Unitary Time Changes of Stationary Processes Yield Oscillatory Processes and a Functional Framework Toward a Hilbert–Pólya Construction

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are square-integrable over compact sets. Applying U_θ to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\ e^{i\lambda\theta(t)}$ and evolutionary spectrum $d\,F_t(\lambda) = \dot{\theta}(t) d\,F(\lambda)$. It is proved that sample paths of any non-degenerate second-order stationary process almost surely lie in $L^2_{\mathrm{loc}}(\mathbb{R})$, making the operator applicable to typical realizations. A zero-localization measure $\mathrm{d}\mu\,(t) = \delta(Z(t))\,|\dot{Z}(t)|\ d\,t$ induces a Hilbert space $L^2(\mu)$ on the zero set of an oscillatory process Z, and the multiplication operator $(L\,f)(t) = t\,f(t)$ has simple pure point spectrum equal to the zero crossing set of Z. This produces a concrete operator scaffold consistent with a Hilbert–Pólya-type viewpoint.

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1 Function Spaces and Unitary Time Change

1.1 σ-compact sets and local L^2

Definition 1. ***[σ-compact sets]*** *A subset $U \subseteq \mathbb{R}$ is σ-compact if*

$$U = \bigcup_{n=1}^{\infty} K_n \tag{1}$$

with each K_n compact.

Definition 2. ***[Locally square-integrable functions]*** *Define*

$$L^2_{\mathrm{loc}}(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} : \int_K |f(t)|^2 \, dt < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \tag{2}$$

Remark 3. Every bounded measurable set in $\mathbb{R}$ is compact or contained in a compact set; hence $L^2_{\mathrm{loc}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

1.2 Unitary time-change operator

Definition 4. ***[Unitary time-change]*** *Let the time-scaling function $\theta : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with $\dot{\theta}(t) > 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero. The function θ maps σ-compact sets to σ-compact sets. Define, for f measurable,*

$$(U_\theta f)(t) = \sqrt{\dot{\theta}(t)} \, f(\theta(t)) \tag{3}$$

Proposition 5. ***[Inversion of Unitary time-change]*** *The inverse of the unitary time-change operator U in Equation (3) is given by*

$$(U_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{4}$$

which is well-defined almost everywhere on every σ-compact set.

Proof. Since $\dot{\theta}(t)=0$ only on sets of measure zero, and θ^{-1} maps sets of measure zero to sets of measure zero because of the fact that absolutely continuous bijective functions preserve measure-zero sets, the denominator $\sqrt{\dot{\theta}(\theta^{-1}(s))}$ is positive almost everywhere. The expression is therefore well-defined almost everywhere on every σ-compact set, which suffices for defining an element of $L^2_{\mathrm{loc}}(\mathbb{R})$. □

Theorem 6. ***[Local unitarity on σ-compact sets]*** *For every σ-compact set $C\subseteq\mathbb{R}$ and $f\in L^2_{\mathrm{loc}}(\mathbb{R})$,*

$$\int_C |(U_\theta f)(t)|^2\,dt=\int_{\theta(C)}|f(s)|^2\,ds \tag{5}$$

Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\mathrm{loc}}(\mathbb{R})$.

Proof. Let $f\in L^2_{\mathrm{loc}}(\mathbb{R})$ and let C be any σ-compact set. The local L^2-norm of $U_\theta f$ over C is:

$$\begin{aligned}\int_C |(U_\theta f)(t)|^2\,dt &=\int_C \left|\sqrt{\dot{\theta}(t)}\,f(\theta(t))\right|^2 dt\\ &=\int_C \dot{\theta}(t)|f(\theta(t))|^2\,dt\end{aligned} \tag{6}$$

Since θ is absolutely continuous and strictly increasing, applying the change of variables $s=\theta(t)$ gives

$$ds=\dot{\theta}(t)\,dt \tag{7}$$

almost everywhere. Since θ maps σ-compact sets to σ-compact sets, as t ranges over C, $s=\theta(t)$ ranges over $\theta(C)$, which is σ-compact. Therefore:

$$\int_C \dot{\theta}(t)\,|f(\theta(t))|^2\,dt=\int_{\theta(C)}|f(s)|^2\,ds \tag{8}$$

To verify that U_θ^{-1} is indeed the inverse, it is seen that:

$$\begin{aligned}(U_\theta^{-1}U_\theta f)(s) &=\left(U_\theta^{-1}\sqrt{\dot{\theta}(s)}\,f(\theta(s))\right)(s)\\ &=\frac{\sqrt{\dot{\theta}(\theta^{-1}(s))}}{\sqrt{\dot{\theta}(\theta^{-1}(s))}}\,f(\theta(\theta^{-1}(s))) \qquad \forall f\in L^2_{\mathrm{loc}}(\mathbb{R})\\ &=f(s)\end{aligned} \tag{9}$$

since

$$\theta(\theta^{-1}(s))=s \tag{10}$$

and similarly, its also plain to see that:

$$\begin{aligned}(U_\theta U_\theta^{-1}\, g)(t) &= \sqrt{\dot{\theta}(t)}\,(U_\theta^{-1}\, g)(\theta(t))\\ &= \frac{\sqrt{\dot{\theta}(t)}}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}}\, g(\theta^{-1}(\theta(t))) \qquad \forall g\in L^2_{\mathrm{loc}}(\mathbb{R})\\ &= \frac{\sqrt{\dot{\theta}(t)}}{\sqrt{\dot{\theta}(t)}}\, g(t)\\ &= g(t)\end{aligned} \tag{11}$$

since

$$\theta^{-1}(\theta(t)) = t \tag{12}$$

Therefore

$$\begin{aligned}(U_\theta U_\theta^{-1}\, f)(t) &= (U_\theta^{-1}\, U_\theta\, f)(t)\\ &= f(t)\end{aligned} \tag{13}$$

on $L^2_{\mathrm{loc}}(\mathbb{R})$. □

2 Oscillatory Processes (Priestley)

Definition 7. ***[Oscillatory process]*** *Let F be a finite nonnegative Borel measure on $\mathbb{R}$. Let*

$$A_t \in L^2(F) \forall t \in \mathbb{R} \tag{14}$$

be the gain function and

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{15}$$

be the corresponding oscillatory function then an oscillatory process *is a stochastic process which can be represented as*

$$\begin{aligned}Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda)\, \mathrm{d}\,\Phi(\lambda)\\ &= \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, \mathrm{d}\,\Phi(\lambda)\end{aligned} \tag{16}$$

where Φ is a complex orthogonal random measure with spectral measure F, that is,

$$\mathbb{E}[\Phi\,(d\,\lambda)\,\overline{\Phi\,(d\,\mu)}] = \delta\,(\lambda-\mu)\; d\,F(\lambda) \tag{17}$$

and corresponding covariance kernel

$$\begin{aligned}R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}]\\ &= \int_{\mathbb{R}} A_t(\lambda)\, \overline{A_s(\lambda)}\, e^{i\lambda(t-s)}\; d\,F(\lambda)\\ &= \int_{\mathbb{R}} \phi_t(\lambda)\overline{\phi_s(\lambda)}\mathrm{d}\,F(\lambda)\end{aligned} \tag{18}$$

Theorem 8. ***[Real-valuedness criterion for oscillatory processes]****Let Z be an oscillatory process with oscillatory function*

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{19}$$

and spectral measure F. Then Z is real-valued if and only if

$$A_t(-\lambda) = \overline{A_t(\lambda)} \tag{20}$$

for F-almost every $\lambda \in \mathbb{R}$, equivalently

$$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \tag{21}$$

for F-almost every $\lambda \in \mathbb{R}$.

Proof. Assume Z is real-valued, i.e.

$$Z(t) = \overline{Z(t)} \quad \forall t \in \mathbb{R} \tag{22}$$

Writing its oscillatory representation,

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, \mathrm{d}\Phi(\lambda) \tag{23}$$

and taking the complex conjugate gives

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t} \mathrm{d}\overline{\Phi(\lambda)} \tag{24}$$

For a real-valued process, the orthogonal random measure Φ must satisfy

$$\mathrm{d}\overline{\Phi(\lambda)} = -\mathrm{d}\Phi(\lambda) \tag{25}$$

which ensures that the spectral representation produces real values. Substituting this identity and using the substitution

$$\mu = -\lambda \tag{26}$$

it is shown that

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t}\, \mathrm{d}\Phi(\mu) \tag{27}$$

Since $Z(t) = \overline{Z(t)}$, comparison of the integrands (which are unique elements of $L^2(F)$) yields

$$A_t(\lambda) = \overline{A_t(-\lambda)} \quad \text{for } F\text{-a.e. } \lambda \tag{28}$$

Equivalently, because the oscillatory function (15) is given by

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{29}$$

we have

$$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \quad \text{for } F\text{-a.e. } \lambda \tag{30}$$

Conversely, if

$$A_t(-\lambda) = \overline{A_t(\lambda)} \tag{31}$$

for F-a.e. λ, then the same substitution shows that

$$\overline{Z(t)} = Z(t) \quad \forall t \in \mathbb{R} \tag{32}$$

so Z is real-valued. This completes the proof. □

Theorem 9. ***[Existence]*** *If F is finite and $(A_t)_{t\in\mathbb{R}}$ is measurable in t with*

$$\int_{\mathbb{R}} |A_t(\lambda)|^2 \, d\,F(\lambda) < \infty \forall t \in \mathbb{R} \tag{33}$$

then there exists a complex orthogonal random measure Φ with spectral measure F such that

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t} \mathrm{d}\Phi\,(\lambda) \tag{34}$$

is well-defined in $L^2(\Omega)$ and has covariance R_Z as in (18) above.

Proof. The proof proceeds by constructing the stochastic integral using the standard extension procedure. First, the integral is defined for simple functions of the form

$$g(\lambda) = \sum_{j=1}^{n} c_j \mathbf{1}_{E_j}(\lambda) \tag{35}$$

where $\{E_j\}$ are disjoint Borel sets with $F(E_j) < \infty$ and $c_j \in \mathbb{C}$:

$$\int_{\mathbb{R}} g(\lambda)\ \Phi\,(d\,\lambda) := \sum_{j=1}^{n} c_j\, \Phi(E_j) \tag{36}$$

For such simple functions, the isometry property holds:

$$\begin{aligned} \mathbb{E}\left[\left|\int_{\mathbb{R}} g(\lambda)\ \Phi\,(d\,\lambda)\right|^2\right] &= \mathbb{E}\left[\left|\sum_{j=1}^{n} c_j\, \Phi(E_j)\right|^2\right] \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \bar{c_k} \mathbb{E}[\Phi(E_j)\overline{\Phi(E_k)}] \\ &= \sum_{j=1}^{n} |c_j|^2\, F(E_j \\ &= \int_{\mathbb{R}} |g(\lambda)|^2 \, d\,F(\lambda) \end{aligned} \tag{37}$$

Since simple functions are dense in $L^2(F)$, the integral is extended by continuity to all $g \in L^2(F)$. For each t, since the oscillatory function (15) is defined by

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{38}$$

and $A_t \in L^2(F)$, $\varphi_t \in L^2(F)$ holds. Therefore

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda)\, \mathrm{d}\Phi\,(\lambda) \tag{39}$$

is well-defined in $L^2(\Omega)$. The covariance is computed as:

$$\begin{aligned} R_Z(t,s) \;&= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\bigg[\int_{\mathbb{R}} \varphi_t(\lambda) \mathrm{d}\Phi\,(\lambda) \int_{\mathbb{R}} \overline{\varphi_s(\mu)} \mathrm{d}\overline{\Phi\,(\mu)}\bigg] \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\mu)}\,\mathbb{E}\,[\mathrm{d}\Phi(\lambda)\mathrm{d}\overline{\Phi(\mu)}] \\ &= \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)}\, d\,F(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)} e^{i\lambda(t-s)}\; d\,F(\lambda) \end{aligned} \tag{40}$$ □

3 Unitarily Time-Changed Stationary Processes

3.1 Stationary processes

Definition 10. ***[Cramér representation]*** *A zero-mean stationary process X with spectral measure F admits the sample path representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\; \Phi\,(d\,\lambda) \tag{41}$$

which has covariance

$$R_X\,(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)}\; d\,F(\lambda) \tag{42}$$

3.2 Stationary → oscillatory via U_θ

Theorem 11. ***[Unitary time change yields oscillatory process]*** *Let X be zero-mean stationary as in Definition 10. For scaling function θ as in Definition 4, define*

$$\begin{aligned} Z(t) &= (U_\theta X)(t) \\ &= \sqrt{\dot{\theta}(t)}\ X(\theta(t)) \end{aligned} \tag{43}$$

Then Z is a realization of an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\ e^{i\lambda\theta(t)} \tag{44}$$

gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\ e^{i\lambda(\theta(t)-t)} \tag{45}$$

and covariance

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\left[\sqrt{\dot{\theta}(t)}\ X(\theta(t))\overline{\sqrt{\dot{\theta}(t)}\ X(\theta(t))}\right] \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\ \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\ R_X(\theta(t)-\theta(s)) \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\ dF(\lambda) \end{aligned} \tag{46}$$

Proof. Applying the unitary time change operator to the spectral representation of $X(t)$:

$$\begin{aligned} Z(t) &= (U_\theta X)(t) \\ &= \sqrt{\dot{\theta}(t)}\ X(\theta(t)) \\ &= \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)}\ \mathrm{d}\,\Phi(\lambda) \\ &= \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}\,\mathrm{d}\,\Phi(\lambda) \\ &= \int_{\mathbb{R}} \varphi_t(\lambda)\,\mathrm{d}\,\Phi(\lambda) \end{aligned} \tag{47}$$

where

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \tag{48}$$

To verify this constitutes an oscillatory representation according to Definition 7, $\varphi_t(\lambda)$ has the form $A_t(\lambda)\, e^{i\lambda t}$:

$$\begin{aligned}\varphi_t(\lambda) &= \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}\\ &= \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)}\, e^{i\lambda t}\\ &= A_t(\lambda)\, e^{i\lambda t}\end{aligned} \tag{49}$$

where

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)} \tag{50}$$

Since $\dot{\theta}(t) \geq 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of measure zero, $A_t(\lambda)$ is well defined almost everywhere. Moreover, $A_t \in L^2(F)$ for each t since:

$$\begin{aligned}\int_{\mathbb{R}} |A_t(\lambda)|^2\, dF(\lambda) &= \int_{\mathbb{R}} \left|\sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)}\right|^2 dF(\lambda)\\ &= \int_{\mathbb{R}} \dot{\theta}(t) |e^{i\lambda(\theta(t)-t)}|^2\, dF(\lambda)\\ &= \dot{\theta}(t) \int_{\mathbb{R}} dF(\lambda)\\ &= \dot{\theta}(t)\, F(\mathbb{R}) < \infty\end{aligned} \tag{51}$$

where $|e^{i\alpha}| = 1$ for all real α is used. The covariance (18) is computed by substituting the spectral representation and applying Fubuni's theorem to interchange the order of operations.

$$\tag{52}$$ □

Corollary 12. ***[Evolutionary spectrum]*** *The evolutionary spectrum is*

$$\begin{aligned}dF_t(\lambda) &= |A_t(\lambda)|^2\, dF(\lambda)\\ &= \dot{\theta}(t)\; dF(\lambda)\end{aligned} \tag{53}$$

Proof. By definition of the evolutionary spectrum and using the gain function from Theorem 11:

$$\begin{aligned}dF_t(\lambda) &= |A_t(\lambda)|^2\, dF(\lambda)\\ &= \left|\sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)}\right|^2 dF(\lambda)\\ &= \dot{\theta}(t) |e^{i\lambda(\theta(t)-t)}|^2\, dF(\lambda)\\ &= \dot{\theta}(t)\; dF(\lambda)\end{aligned} \tag{54}$$

since

$$|e^{i\alpha}| = 1 \forall a \in \mathbb{R} \tag{55}$$ □

3.3 Covariance operator conjugation

Proposition 13. ***[Operator conjugation]*** *Let*

$$(T_K f)(t) := \int_{\mathbb{R}} K(|t-s|)\ f(s)\ d s \tag{56}$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h}\ d F(\lambda) \tag{57}$$

Define the transformed kernel

$$K_\theta(s,t) := \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\ K(|\theta(t) - \theta(s)|) \tag{58}$$

and corresponding integral covariance operator

$$(T_{K_\theta} f)(t) := \int_{\mathbb{R}} K_\theta(s,t)\ f(s)\ d s \tag{59}$$

Then

$$T_{K_\theta} = U_\theta\ T_K\ U_\theta^{-1} \tag{60}$$

on $L^2_{\mathrm{loc}}(\mathbb{R})$.

Proof. For any $g \in L^2_{\mathrm{loc}}(\mathbb{R})$, compute:

$$\begin{aligned} ((U_\theta T_K U_\theta^{-1})\, g)(t) &= (U_\theta\, (T_K U_\theta^{-1}\, g))(t) \\ &= \sqrt{\dot{\theta}(t)}\ (T_K U_\theta^{-1}\, g)(\theta(t)) \\ &= \sqrt{\dot{\theta}(t)}\ \int_{\mathbb{R}} K(|\theta(t) - w|)\, (U_\theta^{-1}\, g)(w)\ d w \end{aligned} \tag{61}$$

Substitute $w = \theta(s)$ with $d w = \dot{\theta}(s)\ d s$:

$$\begin{aligned} &= \sqrt{\dot{\theta}(t)}\ \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|) \frac{g(s)}{\sqrt{\dot{\theta}(s)}}\, \dot{\theta}(s)\ d s \\ &= \sqrt{\dot{\theta}(t)}\ \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|)\ g(s) \sqrt{\dot{\theta}(s)}\ d s \\ &= \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)\dot{\theta}(s)}\ K(|\theta(t) - \theta(s)|)\ g(s)\ d s \\ &= \int_{\mathbb{R}} K_\theta(t,s)\ g(s)\ d s \\ &= (T_{K_\theta} g)(t) \end{aligned} \tag{62}$$

where

$$K_\theta(t,s) = \sqrt{\dot{\theta}(t)\dot{\theta}(s)}\ K(|\theta(t) - \theta(s)|) \tag{63}$$

Therefore

$$T_{K_\theta} = U_\theta T_K U_\theta^{-1} \tag{64}$$ □

4 Sample Paths Live in L^2_{loc}

Theorem 14. ***[Sample paths in $L^2_{\mathrm{loc}}(\mathbb{R})$]*** *Let $\{X(t)\}_{t\in\mathbb{R}}$ be a second-order stationary process with*

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \tag{65}$$

then, almost surely, every sample path $t \mapsto X(\omega, t)$ belongs to $L^2_{\mathrm{loc}}(\mathbb{R})$.

Proof. Fix any bounded interval $[a, b]$ and consider the random variable

$$Y_{[a,b]} := \int_a^b X(t)^2 \, dt \tag{66}$$

By stationarity and Fubini's theorem:

$$\begin{aligned} \mathbb{E}[Y_{[a,b]}] = \mathbb{E}\left[\int_a^b X(t)^2 \, dt\right] &= \int_a^b \mathbb{E}[X(t)^2] \, dt \\ &= \int_a^b \sigma^2 \, dt \\ &= \sigma^2 (b-a) < \infty \end{aligned} \tag{67}$$

By Markov's inequality, for any $M > 0$:

$$P(Y_{[a,b]} > M) \leq \frac{\mathbb{E}[Y_{[a,b]}]}{M} = \frac{\sigma^2 (b-a)}{M} \tag{68}$$

Taking $M \to \infty$, the conclusion is

$$P(Y_{[a,b]} < \infty) = 1 \tag{69}$$

i.e., almost surely the sample path is square-integrable on $[a, b]$. Since $\mathbb{R}$ is the countable union of bounded intervals:

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n] \tag{70}$$

by countable subadditivity of probability:

$$P\left(\bigcap_{n=1}^{\infty} \left\{\int_{-n}^{n} X(t)^2 \, dt < \infty\right\}\right) = 1 \tag{71}$$

Now let K be any compact set. Then K is bounded, so

$$K \subseteq [-N, N] \tag{72}$$

for some N. Therefore:

$$\int_K X(t)^2 \, dt \leq \int_{-N}^{N} X(t)^2 \, dt < \infty \tag{73}$$

almost surely. This holds for every compact set K, so almost surely every sample path lies in $L^2_{\mathrm{loc}}(\mathbb{R})$. □

5 Zero Localization and Hilbert–Pólya Scaffold

5.1 Zero localization measure

Definition 15. ***[Zero localization measure]*** *Let Z be real-valued with $Z \in C^1(\mathbb{R})$ having only simple zeros*

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \tag{74}$$

Define, for Borel $B \subset \mathbb{R}$,

$$\mu(B) = \int_{\mathbb{R}} \mathbf{1}_B(t)\ \delta(Z(t))\,|\dot{Z}(t)|\ d\,t \tag{75}$$

Theorem 16. ***[Atomicity on the zero set]*** *For every $\phi \in C_c^\infty(\mathbb{R})$,*

$$\int_{\mathbb{R}} \phi(t)\ \delta(Z(t))\,|\dot{Z}(t)|\ d\,t = \sum_{t_0 : Z(t_0) = 0} \phi(t_0) \tag{76}$$

hence

$$\mu(t) = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0}(t) \tag{77}$$

Proof. Since all zeros of Z are simple and $Z \in C^1(\mathbb{R})$, by the inverse function theorem each zero t_0 is isolated. Near each zero t_0, Z is locally monotonic, so the one-dimensional change of variables formula for the Dirac delta can be applied. Specifically, near t_0 where $Z(t_0) = 0$ and $\dot{Z}(t_0) \neq 0$, locally

$$Z(t) = (t - t_0)\,\dot{Z}(t_0) + O((t - t_0)^2) \tag{78}$$

holds. The distributional identity for the Dirac delta under smooth changes of variables gives:

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0) = 0} \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|} \tag{79}$$

Therefore:

$$\begin{aligned} \int_{\mathbb{R}} \phi(t)\ \delta(Z(t))\,|\dot{Z}(t)|\ d\,t &= \int_{-\infty}^{\infty} \phi(t)\,|\dot{Z}(t)| \sum_{t_0 : Z(t_0) = 0} \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|}\ d\,t \\ &= \sum_{t_0 : Z(t_0) = 0} \int_{\mathbb{R}} \phi(t) \frac{|\dot{Z}(t)|\,\delta\,(t - t_0)}{|\dot{Z}(t_0)|}\ d\,t \\ &= \sum_{t_0 : Z(t_0) = 0} \frac{|\dot{Z}(t_0)|}{|\dot{Z}(t_0)|}\,\phi(t_0) \\ &= \sum_{t_0 : Z(t_0) = 0} \phi(t_0) \end{aligned} \tag{80}$$

This shows that μ is the discrete measure

$$\mu(t) = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0}(t) \tag{81}$$

assigning unit mass to each zero. □

5.2 Hilbert space on zeros and multiplication operator

Definition 17. ***[Hilbert space on the zero set]*** *Let $\mathcal{H} = L^2(\mu)$ with inner product*

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, \mathrm{d}\mu(t) \tag{82}$$

Proposition 18. ***[Atomic structure]*** *Let*

$$\mu = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0} \tag{83}$$

then

$$\mathcal{H} \cong \left\{ f : \{t_0 : Z(t_0) = 0\} \to \mathbb{C} : \sum_{t_0 : Z(t_0) = 0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \tag{84}$$

with orthonormal basis $\{e_{t_0}\}_{t_0 : Z(t_0) = 0}$ where

$$e_{t_0}(t_1) = \delta_{t_0}(t_1) \tag{85}$$

Proof. By the atomic form of μ, for any $f \in L^2(\mu)$:

$$\|f\|_{\mathcal{H}}^2 = \int |f(t)|^2\, \mathrm{d}\mu(t) \tag{86}$$

$$= \int |f(t)|^2 \sum_{t_0 : Z(t_0) = 0} \delta_{t_0}(t) \tag{87}$$

$$= \sum_{t_0 : Z(t_0) = 0} |f(t_0)|^2 \tag{88}$$

This shows the isomorphism with ℓ^2 where the functions e_{t_0} defined by

$$e_{t_0}(t_1) = \delta_{t_0}(t_1) \tag{89}$$

satisfy the relations

$$\begin{aligned} \langle e_{t_0}, e_{t_1} \rangle &= \int e_{t_0}(t)\overline{e_{t_1}(t)} \mathrm{d}\mu(t) \\ &= \sum_{t : Z(t) = 0} \delta_{t_0}(t)\, \delta_{t_1}(t) \\ &= \delta_{t_0}(t_1) \\ &= \delta t_1(t_0) \end{aligned} \tag{90}$$

thus forming an orthonormal set. Thus, any $f(t)\in\mathcal{H}$ can be written as

$$f(t)=\sum_{t_0:Z(t_0)=0} f(t_0)\,e_{t_0}(t) \tag{91}$$

proving they form a basis. □

Definition 19. ***[Multiplication operator]*** *Define the linear operator*

$$L:\mathcal{D}(L)\subset\mathcal{H}\to\mathcal{H} \tag{92}$$

by

$$(L\,f)(t)=t\,f(t) \tag{93}$$

on the support of μ *with domain*

$$\mathcal{D}(L):=\left\{f\in\mathcal{H}:\int|t\;\;f(t)|^2\mathrm{d}\mu(t)<\infty\right\} \tag{94}$$

Theorem 20. ***[Self-adjointness and spectrum]*** L *is self-adjoint on* $\mathcal{H}$ *and has pure point, simple spectrum*

$$\sigma(L)=\overline{\{t\in\mathbb{R}:Z(t)=0\}} \tag{95}$$

with eigenvalues $\lambda=t_0$ *for each zero* t_0 *and corresponding eigenvectors* e_{t_0}*.*

Proof. First, self-adjointness is verified. For $f,g\in\mathcal{D}(L)$:

$$\begin{aligned}\langle L\,f,g\rangle &=\int(L\,f)(t)\overline{g(t)}\mathrm{d}\mu\,(t)\\ &=\int t\,f(t)\overline{g(t)}\,\mathrm{d}\mu\,(t)\\ &=\int f(t)\overline{t\,g(t)}\,\mathrm{d}\mu\,(t)\\ &=\int f(t)\overline{(L\,g)(t)}\mathrm{d}\mu\,(t)\\ &=\langle f,L\,g\rangle\end{aligned} \tag{96}$$

Thus L is symmetric and acts as

$$(L\,f)(t_0)=t_0\,f(t_0) \tag{97}$$

for each t_0 in the atomic representation where

$$Z(t_0)=0 \tag{98}$$

This is unitarily equivalent to the diagonal operator on ℓ^2 with diagonal entries

$$\{t_0:Z(t_0)=0\} \tag{99}$$

Such diagonal operators are self-adjoint. For the spectrum calculation:

$$L\,e_{t_0}=t_0\,e_{t_0}\forall\{t_0:Z(t_0)=0\} \tag{100}$$

holds, so each t_0 is an eigenvalue of L with eigenvector e_{t_0} and since $\{e_{t_0}\}$ forms an orthonormal basis, L has pure point spectrum. The spectrum of a diagonal operator equals the closure of the set of diagonal entries, hence

$$\sigma(L) = \overline{\{t_0 \colon Z(t_0) = 0\}} \tag{101}$$

The eigenvalues are simple. □

Remark 21. [Operator scaffold] The construction

$$\text{stationary } X \xrightarrow{U_\theta} \text{oscillatory } Z \xrightarrow{\mu = \delta(Z)|\dot{Z}|\,dt} L^2(\mu) \xrightarrow{L:t\cdot} (L, \sigma(L)) \tag{102}$$

produces a concrete self-adjoint operator whose eigenvalues equal the zero set of Z and whose spectrum equals the closure of the zero set, determined by the choice of time-change θ and spectral measure F. This provides an explicit realization consistent with Hilbert–Pólya approaches to encoding arithmetic information in operator spectra.

5.3 Regularity and Simplicity of The Zeros of $Z(t)$

Definition 22. ***[Regularity and simplicity]*** *Assume $Z \in C^1(\mathbb{R})$ and every zero is simple:*

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \tag{103}$$

Lemma 23. ***[Local finiteness and delta decomposition]*** *Under Definition 22, zeros are locally finite and*

$$\delta(Z(t)) = \sum_{t_0 \colon Z(t_0) = 0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \tag{104}$$

whence

$$\mu = \sum_{t_0 \colon Z(t_0) = 0} \delta_{t_0} \tag{105}$$

Proof. Since $Z \in C^1(\mathbb{R})$ and $\dot{Z}(t_0) \neq 0$ at each zero t_0, the inverse function theorem implies that Z is locally invertible near each zero. Specifically, there exists a neighborhood U_{t_0} of t_0 such that $Z|_{U_{t_0}}$ is strictly monotonic and invertible.

This implies zeros are isolated: if $Z(t_0) = 0$ and $\dot{Z}(t_0) \neq 0$, then there exists $\epsilon > 0$ such that $Z(t) \neq 0$ for $0 < |t - t_0| < \epsilon$. Therefore zeros are locally finite (finitely many in any bounded interval).

For the distributional identity, the one-dimensional change of variables formula for the Dirac delta is considered. If $g \colon I \to \mathbb{R}$ is C^1 on interval I with $\dot{g}(x) \neq 0$ for all $x \in I$, then

$$\delta(g(x)) = \sum_{x_0 \colon g(x_0) = 0} \frac{\delta(x - x_0)}{|\dot{g}(x_0)|} \tag{106}$$

Applying this locally around each zero t_0 of Z, and since zeros are isolated, the local results can be patched together to obtain the global identity:

$$\delta(Z(t)) = \sum_{t_0 \colon Z(t_0) = 0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \tag{107}$$

Consequently:

$$\begin{aligned}
\mathrm{d}\mu(t) &= \delta(Z(t))|\dot{Z}(t)|\;dt \\
&= \sum_{t_0: Z(t_0)=0} \frac{|\dot{Z}(t)|}{|\dot{Z}(t_0)|}\,\delta(t-t_0)\;dt \\
&= \sum_{t_0: Z(t_0)=0} \delta_{t_0}(dt)
\end{aligned} \tag{108}$$

where the last equality uses the fact that

$$\frac{|\dot{Z}(t_0)|}{|\dot{Z}(t_0)|} = 1 \tag{109}$$

when evaluating at $t = t_0$. □

5.4 Kac-Rice Formula For The Expected Zero Count

Theorem 24. (Kac-Rice Formula for Zero Crossings) *Let $Z(t)$ be a centered Gaussian process on $[a,b]$ with covariance*

$$\tag{110}$$

then the expected number of zeros in $[a,b]$ is

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{K(t,t)\,K_{\dot{Z}}(t,t) - K_{Z,\dot{Z}}(t,t)^2}}{K(t,t)}\;dt \tag{111}$$

where

$$K(t,t) = \mathbb{E}[Z(t)^2] \tag{112}$$

$$K_{\dot{Z}}(t,t) = -\partial_s^2\,\partial_t\,K(s,t)|_{s=t} \tag{113}$$

and

$$K_{Z,\dot{Z}}(t,t) = \partial_s\,K(s,t)|_{s=t} \tag{114}$$

Proof.

The exact zero counting function is

$$N_{[a,b]} = \int_a^b \delta(Z(t))|\dot{Z}(t)|\;dt \tag{115}$$

so

$$\begin{aligned}
\mathbb{E}[N_{[a,b]}] &= \int_a^b \mathbb{E}[\delta(Z(t))|\dot{Z}(t)|]\;dt \\
&= \int_a^b \int_{-\infty}^{\infty} |v|\;p_{Z,\dot{Z}}(0,v)\;dv\;dt
\end{aligned} \tag{116}$$

The vector $(Z(t), \dot{Z}(t))$ is bivariate Gaussian with covariance matrix

$$\Sigma = \begin{pmatrix} K(t,t) & K_{Z,\dot{Z}}(t,t) \\ K_{Z,\dot{Z}}(t,t) & K_{\dot{Z}}(t,t) \end{pmatrix} \tag{117}$$

whose determinant is given by

$$\det \Sigma = K(t,t)\, K_{\dot{Z}}(t,t) - K_{Z,\dot{Z}}(t,t)^2 \tag{118}$$

the inverse of which satisfies

$$\Sigma^{-1} = \frac{K(t,t)}{\det \Sigma} \tag{119}$$

yielding

$$p_{Z,\dot{Z}}(0,v) = \frac{1}{\sqrt{2\,\pi\, K(t,t)}} \cdot \frac{\exp\left(-\frac{K(t,t)\, v^2}{2 \det \Sigma}\right)}{\sqrt{2\,\pi \det \Sigma / K(t,t)}} \tag{120}$$

which factorizes as $p_Z(0) \cdot p_{\dot{Z}|Z}(v|0)$ where

$$p_Z(0) = \frac{1}{\sqrt{2\,\pi\, K(t,t)}} \tag{121}$$

and

$$\dot{Z} | Z = 0 \sim \mathcal{N}(0, \det \Sigma / K(t,t)) \tag{122}$$

For zero-mean Gaussian $Y \sim \mathcal{N}(0, \sigma^2)$, direct integration gives

$$\begin{aligned}\mathbb{E}[|Y|] &= 2 \int_0^\infty \frac{y}{\sqrt{2\,\pi\,\sigma^2}}\, e^{-y^2/(2\sigma^2)}\, d\,y \\ &= \frac{2\,\sigma}{\sqrt{2\,\pi}} \int_0^\infty e^{-u}\, d\,u \\ &= \sqrt{\frac{2}{\pi}}\, \sigma\end{aligned} \tag{123}$$

so that combining results yields

$$\begin{aligned}\int_{-\infty}^{\infty} |v|\ p_{Z,\dot{Z}}(0,v)\ d\,v &= \frac{\sqrt{\frac{2}{\pi}}\, \sqrt{\frac{\det \Sigma}{K(t,t)}}}{\sqrt{2\,\pi\, K(t,t)}} \\ &= \sqrt{\frac{2}{\pi}}\, \frac{\sqrt{\det \Sigma}}{K(t,t)}\end{aligned} \tag{124}$$ □

Theorem 25. ***[Expected Zero-Counting Function]****Let $\theta \in \mathcal{F}$ and let*

$$K(t,s) = \operatorname{cov}(Z(t), Z(s)) \tag{125}$$

be twice differentiable at $s = 0$ and $t = 0$ then expected number of zeros of the process $Z(t)$ in $[a,b]$ is

$$\mathbb{E}[N_{[a,b]}] = \sqrt{-K(0)}\ (\theta(b) - \theta(a)) \tag{126}$$

Proof. The covariance function of the time-changed process is

$$K_\theta(s,t) = \operatorname{cov}(Z(t), Z(s)) = \sqrt{\theta(s)\theta(t)}\ K(|\theta(t) - \theta(s)|) \tag{127}$$

For the zero-crossing analysis, consider the normalized process. By the Kac-Rice formula:

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{-\lim_{s \to t} \frac{\partial^2}{\partial s \, \partial t} K_\theta(s,t)} \; dt \tag{128}$$

Computing the mixed partial derivative:

$$\frac{\partial}{\partial t} K_\theta(s,t) = \frac{1}{2} \frac{\dot{\theta}(t)}{\sqrt{\theta(t)}} \sqrt{\dot{\theta}(s)} \; K(|\theta(t) - \theta(s)|) \tag{129}$$

$$+ \sqrt{\theta(s)} \sqrt{\theta(t)} \; \dot{K}(|\theta(t) - \theta(s)|) \mathrm{sgn}(\theta(t) - \theta(s)) \; \dot{\theta}(t) \tag{130}$$

Taking the limit as $s \to t$ and using the fact that $\dot{K}(0) = 0$ for stationary processes:

$$\begin{aligned} \lim_{s \to t} \frac{\partial^2}{\partial s \, \partial t} K_\theta(s,t) &= \dot{\theta}(s) \, \dot{\theta}(t) \; \ddot{K}(0) \\ &= \dot{\theta}(t)^2 \; \ddot{K}(0) \end{aligned} \tag{131}$$

Substituting into the Kac-Rice formula we have

$$\begin{aligned} \mathbb{E}[N_{[a,b]}] &= \int_a^b \sqrt{-\dot{\theta}(t)^2 \; \ddot{K}(0)} \; dt \\ &= \sqrt{-\ddot{K}(0)} \int_a^b \dot{\theta}(t) \; dt \\ &= \sqrt{-\ddot{K}(0)} \; (\theta(b) - \theta(a)) \end{aligned} \tag{132}$$

since $\dot{\theta}(t) \geq 0$ almost everywhere. $\square$