The Spectral Jacobi Tau Method for Fractional Riccati Equations

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1 Problem Formulation

The fractional Riccati equation on $[0, T]$:

$$D^{\nu} y(t) = p(t) + q(t)\, y(t) + r(t)\, y^2(t), \quad t \in [0, T] \tag{1}$$

with initial condition $y(0) = y_0$.

2 Spectral Tau Method

The shifted Jacobi polynomials on $[0, 1]$ are defined by:

$$P_i^{(\alpha,\beta)}(t) = \sum_{k=0}^{i} (-1)^{i-k} \binom{i+\alpha}{k} \binom{i+\beta}{i-k} t^k. \tag{2}$$

The solution is expanded as:

$$y_N(t) = \sum_{i=0}^{N} c_i\, P_i^{(\alpha,\beta)}(t/T). \tag{3}$$

Functions $p(t)$, $q(t)$, and $r(t)$ are also expanded:

$$p(t) = \sum_{i=0}^{N} p_i\, P_i^{(\alpha,\beta)}(t/T), \quad q(t) = \sum_{i=0}^{N} q_i\, P_i^{(\alpha,\beta)}(t/T), \quad r(t) = \sum_{i=0}^{N} r_i\, P_i^{(\alpha,\beta)}(t/T). \tag{4}$$

The shifted weight function:

$$w(t) = t^{\alpha} (1-t)^{\beta}. \tag{5}$$

The coefficients are defined by:

$$p_i = \frac{1}{h_i^{(\alpha,\beta)}} \int_0^1 p\,(T t)\, P_i^{(\alpha,\beta)}(t)\, w(t)\; d t, \tag{6}$$

$$q_i = \frac{1}{h_i^{(\alpha,\beta)}} \int_0^1 q\,(T t)\, P_i^{(\alpha,\beta)}(t)\, w(t)\; d t, \tag{7}$$

$$r_i = \frac{1}{h_i^{(\alpha,\beta)}} \int_0^1 r\,(T t)\, P_i^{(\alpha,\beta)}(t)\, w(t)\; d t. \tag{8}$$

The normalization constant is:

$$h_i^{(\alpha,\beta)} = \frac{\Gamma\,(i+\alpha+1)\,\Gamma\,(i+\beta+1)}{(2\,i+\alpha+\beta+1)\,\Gamma\,(i+1)\,\Gamma\,(i+\alpha+\beta+1)}. \tag{9}$$

The fractional derivative matrix $\mathbf{D}^\nu$ has elements:

$$D_{ij}^{\nu} = \frac{1}{T^\nu} \sum_{k=j}^{i} \theta_{i,k}^{\nu}, \tag{10}$$

where

$$\theta_{i,k}^{\nu} = \frac{\Gamma\,(k+\beta+1)\,\Gamma\,(i+\alpha+\beta+1)}{\Gamma\,(k+\alpha+\beta+1)\,\Gamma\,(i+\beta+1)} \binom{i}{k} \frac{\Gamma\,(k+\nu)}{\Gamma\,(k+1)}. \tag{11}$$

The operational matrices $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ have elements:

$$P_{ij} = \frac{1}{h_j^{(\alpha,\beta)}} \int_0^1 P_i^{(\alpha,\beta)}(t)\, P_j^{(\alpha,\beta)}(t)\, w(t)\; d t, \tag{12}$$

$$Q_{ij} = \sum_{k=0}^{N} \frac{q_k}{h_j^{(\alpha,\beta)}} \int_0^1 P_i^{(\alpha,\beta)}(t)\, P_j^{(\alpha,\beta)}(t)\, P_k^{(\alpha,\beta)}(t)\, w(t)\; d t, \tag{13}$$

$$R_{ij} = \sum_{k=0}^{N} \frac{r_k}{h_j^{(\alpha,\beta)}} \int_0^1 P_i^{(\alpha,\beta)}(t)\, P_j^{(\alpha,\beta)}(t)\, P_k^{(\alpha,\beta)}(t)\, w(t)\; d t. \tag{14}$$

The triple product integral has the exact form:

$$\int_0^1 P_i^{(\alpha,\beta)}(t)\, P_j^{(\alpha,\beta)}(t)\, P_k^{(\alpha,\beta)}(t)\, w(t)\; d t = \sum_{m=0}^{i} \sum_{n=0}^{j} \sum_{l=0}^{k} \gamma_{mnl}^{ijk} \frac{\Gamma\,(m+n+l+\alpha+1)\,\Gamma\,(\beta+1)}{\Gamma\,(m+n+l+\alpha+\beta+2)}, \tag{15}$$

where

$$\gamma_{mnl}^{ijk} = (-1)^{i+j+k-m-n-l} \binom{i+\alpha}{m} \binom{i+\beta}{i-m} \binom{j+\alpha}{n} \binom{j+\beta}{j-n} \binom{k+\alpha}{l} \binom{k+\beta}{k-l}. \tag{16}$$

The nonlinear system to solve is:

$$\mathbf{D}^{\nu}\mathbf{c}_m = \mathbf{p} + \mathbf{Q}\mathbf{c}_m + \mathbf{R}\mathbf{c}_m^2, \tag{17}$$

where $\mathbf{c}_m = [c_{m,0}, c_{m,1}, \ldots, c_{m,N}]^{\top}$ is the coefficient vector at iteration m.

The tau condition for the initial value is:

$$\sum_{i=0}^{N} c_{0,i} P_i^{(\alpha,\beta)}(0) = y_0, \quad \text{where} \quad P_i^{(\alpha,\beta)}(0) = (-1)^i \binom{i+\beta}{i}. \tag{18}$$

The Newton iteration update is:

$$\mathbf{c}_{m+1} = \mathbf{c}_m - \mathbf{J}(\mathbf{c}_m)^{-1}\mathbf{F}(\mathbf{c}_m), \tag{19}$$

where

$$\mathbf{F}(\mathbf{c}_m) = \mathbf{D}^{\nu}\mathbf{c}_m - \mathbf{p} - \mathbf{Q}\mathbf{c}_m - \mathbf{R}\mathbf{c}_m^2, \tag{20}$$

and

$$\mathbf{J}(\mathbf{c}_m) = \mathbf{D}^{\nu} - \mathbf{Q} - 2\mathbf{R}\mathbf{c}_m. \tag{21}$$

The reconstructed solution is:

$$y(t) = \sum_{i=0}^{N} c_i^* P_i^{(\alpha,\beta)}\left(\frac{t}{T}\right), \tag{22}$$

where c_i^* are the converged coefficients.