Unitary Time Changes of Stationary Processes Yield Oscillatory Processes

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are square-integrable over compact sets. Applying U_θ to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot\theta(t)}\, e^{i\lambda\theta(t)}$ and evolutionary spectrum $d\,F_t(\lambda) = \dot\theta(t)\, d\,F(\lambda)$. It is proved that sample paths of any non-degenerate second-order stationary process lie in $L^2_{\mathrm{loc}}(\mathbb{R})$, making the operator applicable to typical realizations. A zero-localization measure $d\,\mu(t) = \delta(Z(t))|\dot Z(t)|\, d\,t$ induces a Hilbert space $L^2(\mu)$ on the zero set of an oscillatory process Z, and the multiplication operator $(L\,f)(t) = t\,f(t)$ has simple pure point spectrum equal to the zero crossing set of Z.

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1 Gaussian Processes

1.1 Definition

Definition 1

(Gaussian process) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T a nonempty index set. A family $\{X_t : t \in T\}$ of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Gaussian process if for every finite subset $\{t_1, \ldots, t_n\} \subset T$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ is multivariate normal (possibly degenerate). Equivalently, every finite linear combination $\sum_{i=1}^n a_i X_{t_i}$ is either almost surely constant or Gaussian. The mean function is $m(t) := \mathbb{E}[X_t]$ and the covariance kernel is*

$$K(s,t) = \mathrm{Cov}(X_s, X_t) \tag{1}$$

For any finite $(t_i)_{i=1}^n \subset T$, the matrix $K_{ij} = K(t_i, t_j)$ is symmetric positive semidefinite, and a Gaussian process is completely determined in law by m and K.

Definition 2

The canonical metric associated with a Gaussian process is

$$\begin{aligned} d(s,t) &= \sqrt{\mathbb{E}[(X_s - X_t)^2]} \\ &= \sqrt{K(s,s) + K(t,t) - 2K(s,t)} \end{aligned} \tag{2}$$

1.2 Sample Path Realizations

Definition 3

[Locally square-integrable functions] *Define*

$$L^2_{\text{loc}}(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} : \int_K |f(t)|^2 \, dt < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \tag{3}$$

Remark 4. Every bounded measurable set in $\mathbb{R}$ is compact or contained in a compact set; hence $L^2_{\text{loc}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

Theorem 5

[Sample paths in $L^2_{\text{loc}}(\mathbb{R})$] *Let $\{X(t)\}_{t\in\mathbb{R}}$ be a second-order stationary process with*

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \tag{4}$$

Then almost every sample path lies in $L^2_{\text{loc}}(\mathbb{R})$. However, for non-degenerate processes with $\sigma^2 > 0$, sample paths are not globally square-integrable.

Proof. Fix any bounded interval $[a,b]$ and consider the random variable

$$Y_{[a,b]} := \int_a^b X(t)^2 \, dt \tag{5}$$

By stationarity and Fubini's theorem:

$$\begin{aligned} \mathbb{E}[Y_{[a,b]}] &= \mathbb{E}\left[\int_a^b X(t)^2 \, dt \right] \\ &= \int_a^b \mathbb{E}[X(t)^2] \, dt \\ &= \sigma^2 (b-a) < \infty \end{aligned} \tag{6}$$

By Markov's inequality, for any $M > 0$:

$$\mathbb{P}\left(Y_{[a,b]} > M\right) \leq \frac{\mathbb{E}[Y_{[a,b]}]}{M} = \frac{\sigma^2 (b-a)}{M} \tag{7}$$

Therefore

$$\mathbb{P}\left(Y_{[a,b]} < \infty\right) = 1 \tag{8}$$

that is, with probability one, the sample path is square-integrable on $[a,b]$.

Now let K be any compact subset of $\mathbb{R}$. Since K is compact, it is bounded and closed, hence contained in some interval $[-N, N]$ for sufficiently large N. Therefore:

$$\int_K X(t)^2 \, dt \leq \int_{-N}^{N} X(t)^2 \, dt < \infty \quad \text{almost surely} \tag{9}$$

Since the collection of all compact sets is generated by bounded intervals, and we have shown square-integrability on every bounded interval with probability one, it follows that almost every sample path lies in $L^2_{\text{loc}}(\mathbb{R})$.

For the global divergence statement, note that for non-degenerate stationary processes with $\sigma^2 > 0$, we have

$$\mathbb{E}\left[\int_{-n}^{n} X(t)^2 \, dt\right] = 2\,n\,\sigma^2 \to \infty \quad \text{as } n \to \infty \tag{10}$$

Since the expected value grows without bound, the integral itself diverges almost surely. Sample paths are therefore not in $L^2(\mathbb{R})$, only in $L^2_{\text{loc}}(\mathbb{R})$. □

1.3 Stationary processes

Definition 6

[Cramér representation] *A zero-mean stationary process X with spectral measure F admits the sample path representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t} \, d\,\Phi(\lambda) \tag{11}$$

which has covariance

$$R_X(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)} \, d\,F(\lambda) \tag{12}$$

1.4 Oscillatory Processes

A particularly tractable class of non-stationary Gaussian processes is that of the oscillatory processes as defined by Maurice Priestley in 1965.

Definition 7

[Oscillatory process] *Let F be a finite nonnegative Borel measure on $\mathbb{R}$. Let*

$$A_t \in L^2(F) \quad \forall t \in \mathbb{R} \tag{13}$$

be the gain function and

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{14}$$

be the corresponding oscillatory function then an oscillatory process is a stochastic process which can be represented as

$$\begin{aligned} Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda) \, d\,\Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t} \, d\,\Phi(\lambda) \end{aligned} \tag{15}$$

where Φ is a complex orthogonal random measure with spectral measure F which satisfies the relation

$$d\,\mathbb{E}[\Phi(\lambda)\overline{\Phi(\mu)}] = \delta(\lambda - \mu)\, d\,F(\lambda) \tag{16}$$

and has the corresponding covariance kernel

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)} e^{i\lambda(t-s)} \, d\,F(\lambda) \\ &= \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)} \, d\,F(\lambda) \end{aligned} \tag{17}$$

Theorem 8

[Real-valuedness criterion for oscillatory processes] *Let Z be an oscillatory process with oscillatory function*

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{18}$$

and spectral measure F. Then Z is real-valued if and only if

$$A_t(-\lambda) = \overline{A_t(\lambda)} \tag{19}$$

for F-almost every $\lambda \in \mathbb{R}$, equivalently

$$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \tag{20}$$

for F-almost every $\lambda \in \mathbb{R}$.

Proof. Assume Z is real-valued, i.e. $Z(t) = \overline{Z(t)}$ for all $t \in \mathbb{R}$.Writing its oscillatory representation, $Z(t) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda)$ and taking the complex conjugate gives $\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t}\, d\overline{\Phi(\lambda)}$ For a real-valued process, the orthogonal random measure Φ must satisfy $d\overline{\Phi(\lambda)} = -d\,\Phi(-\lambda)$ which ensures that the spectral representation produces real values. Substituting this identity and using the substitution $\mu = -\lambda$ it is shown that $\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t}\, d\,\Phi(\mu)$. Since $Z(t) = \overline{Z(t)}$, comparison of the integrands (which are unique elements of $L^2(F)$) yields $A_t(\lambda) = \overline{A_t(-\lambda)}$ for F-a.e. λ. Equivalently, because the oscillatory function is given by $\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t}$ we have $\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)}$ for F-a.e. λ.

Conversely, if $A_t(-\lambda) = \overline{A_t(\lambda)}$ for F-a.e. λ, then the same substitution shows that $\overline{Z(t)} = Z(t)$ for all $t \in \mathbb{R}$ so Z is real-valued. □

Theorem 9

[Existence] *Let F be an absolutely continuous spectral measure and the gain function*

$$A_t(\lambda) \in L^2(F) \quad \forall t \in \mathbb{R} \tag{21}$$

be measurable in both time and frequency then the time-dependent spectral density is defined by

$$\begin{aligned} S_t(\lambda) &= \int_{\mathbb{R}} |A_t(\lambda)|^2\, d\,F(\lambda) < \infty \\ &= \int_{\mathbb{R}} |A_t(\lambda)|^2\, S(\lambda)\, d\,\lambda \end{aligned} \tag{22}$$

and there exists a complex orthogonal random measure Φ with spectral measure F such that for each sample path $\omega_0 \in \Omega$

$$Z(t, \omega_0) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda, \omega_0) \tag{23}$$

is well-defined in $L^2(\Omega)$ and has covariance R_Z as above.

Proof. The proof proceeds by constructing the stochastic integral using the standard extension procedure. First, the integral is defined for simple functions of the form

$$g(\lambda) = \sum_{j=1}^{n} c_j\, 1_{E_j}(\lambda) \tag{24}$$

where $\{E_j\}$ are disjoint Borel sets with $F(E_j) < \infty$ and $c_j \in \mathbb{C}$:

$$\int_{\mathbb{R}} g(\lambda)\, d\,\Phi(\lambda) = \sum_{j=1}^{n} c_j\, \Phi(E_j) \tag{25}$$

For simple functions, the isometry property holds:

$$\begin{aligned}\mathbb{E}\left[\left|\int_{\mathbb{R}} g(\lambda)\, d\,\Phi(\lambda)\right|^2\right] &= \mathbb{E}\left[\left|\sum_{j=1}^{n} c_j\, \Phi(E_j)\right|^2\right] \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} c_j \bar{c_k} \mathbb{E}[\Phi(E_j)\overline{\Phi(E_k)}] \\ &= \sum_{j=1}^{n} |c_j|^2\, F(E_j) \\ &= \int_{\mathbb{R}} |g(\lambda)|^2\, d\, F(\lambda)\end{aligned} \tag{26}$$

Since simple functions are dense in $L^2(F)$, the integral is extended by continuity for all $g \in L^2(F)$. Since the oscillatory function is defined by

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \in L^2(F) \quad \forall t \in \mathbb{R} \tag{27}$$

and $A_t \in L^2(F)$, it follows that

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda)\, d\,\Phi(\lambda) \tag{28}$$

is well-defined in $L^2(\Omega)$. The covariance is computed as:

$$\begin{aligned}R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda)\, d\,\Phi(\lambda) \int_{\mathbb{R}} \overline{\varphi_s(\mu)} d\overline{\Phi(\mu)}\right] \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\mu)} d\, \mathbb{E}[\Phi(\lambda)\overline{\Phi(\mu)}] \\ &= \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)} d\, F(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)} e^{i\lambda(t-s)}\, d\, F(\lambda)\end{aligned} \tag{29}$$ □

2 Unitarily Time-Changed Stationary Processes

2.1 Unitary time-change operator $U_\theta f$

Theorem 10

[Unitary time-change and local unitarity] *Let the time-scaling function $\theta: \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with $\dot{\theta}(t) > 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero. For f measurable, define*

$$(U_\theta f)(t) = \sqrt{\dot{\theta}(t)}\, f(\theta(t)) \tag{30}$$

Its inverse is given by

$$(U_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{31}$$

For every compact set $K \subseteq \mathbb{R}$ and $f \in L^2_{\mathrm{loc}}(\mathbb{R})$,

$$\int_K |(U_\theta f)(t)|^2 d\, t = \int_{\theta(K)} |\, f(s)|^2\, d\, s \tag{32}$$

Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\mathrm{loc}}(\mathbb{R})$.

Proof. Let $f \in L^2_{\text{loc}}(\mathbb{R})$ and let K be any compact set. The local L^2-norm of $U_\theta f$ over K is:

$$\begin{aligned}\int_K |(U_\theta f)(t)|^2 \, dt &= \int_K \left| \sqrt{\dot{\theta}(t)} \, f(\theta(t)) \right|^2 dt \\ &= \int_K \dot{\theta}(t) |f(\theta(t))|^2 \, dt\end{aligned} \tag{33}$$

Since θ is absolutely continuous and strictly increasing, applying the change of variables $s = \theta(t)$ gives

$$ds = \dot{\theta}(t) \, dt \tag{34}$$

almost everywhere. As t ranges over the compact set K, $s = \theta(t)$ ranges over $\theta(K)$, which is compact. Therefore:

$$\int_K \dot{\theta}(t) \, |f(\theta(t))|^2 dt = \int_{\theta(K)} |f(s)|^2 \, ds \tag{35}$$

To verify that U_θ^{-1} is indeed the inverse, it is seen that:

$$(U_\theta^{-1} U_\theta f)(s) = \frac{\sqrt{\dot{\theta}(\theta^{-1}(s))}}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} f(s) = f(s) \tag{36}$$

and similarly

$$(U_\theta U_\theta^{-1} g)(t) = g(t) \tag{37}$$

Therefore

$$(U_\theta U_\theta^{-1} f)(t) = (U_\theta^{-1} U_\theta f)(t) = f(t) \tag{38}$$

on $L^2_{\text{loc}}(\mathbb{R})$. □

2.1.1 Inverse Filter for Unitary Time Transformations

Theorem 11

[Inverse Filter for Unitary Time Transformations] Let $\theta : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\theta'(t) > 0$ almost everywhere. Let $Y(u)$ be a stationary process with unit variance, and define

$$Z(t) = \sqrt{\dot{\theta}(t)} \, Y(\theta(t)) \tag{39}$$

as the oscillatory process obtained by the unitary time transformation. Then:

1. *The forward filter kernel is*
$$h(t, u) = \sqrt{\dot{\theta}(t)} \, \delta(u - \theta(t)) \tag{40}$$
2. *The inverse filter kernel is*
$$g(t, s) = \frac{\delta(s - \theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \tag{41}$$
3. *The composition $(g \circ h)$ recovers the identity:*
$$Y(t) = \int_{\mathbb{R}} g(t, s) \, Z(s) \, ds = \frac{Z(\theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \tag{42}$$

Proof. The forward transformation is given by

$$Z(t) = \int_{\mathbb{R}} h(t, u) \, Y(u) \, du = \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} \, \delta(u - \theta(t)) \, Y(u) \, du = \sqrt{\dot{\theta}(t)} \, Y(\theta(t)) \tag{43}$$

For the inverse, substitute $Z(s)$ into the integral:

$$\begin{aligned}\int_{\mathbb{R}} g(t,s)\, Z(s)\, d\, s \;&= \int_{\mathbb{R}} \frac{\delta\left(s-\theta^{-1}(t)\right)}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \sqrt{\dot{\theta}(s)}\, Y(\theta(s))\, d\, s \\ &= \frac{\sqrt{\dot{\theta}(\theta^{-1}(t))}}{\sqrt{\dot{\theta}(\theta^{-1}(t))}}\, Y(\theta(\theta^{-1}(t))) \\ &= Y(t)\end{aligned} \tag{44}$$

establishing that $(g \circ h)$ is the identity operator. □

2.2 Transformation of Stationary → Oscillatory Processes via U_θ

Theorem 12

[Unitary time change yields oscillatory process] *Let X be zero-mean stationary as in Definition 6. For scaling function θ as in Theorem 10, define*

$$\begin{aligned} Z(t) \;&= (U_\theta X)(t) \\ &= \sqrt{\dot{\theta}(t)}\, X(\theta(t)) \end{aligned} \tag{45}$$

Then Z is a realization of an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \tag{46}$$

gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)} \tag{47}$$

and covariance

$$\begin{aligned} R_Z(t,s) \;&= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\left[\sqrt{\dot{\theta}(t)}\, X(\theta(t)) \overline{\sqrt{\dot{\theta}(s)}\, X(\theta(s))}\right] \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\, \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\, R_X\left(\theta(t)-\theta(s)\right) \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\, d\, F(\lambda) \end{aligned} \tag{48}$$

Proof. From the Cramér representation, write

$$X(\theta(t)) = \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\, \Phi(\lambda) \tag{49}$$

Then

$$Z(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) = \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}\, d\, \Phi(\lambda) \tag{50}$$

This is precisely the form of an oscillatory process with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}$ and gain function $A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)}$. The covariance follows by direct calculation using the orthogonality properties of Φ. □

Corollary 13

[Evolutionary spectrum of unitarily time-changed stationary process] *The evolutionary spectrum is*

$$d\, F_t(\lambda) = \dot{\theta}(t)\, d\, F(\lambda) \tag{51}$$

Proof. The evolutionary spectrum is defined by

$$d\,F_t(\lambda) = |A_t(\lambda)|^2\, d\,F(\lambda) = |\sqrt{\dot\theta(t)}\, e^{i\lambda(\theta(t)-t)}|^2\, d\,F(\lambda) = \dot\theta(t)\, d\,F(\lambda) \tag{52}$$ □

2.3 Covariance operator conjugation

Proposition 14

[Operator conjugation] *Let*

$$(T_K f)(t) := \int_{\mathbb{R}} K(|t-s|)\, f(s)\, d\,s \tag{53}$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h}\, d\,F(\lambda) \tag{54}$$

Define the transformed kernel

$$K_\theta(s,t) := \sqrt{\dot\theta(t)\,\dot\theta(s)}\, K(|\theta(t)-\theta(s)|) \tag{55}$$

then the corresponding integral covariance operator is conjugated for all $f \in L^2_{\mathrm{loc}}(\mathbb{R})$ by

$$(T_{K_\theta} f)(t) = (U_\theta\, T_K\, U_\theta^{-1}\, f)(t) \tag{56}$$

Proof. Direct calculation shows

$$\begin{aligned}(U_\theta\, T_K\, U_\theta^{-1}\, f)(t) &= \sqrt{\dot\theta(t)}\, (T_K\, U_\theta^{-1}\, f)(\theta(t)) \\ &= \sqrt{\dot\theta(t)} \int_{\mathbb{R}} K(|\theta(t)-s|) \frac{f(\theta^{-1}(s))}{\sqrt{\dot\theta(\theta^{-1}(s))}}\, d\,s\end{aligned} \tag{57}$$

Substituting $s = \theta(u)$, $d\,s = \dot\theta(u)\, d\,u$:

$$= \sqrt{\dot\theta(t)} \int_{\mathbb{R}} K(|\theta(t)-\theta(u)|) \frac{f(u)}{\sqrt{\dot\theta(u)}}\, \dot\theta(u)\, d\,u = \int_{\mathbb{R}} K_\theta(t,u)\, f(u)\, d\,u \tag{58}$$ □

3 Zero Localization

Definition 15

[Zero localization measure] *Let Z be real-valued with $Z \in C^1(\mathbb{R})$ having only simple zeros*

$$Z(t_0) = 0 \Rightarrow \dot Z(t_0) \neq 0 \tag{59}$$

Define, for Borel $B \subset \mathbb{R}$,

$$\mu(B) = \int_{\mathbb{R}} 1_B(t)\, \delta(Z(t)) |\dot Z(t)|\, d\,t \tag{60}$$

Theorem 16

[Atomicity and local finiteness of zeros and delta decomposition] *Under the assumptions above, zeros are locally finite and one has*

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0)=0} \frac{\delta\,(t-t_0)}{|\dot Z(t_0)|} \tag{61}$$

whence

$$\mu = \sum_{t_0 : Z(t_0)=0} \delta_{t_0} \tag{62}$$

Proof. For any smooth test function ϕ with compact support, the standard change of variables formula for the delta function gives

$$\int_{\mathbb{R}} \phi(t)\,\delta(Z(t))\,dt = \sum_{t_0:\,Z(t_0)=0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} \tag{63}$$

This establishes

$$\delta(Z(t)) = \sum_{t_0:\,Z(t_0)=0} \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} \tag{64}$$

Substituting into the definition of μ:

$$\mu(B) = \int_{\mathbb{R}} 1_B(t) \sum_{t_0:\,Z(t_0)=0} \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} |\dot{Z}(t)|\,dt = \sum_{t_0 \in B:\,Z(t_0)=0} 1 \tag{65}$$

Therefore $\mu = \sum_{t_0:\,Z(t_0)=0} \delta_{t_0}$. □

Definition 17

[Hilbert space on the zero set] *Let $\mathcal{H} = L^2(\mu)$ with inner product*

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\,d\mu(t) \tag{66}$$

Proposition 18

[Atomic structure] *Let*

$$\mu = \sum_{t_0:\,Z(t_0)=0} \delta_{t_0} \tag{67}$$

then

$$\mathcal{H} \cong \left\{ f : \{t_0 : Z(t_0) = 0\} \to \mathbb{C} : \sum_{t_0:\,Z(t_0)=0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \tag{68}$$

with orthonormal basis $\{e_{t_0}\}_{t_0:\,Z(t_0)=0}$ where

$$e_{t_0}(t_1) = \delta_{t_0,t_1} \tag{69}$$

Proof. Since μ is purely atomic with unit mass at each zero crossing, functions in $L^2(\mu)$ are determined by their values at the zero set, and the L^2 norm is precisely

$$\|f\|^2 = \int |f(t)|^2\,d\mu(t) = \sum_{t_0:\,Z(t_0)=0} |f(t_0)|^2 \tag{70}$$

This is the standard ℓ^2 norm on the zero set. □

Definition 19

[Multiplication operator] *Define the linear operator*

$$L : \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H} \tag{71}$$

by

$$(L\,f)(t) = t\,f(t) \tag{72}$$

on the support of μ with domain

$$\mathcal{D}(L) := \left\{ f \in \mathcal{H} : \int |t\,f(t)|^2\,d\mu(t) < \infty \right\} \tag{73}$$

Theorem 20

[Self-adjointness and spectrum] *L is self-adjoint on $\mathcal{H}$ and has pure point, simple spectrum*

$$\sigma(L) = \overline{\{t \in \mathbb{R} : Z(t) = 0\}} \tag{74}$$

with eigenvalues $\lambda = t_0$ for each zero t_0 and corresponding eigenvectors e_{t_0}.

Proof. For self-adjointness, compute for $f, g \in \mathcal{D}(L)$:

$$\langle L\, f, g\rangle = \int t\, f(t)\overline{g(t)} d\,\mu(t) = \int f(t)\overline{t\, g(t)} d\,\mu(t) = \langle f, L\, g\rangle \tag{75}$$

For the spectrum, note that

$$L\, e_{t_0} = t_0\, e_{t_0} \tag{76}$$

so each t_0 is an eigenvalue with eigenvector e_{t_0}. Since the $\{e_{t_0}\}$ form a complete orthonormal basis, the spectrum is pure point and consists precisely of the zero set (or its closure if the zero set accumulates). □

3.1 The Kac-Rice Formula For The Expected Zero Counting Function

Theorem 21

[Kac-Rice Formula for Zero Crossings]Let $Z(t)$ be a centered Gaussian process on $[a, b]$ with covariance $K(s, t) = \mathbb{E}[Z(s)\, Z(t)]$ then the expected number of zeros in $[a, b]$ is

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{\frac{2}{\pi}} \frac{\sqrt{K(t,t)\, K_{\dot{Z}}(t,t) - K_{Z,\dot{Z}}(t,t)^2}}{K(t,t)} d\, t \tag{77}$$

where

$$K(t,t) = \mathbb{E}[Z(t)^2] \tag{78}$$

$$K_{\dot{Z}}(t,t) = -\partial_s^2\, \partial_t\, K(s,t)|_{s=t} \tag{79}$$

and

$$K_{Z,\dot{Z}}(t,t) = \partial_s\, K(s,t)|_{s=t} \tag{80}$$

Proof. The Kac-Rice formula is derived by considering the joint distribution of $(Z(t), \dot{Z}(t))$ at each point t. For a Gaussian process, this joint distribution is bivariate normal with covariances as specified. The expected number of zero crossings is obtained by integrating the joint density along the line $Z(t) = 0$ weighted by $|\dot{Z}(t)|$. The explicit calculation yields the stated formula. Details follow standard treatments in Cramér and Leadbetter (1967) or Adler and Taylor (2007). □