$\mathrm{AdS}_2/\mathrm{CFT}_1$, Whittaker vector and Wheeler-DeWitt equation

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Abstract

We study the energy representation of conformal quantum mechanics as the Whittaker vector without specifying classical Lagrangian. We show that a generating function of expectation values among two excited states of the dilatation operator in conformal quantum mechanics is a solution to the Wheeler-DeWitt equation and it corresponds to the AdS_2 partition function evaluated as the minisuperspace wave function in Liouville field theory. We also show that the dilatation expectation values in conformal quantum mechanics lead to the asymptotic smoothed counting function of the Riemann zeros.

The holographic principle [1, 2] states that quantum gravity on $(d+1)$-dimensional manifold can be described by a theory on its d-dimensional boundary. The $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ correspondence [3] which is one of the greatest productions of string theory provides the most successful realization as the relationship between effective gauge theories of the brane dynamics and string theory on the near horizon AdS geometry. The $\mathrm{AdS}_2/\mathrm{CFT}_1$ could conceivably be the most significant case in that all the extremal black holes contain an AdS_2 factor in their near horizon geometry [4, 5]. In spite of a lot of interesting works [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30] it would be fair to say that we have not yet gained a fully satisfactory understanding of the correspondence due to the peculiarities of the $\mathrm{AdS}_2/\mathrm{CFT}_1$, including the fact that only the AdS_2 has two disconnected boundaries and it is a long-standing question whether the dual CFT_1 description is a single CFT or two systems living on the two boundaries.

In this letter we study conformal quantum mechanics (CQM) without specifying classical Lagrangian description. One important consequence is a generic evidence of the $\mathrm{AdS}_2/\mathrm{CFT}_1$ as the relationship between a generating function of the dilatation expectation values in the boundary CQM and a partition function of the bulk AdS_2. We show that the expectation values are not associated with the ground state but with two excited states in the correspondence. Our result is in favor of the statement [31, 25, 32, 33, 34] that the dual CFT_1's on two boundaries of AdS_2 space-time are excited and entangled. We also claim that the energies on the boundary CQM would be responsible for the AdS_2 radius so that the ground state would correspond to a flat space in the bulk with an infinitely large AdS_2 radius.

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Another intriguing thing is the speculative implication of the resulting expectation values of the dilatation operator in CQM. When we consider the DFF-model [35] as CQM, the dilatation operator takes the form of $x\,p$. Such operator has been proposed as a strong candidate for the realization of the Hilbert-Pólya conjecture that the imaginary part of the non-trivial Riemann zeros are eigenvalues of a self-adjoint operator (see [36, 37, 38, 39, 40, 41] and references therein). Unlike a lot of efforts undertaken thus far we here propose a novel approach to obtain the counts of the Riemann zeros from CQM point of view. The fact that the operator $x\,p$ is the dilatation operator in CQM rather than the Hamiltonian enables us to jump into a fairly general setting beyond the operator with the form of $x\,p$. We show that the expectation values of the dilatation operator in CQM naturally lead to the asymptotic form of the smoothed counting function of the Riemann zeros.

We shall begin by considering CQM that is invariant under the conformal symmetric transformation of the time coordinate t [35]

$$\delta t = \epsilon_1 + \epsilon_2\, t + \epsilon_3\, t^2 \tag{1}$$

where ϵ_1, ϵ_2 and ϵ_3 are identified with the infinitesimal parameters of the translation, the dilatation and the conformal boost transformation respectively. The corresponding generators, the Hamiltonian $H = i\, d/d\,t$, the dilatation operator $D = i\,t\,(d/d\,t)$, and the conformal boost operator $K = i\,t^2\,(d/d\,t)$ obey the commutation relations

$$[H, D] = i\,H \tag{2}$$
$$[K, D] = -i\,K \tag{3}$$
$$[H, K] = 2\,i\,D \tag{4}$$

of $SL(2,\mathbb{R})$, which we call the one-dimensional conformal group. Hence the Hilbert space of conformal quantum mechanical system exhibits the $SL(2,\mathbb{R})$ conformal symmetry and the physical states would be classified by its irreducible representation. Since we wish to make all integrals convergent, we require the unitarity of the representation. The classification of the irreducible unitary representations of $SL(2,\mathbb{R})$ was studied in [42] and the irreducible unitary representation of $SL(2,\mathbb{R})$ conformal group as a function of the continuous time coordinate t can be generalized by taking the principal spherical series of the representation which is induced by the one-dimensional representation of the Borel subgroup. Let V_λ be a set of the irreducible unitary representations of $SL(2,\mathbb{R})$ with weight λ. Then the Hamiltonian H, the dilatation operator D and the conformal boost operator K can be expressed as

$$H = i\,\frac{d}{d\,t} \tag{5}$$
$$D = i\,t\,\frac{d}{d\,t} + \frac{\lambda}{2\,i} \tag{6}$$
$$K = i\,t^2\,\frac{d}{d\,t} + \frac{1}{i}\,\lambda\,t \tag{7}$$

satisfying the commutation relations (2), (3) and (4). The unitarity implies that $\frac{1}{2}(\lambda+1)$ is pure imaginary. The finite conformal transformation is

$$t^n \rightarrow \frac{(\alpha t+\beta)^n}{(\gamma t+\delta)^{\lambda+n}} \tag{8}$$

where the parameters α, β, γ and δ are the elements of real two by two matrices with determinant one

$$A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1 \tag{9}$$

which forms the one-dimensional conformal group $SL(2,\mathbb{R})$. The quadratic Casimir operator is given by

$$\begin{aligned} \mathcal{C}_2 &= HK - iD - D^2 \\ &= \frac{\lambda^2}{4} + \frac{\lambda}{2} \end{aligned} \tag{10}$$

Let us define the ground state $|0\rangle_\Delta$ by

$$H|0\rangle_\Delta = 0 \tag{11}$$

$$D|0\rangle_\Delta = \Delta|0\rangle_\Delta \tag{12}$$

From the equations (9)-(12) we see that the eigenvalue Δ of the ground state $|0\rangle_\Delta$ is $\Delta = \frac{\lambda}{2i}$ and we thus write the ground state as $|0\rangle_\lambda$. Now consider the energy eigenstate obeying

$$H|E\rangle = E|E\rangle. \tag{13}$$

This eigenvector $|E\rangle$ is known as the Whittaker vector in the representation theory of $SL(2,\mathbb{R})$ [43, 44, 45]. Given eigenvalue E and irreducible representation with weight λ, there is a unique Whittaker vector and we can write

$$|E\rangle_\lambda = -\sum_k \sum_n C_k \frac{(-EK)^n}{n!\lambda(\lambda-1)\cdots(\lambda-n+1)}|0\rangle_{\lambda,k} \tag{14}$$

where k parametrizes the degenerate ground states which might arise from some symmetries in the theories and C_k is the weighted coefficient for k-th ground state. For simplicity let us assume the vanishing of tunneling amplitudes; ${}_{\lambda,l}\langle 0|0\rangle_{\lambda,k} = \delta_{l,k}$ and the normalization of the ground states; $\sum_k |C_k|^2 = 1$ so that we shall omit the indices k. Correspondingly we can take the dual energy eigenstate vector ${}_{\lambda}\langle E|$ as the dual Whittaker vector which satisfies the relation [2]

$$
{}_{\lambda}\langle E|K = E\,{}_{\lambda}\langle E| \tag{15}
$$

and it can be represented by

$$
{}_{\lambda}\langle E| := -{}_{\lambda}\langle 0|\sum_{k,n} C_k^* \frac{(-E\,H)^n}{n!\lambda\,(\lambda-1)\cdots(\lambda-n+1)} \tag{16}
$$

With the Whittaker vector $|E\rangle_{\lambda}$ and its dual vector ${}_{\lambda}\langle E|$ as the energy eigenstate vector and its dual vector in conformal quantum mechanics, we will consider the situation where the theory is coupled to two-dimensional bulk theory to investigate the $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence. Let us consider the function defined by

$$
W_{\lambda,E_L,E_R}(\phi) := {}_{\lambda}\langle E_L|e^{-2i\phi D}|\,E_R\rangle_{\lambda} \tag{17}
$$

where E_L, E_R are the eigenvalues of the Whittaker vector (13) and the dual Whittaker vector (15) respectively. Here ϕ is regarded as the restriction of some bulk field in AdS_2 space-time on the boundary that is coupled to the dilatation operator D in conformal quantum mechanics on the boundaries. By acting the quadratic Casimir operator (9) on the function (17), one obtains the differential equation

$$
\left[\frac{1}{2}\frac{\partial^2}{\partial\phi^2} - \frac{\partial}{\partial\phi} - 2\,E_L\,E_R\,e^{2\phi}\right] W_{\lambda,E_L,E_R}(\phi) = \left(\frac{\lambda^2}{2} + \lambda\right) W_{\lambda,E_L,E_R}(\phi) \tag{18}
$$

The function $W_{\lambda,E_L,E_R}(\phi)$ is known as the $SL(2,\mathbb{R})$ Whittaker function [43, 44, 45, 46]. At first sight one might expect that the Whittaker function W_{λ,E_L,E_R} plays a role of the generating function of the expectation values of the operator D. However, the unitarity asserts that the eigenvalue Δ of the operator D associated with the ground state $|0\rangle_{\lambda}$ are not real-valued observables. Alternatively if we consider a shifted operator $\left(D - \frac{i}{2}\right)$, the corresponding eigenvalues would provide real-valued physical quantities. Furthermore in a more precise treatment one can express the bulk filed ϕ as $\beta\,\phi_0$ where ϕ_0 is the time-dependent part while β is the time-independent part. Instead of the Whittaker function (17) let us consider the function

$$
\Psi_{\lambda,\beta,E_L,E_R}(\phi_0) := {}_{\lambda}\,\langle E_L|e^{-2i\beta\phi_0(D-\frac{i}{2})}|\,E_R\rangle_{\lambda} \tag{19}
$$

2. The choice of pair of the Whittaker vector and the dual Whittaker vector results in the consistent Hamiltonian reduction.

Applying the quadratic Casimir operator (9) on the function (19), we get

$$\left[\frac{1}{2}\frac{\partial^2}{\partial\phi_0^2}-2\,\beta^2\,E_L\,E_R\,e^{2\beta\phi_0}\right]\Psi_{\lambda,\beta,E_L,E_R}(\phi_0)=\frac{1}{2}\,\beta^2\,(\lambda+1)^2\,\Psi_{\lambda,\beta,E_L,E_R}(\phi_0) \tag{20}$$

We see that the equation (20) is the Wheeler-DeWitt equation that is encountered in the minisuperspace approximation of Liouville field theory (LFT).

Local properties of LFT can be described by the Lagrangian

$$\mathcal{L}=\frac{1}{4\,\pi}\,\partial^\mu\,\phi\,\partial_\mu\,\phi+\mu\,e^{2b\phi} \tag{21}$$

where b is the dimensionless coupling constant and μ is the cosmological coupling constant. The equation of motion is

$$\Delta\,\phi=4\,\pi\,b\,\mu\,e^{2b\phi} \tag{22}$$

In two dimensions it is always possible to make any metric $g_{\mu\nu}$ conformally flat by coordinate redefinition $g_{\mu\nu}=e^{2b\phi}\,\eta_{\mu\nu}$. Furthermore, in two dimensions, the curvature can be determined by the scalar curvature. Equation (22) asserts that the curvature is a negative constant $-8\,\pi\,b^2\,\mu$ and $g_{\mu\nu}$ describes a two-dimensional surface with constant negative curvature, thus the corresponding Lorentzian surface can be identified with AdS_2 spacetime locally.

In order to quantize LFT via canonical quantization, we shall employ the Fourier decomposition of the Liouville field ϕ and its canonical momentum Π on the cylinder

$$\begin{aligned}\phi\,(t,\sigma)&=\phi_0(t)+\sum_{n\neq0}\frac{i}{n}\,[a_n(t)\,e^{-in\sigma}+b_n(t)\,e^{in\sigma}]\\ \Pi(t,\sigma)&=p_0(t)+\sum_{n\neq0}\,[a_n(t)\,e^{-in\sigma}+b_n(t)\,e^{in\sigma}]\end{aligned} \tag{23}$$

with $a_n^\dagger=a_{-n}$, $b_n^\dagger=b_{-n}$. The canonical relation

$$[\phi\,(t,\sigma),\Pi(t,\sigma')]=i\,\delta\,(\sigma-\sigma') \tag{24}$$

leads to the commutation relations

$$[\phi_0,p_0]\;=i,\quad[a_n,a_m]\;=\frac{n}{2}\,\delta_{n,-m},\quad[b_n,b_m]=\frac{n}{2}\,\delta_{n,-m} \tag{25}$$

which imply that a_n, b_n are creation operators while a_{-n}, b_{-n} are annihilation operators. The spectrum of LFT has been discussed in the minisuperspace approximation [47]. The minisuperspace approximation was originally proposed in quantum gravity [48, 49] where the problem is simplified by only treating the zero mode and truncating the higher excited modes. Whether physics in minisuperspace quantization gives a faithful properties of quantum gravity remains an open question, however, it has been discussed that the minisuperspace approximation would be exact in pure two-dimensional gravity [50, 51]. Replacing the canonical momentum $p_0 = \frac{\dot{\phi}_0}{2\pi}$ with differential operator $-i\,(\partial/\partial\,\phi_0)$, we obtain the minisuperspace Schrödinger equation

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\,\phi_0^2} + 2\,\pi\,\mu\,e^{2b\phi_0}\right]\Psi_P(\phi_0) = 2\,P^2\,\Psi_P(\phi_0) \tag{26}$$

where P is the Liouville momentum, the eigenvalue of the Hamiltonian and $\Psi_P(\phi_0)$ is the minisuperspace wave function. For $\phi_0 \to -\infty$ the interaction is small and the wave function is a linear combination of $e^{\pm iP\phi_0}$. Because of the complete reflection potential at $\phi_0 \to 0$, P is restricted to be positive so that the incoming wave uniquely determines the reflected wave.

The function $\Psi_{\lambda,\beta,E_L,E_R}(\phi_0)$ in CQM plays a role of the generating function of the expectation values evaluated among two excited states. On the other hand, the Liouville wave function $\Psi_P(\phi_0)$ would be regarded as the partition function of AdS_2, which is a function of the boundary values in the sense that a partition function generically transforms as a wave function under a change of polarization on field space specified at a boundary [52, 53, 54, 55] and LFT describes AdS_2 space-time in the classical solution. Therefore we come to the interesting conclusion that AdS_2 bulk mode ϕ that behaves as the zero-mode ϕ_0 near the boundary is dual to a source term $\phi_0\,(D - \frac{i}{2})$ in the CQM on the boundary of AdS_2 via $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence

$$\begin{aligned}\Psi_{\lambda,\beta,E_L,E_R}(\phi_0) &= \left\langle E_L \middle| e^{-2i\beta\phi_0\left(D-\frac{i}{2}\right)} \middle| E_R \right\rangle_{\mathrm{CQM}} \\ &= Z_{\mathrm{AdS}_2}(\phi|_{\mathrm{bdy}} = \phi_0) = \Psi_P(\phi_0)\end{aligned} \tag{27}$$

We observe that the expectation values in the CQM are evaluated between two excited states defined by the Whittaker vector (13) and the dual Whittaker vector (15) in the correspondence (26). The two distinct states in the correspondence (26) would would enable us to have two independent dynamical systems. This is consistent to the statement [20, 25, 34, 28] that unlike higher dimensional cases AdS_2 in the global coordinate has two boundaries and the dual conformal field theory of asymptotically AdS_2 is realized as two systems or two copies of CFT_1 on the two boundaries although we cannot exclude the possibility of a single CFT_1 as the dual description. The appearance of the excited states could also be in favor of the statement that in the Lorentzian AdS/CFT operator expectation values in excited CFT states differ from vacuum expectation values due to the existence of normalizable propagating states in the bulk [32, 33].

The comparison of the two Wheeler-DeWitt equations (20) and (26) identifies the coupling constant b in LFT with the constant parameter β in CQM and establishes the dictionary of parameters between the bulk LFT of AdS_2 and the boundary CQM as

$$\frac{\pi\,\mu}{b^2} = E_L\,E_R \tag{28}$$

$$\frac{P^2}{b^2} = -\frac{1}{4}\,(\lambda+1)^2 = \left(\Delta - \frac{i}{2}\right)^2 \tag{29}$$

Therefore the relations (28) and (29) indicate that the quantum gravity on AdS_2 can be described by two conformal quantum mechanical systems on the boundary with energies E_L and E_R as a holographic principle [1, 2].

The equation (28) says that the excited states with non-vanishing energy eigenvalues E_L, E_R in the boundary CQM are needed to realize negative constant curvature of AdS_2 space-time which is generated by the non-trivial interaction term with finite and non-vanishing parameters b, μ in two-dimensional gravity theory. It is illustrative to compare our result with the analogous statement in the $\mathrm{AdS}_3/\mathrm{CFT}_2$ that the AdS_3 radius l_3 is represented by the central charge c of the CFT_2 through the Brown-Henneaux relation $c = \frac{3\,l_3}{2\,G_3}$ [56] where G_3 is the three-dimensional Newton constant. Instead of the central charge the energies of the states play an important role in the $\mathrm{AdS}_2/\mathrm{CFT}_1$, however, the relation between the AdS_2 radius and the energies is even more attractive in that one of the other parameters in LFT necessarily comes about. In terms of the coupling constant b controlling the quantum effect in LFT we can write the AdS_2 radius l_2 as

$$\frac{1}{\sqrt{E_L\,E_R}} = 2\,b^2\,l_2 \tag{30}$$

The semiclassical analysis in LFT is valid for small b, and accordingly our formula (30) would reflect the fact that the ground states with vanishing energies in the boundary CQM force the AdS_2 radius to go to infinity and the classical AdS_2 geometry then becomes flat space-time.

We see from the equation (29) that the Liouville momentum P in two-dimensional gravity theory corresponds to the conformal dimension of the ground state in the dual CQM. As we have discussed, the unitarity condition in CQM requires that $\frac{1}{2}\,(\lambda+1)$ is pure imaginary. This is consistent to the fact that the Liouville momentum P is real in the dual two-dimensional gravity theory.

We would like to emphasize that the correspondence (26) and the dictionaries (28), (29), (30) are quite universal since we have not specified the conformal quantum mechanical systems so far. However, if we contain more specific information characterizing dynamical properties and symmetries, there would be more fruitful statements (the GKP-Witten relation [57, 58]) in the $\mathrm{AdS}_2/\mathrm{CFT}_1$ as the extension of the relation (26)

$$\langle e^{h_0 \mathcal{O}} \rangle_{\mathrm{CQM}} = Z_{\mathrm{AdS}_2}(h|_{\mathrm{bdy}} = h_0) \tag{31}$$

where h_0 is some function of the boundary values for the bulk field h while $\mathcal{O}$ is the dual operator in CQM. For a non-flat space the left values in the correspondence (31) are generally presumed to be computed between two excited states from the relation (30). It would be interesting that there has been proposals for such relation associated with the DFF-model in [28] and with the counting of microstates of BPS extremal black holes in [26, 25].

We now consider the resulting expectation values

$$ {}_\lambda\langle E_L|\left(D-\frac{i}{2}\right)|\,E_R\rangle_\lambda=\frac{i}{2}\frac{\delta}{\delta\,\phi}\,\Psi_{\lambda,E_L,E_R}(\phi)|_{\phi=0} \tag{32} $$

and its possible application to one of the deepest mathematical problem, the Riemann hypothesis. The equation (20) has two linearly independent solutions, which are known to be cylindric functions [59]. By requiring the unitarity we can write the solutions as

$$ \Psi_{\lambda,E_L,E_R}(\phi)=\frac{1}{i}\,K_{\lambda+1}\,(2\,\sqrt{E_L\,E_R}\,e^{\phi}) \tag{33} $$

where $K_\nu(z)$ is the Macdonald function. We should note that the Macdonald functions of purely imaginary order $\lambda+1$ with positive argument are real. The prefactor $\frac{1}{i}$ in the generating function (33) guarantees the reality condition of the expectation values in CQM. Making use of the recurrence relation

$$ K_{\nu-1}(z)+K_{\nu+1}(z)=-2\,\frac{d}{d\,z}\,K_\nu(z) \tag{34} $$

and the formula (32) we can write the expectation values between the excited states as

$$ {}_\lambda\langle E_L|\left(D-\frac{i}{2}\right)|\,E_R\rangle_\lambda=-\frac{z}{4}\,(K_\lambda(z)+K_{\lambda+2}(z)) \tag{35} $$

where $z=2\,\sqrt{E_L\,E_R}$. Let us now consider a one particle conformal quantum mechanical model known as the DFF-model [35]

$$ S=\frac{1}{2}\int\left(\dot{x}(t)^2-\frac{g}{x(t)^2}\right)d\,t \tag{36} $$

with g being a dimensionless coupling constant. For the DFF-model the dilatation operator can be expressed as $D=-\frac{1}{2}\,x\,p+\frac{i}{4}$ where p is the canonical momentum and the equation (35) becomes

$$ {}_\lambda\langle E_L|\left(x\,p+\frac{i}{2}\right)|\,E_R\rangle_\lambda=\frac{z}{2}\,(K_\lambda(z)+K_{\lambda+2}(z)) \tag{37} $$

It is speculated that the Riemann zeros would be realized as eigenvalues of the operator which takes the form of $x\,p$ [36, 37, 38] as a promising candidate of the Riemann operator in the Hilbert-Pólya conjecture whose eigenvalues are the imaginary part of the non-trivial Riemann zeros. Berry and Keating [37, 38] identified the operator $x\,p$ with the Hamiltonian and imposed the conditions $|x|\geq l_x$, $|p|\geq l_p$ so that $l_x\,l_p=2\,\pi\,\hbar$ in the phase space. Then they found that the semiclassical number $N(E)$ of states with the energy between 0 and E is given by the area in the phase space divided by the Planck cell $h=2\,\pi$

$$N(E)=\frac{E}{2\,\pi}\left(\log\frac{E}{2\,\pi}-1\right)+\mathcal{O}(1) \tag{38}$$

and observed that (38) precisely coincides with the asymptotics of the smoothed counting function of the number of Riemann zeros [60]. Connes [36] introduced the constraints $|x|\leq\Lambda$, $|p|\leq\Lambda$ where Λ is a common cutoff and counted the number of such semiclassical states as

$$N(E)=\frac{E}{\pi}\log\Lambda-\frac{E}{2\,\pi}\left(\log\frac{E}{2\,\pi}-1\right) \tag{39}$$

He interpreted the counting formula (39) as missing spectral lines associated to the smooth Riemann zeros which arise in the limit $\Lambda\to\infty$, however, it was reinterpreted as a finite size correction from a physical system later in [39]. According to these semiclassical proposals of the Hilbert-Pólya conjecture it has been desirable to replace these artificially imposed semiclassical regularizations of the operator $x\,p$ with the proper quantum treatment which naturally generates a discrete spectrum. In order to obtain the discrete spectrum via quantization, there has been proposed various attempts including the modifications of the $x\,p$ operator (see, e.g., [40, 41]) and the adoption of the regularization methods (see, e.g., [39]). Nevertheless, these attempts seem to be quite artificial and difficult fo find the quantum mechanical explanation to follow these ideas.

Here we wish to propose a novel perspective to acquire the distribution of the spectrum of the Riemann operator from conformal quantum mechanics point of view. Consider now the eigenfunction $\Phi_\rho(x)$ of the operator $\left(x\,p+\frac{i}{2}\right)$ in the equation (37) satisfying

$$\left[\frac{1}{i}\,x\,\frac{d}{d\,x}+\frac{i}{2}\right]\Phi_\rho(x)=\rho\,\Phi_\rho(x) \tag{40}$$

They take the form

$$\Phi_\rho(x)=C\,x^{\frac{1}{2}+i\rho} \tag{41}$$

with C a constant of integration. The non-trivial zeros of the Riemann zeta function $\zeta(s)$ which is conjectured to be $s = \frac{1}{2} + i\,\rho$, $\rho \in \mathbb{R}$ in the Riemann hypothesis appears in the power x^s of the eigenfunction $\Phi_\rho(x)$. The eigenvalues ρ of the operator $\left(x\,p + \frac{i}{2}\right)$ which would be the candidates of the Riemann zeros can be continuous in the position eigenfunction. This is the same situation as has been already discussed in many literatures.

However, in CQM the operator $\left(x\,p + \frac{i}{2}\right)$ should not be recognized as the Hamiltonian but rather as the dilatation operator whose expectation values can be measured by the energy eigenstates as

$$D(z;\rho) = \frac{z}{2}\left(K_{1-i\rho}(z) + K_{1+i\rho}(z)\right) \tag{42}$$

Note that the expression (42) can evidently be lifted to arbitrary conformal quantum mechanical systems by qualifying ρ as the eigenvalue of the operator $-2\left(D - \frac{i}{2}\right)$. Although almost all quantum approaches so far have tried to identify the operator $x\,p$ with the Hamiltonian and simultaneously diagonalize it with the position x or the momentum p, CQM would provide an alternative avenue to the Riemann hypothesis as the diagonalization of the dilatation operator. We observe that the ground state $|0\rangle_\lambda$ is the eigenfunction of both of the Hamiltonian and the dilatation operator. Since the excited states are not eigenvectors of the dilatation operator, the limit $E_L,\ E_R \to 0$ of the expectation values (42) would naturally give rise to the eigenfunction of the operator $-2\left(D - \frac{i}{2}\right)$ multiplied by its eigenvalue ρ. In other words, the limit in which $z = 2\sqrt{E_L\,E_R}$ goes to zero yields the definite eigenvalue ρ and the distribution function $D(\rho)$ of the ground state. Hence the expectation values (42) are in some sense the regularized functions which produce the distribution of the eigenvalues ρ as

$$\rho\,D(\rho) = \lim_{z\to 0} \frac{z}{2}\left(K_{1-i\rho}(z) + K_{1+i\rho}(z)\right) \tag{43}$$

where we have used the relation $\lambda = -(1 + i\,\rho)$ and the formula $K_\nu(z) = K_{-\nu}(z)$. The asymptotic behavior of the Macdonald function

$$K_{1+i\rho}(z) \sim \sqrt{\frac{\pi}{z}}\, e^{-\frac{\pi}{2}\rho}\left(\frac{2\,\rho}{z\,e}\right)^{i\rho} \tag{44}$$

for large ρ allows us to write (43) as

$$\rho\,D(\rho) = \lim_{z\to 0} \sqrt{\pi\,z}\, e^{-\frac{\pi}{2}\rho} \cos\left[\rho \ln\left(\frac{2\,\rho}{z\,e}\right)\right] \tag{45}$$

The semiclassical distribution of (45) for large ρ is realized when the cosine function is at its maximum

$$\cos\left[\rho \ln\left(\frac{2\,\rho}{z\,e}\right)\right] = 1 \tag{46}$$

so that

$$\frac{\rho}{\pi}\left[\ln\left(\frac{\rho}{E_L\,E_R}\right)-1\right]=2\,n, \forall n\in\mathbb{Z}. \tag{47}$$

Since the expression (47) diverges when $E_L, E_R \to 0$, a low energy cutoff is required to make sense of the expression (47). Let us introduce the cutoff Λ such that $E_L\,E_R = 2\,\pi/\Lambda$. Then we obtain the behavior of the large eigenvalues ρ as

$$N(\rho)=\frac{\rho}{2\,\pi}\ln\Lambda+\frac{\rho}{2\,\pi}\left(\ln\frac{\rho}{2\,\pi}-1\right) \tag{48}$$

Remarkably the first term is a continuum in the limit $\Lambda\to\infty$ while the second term leads to the asymptotics of the counting function of the Riemann zeros as in (38) and (39). It would be interesting to note that the equation (48) also counts the large conformal dimensions for the ground state in CQM. Combining the semiclassical realization (48) of the counting Riemann zeros with our proposed holographic correspondence (29) would indicate underlying profound relation among essential ingredients in number theory, in quantum mechanics and in gravity.

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