THE TOPOLOGY OF SURFACES OF CONSTANT ENERGY IN INTEGRABLE HAMILTONIAN SYSTEMS, AND OBSTRUCTIONS TO INTEGRABILITY

1987 Math. USSR Izv. 29 629

(http://iopscience.iop.org/0025-5726/29/3/A07)

Изв. Акад. Наук СССР
Сер. Матем. Том 50(1986), № 6

Math. USSR Izvestiya
Vol. 29(1987), No. 3

THE TOPOLOGY OF SURFACES OF CONSTANT ENERGY IN INTEGRABLE HAMILTONIAN SYSTEMS, AND OBSTRUCTIONS TO INTEGRABILITY

UDC 513.944

A. T. FOMENKO

ABSTRACT. The surfaces of constant energy in integrable Hamiltonian systems which possess Bott integrals are classified. A complete topological classification is given of surgery of Liouville tori in general position in integrable Hamiltonian systems.

Bibliography: 28 titles.

§1. The four-dimensional case

1. Introduction. In recent years, many results about the integrability of Hamiltonian systems have been obtained. The problem of searching for stable closed integral curves of Hamiltonian systems is of particular interest. We shall prove that in some cases it is possible to guarantee the existence of at least two such solutions on three-dimensional *surfaces of constant energy* (*isoenergetic surfaces*) of an integrable system solely on the basis of data about the first homology group of these surfaces (Theorem 1). This result is based on the general Theorem 3 which gives *a complete topological classification of constant energy surfaces of integrable systems* (*on four-dimensional manifolds* M^4) *and a canonical representation of such surfaces as the result of pasting together three-dimensional manifolds of three simple types*. In this connection, we shall suppose that the system possesses a second integral which is a smooth Bott function (a Bott function is one in which all critical points lie on nondegenerate smooth submanifolds). We have needed to develop a special new "Morse type" theory for integrable systems. This theory differs from both the usual Morse theory and Bott's theory [19]. In particular, we develop some ideas sketched by Kozlov [6], Anosov [1]–[3], Novikov [10], [11] and Smale [25]. We also pose the question of whether nonsingular isoenergetic surfaces of integrable systems possess specific properties which distinguish them from all smooth three-dimensional manifolds. It is natural to conjecture that most manifolds cannot appear as isoenergetic surfaces in the Bott, analytic, or algebraic cases. Corollary 4 establishes this conjecture for the Bott case. Hence we obtain, in particular, *new topological obstructions* to the integrability of Hamiltonian systems in the class of Bott functions (Corollary 2). Our investigation of geodesic flows on the sphere is based on the well-known work of Anosov [1]–[3] and Klingenberg and Takens [16]. Nowadays, many methods are known for establishing the analytic nonintegrability of systems in general position [6], [7], [12], [13], [8],

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58F05, 58F07, 57R65; Secondary 70H10.

[20]. Our results supplement these investigations and allow us to effectively settle the question of the existence of smooth Bott integrals.

In §2, we construct a topological theory of integrable systems and use it to obtain a *complete classification of general position surgery of Liouville tori in neighborhoods of bifurcation diagrams of moment maps of integrable systems.* We *explicitly* describe the canonical surgery. This allows us to classify the behavior of the integral curves of a system on Liouville tori in a neighborhood of critical energy levels (see [28]). The author is grateful to D. V. Anosov, A. V. Brailov, V. V. Kozlov, Ya. V. Tatarinov, and Heiner Zieschang for useful and stimulating discussions.

2. Formulation of the results. Let $v = \operatorname{sgrad} H$ be a Hamiltonian system with smooth Hamiltonian H on a smooth four-dimensional symplectic manifold M^4. Because H is an integral of the system, v can be restricted to an invariant three-dimensional surface Q of constant energy; that is, $Q = \{x \in M\colon H(x) = \text{const}\}$. Since M^4 is always orientable, Q is also orientable. We shall consider noncritical (nonsingular) surfaces Q; that is, those for which $\operatorname{grad} H \neq 0$. Suppose that the system is Liouville integrable; that is, suppose there exists an additional integral f which is independent of H (almost everywhere) and in involution with H. Restricting this integral to Q gives a smooth function f.

DEFINITION 1. A smooth integral f is said to be a *Bott integral* on Q if the critical points of the function f on Q form nondegenerate critical submanifolds.

The general properties of such functions were studied by Bott [19]. A submanifold consisting of critical points of f is called *nondegenerate* if the Hessian $d^2 f$ is nondegenerate on planes normal to the submanifold. The experience accumulated in investigating various concrete systems (see, for example, [15] and [26]) shows that an overwhelming majority of the integrals discovered to date are Bott functions. Thus, our introduction of the class of Bott integrals seems natural. Throughout the paper, *we shall only consider the integrability or nonintegrability of a system on a single fixed nonsingular level surface Q.*

Let T be a critical nondegenerate submanifold of f on Q. The *separatrix diagram* $P(T)$ is the union of all integral curves of the vector field $\operatorname{grad} f$ which enter or leave T. We let $P_-(T)$ ($P_+(T)$) denote the *entering* (*exiting*) *separatrix diagram.* In a neighborhood of T both diagrams are smooth two-dimensional manifolds with common boundary T.

DEFINITION 2. A Bott integral f on Q is said to be *orientable* if all its critical submanifolds are orientable. Otherwise, the integral f is said to be *nonorientable.*

A hypersurface Q^3 with Bott integral f will sometimes be called an *integral surface.* It turns out that, *without loss of generality, we can restrict ourselves to studying only orientable integrals.* Namely, by considering the surfaces Q to within a double cover we can always assume that the integral f is orientable.

ASSERTION 1. *Let Q^3 be a nonsingular compact surface of constant energy in M^4 and let f be a nonorientable Bott integral on Q. Then all the nonorientable critical submanifolds are homeomorphic to a Klein bottle and f attains either a (local) minimum or maximum on them. Let $U(Q)$ be a sufficiently small tubular neighborhood of Q in M. Then there exists a double cover $\pi(\tilde{U}(\tilde{Q}), \tilde{H}, \tilde{f}) \to (U(Q), H, f)$ (with fiber $\mathbf{Z}_2$), where $\tilde{U}(\tilde{Q})$ is a symplectic manifold with a Hamiltonian system $\tilde{v} = \operatorname{sgrad} \tilde{H}$ (where $\tilde{H} = \pi^*(H)$) which is integrable on $\tilde{Q} = \pi^{-1}(Q)$ by means of an orientable (!) Bott integral $\tilde{f} = \pi^*(f)$. In addition, all critical Klein bottles on Q "develop" into critical tori T^2 on $\tilde{Q}$ (minima or maxima of $\tilde{f}$). The manifold $\tilde{U}(\tilde{Q})$ is a tubular neighborhood of the surface $\tilde{Q}$.*

Thus, if f is a nonorientable integral on Q, then $\pi_1(Q) \neq 0$ and $\pi_1(Q)$ contains a subgroup of index 2. *If, for example,* $Q = S^3$ *(a case which frequently occurs in mechanics), then any Bott integral* f *on* S^3 *is orientable.*

DEFINITION 3. Let γ be a closed integral curve (that is, a periodic solution) of the system v on Q^3. We shall say that γ is stable if it has a tubular neighborhood which is totally (without gaps) fibered into concentric two-dimensional tori which are invariant with respect to v and which envelop γ; that is, all integral curves close to γ are "packed" onto invariant tori with a common axis γ.

An integrable system need not have a stable periodic solution. An example is provided by the geodesic flow on a flat torus. It turns out that there exists a close connection between the following three objects: a) an additional Bott integral f on the surface Q, b) the number of stable periodic solutions on Q, and c) the first integral homology group $H_1(Q, \mathbf{Z})$ (or the fundamental group $\pi_1(Q)$). Let $m = m(Q)$ be the number of stable periodic solutions of the system v on Q. Let $r = r(Q)$ be the number of critical submanifolds of the integral f on Q which are homeomorphic to a Klein bottle.

THEOREM 1. *Let* M^4 *be a smooth symplectic manifold (compact or not) and let* $v = \operatorname{sgrad} H$ *be a Hamiltonian field on* M^4. *Suppose that the system is integrable on some nonsingular compact three-dimensional surface* $Q = (H = \text{const})$ *by means of a Bott integral* f. *Then the number* m *of stable periodic solutions of* v *on* Q *is bounded below in terms of topological invariants of* Q *as follows.*

1) *If the integral* f *is orientable on* Q, *then* a) $m \geq 2$ *if the homology group* $H_1(Q, \mathbf{Z})$ *is finite, and* b) $m \geq 2$ *if the fundamental group* $\pi_1(Q) = \mathbf{Z}$.

2) *If the integral* f *is nonorientable on* Q, *then* a) $m + r \geq 2$ *if* $H_1(Q, \mathbf{Z})$ *is finite,* b) $m \geq 2$ *if* $H_1(Q, \mathbf{Z}) = 0$ *(here the group* $\pi_1(Q)$ *may be infinite),* c) $m \geq 1$ *if* $H_1(Q, \mathbf{Z})$ *is a finite cyclic group,* d) $m \geq 1$ *if* $\pi_1(Q) = \mathbf{Z}$ *or if* $\pi_1(Q)$ *is a finite group, and* e) $m \geq 2$ *if* $H_1(Q, \mathbf{Z})$ *is a finite cyclic group and* Q *does not belong to a small series of manifolds of the form* $Q_0 = (S^1 \times D^2) + sA^3 + K^3$ *which are explicitly described below.*

In both cases 1) *and* 2), *the integral* f *attains a local minimum or maximum on each of the stable periodic solutions of the system (or on the Klein bottles). If* $H_1(Q, \mathbf{Z})$ *is an infinite group (that is, if the rank of* H_1 *is at least* 1), *then the system* v *may have no stable periodic solutions on* Q *at all.*

This criterion is reasonably efficient, since verifying that an integral is a Bott function and computing the rank of $H_1(Q)$ do not usually present difficulties. For many integrable mechanical systems, the surfaces Q are diffeomorphic to either a sphere S^3, or projective space $\mathbf{R}P^3$, or $S^1 \times S^2$. For the equations of motion of a heavy rigid body, we may, after some factorization, assume that some Q are homeomorphic to $\mathbf{R}P^3$ [7]. If the Hamiltonian H has an isolated minimum or maximum (an isolated equilibrium position of the system) on M^4, then all sufficiently close surfaces $Q = (H = \text{const})$ are spheres S^3. Let $L_{p,q}$ be a lens space (the quotient of a sphere S^3 by the action of a cyclic group). We shall single out the cases which are of interest for Hamiltonian mechanics.

PROPOSITION 1. *Suppose that* $v = \operatorname{sgrad} H$ *is integrable by means of a Bott integral* f *on a single constant energy surface* Q *which is homeomorphic to one of the following manifolds:* S^3, $\mathbf{R}P^3$, $S^1 \times S^2$, *or* $L_{p,q}$,

1) *If* f *is orientable, then* $m \geq 2$; *that is, the system* v *necessarily has at least two stable periodic solutions on* Q.

2) *If* f *is nonorientable, then* $m \geq 2$ *if* Q *is homeomorphic to* S^3 *and* $m \geq 1$ *if* Q *is homeomorphic to* $\mathbf{R}P^3$, $S^1 \times S^2$, *or* $L_{p,q}$. *An integrable system on the sphere always has at least two stable periodic solutions.*

Thus, not only does an integrable system have two stable periodic solutions on spheres close to an isolated equilibrium position of the system (a minimum or maximum of H), but it has two such solutions on all expanding level surfaces of H *as long as they remain homeomorphic to* S^3. *The criterion of Theorem* 1 *is sharp* in the sense that there are examples of integrable systems for which $Q = \mathbf{R}P^3$ or $Q = S^3$ has *exactly two* (and no more) stable periodic solutions [13].

It follows from results of Anosov and of Klingenberg and Takens that there is an open, everywhere dense subset of the set of all geodesic flows on a smooth Riemannian manifold which consists of flows with no stable closed integral curves [1], [16]. Thus, the property that a geodesic flow not have stable trajectories is the property of general position.

COROLLARY 1. *The geodesic flow of a Riemannian metric of general position on the sphere* S^2 *(that is, a metric which does not have a stable closed geodesic) is not integrable in the class of smooth Bott integrals.*

Let R be the rank of the group $\pi_1(Q)$; that is, the smallest possible number of generators. If the rank of $\pi_1(Q)$ is 1, then a system which is integrable by means of a Bott integral necessarily has at least one stable periodic solution on Q. In some integrable cases of the problem of inertial motion of a four-dimensional rigid body with a fixed point (see [21] and [22]), the three-dimensional nonsingular integral sufaces Q are diffeomorphic to $S^1 \times S^2$ (a remark due to A. V. Brailov); that is, $\pi_1(S^1 \times S^2) = \mathbf{Z}$ and $R = 1$. In the integrable Kovalevskaya case, some of the surfaces Q (after factorization) are also homeomorphic to $S^1 \times S^2$.

PROPOSITION 2. *Let* v *be a Hamiltonian system which is integrable on a nonsingular compact three-dimensional constant energy surface* Q *by means of a Bott integral. If the system has no stable periodic solutions on* Q, *then* 1) $H_1(Q, \mathbf{Z})$ *is not a finite cyclic group, and* 2) $\operatorname{rank}\pi_1(Q) \geq 2$ *and at least one of the generators of* $\pi_1(Q)$ *has infinite order.*

Consider the geodesic flow on a flat two-dimensional torus (with locally Euclidean metric). *This flow is integrable in the class of Bott integrals and does not have closed stable trajectories.* In view of Proposition 2, we must have $\operatorname{rank}\pi_1(Q) \geq 2$. In fact, in this case, nonsingular surfaces Q are diffeomorphic to the torus T^3 and $H_1(T^3, \mathbf{Z}) = \mathbf{Z}^3$.

COROLLARY 2. *Let* $v = \operatorname{sgrad} H$ *be a system on* M^4 *and* Q *a nonsingular compact three-dimensional constant energy surface. Suppose that the following two conditions are satisfied:* 1) *the system* v *has no stable periodic solutions on* Q, *and* 2) $H_1(Q, \mathbf{Z})$ *is a finite cyclic group of* $\operatorname{rank}\pi_1(Q) \leq 1$. *Then the system* v *is not integrable in the class of smooth Bott integrals on* Q.

A special case of the above is Corollary 1. In the case of a geodesic flow on the flat torus T^2 we have $Q = T^3$, $H_1(T^3, \mathbf{Z}) = \mathbf{Z}^3$, and $R = 1$ (that is, the conditions of Corollary 2 are not satisfied). Although the flow has no closed stable trajectories, it nevertheless is integrable in the class of Bott integrals. In [18] we gave a survey of the methods of noncommutative integration of Hamiltonian systems.

COROLLARY 3. *If a system is integrable in a noncommutative sense by means of Bott integrals, then all the conclusions of Theorem* 1 *hold.*

COROLLARY 4. *Most smooth closed compact orientable three-dimensional manifolds cannot occur as constant energy surfaces in a Hamiltonian system which is integrable by means of Bott integrals. The class of such manifolds will be described below.*

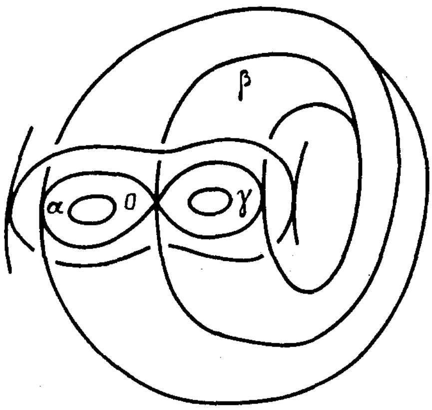

FIGURE 1.

Thus, the topology of the surface Q is a possible obstruction to integrability of the system. It is now easy to construct examples of manifolds that cannot be integral surfaces of integrable systems.

The results enumerated above are based on a general theorem due to the author which provides a *topological classification of the surfaces Q of constant energy in integrable systems.* We first describe five types of simple manifolds, the "elementary blocks", out of which each such surface Q is built.

Type 1. The *solid torus* $S^1 \times D^2$. Its boundary is the torus T^2.

Type 2. The product $T^2 \times D^1$, which will be called a *cylinder.* Its boundary is a union of two tori T^2.

Type 3. The product $N^2 \times S^1$, which will be called an *oriented saddle* (or "*trousers*"), where N^2 is a two-dimensional disk with two holes. Its boundary is a union of three tori T^2.

Type 4. Consider a nontrivial fibration $A^3 \xrightarrow{N^2} S^1$ with base S^1 and fiber N^2. There are only two nonequivalent fibrations over S^1 whose boundaries are tori and whose fibers are N^2. These are $N^2 \times S^1$ (type 3) and A^3. Clearly A^3 is the total space of the oriented "twisted" product $N^2 \tilde{\times} S^1$. It is characterized by the property that after translating the fiber N^2 along the base S^1 it returns to the original position with the position of the holes 1 and 2 in the disk interchanged.

From the point of view of homotopy theory, in type 3 we have a direct product of a figure eight and the circle, while in type 4 the figure eight moves along S^1 in such a way that it is turned over after a complete revolution. A small neighborhood of the base S^1 is homeomorphic (in type 4) to two Möbius strips which intersect along a common axis. It is clear that $\partial A^3 = T^2 \cup T^2$ and that A^3 can be realized in $\mathbf{R}^3$. Consider the usual solid torus from which a thin solid torus, which winds twice around the axis of the bigger torus, has been drilled (Figure 1). We will call A^3 a *nonorientable saddle.* From the topological point of view, a manifold of type 4 is not new. *It is obtained by pasting trousers to a solid torus by a diffeomorphism of the torus.* By convention, we shall write $A^3 = \mathrm{I} + \mathrm{III} = (S^1 \times D^2) + (N^2 \times S^1)$.

Type 5. Let K^2 be a Klein bottle and K^3 the total space of the oriented twisted product of K^2 with the interval; that is, $K^3 = K^2 \tilde{\times} D^1$. Then $\partial K^3 = T^2$.

From a topological point of view, the manifold K^3 is not new either. *It can be obtained by pasting together manifolds of the types already mentioned* (see below for the proof): $K^3 = \mathrm{I} + \mathrm{IV} = (S^1 \times D^2) + A^3 = 2(\mathrm{I}) + \mathrm{III} = 2(S^1 \times D^2) + (N^2 \times S^1)$.

Thus, *only the first three of the five types of manifolds listed above are topologically independent.* The last two are combinations of types 1–3. However, the manifolds A^3 and K^3 are of independent interest for the study of trajectories of the system v.

THEOREM 2 (the topological classification of three-dimensional surfaces of constant energy in integrable systems). *Let M^4 be a smooth symplectic manifold (compact or noncompact) and $v = \operatorname{sgrad} H$ a Hamiltonian system which is Liouville integrable on a nonsingular compact three-dimensional surface Q of constant energy by means of a Bott integral f. Let m be the number of stable periodic solutions of the system v on Q (on which the integral f attains a strict local minimum or maximum), p the number of two-dimensional critical tori of the integral f (minima or maxima of the integral), q the number of critical circles of the integral f (unstable trajectories of the system) with orientable separatrix diagram, s the number of critical circles of the integral f (unstable trajectories of the system) with nonorientable separatrix diagram, and r the number of critical Klein bottles (minima or maxima). This is a complete list of all possible critical submanifolds of the integral f on Q. Then Q can be represented as the result of pasting together (by certain diffeomorphisms of the boundary tori) the following "elementary blocks": $Q = m\,\mathrm{I}+p\,\mathrm{II}+q\,\mathrm{III}+s\,\mathrm{IV}+r\,\mathrm{V} = m(S^1\times D^2)+p(T^2\times D^1)+q(N^2\times S^1)+sA^3+rK^3$. If the integral f is orientable, then the last summand does not appear; that is, $r = 0$.*

In the canonical representation of Q above, the numbers m, p, q, s, and r have a precise interpretation: they tell us how many critical submanifolds of each type a given integral f has on Q. This decomposition of Q will be said to be the *Hamiltonian* decomposition. If, however, we ignore the interpretation of the numbers m, p, q, s, r and pose the question of the simplest *topological* representation of Q, we get Theorem 3.

THEOREM 3. *Let Q be a compact nonsingular surface of constant energy of the system $v = \operatorname{sgrad} H$, and suppose that the system has a Bott integral on Q. Then Q admits the following topological representation: $Q = m'\mathrm{I} + p'\mathrm{II} + q'\mathrm{III} = m'(S^1 \times D^2) + p'(T^2\times D^1)+q'(N^2\times S^1)$, where m', p', and q' are nonnegative integers. They are related to the numbers in Theorem 2 as follows: $m' = m + s + 2r$, $p' = p$, and $q' = q + s + r$.*

We call the above representation the *topological* decomposition of Q. We now give a complete classification of all surgery of Liouville tori which can arise by changing the value of the integral f. Interchanging the roles of f and H, we could speak of bifurcations of Liouville tori *as they intersect a critical level of the energy H* for a fixed second integral f. We shall consider five types of surgery of the torus T^2 corresponding to the manifolds I, II, III, IV, V discussed above. We realize T^2 as one of the components of the boundary of the corresponding manifold. This torus is carried along by changing the value of the integral f and transforms into the union of tori which are the remaining components of the boundary.

1) The torus T^2 contracts to a circle and "disappears" from the level surface of the integral f; $T^2 \to S^1 \to 0$.

2) Two tori merge into one and "disappear"; $2T^2 \to T^2 \to \varnothing$.

3) The torus T^2 splits into two tori T^2 which "remain" on a level surface of the integral f; $T^2 \to 2T^2$.

4) The torus T^2 winds around a torus T^2 twice and remains on a level surface of the integral f; $T^2 \to T^2$.

5) A torus T^2 turns into a Klein bottle (covering it twice) and "disappears".

We do not consider the surgery obtained from the above by replacing the arrows by reverse ones to be new.

THEOREM 4 (Classification of bifurcations of two-dimensional Liouville tori). *Suppose that f is a Bott integral on a nonsingular surface Q^3. Then any surgery of general position of a Liouville torus which occurs in passing through a critical level surface of f is a composition of the elementary surgeries* 1–5 *listed above. Only the first three of these five types are independent (from a topological point of view). Surgeries* 4 *and* 5 *split into a composition of surgeries of the forms* 1 *and* 3.

After the preparation of this paper for publication, some new results, which bear directly on the results above, were obtained. We restrict ourselves to merely formulating the results. We shall consider four classes of closed compact three-dimensional manifolds. Class (H) consists of surfaces Q^3 of constant energy in Hamiltonian systems which are integrable (by means of a Bott integral). Class (Q) consists of manifolds of the form $m(S^1 \times D^2) + p(T^2 \times D^1) + q(N^2 \times S^1)$. As shown by the author in Theorem 3, there is an inclusion $(H) \subset (Q)$.

ASSERTION 2 (A. V. BRAILOV and A. T. FOMENKO). *The equality $(H) = (Q)$ holds; that is, any manifold which can be obtained by pasting together solid tori, cylinders and "trousers" can be realized as an isoenergetic surface of an integrable (by means of a Bott integral) Hamiltonian system on some manifold M^4.*

In [24], based on intrinsic problems of three-dimensional topology, Waldhausen introduced a class (W) of three-dimensional manifolds, which he called *Graphenmannigfaltigkeiten (graphmanifolds)*. A. T. Fomenko and Heiner Zieschang have shown that $(W) = (Q)$. Recently, in the course of further developing the ideas in this paper, A. B. Burmistrova and S. V. Matveev considered the class (S) of three-dimensional manifolds on which there exists a smooth function g all of whose critical points are organized into nondegenerate circles and all of whose nonsingular level surfaces g are unions of tori. They proved that $(S) = (Q)$. Putting together these results gives the following theorem.

ASSERTION 3. *The four classes of three-dimensional manifolds described above coincide; that is, $(H) = (Q) = (S) = (W)$.*

It turns out that the number of critical submanifolds of the integral f on Q can be bounded below by a universal constant which depends only on the group $H_1(Q, \mathbf{Z})$. Let $\beta = \operatorname{rank} H_1(Q, \mathbf{Z})$, and let ε be the number of elementary factors in the finite part $\operatorname{Tor}(H_1)$ of $H_2(Q, \mathbf{Z})$. If $\operatorname{Tor}(H_1)$ is written as an ordered sum of subgroups with the order of each subgroup dividing the order of the preceding, then ε is the number of such summands.

ASSERTION 4 (A. T. FOMENKO and HEINER ZIESCHANG). *Let $Q \in (H)$; that is, suppose that Q^3 is an isoenergetic surface of a system which is integrable (by means of a Bott integral). Let m be the number of stable periodic solutions of the system on Q, s the number of unstable periodic solutions with nonorientable separatrix diagram, and r the number of critical Klein bottles. Then $m' = m + s + 2r \geq \varepsilon - 2\beta + 1$ for $q + s > 0$ and $p + m > 0$; and $\varepsilon - 2\beta \leq 0$ for $p = 1$ and $m = r = 0$ (so $s = q = 0$). If the integral f is orientable and all separatrix diagrams of the critical submanifolds are also orientable, then $s = r = 0$; in particular, we obtain an estimate for the number m of stable periodic solutions in this case: $m \geq \varepsilon - 2\beta + 1$ and $q \geq m - 2$.*

3. Surgery of Liouville tori under changes in the value of f. In what follows, we suppose that the hypotheses of Theorem 2 are satisfied.

LEMMA 1. *A Bott integral f cannot have an isolated critical point on a nonsingular compact surface Q.*

It is clear that there are no critical points of the function H on Q. Therefore, each critical point x_0 of f on Q must lie on a *nondegenerate* integral curve of v consisting entirely of critical points.

LEMMA 2. *The critical points of a Bott integral f on a compact nonsingular surface Q fill out either isolated critical circles, or two-dimensional tori, or Klein bottles.*

PROOF. If the nondegenerate integral curve of v issuing from a critical point x_0 of f is closed, then it is a circle. If the integral curve is not closed, then its closure P is a two-dimensional connected set of critical points. Thus, P lies on a two-dimensional critical level surface L of f on Q. But the critical points of f are organized into nondegenerate manifolds. Thus, P lies on a two-dimensional critical submanifold P'. We claim that the intersection of L with a small neighborhood of P' in Q coincides with P'. Since f is nondegenerate along the normal to P', it is either strictly decreasing or strictly increasing in both directions along the normal to P'. Consequently, a nearby nonsingular level surface $\tilde{P}$ is a double cover of P. Because the system is integrable, $\tilde{P}$ is a Liouville torus. Thus, P is homeomorphic to either a torus or a Klein bottle. The fact of the matter is that there is a nonzero field $\operatorname{sgrad} H$ on P.

Let S^1 be a critical circle of the integral f. We define the *index* of the critical circle to be the index, 0, 1, or 2, of the restriction of f to a two-dimensional disk normal to the circle. A critical circle of index 0 consists of local minima of the integral, one of index 2 consists of local maxima, and one of index 1 consists of saddles.

LEMMA 3. *A critical circle of the integral f can have index* 0, 1 *or* 2, *while a critical torus or Klein bottle can only have index* 0 *or* 1.

The proof is obvious.

We now study surgery of the level surface $B_a = \{x \in Q\colon f(x) = a\} = f^{-1}(a)$ of f as a increases. We put $C_a = \{x \in Q\colon f(x) \leq a\}$. It is clear that $B_a = \partial C_a$. If a is not a critical value, then B_a is a union of Liouville tori.

DEFINITION 4. We define a *circular handle* to be the direct product of a circle and two-dimensional disk on whose boundary two connected nonintersecting arcs l_1 and l_2 are distinguished. A circular handle is a *thickened cylinder with bases (foundations)* $l_1 \times S^1$ and $l_2 \times S^2$.

We define the *operation of attaching a circular handle* to a manifold C_a. Let γ_1 and γ_2 be two nonintersecting and nonselfintersecting circles in B_a, and let N_1 and N_2 be small tubular neighborhoods of them. Since B_a is orientable, these neighborhoods are homeomorphic to $S^1 \times D^1$. We attach the circular handle to B_a by identifying the annuli $l_1 \times S^1$ with N_1 and $l_2 \times S^1$ with N_2 by homeomorphisms. We obtain a new manifold (after smoothing corners). The circles γ_1 and γ_2 are called the *axes of the bases of the circular handle.* We let $\operatorname{sd}(S^1)$ denote the separatrix diagram of the critical circle S^1.

LEMMA 4. *Let a be a critical value of the integral f. Suppose that there is exactly one critical saddle circle S^1 on a critical level surface B_a. Let $\varepsilon > 0$ be such that the interval $[a - \varepsilon, a + \varepsilon]$ contains no critical values of f other than a.*

1) *If $\operatorname{sd}(S^1)$ is orientable, then $C_{a+\varepsilon}$ is obtained from $C_{a-\varepsilon}$ by attaching a circular handle to the boundary of $C_{a-\varepsilon}$. In this case, $C_{a+\varepsilon}$ is homotopy equivalent to a copy of $C_{a-\varepsilon}$ to which both ends of the cylinder $S^1 \times D^1$ have been attached.*

2) *If $\operatorname{sd}(S^1)$ is nonorientable, then $C_{a+\varepsilon}$ is homotopy equivalent to a copy of $C_{a-\varepsilon}$ to which a Möbius strip (that is, a "thick Möbius strip") has been attached.*

PROOF. 1) At each point x of the saddle circle S^1, let $D^2(x)$ be a normal disk of small radius ε. Consider the vector field $\operatorname{grad} f$ on Q. From each point $x \in S^1$ we initiate the separatrices of the vector field grad f. Their union is the set $\mathrm{sd}(S^1)$. Because f is nondegenerate on $D^2(x)$, the separatrix diagram at each $x \in S^1$ is hyperbolic (a saddle). Varying x along S^1, we smoothly deform the separatrices in $D^2(x)$. Consider the intersection $P_- = P^2_-$ of the "entering" part of the separatrix diagram of the circle with the layer $a - \varepsilon \leq f \leq a$. Since ε is small, P_- is a two-dimensional manifold containing S^1 and having boundary homeomorphic to either S^1 or $S^1 \cup S^1$. In the former case P_- is homeomorphic to a Möbius strip; in the latter, to a cylinder $S^1 \times D^1$. Since we are assuming, for the time being, that $\mathrm{sd}(S^1)$ is orientable, we can rule out the case of a Möbius strip. Consequently a tubular neighborhood of P_- is homeomorphic to a circular handle. It is attached to $C_{a-\varepsilon}$ precisely as required by the definition of the operation of attaching a circular handle. The axes of the two bases of the handle are attached to the two circles γ_1 and γ_2 which are traced out in $B_{a-\varepsilon}$ by the ends of the normal interval (of the separatrix diagram) as x slides around S^1.

2) If $\mathrm{sd}(S^1)$ is nonorientable, then it is clear that a "thick Möbius strip" is attached instead of a "thick cylinder". The lemma is proved.

A circular handle can be attached to $B_{a-\varepsilon}$ in only two ways: to a single torus or to two different tori. A "thick Möbius strip" can only be attached to a single torus.

LEMMA 5. *Suppose that exactly one critical saddle circle S^1 lies at the critical level B_a.*

1) *Suppose that* $\mathrm{sd}(S^1)$ *is orientable. Consider the circular handle corresponding to S^1 that is attached by both bases to $B_{a-\varepsilon}$. Then each of the bases lies on a Liouville torus, and the axis of each base is a nonselfintersecting circle which realizes a nonzero element of the fundamental group of the torus. If both bases are attached to the same torus, then the axes γ_1 and γ_2 of both bases of the circular handle do not intersect on the torus. Moreover, they realize the same generator of the fundamental group of the torus and are isotopic to one another on the torus.*

2) *Suppose* $\mathrm{sd}(S^1)$ *is nonorientable. Then the entering separatrix diagram P_-, homeomorphic to a Möbius strip, is attached by its boundary circle to a Liouville torus on which this nonselfintersecting circle realizes one of the generators of the fundamental group of the torus.*

Recall that a tubular neighborhood of P_- in the nonorientable case is called a thickened (or thick) Möbius strip.

LEMMA 6. 1) *Suppose that* $\mathrm{sd}(S^1)$ *is orientable. If a circular handle is attached to two different tori, then these tori merge into one torus under passage through the critical saddle circle S^1 corresponding to the handle. But if the handle is attached to a single torus, then the torus splits into two tori upon passing through the critical circle.*

2) *Suppose that* $\mathrm{sd}(S^1)$ *is nonorientable. Then the torus to which a Möbius strip (in the form of a diagram P_-) is attached transforms anew into a single torus upon passage through a critical saddle circle.*

PROOFS OF LEMMAS 5 AND 6. Suppose that $\mathrm{sd}(S^1)$ is orientable. Consider the first case when the handle is attached to different tori T_1 and T_2. Let γ_1 and γ_2 be simple axes of the bases of the handles and let S^1 be a critical saddle circle at the level B_a. Let $T_{1,\varepsilon}$ and $T_{2,\varepsilon}$ be two tori in $B_{a-\varepsilon}$ that are two of the connected components of $B_{a-\varepsilon}$ (we disregard the remaining components, which do not interest us at the moment). Let γ_1^ε and γ_2^ε be the two circles on $T_{1,\varepsilon}$ and $T_{2,\varepsilon}$ cut out by the entering diagram P_-. We may assume that $T_i = T_{i,\varepsilon_0}$ and $\gamma_i = \gamma_i^{\varepsilon_0}$, $i = 1, 2$, for some sufficiently small fixed number ε_0. As ε decreases, the circles γ_1^ε and γ_2^ε approach one another and merge at $\varepsilon = 0$, becoming a critical saddle circle. Because f is a Bott integral, the circles γ_1^ε and γ_2^ε come together in a hyperbolic way. Thus, we may assume that there are sufficiently small tubular neighborhoods U_1 of γ_1^ε and U_2 of γ_2^ε on T_1 and T_2, respectively, in which the circles γ_1^ε and γ_2^ε "move" without leaving the confines of U_1 and U_2 where γ_i^ε is isotopic to γ_i. The tori $T_{i,\varepsilon}$ are canonically identified by the diffeomorphism along integral curves of the vector field $\operatorname{grad} f$ with the fixed torus $T_i = T_{i,\varepsilon_0}$. Let M be a point of γ_1^ε, and initiate the integral curve τ of the field v from it. Two cases can occur: a) it is closed, or b) it is not closed. In case a), the circle τ lying on $T_{1,\varepsilon}$ is close to the closed trajectory S^1 if ε is small. In this case τ is closed on the nonsingular torus $T_{1,\varepsilon}$. Since the system is integrable, the assertions of Liouville's theorem apply to the torus $T_{1,\varepsilon}$. On each of the tori $T_{i,\varepsilon}$ there exist coordinates with respect to which the field v on the torus determines a conditionally periodic motion. Since the field v has a closed trajectory τ on $T_{1,\varepsilon}$, all its trajectories must be closed. Thus the simple trajectory τ, in making a full revolution on the torus, realizes a nontrivial element of $\pi_1(T_{1,\varepsilon})$ because the trajectory is induced by a rectilinear winding. But τ is arbitrarily close to γ_1^ε, since both are close to S^1. Thus, γ_1^ε also realizes a nonzero element of $\pi_1(T_{1,\varepsilon})$ and, therefore, γ_1 realizes a nonzero element of $\pi_1(T_1)$. A similar argument shows that γ_2 realizes a nonzero element of $\pi_1(T_2)$ in case a). We now consider case b), in which the trajectory τ is not closed in $T_{1,\varepsilon}$. By decreasing ε, we can assume that τ passes arbitrarily close to the saddle circle S^1 over any prescribed (but fixed) time interval. Therefore, by decreasing ε we can ensure that τ returns after a certain time (for the first time) to a small neighborhood of the point M. Let $M_1 \in \tau$ be the point of return near M in Q. In addition, the trajectory τ does not leave a neighborhood U_1 of the circle γ_1^ε on $T_{1,\varepsilon}$. Consequently, τ performs a single revolution on $T_{1,\varepsilon}$ and returns again to a small neighborhood of M. Here we have used the orientability of the separatrix diagram. Joining the points M and M_1 by a small geodesic segment on $T_{1,\varepsilon}$ we obtain a new closed trajectory τ' from τ which lies entirely in U_1. Because it is obtained by a small closing of an almost periodic trajectory which makes a full turn around the torus and returns to a point close to the original, it follows from Liouville's theorem that τ' realizes a nonzero element in $\pi_1(T_{1,\varepsilon})$. Since the simple trajectory τ' is arbitrarily close to γ_1^ε, we have that γ_1^ε and, consequently, γ_1 realize nonzero elements of $\pi_1(T_{1,\varepsilon})$ and $\pi_1(T_1)$, respectively. Thus, if the handle is attached to two different tori, Lemma 5 is proved in the orientable case. We now suppose that the handle is attached to one torus (case 2)). We may assume that the tori $T_{1,\varepsilon}$ and $T_{2,\varepsilon}$ (see above) coincide. Let $T_{-\varepsilon}$ denote this torus. We obtain circles γ_1^ε and γ_2^ε on $T_{-\varepsilon}$. Decreasing ε deforms these circles slightly within prescribed (final) neighborhoods U_1 and U_2 in such a way the circles do not leave the neighborhoods. We may clearly assume that U_1 and U_2 are disjoint on the torus, because γ_1^ε and γ_2^ε fuse in a Morse way. Now consider the cases a) and b). In case a) (closed trajectory τ) the proof is word for word the same as above. In case b), the trajectory τ is not closed. But decreasing ε, we may achieve that after a certain time the trajectory returns (for the first time) to a point M_1 in a small neighborhood of M. We need to show that M_1 is close to M *in the topology of the torus* $T_{-\varepsilon}$. It follows from Lemma 4 that the torus $T_{-\varepsilon}$ intersects a tubular neighborhood of S^1 in two neighborhoods: U_1 and U_2. We may assume that $U_1 \cap U_2 = \varnothing$. But then, in one revolution, the trajectory τ does not leave U_1 and cannot meet U_2, since τ moves close to γ_1^ε during the first revolution, all the while remaining in U_1. Thus, in case 2), both circles γ_1^ε and γ_2^ε again realize nontrivial cycles on the torus $T_{-\varepsilon}$. They do not intersect because the separatrices of the critical points do not intersect outside the critical points. Then the two circles are isotopic. Now, suppose that the diagram $\operatorname{sd}(S^1)$ is nonorientable; that is, suppose, that P_- is homeomorphic to a Möbius strip. Here, after the first revolution along the circle S^1, *the point* M_1 *will not* *be close to the point M in the topology of the torus $T_{-\varepsilon}$.* The trajectory τ only performs *a half revolution* over the torus $T_{-\varepsilon}$. After making one more revolution along S^1, the trajectory τ concludes its motion on $T_{-\varepsilon}$, returning to a point M_2 close to M in the topology of $T_{-\varepsilon}$. The rest of the argument, clearly, is the same as in the orientable case. This completes the proof of Lemma 5.

We turn to the proof of Lemma 6. We begin with the orientable case. Suppose that the circular handle corresponding to the saddle circle is attached to different tori T_1 and T_2 by annuli whose axes are noncontractible (in view of Lemma 5) circles γ_1 and γ_2, respectively. Cutting the tori along these circles, we obtain two annuli (from each torus). Upon attaching the circular handle and examining the boundary of the resulting manifold, we obtain a single torus in the capacity of the top component of the boundary. Turning the argument around, we obtain surgery of two tori into one. In the nonorientable case, both components P_+ and P_- of the separatrix diagram $P = \mathrm{sd}(S^1)$ are homeomorphic to a Möbius strip. Consequently, the boundary of each is connected and homeomorphic to a circle. Hence the torus $T_{-\varepsilon}$ is transformed into a single torus $T_{+\varepsilon}$. This proves Lemma 6.

In proving Lemmas 5 and 6, we used the fact that the saddle circle is an integral curve of the vector field v. Had we restricted our attention instead to a Bott function f whose nonsingular level surfaces are two-dimensional tori (that is, had we dispensed with the requirement that f be an integral), then there would have been a larger number of variants of surgery of tori into tori. There are four possible dispositions of two nonintersecting, nonselfintersecting circles γ_1 and γ_2 on the torus: 1) γ_1 and γ_2 are noncontractible. Then they realize a nontrivial cycle on the torus and are isotopic. 2) γ_1 is noncontractible and γ_2 is contractible. 3) γ_1 and γ_2 are contractible and lie outside one another. 4) γ_1 and γ_2 are contractible and γ_2 lies inside γ_1. Attaching a circular handle in each of the above cases 1)–4) gives rise to a transformation of a torus into: 1) *two tori*, 2) *a single torus*, 3) *a pretzel* (i.e., a sphere with two handles) *and a sphere*, and 4) *two tori*, respectively. Thus we would have qualitatively new cases of surgery of a torus. But cases 2), 3), and 4) are forbidden by Lemmas 5 and 6.

Thus, we discard "75 percent" of all possible cases of surgery of tori and retain only case 1), i.e., those "25 percent" permitted by Liouville's theorem.

4. The description of isoenergetic surfaces of an integrable system. Consider the integral f on Q. Two cases are possible: A) the function f has at least one critical saddle circle S^1 on Q, or B) the function f has no saddle circles. We begin with case A). Let $\mathrm{sd}(S^1)$ be orientable. Consider the two nearby noncritical surfaces $B_{a-\varepsilon}$ and $B_{a+\varepsilon}$. According to Lemma 6, the circle S^1 generates either a splitting of one torus into two or a merging of two tori into one. By replacing f by $-f$, if necessary, we may assume that we study a splitting of one torus into two. Let $U(S^1)$ be the connected component in the layer $a - \varepsilon \leq f \leq a + \varepsilon$ which contains S_1. Its boundary consists of one torus $T_{-\varepsilon}$ in $B_{a-\varepsilon}$ and two tori $T_{1,\varepsilon}$, $T_{2,\varepsilon}$ in $B_{a+\varepsilon}$. The torus $T_{-\varepsilon}$ splits into the tori $T_{1,\varepsilon}$ and $T_{2,\varepsilon}$ under passage through the critical value a. The separatrix diagram P_- leaves the circle S^1 and meets the torus $T_{-\varepsilon}$ along the circles γ_1 and γ_2. It follows from Lemmas 5 and 6 that γ_1 and γ_2 are the boundaries of an annulus of $T_{-\varepsilon}$. We denote by K_1 and K_2 the two annuli into whose union γ_1 and γ_2 partition the torus $T_{-\varepsilon}$. We construct a surface P_1 by appending the annulus K_1 to the diagram P_- and a second surface P_2 by appending K_2 to P_-. It is clear that P_1 and P_2 are homeomorphic to a torus.

LEMMA 7. *Suppose that* $\mathrm{sd}(S^1)$ *is orientable. Then the tori P_1 and P_2 are isotopic in the manifold $U(S^1)$ to the tori $T_{1,\varepsilon}$ and $T_{2,\varepsilon}$, respectively. The circles γ_1, γ_2, and*

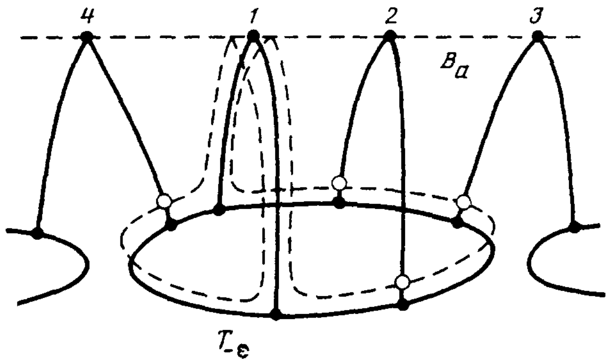

FIGURE 2

S^1 lie on each of the tori P_1 and P_2, and realize generating cycles on them (*they do not intersect and are not contractible*).

PROOF. We construct an isotopy of the torus $T_{i,\varepsilon}$ onto the torus P_i by using the standard Morse theoretic arguments to deform the level surface along the integral curves of the vector field grad f until it lies in a small neighborhood of the separatrix diagram. After this, we contract along the normals to the separatrix diagram. The remaining assertions follow from Lemmas 5 and 6. The lemma is proved.

We assumed above that only one critical saddle circle lies at each critical level B_a. We now consider the general case when several such circles lie on B_a.

LEMMA 8. *It is always possible to assume that there exists exactly one critical saddle circle on each critical level B_a : that is, circular handles or thick Möbius strips can always be attached sequentially instead of simultaneously.*

PROOF. The analogue of this lemma in the usual Morse theory is well known. However, the proof is more delicate in our case because we are dealing with an *integral* and not just a smooth function. We consider a surface $B_{a-\varepsilon}$ which is a union of tori. Suppose that *several* separatrix diagrams of saddle circles meet a single one of these tori. Each diagram intersects the torus $T_{-\varepsilon}$ in a circle, and the circles corresponding to different diagrams do not intersect. The above discussion applies to each of these circles. Consequently, each of them is the axis of a narrow annulus along which the base of a circular handle, or a thick Möbius strip, is attached. It is evident from Figure 2 (which depicts, for simplicity, only the orientable case) that all circular handles and thick Möbius strips can be considered to be attached *independently of one another* to the torus $T_{-\varepsilon}$. Suppose, for example, that the diagrams of saddle circles 1–4 have "left" the torus $T_{-\varepsilon}$ (schematically depicted by dots in Figure 2). We can first perform the surgery of level surfaces along circle 1 only. As a result, the torus $T_{-\varepsilon}$ splits into two tori depicted by dashed lines in Figure 2. The intersections of the dashed-line tori with the former diagrams of the circles 2, 3, and 4 have the same topological type as before (noncontractible generators on the torus). We may assume that the dashed-line tori are close to the torus $T_{-\varepsilon}$ joined with the diagram P_- of circle 1. Thus, the former diagrams cut out isotopic generators on them (indicated by open circles on Figure 2). We can now perform the next surgery along circle 2, and so on. Thus, we may assume that only one circular handle or one thick Möbius strip is attached at each step. It is important in our case that the circular handles and thick Möbius strips are not attached to one another (that would complicate the picture), but to the same nonsingular torus (in different places) or to a small translate of it upwards along the integral curves of the field grad f. In addition, the nonselfintersecting, nonintersecting circles on the torus (in any number) realizing nonzero cycles are pairwise isotopic (and, in fact, realize the same cycle). We can now deform the function f suitably, so that exactly one saddle circle remains at each critical level. Under this deformation, the function f stops being an integral of the field v. However, we have already used all the properties of integrals that will be needed in the proofs of Lemma 5, Lemma 6, and the present lemma. Therefore, we can now treat f as an ordinary function. The lemma is proved.

Suppose that $0 \leq f \leq 1$ on Q. We may assume that all local minima of f are situated at the single level $f^{-1}(0)$ and all local maxima at the single level $f^{-1}(1)$. This can be arranged by changing f in neighborhoods of the minimal and maximal submanifolds only. *It is very convenient to give the manifold Q in the form of a one-dimensional graph* $\Gamma = \Gamma(Q, f)$. We depict the nonsingular two-dimensional Liouville tori by *ordinary points* (one point for each torus). Since each nonsingular layer of the function is now depicted by a set of points (with cardinality equal to the number of Liouville tori), changing the value of f forces the points to shift. As a result, we obtain a one-dimensional graph starting on the plane $\{f = 0\}$ and terminating on the plane $\{f = 1\}$. In addition: 1) We denote a minimal (maximal) circle for f by a *large black dot* (*black circle*) with a single outgoing (incoming) edge of the graph. 2) We denote a minimal (maximal) two-dimensional torus by an *open circle* with two outgoing (incoming) edges of the graph. 3) A connected tubular neighborhood of a critical saddle circle with orientable separatrix diagram P_- will be denoted by a *trefoil* (a "*tripod*"), that is, by a point at which three edges of the graph meet. 4) A tubular neighborhood of a critical saddle circle with nonorientable separatrix diagram P_- will be denoted by an *asterisk* (with an incoming and an outgoing edge). 5) A minimal (maximal) critical Klein bottle will be denoted by a *circle with a dot inside it* with an outgoing (incoming) edge of the graph. In the general case, the graph $\Gamma(Q, f)$ does not depend on f.

Each trefoil (tripod) describes either the splitting of a torus into two tori, or the merging of two tori into one. The indexing of the different vertexes of a graph (see 1–5) chosen above is not fortuitous. It turns out that *there is a one-to-one correspondence between the five types of vertices and the five types of elementary "blocks" of Theorem* 2. Namely, a small neighborhood of a vertex of type i (where $i = 1, 2, 3, 4, 5$) on the graph depicts an elementary manifold of type i in Theorem 2. It is clear that *there is a well-defined continuous map* (*projection*) *of the manifold Q onto the graph* Γ. Let L be a critical submanifold of Q and let $B_{a+\varepsilon}$ and $B_{a-\varepsilon}$ be nearby nonsingular level surfaces. Let $U(L)$ be the connected component of the layer between $B_{a+\varepsilon}$ and $B_{a-\varepsilon}$ containing L.

LEMMA 9. 1) *Let $L = S^1$ be a saddle circle, and suppose that its separatrix diagram P_- is orientable. Then the manifold $U(S^1)$ with boundary $T_{1,\varepsilon} \cup T_{2,\varepsilon} \cup T_{-\varepsilon}$ is homeomorphic to a direct product $N^2 \tilde{\times} S^1$, where N^2 is a two-dimensional disk with two holes.*

2) *If the diagram P_- is nonorientable, then $U(S^1)$ is homeomorphic to the manifold A^3, i.e. to the total space of a nontrivial fibration $A^3 \to S^1$ with fiber S^1; that is, $U(S^1) = N^2 \tilde{\times} S^1$ (a twisted product).*

3) *Let $L = S^1$ be a maximal (or minimal) circle. Then $U(S^1) = S^1 \times D^2$ (a solid torus).*

4) *Let $L = T^2$ be a maximal or minimal torus. Then $U(T^2) = T^2 \times D^1$ (a cylinder).*

5) *Let $L = K^2$ be a maximal (or minimal) Klein bottle. Then $U(K^2) = K^3 = K^2 \tilde{\times} S^1$ (a twisted product).*

PROOF. We begin with the orientable case. It follows from Lemma 5–7 that $U(S^1)$ contains two tori, $P_1 = P_- \cup K_1$ and $P_2 = P_- \cup K_2$, isotopic to the tori $T_{1,\varepsilon}$ and $T_{2,\varepsilon}$, respectively. We shall fix the isotopies. The rising (up along grad f) part of the separatrix diagram of the circle S^1 cuts out the generators γ_1^ε on $T_{1,\varepsilon}$ and γ_2^ε on $T_{2,\varepsilon}$. The isotopy of $T_{1,\varepsilon}$ down onto P_1 carries the circle γ_1^ε into S^1. The isotopy of $T_{2,\varepsilon}$ onto P_2 carries the circle γ_2^ε into S^1. Given any generator on the torus, we can always choose another circle on the torus which intersects the first in only one point. The latter will also be a generator and will be said to "complement" the original one; the choice of complementary generator is not unique. We choose a second such generator on $T_{1,\varepsilon}$. Pushing it down under the isotopy of $T_{1,\varepsilon}$ onto P_1, we obtain a generator γ on P_1 complementary to S^1. The circle γ splits into two arcs: $\gamma = \overline{\gamma} \cup \gamma'$ in the annulus K_1. It is clear that the cycles $\gamma_1^{-\varepsilon}$ and $\gamma_2^{-\varepsilon}$ are isotopic on the torus $T_{-\varepsilon}$ and partition it into a union of annuli K_1 and K_2. We supplement the arc $\overline{\gamma}$ by an arc γ'' in K_2 so that the circle $\overline{\gamma} \cup \gamma''$ is a generator of the torus P_2 complementary to the cycle S^1 and the circle $\tau^{-\varepsilon} = \gamma' \cup \gamma''$ is a generator on the torus $T_{-\varepsilon}$ complementary to $\gamma_1^{-\varepsilon}$ (or, equivalently, to $\gamma_2^{-\varepsilon}$). Recall that the diagram P_- is homeomorphic to a cylinder for small ε. It is clear that the cycle $\overline{\gamma} \cup \gamma''$ is a generator on P_2 complementary to S^1. The isotopy of P_2 onto $T_{2,\varepsilon}$ carries the generator S^1 into γ_2^ε on $T_{2,\varepsilon}$ and the cycle $\overline{\gamma} \cup \gamma''$ into the curve τ_2^ε which is a generator of $T_{2,\varepsilon}$ complementary to γ_2^ε. Thus, we have constructed a coordinate system on $U(S^1)$. We take N^2 to be the surface swept out by the circles $\overline{\gamma} \cup \gamma'$ and $\overline{\gamma} \cup \gamma''$ under the isotopies of the tori P_i onto $T_{i,\varepsilon}$, where $i = 1, 2$. It is clear that $N^2 \cap T_{1,\varepsilon} = \tau_1^\varepsilon$ and $N^2 \cap T_{2,\varepsilon} = \tau_2^\varepsilon$, where τ_i^ε is a generator of $T_{i,\varepsilon}$. Furthermore, $N_2 \cap T_{-\varepsilon} = \tau_{-\varepsilon} = \gamma' \cup \gamma''$, where $T_{-\varepsilon} = K_1 \cup K_2$. It is obvious that the circles isotopic to S^1 under the indicated isotopies of P_i onto $T_{i,\varepsilon}$ give the fibers of the direct product $U(S^1) = N^2 \times S^1$. This proves the lemma in the orientable case.

We consider the nonorientable case. The boundary circle of the diagram P_- (a Möbius strip) is attached to the torus $T_{-\varepsilon}$ by its generator $\gamma^{-\varepsilon}$. We complement it by the second generator $\tau^{-\varepsilon}$ on $T_{-\varepsilon}$. Since it lies on a nonsingular torus, the normal field grad f is defined at each point of $\tau^{-\varepsilon}$. It is clearly possible to define a continuous deformation of $\tau^{-\varepsilon}$ along this field induced by a deformation of the level surfaces. Here, two variants are possible. 1) The circle $\tau^{-\varepsilon}$ first turns into a figure eight, which then splits and turns into a pair of nonintersecting circles. 2) The reverse process occurs: the circle $\tau^{-\varepsilon}$ and its copy obtained after a single circuit around S^1 first meet at a single point, forming a figure eight, which then undergoes surgery and turns into a single circle. In both cases, it is clear that the circle $\tau^{-\varepsilon}$ (or the doubled circle $\tau^{-\varepsilon}$) being deformed sweeps out a *two-dimensional disk with two holes.* After a single rotation along S^1 the disk returns to the same place with its two holes interchanged. This gives the manifold A^3. Suppose, now, that S^1 is a maximal (minimal) circle, and let $U(S^1)$ be a tubular neighborhood of S^1. Because f is a Bott integral, it follows that $U(S^1)$ is fibered into tori which shrink to S^1 and which are level surfaces of f. When K^2 is a critical Klein bottle, a small tubular neighborhood is a manifold bounded by a torus which is obviously a double cover of K^2. Therefore, $U(K^2) \approx K^2 \tilde{\times} D^1$. This completes the proof of the lemma.

LEMMA 10. *There exist homeomorphisms:* 1) $A^3 = N^2 \tilde{\times} S^1 = (S^1 \times D^2) + (N^2 \times S^1)$, *so that* A^3 *is a solid torus with trousers attached; and* 2) $K^3 = K^2 \tilde{\times} D^1 = (S^1 \times D^2) + A^3 = 2(S^1 \times D^2) + (N^2 \times S^1)$, *so that* K^3 *is obtained by sewing together two solid tori and a pair of trousers.*

PROOF. 1) Consider the point 0, which is the center of the figure eight, in Figure 1. Upon removing a tubular neighborhood of the circle β (that is, a thin solid torus), it is clear that we obtain trousers.

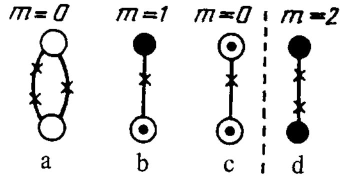

FIGURE 3

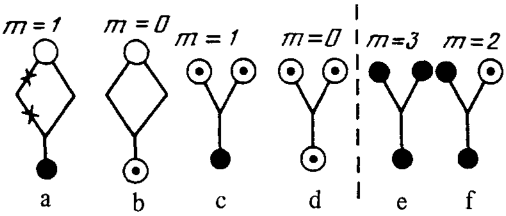

FIGURE 4

2) Consider the fibration $K^2 \to h$, where h is a meridian of the Klein bottle along which the orientation changes. The fiber of the fibration is S^1. Since K^2 is embedded in K^3 as the "zero section", we get a nontrivial fibration $p\colon K^3 \to h$ with fiber $D^1 \times S^1$ (an annulus). Consider the circle $\tilde{h}$ which is a meridian on $K^2 \subset K^3$. Upon removing a small tubular neighborhood of the circle $\tilde{h}$ (that is, a solid torus) from K^3, we obtain a manifold which is easily seen to be homeomorphic to $N^2 \tilde{\times} S^1 = A^3$. Here, $(D^1 \times S^1)\backslash D^2 = N^2$. The lemma is proved.

5. Proof of the main Theorems 2 and 3. We list all critical submanifolds in Q of the integral f. We have: 1) $m \geq 0$ maximal and minimal circles S^1, 2) $p \geq 0$ maximal and minimal tori T^2, 3) $q \geq 0$ saddle circles S^1_+ for which $\mathrm{sd}(S^1)$ is orientable, 4) $s \geq 0$ circles S^1_- for which $\mathrm{sd}(S^1)$ is nonorientable, and 5) $r \geq 0$ maximal and minimal Klein bottles. Then Q admits the Hamiltonian representation $Q = mU(S^1) + pU(T^2) + qU(S^1_+) + sU(S^1_-) + rU(K^2)$, where $U(L)$ denotes a connected regular tubular neighborhood of the critical level surface containing the submanifold L. Lemma 9 completely describes all these manifolds: $U(S^1) = S^1 \times D^2$, $U(T^2) = T^2 \times D^1$, $U(S^1_+) = N^2 \times S^1$, $U(S^1_-) = A^3$, and $U(K^2) = K^3$. It is clear that Theorem 2 follows. Theorem 3 follows from Theorem 2 and Lemma 10.

6. Proof of Assertion 1. Let $K_1^2, \ldots, K_r^2$ be a finite number of Klein bottles which are minima or maxima of f. A level surface of f near K_i^2 is homeomorphic to a torus T^2 doubly covering K_i^2. Cut out of Q^3 all the K_i^3 enveloping K_i^2 (recall that: $K_i^3 = U(K_i^2)$). Take two copies Q_+ and Q_- homeomorphic to $Q^3 \backslash \bigcup_1^r K_i^3$. It is clear that the boundary $\partial(Q_+ \cup Q_-)$ consists of $2r$ tori T^2_{i+} and T^2_{i-} obtained by doubling the tori ∂K_i^3, $1 \leq i \leq r$. We consider $2r$ copies A_i and B_i of the cylinders $T^2 \times D^1$, where we associate two cylinders $A_i = T^2_{i+} \times D^1$ and $B_i = T^2_{i-} \times D^1$ to each pair of tori T^2_{i+}, T^2_{i-}. We identify the base T^2_{i+} (respectively, T^2_{i-}) of each cylinder with the torus T^2_{i+} (respectively, T^2_{i-}) in Q_+ (respectively, Q_-). We identify the two remaining bases of the cylinders so as to get a well-defined double covering $A_i \cup B_i \to K_i^3$. This gives a new manifold $\tilde{Q} = Q_+ \cup Q_- \cup (\bigcup_1^r A_i \cup B_i)$. We have constructed a double cover $\tilde{Q} \to Q$, where Q_+ and Q_- project identically onto $Q \backslash \bigcup_1^r K_i^3$. Since the tubular neighborhood $U(Q)$ of the hypersurface Q^3 in M^4 is homeomorphic to a direct product $Q \times D^1$, it is possible to construct a double cover $\pi\colon \tilde{U}(\tilde{Q}) \to U(Q)$. Since we may assume that the projection $\pi\colon \tilde{Q} \to Q$ is smooth, the symplectic structure ω, the Hamiltonian vector field $\operatorname{sgrad} H$, and the integral f given on Q can be lifted from Q to $\tilde{Q}$. In so doing, all these objects retain their characteristic properties.

7. Proof of Theorem 1. From Theorem 2 we have the Hamiltonian decomposition: $Q = m\mathrm{I} + p\mathrm{II} + q\mathrm{III} + sA^3 + rK^3$. A corepresentation of a finitely generated group G with generators $a_1, \dots, a_n$ and relations $W_1, \dots, W_m$ will be denoted by $(a_1, \dots, a_n | W_1 = 1, \dots, W_m = 1)$. It is easy to compute the fundamental groups of the "elementary blocks". We have:

1) $\pi_1(S^1 \times D^2) = \mathbf{Z}$;

2) $\pi_1(T^2 \times D^1) = \mathbf{Z} \oplus \mathbf{Z}$;

3) $\pi_1(N^2 \times S^1) = F_2(a, b) \oplus \mathbf{Z}(c)$, where F_2 is the free group and $H_1(N^2 \times S^1) = \mathbf{Z}^3$;

4) $\pi_1(A^3) = F_2(a, b)/(ab^2 = b^2 a)$ and $H_1(A^3) = \mathbf{Z} \oplus \mathbf{Z}$;

5) $\pi_1(K^3) = \pi_1(K^2) = (a, b | \, aba^{-1}b = 1)$ and $H^1(K^3) = \mathbf{Z} \oplus \mathbf{Z}_2$.

Consider Q^3 and the graph $\Gamma(Q, f)$ corresponding to the integral f. We distinguish four cases: 1) $q = 0$ (no trousers), 2) $q = 1$, 3) $q = 2$, and 4) $q > 2$. We shall describe all possible connected graphs $\Gamma(Q, f)$ in cases 1)–3).

LEMMA 11. *If $q = 0$, then all possible connected graphs $\Gamma(Q, f)$ are exhibited in Figure* 3. *Under the condition that $m \leq 1$, the possible graphs $\Gamma(Q, f)$ are those in Figures* 3a, 3b, *and* 3c. *The number of copies of A^3 (that is, the number of asterisks on the edges of the graph) can be arbitrary.*

The proof reduces to examination of all cases. To each angle (vertex) of the graph Γ we associate two numbers: the number of free generators (indicated by the subscript ∞) and the number of all generators of the group H_1 of the corresponding elementary manifold. For example, to the solid torus (a black vertex) we associate the numbers 1_∞ and 1. Let β be the first Betti numbers and μ the number of all generators of H_1.

LEMMA 12. *If $q = 1$, then all possible connected graphs $\Gamma(Q, f)$ are exhibited in Figure* 4. *The graphs Γ for which $m \leq 1$ are those in Figures* 4a–4d. *The number s (that is, the number of asterisks) is arbitrary.*

LEMMA 13. *If $q = 2$, then all possible graphs $\Gamma(Q, f)$ for which $m \leq 1$ are exhibited in Figure* 5. *The number of asterisks is arbitrary.*

We first prove the assertions of Theorem 1 which involve the homology group $H_1(Q, \mathbf{Z})$.

LEMMA 14. 1) *Suppose that the graph $\Gamma(Q, f)$ is connected and $q = 0$. For the graph in Figure* 3a, *the group H_1 is always infinite; that is, $\beta \geq 1$. For the graph in Figure* 3b, *the group H_1 has at least one (possibly finite) generator; that is, $\mu \geq 1$ and $\beta \geq 0$. In particular, H_1 can be a finite cyclic group if $Q_0 = (S^1 \times D^2) + sA^3 + K^3$. For the graph Γ in Figure* 3c, *the group H_1 has no fewer than two independent (possibly finite) generators; that is, $\mu \geq 2$ and $\beta \geq 0$.*

2) *Suppose that the graph Γ is connected and $q = 1$. For the graphs Γ in Figures* 4a *and* 4b, *the groups H_1 are always infinite; that is, $\beta \geq 1$. For the graphs Γ in Figures* 4c *and* 4d, *we have $\beta \geq 0$ and $\mu \geq 2$ (for Figure* 4c*), and $\beta \geq 0$ and $\mu \geq 3$ (for Figure* 4d*). In particular, in the cases of Figures* 4c *and* 4d, *the group H_1 is always different from zero and cannot be finite cyclic.*

3) *Suppose that the graph Γ is connected, and $q = 2$ and $m \leq 1$. If the graph Γ contains at least one cycle (Figure* 5a*), then H_1 is always infinite; that is, $\beta \geq 1$. For the graph in Figure* 5b, *we have $\beta \geq 0$ and $\mu \geq 4$, and for the graph in Figure* 5c *we have $\beta \geq 0$ and $\mu \geq 3$. In the last two cases, H_1 is always different from zero and cannot be finite cyclic.*

4) *Suppose that the graph Γ is connected and $q > 2$. If the graph Γ contains at least one cycle, then the group H_1 is infinite. If there are no cycles, then the graph Γ is a tree. In this case $\mu \geq 1 + q > 3$, so that the group H_1 is always different from zero and cannot be finite cyclic.*

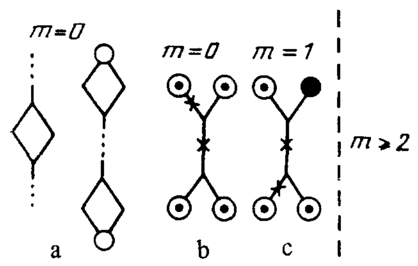

FIGURE 5

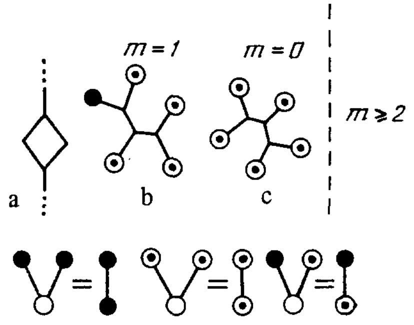

FIGURE 6

PROOF. Suppose that the complex X is obtained by gluing two subcomplexes Y and T which intersect in a connected subcomplex R. Let β_Y and μ_Y (β_T and μ_T) be the Betti number and the number of all generators for the homology group $H_1(Y, \mathbf{Z})$ ($H_1(T, \mathbf{Z})$). Let μ_R be the number of generators of $H_1(R, \mathbf{Z})$. The van Kampen theorem implies that $\beta_X \geq \beta_Y + \beta_T - \mu_R$ and $\mu_X \geq \mu_Y + \mu_T - \mu_R$. Furthermore, if the graph $\Gamma(Q, f)$ contains at least one closed cycle, then another application of the van Kampen theorem (but now to the case when the intersection R is disconnected) implies that the group $H_1(X, \mathbf{Z})$ contains at least one infinite generator. We combine these two observations to sort out the cases listed in Figures 3–5. 1) In the case of Figure 3a, the graph Γ contains a cycle; that is, $H_1 = \infty$. In the case of Figure 3b, we have

$$\beta(Q) \geq \beta(S^1 \times D^2) + \beta(K^3) + s \cdot \beta(A^3) - (s+1)\beta(T^2) = 1 + 1 + 2s - 2(s+1) = 0;$$

that is, $\beta \geq 0$. Similarly,

$$\mu(Q) \geq \mu(S^1 \times D^2) + \mu(K^3) + s\mu(A^3) - (s+1)\mu(T^2) = 1 + 2 + 2s - 2(s+1) = 1.$$

It is easy to construct an example (by choosing the diffeomorphisms of the two-dimensional boundary tori appropriately) where $H_1 = \mathbf{Z}_\alpha$. Hence, H_1 may be a finite cyclic group here, but we have $m = 1$ in the case under consideration. This happens if

$$Q_0 = (S^1 \times D^2) + sA^3 + K^3 = (s+4)(S^1 \times D^2) + (s+2)(N^2 \times S^1).$$

But if we know beforehand that Q is not diffeomorphic to Q_0, then this exceptional case (when H_1 is finite cyclic) is excluded. All the remaining cases are considered similarly.

Suppose $q > 2$ and $m \leq 1$ (Figure 6). If Γ contains a cycle, then $H_1 = \infty$. Suppose that Γ is a tree. Then only two cases are possible:

$$Q = (S^1 \times D^2) + q(N^2 \times S^1) + sA^3 + rK^3; \qquad Q = q(N^2 \times S^1) + sA^3 + rK^3.$$

In the first case we have

$$\begin{aligned}\mu(Q) &\geq \mu(S^1 \times D^2) + q\mu(N^2 \times S^1) + sA^3 + r\mu(K^3) - (r + 1 + q - 1 + s)\mu(T^2) \\ &= 1 + 3q + 2s + 2r - 2(r + q + s) = 1 + q \geq 3.\end{aligned}$$

In the second case, $\mu(Q) \geq q + 2 \geq 4$. Lemma 14 is proved.

The proof of the homological part of Theorem 1 is completed as follows. If the integral is orientable, then $r = 0$. Consequently, if we demand that H_1 be finite, the only graphs remaining in Figures 3–5 are those to the right of the vertical line; that is, $m \geq 2$. If the integral is not orientable, then the homological part of the assertion follows from Lemma 14. The inequality $m + r \geq 2$ (under the condition that H_1 be finite) is clearly equivalent to the inequality $m \geq 2$ (in the orientable case) if we count each K^3 as a solid torus (see Lemma 14). It remains only to prove the assertions of Theorem 1 connected with $\pi_1(Q)$. Suppose that $\pi_1(Q) = \mathbf{Z}$ in the case of an orientable integral. For the graphs in Figures 3b, 3c, 4c, 4d, 5b, 5c, and 6b, it follows from Lemma 14 that the group $H_1(Q, \mathbf{Z})$ has no fewer than two independent generators. Hence, $\pi_1(Q)$ has no fewer than two independent generators in this case and, hence, $\pi_1(Q) \neq \mathbf{Z}$. Thus, these graphs do not interest us. It remains to check that *if the graph $\Gamma(Q, f)$ contains at least one cycle, then $\pi_1(Q)$ has at least two independent generators.* We shall consider the case of Figure 3a; that is, two cylinders attached together. We need to study the cases given by Figures 3a, 4a, 4b, 5a, and 6a. In the case of 5a, two saddles can be attached along two (out of the three) tori in the boundary. A calculation gives $\mu \geq 3$ and, hence, $\pi_1(Q) \neq \mathbf{Z}$. If two saddles (tripods) are attached in the form of a tree (Figure 5a to the right), then it is necessary to glue in four boundary tori. Two variants are possible: two cylinders are attached (then $\beta \geq 2$ and $\pi_1 \neq \mathbf{Z}$) or one cylinder and two solid tori are attached (but $m = 2$ in this case). A similar argument shows that in the case of Figure 6a, that is, $q > 2$, we have a fortiori $\pi_1 \neq \mathbf{Z}$. It remains only to examine the case of the left part of Figure 4a and Figure 3a. It is clearly sufficient to study the case of Figure 3a, since in the left-hand part of Figure 4a gluing a solid torus into trousers gives, at worst, a commutative group with two generators, and this is exactly the case of the lower cylinder in Figure 3a. Thus, we consider the case $Q = 2(T^2 \times D^1) + sA^3$. The graph Γ contains one cycle. We denote the corresponding generator in $\pi_1(Q)$ by t. We represent Q in the following form. We cut one edge of the graph Γ and obtain a cylinder $T^2 \times D^1$ to which several nonorientable saddles (asterisks) have been attached. Suppose first that there are none; that is, $s = 0$. Let $G = \pi_1(T^2) = \mathbf{Z} \oplus \mathbf{Z}$ and let i_1 and i_2 be the embeddings of T^2 into the upper and lower boundaries of the cylinder $T^2 \times D^1$. We obtain homomorphisms $i_{1*}\colon G \to \pi_1(T^2 \times D^1) = \mathbf{Z} \oplus \mathbf{Z}$ and $i_{2*}\colon G \to \pi_1(T^2 \times D^1)$. It is clear that i_{1*} and i_{2*} are isomorphisms. Let G_1 and G_2 be images of the group G under these homomorphisms. Then $\pi_1(G)$ is obtained from the free product $\mathbf{Z}(t) * G$ by introducing the relations $ti_{1*}(h)t^{-1} = i_{2*}(h)$, where $h \in G$. For brevity, we denote the collection of all these relations by $tG_1t^{-1} = G_2$. In our case, $G_1 \approx G_2 \approx G \approx \mathbf{Z} \oplus \mathbf{Z}$ since i_1 and i_2 are homeomorphisms of tori. We can assume from the outset that $G_1 = G$; that is, i_1 is the identity and i_2 some homeomorphism. Then i_2 is given by a unimodular matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and the relations take the form $t\alpha t^{-1} = \alpha^a\beta^c$ and $t\beta t^{-1} = \alpha^b\beta^d$, where α and β are generators of the group $G_1 = G = \mathbf{Z}\oplus\mathbf{Z}$. It follows that $\operatorname{rank}\pi_1(Q) \geq 2$, where at least the generator t has infinite order. Suppose that exactly one nonoriented saddle is attached to the cylinder $T^2 \times D^1$. Proceeding as above, we obtain a lower bound for the rank of $\pi_1(Q)$. We shall find $\pi_1(\tilde{Q})$, where $\tilde{Q}$ is obtained from Q by cutting along a Liouville torus; that is, an edge is cut in the graph Γ. It is clear that $\tilde{Q}$ is homeomorphic to A^3. It follows from Figure 1 that

$$\begin{aligned} \pi_1(\tilde{Q}) &= F_2(\alpha, \gamma) * \mathbf{Z}(\beta)/(\beta\alpha\beta^{-1} = \gamma; \beta\gamma\beta^{-1} = \alpha) \\ &= F_1(\alpha) * \mathbf{Z}(\beta)/(\beta^2\alpha\beta^{-2} = \alpha) = F_2(\alpha, \beta)/(\beta^2\alpha = \alpha\beta^2). \end{aligned}$$

Here F_2 is the free group on two generators. We now compute $\pi_1(\tilde{Q})$. Let i_1 and i_2 be the embeddings of T^2 into the upper and lower boundary of $\tilde{Q} = A^3$. We assume that i_1 is the identity. Here i_{1*} and i_{2*} are the maps of $\mathbf{Z} \oplus \mathbf{Z}$ into $\pi_1(A^3)$ constructed as follows. We may assume that the i_{1*} carries the generators of $\pi_1(T^2) = \mathbf{Z} \oplus \mathbf{Z}$ to α and β^2 in $\pi_1(A^3)$. We only need to understand what happens to α and β^2 after they are carried along t. From Figure 1, clearly it is necessary to compare the generators on the two tori situated on different sides of the singular layer (which is homeomorphic to A^2 which, in turn, is a deformation retract of A^3). We can calculate that $t\alpha t^{-1} = \alpha\beta\alpha\beta^{-1}$,

$t\beta^2t^{-1} = \beta$. Thus,

$$\pi_1(Q) = F_3(\alpha, \beta, t)/(t\beta t^{-1} = \alpha\beta\alpha\beta^{-1},\ t\beta^2 t^{-1} = \beta).$$

From this we have $\operatorname{rank} \pi_1(Q) \geq 2$ where at least one generator, namely t, has infinite order in $\pi_1(Q)$. It is clear that increasing the number of nonoriented saddles in Q cannot lower the rank below two. This proves Theorem 1 in the orientable case.

Now suppose that the integral is not orientable. We shall prove that $m \geq 1$ if $\pi_1(Q) = \mathbf{Z}$. Consider the double cover $\pi\colon \tilde{Q} \to Q$ (see Assertion 1). Since $\pi_1(\tilde{Q})$ is a subgroup of index 2 in $\pi_1(Q)$, we have $\pi_1(\tilde{Q}) = \mathbf{Z}$. But now we have an orientable integral on $\tilde{Q}$ and, according to the preceding discussion, we have $\tilde{m} \geq 2$ for $\tilde{f} = \pi^* f$. Under the projection down onto Q, two stable periodic trajectories can map onto one such trajectory. Thus, $m \geq 1$. Theorem 1 is proved.

It is easy to see that *in the case of the graph in Figure* 3a, *we can arrange that* $H_1(Q, \mathbf{Z}) = \mathbf{Z}$ *by attaching the cylinder appropriately, but* $\pi_1(Q)$ *will continue to have at least two generators.* Indeed, the relations in $\pi_1(Q)$ were written out above. Passing to homology, we obtain the relations $\alpha = a\alpha + c\beta$ and $\beta = b\alpha + d\beta$; that is, $\alpha(a-1) + \beta c = 0$ and $\alpha b + \beta(d-1) = 0$. We consider the matrix

$$I = \begin{pmatrix} a-1 & c \\ b & d-1 \end{pmatrix}$$

of this system. We take the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$I = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \det I = 1 \neq 0.$$

Consequently, the manifold Q obtained from the cylinder $T^2 \times D^1$ by gluing two of its boundary tori by a diffeomorphism with matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ has $H_1(Q, \mathbf{Z}) = \mathbf{Z}$.

The manifolds A^3 and K^3 actually arise in concrete mechanical integrable systems. For example, in [26] Kharlamov exhibited topological surgery in the dynamics of a heavy rigid body equivalent to A^3. The geodesic flow of a flat metric on the Klein bottle K^2 possesses an integral surface $Q^3 = 2K^3$. Passing to the double cover, K^2 is transformed into a torus T^2 and we have $\tilde{Q}^3 = 2(T^2 \times D^1)$. Recently a student, A. Oshemkov, described the graphs Γ corresponding to integral surfaces in the dynamics of a four-dimensional rigid body (with fixed center).

We shall now prove Corollary 4. We draw on results of Waldhausen [24], to which Heiner Zieschang kindly called my attention. The manifolds W considered in [24] are given as follows. There must exist a family T of nonintersecting tori T^2 in W whose removal results in a manifold, each connected component of which fibers over a two-dimensional manifold with fiber S^1. It is easy to see that the manifolds W contain all constant energy surfaces of integrable systems (Theorems 2 and 3). This follows from Theorem 2. *Each of the "elementary blocks" described in Theorem* 2 *admits a representation required in* [24]. Furthermore, it is proved in [24] that the class of manifolds of the form W does not coincide with the class of all three-dimensional manifolds. This proves Corollary 4.

Some examples of three-dimensional manifolds (with boundary) which do not belong to the class W (and, hence, not to the class Q) were mentioned in [24]. Consider the sphere S^3 and remove a solid torus $U(K)$ whose axis is given by any knot K. If the manifold $S^3\backslash U(K)$ is of type W, then it turns out that the knot K must be related in a certain way to the so-called torus knots. We will omit the precise definitions. Since such knots comprise a "small percent" of the set of all knots, *the manifolds of type* W *(and, a fortiori, the isoenergetic surfaces of integrable systems) are "very sparse"; they fill out a subset of "measure zero" in the class of all three-dimensional manifolds.* Zieschang has remarked that the above example can be modified to obtain *closed three-dimensional manifolds which do not belong to the class* W. It is sufficient to glue up the boundary torus of the manifold mentioned above in such a way as to obtain a homology sphere.

2. The higher-dimensional case. Classification of surgery of Liouville tori in a neighborhood of bifurcation sets (of the moment map)

Let $v = \operatorname{sgrad} H$ be a smooth system on a symplectic manifold M^{2n}. Suppose that v is integrable; that is, suppose there exist n independent (almost everywhere) smooth integrals $f_1, \dots, f_n$ in involution. Let $f_1 = H$. Let $F\colon M^{2n} \to \mathbf{R}^n$ be the *moment map* corresponding to these integrals; that is, $F(x) = (f_1(x), \dots, f_n(x))$. A point $x \in M$ is said to be a *regular* point of F if $\operatorname{rank} dF(x) = n$; that is, if $dF(x)\colon T_xM \to T_x\mathbf{R}^n = \mathbf{R}^n$ is an epimorphism. Otherwise, x is said to be a *critical* point and its image $F(x)$ is called a *critical value.* Let $N \subset M$ be the set of all critical points and $\Sigma = F(N)$ the *set of critical values (the bifurcation diagram).* Because F is smooth, $\dim \Sigma \leq n-1$. If $a \in \mathbf{R}^n$ is not a critical value (that is, $a \in \mathbf{R}^n\backslash\Sigma$), then its preimage $B_a = F^{-1}(a) \subset M$ (called a nonsingular fiber) does not contain critical points of F, and therefore, by the Liouville theorem, each of its connected components is diffeomorphic to a torus T^n. Suppose, for simplicity, that the whole fiber B_a is compact. The corresponding assertions for noncompact fibers B_a follow easily from the results obtained below. If $a \in \Sigma$, then the joint level surface (that is, the fiber) B_a of the integrals is singular (critical) and $\dim B_a \leq n$. As the point a is moved in $\mathbf{R}^n$, its preimage (that is, B_a) is deformed in some way. As long as a does not meet Σ, the fiber B_a is transformed by diffeomorphisms. In particular, any two fibers B_a and B_b for which the points a and b can be joined by a smooth curve $\gamma \subset \mathbf{R}^n\backslash\Sigma$ are diffeomorphic. But if the curve γ meets the set Σ at some point, then the fibers B_a and B_b can be different. *If the point a pierces Σ then the fiber B_a undergoes topological surgery.* The following general problem arises. *Describe the topological surgery of Liouville tori arising at the moment when the point intersects the set* Σ. We will show below that *it is possible to classify the surgery in general position; they all have a relatively simple form.* It is clear that we must distinguish between two cases: 1) $\dim \Sigma < n-1$ and 2) $\dim \Sigma = n-1$. In case 1), the set Σ does not separate $\mathbf{R}^n$; that is, any two points $a, b \in \mathbf{R}^n$ can be joined by a smooth curve $\gamma \subset \mathbf{R}^n\backslash\Sigma$. Consequently, all compact nonsingular fibers are diffeomorphic to one another and, in particular, consist of the same number of Liouville tori. Case 2) is complicated. Here, Σ will, generally speaking, partition $\mathbf{R}^n$ into several open disjoint domains. Within each, the topology of the nonsingular fiber is its own. Thus, suppose $\dim \Sigma = n-1$. We fix a point c on Σ and study the surgery of Liouville tori as a smooth curve γ pierces Σ at c. It suffices to consider a small neighborhood $U = U(c)$ of the point c in $\mathbf{R}^n$. We shall study the "general position" case; that is, the case when γ pierces Σ *transversally* at a point c on an $(n-1)$-dimensional smooth stratum (sheet) of Σ; that is, we shall suppose that $U \cap \Sigma$ is a smooth $(n-1)$-dimensional submanifold of $\mathbf{R}^n$. In the general position case, we may assume that the set $N \cap F^{-1}(U)$ of critical points is a union of a finite number of smooth submanifolds of M stratified by the rank of dF. The concept of *general position* may be further sharpened as follows. Since we assume that Σ is an $(n-1)$-dimensional submanifold in a neighborhood of c, we may assume that, in the neighborhood of some *connected* component B_c^0 of the singular fiber B_c, the last $n-1$ integrals $f_2, \dots, f_n$ are independent and the first integral $f_1 = H$ (the energy) becomes dependent on them on the submanifold $N \cap B_c^0$ of critical points. Indeed, we restrict F to the submanifold $N \cap F^{-1}(U)$ which is, by the general position requirement, a union of a finite number of smooth submanifolds. Since the restriction of F to each stratum $N' \cap F^{-1}(U)$, including maximal ones, is a smooth map of a smooth submanifold, it follows that the map $dF(x)\colon T_xN' \to T_{F(x)}\Sigma$ is an epimorphism, and $\operatorname{rank} dF(x) \geq n-1$ because $\dim U \cap \Sigma = n-1$. At the same time, $\operatorname{rank} dF(x) \leq n-1$ because $x \in N$ is a critical point. Consequently, $\operatorname{rank} dF(x) = n-1$. Therefore, we may assume (by changing the basis of the set of integrals of the system if necessary) that $f_2, \dots, f_n$ are independent on B_c^0 (see [4] and [5]). Consequently, f_1 becomes dependent on them on $T = N \cap B_c^0$.

Now, we shall consider five types of $(n+1)$-dimensional manifolds whose boundaries are tori.

1) We consider the "solid torus" $D^2 \times T^{n-1}$ with the torus T^{n-1} as "axis". Its boundary is the torus T^n. We call $D^2 \times T^{n-1}$ a *dissipative solid torus* (for the terminology see the mechanical example below).

2) We consider the product $T^n \times D^1$, which we shall call a *cylinder.* Its boundary is two tori T^n.

3) Let N^2 be a disk with two holes. We shall call the direct product $N^2 \times T^{n-1}$ an *oriented toroidal saddle* (or *trousers*). Its boundary is three tori.

4) We consider all nonequivalent fibrations over T^{n-1} whose fiber is the interval $D^1 = [-1, 1]$. They are classified by elements α of the homology group $H_1(T^{n-1}, \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \cdots \oplus \mathbf{Z}_2$ ($n-1$ times). We let Y_α^n denote the total space of the fibration corresponding to α. We let T^{n-1} denote the trivial section and write $Y_\alpha \xrightarrow{D^1} T^{n-1}$. We consider fibrations $A_\alpha \xrightarrow{N^2} T^{n-1}$ with base T^{n-1} and fiber N^2. To construct them, we consider an interval D^1 on the disk N^2 which passes through the center of the disk and joins the centers of the deleted disks (holes). Then, to each fibration $Y_\alpha^n \to T^{n-1}$ we associate a new fibration $A_\alpha^{n+1} \xrightarrow{N^2} T^{n-1}$ by replacing the fiber D^1 by the fiber N^2. *In addition, we require that the boundary of the manifolds A_α^{n+1} be a union of tori.* A particular case of such a fibration is the direct product $N^2 \times T^{n-1}$; that is, a manifold of type 3) (see above). It is obtained only if $\alpha = 0$. But if $\alpha \neq 0$, the fibration A_α is nontrivial. For $\alpha \neq 0$, the boundary of the manifold Y_α^n is a torus T^{n-1}. We call A_α^{n+1} a *nonoriented toroidal saddle* if $\alpha \neq 0$. The boundary of A_α consists of two tori. It is easy to see that, *for $\alpha \neq 0$, all the manifolds A_α^{n+1} are diffeomorphic to one another.* Therefore, we will write them as follows: $A_\alpha^{n+1} = N^2 \tilde{\times} T^{n-1}$ (a twisted product). Thus, as in the four-dimensional case, we obtain only two topologically distinct manifolds: $N^2 \times T^{n-1}$ and $N^2 \tilde{\times} T^{n-1}$. However, from the point of view of Hamiltonian mechanics, the manifolds A_α should be considered independently for different α. The fact that the A_α are diffeomorphic (see above) follows from the fact that any circle realizing the cycle α (mod 2) on the torus T^{n-1} can be included in a complete system of generators of the torus T^{n-1}. The manifolds of type 4 admit a more direct description. For example, for $n = 2$, we obtain the manifold A^3 already familiar to us (see §1). A similar realization exists in the higher-dimensional case.

5) Let $p\colon T^n \to K^n$ be a double cover of the nonorientable manifold K^n. All such coverings p can be classified. We let K_p^{n+1} denote the mapping cylinder of p. It is clear that $\dim K_p^{n+1} = n+1$ and $\partial K_p^{n+1} = T^n$. These manifolds are higher-dimensional analogues of the manifolds K^3 (see §1).

LEMMA 15. *The manifolds A_α^{n+1} and K_p^{n+1} can be formed by pasting together (along the boundary tori) manifolds of the first three types; that is, from a topological point of view, the "elementary blocks" consist only of the manifolds $D^2 \times T^{n-1}$, $T^n \times D^1$, and $N^2 \times T^{n-1}$.*

This lemma is similar to Lemma 10 in §1, and so we omit the proof. We now describe the five types of surgery of the torus T^n.

1) The torus is given as the boundary of a dissipative solid torus $D^2 \times T^{n-1}$ and then contracts to its "axis" T^{n-1}. We call this operation a *limiting degeneration* and use the notation $T^n \to T^{n-1} \to \varnothing$ to denote it.

2) The two tori T_1^n and T_2^n that constitute the boundary of the cylinder $T^n \times D^1$ move along it towards one another and in the middle of the cylinder merge into one torus T^n. We denote this situation by $2T^n \to T^n \to \varnothing$.

3) The torus T^n that constitutes the lower boundary of the oriented saddle $N^2 \times T^{n-1}$ (trousers) rises upwards, and in accordance with the topology of the manifold $N^2 \times T^{n-1}$ (see above and §1) splits into two tori T_1^n and T_2^n. This will be denoted by: $T^n \to 2T^n$.

4) The torus T^n which is one of the boundaries of the manifold A_α where $\alpha \neq 0$ (realized, for example, as the boundary of the inner thin solid torus—see Figure 1) rises "upwards" through A_α and undergoes surgery in the middle, turning into a single torus that is the upper boundary of the manifold A_α. We denote this by $T^n \xrightarrow{\alpha} T^n$.

5) The torus T^n is realized as the boundary of K_p^{n+1}. Upon deforming it inside K_p^{n+1} along the projection p, the torus T^n finally doubly covers the nonorientable manifold K^n after which it "disappears". We denote this situation by $T^n \to K^n \to \varnothing$.

We now formulate the definitive definition of general position surgery of a Liouville torus. We fix the values of the last $n-1$ integrals $f_2, \dots, f_n$ and consider the resulting $(n+1)$-dimensional surface X^{n+1}. The first integral $f_1 = H$ restricts to a smooth function f on the manifold X^{n+1}.

DEFINITION 5. We say that surgery of the Liouville tori constituting a nonsingular fiber B_a is *general position surgery* if there is a neighborhood of the torus T^n undergoing surgery in which X^{n+1} is compact and nonsingular and the restriction of the energy function $f_1 = H$ to X^{n+1} is a *Bott function* (see §1).

THEOREM 5 (The classification theorem for surgery of Liouville tori). 1) *Fix an integrable system and let Σ be the bifurcation diagram of its moment map. If* $\dim \Sigma < n-1$, *then all the nonsingular fibers B_a are diffeomorphic.*

2) *Suppose that* $\dim \Sigma = n-1$. *Suppose that a nondegenerate Liouville torus T^n is moving along the joint nonsingular level surface X^{n+1} of the integrals $f_2, \dots, f_n$, carried along by the changing value of the energy integral $f_1 = H$. This is equivalent to moving the point $a = F(T^n) \in \mathbf{R}^n$ along a smooth segment γ towards Σ. Suppose that the torus T^n undergoes topological surgery at some moment of time. This happens when (and only when) the torus T^n encounters critical points N of the moment map $F\colon M^{2n} \to \mathbf{R}^n$ along its path. In other words, the path γ transversally pierces an $(n-1)$-dimensional stratum of Σ with nonzero velocity at a point c. If the surgery is general position surgery, then it is a composition of the five canonical types* 1), 2), 3), 4), 5) *of surgery listed above. In fact, from the topological point of view, only the first three types of surgery are independent; surgeries* 4) *and* 5) *are compositions of them.*

In case 1) (the surgery $T^n \to T^{n-1} \to \varnothing$) the torus T^n first turns into (that is, degenerates to) the torus T^{n-1} as the energy increases, and then disappears from the surface of constant energy $H = \text{const}$ (a limiting degeneration). In case 2) (the surgery $2T^n \to T^n \to \varnothing$) the two tori T_1^n and T_2^n first merge into one torus as the energy increases, and then disappear from the surface H = const. In case 3) (the surgery $T^n \to 2T^n$), the torus T^n "breaks through" the critical energy level as the energy H increases and splits into two tori T_1^n and T_2^n on the surface H = const (that is, it "survives" the passage through the critical level). In case 4) (the surgery $T^n \to T^n$) the torus T^n "breaks through" the critical energy level as the energy H increases and turns again into a torus T^n (a nontrivial transformation by a double winding; see above). In case 5) (the surgery $T^n \to K^n \to \varnothing$), the torus T^n double covers the nonorientable manifold K^n, after which it disappears from the surface H = const. Changing the direction of motion of a Liouville torus gives the five *inverse* processes of bifurcation of the torus T^n.

Thus, in cases 1) and 5) we may assume that the torus "colliding" with the critical energy level is "absorbed" by it. In case 2) we can also assume that a torus T_1^n "collides" with the critical energy level and is "reflected" from it in the form of a second torus T_2^n. Some of the surgery was discovered earlier in concrete mechanical systems (see Kharmalov [26] and Kharlamov and Pogosyan [15]). For example, such is the surgery of tori in the Kovalevskaya and Goryachev-Chaplygin cases (see [26] and [17]). It is possible to show that the torus surgery found in [26] and denoted there by $\varnothing \to S^1 \to T^2$, $T^2 \to R \to 2T^2$, $2T^2 \to Q \to 2T^2$, and $T^2 \to P \to T^2$ is a composition of the surgery found in Theorem 5. The first surgery in [26] is our surgery 1), the second our surgery 3), the third is the surgery generated by two oriented saddles (that is, a composition of two of our type 3) surgery operations, and the fourth is generated by a nonoriented saddle and is our type 4) surgery.

As in the four-dimensional case, we could distinguish between orientable and nonorientable Hamiltonians H. A Hamiltonian is said to be *orientable* if all its critical submanifolds (in X^{n+1}) are orientable (that is, if none of them is K^n). If we consider manifolds X^{n+1} up to a double cover, then the Hamiltonian H can always be considered to be orientable.

PROPOSITION 3. *Any integrable Hamiltonian system* $(U(X^{n+1}); \operatorname{sgrad} H; f_2, \dots, f_n)$ *with a nonorientable Hamiltonian H on X^{n+1} which is a Bott function can be doubly covered by a Hamiltonian system* $(\tilde{U}(\tilde{X}^{n+1}); \operatorname{sgrad} \tilde{H}; \tilde{f}_2, \dots, \tilde{f}_n)$ *with an orientable Hamiltonian $\tilde{H}$ on $\tilde{X}^{n+1}$.*

The proof is similar to that of Assertion 1.

Let v be an integrable system on M^{2n}. Fix values of the last $n-1$ integrals $f_2, \dots, f_n$ and suppose that the resulting $(n+1)$-dimensional surface X^{n+1} is compact and nonsingular; that is, that the integrals $f_2, \dots, f_n$ are independent on X^{n+1}. The surface X^{n+1} is an invariant submanifold of the system v. Changing the value of the energy H forces the Liouville torus T^n to move along X^{n+1}. Sometimes in the process limiting degenerations occur; that is, the torus T^n contracts to the torus T^{n-1} (see surgery 1)). As V. V. Kozlov has informed the author, limiting degenerations really occur in dissipative mechanical systems. If a small amount of friction is introduced into an integrable system, then we may assume in the first approximation that the dissipation of energy is modelled by a decrease in the value of the energy $f_1 = H$ which causes, therefore, a slow evolution (drift) of Liouville tori along the surface X^{n+1}. For $n = 1$ the surgeries $T^1 \to 2T^1$ and $T^1 \to \varnothing$ can be seen in the problem of motion of a heavy point (a bead) in a "two-humped" well. As a small amount of energy is dissipated, the motion of the point in phase space occurs over "one-dimensional tori" (circles) which evolve slightly and, finally meeting the critical energy level, undergo surgery in which the circle turns into a figure eight which then splits into two circles. Later both circles contract to points and disappear.

THEOREM 6. *Let M^{2n} be a symplectic manifold and let $v = \operatorname{sgrad} H$ be a system which is integrable by means of smooth independent commuting integrals $H = f_1, \dots, f_n$. Let X^{n+1} be a fixed, compact, nonsingular joint level surface of the last $n-1$ integrals $f_2, \dots, f_n$ (an integral surface). Suppose that the restriction of H to X^{n+1} is a Bott function. Then X^{n+1} has the form*

$$X^{n+1} = m(D^2 \times T^{n-1}) + p(T^n \times D^1) + q(N^2 \times T^{n-1}) + sA_\alpha^{n+1} + rK_p^{n+1};$$

that is, it is obtained by pasting together the "elementary blocks" described above by some diffeomorphisms of their boundary tori. The number m is the number of limiting degenerations of the system v on the surface X^{n+1}.

The nonorientable manifolds K^n which are minima or maxima of the Bott integral f on X^{n+1} can be explicitly described. This result was obtained by the author and A. V. Brailov after the present paper had been submitted for publication. Therefore, we shall only announce the result here. For each $\alpha = 0, 1$, we let G_α denote the group of transformations of the torus $T^n = \mathbf{R}^n/\mathbf{Z}^n$ generated by the involution

$$R_\alpha(a) = \begin{cases} (-a_1, a_2 + \frac{1}{2}, a_3, \dots, a_n), & \alpha = 0, \\ (a_2, a_1, a_3 + \frac{1}{2}, a_4, \dots, a_n), & \alpha = 1, \end{cases}$$

where $a = (a_1, \dots, a_n) \in \mathbf{R}^n/\mathbf{Z}^n$. Here we suppose that $n \geq 2$ for $\alpha = 0$ and $n \geq 3$ for $\alpha = 1$. The group G_α acts on T^n without fixed points. Consequently, the quotient set $K_\alpha^n = T^n/G_\alpha$ is a smooth manifold. The transformation R_α reverses orientation, so the manifold K_α^n is nonorientable. The manifolds K_0^n and K_1^n are not homeomorphic, because $H_1(K_0^n, \mathbf{Z}) = \mathbf{Z}^{n-1} \oplus \mathbf{Z}_2$ and $H_1(K_1^n, \mathbf{Z}) = \mathbf{Z}^{n-1}$. It follows from the definition of K_α^n that $K_0^n = K_0^2 \times T^{n-2}$ and $K_1^n = K_1^3 \times T^{n-3}$, where K_0^2 is the usual Klein bottle and K_1^3 is its generalization.

THEOREM 7 (A. V. BRAILOV and A. T. FOMENKO). *Let $f_1, \dots, f_n$ be a complete, involutory set of smooth functions on M^{2n} and $F\colon M^{2n} \to \mathbf{R}^n$ the moment map. Let $X^{n+1} = \{f_i(x) = c_i,\ i = 2, \dots, n\}$ be an integral submanifold. Suppose that the restriction $f = f_1|_{X^{n+1}}$ is a Bott function and $K_p^n = \{f^i(x) = c_i,\ i = 1, \dots, n\}$ is a nonorientable minimum or maximum manifold of the function f (that is, a critical fiber of the moment map). Then K_p^n is diffeomorphic to either K_0^n $(n \geq 2)$ or K_1^n $(n \geq 3)$.*

The proof will be given in another paper.

We now turn to the proofs of Theorems 5 and 6. We shall not repeat arguments which have been set forth in §1; rather we shall supplement them by pointing out the new features that are caused by the fact that the system is multidimensional. Suppose that $c \in \Sigma$ and that a small neighborhood $U(c) \cap \Sigma$ of c on Σ is an $(n-1)$-dimensional submanifold. Let γ be a path which pierces Σ transversally at the point c and joins two noncritical values a and b on different sides of Σ. Consider the connected component B_c^0 of the singular fiber $B_c = F^{-1}(c)$, and let $T = B_c^0 \cap N$. Let X_0^{n+1} be the *connected* component of $F^{-1}(U(c))$ included between two nearby nonsingular fibers B_a and B_b. Since a and b are not critical, B_a and B_b are unions of Liouville tori. Let $B_a^0 = X_0^{n+1} \cap B_a$ and $B_b^0 = X_0^{n+1} \cap B_b$; that is, $\partial X_0^{n+1} = B_a^0 \cup B_b^0$. Let $f = f_1|_{X_0^{n+1}}$.

LEMMA 16. *A point $x \in X_0$ is a critical point of f if and only if the first integral f_1 at x is dependent (on M) on the last $n-1$ integrals $f_2, \dots, f_n$.*

PROOF. The gradients of $f_2, \dots, f_n$ form a basis of the normal plane to X_0 in M. The dependence of f_1 on $f_2, \dots, f_n$ means that $\operatorname{grad} f_1$ is a linear combination of $\operatorname{grad} f_i$, $2 \leq i \leq n$. It is clear that $\operatorname{grad} f$ on X_0 is obtained by orthogonal projection of $\operatorname{grad} f_1$ onto X_0. The lemma is proved.

LEMMA 17. *The set T of critical points of f on X_0^{n+1} is a disjoint union of a (finite) number of tori T^n, tori T^{n-1}, and the n-dimensional (nonorientable) manifolds K_p^n.*

PROOF. If $T = B_c^0$, then T is the (singular) joint level surface of all n integrals. The nearby level surfaces R are nonsingular compact tori. Clearly R is the boundary of a tubular neighborhood V^{n+1} of T in X_0. If $\dim T = k$, then $R \approx T^n$ fibers over T with base S^{n-k}. This can only happen in the case when $n-k$ is 0 or 1; that is, in the case when $\dim T = n$ or $\dim T = n-1$. If T_0^n is the connected component of T, then T_0^n is either T^n or K_p^n. Suppose that $B_c^0 \neq T$. In this case, $\dim T < \dim B_c^0 = n$, so that $\dim T \leq n-1$. It follows from the condition imposed on the integrals of the system that the integrals $f_2, \ldots, f_n$ are independent on T (they are independent on all of B_c^0). Consequently, there exist $n-1$ independent commuting vector fields $\operatorname{sgrad} f_i$, $2 \leq i \leq n$, on T_0. It follows immediately that $T_0 = T^{n-1}$. The lemma is proved.

Let $P_-^n = P_-^n(T^{n-1})$ and $P_+^n = P_+^n(T^{n-1})$ be the entering and exiting separatrix diagrams, respectively, of the critical submanifold T^{n-1}.

DEFINITION 6. The direct product $T^k \times D^\lambda \times D^{n+1-k-\lambda}$ will be called a *toroidal handle of index λ and degeneracy degree k*. The part $(T^k \times S^{\lambda-1}) \times D^{n+1-k-\lambda}$ of its boundary will be called the *base* of the handle, and $T^k \times S^{\lambda-1}$ will be called the *axis* of the base.

We shall define the operation of *attaching a toroidal handle* to the boundary V^n of the manifold W^{n+1}. Suppose that the boundary contains an embedded submanifold $T^k \times S^{\lambda-1}$ and that its tubular neighborhood is homeomorphic to the direct product $(T^k \times S^{\lambda-1}) \times D^{n+1-k-\lambda}$. We can remove the tubular neighborhood; its boundary is homeomorphic to $T^k \times S^{\lambda-1} \times S^{n-k-\lambda}$. On the other hand, the boundary of the base of a toroidal handle is also homeomorphic to $T^k \times S^{\lambda-1} \times S^{n-k-\lambda}$. We identify this boundary with the boundary of the removed neighborhood in W^{n+1} to obtain a new manifold. Its boundary will be said to be the result of *toroidal surgery of the boundary* V^n. We may assume that the path $\gamma = \gamma(t) \subset \mathbf{R}^n$ is modelled by an interval on the real line $\mathbf{R}^1$ containing the three points $a < c < b$, where c is a critical value and a and b are close to c. We set $C_a = F^{-1}(t \leq a)$ and $C_b = F^{-1}(t \leq b)$. Then $C_a \subset C_b$. We may assume that $f\colon X_0^{n+1} \to \mathbf{R}^1$ and $C_a = (f \leq a)$, $C_b = (f \leq b)$, $B_c^0 = f^{-1}(c)$.

LEMMA 18. *Suppose that exactly one critical (saddle) torus T^{n-1} lies on the singular fiber B_c^0 of the moment map.*

1) *Suppose that the diagram $P_-^n(T^{n-1})$ is orientable. Then C_b is obtained from C_a by attaching a toroidal handle of index* 1 *and degeneracy degree $n-1$ to the boundary B_a. In addition, C_b is homotopy equivalent to a copy of C_a to which the manifold $T^{n-1} \times D^1$ has been attached along two nonintersecting tori $T_{1,a}^{n-1}$ and $T_{2,a}^{n-1}$.*

2) *Suppose that $P_-^n(T^{n-1})$ is nonorientable. Let Y_α^n be the manifold with boundary T^{n-1} which is the total space of the fibration $Y_\alpha^n \xrightarrow{D^1} T^{n-1}$ corresponding to the nonzero element $\alpha \in H_1(T^{n-1}, \mathbf{Z}_2)$. Then C_b is homotopy equivalent to a copy of C_a to which a copy of Y_α^n has been attached along the torus T_a^{n-1} on the boundary B_c^0.*

The proof proceeds along the same lines as the proof of Lemma 4.

LEMMA 19. *Suppose that a torus T_a^{n-1} embedded in a nonsingular Liouville torus $T_a^n \subset B_a^0$ is either a base of a toroidal handle of index* 1 *and degeneracy index $n-1$ or the boundary of a manifold Y_α^n (in the case when $P_-^n(T^{n-1})$ is nonorientable). Then the torus T_a^{n-1} always realizes one of the generators of the homology group $H_{n-1}(T_a^n, \mathbf{Z}) = \mathbf{Z}^{n-1}$. If both bases of the toroidal handle are attached to the same torus T_a^n, then the corresponding axes of these bases (that is, the tori $T_{1,a}^{n-1}$ and $T_{2,a}^{n-1}$) do not intersect,* *realize the same generator of the group $H_{n-1}(T^n_a, \mathbf{Z})$, and, consequently, are isotopic inside the torus T^n_a.*

PROOF. Consider the critical saddle torus T^{n-1}. By Lemma 17 it is the orbit of the action of an abelian subgroup $\mathbf{R}^{n-1}$ embedded in the group $\mathbf{R}^n$ generated by the fields $v_i = \operatorname{sgrad} f_i$, $1 \leq i \leq n$. The fields $v_2, \ldots, v_n$ are a basis of $\mathbf{R}^{n-1}$. Fix this subgroup. Since the action of $\mathbf{R}^n$ is defined on all of M, we can consider the orbits of $\mathbf{R}^{n-1}$ close to the orbit T^{n-1}. Consider a nonsingular Liouville torus T^n_a close enough to the fiber B^0_c so that the separatrix diagram P^n_- cuts out a nonsingular torus T^{n-1}_a. This torus will not, in general, be an orbit of $\mathbf{R}^{n-1}$ on T^n_a. However, it is possible to *approximate* T^{n-1}_a by an orbit of $\mathbf{R}^{n-1}$. To do this, we consider an element $\alpha \in H_{n-1}(T^{n-1}, \mathbf{Z}_2)$. We know from Lemma 18 that the torus T^{n-1}_a is one of the components of the boundary of the manifold Y^n_a attached to T^n_a. If $\alpha \neq 0$, then $T^{n-1}_a = \partial Y^n_a$; if $\alpha = 0$, then $\partial Y^n_0 = \partial(T^{n-1} \times D^1) = T^{n-1}_{1,a} \cup T^{n-1}_{2,a}$ and $T^{n-1}_{1,a} = T^{n-1}_a$. The element α determines a certain number k of generators of the torus T^{n-1}, in the circuit around which the normal segment of the separatrix diagram P^n_- changes its orientation. We single out these generators. If P^n_- is orientable, then $k = 0$ because $\alpha = 0$. Since T^{n-1} is an orbit of $\mathbf{R}^{n-1}$, we may, by replacing the generators in $\mathbf{R}^{n-1}$ (if necessary), assume that if P^n_- ($k \geq 1$) is nonorientable, then there are exactly k fields $v_2, \ldots, v_{k+1}$ from among the fields v_i, $2 \leq i \leq n$, which are such that a single circuit along the orbits of a point $x \in T^{n-1}$, generated by the corresponding one-dimensional subgroups $\mathbf{R}^1_2, \ldots, \mathbf{R}^1_{k+1}$, changes the orientation of the normal segment of the separatrix diagram of the critical torus. First suppose that P^n_- is orientable; that is, $k = 0$. Let Π be an $(n-1)$-dimensional parallelepiped which is a fundamental domain for the action of $\mathbf{R}^{n-1}$ on T^{n-1}. Under the natural map of $\mathbf{R}^{n-1}$ onto T^{n-1}, this parallelepiped covers the entire torus; that is, T^{n-1} is obtained by identifying opposite faces of the parallelepiped. Since Π consists of mappings onto M, we consider the orbit of this parallelepiped under its action on a point $h \in T^{n-1}_{1,a} \subset T^n_a$. This orbit will not be an $(n-1)$-dimensional torus in T^n_a. However, since the point h is close to the point $x \in T^{n-1}$, the orbit $\Pi(h)$ is an "almost-torus"; that is, each of the generators of the parallelepiped Π is taken to a segment whose endpoints are close on the torus T^n_a (that is, we obtain an "almost-circle"). We choose coordinates $\varphi_1, \ldots, \varphi_n$ on T^n_a in accord with the Liouville theorem. We will also use the fact that the points of Π are represented by symplectic transformations. In these coordinates the "almost torus" $\Pi(h)$ is a linear totally geodesic submanifold with possibly nonempty boundary. Representing the torus T^n_a (in these coordinates) as a cube with opposite faces identified, we obtain in it a plane Π' whose intersections with opposite faces are $(n-2)$-dimensional subspaces which turn out to be close after the opposite faces of the cube are identified. The plane Π' can be turned slightly in the cube to that it becomes (after the cube is factorized into a torus) an $(n-1)$-dimensional linear (totally geodesic) torus T^{n-1}_* *in* T^n_a. It is clear that the torus T^{n-1}_* is close to the "almost-torus" $\Pi(h)$ and, at the same time, close to $T^{n-1}_{1,a}$. It follows that they are isotopic. We have proved that there exists a small isotopy of $T^{n-1}_{1,a}$ in T^n_a carrying it into a linear torus. But then $T^{n-1}_{1,a}$ realizes a generator of $H_{n-1}(T^n_a, \mathbf{Z})$, as required. Thus, the lemma is proved for orientable P^n_-. We remark that by slightly moving Π we obtain a new plane Π_* (a parallelepiped) whose generators may already include the generator $v_1 = v$ which was excluded earlier from Π. It is clear that $T^{n-1}_* = \Pi_*(h)$.

Now suppose that P^n_- is nonorientable. Here the argument is more delicate. Here the parallelepiped Π does not suffice. Indeed, from the definition of a nonorientable separatrix diagram, it follows that the orbits $(\Pi \cap \mathbf{R}^1_2)h, \ldots, (\Pi \cap \mathbf{R}^n_{k+1})h$ of the generators of $\mathbf{R}^1_2, \dots, \mathbf{R}^1_{k+1}$ (corresponding to the fields v_i, $2 \le i \le k+1$) are not "almost-closed" trajectories on T^n_a. We denote the corresponding edges of the parallelepiped Π by Π_i; that is, $\Pi_i = \Pi \cap \mathbf{R}^1_i$, $2 \le i \le k+1$. Under the action of Π_i on the point h it manages to make only half a complete revolution on T^n_a. In order to make an almost-complete revolution, the transformations in Π_i need to act on the point again. In other words, in order to force the point h to make an almost-complete revolution on T^n_a, we should apply to it the transformations of $2\Pi_i$, that is, double the corresponding side of Π. Thus, it is necessary to double all the sides $\Pi_2, \dots, \Pi_{k+1}$ of the parallelepiped. We obtain a new (elongated) parallelepiped $\tilde{\Pi}$ stretched to twice its length in k directions. We let $\tilde{\Pi}$ act on the point h to obtain the orbit $\tilde{\Pi}(h)$. Clearly it is represented (in the action-angle variables on the Liouville torus) by a linear plane which is "almost-closed" (after the cube has been factorized into a torus). The rest of the argument goes by repeating arguments for orientable P^n_-. The lemma is proved.

The remaining arguments proceed along the lines outlined in §1.

Theorems 5 and 6 are proved. Each of the five elementary types of surgery of Liouville tori described in Theorem 5 is actually realized in concrete mechanical integrable systems.

Let $v = \operatorname{sgrad} H$ be an integrable system on the manifold M^{2n} and let $f_1 = H$ and $f_2, \dots, f_n$ be a complete commuting set of integrals such that the restriction $f = f_1|_{X^{n+1}}$ to the joint compact nonsingular level surface of the remaining $n-1$ integrals $X^{n+1} = \{x \in M : f_2(x) = c_2, \dots, f_n(x) = c_n\}$ is a Bott function. Let c_1 be a critical value of the function f_1 on X^{n+1}; that is, let $c = (c_1, \dots, c_n)$ be a critical value of the moment map $F: M \to \mathbf{R}^n$, $c \in \Sigma$. Let $B_c = F^{-1}(c)$ be the cricital fiber of the moment map; that is, let $B_c = \{f_1(x) = c_1\}$ be a critical level surface of f_1 on X^{n+1}. A critical fiber B_c necessarily contains critical points of f.

THEOREM 8. *Each connected compact component B^0_c of the critical fiber B_c is homeomorphic to a set which is of one of the following five types:* 1) *a torus T^n;* 2) *and* 3) *the nonorientable manifolds K^n_0 and K^n_1 described in Theorem* 7; 4) *a torus T^{n-1}; or* 5) *a cell complex $T^n_1 \cup T^n_2$ obtained by removing $(n-1)$-dimensioinal tori T^{n-1}_1 from T^n_1 and T^{n-1}_2 from T^n_2, which realize nonzero generators of the homology groups $H_{n-1}(T^n_1, \mathbf{Z})$ and $H_{n-1}(T^n_2, \mathbf{Z})$, and gluing T^n_1 and T^n_2 together by identifying only the tori T^{n-1}_1 and T^{n-1}_2 (by means of a diffeomorphism). In cases* 1)–4) *the critical fibers consist entirely of critical points of the function f on which a minimum or a maximum is attained. In case* 5) *the critical points of f in the critical fiber B^0_c form a torus T^{n-1} (the result of gluing together the tori T^{n-1}_1 and T^{n-1}_2) which is a "saddle" for the function f.*

For a four-dimensional manifold M^4 the connected critical level surfaces of a Bott integral $f = f_2|_{Q^3}$ on a compact nonsingular isoenergetic surface Q^3 have the form: 1) a torus T^2, 2) a Klein bottle K^2, 3) a circle S^1, or 4) a complex obtained by gluing together two two-dimensional tori along distinguished circles realizing nontrivial elements of the fundamental group of a torus. We can now describe the behavior of the trajectories of the field $\operatorname{sgrad} H$ on each critical level of the function f. For example, in the general position case for M^4, the integral curves on a singular fiber B_c homeomorphic to the result of attaching two tori along a nontrivial circle S^1 either asymptotically wind around the nontrivial cycle S^1 or coincide with it. In the higher-dimensional case, the trajectories asymptotically tend to a critical torus T^{n-1} along which the two tori, T^n_1 and T^n_2, are attached (forming the fiber B^0_c). On critical fibers of types 1), 2), 3), and 4) (all of which are submanifolds of X^{n+1}) the trajectories of the field $\operatorname{sgrad} H$ form, in the general position case, everywhere dense windings (on the tori T^n and T^{n-1}, or on the "generalized Klein bottles" K^n_0 and K^n_1).

We consider three-dimensional isoenergetic surfaces Q^3 and mention another corollary of Theorem 2. Suppose that a Bott integral f on Q is "totally orientable"; that is, all its critical submanifolds and all its separatrix diagrams are orientable, i.e. the Hamiltonian decomposition of Q has the form $Q = m\mathrm{I} + p\mathrm{II} + q\mathrm{III}$, $s = r = 0$. Let M_g^2 be a two-dimensional manifold homeomorphic to a sphere with g handles, and consider the direct product $M_g^2 \times S^1$. Select a finite set of nonintersecting circles α_i on M_g^2 with the property that exactly m of them are contractible (and the others are noncontractible in M_g^2). In the product $M_g^2 \times S^1$, the circles α_i determine tori $T_i^2 = \alpha_i \times S^1$. We cut $M_g^2 \times S^1$ along these tori and identify opposite sides of the cuts (that is, the tori) with the help of some diffeomorphisms. A new three-dimensional manifold is obtained. *The original manifold Q has precisely such a form.* According to the main classification, Theorem 5, any general position surgery of a Liouville torus splits into a composition of the three types of elementary surgery. Question: *can any such composition be realized for an appropriate moment map* on some manifold? An affirmative answer has been obtained by A. V. Brailov and A. T. Fomenko. Let X^{n+1} be a smooth closed compact orientable manifold, obtained by pasting together an arbitrary number of elementary manifolds of types I, II, and III (solid tori, cylinders, and trousers) by arbitrary diffeomorphisms of their boundary tori. Then there always exists a smooth compact symplectic manifold M^{2n} with boundary diffeomorphic to a disjoint union of a number of manifolds $S^{n-1} \times T^n$ and a complete set of smooth functions $f_1, \dots, f_n$ on M^{2n} such that $X^{n+1} = \{x \in M^{2n} \colon f_2(x) = \cdots = f_n(x) = 0\}$. The proof will be given in another paper.

In §1, it was proved that the class (H) of isoenergetic three-dimensional surfaces of integrable systems (which coincides with the classes (Q), (S), and (W)) forms a "meager" subset, in some sense, of the class of all closed orientable three-dimensional manifolds. It is a curious fact that, nevertheless, *it is possible to construct three-dimensional oriented manifolds with any possible (for three-dimensional oriented manifolds) integral homology groups by pasting together "trousers" and solid tori* (a remark of G. Mamedov). This assertion can also be deduced from the theory of Seifert manifolds. Thus, from the point of view of homology classes, (H) "coincides" with the class of all three-dimensional manifolds. In other words, it is not possible to distinguish the isoenergetic surfaces from among three-dimensional manifolds on the basis of homology groups only.

We mention some problems which have not yet been completely solved.

1) What happens to the elementary surgery of Liouville tori (of integrable systems) under a small perturbation of the Bott integral? It is necessary to study their stability.

2) Can the Bott integrals be considered to be "everywhere dense", in some sense, in the set of smooth integrals of an integrable system?

3) What is the structure of the surgery of Liouville tori when the path γ pierces the set Σ at its singular points? That is, how can one classify the "singular surgery" of Liouville tori (see [9], [14], and [23])?

Added in translation. Let M^4 be a symplectic manifold, and v a Hamiltonian system with Hamiltonian H; assume v is completely integrable on the compact regular isoenergetic surface $Q^3 = (H = \text{const})$. Let $f\colon Q \to R$ be a second independent Bott integral on Q. The critical submanifolds of f are nondegenerate in Q. The Hamiltonian H is called *nonresonance* if the set of Liouville tori with irrational trajectories of v is dense in Q. We denote by $f^{-1}(a)$ the set of tori in the case when $a \in R$ is regular.

THEOREM. *There exist a one-dimensional graph $Z(Q, f)$, a two-dimensional closed compact surface $P(Q, f)$, and an embedding $h\colon Z(Q, f) \to P(Q, f)$ which are naturally and uniquely defined by the integrable nonresonance Hamiltonian H with the Bott integral f on Q. The triple (Z, P, h) does not depend on the choice of the second integral f. This* *means that if f and f' are two arbitrary Bott integrals of a given system, then the graphs Z and Z' are homeomorphic, the surfaces P and P' are homeomorphic, and the diagram*

$$\begin{array}{rccc} h\colon & Z & \to & P \\ & \wr\wr & & \wr\wr \\ h'\colon & Z' & \to & P' \end{array}$$

is commutative. Consequently, the graph $Z(Q)$, the surface $P(Q)$, and the embedding $h(Q)\colon Z(Q) \to P(Q)$ are the topological invariants of the integrable case (of the Hamiltonian H) proper. The triple $Z(Q)$, $P(Q)$, $h(Q)$ allows us to classify the integrable Hamiltonians corresponding to their topological types.

In particular, we can now demonstrate the transparent difference between the invariant topological structure of the Kovalevskaya case, the Goryachev-Chaplygin case, and so on. The subdivision of the surface $P(Q)$ into the sum of domains is also a topological invariant of the Hamiltonian H, and describes its topological complexity. The graph Z^* dual to Z on P has vertices of multiplicity at most 4. The collection of the graphs $Z(Q)$, surfaces $P(Q)$, and embeddings $h(Q)$ is the total topological invariant of H.

Received 14/JAN/85

BIBLIOGRAPHY

1. D. V. Anosov, *On generic properties of closed geodesics*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 675–709; English transl. in Math. USSR Izv. **21** (1983).

2. ——, *Closed geodesics*, Quantitative Methods for Investigating Nonlinear Differential Equations and Nonlinear Oscillations (Papers, School-Sem. Differential Equations and Appl., Uzhgorod, 1980; Yu. A. Mitropol'skiĭ, editor), Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1981, pp. 5–24. (Russian)

3. ——, *Some homology classes in the space of closed curves in the n-dimensional sphere*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), 467–490; English transl. in Math. USSR Izv. **18** (1982).

4. V. I. Arnol'd, *Mathematical methods in classical mechanics*, "Nauka", Moscow 1974; English transl., Springer-Verlag, 1978.

5. ——, *Proof of a theorem of A. N. Kolmogorov on the preservation of conditionally periodic motions under a small perturbation of the Hamiltonian*, Uspekhi Mat. Nauk **18** (1963), no. 5(113), 13–40; English transl. in Russian Math. Surveys **18** (1963).

6. V. V. Kozlov, *Integrability and nonintegrability in Hamiltonian mechanics*, Uspekhi Mat. Nauk **38** (1983), no. 1(229), 3–67; English transl. in Russian Math. Surveys **38** (1983).

7. ——, *Methods of qualitative analysis in the dynamics of a rigid body*, Izdat. Moskov. Gos. Univ., Moscow, 1980. (Russian)

8. V. N. Kolokol'tsov, *Geodesic flows on two-dimensional manifolds with an additional first integral that is polynomial in the velocities*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 994–1010; English transl. in Math. USSR Izv. **21** (1983).

9. A. M. Stepin, *Integrable Hamiltonian systems*. I, II, Qualitative Methods for Investigating Nonlinear Differential Equations and Nonlinear Oscillations (Papers, School-Sem. Differential Equations and Appl., Uzhgorod, 1980; Yu. A. Mitropol'skiĭ, editor), Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1981, pp. 116–142, 143–170. (Russian)

10. S. P. Novikov and I. Shmel'tser, *Periodic solutions of Kirchhoff equations for the free motion of a rigid body in a fluid, and the extended Lyusternik-Schnirelmann-Morse theory*. I, Funktsional. Anal. i Prilozhen. **15** (1981), no. 3, 54–66; English transl. in Functional Anal. Appl. **15** (1981).

11. S. P. Novikov, *Variational methods and periodic solutions of equations of Kirchhoff type*. II, Funktsional. Anal. i Prilozhen. **15** (1981), no. 4, 37–52; English transl. in Functional Anal. Appl. **15** (1981).

12. S. V. Bolotin, *Nonintegrability of the problem of n centers for n > 2*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1984**, no. 3, 65–68; English transl. in Moscow Univ. Mech. Bull. **39** (1984).

13. V. V. Kozlov, *Calculus of variations in the large and classical mechanics*, Uspekhi Mat. Nauk **40** (1985), no. 2(242), 33–60; English transl. in Russian Math. Surveys **40** (1985).

14. A. T. Fomenko, *Algebraic structure of certain integrable Hamiltonian systems*, Global Analysis: Studies and Applications. I, Lecture Notes in Math., vol. 1108, Springer-Verlag, 1984, pp. 103–127.

15. T. I. Pogosyan and M. P. Kharlamov, *Bifurcation set and integral manifolds of the problem on the motion of a rigid body in a linear force field*, Prikl. Mat. Mekh. **43** (1979), 419–428; English transl. in J. Appl. Math. Mech. **43** (1979).

16. Wilhelm Klingerberg, *Lectures on closed geodesics*, Springer-Verlag, 1978.

17. Ya. V. Tatarinov, *Portraits of the classical integrals of the problem of the rotation of a rigid body about a fixed point*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1974**, no. 6, 99–105; English transl. in Moscow Univ. Mech. Bull. **29** (1974).

18. V. V. Trofimov and A. T. Fomenko, *Liouville integrability of Hamiltonian systems on Lie algebras*, Uspekhi Mat. Nauk **39** (1984), no. 2(236), 3–56; English transl. in Russian Math. Surveys **39** (1984).

19. Raoul Bott, *Nondegenerate critical manifolds*, Ann. of Math. (2) **60** (1954), 248–261.

20. M. Adler and P. van Moerbeke, *The algebraic integrability of geodesic flow on* SO(4), Invent. Math. **67** (1982), 297–331.

21. A. T. Fomenko, *The integrability of some Hamiltonian systems*, Ann. Global Anal. Geom. **1** (1983), no. 2, 1–10.

22. ——, *Algebraic properties of some integrable Hamiltonian systems*, Topology (Leningrad, 1982), Lecture Notes in Math., vol. 1060, Springer-Verlag, 1984, pp. 246–257.

23. Charles Conley and Eduard Zehnder, *Morse-type index theory for flows and periodic solutions for Hamiltonian equations*, Comm. Pure Appl. Math. **37** (1984), 207–253.

24. Friedhelm Waldhausen, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten*. I, Invent. Math. **3** (1967), 308–333.

25. S. Smale, *Topology and mechanics*. I, II, Invent. Math. **10** (1970), 305–331; **11** (1970), 45–64.

26. M. P. Kharlamov, *Topological analysis of classical integrable systems in the dynamics of a rigid body*, Dokl. Akad. Nauk SSSR **273** (1983), 1322–1325; English transl. in Soviet Math. Dokl. **28** (1983).

27. A. T. Fomenko, *Topology of three-dimensional manifolds and integrable mechanical Hamiltonian systems*, Fifth Tiraspol Sympos. General Topology Appl., Abstracts of Reports, Kishinev, 1985, pp. 235–237. (Russian)

28. ——, *Morse theory of integrable Hamiltonian systems*, Dokl. Akad. Nauk SSSR **287** (1986), 1071–1075; English transl. in Soviet Math. Dokl. **33** (1986).

Translated by D. B. O'SHEA