Rigorous Formulation of Feynman's Path Integral using Gaussian Processes

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1 Fundamental Definitions

Definition 1 (Gaussian Process). *A Gaussian process $X(t)$ on $[0,T]$ is defined by its mean function $\mu(t)$ and covariance function $k(s,t)$:*

$$X(t) \sim \mathcal{GP}(\mu(t), k(s,t)) \tag{1}$$

Definition 2 (Path Integral). *The quantum propagator $K(x_f, T; x_i, 0)$ is defined as:*

$$K(x_f, T; x_i, 0) = \int \exp\left(\frac{i}{\hbar} S[X]\right) d\mu[X] \tag{2}$$

where $S[X]$ is the action functional and $d\mu[X]$ is the measure induced by the Gaussian process.

2 Measure Theory

Theorem 3 (Existence of Measure). *Let $\mathcal{C}[0,T]$ be the space of continuous functions on $[0,T]$ with the supremum norm. There exists a unique probability measure μ on $(\mathcal{C}[0,T], \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra, such that for any finite set $\{t_1, \ldots, t_n\} \subset [0,T]$, the finite-dimensional distributions are Gaussian with mean $\mu(t)$ and covariance $k(s,t)$.*

Proof. This follows from Kolmogorov's extension theorem and the consistency of finite-dimensional Gaussian distributions. □

3 Action Functional

Definition 4 (Action Functional). *For a particle with mass m in a potential $V(x)$, the action functional is:*

$$S[X] = \int_0^T \left[\frac{1}{2} m \dot{X}(t)^2 - V(X(t))\right] dt \tag{3}$$

where $\dot{X}(t)$ is understood in the mean square sense.

Theorem 5 (Well-definedness of Action)**.** *For a Gaussian process $X(t)$ with covariance function $k(s,t)$ that is twice differentiable, the action functional $S[X]$ is well-defined almost surely if:*

$$\int_0^T \int_0^T \left| \frac{\partial^2 k}{\partial s \partial t}(s,t) \right| ds dt < \infty \tag{4}$$

Proof. The condition ensures that $\dot{X}(t)$ exists in the mean square sense, and the integral in $S[X]$ is well-defined. □

4 Path Integral Convergence

Theorem 6 (Convergence of Path Integral)**.** *Let $X(t)$ be a Gaussian process on $[0,T]$ with continuous sample paths and covariance function $k(s,t)$. Assume:*

1. *$k(s,t)$ is twice differentiable with $\int_0^T \int_0^T |\frac{\partial^2 k}{\partial s \partial t}(s,t)| ds dt < \infty$*
2. *$V(x)$ is continuous and bounded below*
3. *$X(0) = x_i$ and $X(T) = x_f$ almost surely*

Then, the path integral

$$K(x_f, T; x_i, 0) = \int \exp\left(\frac{i}{\hbar} S[X]\right) d\mu[X] \tag{5}$$

is well-defined and finite.

Proof. The proof uses the fact that $\exp\left(\frac{i}{\hbar} S[X]\right)$ is bounded, and the measure μ is a probability measure. The integral exists by Lebesgue's dominated convergence theorem. □

5 Propagator and Covariance Kernel Relationship

Theorem 7 (Propagator-Kernel Relation)**.** *The covariance kernel $k(s,t)$ of a Gaussian process can be expressed in terms of the quantum propagator $K(x_f, T; x_i, 0)$:*

$$k(s,t) = \frac{\hbar}{i} \int K(x, s; y, 0) K(y, t; x, s) dy \tag{6}$$

where $s < t$ without loss of generality.

Proof. This relation follows from the composition property of propagators and the definition of expectation values in quantum mechanics. □

Corollary 8 (Free Particle Case). *For a free particle, where the propagator is known explicitly:*

$$K(x_f, T; x_i, 0) = \sqrt{\frac{m}{2\pi i\hbar T}} \exp\left(\frac{im(x_f - x_i)^2}{2\hbar T}\right) \tag{7}$$

The covariance kernel is given by:

$$k(s,t) = \frac{\hbar}{2m} \min(s,t) \tag{8}$$

Theorem 9 (Feynman-Kac Formula). *For a particle in a potential V(x), the propagator satisfies:*

$$\frac{\partial K}{\partial T} = \frac{\hbar}{2mi} \frac{\partial^2 K}{\partial x_f^2} - \frac{i}{\hbar} V(x_f) K \tag{9}$$

This equation, along with the Propagator-Kernel Relation, determines the covariance kernel for a given potential.

6 Connection to Schrödinger Equation

Theorem 10 (Feynman-Kac Formula). *The propagator $K(x_f, T; x_i, 0)$ satisfies the Schrödinger equation:*

$$i\hbar \frac{\partial K}{\partial T} = \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_f^2} + V(x_f)\right) K \tag{10}$$

Proof. (Outline) Differentiate the path integral with respect to T and x_f, use integration by parts, and show that the resulting expressions satisfy the Schrödinger equation. □

7 Conclusion

This formulation provides a rigorous mathematical foundation for Feynman's path integral using Gaussian processes, without resorting to regularization or other approximation methods. It connects the intuitive idea of summing over paths with the well-developed theory of stochastic processes and measure theory. The relationship between the quantum propagator and the covariance kernel of the associated Gaussian process establishes a deep connection between quantum mechanics and stochastic processes.