Unitary Time Changes of Stationary Processes Yield Oscillatory Processes and a Functional Framework Toward a Hilbert–Pólya Construction

by Stephen Crowley

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1 Unitary Time Change on $L^2(\mathbb{R})$

Definition 1. *[Unitary time change operator on $L^2(\mathbb{R})$] Let $\theta: \mathbb{R} \to \mathbb{R}$ be absolutely continuous with $\theta'(t) \neq 0$ almost everywhere. Define $U_\theta: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by*

$$(U_\theta f)(t) := \sqrt{|\theta'(t)|}\; f(\theta(t)) \qquad (f \in L^2(\mathbb{R})). \tag{1}$$

Theorem 2. *[Unitarity of U_θ] U_θ is unitary on $L^2(\mathbb{R})$.*

Proof. By absolute continuity and $\theta'(t) \neq 0$ a.e., the change-of-variables formula gives

$$\int_{\mathbb{R}} |(U_\theta f)(t)|^2\, dt = \int_{\mathbb{R}} |\theta'(t)|\, |f(\theta(t))|^2\, dt = \int_{\mathbb{R}} |f(u)|^2\, du,$$

so U_θ is an isometry. Since θ admits an a.e. inverse θ^{-1} with the same regularity and non-vanishing derivative a.e., one has $U_{\theta^{-1}} U_\theta = \mathrm{Id}$ and $U_\theta U_{\theta^{-1}} = \mathrm{Id}$ a.e., hence U_θ is unitary. □

2 Oscillatory Processes in the Sense of Priestley

Definition 3. *[Oscillatory process, gain and oscillatory function] Let F be a finite non-negative Borel measure on $\mathbb{R}$. For each $t \in \mathbb{R}$ let $A_t: \mathbb{R} \to \mathbb{C}$ be measurable and square-integrable with respect to F. Define*

$$\varphi_t(\lambda) := A_t(\lambda)\ e^{i\lambda t} \tag{2}$$

An oscillatory process *Z is a stochastic process with spectral representation*

$$Z(t) := \int_{\mathbb{R}} \varphi_t(\lambda)\ \Phi(d\lambda) = \int_{\mathbb{R}} A_t(\lambda)\ e^{i\lambda t}\ \Phi(d\lambda) \tag{3}$$

where Φ is a complex orthogonal random measure with spectral measure F satisfying the orthogonality of infinitesimal increments

$$\mathbb{E}[\Phi(d\lambda)\,\overline{\Phi(d\mu)}] = \delta(\lambda - \mu)\ dF(\lambda) \tag{4}$$

The covariance kernel is

$$R_Z(t,s) := \mathbb{E}[Z(t)\,\overline{Z(s)}] = \int_{\mathbb{R}} A_t(\lambda)\,\overline{A_s(\lambda)}\, e^{i\lambda(t-s)}\ dF(\lambda) \tag{5}$$

Remark 4. [Real-valuedness] Z is real-valued if and only if, for each fixed t, $A_t(-\lambda) = \overline{A_t(\lambda)}$ for F-a.e. λ, equivalently $\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)}$ for F-a.e. λ.

Theorem 5. *[Existence of oscillatory processes with prescribed $(A_t)_t$] Let F be finite and $(A_t)_t$ measurable with $\int |A_t(\lambda)|^2\, dF(\lambda) < \infty$ for each t. There exists a complex orthogonal random measure Φ on $\mathbb{R}$ with spectral measure F such that $Z(t) = \int \varphi_t(\lambda)\ \Phi(d\lambda)$ is well-defined in $L^2(\Omega)$ and has covariance*

$$R_Z(t,s) = \int_{\mathbb{R}} \varphi_t(\lambda)\,\overline{\varphi_s(\lambda)}\, dF(\lambda) = \int_{\mathbb{R}} A_t(\lambda)\,\overline{A_s(\lambda)}\, e^{i\lambda(t-s)}\ dF(\lambda) \tag{6}$$

Proof. Construct the stochastic integral first for simple functions in $L^2(\mathbb{R}, F)$ and extend by isometry using

$$\mathbb{E}\left[\left|\int g(\lambda)\ \Phi(d\lambda)\right|^2\right] = \int |g(\lambda)|^2\ dF(\lambda) \tag{7}$$

Apply with $g = \varphi_t$ to obtain $Z(t)$ and the stated covariance. $\square$

3 Unitary Time Changes Map Stationary to Oscillatory

Definition 6. *[Stationary process via Cramér representation] A zero-mean stationary process X with spectral measure F admits*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, \Phi\,(d\,\lambda) \tag{8}$$

with covariance

$$R_X\,(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)}\, d\,F(\lambda) \tag{9}$$

Theorem 7. *[Unitary time change yields an oscillatory process] Let X be zero-mean stationary with*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, \Phi\,(d\,\lambda) \tag{10}$$

Let θ satisfy the hypotheses of the unitary time change and set

$$Z(t) := (U_\theta\, X)(t) = \sqrt{|\theta'(t)|}\; X(\theta(t)) \tag{11}$$

Then Z is an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = \sqrt{|\theta'(t)|}\; e^{i\lambda\theta(t)} \tag{12}$$

and gain

$$A_t(\lambda) = \sqrt{|\theta'(t)|}\; e^{i\lambda(\theta(t)-t)} \tag{13}$$

The covariance is

$$\begin{aligned} R_Z(t,s) &= \int_{\mathbb{R}} A_t(\lambda)\,\overline{A_s(\lambda)}\,e^{i\lambda(t-s)}\;d\,F(\lambda) \\ &= \int_{\mathbb{R}} \sqrt{|\theta'(t)\,\theta'(s)|}\; e^{i\lambda(\theta(t)-\theta(s))}\;d\,F(\lambda) \end{aligned} \tag{14}$$

Proof. Compute

$$
\begin{aligned}
Z(t) &= \sqrt{|\theta'(t)|}\; X(\theta(t)) \\
&= \sqrt{|\theta'(t)|} \int_{\mathbb{R}} e^{i\lambda\theta(t)}\; \Phi\,(d\,\lambda) \\
&= \int_{\mathbb{R}} \sqrt{|\theta'(t)|}\; e^{i\lambda\theta(t)}\; \Phi\,(d\,\lambda)
\end{aligned} \tag{15}
$$

Thus

$$
\varphi_t(\lambda) = \sqrt{|\theta'(t)|}\; e^{i\lambda\theta(t)} \tag{16}
$$

and

$$
A_t(\lambda) = \varphi_t(\lambda)\, e^{-i\lambda t} \tag{17}
$$

The covariance follows from orthogonality of Φ. □

Remark 8. [Real-valuedness under time change] If X is real-valued and θ is real with $\theta'(t) > 0$ a.e., then Z is real-valued by the Hermitian symmetry of A_t.

4 Zero Localization by a Functional Measure

Definition 9. *[Zero localization measure] Let Z be real-valued, with sample paths in $C^1(\mathbb{R})$, and such that every zero of Z is simple (i.e. $Z(t_0) = 0 \Longrightarrow Z'(t_0) \neq 0$). Define the measure on Borel $B \subset \mathbb{R}$ by*

$$
\mu(B) := \int_{\mathbb{R}} 1_B(t)\; \delta(Z(t))\, |Z'(t)|\; d\,t \tag{18}
$$

Theorem 10. *[Support and mass on the zero set] For any test function $\phi \in C_c^\infty(\mathbb{R})$,*

$$
\int_{\mathbb{R}} \phi(t)\; \delta(Z(t))\, |Z'(t)|\; d\,t = \sum_{t_0 : Z(t_0) = 0} \phi(t_0) \tag{19}
$$

and hence $\mu = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0}$ is a discrete measure assigning unit mass to each simple zero of Z.

Proof. At a simple zero t_0, the distributional identity holds:

$$
\delta(Z(t)) = \frac{\delta\,(t - t_0)}{|Z'(t_0)|} + \sum_{t_1 \neq t_0 : Z(t_1) = 0} \frac{\delta\,(t - t_1)}{|Z'(t_1)|} \tag{20}
$$

Multiplying by $|Z'(t)|$ and integrating against ϕ yields the stated identity and the atomic form of μ. $\square$

5 Hilbert Space on the Zero Set and Multiplication Operator

Definition. *[Hilbert space on the zero set via μ] Define*

$$\mathcal{H} := L^2(\mu) = \left\{ f: \mathbb{R} \to \mathbb{C} : \|f\|_{\mathcal{H}}^2 = \int |f(t)|^2 \, \delta(Z(t)) \, |Z'(t)| \, dt < \infty \right\} \tag{21}$$

The inner product is

$$\langle f, g \rangle = \int f(t) \overline{g(t)} \, \delta(Z(t)) \, |Z'(t)| \, dt \tag{22}$$

Proposition 11. *[Atomic structure] With $\mu = \sum_{t_0: Z(t_0) = 0} \delta_{t_0}$, one has*

$$\mathcal{H} = \left\{ f: \{t_0: Z(t_0) = 0\} \to \mathbb{C} : \sum_{Z(t_0) = 0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \tag{23}$$

and the functions e_{t_0} defined by $e_{t_0}(t_1) = \delta_{t_0 t_1}$ form an orthonormal basis.

Proof. Substitute the atomic form of μ into the L^2-definition to obtain the ℓ^2-structure; the canonical coordinate functions form an ONB. $\square$

Definition 12. *[Multiplication operator] Define $L: \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H}$ by $(L f)(t) = t \; f(t)$ on* $\sup(\mu)$*, with*

$$\mathcal{D}(L) = \left\{ f \in \mathcal{H} : \int |t \; f(t)|^2 \, \delta(Z(t)) \, |Z'(t)| \, dt < \infty \right\} \tag{24}$$

Theorem 13. *[Self-adjointness and spectrum] L is self-adjoint on $\mathcal{H}$, and its spectrum is exactly*

$$\sigma(L) = \{ t \in \mathbb{R} : Z(t) = 0 \} \tag{25}$$

with pure point spectrum consisting of simple eigenvalues $\lambda = t_0$ (for each zero t_0) and eigenvectors e_{t_0}.

Proof. For $f, g \in \mathcal{D}(L)$,

$$\langle L f, g\rangle = \int t\ f(t)\,\overline{g(t)}\,\delta(Z(t))\,|Z'(t)|\ dt = \int f(t)\,\overline{t\ g(t)}\,\delta(Z(t))\,|Z'(t)|\ dt = \langle f, L g\rangle \qquad (26)$$

so L is symmetric. On the atomic space, L is unitarily equivalent to the diagonal operator $(c_{t_0}) \mapsto (t_0 c_{t_0})$ on ℓ^2, which is self-adjoint with spectrum equal to the set of diagonal entries $\{t_0 : Z(t_0) = 0\}$, each simple, with eigenvectors the coordinate basis identified with e_{t_0}. □

6 Time-Changed Stationary Processes and a Hilbert–Pólya Scaffold

Definition 14. *[Arithmetic phase time change] Let $\theta : \mathbb{R} \to \mathbb{R}$ be an absolutely continuous phase with $\theta'(t) > 0$ a.e. encoding the target arithmetic structure (e.g. a Riemann–Siegel-type phase). Let X be zero-mean stationary with spectral measure F and orthogonal random measure Φ. Define the time-changed oscillatory process*

$$Z(t) = \int_{\mathbb{R}} \sqrt{|\theta'(t)|}\ e^{i\lambda\theta(t)}\ \Phi(d\lambda) \qquad (27)$$

Proposition 15. *[Covariance under time change]*

$$R_Z(t, s) = \int_{\mathbb{R}} \sqrt{|\theta'(t)\,\theta'(s)|}\ e^{i\lambda(\theta(t) - \theta(s))}\ dF(\lambda) \qquad (28)$$

In particular, if F is chosen so that R_Z concentrates along $\theta(t) = \theta(s)$, then the correlation structure of Z is phase-aligned with θ.

Proof. Insert the oscillatory function into the covariance integral and use the orthogonality relation for Φ. □

Definition 16. *[Zero-localized Hilbert space and operator] With the zero localization measure $\mu(dt) = \delta(Z(t))\,|Z'(t)|\ dt$, define $\mathcal{H} = L^2(\mu)$ and L as multiplication by t on $\mathcal{H}$.*

Theorem 17. *[Spectral encoding of zero set] The spectrum of L is the zero set of Z:*

$$\sigma(L) = \{t : Z(t) = 0\} \qquad (29)$$

and L has simple pure point spectrum with eigenvectors supported at individual zeros.

Proof. Follows from the established atomic structure of μ and the diagonal form of L on $L^2(\mu)$. $\square$

Remark 18. [Operator scaffold] The sequence

$$\text{stationary } X \xrightarrow{U_\theta} \text{oscillatory } Z \xrightarrow{\delta(Z)\,|Z'|\,dt} \mu \xrightarrow{L^2(\mu)} \mathcal{H} \xrightarrow{t\cdot} L \tag{30}$$

produces a concrete self-adjoint operator whose spectrum equals the (constructed) zero set governed by the choice of θ and F. Aligning θ and F to a prescribed arithmetic target sets the stage for a Hilbert–Pólya-type identification.

7 Appendix: Regularity and Simple Zeros

Definition 19. *[Regularity and simplicity] Assume $Z \in C^1(\mathbb{R})$ and every zero of Z is simple: $Z(t_0) = 0 \Longrightarrow Z'(t_0) \neq 0$.*

Lemma 20. *[Local finiteness and decomposition] Under the above condition, zeros are locally finite and the distributional identity*

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0) = 0} \frac{\delta\,(t - t_0)}{|Z'(t_0)|} \tag{31}$$

holds, yielding

$$\mu = \sum_{t_0} \delta_{t_0} \tag{32}$$

Proof. Continuity and $Z'(t_0) \neq 0$ imply isolated zeros by the inverse function theorem; the distributional identity is standard from the one-dimensional change-of-variables formula for the Dirac delta under monotone C^1 maps near each zero. $\square$