Injective Measure-Preserving Time-Changes of Stationary Processes are Oscillatory

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Oscillatory Processes and Normalized Injective Time-Changes

Definition 1. *[Oscillatory Process] A complex-valued second-order stochastic process $\{X_t\}_{t \in I}$ is said to be oscillatory if there exists a family of functions $\phi_t(\omega)$ and a complex orthogonal increment process $Z(\omega)$ with*

$$E\,|d\,Z(\omega)|^2 = d\,\mu(\omega) \tag{1}$$

such that

$$X_t = \int_{-\infty}^{\infty} \phi_t(\omega)\ d\,Z(\omega) \tag{2}$$

where

$$\phi_t(\omega) = A_t(\omega)\, e^{i\omega t} \tag{3}$$

and $A_t(\omega)$ is a quadratically integrable gain function with respect to $d\,\mu$.

Definition. *[Stationary Process] A second-order process $\{S_t\}_{t \in J}$ is stationary if it admits the spectral representation*

$$S_t = \int_{-\infty}^{\infty} e^{i\omega t}\ d\,Z(\omega) \tag{4}$$

for some orthogonal increment process $Z(\omega)$ with

$$E\,|d\,Z(\omega)|^2 = d\,\mu(\omega) \tag{5}$$

Theorem 2. *[Time-Varying Filter for Injective Time-Change] Let S_t be a stationary process and $\theta\colon\mathbb{R}\to\mathbb{R}$ be smooth and strictly increasing with $\theta'(t)>0$. To achieve the transformation*

$$X_t = \sqrt{\theta'(t)}\, S_{\theta(t)} \tag{6}$$

via convolution

$$X_t = \int_{-\infty}^{\infty} S_{t-u}\, h_t(u)\, d\,u \tag{7}$$

the time-varying impulse response must be

$$h_t(u) = \sqrt{\theta'(t)}\, \delta\,(u-(t-\theta(t))) \tag{8}$$

Proof. For the convolution to yield

$$X_t = \sqrt{\theta'(t)}\, S_{\theta(t)} \tag{9}$$

the argument of S in the integrand must equal $\theta(t)$ when the delta function is activated. This requires:

$$t-u=\theta(t) \tag{10}$$

Solving for u:

$$u=t-\theta(t) \tag{11}$$

Therefore:

$$h_t(u) = \sqrt{\theta'(t)}\, \delta\,(u-(t-\theta(t))) \tag{12}$$

Verification by direct computation:

$$\begin{aligned}
X_t &= \int_{-\infty}^{\infty} S_{t-u}\, h_t(u)\, d\,u && (13)\\
&= \int_{-\infty}^{\infty} S_{t-u}\, \sqrt{\theta'(t)}\, \delta\,(u-(t-\theta(t)))\, d\,u && (14)\\
&= \sqrt{\theta'(t)}\, S_{t-(t-\theta(t))} && (15)\\
&= \sqrt{\theta'(t)}\, S_{\theta(t)} && (16)
\end{aligned}$$

□

Theorem 3. *[Oscillatory Representation of Injective Time-Change] The process*

$$X_t = \sqrt{\theta'(t)}\, S_{\theta(t)} \tag{17}$$

has the oscillatory representation

$$\begin{aligned} X_t &= \int_{-\infty}^{\infty} \phi_t(\omega)\, d\, Z(\omega) \\ &= \int_{-\infty}^{\infty} \sqrt{\theta'(t)}\, e^{i\omega\theta(t)} d\, Z(\omega) \end{aligned} \tag{18}$$

where

$$\phi_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega\theta(t)} \tag{19}$$

Proof. Starting from the spectral representation of S_t:

$$X_t = \sqrt{\theta'(t)}\, S_{\theta(t)} \tag{20}$$

$$= \int_{-\infty}^{\infty} \sqrt{\theta'(t)}\, e^{i\omega\theta(t)}\, d\, Z(\omega) \tag{21}$$

$$= \sqrt{\theta'(t)} \int_{-\infty}^{\infty} e^{i\omega\theta(t)}\, d\, Z(\omega) \tag{22}$$

Thus

$$\phi_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega\theta(t)} \tag{23}$$ □

Corollary 4. *[Envelope in Standard Form] The oscillatory functions can be written in the standard for*

$$\phi_t(\omega) = A_t(\omega)\, e^{i\omega t} \tag{24}$$

where

$$A_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega(\theta(t)-t)} \tag{25}$$

Proof. Substitute and combine the exponentials

$$\phi_t(\omega) = A_t(\omega)\, e^{i\omega t} \tag{26}$$

$$= \sqrt{\theta'(t)}\, e^{i\omega(\theta(t)-t)}\, e^{i\omega t} \tag{27}$$

$$= \sqrt{\theta'(t)}\, e^{i\omega(\theta(t)-t+t)} \tag{28}$$

$$= \sqrt{\theta'(t)}\, e^{i\omega\theta(t)}$$

where

$$A_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega(\theta(t) - t)} \tag{29}$$ □

Theorem 5. *[Evolutionary Power Spectrum] For the oscillatory process*

$$X_t = \sqrt{\theta'(t)}\, S_{\theta(t)} \tag{30}$$

with envelope

$$A_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega(\theta(t) - t)} \tag{31}$$

the evolutionary power spectrum at time t is

$$d\, F_t(\omega) = |A_t(\omega)|^2\, d\,\mu(\omega) = \theta'(t)\, d\,\mu(\omega) \tag{32}$$

Proof. The evolutionary power spectrum is defined as

$$d\, F_t(\omega) = |A_t(\omega)|^2\, d\,\mu(\omega) \tag{33}$$

Computing the magnitude squared:

$$\begin{aligned}
|A_t(\omega)|^2 &= \left|\sqrt{\theta'(t)}\, e^{i\omega(\theta(t) - t)}\right|^2 && (34)\\
&= \theta'(t) |e^{i\omega(\theta(t) - t)}|^2 && (35)\\
&= \theta'(t) \cdot 1 && (36)\\
&= \theta'(t) && (37)
\end{aligned}$$

Therefore

$$d\, F_t(\omega) = \theta'(t)\, d\,\mu(\omega) \tag{38}$$ □

Theorem 6. *[L^2-Norm Preservation] The transformation/*

$$S_t \mapsto \sqrt{\theta'(t)}\, S_{\theta(t)} \tag{39}$$

preserves the L^2-norm in the sense that

$$\int_I E |X_t|^2\, d\, t = \int_J E |S_s|^2\, d\, s \tag{40}$$

where I is the domain of t and $J = \theta(I)$.

Proof. Using the change of variables $s = \theta(t)$, so $d\,s = \theta'(t)\,d\,t$:

$$\int_I E|X_t|^2\,d\,t = \int_I E\left|\sqrt{\theta'(t)}\,S_{\theta(t)}\right|^2 d\,t \tag{41}$$

$$= \int_I \theta'(t)\,E|S_{\theta(t)}|^2\,d\,t \tag{42}$$

$$= \int_J E|S_s|^2\,d\,s \tag{43}$$

□