perplexity

Spectral Representation of Itô Processes: Bridging Time and Frequency Domains

Core Principle: Convolutional Duality Between White Noise and Stochastic Dynamics

The fundamental equivalence between Itô processes and oscillatory processes arises from the **convolutional representation theorem**: every mean-square continuous stochastic process can be expressed as a filtered white noise process through either:

1. **Time-domain convolution**:

$$X(t) = \int_{\mathbb{R}} K(t-\tau) dW(\tau)$$

where K is a smoothing kernel and W is Brownian motion.

2. **Frequency-domain modulation**:

$$X(t) = \int_{\mathbb{R}} e^{i\omega t} A(t,\omega) dZ(\omega)$$

where $A(t,\omega)$ is a slowly varying amplitude and $Z(\omega)$ has orthogonal increments.

Mathematical Equivalence Framework

Itô Process Definition

An Itô process satisfies:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

Spectral Representation Theorem

For any Itô process, there exists:

1. A complex orthogonal measure $dZ(\omega)$ with $\mathbb{E}[|dZ(\omega)|^2] = S(\omega)d\omega$
2. A time-frequency kernel $K(t,\omega)$

Such that:

$$X_t = \int_{-\infty}^{\infty} K(t,\omega) e^{i\omega t} dZ(\omega)$$

Key Equations and Transformations

From Itô to Oscillatory Representation

1. **Lamperti Transformation**:
   For state-dependent volatility $\sigma(X_t, t)$, define:

$$Z_t = \int_0^{X_t} \frac{1}{\sigma(x,t)} dx$$

Transforms original SDE to:

$$dZ_t = \left(\frac{\mu(X_t,t)}{\sigma(X_t,t)} - \frac{1}{2} \frac{\partial \sigma}{\partial x} \Big|_{X_t} \right) dt + dW_t$$

2. **Cramér Representation**:
   The transformed process admits:

$$Z_t = \int_{-\infty}^{\infty} \frac{e^{i\omega t} - 1}{i\omega} \Phi(\omega) d\widetilde{W}(\omega)$$

where $\Phi(\omega)$ is the characteristic function of the drift-adjusted terms.

Time-Frequency Correspondence

Time Domain	Frequency Domain
$dX_t = \mu dt + \sigma dW_t$	$X(\omega) = \frac{\sigma}{i\omega+\theta} \widetilde{W}(\omega)$
Quadratic Variation $[X]_t$	Power Spectrum $S(\omega) = \frac{\sigma^2}{\omega^2+\theta^2}$
Itô Isometry	Parseval's Identity

Canonical Example: Ornstein-Uhlenbeck Process

Time Domain

$$dX_t = -\theta X_t dt + \sigma dW_t$$

Spectral Representation

$$X_t = \sigma \int_{-\infty}^{t} e^{-\theta(t-s)} dW_s$$

Frequency Domain

Power spectral density:

$$S(\omega) = \frac{\sigma^2}{\omega^2 + \theta^2}$$

General Construction Principle

For any Itô process with coefficients (μ, σ):

1. **Volatility Normalization**: Apply Lamperti transform to remove state-dependent volatility
2. **Drift Decomposition**: Express adjusted drift as potential function $\nabla V(Z_t)$
3. **Spectral Expansion**:

$$Z_t = \int_{\mathbb{R}} \frac{e^{i\omega t}}{\sqrt{\omega^2 + \lambda^2}} d\widetilde{W}(\omega)$$

where λ controls mean-reversion strength.

Deep Structural Correspondence

Itô's Lemma ⇄ Modulation Theorem

The chain rule for stochastic calculus mirrors the frequency modulation property:

Itô:

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d[X]_t$$

Spectral:

$$\mathcal{F}\{tf(t)\} = i\frac{d}{d\omega}\mathcal{F}\{f\}$$

Martingale Representation ⇄ Analytic Signal

The Girsanov theorem finds its spectral counterpart in the Hilbert transform:

$$H[X](t) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty} \frac{X(\tau)}{t - \tau} d\tau$$

Limitations and Boundary Cases

While theoretically elegant, practical equivalence requires:

1. **Slow Variation Condition**: $A(t, \omega)$ must satisfy Priestley's extremal slow variation
2. **Adaptivity Constraint**: Path-dependent coefficients induce frequency modulation violating strict oscillatory definitions
3. **Non-Gaussian Extensions**: Multiplicative noise processes require Volterra series expansions

Conclusion

The spectral representation of Itô processes reveals a profound duality:

Every stochastic differential equation is fundamentally a filtered white noise process, with:

- Temporal drift/variance ⇄ Frequency-dependent attenuation
- Path dependence ⇄ Non-stationary spectral correlations
- Quadratic variation ⇄ Power spectrum density

This correspondence enables simultaneous analysis of stochastic systems through both probabilistic and harmonic lenses, unifying the Itô calculus and spectral theory frameworks.

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