Zippers and Univalent Functions

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1. Introduction. Every simply connected domain **C**, with the exception of **C** itself, can be parametrized by a conformal map from the unit disk, that is, by a univalent function. Families of domains which vary continuously with parameters arise in several mathematical contexts, particularly in the theory of Kleinian groups and in the theory of other 1-dimensional complex dynamical systems. A family of domains gives rise to a family of univalent functions. There are various topologies on the set of univalent functions. One topology which has proven valuable for the theory of Kleinian groups is the topology of uniform convergence of the Schwarzian derivative (to be described presently), uniform as measured using the hyperbolic metric on the disk. In this note, we will construct simple examples of domains whose corresponding univalent functions are isolated points in this topology. These domains have complements which are arcs, in fact, quasi-intervals, and they are reminiscent of zippers.

The construction and result here are related to that of Gehring [**1**].

2. Preliminaries. For the benefit of people to whom the Schwarzian derivative may seem a mystery, we will set the stage by discussing the Schwarzian derivative. Actually, the Schwarzian derivative is not really essential to the current discussion. We will use it only for the definition of a topology on the set of univalent maps, and logically we really use an equivalent description of the topology (to be explained later) which makes no mention of Schwarzian derivatives. This is not meant to downplay the significance of the Schwarzian derivative; by analogy, many qualitative constructions in differential geometry can be done without ever mentioning curvature, even though curvature is central to differential geometry.

The Schwarzian derivative is very much like a kind of curvature: the various kinds of curvature in differential geometry measure deviation of curves or manifolds from being flat, while the Schwarzian derivative measures the deviation of a conformal map from being a Mœbius transformation. A Mœbius transformation is determined by its two-jet (its value and first two derivatives) at any point. The two-jet is arbitrary, except for the proviso that the first derivative be nonsingular. Thus, for any locally univalent f and any point w in the domain of f there is associated a unique Mœbius transformation $M(f,w)$ which agrees with f through second order. Note that if A is a Mœbius transformation, then

$$M(A \circ f, w) = A \circ M(f,w).$$

More generally, if f and g are two locally univalent functions, then

$$M(g \circ f, w) = M(g, f(w)) \circ M(f,w).$$

Conformal mappings up to postcomposition (composition on the left) with Mœbius transformations are therefore conveniently described by their three-jets, up to the action of Mœbius transformations on three-jets. The quotient space of this action, at any point z in the domain of the mapping, is isomorphic to $\mathbf{C}$. To see the isomorphism, normalize a map f by postcomposing with the Mœbius transformation $M^{\circ^{-1}}(f,w)$ to obtain a new map having the two-jet of the identity at w. The information that remains in the three-jet is the third derivative, which is an arbitrary element of $\mathbf{C}$. Note that if the original map is a Mœbius transformation, it will have been normalized to be the identity, so its third derivative will be 0.

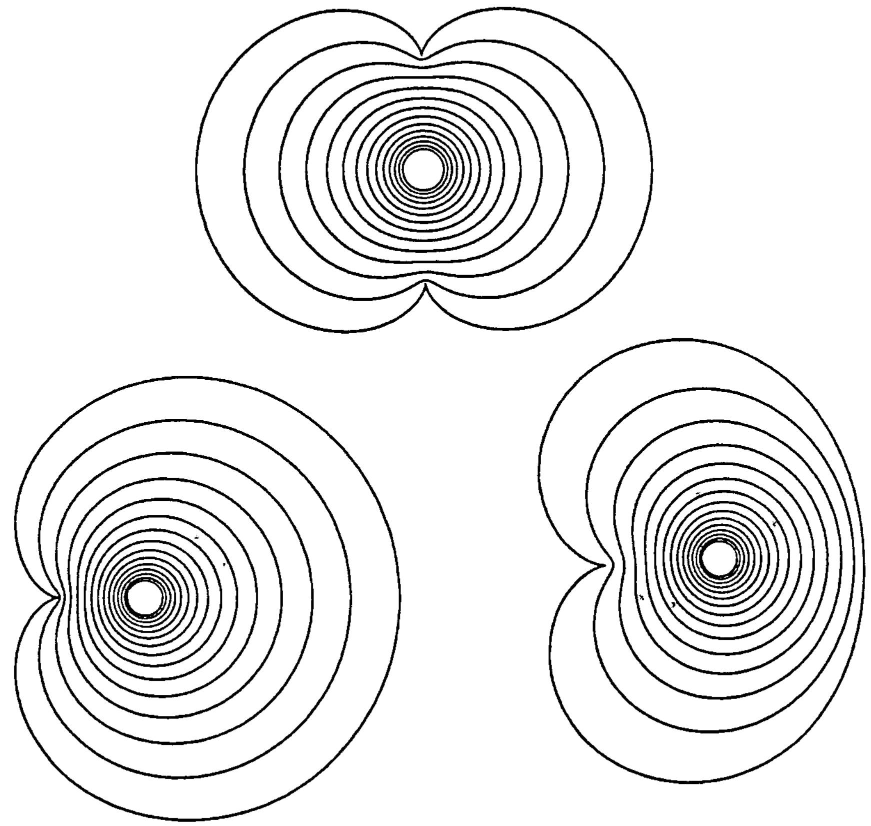
As z varies, this determines a complex vector bundle over the domain of the map whose fiber consists of the bundle of three-jets modulo the chain rule action of Mœbius transformations on three-jets. The linear structure in the fiber arises not because the action is linear (which it isn't), but because a linear structure is inherent in the complex structure of $\mathbf{C}$, once the origin is determined.

What vector bundle is it? A third derivative is a cubic mapping from the tangent space of the domain to the tangent space of the range. However, for a nonsingular mapping, the tangent space of the domain is identified with the tangent space of the range via the first derivative, so we may describe the third derivative in terms of a cubic map P from the tangent space of the domain to itself. We can form the quotient $P(w)/w$ to replace this cubic map by a quadratic map from the tangent bundle of the domain to $\mathbf{C}$ which contains the same information.

It follows that this invariant for locally univalent functions up to Mœbius transformations, called the Schwarzian derivative, is a holomorphic quadratic differential form, usually referred to as a quadratic differential.

Sketching the picture of a conformal map which has the two-jet of the identity at a point (Figure 2.1: jets of locally univalent mappings), one can see that the third derivative tends to make the image of a small circle around a point in the domain look elliptic in the range. Since Mœbius transformations preserve circles, the Schwarzian derivative may be interpreted as measuring the third-order distortion of the shape of the image of a small circle.

A formula for the Schwarzian derivative can be readily determined from the information above, or it may be looked up—someplace else. Like the formula for the curvature of a curve in the plane, the formula looks somewhat mystical at first, and in a qualitative discussion, the formula tends to be a distraction from the real issues.



2.1. FIGURE: JETS OF LOCALLY UNIVALENT FUNCTIONS. Here are three univalent mappings illustrating the effects of second and third derivatives of a locally univalent mapping on a family of concentric circles. In (a) the mapping is $z \mapsto z + z^3/3$, and in (b), $z \mapsto z + z^2/2$. Thus, mapping (a) has the two-jet of the identity, and the third-order effect on the eccentricity of circles, as measured by the Schwarzian derivative, can be seen. Mapping (b) has the 1-jet but not the 2-jet of the identity, so even though its third derivative is 0, its Schwarzian derivative is not 0. The second derivative can be seen in the way that the centers of the images of circles are displaced toward the right. The 2-jet can be corrected by composing with the Mœbius transformation $z \mapsto z/(1 - .5z)$. The composition is illustrated in (c). The third-order term in the Taylor expansion of the composition is $-z^3/4$, and again it can be seen in the third-order effect on eccentricity.

How does the Schwarzian derivative behave under composition? If f and g are two locally univalent mappings which have order two contact with the identity at a point z, then the third derivative of the composition is the sum of their third derivatives, so the Schwarzian derivative of the composition is the sum of the Schwarzian derivatives, at least in this case. The same formula holds in general, provided we are careful to treat the Schwarzian derivative as a quadratic form. This easily follows from the fact that the Schwarzian derivative is invariant under postcomposition by a Mœbius transformation, and it transforms as a quadratic form under precomposition by a Mœbius tranformation. More explicitly, for the Schwarzian derivative of an arbitrary composition $f \circ g$ evaluated at a point z, one can replace it by $(f \circ M_1) \circ (M_1^{\circ -1} \circ g \circ M_2)$, where M_1 and M_2 are the Mœbius transformations chosen so that the terms in parentheses have the 2-jet of the identity at $f \circ g(z)$.

The Schwarzian derivative together with the 2-jet of a locally univalent mapping determines the 3-jet, which is to say that the mapping satisfies a third-order differential equation in terms of its Schwarzian derivative. Since it is an ordinary differential equation, the equation can be locally integrated no matter what the holomorphic quadratic differential. Thus, every quadratic differential occurs, at least locally, as the Schwarzian derivative of a conformal map. Since Mœbius transformations act transitively on nonsingular 2-jets, the three degrees of freedom for the solution are accounted for by compositions with Mœbius transformations. If the quadratic differential is defined over a simply connected domain, local solutions can be pasted together using Mœbius transformations to give a locally univalent mapping defined from the entire domain to the Riemann sphere $\hat{\mathbf{C}}$ and having the prescribed Schwarzian derivative.

Here is a slightly different way to think of the Schwarzian derivative. As a point w varies in the domain of f, the approximating Mœbius transformation $M(f,w)$ varies. The derivative of $M(f,w)$ with respect to w associates to any tangent vector V at w an infinitesimal Mœbius transformation, which is a holomorphic vector field on the Riemann sphere

$$X(f,w,V) = \frac{\partial}{\partial t} M^{\circ -1}(f,w) \circ M(f, w+tV).$$

The order of composition has been chosen so that $X(A \circ f, w, V) = X(f,w,V)$, so we may normalize f to have order 2 contact with the identity at w without changing X. Then it is clear that $X(f,w,V)$ vanishes through order 1 at w. X is thus determined by its second derivative, which is an element of $\hom(T \otimes T, T)$, where T is the tangent space to $\hat{\mathbf{C}}$ at w. Therefore we have as an invariant of f at w an element of $\hom(T, (T \otimes T, T))$, which is canonically isomorphic to $\hom(T \otimes T \otimes T, T)$, which in turn is isomorphic to $\hom(T \otimes T, \mathbf{C})$ (taking into account the fact that T is 1-dimensional).

This invariant is, of course, the same as the Schwarzian derivative of f, but it has perhaps a clearer interpretation as measuring the rate of change of the best approximating Mœbius transformation.

Consider a univalent mapping f, that is, a conformal embedding of the open unit disk in $\mathbf{C}$. We can normalize f by postcomposition with an affine transformation until it has the 1-jet of the identity at the origin. Once this is done, it is easy to see that each derivative of f is bounded in a way which is independent of f. It follows that the Schwarzian derivatives of univalent functions are uniformly bounded with respect to the hyperbolic metric on the domain disk.

One useful topology on the set of univalent functions is the topology of uniform convergence on compact subsets of the open disk; this topology is used in the theory of normal families, and in particular, in the standard proof of the Riemann mapping theorem. The group of complex affine transformations acts on univalent functions, and the quotient space is contractible with respect to this topology: if f is normalized to have derivative 1 at the origin, then the isotopy

$$f_t(z) = \frac{1}{t} f(tz) \qquad [0 < t \le 1]$$

connects f to the identity map (which is the limit as t approaches 0).

This topology is much too weak for many other purposes, however. Since Schwarzian derivatives are uniformly bounded, another natural topology comes from uniform convergence on quadratic differentials in the hyperbolic plane. This works best on the set of univalent maps up to postcomposition with Mœbius transformations. Denote this space by S. An element of S may be represented by any conformal embedding of the unit disk on the Riemann sphere $\hat{\mathbf{C}}$—up to Mœbius transformations; it is pointless to require that infinity is in the complement of the image.

Because of the chain rule above for Schwarzian derivatives, an ε neighborhood of a univalent mapping f consists precisely of mappings which have the form $g \circ f$, where g is a univalent mapping defined on the image of f whose Schwarzian derivative has norm less than ε as measured in the Poincaré metric of the image of f.

Here is a macroscopic (rather than infinitesimal) description of the topology of S. If f_1 and f_2 are two univalent mappings, we can compare them by considering the disk of radius R with respect to the hyperbolic metric about a point z in the domain of the mappings. Change coordinates to put z at the origin, and normalize the mappings to have the 2-jet of the identity at 0, i.e., let B be a conformal automorphism of the unit disk sending the origin to z, and let the normalized transformation be $g_i = M^{\circ -1}(f_i \circ B, 0) \circ f_i \circ B$. Then define $d_{z,R}(f_1, f_2)$ to be the maximum distance between g_1 and g_2 in the hyperbolic disk of radius R about the origin, measured with respect to the spherical metric in the range. We can define f_1 and f_2 to be (ε, R) close if $d_{z,R} < \varepsilon$ for all z in the open unit disk. This determines a neighborhood basis for a topology. Actually, we would arrive at a neighborhood basis for the same topology if we picked any particular R and held it fixed.

In fact, the resulting topology would be the same if the normalized maps were prescribed to be not only pointwise close, but C^r close, since derivatives of holomorphic maps are estimated in terms of values, on account of the Cauchy integral formula. Clearly, this implies the Schwarzian derivatives are close. On the other hand, solutions of differential equations depend continuously on the data, so if the Schwarzian derivatives of f_1 and f_2 are uniformly close, then f_1 and f_2 are (ε, R) close. Thus, the topology is the same as that of S.

What is the macroscopic analogue of the description of neighborhoods of univalent functions in terms of compositions with functions having small Schwarzian derivatives? To construct such a description, we must modify the definition of $d_{z,R}$ so that it can be applied to functions defined on a domain U other than the unit disk. The difficulty is that we need a way to measure the distance between the values of two mappings which is not affected by postcomposition with a Mœbius transformation. In the infinitesimal setting, we used the Poincaré metric of the domain, but this will not do in the macroscopic setting since the mapping is unlikely to have image contained in U.

The solution is to measure using the unique round metric $R(z, U)$ on $\hat{\mathbf{C}}$ which has first-order contact to the Poincaré metric at a point z. By round, we mean the metric is equivalent to the standard spherical metric by a Mœbius transformation. If B is a conformal mapping of the unit disk to U which sends 0 to z, then this round metric is the image of the standard spherical metric under $M(B, 0)$. More generally, the round metric associated to a point z in a domain U transforms by $M(g, z)$ if g is a conformal embedding of U in $\hat{\mathbf{C}}$. (Note that the transformation of the 1-jet of a metric depends on the 2-jet of a mapping.)

One way to visualize the associated round metric is to observe that a choice of a round metric is equvialent to a choice of a point p in the $\mathbf{H}^3$ bounded by $\hat{\mathbf{C}}$. The round metric is the visual metric at p, that is, it is induced from the metric on the unit tangent space to $\mathbf{H}^3$ at p via the homeomorphism which sends a tangent vector to the endpoint of its geodesic on the sphere at infinity. Thus, to a point z in a domain U there is canonically associated a point $p(z, U)$ in $\mathbf{H}^3$. In the case of a round disk, the associated point is the perpendicular projection to the hyperbolic plane with the same bounding circle. The point $p(z, U)$ is transformed by the formula $p(f(z), f(U)) = M(f, z) \circ p(z, U)$ whenever U is mapped to another domain by an embedding (or even a covering map).

Now it is clear that another neighborhood basis for an element f of S consists of the sets of maps of the form $g \circ f$, where g is a conformal embedding of the image U of the unit disk which has the property that for each $z \in U$, $M^{\circ^{-1}}(g, z) \circ g$ is within ε of the identity when restricted to the Poincaré ball of radius R about z and measured using $R(z, U)$.

We shall prove that S has uncountably many isolated points. This contrasts strongly with the contractibility of the set S when it is equipped with the topology of uniform convergence on compact subsets.

In passing, we point out that there is a close connection developed by Lipman Bers between the space S and Teichmüller spaces. In fact, denote by U the space of locally univalent functions with the topology of uniform convergence of

Schwarzian derivatives, so that $S \subset U$ has the topology induced from U. The interior T of S is called the universal Teichmüller space. All ordinary Teichmüller spaces are embedded in the universal Teichmüller space in a natural way. T is also characterized as consisting of those univalent functions whose image is bounded by a quasicircle (see definition below). The main result of Gehring [**1**] is that S is not the closure of T.

3. Zippers.

The image of a univalent mapping determines the mapping, up to precomposition with an isometry of the domain hyperbolic plane. The isometries of $\mathbf{H}^2$ do not act continously on S. Still, we will analyze univalent functions in terms of the shapes of their images.

We will say that an open set U is *rigid* if there is some ε such that any conformal embedding U in $\hat{\mathbf{C}}$ with Schwarzian derivative less than ε is a Mœbius transformation. When U is simply connected, this is equivalent to the condition that *every* Riemann mapping from the open disk to the domain represents an isolated point in S. Note that the condition of rigidity makes sense even when the domain is not simply connected.

A *quasi-interval* in $\hat{\mathbf{C}}$ is an arc α in $\hat{\mathbf{C}}$ with the property that there is a constant K such that for any two points $a, b \in \alpha$, the subarc between a and b lies in a disk of radius $K\, d(a, b)$ about a. Ahlfors showed that quasi-intervals are exactly the quasiconformal images of a unit interval in the real line. (Note: this follows from a similar characterization of the quasiconformal images of circles, using a square root construction.) The main point of this paper will be to prove

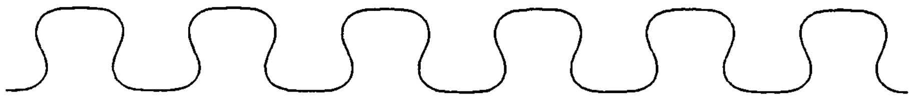
3.1. THEOREM: RIGID QUASI-INTERVALS. *There are many quasi-intervals with rigid complement.*

A more precise formulation of the statement will be given in Theorem 4.3: incorrigible implies rigid, where "many" will be replaced by a specific definition.

The proof of this theorem may make it sound like S has such a strong topology that almost no domains should be nonrigid, except for cases such as Jordan domains whose complement contains an open set, and geometrically simple cases such as domains with piecewise-smooth frontier. Perhaps so. On the other hand, there are many interesting nonrigid examples which arise in the theory of Kleinian groups: the complements of the limit sets of all known degenerate groups are nonrigid. Maybe the technique here can be turned around to help give information about the geometry and topology of these limit sets.

We will now give an intuitive description of a construction for rigid quasi-intervals. Rather than making the specific estimates needed to demonstrate the particular example rigorously, we will in the next section give a general proof that a general class of quasi-intervals (including the ones we construct) have rigid complements.

The idea is based on zippers. Zippers have the property that they cannot be pulled apart in the middle without a lot of distortion. In the same way, the



3.2. FIGURE. A simple zipper curve.

simple zipper curve of Figure 3.2 has the property that it separates the plane into two components which cannot be moved apart disjointly without first making a definite amount of distortion, because the zipper teeth on the two sides of the curve interlock. The usual custom in the physical world is to make zippers with rectilinear teeth, but these smooth teeth will serve the same purpose while being more convenient for constructing higher-order zippers.

The curve of Figure 3.2 is part of an embedded line in $\mathbf{C}$ invariant by a translation, which is a parabolic Mœbius transformation. Similarly, we can construct a simple zipper arc Z_1 which is invariant by a hyperbolic Mœbius transformation. Any conformal embedding of the complement of Z_1 with small Schwarzian derivative (measured as a quadratic differential using the Poincaré metric of the complement of Z_1) will have zipper teeth of approximately the same shape. If ε is small enough that a few of the teeth are interlocked, there is a cascade effect: all the other teeth will be forced to be interlocked.

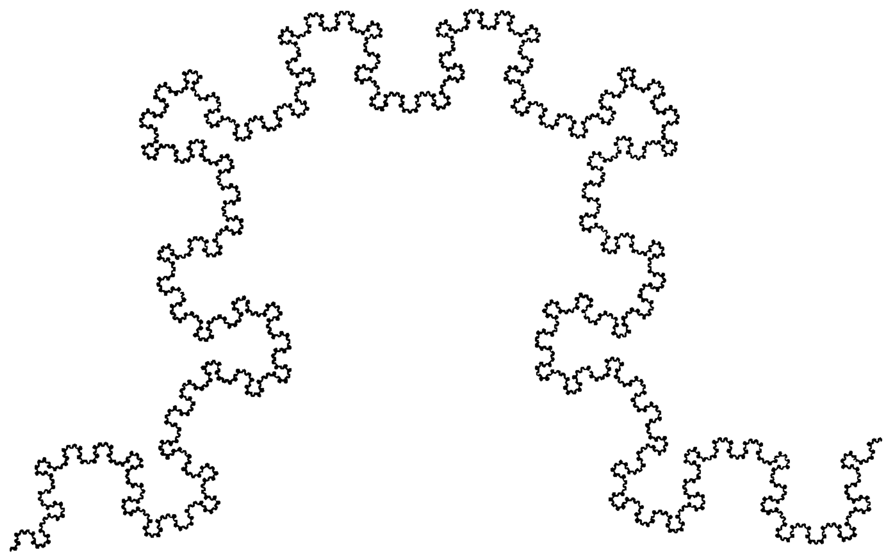
The complement of Z_1 is not rigid. For one thing, the Schwarzian derivative varies continuously if the curve is wiggled in the C^1 topology. To take care of this, we iterate the zipper construction. Inductively replace the curve Z_k by a new curve Z_{k+1} lying in a thin band about the original and having teeth of a smaller scale. There is a limit curve Z_∞.

Any conformal embedding of the complement of Z_∞ with Schwarzian derivative uniformly near 0 must have the largest of the top level teeth interlocked. Because of the cascade effect, all the top level teeth are interlocked. Also, this forces the level 2 teeth to be approximately interlocked; hence they too are interlocked. The cascade effect proceeds to finer and finer levels, forcing all teeth at all levels to be interlocked.

This implies that the new conformal embedding extends continuously over Z_∞, giving a homeomorphism of $\hat{\mathbf{C}}$ which is conformal at least in the complement of Z_∞.

3.3. PROPOSITION: EXTENSION CONFORMAL. *Any homeomorphism of* $\hat{\mathbf{C}}$ *which is conformal in the complement of a quasi-interval* α *is a Mœbius transformation.*

PROOF. By the theorem of Ahlfors mentioned above, α can be straightened to a line segment by a quasiconformal homeomorphism. The proposition is easy to prove in the case of a line segment, using the Cauchy integral formula. This implies that the original map f is at least quasiconformal. Its quasiconformal distortion is zero almost everywhere; therefore f is conformal. Consequently, it is a Mœbius transformation. □



3.4. FIGURE: ITERATED ZIPPER CURVE. A detail of the iterated zipper arc Z_∞. The picture will look much the same if any small tooth is enlarged to the size of this large tooth. On a large scale, the entire zipper arc fits inside a banana-shaped region of the $\mathbf{C}$, and it is invariant by a hyperbolic transformation.

This gives a demonstration, at least on the intuitive level, that the complement of Z_∞ is rigid.

4. Compactness.

We will make a more rigorous and more general analysis by considering all possible quasi-intervals microscopically. For this, it is convenient to consider quasi-intervals in $\mathbf{C}$, so that a K-quasi-interval when magnified by any constant is still a K-quasi-interval. (On the 2-sphere, the magnification would bring the two endpoints close together, so that the constant would have to go to infinity.) As a limiting case, there are also quasilines and quasihalf-lines, which are properly embedded copies of $(-\infty, \infty)$ or $[0, \infty)$.

The Hausdorff metric is a metric on the set of closed subsets of a metric space which says that two sets are within ε if every point in either set is within ε of some point of the other set. Since $\mathbf{C}$ is noncompact, it is convenient to extend closed subsets of $\mathbf{C}$ to subsets of $\hat{\mathbf{C}}$ by forming the closure, and to measure the Hausdorff distance of the resulting closed subsets of $\hat{\mathbf{C}}$ using the spherical metric. The set of closed subsets of $\mathbf{C}$ with this topology forms a compact metric space.

Let $Q(K)$ be the set of closed sets consisting of the empty set and all singletons, together with all K-quasi-intervals, K-quasilines, and K-quasihalf-lines.

4.1. PROPOSITION: CLOSURE QUASIARCS. *The set* $Q(K)$ *is closed and hence compact.*

PROOF. This is left as a nice exercise for the reader. □

Given any K-quasi-interval α, let $E(\alpha)$ consist of all enlargements of α, that is, the set of all images of α under transformations $z \mapsto az+b$, where $|a| \geq 1$.

4.2. DEFINITION. A K-quasi-interval α is *incorrigible* if the closure of $E(\alpha)$ contains no straight lines.

It is easy to see that incorrigibility is equivalent to the condition that the closure of $E(\alpha)$ contains no circles.

4.3. THEOREM: INCORRIGIBLE IMPLIES RIGID. *The complement of any incorrigible K-quasiarc is rigid.*

4.4. COROLLARY: RIGID DOMAINS EXIST. *There are domains which are rigid.*

PROOF. For one example, use the complement of Z_∞. There will be no lines in the closure of $E(Z_\infty)$ provided the construction is carried out in such a way that the ratio of the size of the teeth of the kth level to the teeth of the $(k+1)$st level is bounded.

A more interesting example comes from the limit set of any quasi-Fuchsian group which has no parabolic elements and which is not Fuchsian. Let α be any arc on such a limit set. The limit set has the property that the set of subsets of $\mathbf{C}$ obtained by conjugating the limit set by any Mœbius transformation which makes it pass through 0 and ∞ is already a closed subset of $Q(K)$, for some K. In fact, the set of such conjugating transformations, up to the action of the quasi-Fuchsian group, is compact. This implies that $E(\alpha)$ contains no lines. □

Note that if a quasi-Fuchsian group contains parabolics, the closure of $E(\alpha)$ in fact does contain lines, although a refinement of the proof of the theorem would show that the complement of any subarc of such a limit set is rigid anyway.

PROOF OF THEOREM 4.3. We will prove the theorem by proving a more positive form of the statement: If α is a K-quasi-interval and f_i a sequence of conformal embeddings which are not Mœbius transformations but whose Schwarzian derivatives converge uniformly to 0, then there are lines (in fact, many lines) in the closure of $E(\alpha)$.

Note that the closure of $E(\alpha)$ is invariant under the process of enlargement. What we will do is pass to a limit of enlargements of α, then a limit of enlargements of this, and so on until we find a line.

We may assume that f_i is normalized at some base point on $\hat{\mathbf{C}}$, to have the 2-jet of the identity transformation at that point. It is convenient to take as base point the point at infinity. The f_i converge uniformly on compact sets of the domain $U = \hat{\mathbf{C}} - \alpha$ to the identity.

We need a bit of information about the qualitative properties of a quasi-interval and the behavior of maps with small Schwarzian derivatives.

4.5. LEMMA. *Let t be any point in the interior of α, and let D be the disk of radius $3Kr$ about t. There must be points within D which are at distance exactly r from α.*

PROOF. Indeed, t separates α into two subarcs α_1 and α_2. Let A be the annulus about t with inner radius $2Kr$ and outer radius $3Kr$. The r-neighborhoods of the α_i intersected with A must be disjoint, since α is a K-quasiarc.

In the typical case that these intersections are nonempty, since the annulus is connected we can find points in the frontier of their union, which is the r-neighborhood of α, as claimed. If each α_i intersects the circle of radius $3Kr$, we can even find a point as claimed on both homological sides of the intersection of α with the $3Kr$-disk.

Even near the ends of α, where only one of the pieces (say α_1) of α intersects the annulus, a related argument still works. To see this, consider the double cover of $\mathbf{C}$ branched at t. The inverse image of the r-neighborhood of α_1 intersect A must have two components, so there are points in their frontier. □

The lemma implies that the collection of disks centered about points in the complement of α whose radius is $3K$ times the distance to α cover $\mathbf{C}$. In fact, the subcollection consisting only of those disks of any fixed radius cover some neighborhood of α.

We will define a quantity $D(f,z)$ which is a measure of how far f is distorted from a Mœbius transformation in the vicinity of $z \in U$. D will measure distortion roughly on the scale of the distance $r(z)$ of z from α. The definition is that $D(f,z)$ is the maximum distance of $M^{\circ^{-1}}(f,z) \circ M(f,w)$ from the identity, measured with respect to the round metric $R(z,U)$ on $\hat{\mathbf{C}}$, as w ranges over the set of points a distance at least $r(z)/2$ from α but distance at most $10Kr(z)$ from z.

Note that $D(f,z)$ goes to 0 as z goes to infinity. What happens for z near α?

4.6. PROPOSITION: SMALL DISTORTION IMPLIES CONTINUOUS. *There is a constant $a > 0$ such that if $D(f,z)$ is less than a in some neighborhood of α, then f extends continuously to α, and hence f is a Mœbius transformation.*

PROOF. The condition that a point w is within the disk of radius $10Kr(z)$ about z but has distance more than $r(z)/2$ from α implies that the round metrics $R(z,U)$ and $R(w,U)$ are roughly compatible, or in other words, there is an a priori bound for $d(p(z,U),p(w,U))$. In fact, among points z such that $r(z)$ is less than a bounded fraction of the diameter of α, the metric $R(z,U)$ deviates by a bounded amount from the metric obtained by enlarging the picture until $r(z)=1$, then mapping to a sphere via the stereographic projection which sends the unit circle about z to the equator of the sphere. To see this, consider the case that $r(z)=1$. Make a parabolic transformation fixing z so that α goes through ∞. The first two derivatives of a univalent function which sends the origin to z are bounded, which means that the 1-jet of the Poincaré metric deviates by a bounded amount from the 1-jet of the Euclidean metric, which is the same as the 1-jet of the round metric defined by stereographic projection.

The proof of the proposition has to do with the nesting properties of the disks of radius $10Kr(z)$ about points z of distance $r(z)$ from α.

For any point z' with $r(z') = r(z)/2$ and such that the $3Kr$ disks for z and z' intersect, then they are at distance no more than $3Kr(z) + 3Kr(z') = 4.5Kr(z)$ from each other. Then the $10Kr(z') = 5Kr(z)$ disk about z' is contained in the $10Kr(z)$ disk about z, leaving a $r(z)/2$ margin of safety.

Under the hypothesis that $D(f,z)$ is small, the transformation $M^{\circ^{-1}}(f,z) \circ M(f,z')$ sends the $10Kr(z')$ disk safely inside the $9.9Kr(z)$ disk about z.

Consider a pair $A \supset B$ of nested disks and all possible images of them under Mœbius transformations taking A to the finite complex plane. There is a maximum for the ratio of the radii of these images; this upper bound is attained when the two disks are concentric. It follows that there is an upper bound $\beta < 1$ to the ratio of the radius of the image of the $10Kr(z')$ disk about z' by $M(f,z')$ to that of the $10Kr(z)$ disk about z under $M(f,z)$, where z and z' range over pairs of points chosen as above. In fact, β can be found as a function only of $D(f,z)$; in the circumstances above, certainly $\beta < .99$.

Renormalize f for convenience to make sure that $M(f,z)$ maps the $10Kr(z)$ disk about z to a disk E in the finite complex plane. Let R be the radius of E. The image of the $10Kr(z')$ disk about z' by $M(f,z')$ is contained in E and has a radius less than βR.

Inductively, it follows that for any point w such that $r(w) = 2^{-k}r(z)$ whose $3Kr(w)$ disk intersects the $3Kr(z)$ disk about z, the diameter of its image under $M(f,w)$ is a disk of radius less than $R\beta^k$ contained in the original disk E. In particular, the image by f of the entire $3Kr(z)$ disk about z is contained in the image by $M(f,z)$ of the $10Kr(z)$ disk.

Consequently, for any point t on α, there are neighborhoods of t in $\mathbf{C}$ whose images under f (where f is defined) have arbitrarily small diameter. Therefore f converges along α to a continuous map. □

It is interesting to remark here that when f is a mapping with small Schwarzian derivative, the hypothesis of the proposition would be satisfied if the measure of distortion $D(f,z)$ were modified so that only points w are considered which are on the same side of α as z. The proof of the proposition therefore shows that f has a continuous limit on each side of α—but in general, the limits from the two sides will be different.

PROOF OF THEOREM 4.3 (CONTINUED). Now back to our sequence f_i. By assumption, the f_i are not Mœbius transformations, so there is a constant a such that $D(f_i,z) > a$ for points z arbitrarily close to α.

By the intermediate value theorem, for each sufficiently large integer $n > 0$, there is a point $x_{i,n}$ where D_i takes the value $1/n$. Let $f_{i,n}$ be a rescaling of f_i so that $x_{i,n}$ is at the origin, and α becomes a curve $\alpha_{i,n}$ at distance one from the origin. We renormalize $f_{i,n}$ to have the 2-jet of the identity at the origin.

If we fix n and let i tend to infinity, there is a subsequence of the i's so that $\alpha_{i,n}$ converges to a quasiline or quasihalf-line $\alpha_{\infty,n}$, and so that the $f_{i,n}$ converge to a transformation $f_{\infty,n}$ which is the identity near 0 and is locally Mœbius (since its Schwarzian derivative is 0) but not Mœbius, since $D(f_{i,n},0)$ is constant. Obviously, the complement of $\alpha_{\infty,n}$ cannot be connected. Therefore, $\alpha_{\infty,n}$ must be a quasiline which separates $\mathbf{C}$ into two components, U_0 which contains the origin and U_1 which does not. The transformation $f_{\infty,n}$ is the identity on U_0, and on U_1 it is a nontrivial Mœbius transformation taking U_1 into itself.

Now we let n go to infinity. There is some subsequence such that the $\alpha_{\infty,n}$ converge to a quasiline α_∞. The transformations $f_{\infty,n}$ converge to the identity on both sides of the limiting quasiline, since $D(f_{\infty,n},0)$ converges to 0.

Let X be any vector field which represents a limiting direction for the Mœbius transformations $f_{\infty,n}$. By "direction," we refer to the direction from the identity in the Lie group of Mœbius transformations. More explicitly, let

$$X(z) = \left[\lim a_{i_j}(f_{\infty,i_j}(z) - z)\right] \frac{\partial}{\partial z},$$

where the subsequence $\{i_j\}$ and sequence of constants a_{i_j} are chosen to make the limit exist and be nonzero.

We claim that X carries U_1 into itself. To see this, consider any flow line of X through a point x in U_1. For any point y downstream from x on the flow line, there is some $f_{\infty,n}$ and some integer q such that its qth iterate carries x approximately to y. It follows that y is at least in the closure of U_1. Since the image of an open set by a time t map of a flow is open, y must be in U_1 itself.

Choose a point t on α where the flow is not zero, and conjugate by a sequence of enlargements centered about t so that in the limit X becomes an infinitesimal translation in the i direction. Passing to the limit of a subsequence, we obtain a new quasiline β, which intersects each line parallel to the y-axis in an interval or a point.

If any of these intersections are intervals, we immediately obtain a line in the closure of $E(\alpha)$. In fact, if the slopes of secant lines connecting pairs of points of β are unbounded (either above or below), there is a sequence of enlargements carrying such pairs of points to pairs at distance n from each other. The limit is a vertical line, so we are done in this case also.

If the slopes of the secant lines are bounded, then β is the graph of a Lipschitz function. A Lipschitz function is differentiable almost everywhere. A sequence of enlargements centered at any point where the function is differentiable converges to a line whose slope is the derivative. □

REFERENCES

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