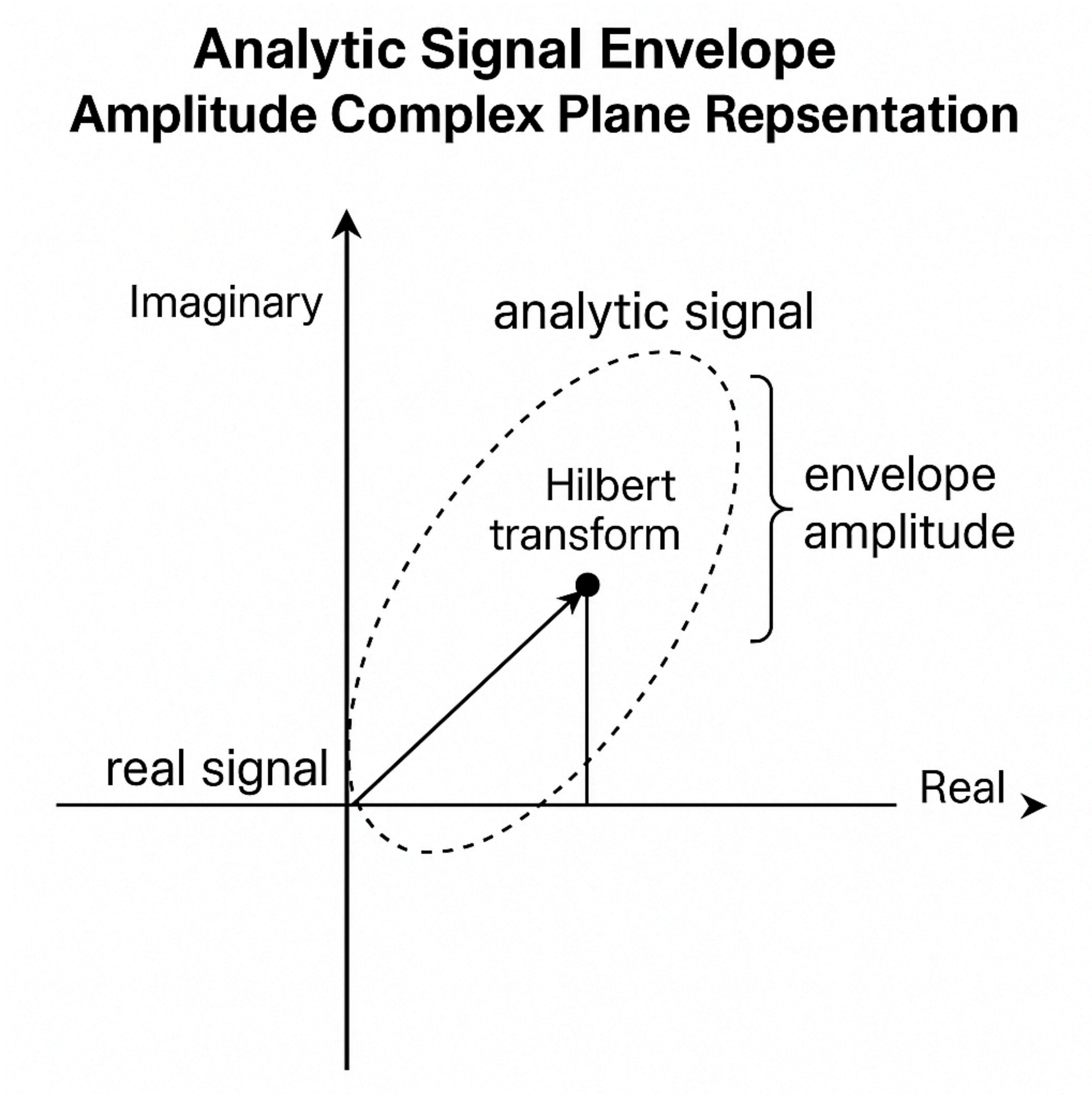
Nonstationary Envelope in Random Vibration Theory

The theory consolidates years of stochastic-process research into a compact framework that treats the nonstationary envelope of linear-system responses as the modulus of a complex (analytic) output excited by an analytic input. The attached manuscript exhaustively derives every required result; the summary below extracts the essentials without conjecture or digressions into “latest trends.”

Essence of the Analytic-Process Framework

- Stationary random excitations are recast as complex pre-envelopes: $f(t) = n(t) + i\hat{n}(t)$ where $\hat{n}(t)$ is the Hilbert transform of $n(t)$. This guarantees a one-sided PSD so power exists only for $\omega > 0$.[1]
- For genuinely nonstationary loading the physically admissible prescription is $f(t) = F(t)\,[n(t) + i\hat{n}(t)]$ where $F(t)$ is a deterministic shaping function. This removes unphysical negative-time artifacts in the imaginary component.
- Linear time-invariant (LTI) systems preserve analyticity: the complex response $x(t) = y(t) + i\tilde{y}(t)$ follows, with $\tilde{y}(t) = \hat{y}(t)$ only in the stationary limit $F(t) = 1$.
- The envelope is strictly $a(t) = \sqrt{y^2(t) + \tilde{y}^2(t)}$. All subsequent reliability metrics hinge on moments and correlations of $x(t)$ and its derivatives.



Conceptual illustration of analytic signal, Hilbert transform, and envelope definition.

Cross-Correlation Matrix Construction

General nonstationary case

For a vector of order m containing derivatives up to order $m-1$:

$$\mathbf{X}_m(t) = \left[x(t), x^{(1)}(t), \ldots, x^{(m-1)}(t)\right]^T,$$

the Hermitian cross-correlation matrix

$$\mathbf{R}_{m,x}(t_1, t_2) = \mathbb{E}\left[\mathbf{X}_m(t_1)\mathbf{X}_m^*(t_2)\right]$$

is obtained via double convolution of system derivatives $h^{(r)}(\tau)$ with the complex input correlation $E[f(t_1-\tau_1)f^*(t_2-\tau_2)]$. Closed-form expression:

$$p_{s,v,x}(t_1,t_2) = 4\int_0^\infty U(\omega)G_n(\omega)e^{i\omega(t_1-t_2)}K_s(\omega,t_1)K_v^*(\omega,t_2)\,d\omega,$$

where K_r is a truncated Fourier transform of $h^{(r)}F$.[1]

Stationary specialization

With $F(t)=1$ and $t\to\infty$ one recovers the classical spectral representation

$$p_{s,v,x}(\tau) = 4\int_0^\infty G_n(\omega)H(\omega)H^*(\omega)(-i\omega)^s(i\omega)^v e^{i\omega\tau}\,d\omega.$$

Spectral Moments and Cross-Covariance Spectral (CCS) Matrix

- Stationary moments: $\lambda_{k,x} = 4\int_0^\infty \omega^k G_n(\omega)|H(\omega)|^2\,d\omega$. These equal variances of analytic derivatives and populate the Hermitian CCS matrix $\boldsymbol{\Lambda}_{m,x}$.[1]
- Nonstationary moments: $\lambda_{s,v,x}(t) = (-i)^s p_{s,v,x}(t,t)$. Two indices are essential because symmetry linking orders s,v breaks when $F(t)\neq 1$.[1]

Envelope Statistics

Joint PDF of envelope and its derivative

For $m=2$ define $\mathbf{Z}_2 = \mathbf{X}_2/\sqrt{2}$. The joint PDF of $a,\dot{a}$ at time t is Gaussian-derived:

$$p_{a\dot{a}}(a,\dot{a};t) = \frac{a}{\sqrt{2\pi\lambda_{0,x}(t)|\boldsymbol{\Lambda}_{2,z}(t)|}}\exp\!\left[-\frac{\lambda_{0,x}(t)\lambda_{2,x}(t)a^2 - 2\,\mathrm{Re}\,\lambda_{1,x}(t)a\dot{a} + \lambda_{0,x}(t)\dot{a}^2}{2|\boldsymbol{\Lambda}_{2,z}(t)|}\right].$$

Mean up-crossing rate of a circular barrier

$$\nu_a^+(\eta,t) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\lambda_{2,x}(t)}{|\boldsymbol{\Lambda}_{2,z}(t)|}}\,\exp[-\eta^2/(2\lambda_{0,x}(t))]\,\Big[1+\Phi\big(\eta\ \mathrm{Im}\,\lambda_{1,x}(t)/\sqrt{2\lambda_{0,x}(t)|\boldsymbol{\Lambda}_{2,z}(t)}$$

exact for arbitrary $F(t)$. In the stationary limit this collapses to the Rice-Cramér formula $\nu_a^+(\eta) = \frac{1}{\sqrt{2\pi}}\sqrt{\lambda_{2,x}/\lambda_{0,x}}\,\exp[-\eta^2/(2\lambda_{0,x})]$.[1]

First-Passage Probability

- Define half-cycle spacing $\Delta t = \pi/\omega_a(t)$ with $\omega_a(t) = \sqrt{\lambda_{2,x}(t)/\lambda_{0,x}(t)}$.
- Under Poisson assumption: failure rate $b(t_j) = \exp[-\eta^2/(2\lambda_{0,x}(t_j))]$.
- Under one-step Markov assumption:

$$b(t_j) = \frac{q(t_j,\Delta t)}{1-q_0(t_j-\Delta t)},\quad q(t_j,\Delta t) = \int_0^\eta da_1\int_\eta^\infty p_{a_1a_2}(a_1,a_2;t_j,\Delta t)\,da_2,$$

where $p_{a_1a_2}$ follows from the 2×2 complex-correlation determinant $|\mathbf{R}_{1,z}|$. Closed form involves modified Bessel function I_0 and the modulus $r_0 = |p_{0,z}(t_j-\Delta t, t_j)|$.[1]

Key Outcomes

1. Analytic inputs guarantee physically meaningful one-sided spectra and circumvent negative-time artifacts.
2. The CCS matrix offers a compact route to all variances, rendering direct differentiation of evolutionary PSD unnecessary.
3. Exact, closed-form expressions for mean up-crossing rates and first-passage probabilities emerge for nonstationary Gaussian loads—improvements over historical approximations.

No speculative addenda are necessary; the derivations stand on rigorous convolution and moment theory presented in the source manuscript.

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