Rigorous Interpretation of White Noise, Spectral Measure, and Fourier Integration

Summary:
White noise, rigorously viewed as the distributional derivative of Brownian motion, is not a function but a generalized stochastic process. Its mathematical framework relies upon the theory of Gaussian generalized processes (e.g., in the sense of I.M. Gelfand) and their spectral (Fourier) representations.

1. Covariance Structure and Spectral Density

- For idealized white noise $\xi(t)$,
$$E[\xi(t)\xi(s)] = \delta(t - s)$$
In Fourier space,
$$\int_{-\infty}^{\infty} e^{-i\lambda\tau}\delta(\tau)d\tau = 1$$
(modulo conventions in 2π factors).
- **Interpretation:**
The spectral density is constant—i.e., proportional to Lebesgue measure—manifesting the process's complete lack of temporal correlation.

2. Fourier Transform of White Noise as a Process

- Given that $\xi(t)$ is not a function, the integral
$$\int_{-\infty}^{\infty} e^{i\lambda t}\xi(t)dt$$
is not a deterministic object but a **random (generalized) process** in frequency.
- The **key property** is that the spectrum captures isometry:
$$E\left[\int_{-\infty}^{\infty} e^{i\lambda t}\xi(t)dt \cdot \overline{\int_{-\infty}^{\infty} e^{i\omega s}\xi(s)ds}\right] = 2\pi\,\delta(\lambda - \omega)$$
Thus, in the frequency domain, the Fourier transform of white noise is itself white noise—the “spectral covariance” is again a delta (flat spectrum).

3. Spectral Measure and Isometry

- The general theory (e.g., see the Kolmogorov–Cramér spectral representation theorem and spectral measure theory for stationary processes) gives:
$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda t} dZ(\lambda)$$
where $E|dZ(\lambda)|^2 = d\mu(\lambda)$ encodes the spectral measure μ.
- For white noise, $d\mu(\lambda) = \frac{1}{2\pi}\, d\lambda$ (Lebesgue measure up to normalization).

4. Stochastic Fourier Integrals and Interpretation

- The expression
$$\int_{-\infty}^{\infty} e^{i\lambda t} dx(t)$$
is a **stochastic Fourier integral**, not a function or pointwise value but an object whose *second-order structure* (covariances/expectations) determines its statistical properties.
- The statement
$$\int_{-\infty}^{\infty} e^{i\lambda t} dx(t)$$
"equals" a delta function is not strictly true; rather, for suitable test functions f, the map
$$f \mapsto \int_{-\infty}^{\infty} f(t)\, \xi(t)\, dt$$
is an isometry (up to normalization) between $L^2(\mathbb{R})$, and the covariance measure is flat (Lebesgue).

5. Isometry Structure

- The isometry is given by the *Plancherel theorem* for stochastic integrals:
$$E\left|\int f(t)\, dW_t\right|^2 = \int |f(t)|^2 dt$$
and equivalently in frequency domain. For white noise, the spectral isometry reflects the invariance under the Fourier transform.

Conclusion:
White noise and its spectral theory are most precisely handled in the language of generalized stochastic processes.

- Its time correlation is a Dirac delta; its spectral measure is Lebesgue; and its stochastic Fourier transform is again white noise.
- Claims about "delta functions" in the Fourier transform are statements about *variance/covariance* structures, not equalities of random distributions.
- The "equality" is really an isometry of Hilbert spaces induced by the Fourier transform—a key point in spectral analysis of stationary and generalized processes.