\title{Schoenberg's Theory of Totally Positive Functions and the Riemann Zeta

Function}

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Abstract

We review Schoenberg's characterization of totally positive functions and its connection to the Laguerre-Polya class. This characterization yields a new condition that is equivalent to the truth of the Riemann hypothesis.

In a series of papers in the 1950s Schoenberg investigated the properties of {\tp} functions~{\cite{CS66,sch47,Sch50,sch51,SW53}}. He found several characterizations and used total positivity to prove fundamental properties of splines~{\cite{SW53,Sch73}}. The purpose of this note is to survey some aspects of Schoenberg's work on {\tp} functions, to advertize the connection between {\tp} functions and the Riemann hypothesis, and to provide some mathematical entertainment.

One may speculate whether Schoenberg himself thought about the Riemann zeta function. He was the son-in-law of the eminent number theorist Edmund Landau, he collaborated with Polya, he knew deeply the work of Polya and Schur about the {\lpc} of entire functions that remains influential in the study of the Riemann hypothesis. Yet, to my knowledge, he never mentioned any number theory in his work on {\tp} functions and splines; by the same token, Schoenberg's name is not mentioned in analytic number theory.

{\tmem{Totally positive functions.}} A measurable function Λ on
is {\tp}, if for every and every two sets of increasing numbers

$x_1 < x_2 < \ldots < x_n$ and $y_1 < y_2 < \ldots < y_n$ the matrix $(\Lambda(x_j - y_k))_{j,k=1,\ldots,n}$ has non-negative determinant:

$$\det(\Lambda(x_j - y_k))_{j,k=1,\ldots,n} \geq 0\,. \qquad (1)$$

If in addition Λ is integrable, then Λ is called a Polya frequency function.

If Λ is {\tp} and not equal to e^{ax+b}, there exist an exponential e^{cx}, such that $\Lambda_1(x) = e^{cx}\Lambda(x)$ is a {\pff}, i.e., Λ_1 is {\tp} and integrable$_{\text{{\textbackslash cite[Lemma}}}$4]{sch51}}. It is usually no loss of generality to restrict to {\pff}s.

The class of {\tp} functions played and plays an important role in approximation theory, in particular in spline theory~{\cite{SW53}}, and in statistics~{\cite{Efr65,karlin}}. In a different and rather surprising direction, {\tp} functions appear in the representation theory of infinite dimensional motion groups~{\cite{Pick91}}. Recently, {\tp}

functions appeared in sampling theory and in {\tfa} {\cite{GS13,GRS18,GRS20}}, where they were instrumental in the derivation of optimal results.

{\tmem{The Laguerre-Polya class.}} An entire function Ψ of order at most 2 belongs to the Laguerre-Polya class, if its Hadamard factorization is of the form

where are the zeros of Ψ, m is the order of the zero at 0, $\gamma \geq 0$, , and

$$0 < \gamma + \sum_{j=1}^{\infty} \delta_j^2 < \infty\,. \qquad (2)$$

Thus the {\lpc} consists of entire functions of order two with convergence exponent two with only real zeros. While the study of the distribution of zeros of entire functions is a perennial topic in complex analysis and of interest in its own right~{\cite{Levin80}}, the {\lpc} has gained special prominence in analytic number theory: the Riemann hypothesis says that a relative of the Riemann zeta function belongs to the {\lpc}.

{\tmem{Schoenberg's characterization of {\tp} functions.}} The fundamental results about {\tp} functions were derived by Schoenberg in a series of papers~{\cite{sch47,Sch50,sch51,SW53}}. A comprehensive treatment is contained in Karlin's monograph$_{\text{{\textbackslash cite[Ch.}}}$7]{karlin}}.

The notions of {\tp} functions and {\lpc} are seemingly unrelated, yet there is a deep connection between them through the following characterization of Schoenberg~{\cite{sch51}}.

{\tm{(i) If Λ is a Polya frequency function, then its
(two-sided) Laplace transform converges in a vertical strip , and

$$\int_{-\infty}^{\infty} \Lambda(x) e^{-sx}\, dx = \frac{1}{\Psi(s)} \qquad (3)$$

is the reciprocal of a function Ψ in the {\lpc} with $\Psi(0) > 0$.

(ii) Conversely, if Ψ is in the {\lpc} with $\Psi(0) > 0$, then its
reciprocal $1/\Psi$ is the Laplace transform of a Polya frequency function
Λ.}}

This is a fascinating theorem, because it relates two function classes that seem to bear absolutely no resemblance to each other. Schoenberg's theorem establishes a bijection between the class of {\pff}s, the {\lpc}, and yields a parametrization by the set .

By using the Fourier transform instead of the Laplace transform, Schoenberg's theorem can be recast as follows: A function Λ is {\tp} and integrable, {\fif} its Fourier transform possesses the factorization

where $C > 0$, $\gamma \geq 0$, and $\sum_{j=1}^{\infty} \delta_j^2 < \infty$ (and the product in [eq:6] may also be finite).

If we drop the condition of integrability and exclude exponential functions, then the representation [eq:6] still holds for every {\tp} function, but their Laplace transform of Λ converges in some vertical strip that does not contain 0.

A similar result holds for one-sided {\tp} functions{\cite[Thm.2]{sch51}}.

{\tm{(i) If Λ is a Polya frequency function with
support in $[0, \infty)$, then its Laplace transform converges in a half-plane
, and

$$\int_{0}^{\infty} \Lambda(x) e^{-sx}\, dx = \frac{1}{\Psi(s)} \qquad (4)$$

is the reciprocal of an entire function Ψ with Hadamard factorization

$$\Psi(s) = C e^{\delta s} \prod_{j=1}^{\infty} (1 + \delta_j s)\,, \qquad (5)$$

with .

(ii) Conversely, if Ψ possesses the factorization (5), then
its reciprocal $1/\Psi$ is the Laplace transform of a Polya frequency
function Λ with support in $[0, \infty)$.}}

{\vspace{3mm}}

{\tmem{Elementary examples.}} If , then is the one-sided exponential function. For $\hat{\Lambda}(\tau) = e^{-\pi\gamma\tau^2}$ for $\gamma > 0$, we obtain the Gaussian $\Lambda(x) = \gamma^{-1/2} e^{-\pi x^2/\gamma}$. In both cases, it is easy to
check directly that these functions are {\tp}.

The proof of the implication (ii) of Theorem~[tm:tp] is based on the (non-trivial) fact that the convolution $\Lambda = \Lambda_1 * \Lambda_2$ of two {\pff}s Λ_1, Λ_2 is again a {\pff}. The converse in Theorem~[tm:tp] lies much deeper, and Schoenberg used heavily several results of Polya about functions in the {\lpc}~{\cite{Pol15,PS1914}}. See the end of this note for the essential step of the argument.

Schoenberg's motivation was the characterization and deeper understanding of {\tp} functions, and thus the implication (i) and the factorization [eq:6] can be considered his main insight about {\tp} functions. However, instead of reading Schoenberg's theorem as a characterization of {\tp} functions, one may read it as a characterization of the {\lpc}. {\tmem{A function Ψ with $\Psi(0) > 0$ and $\Psi \neq e^{as+b}$ is in the {\lpc}, {\fif} the Fourier transform of $1/\Psi$ is a {\pff}.}}

{\tmem{The Riemann hypothesis and {\tp} functions.}} Let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for , be the Riemann zeta function and let

$$\xi(s) = \tfrac{1}{2} s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad (6)$$

$$\Xi(s) = \xi\left(\frac{1}{2} + is\right) \quad (7)$$

be the Riemann xi-functions (where Γ is the usual gamma function). Then the functional equation for the Riemann zeta function is expressed by the symmetry

$$\xi(s) = \xi(1-s) \qquad \text{and} \qquad \Xi(s) = \Xi(-s) \quad (8)$$

for the xi-functions.

The Riemann hypothesis conjectures that all non-trivial zeros of the zeta function lie on the critical line $1/2 + it$. See the monographs~{\cite{Iwa14,Ivi03,Tit86}}, the two volumes about equivalents of the Riemann hypothesis~{\cite{Br17-1,Br17-2}} or the survey articles~{\cite{Bom10,Conrey03}}.

Expressed in terms of the xi-functions, the Riemann hypothesis states that Ξ has only real zeros, in other words, {\tmem{Ξ belongs to the {\lpc}.}} Thus many investigations of the zeta function involve complex analysis related to the {\lpc}. Schoenberg's theorem immediately leads to the following equivalent condition for the Riemann hypothesis to hold.

{\tm{The Riemann hypothesis holds, {\fif} there exists a
{\pff} Λ, such that

where $1/2+it_0$ is the first zero of the zeta function on the critical line.}}

Let us make this statement a bit more explicit by taking the Fourier transform instead of the Laplace transform.

{\tm{The Riemann hypothesis holds, {\fif}

$$\Lambda(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\xi(\frac{1}{2}+\tau)} e^{-ix\tau}\, d\tau \quad (9)$$

is a Polya frequency function.}}

The growth of ξ in the complex plane is~{\cite{Tit86}} , and on the positive real line

$$\ln \xi(\sigma) \asymp \tfrac{1}{2}\sigma \log \sigma \qquad \sigma > 1\,.$$

Consequently $\frac{1}{\xi(\sigma)} \leq Ce^{-|\sigma|\log|\sigma|/2}$ decays super-exponentially. Since ζ and thus ξ do not have any
real zeros in the interval $[0,1]$ and $\zeta > 0$ on $(1,\infty)$, the function ξ is therefore strictly positive on and $1/\xi$ is integrable. Thus its Fourier transform is well-defined.

Using $s = 2\pi i\tau$, we can rewrite [eq:9] as a Fourier transform. The inversion formula for the Fourier transform now yields

$$\begin{aligned}\Lambda(x) &= \int_{-\infty}^{\infty} \frac{1}{\Xi(2\pi i\tau)} e^{2\pi ix\tau}\, d\tau \\ &= \int_{-\infty}^{\infty} \frac{1}{\xi(1/2-2\pi\tau)} e^{2\pi ix\tau}\, d\tau\,,\end{aligned}$$

which is (9).

Using the symmetry of Ξ, there is an alternative formulation of Theorem~[tm:equi1] with the restricted {\lpc} defined in (5). Since Ξ is symmetric, it can be written as $\Xi(s) = \Xi_1(-s^2)$ for an entire function Ξ_1 of order $1/2$. Furthermore, Ξ has only real zeros, {\fif} Ξ_1 has only negative zeros (with convergence exponent at most 1). The characterization of one-sided {\pff}s yields the following equivalence.

{\tm{The Riemann hypothesis holds, {\fif} there exists a {\pff} Λ with support in $[0,\infty)$, such that

for some $\alpha < 0$.}}

These equivalences seem to be new. Schoenberg's name is not even mentioned in ~{\cite{Br17-1,Br17-2}} on equivalents of the Riemann hypothesis.

It is interesting that the characterization of Theorem~[tm:equi2] is ``orthogonal'' to most research on ζ and to the well-known criteria for the Riemann hypothesis. Theorem~[tm:equi2] requires only {\tmem{the values of ζ on the real line}} to probe the secrets of ζ in the critical strip. This fact is remarkable, but the price to pay is the added difficulty to extract any meaningful statements about ξ on the critical strip from its restriction to . This seems much harder, if not impossible.

To work with Theorem~[tm:equi2], one would need a viable expression for the Fourier-Laplace transform of $1/\xi$, but there seems to be none. The 1-positivity in (1) says that $\Lambda > 0$, which is equivalent to the Fourier transform $\hat{\Lambda} = 1/\xi$ to be positive definite by Bochner's theorem. Explicitly, we would need to know that, for all choices of , and all , we have . Not even this property of $1/\xi$ seems to be known. It is therefore unlikely that much is gained by Theorems~[tm:equi1] -- [tm:ones].

By contrast, the Fourier transform of $\Xi(x)$ on the critical line (!) was already known to Riemann (see~{\cite[2.16.1]{Tit86}}) and is the starting point of a program to prove the Riemann hypothesis that goes back to Polya~{\cite{Pol26}}. After important work of de Bruijn, Hejhal, and Newman this line of thought has recently culminated in the resolution of the Newman conjecture by Rodgers and Tao~{\cite{RT20}}.

{\tmem{Some non-trivial {\pff}s.}} Perhaps Schoenberg had also the Riemann hypothesis in mind, when he investigated Polya frequency functions. The examples in~{\cite{sch47,sch51}} of {\tp} functions smell of the zeta function.

(i) The zero set $\{0, -1, -2, \ldots\}$ with multiplicity one yields the entire function

$$\Psi(s) = e^{\gamma s} s \prod_{n=1}^{\infty} (1 + \frac{s}{n})\, e^{-s/n}\,, \quad (10)$$

where γ is the Euler constant. By a classical result Ψ is the reciprocal of the Γ-function $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx$. Consequently, the Laplace transform of is a {\tp} function. Indeed, using the substitution $x = e^{-t}$ in the definition of Γ, one obtains

$$\Gamma(s) = \int_{-\infty}^{\infty} e^{-e^{-x}} e^{-sx}\, dx \qquad \mathrm{Re}\, s > 0\,. \quad (11)$$

Theorem~[tm:tp] implies that

$$\Lambda(x) = e^{-e^{-x}}$$

is {\tp}. By removing the pole of Γ at 0, we obtain

$$s\Gamma(s) = \int_{-\infty}^{\infty} \Lambda'(x) e^{-sx}\, dx = \int_{-\infty}^{\infty} e^{-x}\, e^{-e^{-x}} e^{-sx}\, dx\,, \qquad \mathrm{Re} s > -1\,.$$

Consequently $\Lambda_1(x) = e^{-x-e^{-x}}$ is a {\pff}.

(ii) The zero set with simple zeros yields $\Psi(s) = \frac{\sin \pi s}{\pi}$. By Theorem~[tm:tp], $1/\Psi$ is the Laplace transform of a
{\tp} function on a suitable strip of convergence. Schoenberg's calculation yields the {\tp} function

$$\Lambda(x) = \frac{1}{1+e^{-x}}\,.$$

(iii) Finally the zero set yields the entire function

$$\Psi(s) = s \prod_{n=1}^{\infty} (1+\frac{s}{n^2}) = -\frac{1}{\pi}\sqrt{-s}\sin \pi\sqrt{-s}\,.$$

The associated {\tp} function is the Jacobi theta function

$$\Lambda(x) = \begin{cases} \sum_{j=-\infty}^{\infty} (-1)^j e^{-j^2 x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0\,. \end{cases}$$

All three functions show up prominently in the treatment of the functional equation of the zeta function: Γ is contained in the definition of the xi-function, $\sin$ in the formulation of the functional equation, and a Jacobi theta function is used in several proofs of the functional equation (Riemann's original proof, see~{\cite{Tit86}}).

{\tmem{Intrinsic characterization of {\pff}s.}} The fundamental property of {\pff}s is their smoothing property or {\tmem{variation diminishing property}}. The relevance of smoothing properties for many applications is outlined in Schoenberg's survey~{\cite{Sch53}}. In this context the variation of a real-valued function on is measured either by the number of sign changes or by the number of {\tmem{real}} zeros. Formally, given let

and let $N(f)$ be the number of {\tmem{real}} zeros of f counted with multiplicity.

Given a function Λ, let T_Λ be the convolution operator $T_\Lambda f = f * \Lambda$. Schoenberg's second characterization of {\pff}s is as follows~{\cite{Sch50}}.

{\tm{Let Λ be integrable and continuous. Then Λ is variation diminishing, i.e.,

$$v(T_\Lambda f) \leq v(f)$$

for all functions that are locally Riemann integrable, {\fif} either Λ or $-\Lambda$ is a {\pff}.}}

This characterization is ``intrinsic'' in the sense that it uses only the properties of the matrices occurring in the definition (1) of total positivity.

With a perturbation argument one can replace sign changes with zeros and obtains the following consequence.

{\vspace{3mm}}
{\tmem{Intrinsic characterizations of the {\lpc}.}} There are several characterizations of the {\lpc} that require only their properties as entire functions. This is part of classical complex analysis and the results are due to Polya and Schur~{\cite{Pol15,PS1914}} building on work of Laguerre, Hadamard, and many others. These results relate the properties of the zero set to properties of the power series expansion of an entire function. Before formulating a sequence of equivalences, we note that every formal power series $F(s) \sim \sum_{j=0}^\infty a_j s^j$ yields a differential operator $F(D)p(x) = \sum_{j=0}^\infty a_j D^j p(x)$ with $D = \frac{d}{dx}$. The
differential operator is well-defined at least on polynomials, and the mapping $F \mapsto F(D)$ is an algebra homomorphism and thus provides a simple functional calculus.
{\tm{Let $\Psi(s) = \sum_{j=0}^\infty \frac{\beta_j}{j!} s^j$ be an entire function. Then the following are equivalent:
(i) Ψ belongs to the {\lpc}.
(ii) Ψ can be approximated uniformly on compact sets by polynomials with only real zeros.
(iii) For all the polynomials $p_n(x) = \sum_{j=0}^n \beta_j \binom{n}{j} x^j$ and $q_n(x) = \sum_{j=0}^n \beta_j \binom{n}{j} x^{n-j}$ have only real zeros.
(iv) If $p(x) = \sum_{j=0}^m c_j x^j$ is a polynomial with only real,
{\tmem{non-positive}} zeros, then the polynomial $q(x) = \sum \beta_j c_j x^j$ has only real zeros.
If, in addition, $\Psi(0) > 0$ and $\frac{1}{\Psi(s)} = \sum_{j=0}^\infty \frac{\gamma_j}{j!} s^j$, then the following property is equivalent to (i) -- (iv).
(v) The transform $p \mapsto \frac{1}{\Psi(D)} p$ is zero-decreasing, i.e.,
the polynomial $q(x) = \frac{1}{\Psi(D)} p(x) = \sum_{j=0}^\infty \frac{\gamma_j}{j!} p^{(j)}(x)$ has at most as many real zeros as p (real-valued):

$$N\left(\frac{1}{\Psi(D)} p\right) \leq N(p)\,.$$

}}
Applying condition (iv) to the polynomials $x^{n-1}(1+x)^2$, one obtains a necessary condition on the Taylor coefficients of a function in the {\lpc}, namely the so-called Turan inequalities.
Applying condition (v) to polynomials of the form $p(x) = (\sum_{k=1}^n a_k x^k)^2$ and working out , one obtains the following necessary

condition for the {\lpc}{\cite[p.235]{Pol15}}.
However, the positivity of the Hankel matrices is not sufficient for Ψ to be in the {\lpc}, as was proved already by Hamburger~{\cite{Ham20}}.
Theorem~[tm:lpcc] and its corollaries are all contained in the seminal papers of Polya and Schur~{\cite{Pol15,PS1914}} from 1914 and 1915 and have inspired a century of exciting mathematics. Each of the equivalent conditions in Theorem~[tm:lpcc] is a point of departure for the study of the Riemann hypothesis.
No list can do justice to all contributions between 1914 and 2020, so let us mention only a few directions whose origin is in Polya's work. Further references and more detailed history can be found in the cited articles.
Condition (iii) applied to the Riemann function Ξ yields an important equivalence of the Riemann hypothesis. The polynomials in condition (iii) are nowadays called Jensen polynomials. In modern language (iii) says that ``the Jensen polynomials for the Riemann function $\Xi(s)$ must be hyperbolic''.
Significant recent progress on this equivalence is reported in~{\cite{GORZ}}.
The relations between the Jensen polynomials, the multiplier sequences of condition (iv), and the Turan inequalities and their generalizations have been studied in depth by Craven, Csordas, and Varga~{\cite{Cso15,CC89,CV90}} who found many additional equivalences to the Riemann hypothesis. A particular highlight is their proof that Ξ, or rather the Taylor coefficients of $\Xi(\sqrt{s})$ satisfy the Turan inequalities~{\cite{CNV86}}, thereby resolving a
60 year old conjecture going back to --- Polya.
Finally let us mention that total positivity enters the investigation of the {\lpc} in yet another way. A entire function belongs to the restricted {\lpc} defined by (5), {\fif} the sequence of its Taylor coefficients (a_n) is a Polya frequency {\tmem{sequence}}~{\cite{AS52}}. This means that the infinite upper triangular Toeplitz matrix A with entries \$A_{jk} = a_{k

- j}\$, if $k \geq j$ and $A_{jk} = 0$, if $k < j$ has only positive minors.
  This aspect of total positivity has been used in~{\cite{Kat07,Nut13}} for the investigation of the zeta function.

{\tmem{From total positivity to the {\lpc}.}} By comparing the two intrinsic characterizations in Theorems~[tm:vd] and [tm:lpcc] one may guess that the respective conditions on zero diminishing must play the decisive role in the proof of Theorem~[tm:tp](i). To give the gist of this argument, we cannot do better than repeat Schoenberg's beautiful argument.
First, since Λ is assumed to be a {\pff}, Λ must decay exponentially{\cite[Lemma2]{sch51}}, therefore its moments of all orders exist. Let
be the n-th moment. By expanding the exponential $e^{-sx} = \sum_{j=1}^{\infty} \frac{(-s)^j}{j!} x^j$ we express the Laplace transform of
Λ as a power series
Since and $\Lambda \geq 0$, we have $F(0) > 0$, and its
reciprocal also possesses a power series expansion around 0 with a positive radius of convergence

$$\Psi(s)=\frac{1}{F(s)}=\sum_{j=0}^{\infty}\frac{\beta_j}{j!}\,s^j\,.$$

Next, we consider the convolution of Λ with a polynomial p of degree N and relate it to the moments of Λ:

By Corollary~[zdim] the number of real zeros of q (counted with multiplicity) does not exceed the number of real zeros of p,

$$N(q)=N(F(D)p)\le N(p)\,.\quad (12)$$

Using the functional calculus, we can invert $F(D)$ and recover p from $q=\Lambda*p$ via

$$p(x)=\frac{1}{F(D)}q(x)=\Psi(D)q(x)=\sum_{j=0}^{\infty}\frac{\beta_j}{j!}\,q^{(j)}(x)\,.$$

For the monomial $q(x)=x^n$ we obtain the polynomial

$$q_n(x)=\Psi(D)x^n=\sum_{j=0}^{n}\beta_j\binom{n}{j}x^{n-j}$$

of degree n. Since $x^n=F(D)q_n$, (12) implies the count of zeros (with multiplicities)

$$n=N(x^n)\le N(q_n)\le n\,.$$

For every n, q_n therefore has only real zeros. This is precisely condition (iii) of Theorem~[tm:lpcc], and we conclude that Ψ is in the {\lpc}.

{\tmem{Summary.}} Schoenberg's characterization of {\tp} functions implies a condition equivalent to the Riemann hypothesis. The characterization is interesting in itself because it involves only the values of the Riemann zeta function on the real axis. To the best of our knowledge, the characterization of the {\lpc} by means of {\tp} functions has not yet been tested on the Riemann zeta function.

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The very same results of Polya~ were also the point of departure for the many investigations of the zeros of the Riemann zeta function after Polya, de Bruijn, Newman, Hejhal and many others~. Yet, in none of the investigations of the zeros of the Riemann zeta function via zeros of integral transforms and the factorization~[eq:2] one finds a hint to {\tp} functions or the work of Schoenberg.

The LaTeX provided is a complete academic article by Karlheinz Gröchenig titled *"Schoenberg's Theory of Totally Positive Functions and the Riemann Zeta Function."* It reviews Schoenberg's characterization of totally positive (TP) and Pólya frequency functions, connects them to the Laguerre–Pólya class (LPC), and derives a functional equivalence linking the Riemann hypothesis to the existence of a corresponding TP (or Pólya frequency) function whose Laplace transform is the reciprocal of the Riemann ξ-function. The article's structure can be summarized as follows:

Summary of the Paper

1. Core concept

A function Λ is *totally positive* if all minors of the kernel matrix (Λ(xj − yk)) are nonnegative for increasing sequences (xj), (yk). The integrable subclass forms the *Pólya frequency functions* (PFFs).

2. The Laguerre–Pólya class

An entire function Ψ of order ≤ 2 with real zeros and Hadamard factorization

$$\Psi(s) = Cs^m e^{-\gamma s^2+\delta s}\prod_{j=1}^{\infty}(1+\delta_j s)e^{-\delta_j s}$$

belongs to the LPC. The Riemann Hypothesis asserts that the Riemann ξ-function lies in this class.

3. Schoenberg's characterization

Schoenberg (1947–1951) established a duality:

- Λ is a PFF if and only if its Laplace transform is the reciprocal of a function Ψ in LPC with Ψ(0)>0:

$$\int_{-\infty}^{\infty}\Lambda(x)e^{-sx}dx = \frac{1}{\Psi(s)}.$$

Equivalently, the Fourier transform of an integrable TP function has an LPC-type product expansion.

4. Equivalence to the Riemann hypothesis

The Riemann hypothesis holds if and only if there exists a PFF Λ satisfying

$$\frac{1}{\Xi(s)} = \int_{-\infty}^{\infty}\Lambda(x)e^{-sx}dx,$$

where Ξ(s)=ξ(½+is). Hence RH ⇔ 1/Ξ(s) is the Laplace transform of a PFF ⇔ 1/ξ(½+iτ) is positive definite.

5. One-sided version

For Λ supported on $[0,\infty)$, the equivalence involves the restricted LPC with product

$$\Psi(s)=Ce^{\delta s}\prod_{j=1}^{\infty}(1+\delta_j s),\quad \delta_j\ge 0.$$

6. Examples

- $\Lambda(x)=e^{\{-e}\{-x\}\}$ corresponds to $\Psi(s)=1/\Gamma(s)$.
- $\Lambda(x)=(1+e^{\{-x\})}\{-1\}$ corresponds to $\Psi(s)=\sin(\pi s)/\pi$.
- $\Lambda(x)=\sum\_\{j\in\mathbb{Z}\}(-1)\hat{}j\ e^{\{-j}2x\}$ corresponds to $\Psi(s)=-\pi^{-1}\sqrt{(-s)}\sin(\pi\sqrt{(-s)})$.
  Each connects directly to functions appearing in the ξ functional equation.

7. Intrinsic characterizations

(a) Variation-diminishing property: Λ is PFF ⇔ convolution $T\Lambda f=f*\Lambda$ reduces number of sign changes of real-valued f.
(b) LPC equivalence: $\Psi\in$LPC ⇔ $1/\Psi(D)$ is zero-decreasing on real polynomials.
This links Schoenberg's operator-theoretic view with Pólya–Schur multiplier theory.

8. Combinatorial and analytic consequences

The Taylor coefficients of LPC functions satisfy Turán inequalities

$$\beta_n^2-\beta_{n-1}\beta_{n+1}\ge 0,$$

and the Hankel matrices of coefficients of $1/\Psi$ are positive definite.

9. Final synthesis

By constructing moments μn of Λ and using zero-monotonicity of convolution operators, Schoenberg proved that the generating function $\Psi(s)=1/F(s)$, where $F(s)=\int\Lambda(x)e\hat{}\{-sx\}dx$, lies in LPC. This bridges total positivity (a real-variable concept) and the analytic properties of entire functions central to the Riemann hypothesis.

Would you like a distilled formal statement of the *precise equivalence theorem* between total positivity and the Riemann hypothesis written in theorem-proof LaTeX format?