Measure-Preserving Bijective Time Changes of Stationary Gaussian Processes Constitute a Subclass of Oscillatory Gaussian Processes

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Abstract

This article establishes that Gaussian processes obtained through measure-preserving bijective unitary time transformations of stationary processes constitute a subclass of oscillatory processes in the sense of Priestley. The transformation $Z(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t))$, where $X(t)$ is a realization of stationary Gaussian process and θ is a strictly increasing C^1 differentiable monotonic function, yields an oscillatory process with evolutionary power spectrum $d\,F_t(\omega) = \dot{\theta}(t)\, d\,\mu(\omega)$. An explicit unitary transformation between the input stationary process and the transformed oscillatory process is established, preserving the L^2-norm and providing a complete spectral characterization.

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1 Scaling Functions

Definition 1. *[Scaling Functions] Let $\mathcal{F}$ denote the set of functions $\theta: \mathbb{R} \to \mathbb{R}$ satisfying*

1. *θ is absolutely continuous with*

$$\dot{\theta}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\theta(t) \geq 0 \tag{1}$$

almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero

2. *θ is strictly increasing and bijective.*

Remark 2. The conditions in Definition 1 ensure that $\theta^{-1}(s)$ exists and is absolutely continuous. By the inverse function theorem for absolutely continuous functions,

$$\frac{\mathrm{d}}{\mathrm{d}s}(\theta^{-1})(s) = \frac{1}{\dot{\theta}(\theta^{-1}(s))} = \dot{\theta}(\theta^{-1}(s))^{-1} \tag{2}$$

for almost all s in the range of θ. The condition that $\dot{\theta}(t) = 0$ only on sets of measure zero ensures that $\frac{1}{\dot{\theta}(\theta^{-1}(s))}$ is well-defined almost everywhere.

2 Oscillatory Processes

An oscillatory process can be represented as a time-dependent filter applied to a stationary process

Definition 3. *[Stationary Process] A real–valued process $\{X(t)\}_{t \in \mathbb{R}}$ is a stationary Gaussian process if it can be represented by the continuous spectral representation*

$$X(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, d\,\Phi(\omega) \tag{3}$$

where $\Phi(\omega)$ is an orthogonal-increment process with spectral density

$$E\,|d\,\Phi(\omega)|^2 = d\,\mu(\omega) = S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(u) e^{-i\omega u} \mathrm{d}u = \dot{\mu}(\omega) = \frac{\mathrm{d}}{\mathrm{d}\omega}\mu(\omega) \tag{4}$$

and μ is an absolutely continuous Lebesgue measure on $\mathbb{R}$

Definition 4. *[Oscillatory Process] A complex-valued, second-order process* $\{Z(t)\}_{t\in\mathbb{R}}$ *is called* oscillatory *if there exist*

1. *a family of oscillatory basis functions* $\{\phi_t(\omega)\}_{t\in\mathbb{R}}$ *with*

$$\begin{aligned}\phi_t(\omega) &= A_t(\omega)\, e^{i\omega t}\\ &= \int_{-\infty}^{\infty} h(t,u) e^{i\omega u} \mathrm{d}u\end{aligned} \tag{5}$$

   and a given family of gain functions

$$A_t(\omega) = \frac{\phi_t(\omega)}{e^{i\omega t}} \in L^2(\mu) \tag{6}$$

   with time-dependent filter given by

$$h(t,u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_t(\omega) e^{-i\omega u} \mathrm{d}\lambda \tag{7}$$

2. *and a complex orthogonal random measure* $\Phi(\omega)$ *with*

$$E\,|d\,\Phi(\omega)|^2 = d\,\mu(\omega) = S(\omega) \tag{8}$$

such that

$$\begin{aligned}Z(t) &= \int_{-\infty}^{\infty} \phi_t(\omega)\ d\,\Phi(\omega)\\ &= \int_{-\infty}^{\infty} A_t(\omega)\, e^{i\omega t} d\,\Phi(\omega)\\ &= \int_{-\infty}^{\infty} h(t,u) X(u) \mathrm{d}u\end{aligned} \tag{9}$$

where

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \mathrm{d}\Phi(\lambda) \tag{10}$$

3 Unitarily Time-Changed Process

3.1 Definition and Unitary Operator

Definition 5. *[Unitary Time-Change Operator]For* $\theta \in \mathcal{F}$*, define the operator* $M_\theta: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *by*

$$(M_\theta f)(t) = \sqrt{\dot{\theta}(t)}\ f(\theta(t)) \tag{11}$$

Definition 6. *[Unitarily Time-Changed Stationary Process] For $\theta \in \mathcal{F}$, apply the unitary time change operator M_θ from Definition-? to a realization of a stationary process $X(t)$ from the ensemble $\{X(t)\}$ to define a realization of the unitarily time-changed process*

$$Z(t) = \sqrt{\dot{\theta}(t)}\; X(\theta(t)) \forall t \in \mathbb{R} \tag{12}$$

Definition 7. *[Inverse Unitary Time-Change Operator] The inverse operator M_θ^{-1}: $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ corresponding to the unitary time-change operator $(M_\theta f)(t)$ defined in Equation-11 is given by*

$$(M_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{13}$$

Lemma 8. *[Well-Definedness of Inverse Operator] The operator M_θ^{-1} in Definition 7 is well-defined $\forall \theta \in \mathcal{F}$.*

Proof. Since $\dot{\theta}(t) = 0$ only on sets of measure zero by Definition 1, and θ^{-1} maps sets of measure zero to sets of measure zero (as it preserves absolute continuity), the denominator $\sqrt{\dot{\theta}(\theta^{-1}(s))}$ is positive almost everywhere. The expression in equation (13) is therefore well-defined almost everywhere, which is sufficient for defining an element of $L^2(\mathbb{R})$. ☐

Theorem 9. *[Unitarity of Transformation Operator] The operator M_θ defined in equation (11) is unitary, i.e.,*

$$\int_{\mathbb{R}} |(M_\theta f)(t)|^2 \, dt = \int_{\mathbb{R}} |f(s)|^2 \, ds \forall f \in L^2(\mathbb{R}) \tag{14}$$

Proof. Let $f \in L^2(\mathbb{R})$. The L^2-norm of $M_\theta f$ is computed as follows:

$$\int_{\mathbb{R}} |(M_\theta f)(t)|^2 \, dt = \int_{\mathbb{R}} \left| \sqrt{\dot{\theta}(t)}\; f(\theta(t)) \right|^2 dt \tag{15}$$

$$= \int_{\mathbb{R}} \dot{\theta}(t) \, |f(\theta(t))|^2 \, dt \tag{16}$$

Apply the change of variables $s = \theta(t)$. Since θ is absolutely continuous and strictly increasing, its Jacobian is given by

$$ds = \dot{\theta}(t) \; dt \tag{17}$$

almost everywhere. As t ranges over $\mathbb{R}$, $s=\theta(t)$ ranges over $\mathbb{R}$ due to the bijectivity of θ. Therefore:

$$\int_{\mathbb{R}} \dot{\theta}(t)\,|f(\theta(t))|^2\,dt = \int_{\mathbb{R}} |f(s)|^2\,ds \tag{18}$$

This establishes equation (14). To complete the proof of unitarity, it remains to show that M_θ^{-1} is indeed the inverse of M_θ. For any $f\in L^2(\mathbb{R})$:

$$\begin{aligned}
(M_\theta^{-1}\,M_\theta\,f)(s) &= (M_\theta^{-1})\left[\sqrt{\dot{\theta}(\cdot)}\;f(\theta(\,\cdot\,))\right](s) &(19)\\
&= \frac{\sqrt{\dot{\theta}(\theta^{-1}(s))}\;f(\theta(\theta^{-1}(s)))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} &(20)\\
&= f(s) &(21)
\end{aligned}$$

where the last equality uses $\theta(\theta^{-1}(s))=s$. Similarly, for any $g\in L^2(\mathbb{R})$:

$$\begin{aligned}
(M_\theta\,M_\theta^{-1}\,g)(t) &= \sqrt{\dot{\theta}(t)}\;(M_\theta^{-1}\,g)(\theta(t)) &(22)\\
&= \sqrt{\dot{\theta}(t)}\;\frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} &(23)\\
&= \sqrt{\dot{\theta}(t)}\;\frac{g(t)}{\sqrt{\dot{\theta}(t)}} &(24)\\
&= g(t) &(25)
\end{aligned}$$

Therefore

$$M_\theta\,M_\theta^{-1} = M_\theta^{-1}\,M_\theta = I \tag{26}$$

proving that M_θ is unitary. ☐

Corollary 10. *[Measure Preservation] The transformation M_θ preserves the L^2-measure in the sense that for any measurable set $A\subseteq\mathbb{R}$*

$$\int_A |(M_\theta\,f)(t)|^2\,dt = \int_{\theta(A)} |f(s)|^2\,ds \tag{27}$$

Proof. The proof follows the same change of variables argument as in Theorem 9, applied to the characteristic function of the set A. ☐

3.2 L^2-Norm Preservation

Theorem 11. *[Measure Preservation] The transformation defined in equation (12) preserves the L^2-norm in the sense that*

$$\int_I \operatorname{var}(Z(t))\,dt = \int_{\theta(I)} \operatorname{var}(X(s))\,ds \tag{28}$$

for any measurable set $I \subseteq \mathbb{R}$.

Proof. Using the change of variables $s = \theta(t)$ with $ds = \dot{\theta}(t)\,dt$:

$$\int_I \operatorname{var}(X(t))\,dt = \int_I \operatorname{var}\Big(\sqrt{\dot{\theta}(t)}\;X(\theta(t))\Big)\,dt \tag{29}$$

$$= \int_I \dot{\theta}(t)\operatorname{var}(X(\theta(t)))\,dt \tag{30}$$

$$= \int_{\theta(I)} \operatorname{var}(X(s))\,ds \tag{31}$$

□

3.3 Oscillatory Representation

Theorem 12. *[Oscillatory Form] The process $\{Z(t)\}$ defined in equation (12) is oscillatory with oscillatory functions*

$$\phi_t(\omega) = A_t(\omega)\,e^{i\omega t} = \sqrt{\dot{\theta}(t)}\;e^{i\omega\theta(t)} \tag{32}$$

and gain functions

$$A_t(\omega) = \sqrt{\dot{\theta}(t)}\;e^{i\omega(\theta(t)-t)} \tag{33}$$

Proof. Apply the unitary time change operator $(M_\theta f)(t)$ in Defintion (5) then substitute the spectral representation (3) of the stationary process $X(t)$:

$$Z(t) = (M_\theta X)(t)$$

$$= \sqrt{\dot{\theta}(t)}\,X(\theta(t)) \tag{34}$$

$$= \sqrt{\dot{\theta}(t)}\int_{-\infty}^{\infty} e^{i\omega\theta(t)}\,d\Phi(\omega) \tag{35}$$

$$= \int_{-\infty}^{\infty} \sqrt{\dot{\theta}(t)}\;e^{i\omega\theta(t)}\,d\phi(\omega) \tag{36}$$

$$= \int_{-\infty}^{\infty} \phi_t(\omega)\,d\Phi(\omega) \tag{37}$$

where

$$\phi_t(\omega) = \sqrt{\dot{\theta}(t)}\ e^{i\omega\theta(t)} \tag{38}$$

To verify this is an oscillatory representation according to Definition 4, express $\phi_t(\omega)$ in the form of a function of the time-dependent gain $A_t(\lambda)$ as required

$$\begin{aligned}\phi_t(\omega)\ &= A_t(\omega)\, e^{i\omega t}\\ &= \sqrt{\dot{\theta}(t)}\ e^{i\omega(\theta(t)-t)}\ e^{i\omega t}\\ &= \sqrt{\dot{\theta}(t)} e^{i\omega(\theta(t)-t+t)}\\ &= \sqrt{\dot{\theta}(t)}\ e^{i\omega\theta(t)}\end{aligned} \tag{39}$$

where

$$A_t(\omega) = \sqrt{\dot{\theta}(t)}\ e^{i\omega(\theta(t)-t)} \tag{40}$$

Since $\dot{\theta}(t) \geq 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of measure zero, the function $A_t(\omega)$ is well-defined almost everywhere. Moreover, $A_t(\cdot) \in L^2(\mu)$ for each t since:

$$\int_{-\infty}^{\infty} |A_t(\omega)|^2\, d\,\mu(\omega) = \int_{-\infty}^{\infty} \dot{\theta}(t)\ d\,\mu(\omega) \tag{41}$$

$$= \dot{\theta}(t) \int_{-\infty}^{\infty} d\,\mu(\omega)$$

$$= \dot{\theta}(t)\, \mu(\mathbb{R}) < \infty \tag{42}$$

where the finiteness follows from μ being a finite measure and $\dot{\theta}(t)$ being finite almost everywhere. □

3.4 Envelope and Evolutionary Spectrum

Corollary 13. *[Evolutionary Spectrum] The evolutionary power spectrum is*

$$\begin{aligned}d\,F_t(\omega)\ &= |A_t(\omega)|^2\, d\,\mu(\omega)\\ &= \dot{\theta}(t)\ d\,\mu(\omega)\end{aligned} \tag{43}$$

Proof. By Definition 4 and the envelope from Equation 6, the evolutionary power spectrum is:

$$
\begin{aligned}
d\,F_t(\omega) =& |A_t(\omega)|^2 \; d\,\mu(\omega) && (44)\\
=& \left|\sqrt{\dot{\theta}(t)}\; e^{i\omega(\theta(t)-t)}\right|^2 \; d\,\mu(\omega) && (45)\\
=& \dot{\theta}(t)\,|e^{i\omega(\theta(t)-t)}|^2 \; d\,\mu(\omega) && (46)\\
=& \dot{\theta}(t)\; d\,\mu(\omega) && (47)
\end{aligned}
$$

since

$$|e^{i\alpha}| = 1 \forall \alpha \in \mathbb{R} \tag{48}$$ □

4 Operator Conjugation

Theorem 14. *[Operator Conjugation] Let T_K be the integral covariance operator defined by*

$$(T_K\, f)(t) \; = \int_{-\infty}^{\infty} K(|t-s|)\; f(s)\; d\,s \tag{49}$$

where $K(h)$ is the stationary kernel

$$K(h) = \int_{-\infty}^{\infty} S(\lambda) e^{i\lambda h} \mathrm{d}\lambda \tag{50}$$

and let T_{K_θ} be the integral covariance operator defined by

$$
\begin{aligned}
(T_{K_\theta}\, f)(t) \; &= \int_{-\infty}^{\infty} K_\theta(s,t)\, f(s)\; d\,s\\
&= \int_{-\infty}^{\infty} K(|\theta(t)-\theta(s)|)\; \sqrt{\dot{\theta}(t)\dot{\theta}(s)}\, f(s)\; d\,s
\end{aligned}
\tag{51}
$$

for the unitarily time-changed kernel

$$K_\theta(s,t) = K(|\theta(t)-\theta(s)|) \sqrt{\dot{\theta}(t)\dot{\theta}(s)} \tag{52}$$

Then

$$T_{K_\theta} = M_\theta\, T_K\, M_\theta^{-1} \tag{53}$$

Proof. For any $g \in L^2(\mathbb{R})$, compute $(M_\theta T_K M_\theta^{-1} g)(t)$:

$$(M_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\theta'(\theta^{-1}(s))}} \tag{54}$$

$$(T_K M_\theta^{-1} g)(t) = \int_{-\infty}^{\infty} K(|t-s|) \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \, d s \tag{55}$$

Apply the change of variables $u = \theta^{-1}(s)$, so $s = \theta(u)$ and $d s = \dot{\theta}(u) \, d u$:

$$(T_K M_\theta^{-1} g)(t) = \int_{-\infty}^{\infty} K(|t-\theta(u)|) \frac{g(u)}{\sqrt{\dot{\theta}(u)}} \dot{\theta}(u) \, d u \tag{56}$$

$$= \int_{-\infty}^{\infty} K(|t-\theta(u)|) \, g(u) \sqrt{\dot{\theta}(u)} \, d u \tag{57}$$

Now apply M_θ:

$$(M_\theta T_K M_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)} \, (T_K M_\theta^{-1} g)(\theta(t)) \tag{58}$$

$$= \sqrt{\dot{\theta}(t)} \int_{-\infty}^{\infty} K(|\theta(t)-\theta(u)|) \, g(u) \sqrt{\dot{\theta}(u)} \, d u \tag{59}$$

Apply the change of variables $s = \theta(u)$ in the reverse direction:

$$(M_\theta T_K M_\theta^{-1} g)(t) = \int_{-\infty}^{\infty} K(|\theta(t)-\theta(s)|) \, g(s) \, d s \tag{60}$$

$$= (T_{K_\theta} g)(t) \tag{61}$$

This establishes the conjugation relation (53). □

5 Expected Zero Count

Theorem 15. *[Expected Zero-Counting Function] Let $\theta \in \mathcal{F}$ and let*

$$K(\tau) = \operatorname{cov}(X(t), X(\tau)) \tag{62}$$

be twice differentiable at $\tau = 0$. The expected number of zeros of the process X_t in $[a, b]$ is

$$\mathbb{E}[N_{[a,b]}] = \sqrt{-\ddot{K}(0)} \, (\theta(b) - \theta(a)) \tag{63}$$

Proof. The covariance function of the time-changed process is

$$K_\theta(s,t) = \operatorname{cov}(X_s, X_t) = \sqrt{\dot\theta(s)\,\dot\theta(t)}\;K(|\theta(t)-\theta(s)|) \tag{64}$$

For the zero-crossing analysis, consider the normalized process. By the Kac-Rice formula:

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{-\lim_{s\to t}\frac{\partial^2}{\partial s\,\partial t}K_\theta(s,t)}\;dt \tag{65}$$

Computing the mixed partial derivative:

$$\frac{\partial}{\partial t}K_\theta(s,t) = \frac{1}{2}\frac{\ddot\theta(t)}{\sqrt{\dot\theta(t)}}\sqrt{\theta'(s)}\,K(|\theta(t)-\theta(s)|) \tag{66}$$

$$+\sqrt{\dot\theta(s)\,\dot\theta(t)}\,\dot K(|\theta(t)-\theta(s)|)\operatorname{sgn}(\theta(t)-\theta(s))\,\dot\theta(t) \tag{67}$$

Taking the limit as $s\to t$ and using the fact that $\dot K(0)=0$ for stationary processes:

$$\lim_{s\to t}\frac{\partial^2}{\partial s\,\partial t}K_\theta(s,t) = \dot\theta(s)\,\dot\theta(t)\,\ddot K(0) \tag{68}$$

$$= \dot\theta(t)^2\,\ddot K(0) \tag{69}$$

Substituting into the Kac-Rice formula:

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{-\dot\theta(t)^2\,\ddot K(0)}\;dt \tag{70}$$

$$= \sqrt{-\ddot K(0)}\int_a^b \dot\theta(t)\;dt \tag{71}$$

$$= \sqrt{-\ddot K(0)}\;(\theta(b)-\theta(a)) \tag{72}$$

Here the second equality uses $\dot\theta(t)\geq 0$ almost everywhere. □

6 Conclusion

This analysis establishes that Gaussian processes generated by measure-preserving bijective time changes of stationary processes form a well-defined subclass of oscillatory processes. The key contributions include:

1. The construction of the unitary operator M_θ and its inverse, with proper treatment of the case where

$$\dot\theta(t) = 0 \tag{73}$$

on sets of measure zero

2. The explicit oscillatory representation with envelope function

$$A_t(\omega) = \sqrt{\dot{\theta}(t)}\, e^{i\omega(\theta(t) - t)} \tag{74}$$

3. The evolutionary power spectrum formula

$$d\,F_t(\omega) = \dot{\theta}(t)\, d\,\mu(\omega) \tag{75}$$

4. The operator conjugation relationship

$$T_{K_\theta} = M_\theta\, T_K\, M_\theta^{-1} \tag{76}$$

5. A closed-form expression for the expected zero count in terms of the range of the time scaling function

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