Uniformly Convergent Orthonormal Expansions for Positive Definite Functions

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Abstract

A method for deriving uniformly convergent orthonormal expansions for positive-definite functions in the context of covariance functions of stationary(translation-invariant) Gaussian processes is presented.

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1 Introduction: Stationary Gaussian Processes

1.1 Polynomials Orthogonal wth Respect to the Spectral Density

Let $C(x)$ be the covariance function of a stationary Gaussian process on $[0, \infty)$ which by definition must be positive definite. The integral covariance operator T is defined by:

$$(Tf)(x) = \int_0^\infty C(x - y)\, f(y)\, \mathrm{d}y \tag{1}$$

Let $S(\omega)$ be the spectral density related to $C(x)$ by the Wiener-Khinchin theorem:

$$C(x) = \frac{1}{\pi} \int_{-\infty}^\infty e^{i\omega x}\, S(\omega)\, \mathrm{d}\omega \tag{2}$$

$$S(\omega) = \int_0^\infty C(x)\, e^{-ix\omega}\, \mathrm{d}x \tag{3}$$

Consider polynomials $\{p_n(\omega)\}$ orthogonal with respect to $S(\omega)$:

$$\int_{-\infty}^{\infty} p_n(\omega)\, p_m(\omega)\, S(\omega)\, \mathrm{d}\omega = \delta_{nm} \tag{4}$$

1.2 Null Space of the Inner Product Operator

Define $r_n(x)$ as the inverse Fourier transforms of $p_n(\omega)$:

$$r_n(x) = \int_{-\infty}^{\infty} p_n(\omega)\, e^{i\omega x}\, \mathrm{d}\omega \tag{5}$$

Lemma 1. *The functions $r_n(x)$ form the null space of the kernel inner product:*

$$\int_0^{\infty} C(x)\, r_n(x)\, \mathrm{d}x = 0 \tag{6}$$

Proof. Let $C(x)$ and $r_n(x)$ be defined as:

$$C(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{iax}\, S(a)\, \mathrm{d}a \tag{7}$$

$$r_n(x) = \int_{-\infty}^{\infty} p_n(b)\, e^{ibx}\, \mathrm{d}b \tag{8}$$

where $S(a)$ is the spectral density and $p_n(b)$ are orthogonal polynomials with respect to $S(a)$. Note that $C(x)$ and $r_n(x)$ are even functions, as they depend on the difference between two variables.

Substitute the definitions of $C(x)$ and $r_n(x)$, and apply Fubini's theorem:

$$\begin{aligned}\int_0^{\infty} C(x)\, r_n(x)\, \mathrm{d}x &= \int_0^{\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} e^{iax}\, S(a)\, \mathrm{d}a \int_{-\infty}^{\infty} p_n(b)\, e^{ibx}\, \mathrm{d}b\, \mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_n(b)\, S(a) \int_0^{\infty} e^{i(a+b)x}\, \mathrm{d}x\, \mathrm{d}b\, \mathrm{d}a\end{aligned}$$

Since $C(x)$ and $r_n(x)$ are even functions, we can write:

$$\int_0^{\infty} C(x)\, r_n(x)\, \mathrm{d}x = \frac{1}{2} \int_{-\infty}^{\infty} C(x)\, r_n(x)\, \mathrm{d}x \tag{9}$$

Now we have:

$$
\begin{aligned}
\frac{1}{2}\int_{-\infty}^{\infty} C(x)\, r_n(x)\,\mathrm{d}x =& \frac{1}{2\,\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_n(b)\, S(a)\int_{-\infty}^{\infty} e^{i(a+b)x}\,\mathrm{d}x\,\mathrm{d}b\,\mathrm{d}a\\
=& \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_n(b)\, S(a)\,\delta\,(a+b)\,\mathrm{d}b\,\mathrm{d}a\\
=& \frac{1}{2}\int_{-\infty}^{\infty} p_n\,(-a)\, S(a)\,\mathrm{d}a
\end{aligned}
$$

By the orthogonality of $p_n(a)$ with respect to $S(a)$, we conclude:

$$
\frac{1}{2}\int_{-\infty}^{\infty} p_n\,(-a)\, S(a)\,\mathrm{d}a = 0 \tag{10}
$$

Thus, $\int_0^{\infty} C(x)\, r_n(x)\,\mathrm{d}x = 0\forall$n. □

2 Orthonormal Expansion for the Kernel

By orthogonalizing the null space $\{r_n(x)\}$ to form its orthogonal complement, a pointwisely coverging expansion for $C(x)$ is obtained

$$
r_n^{\perp}(y) = \psi_n(x) = \sum_{k=0}^{n} a_{nk}\, r_k(x) = r_n(y) - \sum_{m=0}^{n-1} \frac{\langle r_n(y), r_m^{\perp}(y)\rangle}{\langle r_m^{\perp}(y), r_m^{\perp}(y)\rangle}\, r_m^{\perp}(y)
$$

where the coefficients a_{nk} are given by:

$$
a_{nk} = \begin{cases} 1 & \text{if } k = n\\ -\sum_{j=k}^{n-1} a_{nj}\,\langle r_n, \psi_j\rangle & \text{if } k\, n\\ 0 & \text{if } k > n \end{cases} \tag{11}
$$

Lemma 2. *The eigenvalues of the integral covariance operator (1) are given by*

$$
\lambda_n = \int_0^{\infty} C(z)\,\psi_n(z)\,\mathrm{d}z \tag{12}
$$

Theorem. *The expansion*

$$
C(x) = \sum_{k=0}^{\infty} \psi_k(x)\lambda_k \tag{13}
$$

converges uniformly.

Example 3. Let the kernel function be given by $K(t,s) = J_0\,(t-s)$ then identify the orthogonal polynomial sequence associated with the spectral density

$$S(\omega) = \int_0^{\infty} J_0(x)\, e^{i x \omega}\, \mathrm{d}x = \begin{cases} \dfrac{1}{\sqrt{1-\omega^2}} & \omega \in (-1,1) \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

as being the orthogonality measure of the Type-I Chebyshev polynomials $T_n(x)$ so that the orthogonal polynomial sequence is identified as

$$S_n(x) = T_n(x) \tag{15}$$

so that

$$\int_{-1}^{1} S_n(\omega)\, S_m(\omega)\, S(\omega)\, \mathrm{d}\omega = \begin{cases} 0 & n \neq m \\ 2\,\pi & n = m = 0 \\ \pi & n = m \neq 0 \end{cases} \tag{16}$$

The finite Fourier transforms of the Chebyshev polynomials[1] are just the usual infinite Fourier transforms with the integration restricted to the range $-1\ldots 1$ since $T_n(x) = 0 \forall x \notin [-1,1]$). Equivalently, the spectral density function can be extended to take the value 0 outside the interval $[-1,1]$.

$$\begin{aligned} \hat{T}_n(y) &= \int_{-\infty}^{\infty} e^{-ixy}\, T_n(x)\, \mathrm{d}y = \int_{-1}^{1} e^{-ixy}\, T_n(x)\, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} e_2^{-ixy}\, F_1\left(\begin{matrix} n, & -n \\ & \frac{1}{2} \end{matrix} \,|\frac{1}{2} - \frac{x}{2} \right) \mathrm{d}x \\ &= \frac{i}{y}\, (e^{-iy}\, F_n^+(y) - e^{i(\pi n + y)}\, F_n^-(y)) \end{aligned} \tag{17}$$

where

$$F_n^{\pm}(y) = {}_3F_1\left(\begin{matrix} 1, & n, & -n \\ & & \frac{1}{2} \end{matrix} \,|\frac{\pm i\, y}{2} \right) \tag{18}$$

the spectral polynomials S_n are given by the Type-I Chebyshev polynomials

$$S_n(x) = T_n(x) \tag{19}$$

and their normalization is

$$\begin{aligned} Y_n(y) &= \frac{\hat{T}_n(y)}{|\hat{T}_n|} \\ &= \frac{i}{y} \left(\frac{e^{-iy}\, F_n^+(y) - e^{i(\pi n + y)}\, F_n^-(y)}{\sqrt{\frac{4\,(-1)^n\,\pi - (2\,n^2 - 1)}{4\,n^2 - 1}}} \right) \end{aligned} \tag{20}$$

where the L^2 norm of $\hat{T}_n(y)$ is given by

$$\begin{aligned} |\hat{T}_n| &= \sqrt{\int_{-\infty}^{\infty} \hat{T}_n(y)^2 \,\mathrm{d}y} \\ &= \sqrt{\frac{4\,(-1)^n\,\pi - (2\,n^2 - 1)}{4\,n^2 - 1}} \end{aligned} \tag{21}$$

Definition 4. *Let $j_n(x)$ is the spherical Bessel function of the first kind,*

$$\begin{aligned} j_n(z) &= \sqrt{\frac{\pi}{2\,z}}\, J_{n+\frac{1}{2}}(x) \\ &= \frac{1}{\sqrt{z}} \left(\sin(z)\, R_{n,\frac{1}{2}}(z) - \cos(z)\, R_{n-1,\frac{3}{2}}(z) \right) \end{aligned} \tag{22}$$

where $R_{n,v}(z)$ are the (misnamed) Lommel polynomials

$$R_{n,v}(z) = \frac{\Gamma(n+v)}{\Gamma(v)} \left(\frac{2}{z}\right)^n {}_2F_3\!\left(\left[-\frac{n}{2}, \frac{1}{2} - \frac{n}{2} \right]; [v, -n, 1 - v - n]; -z^2 \right) \tag{23}$$

where ${}_2F_3$ is a generalized hypergeometric function. The "Lommel polynomials" are actually rational functions of z, not polynomial; but rather "polynomial in $\frac{1}{z}$".

Theorem 5. *The expansion*

$$\begin{aligned} J_0(x) &= \sum_{n=0}^{\infty} \lambda_n\, Y_n^{\perp}(x) \\ &= \sum_{k=0}^{\infty} \sqrt{\frac{2\,n + \frac{1}{2}}{\pi}}\, (n+1)^2_{-\frac{1}{2}}\, (-1)^n \sqrt{\frac{8\,n + 2}{\pi}}\, j_{2n}(x) \\ &= \sum_{k=0}^{\infty} \frac{4n+1}{\pi}\, (n+1)^2_{-\frac{1}{2}}\, (-1)^n\, j_{2n}(x) \end{aligned} \tag{24}$$

converges uniformly $\forall x \in \mathbb{C} \backslash \{0\}$ where $\lim_{x \to 0} = J_0(x) = 1$.

Proof. The proof is left as an excercise for the reader. ☐

Remark 6. It would fantastic to know if the orthonormal basis for $J_0(x)$ could be used to find a Mercer expansion as in [3]

3 Appendix

3.1 Proofs & Lemmas

3.1.1 Unique Uniform Convergence of Eigenfunctions to Kernel

Theorem 7. *Let $K(\alpha, \gamma)$ be a symmetric kernel defined on $[0, \infty) \times [0, \infty)$, and let $\{\phi_n(\alpha)\}_{n=0}^{\infty}$ be the set of orthonormal eigenfunctions of the integral equation*

$$\phi(\alpha) = \lambda \int_0^{\infty} K(\alpha, \xi)\, \phi(\xi)\, \mathrm{d}\xi \tag{25}$$

with corresponding eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$. The eigenfunctions satisfy the orthonormality condition:

$$\int_0^{\infty} \phi_n(\alpha)\, \phi_m(\alpha)\, \mathrm{d}\alpha = \delta_{nm} \tag{26}$$

where δ_{nm} is the Kronecker delta. Then, if the series

$$\sum_{n=0}^{\infty} \frac{\phi_n(\alpha)\, \phi_n(\gamma)}{\lambda_n} \tag{27}$$

is uniformly convergent for $0 \leq \alpha, \gamma < \infty$, we have

$$K(\alpha, \gamma) = \sum_{n=0}^{\infty} \frac{\phi_n(\alpha)\, \phi_n(\gamma)}{\lambda_n} \tag{28}$$

Proof. Let

$$S(\alpha, \gamma) = \sum_{n=0}^{\infty} \frac{\phi_n(\alpha)\, \phi_n(\gamma)}{\lambda_n} \tag{29}$$

Consider the action of $S(\alpha, \gamma)$ on an eigenfunction $\phi_m(\gamma)$:

$$\int_0^{\infty} S(\alpha, \gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma = \int_0^{\infty} \sum_{n=0}^{\infty} \frac{\phi_n(\alpha)\, \phi_n(\gamma)}{\lambda_n}\, \phi_m(\gamma)\, \mathrm{d}\gamma \tag{30}$$

$$= \sum_{n=0}^{\infty} \frac{\phi_n(\alpha)}{\lambda_n} \int_0^{\infty} \phi_n(\gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma \tag{31}$$

$$= \sum_{n=0}^{\infty} \frac{\phi_n(\alpha)}{\lambda_n}\, \delta_{nm} \tag{32}$$

$$= \frac{\phi_m(\alpha)}{\lambda_m} \tag{33}$$

The interchange of summation and integration is justified by the uniform convergence of the series. For the eigenfunction $\phi_m(\alpha)$:

$$\phi_m(\alpha) = \lambda_m \int_0^\infty K(\alpha, \gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma \tag{34}$$

Comparing this with our result for $S(\alpha, \gamma)$, we see that

$$\int_0^\infty S(\alpha, \gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma = \int_0^\infty K(\alpha, \gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma \tag{35}$$

for all eigenfunctions $\phi_m(\alpha)$. For any square-integrable function $f(\alpha)$:

$$f(\alpha) = \sum_{m=0}^{\infty} c_m\, \phi_m(\alpha) \tag{36}$$

where $c_m = \int_0^\infty f(\gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma$. Then:

$$\begin{aligned}
\int_0^\infty S(\alpha, \gamma)\, f(\gamma)\, \mathrm{d}\gamma &= \int_0^\infty S(\alpha, \gamma) \sum_{m=0}^{\infty} c_m\, \phi_m(\gamma)\, \mathrm{d}\gamma && (37)\\
&= \sum_{m=0}^{\infty} c_m \int_0^\infty S(\alpha, \gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma && (38)\\
&= \sum_{m=0}^{\infty} c_m \int_0^\infty K(\alpha, \gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma && (39)\\
&= \int_0^\infty K(\alpha, \gamma) \sum_{m=0}^{\infty} c_m\, \phi_m(\gamma)\, \mathrm{d}\gamma && (40)\\
&= \int_0^\infty K(\alpha, \gamma)\, f(\gamma)\, \mathrm{d}\gamma && (41)
\end{aligned}$$

Since this equality holds for all square-integrable functions $f(\alpha)$, we conclude that

$$S(\alpha, \gamma) = K(\alpha, \gamma) \tag{42}$$

To prove uniqueness, suppose there exists another expansion

$$K(\alpha, \gamma) = \sum_{n=0}^{\infty} \frac{\phi_n(\alpha)\, \phi_n(\gamma)}{\lambda_n} + H(\alpha, \gamma) \tag{43}$$

where $H(\alpha, \gamma)$ is a non-zero symmetric function. Then for any eigenfunction $\phi_m(\alpha)$:

$$\int_0^\infty H(\alpha, \gamma)\, \phi_m(\gamma)\, \mathrm{d}\gamma = 0 \tag{44}$$

This implies $H(\alpha, \gamma)$ must be identically zero, contradicting our assumption. Therefore, the expansion is unique. [2, ..?] $\square$

Bibliography

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