11.70. The development of a symmetric nucleus.

Let $\phi_n(\alpha)$ be a complete set of orthogonal functions satisfying the homogeneous integral equation with symmetric nucleus

$$\phi(\alpha)=\lambda\int_a^b K(\alpha,\xi)\,\phi(\xi)\,d\,\xi \tag{1}$$

the corresponding characteristic numbers being $\lambda_1,\lambda_2,\lambda_3,\ldots$

Now suppose that the series $\sum_{n=1}^{\infty}\frac{\phi_n(\alpha)\,\phi_n(\gamma)}{\lambda_n}$ is uniformly convergent when $a\le\alpha\le b$, $a\le\gamma\le b$. Then it will be shown that

$$K(\alpha,\gamma)=\sum_{n=1}^{\infty}\frac{\phi_n(\alpha)\,\phi_n(\gamma)}{\lambda_n} \tag{2}$$

For consider the symmetric nucleus

$$H(\alpha,\gamma)=K(\alpha,\gamma)-\sum_{n=1}^{\infty}\frac{\phi_n(\alpha)\,\phi_n(\gamma)}{\lambda_n} \tag{3}$$

If this nucleus is not identically zero, it will possess at least one characteristic number μ. Let $\psi(\gamma)$ be any solution of the equation

$$\psi(\alpha)=\mu\int_a^b H(\alpha,\xi)\,\psi(\xi)\,d\,\xi \tag{4}$$

which does not vanish identically.

Multiply by $\phi_m(\alpha)$ and integrate and we get, since the series converges uniformly, we may integrate term by term and get

$$\int_a^b\psi(\alpha)\,\phi_m(\alpha)\,d\,\alpha=\mu\int_a^b\int_a^b\left[K(\alpha,\xi)-\sum_{n=1}^{\infty}\frac{\phi_n(\alpha)\,\phi_n(\xi)}{\lambda_n}\right]\psi(\xi)\,\phi_m(\alpha)\,d\,\xi\,d\,\alpha=0 \tag{5}$$

Therefore $\psi(\alpha)$ is orthogonal to $\phi_1(\alpha),\phi_2(\alpha),\ldots$; and so taking the equation

$$\psi(\alpha)=\mu\int_a^b\left[K(\alpha,\xi)-\sum_{n=1}^{\infty}\frac{\phi_n(\alpha)\,\phi_n(\xi)}{\lambda_n}\right]\psi(\xi)\,d\,\xi \tag{6}$$

we have

$$\psi(\alpha)=\mu\int_a^b K(\alpha,\xi)\,\psi(\xi)\,d\,\xi \tag{7}$$

Therefore μ is a characteristic number of $K(\alpha,\gamma)$, and so $\psi(\alpha)$ must be a linear combination of the functions $\phi_n(\alpha)$ corresponding to this number; let

$$\psi(\alpha) = \sum_m a_m\,\phi_m(\alpha) \tag{8}$$

Multiply by $\phi_m(\gamma)$ and integrate; then since $\psi(\alpha)$ is orthogonal to all the functions $\phi_m(\alpha)$, we see that $a_m = 0$, so, contrary to hypothesis, $\psi(\alpha) = 0$.

The contradiction implies that the nucleus $H(\alpha,\gamma)$ must be identically zero; that is to say, $K(\alpha,\gamma)$ can be expanded in the given series, if it is uniformly convergent.

1 Explanation

Given:

- An orthogonal set of functions $\phi_n(\alpha)$ defined over an interval $[a,b]$.
- A symmetric kernel $K(\alpha,\gamma)$ defined over $[a,b]\times[a,b]$.
- The series $\sum_{n=1}^{\infty}\frac{\phi_n(\alpha)\,\phi_n(\gamma)}{\lambda_n}$ uniformly converges to $K(\alpha,\gamma)$.

To Prove:

- The functions $\phi_n(\alpha)$ are the unique eigenfunctions of the integral operator with kernel $K(\alpha,\gamma)$.

Proof:

1. **Uniform Convergence of Series Representation:**
    - By hypothesis, the series

    $$K(\alpha,\gamma) = \sum_{n=0}^{\infty}\frac{\phi_n(\alpha)\,\phi_n(\gamma)}{\lambda_n} \tag{9}$$

    converges uniformly to $K(\alpha,\gamma)$.
    - This uniform convergence ensures that the series represents K accurately over the entire domain $[a,b]\times[a,b]$.

2. **Orthogonality and Completeness:**
    - The functions ϕ_n are orthogonal, meaning:

    $$\int_a^b \phi_m(\alpha)\,\phi_n(\alpha)\,d\alpha = 0 \quad \text{for all } m \neq n \tag{10}$$

- Orthogonality implies that no ϕ_n can be represented by a linear combination of other ϕ_ms in the set.

3. **Eigenfunction Equation:**

    - Each function ϕ_n satisfies the integral equation:

    $$\phi_n(\alpha) = \lambda_n \int_a^b K(\alpha, \xi)\, \phi_n(\xi)\, d\,\xi \tag{11}$$

    defining them as eigenfunctions of K with corresponding eigenvalues λ_n.

4. **Uniqueness:**

    - Assume there exists another function $\psi(\alpha)$ that is not a linear combination of ϕ_n and also satisfies the integral equation for some λ:

    $$\psi(\alpha) = \lambda \int_a^b K(\alpha, \xi)\, \psi(\xi)\, d\,\xi \tag{12}$$

    - Multiply both sides by $\phi_m(\alpha)$ and integrate:

    $$\int_a^b \psi(\alpha)\, \phi_m(\alpha)\, d\,\alpha = \lambda \int_a^b \int_a^b K(\alpha, \xi)\, \psi(\xi)\, \phi_m(\alpha)\, d\,\xi\, d\,\alpha \tag{13}$$

    - Since ψ is orthogonal to all ϕ_n, the left-hand side is zero, implying $\psi(\alpha)$ must be zero by the completeness of ϕ_n.

5. **Conclusion:**

    - The set ϕ_n uniquely represents the kernel K via their series expansion. No other function set orthogonal to ϕ_n can satisfy the kernel's integral equation unless it is zero.

    - Therefore, $\phi_n(\alpha)$ are the unique eigenfunctions of the integral operator defined by $K(\alpha, \gamma)$.