$\mathbf{L^2}$ Norm Preservation Under Smooth Bijective Unbounded Substitutions

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1 Introduction

This document concerns the structure of L^2-norm-preserving operators induced on L^2 spaces by smooth, bijective, orientation-preserving substitutions $g: I \to J$ on (possibly unbounded) intervals $I, J \subseteq \mathbb{R}$. The topic is fundamental in ergodic theory and operator theory, as it precisely characterizes when a substitution operator corresponds to a unitary operator, and relates directly to the behavior of measures under measure-preserving bijections. The classical result is also crucial for understanding the behavior of the L^2 norm under change of variables. Canonicity and necessity of the Jacobian factor is established, and the role of unboundedness is treated from the start.

2 Smooth Bijective Transformations and L^2 Norm Preservation

Definition 1. *Let $I, J \subseteq \mathbb{R}$ be (possibly unbounded) open intervals. A map $g: I \to J$ is called a* smooth bijection *if g is:*

1. *Bijection between I and J,*
2. *Differentiable on I with $g'(y) > 0$ for almost every $y \in I$ (i.e., g is strictly increasing except possibly on a set of Lebesgue measure zero).*

Lemma 2. *[Bijectivity of Strictly Increasing Unbounded C^1 Maps] Let $I, J \subseteq \mathbb{R}$ be (possibly unbounded) open intervals. Suppose $g: I \to J$ is a C^1 function with $g'(y) > 0$ for all $y \in I$ except possibly a Lebesgue null set, and g is unbounded above and below on I. Then g is bijective onto $J = g(I)$, g^{-1} exists and is also strictly increasing and differentiable a.e.*

Proof. The function g is strictly increasing on every subset of I where $g'(y) > 0$; on the (at most measure-zero) set where $g'(y) = 0$, g remains monotonic and continuous by C^1 regularity. Since I is an interval and g is continuous and strictly increasing almost everywhere, g is injective by the intermediate value property of continuous strictly increasing functions.

Unboundedness of g on I implies that $g(I)$ is also an open interval in $\mathbb{R}$ (possibly the whole real line), so $g: I \to J$ is surjective. Thus, g is bijective from I onto $J = g(I)$. Its inverse $g^{-1}: J \to I$ is again continuous, strictly increasing (except possibly on a null set), and differentiable almost everywhere by the inverse function theorem. □

3 Norm-Preserving Substitution Operators: Measure-Preservation and Unitarity

Theorem 3. *[L^2 Norm Preservation via Jacobian Factor] Let $g: I \to J$ be a smooth bijection in the sense of Definition 1. For any $f \in L^2(J, dx)$, define*

$$\tilde{f}(y) := f(g(y))\sqrt{g'(y)}. \tag{1}$$

Then $\tilde{f} \in L^2(I, dy)$ and

$$\|\tilde{f}\|_{L^2(I,dy)} = \|f\|_{L^2(J,dx)}. \tag{2}$$

Proof. Since $g: I \to J$ is bijective, strictly increasing and differentiable almost everywhere with $g'(y) > 0$ a.e., the change of variables theorem applies (see e.g., [RoydenFitzpatrick], [Folland]).

For any $f \in L^2(J, dx)$,

$$\|\tilde{f}\|^2_{L^2(I,dy)} = \int_I |f(g(y))\sqrt{g'(y)}|^2 \, dy \tag{3}$$

$$= \int_I |f(g(y))|^2 g'(y) \, dy \tag{4}$$

By the change of variables formula for Lebesgue integrals, for any measurable function φ and bijective, strictly increasing g as in Lemma 2:

$$\int_I \varphi(g(y)) \, g'(y) \, dy = \int_J \varphi(x) \, dx. \tag{5}$$

Setting $\varphi(x) = |f(x)|^2$, one obtains

$$\int_I |f(g(y))|^2 g'(y)\ dy = \int_J |f(x)|^2\ dx = \|f\|^2_{L^2(J,dx)} \tag{6}$$

Thus, $\|\tilde{f}\|_{L^2(I,dy)} = \|f\|_{L^2(J,dx)}$ as claimed. ☐

4 Necessity and Canonicality of the Jacobian Weight

Lemma 4. *[Density of Substitution Images] Let $g: I \to J$ be as in Theorem 3. Then the collection $\{f \circ g: f \in L^2(J, dx)\}$ is dense in $L^2(I, g'(y)\ dy)$.*

Proof. The transformation $T: L^2(J, dx) \to L^2(I, g'(y)\ dy)$ defined by $T(f) = f \circ g$ is an isometric isomorphism by the change of variables (5). The image of an isomorphism from a complete space is itself complete and thus dense. ☐

Theorem 5. *[Necessity of the Square Root Jacobian Factor] Let $g: I \to J$ be as above. Suppose $\psi: I \to \mathbb{R}^+$ is measurable and for every $f \in L^2(J, dx)$,*

$$|f(g(\cdot)) \cdot \psi(\cdot)|_{L^2(I,dy)} = \|f\|_{L^2(J,dx)}. \tag{7}$$

Then $\psi(y) = \sqrt{g'(y)}$ for almost every $y \in I$.

Proof. Suppose (7) holds for all $f \in L^2(J, dx)$. Compute:

$$\int_I |f(g(y))|^2 |\psi(y)|^2\ dy = \|f\|^2_{L^2(J,dx)} \tag{8}$$

$$= \int_I |f(g(y))|^2\, g'(y)\ dy \tag{9}$$

Subtracting, for every f,

$$\int_I |f(g(y))|^2 \left(|\psi(y)|^2 - g'(y)\right)\ dy = 0 \tag{10}$$

By Lemma 4, the set $\{f(g(y))\}$ is dense in $L^2(I, g'(y)\, dy)$. Thus, for every $u \in L^2(I, g'(y)\, dy)$,

$$\int_I |u(y)|^2 \left(|\psi(y)|^2 - g'(y)\right)\ dy = 0 \tag{11}$$

By standard measure-theoretic arguments (cf. [Folland], p. 70), the only way for this to be true for all u is for $|\psi(y)|^2 = g'(y)$ almost everywhere. Since ψ is taken as non-negative, $\psi(y) = \sqrt{g'(y)}$ a.e. $\square$

5 Unitary Operators, Invariant Measures, and Measure-Preservation

Definition 6. *[Koopman Operator] Let $(X, \mathcal{B}, \mu)$ be a probability measure space, $T: X \to X$ a measurable bijection, and μ a T-invariant measure: for all $A \in \mathcal{B}$, $\mu(T^{-1}A) = \mu(A)$. The* Koopman operator *U_T is defined for measurable $f: X \to \mathbb{C}$ by*

$$(U_T f)(x) = f(Tx). \tag{12}$$

Theorem 7. *[Unitarity Corresponds to Measure-Preservation] The Koopman operator U_T on $L^2(X, \mu)$ is unitary if and only if T is invertible and both T and T^{-1} preserve the measure μ.*

Proof. If T is invertible and μ is T-invariant,

$$\|U_T f\|_{L^2(X,\mu)}^2 = \int_X |f(Tx)|^2 \, d\mu(x) = \int_X |f(x)|^2 \, d\mu(x)$$

where the last equality is by change of variables $x = T^{-1}(y)$ and measure-preservation, so U_T is an isometry. Surjectivity follows from invertibility of T and surjectivity of L^2 composition. Conversely, if U_T is unitary, then the above identity must hold for all f. Choosing indicator functions of sets A, it follows that $\mu(T^{-1}(A)) = \mu(A)$, so T preserves the measure. $\square$

6 Bibliography

Bibliography

[RoydenFitzpatrick] H. L. Royden and P. M. Fitzpatrick, *Real Analysis*, Fourth Edition, Pearson, 2010.

[Folland] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, Second Edition, Wiley, 1999.

[Walters] P. Walters, *An Introduction to Ergodic Theory*, Springer, 1982.

[Halmos] P. R. Halmos, *Measure Theory*, Springer, 1974.

[EinsiedlerWard] M. Einsiedler and T. Ward, *Ergodic Theory with a View Towards Number Theory*, Springer, 2011.