Lagrange Inversion Formula for the Riemann-Siegel Theta Function

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Introduction

The Riemann-Siegel theta function admits a convergent Taylor series expansion in a neighborhood of the origin. The inversion of this series via the Lagrange-Bürmann formula yields the compositional inverse. This exposition develops the general theory and applies it to the Riemann-Siegel theta function.

1 General Lagrange-Bürmann Inversion Formula

Theorem 1. *[Lagrange-Bürmann Inversion] Let $\phi: U \to \mathbb{C}$ be an analytic function on the disk $U = \{t \in \mathbb{C}: |t| < R\}$ with $\phi(0) = 1$ and Taylor series $\phi(t) = \sum_{j=0}^{\infty} \phi_j t^j$. Define $w(t) = \frac{t}{\phi(t)}$. The compositional inverse $t(w)$ satisfying $w(t(w)) = w$ has the representation*

$$t(w) = \sum_{n=1}^{\infty} c_n w^n$$

where

$$c_n = \frac{1}{n\,(n-1)!} \frac{d^{n-1}}{d\,t^{n-1}} (\phi(t))^n|_{t=0}$$

for each positive integer n.

Proof. Consider the functional equation $w = \frac{t}{\phi(t)}$. The Cauchy residue formula applied to

$$t(w) = \frac{1}{2\,\pi\,i} \oint_C \frac{t\,\phi'(t)}{w\,\phi(t) - t}\,d\,t$$

where C is a positively oriented circle of radius ϵ centered at the origin with $0 < \epsilon < R$ and $|w|$ small enough that $|w\,\phi(t)| < |t|$ on C.

Expanding the denominator,

$$\frac{1}{w\,\phi(t) - t} = -\frac{1}{t}\sum_{n=0}^{\infty}\left(\frac{w\,\phi(t)}{t}\right)^n = -\sum_{n=0}^{\infty}\frac{w^n(\phi(t))^n}{t^{n+1}}$$

gives

$$t(w) = -\frac{1}{2\,\pi\,i}\sum_{n=0}^{\infty} w^n \oint_C \frac{\phi'(t)(\phi(t))^n}{t^{n+1}}\,d\,t$$

The residue at $t = 0$ of $\frac{\phi'(t)(\phi(t))^n}{t^{n+1}}$ equals $\frac{1}{n!}\frac{d^n}{d\,t^n}(\phi(t))^{n+1}|_{t=0}$. By the residue theorem,

$$\oint_C \frac{\phi'(t)(\phi(t))^n}{t^{n+1}}\,d\,t = 2\,\pi\,i\cdot\frac{1}{n!}\frac{d^n}{d\,t^n}(\phi(t))^{n+1}|_{t=0}$$

Substituting yields

$$t(w) = \sum_{n=0}^{\infty}\frac{w^n}{n!}\frac{d^n}{d\,t^n}(\phi(t))^{n+1}|_{t=0}$$

Reindexing with $m = n + 1$ gives $n = m - 1$, so

$$t(w) = \sum_{m=1}^{\infty}\frac{w^{m-1}}{(m-1)!}\frac{d^{m-1}}{d\,t^{m-1}}(\phi(t))^m|_{t=0} = \sum_{m=1}^{\infty}\frac{1}{m\,(m-1)!}\frac{d^{m-1}}{d\,t^{m-1}}(\phi(t))^m|_{t=0}w^m$$

Define $c_m := \frac{1}{m\,(m-1)!}\frac{d^{m-1}}{d\,t^{m-1}}(\phi(t))^m|_{t=0}$ for each positive integer m. □

2 Application to the Riemann-Siegel Theta Function

2.1 Definition and Properties

The Riemann-Siegel theta function is defined for real t by

$$\theta(t) = \frac{1}{2}\arg\Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right) - \frac{t}{2}\log\pi$$

where the argument is continuous with $\arg\Gamma(1/4) = 0$. The function $\theta(t)$ is real-valued, odd, and satisfies $\theta(0) = 0$.

The Taylor series expansion is

$$\theta(t) = \sum_{k=0}^{\infty} a_k\, t^k$$

where

$$a_k = \frac{1}{k!}\frac{d^k}{d\,t^k}\theta(t)|_{t=0}$$

for each nonnegative integer k. Since $\theta(t)$ is odd, $a_k = 0$ whenever k is even. For odd k, the coefficient a_k is nonzero in general, with $a_1 = -\frac{1}{2}\log\pi \neq 0$.

2.2 Construction of the Auxiliary Function

Define the auxiliary function $\phi: \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) = \frac{t}{\theta(t)}$$

with $\phi(0) := a_1^{-1}$. Since $\theta(t) = t\sum_{k=0}^{\infty} a_{2k+1}\, t^{2k}$, the function $\phi(t)$ is given by

$$\phi(t) = \frac{1}{\sum_{k=0}^{\infty} a_{2k+1}\, t^{2k}} = \sum_{j=0}^{\infty} \phi_j\, t^j$$

where

$$\phi_j = \frac{1}{j!}\frac{d^j}{d\,t^j}\phi(t)|_{t=0}$$

for each nonnegative integer j.

Since the denominator $\sum_{k=0}^{\infty} a_{2k+1}\, t^{2k}$ contains only even powers of t, the reciprocal $\phi(t)$ contains only even powers. Define the subsequence

$$\psi_i = \phi_{2i}$$

for each nonnegative integer i, so $\phi(t) = \sum_{i=0}^{\infty} \psi_i\, t^{2i}$ with $\psi_0 = a_1^{-1}$

2.3 Inversion via Lagrange-Bürmann Formula

Apply Theorem 1 with $\phi(t) = \frac{t}{\theta(t)}$ and $w = \theta(t)$. The compositional inverse satisfies

$$t(w) = \sum_{n=1}^{\infty} c_n\, w^n$$

where

$$c_n = \frac{1}{n\,(n-1)!}\,\frac{d^{n-1}}{d\,t^{n-1}}(\phi(t))^n|_{t=0}$$

Since $\phi(t) = \sum_{i=0}^{\infty} \psi_i\, t^{2i}$ contains only even powers, the power $(\phi(t))^n$ contains only even powers of t. When n is even, the function $(\phi(t))^n$ is even, so all odd derivatives vanish at $t = 0$. For even n, write $n = 2\,m$, so $n - 1 = 2\,m - 1$ is odd, giving $c_{2m} = 0$.

When n is odd, write $n = 2\,m - 1$ for a positive integer m. Then $n - 1 = 2\,m - 2$ is even, and the derivative is nonzero. Define

$$b_m = c_{2m-1} = \frac{1}{(2\,m-1)\,(2\,m-2)!}\,\frac{d^{2m-2}}{d\,t^{2m-2}}(\phi(t))^{2m-1}|_{t=0}$$

for each positive integer m.

2.4 Explicit Coefficient Formula

The derivative $\frac{d^{2m-2}}{d\,t^{2m-2}}(\phi(t))^{2m-1}|_{t=0}$ equals $(2\,m-2)!$ times the coefficient of t^{2m-2} in the Taylor expansion of $(\phi(t))^{2m-1}$.

Write $(\phi(t))^{2m-1} = (\sum_{i=0}^{\infty} \psi_i\, t^{2i})^{2m-1}$. Define $\mathcal{K}(m)$ as the set of all sequences $\kappa = (\kappa_0, \kappa_1, \kappa_2, \ldots)$ of nonnegative integers with finite support satisfying

$$\sum_{i=0}^{\infty} \kappa_i = 2\,m - 1 \quad \text{and} \quad \sum_{i=0}^{\infty} 2\,i\,\kappa_i = 2\,m - 2$$

For each sequence $\kappa \in \mathcal{K}(m)$, the multinomial coefficient is

$$\frac{(2\,m-1)!}{\prod_{i=0}^{\infty} \kappa_i!}.$$

The derivative is

$$\frac{d^{2m-2}}{d\,t^{2m-2}}(\phi(t))^{2m-1}|_{t=0} = (2\,m-2)! \sum_{\kappa \in \mathcal{K}(m)} \frac{(2\,m-1)!}{\prod_{i=0}^{\infty} \kappa_i!} \prod_{i=0}^{\infty} \psi_i^{\kappa_i}$$

Therefore

$$b_m = \frac{1}{2\,m-1} \sum_{\kappa \in \mathcal{K}(m)} \frac{(2\,m-1)!}{\prod_{i=0}^{\infty} \kappa_i!} \prod_{i=0}^{\infty} \psi_i^{\kappa_i}$$

2.5 Summary

The compositional inverse of the Riemann-Siegel theta function is

$$t(w) = \sum_{m=1}^{\infty} b_m \, w^{2m-1}$$

where each coefficient b_m is defined by

$$b_m = \frac{1}{(2\,m-1)\,(2\,m-2)!}\,\frac{d^{2m-2}}{d\,t^{2m-2}}\left(\frac{t}{\theta(t)}\right)^{2m-1}\Big|_{t=0}$$

for each positive integer m.