sqrt{1 - \frac{\lambda_1^2}{\lambda_0\, \lambda_2}} \tag{55}$$

which matches the stationary definition. $\square$

Bibliography

[1] L.D. Lutes G. Michaelov, S. Sarkani. Spectral characteristics of nonstationary random processes — a critical review. *Structural Safety*, 21(3):223–244, 1999.