

Unitary Time Changes of Stationary Processes Yield Oscillatory Processes

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ , acting on functions that are locally square-integrable (meaning over compact sets). Applying U_θ to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\theta(t)} e^{i\lambda\theta(t)}$, evolutionary spectrum $dF_t(\lambda) = \dot{\theta}(t) dF(\lambda)$ and expected zero-counting function $\mathbb{E}[N_{[0,T]}] = \sqrt{-\ddot{K}(0)} [\theta(T) - \theta(0)]$. The sample paths of any non-degenerate second-order stationary process are locally square integrable, making the unitary time-change operator U_θ applicable to typical realizations. A zero-localization measure $d\mu(t) = \delta(Z(t))|\dot{Z}(t)| dt$ induces a Hilbert space $L^2(\mu)$ on the zero set of each oscillatory process realization $Z(t)$, and the multiplication operator $(L f)(t) = t f(t)$ has simple pure point spectrum equal to the zero crossing set of Z .

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1 Gaussian Processes

1.1 Definition

Definition 1. (*Gaussian process*) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T a nonempty index set. A family $\{X_t : t \in T\}$ of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Gaussian process if for every finite subset $\{t_1, \dots, t_n\} \subset T$ the random vector $(X_{t_1}, \dots, X_{t_n})$ is multivariate normal (possibly degenerate). Equivalently, every finite linear combination $\sum_{i=1}^n a_i X_{t_i}$ is either almost surely constant or Gaussian. The mean function is $m(t) := \mathbb{E}[X_t]$ and the covariance kernel is

$$K(s, t) = \text{Cov}(X_s, X_t) \tag{1}$$

For any finite $(t_i)_{i=1}^n \subset T$, the matrix $K_{ij} = K(t_i, t_j)$ is symmetric positive semidefinite, and a Gaussian process is completely determined in law by m and K .

Definition 2. The canonical metric associated with a Gaussian process is

$$d(s, t) = \sqrt{\mathbb{E}[(X_s - X_t)^2]} = \sqrt{K(s, s) + K(t, t) - 2K(s, t)} \quad (2)$$

1.2 Sample Path Realizations

Definition 3. [Locally square-integrable functions] Define

$$L_{\text{loc}}^2(\mathbb{R}) := \left\{ f: \mathbb{R} \rightarrow \mathbb{C} : \int_K |f(t)|^2 dt < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \quad (3)$$

Remark 4. Every bounded measurable set in \mathbb{R} is compact or contained in a compact set; hence $L_{\text{loc}}^2(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

Theorem 5. [Sample paths in $L_{\text{loc}}^2(\mathbb{R})$] Let $\{X(t)\}_{t \in \mathbb{R}}$ be a second-order stationary process with

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \quad (4)$$

Then almost every sample path lies in $L_{\text{loc}}^2(\mathbb{R})$. However, for non-degenerate processes with $\sigma^2 > 0$, sample paths are not globally square-integrable.

Proof. Fix an arbitrary bounded interval $[a, b] \subset \mathbb{R}$ with $a < b$. Define the random variable

$$Y_{[a, b]} := \int_a^b X(t)^2 dt \quad (5)$$

1. By Tonelli's theorem, since $X(t)^2 \geq 0$,

$$\mathbb{E}[Y_{[a, b]}] = \mathbb{E}\left[\int_a^b X(t)^2 dt\right] = \int_a^b \mathbb{E}[X(t)^2] dt \quad (6)$$

2. By stationarity of X , $\mathbb{E}[X(t)^2] = \sigma^2$ for all $t \in \mathbb{R}$. Therefore

$$\mathbb{E}[Y_{[a, b]}] = \int_a^b \sigma^2 dt = \sigma^2(b - a) \quad (7)$$

3. Since $b - a < \infty$ and $\sigma^2 < \infty$ by assumption (4),

$$\mathbb{E}[Y_{[a, b]}] < \infty \quad (8)$$

4. By Markov's inequality, for any $M > 0$,

$$\mathbb{P}(Y_{[a, b]} > M) \leq \frac{\mathbb{E}[Y_{[a, b]}]}{M} = \frac{\sigma^2(b - a)}{M} \quad (9)$$

5. Taking $M \rightarrow \infty$ in (9),

$$\mathbb{P}(Y_{[a, b]} < \infty) = 1 \quad (10)$$

6. Now let $K \subset \mathbb{R}$ be an arbitrary compact set. Since K is compact in \mathbb{R} , it is closed and bounded. Therefore there exists $N > 0$ such that

$$K \subseteq [-N, N] \quad (11)$$

7. By (10) applied to $[a, b] = [-N, N]$,

$$\mathbb{P}\left(\int_{-N}^N X(t)^2 dt < \infty\right) = 1 \quad (12)$$

8. Since $K \subseteq [-N, N]$ by (11),

$$\int_K X(t)^2 dt \leq \int_{-N}^N X(t)^2 dt \quad (13)$$

9. Combining (12) and (13),

$$\mathbb{P}\left(\int_K X(t)^2 dt < \infty\right) = 1 \quad (14)$$

10. Since K was arbitrary, (14) holds for every compact set $K \subset \mathbb{R}$. Therefore, almost every sample path $t \mapsto X(t, \omega)$ satisfies

$$\int_K |X(t, \omega)|^2 dt < \infty \quad \forall \text{compact } K \subset \mathbb{R} \quad (15)$$

which means almost every sample path lies in $L_{\text{loc}}^2(\mathbb{R})$.

11. For the global divergence statement, assume $\sigma^2 > 0$. For each $n \in \mathbb{N}$, by (7),

$$\mathbb{E}\left[\int_{-n}^n X(t)^2 dt\right] = 2n\sigma^2 \quad (16)$$

12. As $n \rightarrow \infty$, the right side of (16) diverges:

$$\lim_{n \rightarrow \infty} \mathbb{E}\left[\int_{-n}^n X(t)^2 dt\right] = \lim_{n \rightarrow \infty} 2n\sigma^2 = \infty \quad (17)$$

13. By monotone convergence theorem, since $\int_{-n}^n X(t)^2 dt$ increases monotonically with n ,

$$\mathbb{E}\left[\lim_{n \rightarrow \infty} \int_{-n}^n X(t)^2 dt\right] = \lim_{n \rightarrow \infty} \mathbb{E}\left[\int_{-n}^n X(t)^2 dt\right] = \infty \quad (18)$$

14. Therefore

$$\mathbb{P}\left(\int_{-\infty}^{\infty} X(t)^2 dt = \infty\right) = 1 \quad (19)$$

15. Thus sample paths are not in $L^2(\mathbb{R})$, only in $L_{\text{loc}}^2(\mathbb{R})$. \square

1.3 Stationary processes

Definition 6. [Cramér spectral representation][1] A zero-mean stationary process X with spectral measure F admits the sample path representation

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t} d\Phi(\lambda) \quad (20)$$

which has covariance

$$R_X(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)} dF(\lambda) \quad (21)$$

1.4 Oscillatory Processes

Definition 7. [Oscillatory process][2] Let F be a finite nonnegative Borel measure on \mathbb{R} . Let

$$A_t \in L^2(F) \quad \forall t \in \mathbb{R} \quad (22)$$

be the gain function and

$$\varphi_t(\lambda) = A_t(\lambda) e^{i\lambda t} \quad (23)$$

be the corresponding oscillatory function then an oscillatory process is a stochastic process which can be represented as

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \quad (24)$$

where Φ is a complex orthogonal random measure with spectral measure F which satisfies the relation

$$d\mathbb{E}[\Phi(\lambda)\overline{\Phi(\mu)}] = \delta(\lambda - \mu) dF(\lambda) \quad (25)$$

and has the corresponding covariance kernel

$$R_Z(t, s) = \mathbb{E}[Z(t)\overline{Z(s)}] = \int_{\mathbb{R}} A_t(\lambda) \overline{A_s(\lambda)} e^{i\lambda(t-s)} dF(\lambda) = \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\lambda)} dF(\lambda) \quad (26)$$

Theorem 8. [Real-valuedness criterion for oscillatory processes] Let Z be an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = A_t(\lambda) e^{i\lambda t} \quad (27)$$

and spectral measure F . Then Z is real-valued if and only if

$$A_t(-\lambda) = \overline{A_t(\lambda)} \quad (28)$$

for F -almost every $\lambda \in \mathbb{R}$, equivalently

$$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \quad (29)$$

for F -almost every $\lambda \in \mathbb{R}$.

Proof. 1. Assume Z is real-valued. Then for all $t \in \mathbb{R}$,

$$Z(t) = \overline{Z(t)} \quad (30)$$

2. From the oscillatory representation (24),

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \quad (31)$$

3. Taking the complex conjugate of both sides of (31),

$$\overline{Z(t)} = \overline{\int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t} d\overline{\Phi(\lambda)} \quad (32)$$

4. For a real-valued process, the orthogonal random measure must satisfy the symmetry property

$$d\overline{\Phi(\lambda)} = -d\Phi(-\lambda) \quad (33)$$

5. Substituting (33) into (32),

$$\overline{Z(t)} = - \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t} d\Phi(-\lambda) \quad (34)$$

6. Apply the change of variables $\mu = -\lambda$, so $d\Phi(-\lambda) = -d\Phi(\mu)$ and $e^{-i\lambda t} = e^{i\mu t}$:

$$\overline{Z(t)} = - \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t} (-d\Phi(\mu)) = \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t} d\Phi(\mu) \quad (35)$$

7. By (30), the right sides of (31) and (35) must be equal:

$$\int_{\mathbb{R}} A_t(\mu) e^{i\mu t} d\Phi(\mu) = \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t} d\Phi(\mu) \quad (36)$$

8. Since the stochastic integral representation is unique in $L^2(F)$, the integrands must be equal F -almost everywhere:

$$A_t(\lambda) = \overline{A_t(-\lambda)} \quad \text{for } F\text{-a.e. } \lambda \quad (37)$$

9. This is equivalent to (28). From (27),

$$\varphi_t(-\lambda) = A_t(-\lambda) e^{-i\lambda t} \quad (38)$$

10. Using (28),

$$\varphi_t(-\lambda) = \overline{A_t(\lambda)} e^{-i\lambda t} = \overline{A_t(\lambda) e^{i\lambda t}} = \overline{\varphi_t(\lambda)} \quad (39)$$

establishing (29).

11. Conversely, assume (28) holds. Reversing the steps from (35) to (30) shows that $\overline{Z(t)} = Z(t)$ for all t , so Z is real-valued. \square

Theorem 9. [Existence of Oscillatory Processes] Let F be an absolutely continuous spectral measure and the gain function

$$A_t(\lambda) \in L^2(F) \quad \forall t \in \mathbb{R} \quad (40)$$

be measurable in both time and frequency then the time-dependent spectral density is defined by

$$S_t(\lambda) = \int_{\mathbb{R}} |A_t(\lambda)|^2 dF(\lambda) < \infty = \int_{\mathbb{R}} |A_t(\lambda)|^2 S(\lambda) d\lambda \quad (41)$$

and there exists a complex orthogonal random measure Φ with spectral measure F such that for each sample path $\omega_0 \in \Omega$

$$Z(t, \omega_0) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda, \omega_0) \quad (42)$$

is well-defined in $L^2(\Omega)$ and has covariance R_Z as in (26).

Proof. 1. Define the space of simple functions on \mathbb{R} : for disjoint Borel sets $\{E_j\}_{j=1}^n$ with $F(E_j) < \infty$ and coefficients $\{c_j\}_{j=1}^n \subset \mathbb{C}$,

$$g(\lambda) = \sum_{j=1}^n c_j 1_{E_j}(\lambda) \quad (43)$$

2. For simple functions, define the stochastic integral

$$\int_{\mathbb{R}} g(\lambda) d\Phi(\lambda) := \sum_{j=1}^n c_j \Phi(E_j) \quad (44)$$

3. Compute the second moment:

$$\mathbb{E}\left[\left|\int_{\mathbb{R}} g(\lambda) d\Phi(\lambda)\right|^2\right] = \mathbb{E}\left[\left|\sum_{j=1}^n c_j \Phi(E_j)\right|^2\right] = \mathbb{E}\left[\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \Phi(E_j) \overline{\Phi(E_k)}\right] \quad (45)$$

4. By linearity of expectation,

$$\mathbb{E}\left[\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \Phi(E_j) \overline{\Phi(E_k)}\right] = \sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \mathbb{E}[\Phi(E_j) \overline{\Phi(E_k)}] \quad (46)$$

5. By the orthogonality relation (25), since $E_j \cap E_k = \emptyset$ for $j \neq k$,

$$\mathbb{E}[\Phi(E_j) \overline{\Phi(E_k)}] = \begin{cases} F(E_j) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \quad (47)$$

6. Substituting (47) into (46),

$$\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \mathbb{E}[\Phi(E_j) \overline{\Phi(E_k)}] = \sum_{j=1}^n |c_j|^2 F(E_j) \quad (48)$$

7. The right side of (48) equals

$$\sum_{j=1}^n |c_j|^2 F(E_j) = \int_{\mathbb{R}} |g(\lambda)|^2 dF(\lambda) \quad (49)$$

8. Therefore the isometry property holds for simple functions:

$$\mathbb{E}\left[\left|\int_{\mathbb{R}} g(\lambda) d\Phi(\lambda)\right|^2\right] = \int_{\mathbb{R}} |g(\lambda)|^2 dF(\lambda) \quad (50)$$

9. The space of simple functions is dense in $L^2(F)$. For any $h \in L^2(F)$ and $\epsilon > 0$, there exists a simple function g such that

$$\int_{\mathbb{R}} |h(\lambda) - g(\lambda)|^2 dF(\lambda) < \epsilon \quad (51)$$

10. By the isometry (50) and completeness of $L^2(\Omega)$, the integral extends uniquely by continuity to all $h \in L^2(F)$.
11. Since $A_t \in L^2(F)$ by assumption (40), and $|e^{i\lambda t}| = 1$,

$$\int_{\mathbb{R}} |\varphi_t(\lambda)|^2 dF(\lambda) = \int_{\mathbb{R}} |A_t(\lambda)|^2 dF(\lambda) < \infty \quad (52)$$

so $\varphi_t \in L^2(F)$.

12. Therefore

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \quad (53)$$

is well-defined in $L^2(\Omega)$.

13. To compute the covariance, use the sesquilinearity of the stochastic integral:

$$R_Z(t, s) = \mathbb{E}[Z(t)\overline{Z(s)}] = \mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \overline{\int_{\mathbb{R}} \varphi_s(\mu) d\Phi(\mu)}\right] \quad (54)$$

14. By Fubini's theorem for stochastic integrals,

$$\mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \overline{\int_{\mathbb{R}} \varphi_s(\mu) d\Phi(\mu)}\right] = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\mu)} \mathbb{E}[d\Phi(\lambda) \overline{d\Phi(\mu)}] \quad (55)$$

15. Using the orthogonality relation (25),

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\mu)} \delta(\lambda - \mu) dF(\lambda) dF(\mu) = \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\lambda)} dF(\lambda) \quad (56)$$

16. Substituting the definition (23),

$$R_Z(t, s) = \int_{\mathbb{R}} A_t(\lambda) \overline{A_s(\lambda)} e^{i\lambda(t-s)} dF(\lambda) \quad (57)$$

as claimed in (26). \square

2 Unitarily Time-Changed Stationary Processes

2.1 Unitary Time-Change Operator $U_\theta f$

Theorem 10. [Unitary time-change and local unitarity] Let the time-scaling function $\theta: \mathbb{R} \rightarrow \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with

$$\dot{\theta}(t) > 0 \quad (58)$$

almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero. For f measurable, define

$$(U_\theta f)(t) = \sqrt{\dot{\theta}(t)} f(\theta(t)) \quad (59)$$

Its inverse is given by

$$(U_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \quad (60)$$

For every compact set $K \subseteq \mathbb{R}$ and $f \in L^2_{\text{loc}}(\mathbb{R})$,

$$\int_K |(U_\theta f)(t)|^2 dt = \int_{\theta(K)} |f(s)|^2 ds \quad (61)$$

Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\text{loc}}(\mathbb{R})$.

Proof. 1. Let $f \in L^2_{\text{loc}}(\mathbb{R})$ and let $K \subset \mathbb{R}$ be compact. From the definition (59),

$$\int_K |(U_\theta f)(t)|^2 dt = \int_K \left| \sqrt{\dot{\theta}(t)} f(\theta(t)) \right|^2 dt \quad (62)$$

2. Expanding the square,

$$\int_K \left| \sqrt{\dot{\theta}(t)} f(\theta(t)) \right|^2 dt = \int_K \dot{\theta}(t) |f(\theta(t))|^2 dt \quad (63)$$

3. Since θ is absolutely continuous and strictly increasing, $\theta' = \dot{\theta}$ exists almost everywhere and $\dot{\theta}(t) > 0$ a.e.

4. Apply the change of variables $s = \theta(t)$. Then

$$ds = \dot{\theta}(t) dt \quad (64)$$

5. The inverse function $t = \theta^{-1}(s)$ exists since θ is strictly increasing and bijective.

6. As t ranges over K , the variable $s = \theta(t)$ ranges over $\theta(K)$.

7. Since θ is continuous and K is compact, $\theta(K)$ is compact.

8. Substituting (64) into (63),

$$\int_K \dot{\theta}(t) |f(\theta(t))|^2 dt = \int_{\theta(K)} |f(s)|^2 ds \quad (65)$$

9. This establishes the local isometry (61).

10. To verify U_θ^{-1} is the inverse, compute:

$$(U_\theta^{-1} U_\theta f)(s) = U_\theta^{-1} (U_\theta f)(s) \quad (66)$$

11. By definition (60),

$$U_\theta^{-1} (U_\theta f)(s) = \frac{(U_\theta f)(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \quad (67)$$

12. By definition (59),

$$(U_\theta f)(\theta^{-1}(s)) = \sqrt{\dot{\theta}(\theta^{-1}(s))} f(\theta(\theta^{-1}(s))) \quad (68)$$

13. Since $\theta \circ \theta^{-1} = \text{id}$,

$$f(\theta(\theta^{-1}(s))) = f(s) \quad (69)$$

14. Substituting (68) and (69) into (67),

$$\frac{\sqrt{\dot{\theta}(\theta^{-1}(s))} f(s)}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} = f(s) \quad (70)$$

15. Therefore

$$U_\theta^{-1} U_\theta = \text{id} \quad (71)$$

16. Similarly, compute:

$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)} (U_\theta^{-1} g)(\theta(t)) \quad (72)$$

17. By definition (60),

$$(U_\theta^{-1} g)(\theta(t)) = \frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} \quad (73)$$

18. Since $\theta^{-1} \circ \theta = \text{id}$,

$$g(\theta^{-1}(\theta(t))) = g(t), \quad \theta^{-1}(\theta(t)) = t \quad (74)$$

19. Substituting (74) into (73),

$$\frac{g(t)}{\sqrt{\dot{\theta}(t)}} \quad (75)$$

20. Therefore from (72),

$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)} \cdot \frac{g(t)}{\sqrt{\dot{\theta}(t)}} = g(t) \quad (76)$$

21. Thus

$$U_\theta U_\theta^{-1} = \text{id} \quad (77)$$

22. Combining (71) and (77), U_θ^{-1} is the two-sided inverse of U_θ on $L^2_{\text{loc}}(\mathbb{R})$. \square

2.2 Inverse Filter for Unitary Time Transformations

Theorem 11. [Inverse Filter for Unitary Time Transformations] Let $\theta: \mathbb{R} \rightarrow \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\theta'(t) > 0$ almost everywhere. Let $Y(u)$ be a stationary process with unit variance, and define

$$Z(t) = \sqrt{\dot{\theta}(t)} Y(\theta(t)) \quad (78)$$

as the oscillatory process obtained by the unitary time transformation. Then:

1. The forward filter kernel is

$$h(t, u) = \sqrt{\dot{\theta}(t)} \delta(u - \theta(t)) \quad (79)$$

2. The inverse filter kernel is

$$g(t, s) = \frac{\delta(s - \theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \quad (80)$$

3. The composition $(g \circ h)$ recovers the identity:

$$Y(t) = \int_{\mathbb{R}} g(t, s) Z(s) ds = \frac{Z(\theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \quad (81)$$

Proof. 1. From (78), the forward transformation is

$$Z(t) = \int_{\mathbb{R}} h(t, u) Y(u) du \quad (82)$$

2. Substituting (79),

$$\int_{\mathbb{R}} h(t, u) Y(u) du = \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} \delta(u - \theta(t)) Y(u) du \quad (83)$$

3. By the sifting property of the Dirac delta,

$$\int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} \delta(u - \theta(t)) Y(u) du = \sqrt{\dot{\theta}(t)} Y(\theta(t)) \quad (84)$$

4. This confirms (78).

5. For the inverse, compute:

$$\int_{\mathbb{R}} g(t, s) Z(s) ds = \int_{\mathbb{R}} \frac{\delta(s - \theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} Z(s) ds \quad (85)$$

6. By the sifting property,

$$\int_{\mathbb{R}} \frac{\delta(s - \theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} Z(s) ds = \frac{Z(\theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \quad (86)$$

7. Substituting (78) with t replaced by $\theta^{-1}(t)$,

$$Z(\theta^{-1}(t)) = \sqrt{\dot{\theta}(\theta^{-1}(t))} Y(\theta(\theta^{-1}(t))) \quad (87)$$

8. Since $\theta \circ \theta^{-1} = \text{id}$,

$$Y(\theta(\theta^{-1}(t))) = Y(t) \quad (88)$$

9. Substituting (87) and (88) into (86),

$$\frac{Z(\theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} = \frac{\sqrt{\dot{\theta}(\theta^{-1}(t))} Y(t)}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} = Y(t) \quad (89)$$

10. This establishes (81), confirming that $g \circ h = \text{id}$. \square

2.3 Transformation of Stationary \rightarrow Oscillatory Processes via U_θ

Theorem 12. [Unitary time change yields oscillatory process] Let X be zero-mean stationary as in Definition 6. For scaling function θ as in Theorem 10, define

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) \quad (90)$$

Then Z is a realization of an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \quad (91)$$

gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \quad (92)$$

and covariance

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\left[\sqrt{\dot{\theta}(t)} X(\theta(t)) \overline{\sqrt{\dot{\theta}(s)} X(\theta(s))}\right] \\ &= \sqrt{\dot{\theta}(t) \dot{\theta}(s)} \mathbb{E}[X(\theta(t)) \overline{X(\theta(s))}] \quad R_Z(t, s) = \sqrt{\dot{\theta}(t) \dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} dF(\lambda) \\ &= \sqrt{\dot{\theta}(t) \dot{\theta}(s)} R_X(\theta(t) - \theta(s)) \\ &= \sqrt{\dot{\theta}(t) \dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} dF(\lambda) \end{aligned} \quad (93)$$

Proof. 1. From the Cramér representation (20),

$$X(u) = \int_{\mathbb{R}} e^{i\lambda u} d\Phi(\lambda) \quad (94)$$

2. Substituting $u = \theta(t)$ into (94),

$$X(\theta(t)) = \int_{\mathbb{R}} e^{i\lambda\theta(t)} d\Phi(\lambda) \quad (95)$$

3. From the definition (90),

$$Z(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)} d\Phi(\lambda) \quad (96)$$

4. By linearity of the stochastic integral,

$$Z(t) = \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} d\Phi(\lambda) \quad (97)$$

5. Define

$$\varphi_t(\lambda) := \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \quad (98)$$

6. Then (97) becomes

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda) dF(\lambda) \quad (99)$$

which is the oscillatory representation (24).

7. To express this in terms of the standard oscillatory function form, write

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} = \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} e^{i\lambda t} \quad (100)$$

8. Define the gain function

$$A_t(\lambda) := \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \quad (101)$$

9. Then

$$\varphi_t(\lambda) = A_t(\lambda) e^{i\lambda t} \quad (102)$$

confirming the oscillatory function form (23).

10. To compute the covariance, use (26):

$$R_Z(t, s) = \mathbb{E}[Z(t)\overline{Z(s)}] \quad (103)$$

11. Substituting (90),

$$R_Z(t, s) = \mathbb{E}\left[\sqrt{\dot{\theta}(t)} X(\theta(t)) \overline{\sqrt{\dot{\theta}(s)} X(\theta(s))}\right] \quad (104)$$

12. Since $\dot{\theta}$ is deterministic,

$$R_Z(t, s) = \sqrt{\dot{\theta}(t)} \sqrt{\dot{\theta}(s)} \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \quad (105)$$

13. By stationarity of X , using (21),

$$\mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] = R_X(\theta(t) - \theta(s)) = \int_{\mathbb{R}} e^{i\lambda(\theta(t) - \theta(s))} dF(\lambda) \quad (106)$$

14. Substituting (106) into (105),

$$R_Z(t, s) = \sqrt{\dot{\theta}(t) \dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t) - \theta(s))} dF(\lambda) \quad (107)$$

establishing (93). \square

Corollary 13. [Evolutionary spectrum of unitarily time-changed stationary process] The evolutionary spectrum is

$$dF_t(\lambda) = \dot{\theta}(t) dF(\lambda) \quad (108)$$

Proof. 1. The evolutionary spectrum is defined by

$$dF_t(\lambda) = |A_t(\lambda)|^2 dF(\lambda) \quad (109)$$

2. From (92),

$$|A_t(\lambda)|^2 = \left| \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \right|^2 \quad (110)$$

3. Since $|e^{i\alpha}| = 1$ for all real α ,

$$|e^{i\lambda(\theta(t)-t)}|^2 = 1 \quad (111)$$

4. Therefore

$$|A_t(\lambda)|^2 = \left(\sqrt{\dot{\theta}(t)} \right)^2 \cdot 1 = \dot{\theta}(t) \quad (112)$$

5. Substituting (112) into (109),

$$dF_t(\lambda) = \dot{\theta}(t) dF(\lambda) \quad (113) \quad \square$$

2.4 Covariance operator conjugation

Proposition 14. [Operator conjugation] Let

$$(T_K f)(t) := \int_{\mathbb{R}} K(|t-s|) f(s) ds \quad (114)$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h} dF(\lambda) \quad (115)$$

Define the transformed kernel

$$K_\theta(s, t) := \sqrt{\dot{\theta}(t) \dot{\theta}(s)} K(|\theta(t) - \theta(s)|) \quad (116)$$

then the corresponding integral covariance operator is conjugated for all $f \in L^2_{loc}(\mathbb{R})$ by

$$(T_{K_\theta} f)(t) = (U_\theta T_K U_\theta^{-1} f)(t) \quad (117)$$

Proof. 1. From (117), expand the right side:

$$(U_\theta T_K U_\theta^{-1} f)(t) = \sqrt{\dot{\theta}(t)} (T_K U_\theta^{-1} f)(\theta(t)) \quad (118)$$

2. By definition (114),

$$(T_K U_\theta^{-1} f)(\theta(t)) = \int_{\mathbb{R}} K(|\theta(t) - s|) (U_\theta^{-1} f)(s) ds \quad (119)$$

3. By definition (60),

$$(U_\theta^{-1} f)(s) = \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \quad (120)$$

4. Substituting (120) into (119),

$$\int_{\mathbb{R}} K(|\theta(t) - s|) \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} ds \quad (121)$$

5. Apply the change of variables $s = \theta(u)$, so $ds = \dot{\theta}(u) du$ and $\theta^{-1}(s) = u$:

$$\int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \frac{f(u)}{\sqrt{\dot{\theta}(u)}} \dot{\theta}(u) du \quad (122)$$

6. Simplify:

$$\int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \frac{\dot{\theta}(u)}{\sqrt{\dot{\theta}(u)}} f(u) du = \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \sqrt{\dot{\theta}(u)} f(u) du \quad (123)$$

7. Substituting (123) into (118),

$$\sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \sqrt{\dot{\theta}(u)} f(u) du \quad (124)$$

8. Bring the constant inside the integral:

$$\int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} \sqrt{\dot{\theta}(u)} K(|\theta(t) - \theta(u)|) f(u) du \quad (125)$$

9. By definition (116),

$$\sqrt{\dot{\theta}(t)} \sqrt{\dot{\theta}(u)} K(|\theta(t) - \theta(u)|) = K_\theta(u, t) \quad (126)$$

10. Therefore

$$\int_{\mathbb{R}} K_\theta(u, t) f(u) du = (T_{K_\theta} f)(t) \quad (127)$$

establishing (117). \square

3 Zero Localization

Definition 15. [Zero localization measure] Let Z be real-valued with $Z \in C^1(\mathbb{R})$ having only simple zeros

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \quad (128)$$

Define, for Borel $B \subset \mathbb{R}$,

$$\mu(B) = \int_{\mathbb{R}} 1_B(t) \delta(Z(t)) |\dot{Z}(t)| dt \quad (129)$$

Theorem 16. [Atomicity and local finiteness of zeros and delta decomposition] Under the assumptions of Definition 15, zeros are locally finite and one has

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \quad (130)$$

whence

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (131)$$

Proof. 1. For any smooth test function ϕ with compact support, apply the standard change of variables formula for the delta function. Let $\{t_0^{(1)}, t_0^{(2)}, \dots\}$ denote the zeros of Z .

2. By the change of variables formula for distributions,

$$\int_{\mathbb{R}} \phi(t) \delta(Z(t)) dt = \sum_{t_0: Z(t_0)=0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} \quad (132)$$

3. The right side of (132) equals

$$\sum_{t_0: Z(t_0)=0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} = \sum_{t_0: Z(t_0)=0} \int_{\mathbb{R}} \phi(t) \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} dt \quad (133)$$

4. By Fubini's theorem (justified since the sum has locally finite terms due to C^1 regularity and simple zeros),

$$\sum_{t_0: Z(t_0)=0} \int_{\mathbb{R}} \phi(t) \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} dt = \int_{\mathbb{R}} \phi(t) \sum_{t_0: Z(t_0)=0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} dt \quad (134)$$

5. Comparing (132) and (134),

$$\int_{\mathbb{R}} \phi(t) \delta(Z(t)) dt = \int_{\mathbb{R}} \phi(t) \sum_{t_0: Z(t_0)=0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} dt \quad (135)$$

6. Since ϕ is arbitrary,

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \quad (136)$$

establishing (130).

7. Substituting (136) into the definition (129),

$$\mu(B) = \int_{\mathbb{R}} 1_B(t) \sum_{t_0: Z(t_0)=0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} |\dot{Z}(t)| dt \quad (137)$$

8. By the sifting property of the delta function, $|\dot{Z}(t)|$ evaluated at $t = t_0$ gives $|\dot{Z}(t_0)|$:

$$\int_{\mathbb{R}} 1_B(t) \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} |\dot{Z}(t)| dt = \frac{1_B(t_0) |\dot{Z}(t_0)|}{|\dot{Z}(t_0)|} = 1_B(t_0) \quad (138)$$

9. Summing over all zeros,

$$\mu(B) = \sum_{t_0: Z(t_0)=0} 1_B(t_0) = \sum_{t_0 \in B: Z(t_0)=0} 1 \quad (139)$$

10. This is precisely the atomic measure

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (140)$$

establishing (131). \square

Definition 17. [Hilbert space on the zero set] Let $\mathcal{H} = L^2(\mu)$ with inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} d\mu(t) \quad (141)$$

Proposition 18. [Atomic structure] Let

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (142)$$

then

$$\mathcal{H} \cong \left\{ f: \{t_0: Z(t_0)=0\} \rightarrow \mathbb{C}: \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \quad (143)$$

with orthonormal basis $\{e_{t_0}\}_{t_0: Z(t_0)=0}$ where

$$e_{t_0}(t_1) = \delta_{t_0, t_1} \quad (144)$$

Proof. 1. By (142), μ is a purely atomic measure with atoms at the zero set.

2. For any $f \in L^2(\mu)$, the L^2 norm is

$$\|f\|_{L^2(\mu)}^2 = \int_{\mathbb{R}} |f(t)|^2 d\mu(t) \quad (145)$$

3. Substituting (142),

$$\int_{\mathbb{R}} |f(t)|^2 d\mu(t) = \int_{\mathbb{R}} |f(t)|^2 \sum_{t_0: Z(t_0)=0} \delta_{t_0}(dt) = \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 \quad (146)$$

4. Therefore

$$\|f\|_{L^2(\mu)}^2 = \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 \quad (147)$$

5. This is precisely the ℓ^2 norm on the zero set.

6. Define the map $\Psi: L^2(\mu) \rightarrow \ell^2$ by

$$\Psi(f) = (f(t_0))_{t_0: Z(t_0)=0} \quad (148)$$

7. From (147), Ψ is an isometry:

$$\|\Psi(f)\|_{\ell^2}^2 = \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 = \|f\|_{L^2(\mu)}^2 \quad (149)$$

8. Ψ is surjective: for any sequence $(c_{t_0}) \in \ell^2$, define $f(t) = \sum_{t_0} c_{t_0} \delta_{t_0}(t)$, which is in $L^2(\mu)$.

9. Therefore Ψ is a Hilbert space isomorphism, establishing (143).

10. For the orthonormal basis, define e_{t_0} by (144).

11. Then

$$\langle e_{t_0}, e_{t_1} \rangle = \int_{\mathbb{R}} e_{t_0}(t) \overline{e_{t_1}(t)} d\mu(t) = \sum_{s: Z(s)=0} \delta_{t_0, s} \delta_{t_1, s} = \delta_{t_0, t_1} \quad (150)$$

12. Therefore $\{e_{t_0}\}$ is an orthonormal set.

13. Since every $f \in L^2(\mu)$ can be written as

$$f = \sum_{t_0: Z(t_0)=0} f(t_0) e_{t_0} \quad (151)$$

the set $\{e_{t_0}\}$ is complete, hence an orthonormal basis. \square

Definition 19. [Multiplication operator] Define the linear operator

$$L: \mathcal{D}(L) \subset \mathcal{H} \rightarrow \mathcal{H} \quad (152)$$

by

$$(L f)(t) = t f(t) \quad (153)$$

on the support of μ with domain

$$\mathcal{D}(L) := \left\{ f \in \mathcal{H}: \int |t f(t)|^2 d\mu(t) < \infty \right\} \quad (154)$$

Theorem 20. [Self-adjointness and spectrum] L is self-adjoint on \mathcal{H} and has pure point, simple spectrum

$$\sigma(L) = \overline{\{t \in \mathbb{R}: Z(t) = 0\}} \quad (155)$$

with eigenvalues $\lambda = t_0$ for each zero t_0 and corresponding eigenvectors e_{t_0} .

Proof. 1. For $f, g \in \mathcal{D}(L)$, compute the inner product:

$$\langle L f, g \rangle = \int_{\mathbb{R}} (L f)(t) \overline{g(t)} d\mu(t) \quad (156)$$

2. By definition (153),

$$\int_{\mathbb{R}} t f(t) \overline{g(t)} d\mu(t) \quad (157)$$

3. Since t is real-valued, $\bar{t} = t$, so

$$\int_{\mathbb{R}} t f(t) \overline{g(t)} d\mu(t) = \int_{\mathbb{R}} f(t) \overline{t g(t)} d\mu(t) \quad (158)$$

4. The right side of (158) is

$$\int_{\mathbb{R}} f(t) \overline{(L g)(t)} d\mu(t) = \langle f, L g \rangle \quad (159)$$

5. Therefore

$$\langle L f, g \rangle = \langle f, L g \rangle \quad (160)$$

for all $f, g \in \mathcal{D}(L)$, establishing that L is symmetric.

6. Since L is a multiplication operator on $L^2(\mu)$, it is self-adjoint (by standard functional analysis).

7. To determine the spectrum, compute the action on basis vectors. From (153) and (144),

$$(L e_{t_0})(t) = t e_{t_0}(t) = t \delta_{t_0}(t) \quad (161)$$

8. By the sifting property,

$$t \delta_{t_0}(t) = t_0 \delta_{t_0}(t) = t_0 e_{t_0}(t) \quad (162)$$

9. Therefore

$$L e_{t_0} = t_0 e_{t_0} \quad (163)$$

10. This shows that each t_0 is an eigenvalue with eigenvector e_{t_0} .

11. Since the $\{e_{t_0}\}$ form a complete orthonormal basis (Proposition 18), the spectrum is pure point.

12. Each eigenspace is one-dimensional (spanned by e_{t_0}), so the spectrum is simple and given by the closure of the zero set

$$\sigma(L) = \{t_0 : Z(t_0) = 0\} = \overline{\{t \in \mathbb{R} : Z(t) = 0\}} \quad (164) \quad \square$$

3.1 The Kac-Rice Formula For The Expected Zero Counting Function

Theorem 21. [Kac-Rice Formula for Zero Crossings] Let $Z(t)$ be a centered Gaussian process on $[a, b]$ with covariance $K(s, t) = \mathbb{E}[Z(s)Z(t)]$ then the expected number of zeros in $[a, b]$ is

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{\frac{2}{\pi} \frac{\sqrt{K(t,t)K_{\dot{Z}}(t,t) - K_{Z,\dot{Z}}(t,t)^2}}{K(t,t)}} dt \quad (165)$$

where

$$K(t,t) = \mathbb{E}[Z(t)^2] \quad (166)$$

$$K_{\dot{Z}}(t,t) = -\partial_s^2 \partial_t K(s,t)|_{s=t} \quad (167)$$

and

$$K_{Z,\dot{Z}}(t,t) = \partial_s K(s,t)|_{s=t} \quad (168)$$

Proof. 1. The expected number of zeros is given by the Kac-Rice formula for the level crossing density at level zero.

2. For a Gaussian process $Z(t)$, the joint distribution of $(Z(t), \dot{Z}(t))$ is bivariate normal with covariance matrix

$$\Sigma(t) = \begin{pmatrix} K(t,t) & K_{Z,\dot{Z}}(t,t) \\ K_{Z,\dot{Z}}(t,t) & K_{\dot{Z}}(t,t) \end{pmatrix} \quad (169)$$

3. The determinant of $\Sigma(t)$ is

$$\det \Sigma(t) = K(t,t)K_{\dot{Z}}(t,t) - K_{Z,\dot{Z}}(t,t)^2 \quad (170)$$

4. The Kac-Rice formula states that the expected number of zeros in $[a, b]$ is

$$\mathbb{E}[N_{[a,b]}] = \int_a^b p_{Z(t),\dot{Z}(t)}(0,v)|v|dvdt \quad (171)$$

where $p_{Z(t),\dot{Z}(t)}$ is the joint density of $(Z(t), \dot{Z}(t))$.

5. For a centered bivariate normal distribution with covariance $\Sigma(t)$,

$$p_{Z(t),\dot{Z}(t)}(z,v) = \frac{1}{2\pi\sqrt{\det\Sigma(t)}} \exp\left(-\frac{1}{2}\begin{pmatrix} z \\ v \end{pmatrix}^T \Sigma(t)^{-1} \begin{pmatrix} z \\ v \end{pmatrix}\right) \quad (172)$$

6. At $z=0$,

$$p_{Z(t),\dot{Z}(t)}(0,v) = \frac{1}{2\pi\sqrt{\det\Sigma(t)}} \exp\left(-\frac{v^2 K(t,t)}{2\det\Sigma(t)}\right) \quad (173)$$

7. Integrating (173) against $|v|$,

$$\int_{-\infty}^{\infty} p_{Z(t),\dot{Z}(t)}(0,v)|v|dv = \frac{1}{2\pi\sqrt{\det\Sigma(t)}} \int_{-\infty}^{\infty} |v| \exp\left(-\frac{v^2 K(t,t)}{2\det\Sigma(t)}\right) dv \quad (174)$$

8. Using symmetry, $\int_{-\infty}^{\infty} |v| e^{-av^2} dv = 2 \int_0^{\infty} v e^{-av^2} dv = \frac{1}{a} \sqrt{\frac{\pi}{a}}$:

$$\int_{-\infty}^{\infty} |v| \exp\left(-\frac{v^2 K(t,t)}{2\det\Sigma(t)}\right) dv = \sqrt{\frac{2\pi\det\Sigma(t)}{K(t,t)}} \quad (175)$$

9. Substituting (175) into (174),

$$\frac{1}{2\pi\sqrt{\det\Sigma(t)}} \cdot \sqrt{\frac{2\pi\det\Sigma(t)}{K(t,t)}} = \sqrt{\frac{2}{\pi}} \frac{\sqrt{\det\Sigma(t)}}{K(t,t)} \quad (176)$$

10. Using (170),

$$\sqrt{\frac{2}{\pi}} \frac{\sqrt{K(t,t)K_{\dot{Z}}(t,t) - K_{Z,\dot{Z}}(t,t)^2}}{K(t,t)} \quad (177)$$

11. Integrating (177) over $[a,b]$ yields (165). \square

3.1.1 The Expected Zero Counting Function for Unitarily Time-Changed Stationary Processes

Theorem 22. (Expected Zero-Counting Function Of The Oscillatory Process Subclass of Unitarily Time-Changed Stationary Processes) Let $\theta: \mathbb{R} \rightarrow \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\dot{\theta}(t) > 0$ almost everywhere. Let X be a centered stationary Gaussian process with spectral measure F and covariance function

$$K(h) = \int_{\mathbb{R}} e^{i\omega h} dF(\omega) \quad (178)$$

twice differentiable at $h=0$. Define the unitarily time-changed process

$$Z(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) \quad (179)$$

Then Z is a centered Gaussian process with covariance

$$K_Z(t,s) = \sqrt{\dot{\theta}(t)\dot{\theta}(s)} K(\theta(t) - \theta(s)) \quad (180)$$

and the expected number of zeros in $[0,T]$ is

$$\mathbb{E}[N_{[0,T]}] = \sqrt{\frac{-\ddot{K}(0)}{\pi K(0)}} [\theta(T) - \theta(0)] \quad (181)$$

Proof. 1. By the Kac-Rice formula:

$$\mathbb{E}[N_{[0,T]}] = \int_0^T \sqrt{-\lim_{s \rightarrow t} \frac{\partial^2}{\partial s \partial t} K_Z(s,t)} dt \quad (182)$$

2. Differentiate (180) with respect to s :

$$\frac{\partial}{\partial s} K_Z(s,t) = \frac{\ddot{\theta}(s)}{2\sqrt{\dot{\theta}(s)\dot{\theta}(t)}} K(\theta(t) - \theta(s)) - \dot{\theta}(s) \sqrt{\dot{\theta}(s)\dot{\theta}(t)} \dot{K}(\theta(t) - \theta(s)) \quad (183)$$

3. Differentiate (183) with respect to t and take $s \rightarrow t$. Since $\dot{K}(0) = 0$ by stationarity:

$$\lim_{s \rightarrow t} \frac{\partial^2}{\partial s \partial t} K_Z(s,t) = -\dot{\theta}(t)^2 \ddot{K}(0) \quad (184)$$

4. Substitute (184) into (182):

$$\mathbb{E}[N_{[0,T]}] = \int_0^T \sqrt{-(-\dot{\theta}(t)^2 \ddot{K}(0))} dt = \int_0^T \sqrt{\dot{\theta}(t)^2 (-\ddot{K}(0))} dt \quad (185)$$

5. Since $\dot{\theta}(t) > 0$ and $\ddot{K}(0) < 0$:

$$\mathbb{E}[N_{[0,T]}] = \int_0^T \dot{\theta}(t) \sqrt{-\ddot{K}(0)} dt = \sqrt{-\ddot{K}(0)} \int_0^T \dot{\theta}(t) dt \quad (186)$$

6. Evaluate the integral:

$$\mathbb{E}[N_{[0,T]}] = \sqrt{-\dot{K}(0)} [\theta(T) - \theta(0)] \quad (187) \quad \square$$

Theorem 23. [Deterministic zero-crossing at vanishing derivative] Let X be a zero-mean stationary process with spectral measure F as in Definition 6 and finite variance $\sigma^2 = \mathbb{E}[X(t)^2] < \infty$. Let $\theta: \mathbb{R} \rightarrow \mathbb{R}$ be the time-change function from Theorem 10, which is absolutely continuous (has derivative $\dot{\theta}$ that exists almost everywhere and is Lebesgue integrable), strictly increasing (so $\theta(t_1) < \theta(t_2)$ whenever $t_1 < t_2$), and bijective (one-to-one and onto). The derivative $\dot{\theta}(t)$ is strictly positive almost everywhere, meaning $\dot{\theta}(t) > 0$ for all t except possibly on a set of Lebesgue measure zero. Define the transformed process

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) \quad (188)$$

as in equation (59). Consider a point $t_0 \in \mathbb{R}$ where the derivative vanishes: $\dot{\theta}(t_0) = 0$. Then **every sample path of Z passes through zero at t_0** : for all $\omega \in \Omega$,

$$Z(t_0, \omega) = 0 \quad (189)$$

This is a **deterministic zero-crossing**: unlike the random zero-crossings of the stationary process X , which occur probabilistically according to Bulinskaya's statistics, the zero at t_0 occurs with certainty in every realization of Z . The randomness of X is completely suppressed at t_0 by the vanishing amplitude factor $\sqrt{\dot{\theta}(t_0)} = 0$.

Proof. 1. Consider a point $t_0 \in \mathbb{R}$ where $\dot{\theta}(t_0) = 0$.

2. From the definition (188), the value of Z at t_0 for any sample path $\omega \in \Omega$ is

$$Z(t_0, \omega) = \sqrt{\dot{\theta}(t_0)} \cdot X(\theta(t_0), \omega) \quad (190)$$

3. Since $\dot{\theta}(t_0) = 0$ by hypothesis,

$$\sqrt{\dot{\theta}(t_0)} = \sqrt{0} = 0 \quad (191)$$

4. Substituting (191) into (190),

$$Z(t_0, \omega) = 0 \cdot X(\theta(t_0), \omega) = 0 \quad (192)$$

regardless of the value of $X(\theta(t_0), \omega)$.

5. Since $\omega \in \Omega$ was arbitrary, equation (192) holds for every sample path:

$$Z(t_0, \omega) = 0 \quad \forall \omega \in \Omega \quad (193)$$

6. Therefore t_0 is a deterministic zero-crossing: the process Z reaches zero at t_0 in every realization, not probabilistically.

7. As a direct consequence, the variance of Z at t_0 is zero:

$$\text{Var}[Z(t_0)] = \mathbb{E}[(Z(t_0) - \mathbb{E}[Z(t_0)])^2] = \mathbb{E}[0^2] = 0 \quad (194)$$

8. By Corollary 13, the evolutionary spectrum at t_0 vanishes:

$$dF_{t_0}(\lambda) = \dot{\theta}(t_0) dF(\lambda) = 0 \cdot dF(\lambda) = 0 \quad (195)$$

meaning there is no spectral energy at t_0 .

9. The point t_0 belongs to the zero set $\{t \in \mathbb{R}: Z(t, \omega) = 0\}$ for every $\omega \in \Omega$. By Definition 3, this deterministic zero-crossing differs fundamentally from the random zero-crossings governed by the statistics of the stationary process X : it occurs because the amplitude factor $\sqrt{\dot{\theta}(t_0)}$ vanishes, completely eliminating the influence of the random process X at that instant. \square

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