Equivalence of Egorov's Theorem and Vitali-Hahn-Saks Theorem for σ-compact Spaces

by Assistant

Theorem 1. *For σ-compact spaces, the following are equivalent:*

(a). Egorov's Theorem

(b). Vitali-Hahn-Saks Theorem

Preliminaries: This proof relies on several advanced measure-theoretic results, including the Hahn Decomposition Theorem, Radon-Nikodym Theorem, and properties of weak convergence in L^1 spaces. Familiarity with these concepts is assumed.

We will prove both directions of the equivalence.

(a) ⇒ (b):

Assume Egorov's Theorem holds for σ-compact spaces. Let (X, Σ) be a σ-compact measurable space and $\{\mu_n\}$ a sequence of finite signed measures on (X, Σ) such that for each $E \in \Sigma$, $\lim_{n\to\infty} \mu_n(E)$ exists and is finite.

1. Define $\mu(E) = \lim_{n\to\infty} \mu_n(E)$ for $E \in \Sigma$. Note that μ is a finite signed measure.

2. Write $X = \bigcup_{i=1}^{\infty} K_i$, where each K_i is compact and $K_i \subset K_{i+1}$.

3. For each i, consider the restrictions of μ_n and μ to K_i, denoted $\mu_{n,i}$ and μ_i. These are finite signed measures on compact sets.

4. Apply the Hahn Decomposition Theorem to each $\mu_{n,i}$ and μ_i. Then, by the Radon-Nikodym theorem, there exist measurable functions $f_{n,i}$ such that:

$$\mu_{n,i}(E) = \int_E f_{n,i}\, d|\mu_i| \text{ for all } E \subset K_i, E \in \Sigma$$

5. The assumption $\lim_{n\to\infty} \mu_n(E) = \mu(E)$ for all $E \in \Sigma$ implies that $\{f_{n,i}\}$ converges weakly to f_i in $L^1(K_i, |\mu_i|)$ for each i. This follows from the definition of weak convergence in L^1 spaces: for any $g \in L^\infty(K_i, |\mu_i|)$,

$$\lim_{n\to\infty} \int_{K_i} f_{n,i}\, g\, d|\mu_i| = \lim_{n\to\infty} \mu_{n,i}(g) = \mu_i(g) = \int_{K_i} f_i\, g\, d|\mu_i|$$

6. Apply Egorov's Theorem to $\{f_{n,i}\}$ on each K_i: For any $\varepsilon > 0$, there exists $A_i \subset K_i$, $A_i \in \Sigma$ such that $|\mu_i|(K_i \setminus A_i) < \varepsilon / 2^i$ and $f_{n,i}$ converges uniformly to f_i on A_i.

7. Define $A = \bigcup_{i=1}^{\infty} A_i$. A is measurable as it is a countable union of measurable sets. Note that $|\mu|(X \setminus A) \leq \sum_{i=1}^{\infty} |\mu_i|(K_i \setminus A_i) < \varepsilon$.

8. For any $\delta > 0$ and each i, choose N_i such that for all $n \geq N_i$ and $x \in A_i$, $|f_{n,i}(x) - f_i(x)| < \delta$.

9. For any $E \subset A$, $E \in \Sigma$, and for all $n \geq \max\{N_i : i \in \mathbb{N}\}$:

$$\begin{aligned}
|\mu_n(E) - \mu(E)| &\leq \sum_{i=1}^{\infty} |\mu_{n,i}(E \cap A_i) - \mu_i(E \cap A_i)| \\
&= \sum_{i=1}^{\infty} \left| \int_{E \cap A_i} (f_{n,i} - f_i)\, d|\mu_i| \right| \\
&\leq \sum_{i=1}^{\infty} \int_{E \cap A_i} |f_{n,i} - f_i|\, d|\mu_i| \\
&< \delta \sum_{i=1}^{\infty} |\mu_i|(E \cap A_i) = \delta |\mu|(E)
\end{aligned}$$

   This establishes uniform convergence for sufficiently large n.

10. This establishes uniform absolute continuity of $\{\mu_n\}$ with respect to $|\mu|$ on A, and thus on X since $|\mu|(X \setminus A) < \varepsilon$.

Therefore, the Vitali-Hahn-Saks Theorem holds for σ-compact spaces.

(b) ⇒ (a):

Assume the Vitali-Hahn-Saks Theorem holds for σ-compact spaces. Let (X, Σ, λ) be a σ-compact measure space and $\{f_n\}$ a sequence of measurable functions converging pointwise λ-almost everywhere to f.

1. Write $X = \bigcup_{i=1}^{\infty} K_i$, where each K_i is compact and $K_i \subset K_{i+1}$.

2. Define measures ν_n on (X, Σ) by:

$$\nu_n(E) = \int_E \min(1, |f_n - f|)\, d\lambda \text{ for } E \in \Sigma$$

   Note that ν_n are indeed finite measures:

   - Non-negative: $\min(1, |f_n - f|) \geq 0$

- Countably additive: follows from the countable additivity of the integral
- $\nu_n(\emptyset) = 0$: integral over empty set is zero
- $\nu_n(X) \leq \lambda(X) < \infty$: since $\min(1, |f_n - f|) \leq 1$

3. For each $E \in \Sigma$, by the Dominated Convergence Theorem:

$$\lim_{n\to\infty} \nu_n(E) = \int_E \lim_{n\to\infty} \min(1, |f_n - f|)\, d\lambda = 0$$

This holds because $\min(1, |f_n - f|)$ is bounded by 1 and converges pointwise to 0 λ-almost everywhere.

4. Apply the Vitali-Hahn-Saks Theorem to $\{\nu_n\}$: For any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $E \in \Sigma$ with $\lambda(E) < \delta$, we have $\nu_n(E) < \varepsilon$ for all n.

5. For each $k \in \mathbb{N}$ and $i \in \mathbb{N}$, define sets:

$$A_{k,i} = \{x \in K_i : |f_n(x) - f(x)| \geq 1/k \text{ for infinitely many } n\}$$

These sets are measurable as they are countable intersections and unions of measurable sets.

6. By pointwise convergence, $\lambda(A_{k,i}) \to 0$ as $k \to \infty$ for each i. This is true because for each x where $f_n(x)$ converges to $f(x)$, there exists a k large enough such that $|f_n(x) - f(x)| < 1/k$ for all but finitely many n. As k increases, fewer points fail to meet this criterion, so the measure of $A_{k,i}$ decreases to zero.

7. For each i, choose K_i such that $\lambda(A_{K_i,i}) < \delta/2^i$.

8. Define $B = \bigcup_{i=1}^{\infty} A_{K_i,i}$. Note that $\lambda(B) < \delta$.

9. By the uniform absolute continuity from step 4:

$$\int_B \min(1, |f_n - f|)\, d\lambda < \varepsilon \text{ for all } n$$

10. This implies that for any $\eta > 0$, there exists a set $C_\eta \subset X \setminus B$ with $\lambda(C_\eta) < \eta$ such that:

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \text{ sufficiently large and all } x \in X \setminus (B \cup C_\eta)$$

Detailed explanation: Suppose this were not true. Then there would exist a set $D \subset X \setminus B$ of positive measure such that for each $x \in D$, $|f_n(x) - f(x)| \geq \varepsilon$ for infinitely many n. This would imply:

$$\int_D \min(1, |f_n - f|)\, d\lambda \geq \int_D \min(1, \varepsilon)\, d\lambda = \varepsilon\, \lambda(D) > 0$$

for infinitely many n. However, this contradicts the uniform absolute continuity established in step 4, which implies that for any set E with $\lambda(E) < \delta$, we have $\int_E \min(1, |f_n - f|)\, d\lambda < \varepsilon$ for all n. We can choose η small enough so that $\lambda(D) < \delta$, leading to this contradiction.

11. Since $\lambda(B \cup C_\eta) < \delta + \eta$, which can be made arbitrarily small, we have established Egorov's Theorem for σ-compact spaces.

This completes the proof of the equivalence between Egorov's Theorem and the Vitali-Hahn-Saks Theorem for σ-compact spaces.

Note: This equivalence is specific to σ-compact spaces. The σ-compactness property is crucial for this proof as it allows us to decompose the space into a countable union of compact sets. This decomposition enables us to apply Egorov's Theorem on each compact set in the (a) $\Rightarrow$ (b) direction, and to construct the set B in the (b) $\Rightarrow$ (a) direction. For more general spaces, this approach might not work, and the relationship between these theorems could be different or require alternative methods of proof.