Hamburger-Siegel Theorem

The **Hamburger-Siegel theorem** (more commonly known as **Hamburger's theorem** for the moment problem) characterizes when a sequence of real numbers is a moment sequence of a positive measure on the real line.[1] [2] [3] [4]

Statement of the Theorem

Given a sequence $(s_n)_{n\geq 0}$ of real numbers, the **Hamburger moment problem** asks whether there exists a positive Borel measure μ on $\mathbb{R}$ such that

$$s_n = \int_{-\infty}^{\infty} x^n \, d\mu(x), \quad n = 0, 1, 2, \ldots$$

Hamburger's Theorem (Existence): The sequence $(s_n)_{n\geq 0}$ is a moment sequence if and only if the **Hankel matrix**

$$H = \begin{pmatrix} s_0 & s_1 & s_2 & \cdots \\ s_1 & s_2 & s_3 & \cdots \\ s_2 & s_3 & s_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is **positive semidefinite**. That is, for every finite sequence $(c_j)_{j\geq 0}$ of complex numbers with only finitely many nonzero terms,[3] [4] [1]

$$\sum_{j,k\geq 0} s_{j+k} c_j \overline{c_k} \geq 0.$$

Equivalently, all principal minors (finite Hankel determinants) must be nonnegative:

$$\det \begin{pmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-2} \end{pmatrix} \geq 0$$

for all $n \geq 1$.[1]

Uniqueness (Determinacy)

Hamburger also addressed the uniqueness question. The moment problem is **determinate** (the measure μ is unique) if and only if

$$\lim_{n\to\infty} \frac{\det(H_n)}{\det(H_n')} = 0,$$

where H_n and H_n' are certain finite sections of the Hankel matrix. A more practical sufficient condition for determinacy is **Carleman's criterion**: if [1]

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}} = \infty,$$

then the moment problem is determinate. [2] [1]

Hamburger's Converse Theorem for the Riemann Zeta Function

A different but related result, also due to Hamburger, characterizes the Riemann zeta function via its functional equation. [5] [6] [7] [8]

Hamburger's Converse Theorem (1921): Let $h(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ be absolutely convergent for $\Re(s) > 1$, and suppose both $(s-1)h(s)$ and $(s-1)g(s)$ are entire functions of finite order. If the functional equation [6] [5]

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)h(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)g(1-s)$$

holds, then $h(s) = g(s) = a_1\zeta(s)$.

This says the Riemann zeta function is **uniquely determined** by its functional equation (up to a constant factor), provided suitable regularity conditions hold. [5] [6]

Detailed Proof of Hamburger's Theorem

The proof uses orthogonal polynomials, functional analysis, and the theory of linear functionals on polynomial algebras.

Step 1: Reformulation via Linear Functionals

For a sequence $s = (s_n)_{n\geq 0}$, define a linear functional L_s on the polynomial algebra $\mathbb{R}[x]$ by

$$L_s(x^n) = s_n, \quad n = 0, 1, 2, \ldots$$

By linearity of integration, (s_n) is a moment sequence for measure μ if and only if

$$L_s(p) = \int_{-\infty}^{\infty} p(x)\, d\mu(x)$$

for all polynomials $p \in \mathbb{R}[x]$. [4]

Step 2: Positive Definiteness

The measure μ is positive if and only if

$$L_s(p^2) = \int_{-\infty}^{\infty} p(x)^2 \, d\mu(x) \geq 0$$

for all $p \in \mathbb{R}[x]$. Writing $p(x) = \sum_{j=0}^{m} c_j x^j$, we have [4]

$$L_s(p^2) = L_s\left(\left(\sum_j c_j x^j\right)^2\right) = \sum_{j,k} c_j c_k s_{j+k}.$$

Thus, μ is positive if and only if the associated Hankel form is positive semidefinite. [3] [4]

Step 3: Construction via Orthogonal Polynomials

Given a positive definite sequence (s_n), construct orthonormal polynomials (P_n) via Gram-Schmidt orthogonalization with respect to the inner product

$$\langle p, q \rangle = L_s(pq).$$

These polynomials satisfy a three-term recurrence relation

$$xP_n(x) = b_n P_{n+1}(x) + a_n P_n(x) + b_{n-1} P_{n-1}(x),$$

with $b_n > 0$. [2] [1]

By **Favard's theorem**, such a recurrence relation determines a positive measure μ for which (P_n) are orthonormal. [2] [1]

Step 4: Spectral Theorem

The multiplication operator $Mf(x) = xf(x)$ on $L^2(\mu)$ is self-adjoint. By the spectral theorem, μ is the spectral measure of this operator, uniquely determined by the moments. [4] [1]

Step 5: Uniqueness (Determinacy)

Hamburger proved that the measure μ is unique if and only if the polynomials are dense in $L^2(\mu)$, or equivalently,

$$\sum_{n=0}^{\infty} \left(P_n^2(0) + Q_n^2(0)\right) < \infty,$$

where Q_n are the polynomials of the second kind. Carleman's sufficient condition follows from analyzing growth rates of moments. [1] [2]

Conclusion

Hamburger's theorem provides a complete solution to the existence and uniqueness questions for the moment problem on $\mathbb{R}$. The positive definiteness criterion is both necessary and sufficient for existence, and various criteria (Carleman, Krein) determine uniqueness.[3] [2] [4] [1]

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