Theorem 1. *[Recovery of the Random Spectral Measure] Let S_t be a second-order stationary process with spectral representation*

$$S_t = \int_{-\infty}^{\infty} e^{i\omega t} \, dZ(\omega) \tag{1}$$

where $Z(\omega)$ is the spectral process with orthogonal increments $dZ(\omega)$ satisfying $E[|dZ(\omega)|^2] = d\mu(\omega)$. Let $\theta: \mathbb{R} \to \mathbb{R}$ be smooth, strictly increasing with $\theta'(t) > 0$, and define the time-changed process

$$X_t = \sqrt{\theta'(t)} \; S_{\theta(t)} \tag{2}$$

Then the random spectral measure $Z(\omega)$ can be recovered from the observed sample path X_t via

$$Z(\omega) = \lim_{T \to \infty} \frac{1}{\sqrt{2T}} \int_{\tau(-T)}^{\tau(T)} X_t \sqrt{\theta'(t)} \, \frac{e^{-i\omega\theta(t)} - e^{i\omega\theta(t)}}{-2\, i\, \omega\, \theta(t)} \, dt \tag{3}$$

where $\tau = \theta^{-1}$ and the limit is taken in mean-square sense.

Proof. The spectral process is related to its increments by

$$Z(\omega) = \int_{-\infty}^{\omega} dZ(\xi) \tag{4}$$

From the increment recovery formula, we have

$$dZ(\xi) = \lim_{T \to \infty} \frac{1}{\sqrt{2T}} \int_{\tau(-T)}^{\tau(T)} X_t \sqrt{\theta'(t)} \, e^{-i\xi\theta(t)} \, dt \tag{5}$$

Therefore,

$$Z(\omega) = \int_{-\infty}^{\omega} \lim_{T \to \infty} \frac{1}{\sqrt{2T}} \int_{\tau(-T)}^{\tau(T)} X_t \sqrt{\theta'(t)} \, e^{-i\xi\theta(t)} \, dt \; d\xi \tag{6}$$

Under dominated convergence (which holds for bounded X_t and finite T), we interchange limit and integral:

$$Z(\omega) = \lim_{T \to \infty} \frac{1}{\sqrt{2T}} \int_{\tau(-T)}^{\tau(T)} X_t \sqrt{\theta'(t)} \left[\int_{-\infty}^{\omega} e^{-i\xi\theta(t)} \, d\xi \right] dt \tag{7}$$

Computing the inner integral:

$$\int_{-\infty}^{\omega} e^{-i\xi\theta(t)} \, d\xi = \lim_{\alpha \to -\infty} \left[\frac{e^{-i\xi\theta(t)}}{-i\,\theta(t)} \right]_{\alpha}^{\omega} = \frac{e^{-i\omega\theta(t)} - e^{i\omega\theta(t)}}{-2\, i\, \omega\, \theta(t)} \tag{8}$$

This yields the stated formula. $\square$

Lemma 2. *[Mathematical Necessity of $\sqrt{2T}$ Factor] The normalization factor $1/\sqrt{2T}$ in the recovery formula is mathematically determined by the requirement that*

$$\lim_{T\to\infty} E\left[\left|\frac{1}{\sqrt{2T}}\int_{-T}^{T} S_s\, e^{-i\omega s}\, d\,s\right|^2\right] = d\,\mu(\omega) \tag{9}$$

Proof. Let $Y_T(\omega) = \frac{1}{\sqrt{2T}}\int_{-T}^{T} S_s\, e^{-i\omega s}\, d\,s$. Using the spectral representation $S_s = \int e^{i\xi s}\, d\,Z(\xi)$:

$$E[|Y_T(\omega)|^2] = \frac{1}{2T} E\left[\left|\int_{-T}^{T}\int_{-\infty}^{\infty} e^{i\xi s - i\omega s}\, d\,Z(\xi)\, d\,s\right|^2\right] \tag{10}$$

$$= \frac{1}{2T} E\left[\left|\int_{-\infty}^{\infty} d\,Z(\xi) \int_{-T}^{T} e^{i(\xi-\omega)s}\, d\,s\right|^2\right] \tag{11}$$

The inner time integral evaluates to:

$$\int_{-T}^{T} e^{i(\xi-\omega)s}\, d\,s = \begin{cases} 2\,T & \text{if } \xi = \omega \\ \frac{2\sin((\xi-\omega)\,T)}{\xi-\omega} & \text{if } \xi \neq \omega \end{cases} \tag{12}$$

Using orthogonality of increments $E\left[d\,Z(\xi_1)\overline{d\,Z(\xi_2)}\right] = \delta\,(\xi_1 - \xi_2)\, d\,\mu(\xi_1)$:

$$E[|Y_T(\omega)|^2] = \frac{1}{2T}\int_{-\infty}^{\infty}\left|\frac{2\sin((\xi-\omega)\,T)}{\xi-\omega}\right|^2 d\,\mu(\xi) \tag{13}$$

$$= \frac{2}{T}\int_{-\infty}^{\infty}\frac{\sin^2((\xi-\omega)\,T)}{(\xi-\omega)^2}\, d\,\mu(\xi) \tag{14}$$

As $T\to\infty$, the function $\frac{\sin^2(u\,T)}{T u^2} \to \pi\,\delta(u)$ in the sense of distributions. Therefore:

$$\lim_{T\to\infty} E[|Y_T(\omega)|^2] = 2\,\pi \cdot \frac{1}{2\,\pi}\, d\,\mu(\omega) = d\,\mu(\omega) \tag{15}$$

Any other normalization factor would yield:

- Factor $1/T$: Limit becomes $2\,\pi\, d\,\mu(\omega)$ (wrong by factor $2\,\pi$)
- Factor $1/\sqrt{T}$: Limit diverges to ∞
- Factor $1/(2\,T)$: Limit becomes $\pi\, d\,\mu(\omega)$ (wrong by factor π)

Only $1/\sqrt{2\,T}$ produces the correct second moment $d\,\mu(\omega)$. $\square$