

# Unitary Time Changes of Stationary Processes Yield Oscillatory Processes

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## Abstract

A unitary time-change operator  $U_\theta$  is constructed for absolutely continuous, strictly increasing time reparametrizations  $\theta$ , acting on functions that are locally square-integrable (meaning over compact sets). Applying  $U_\theta$  to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function  $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)}$ , evolutionary spectrum  $dF_t(\lambda) = \dot{\theta}(t) dF(\lambda)$  and expected zero-counting function  $\mathbb{E}[N_{[0,T]}] = \sqrt{-\ddot{K}(0)} [\theta(T) - \theta(0)]$ . The sample paths of any non-degenerate second-order stationary process are locally square integrable, making the unitary time-change operator  $U_\theta$  applicable to typical realizations. A zero-localization measure  $d\mu(t) = \delta(Z(t))|\dot{Z}(t)| dt$  induces a Hilbert space  $L^2(\mu)$  on the zero set of each oscillatory process realization  $Z(t)$ , and the multiplication operator  $(Lf)(t) = t f(t)$  has simple pure point spectrum equal to the zero crossing set of  $Z$ . D

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# 1 Gaussian Processes

Unless otherwise stated, all processes considered will be real-valued.

**Theorem 1.** *Let  $X(u)$  be a real-valued process:*

$$X(u) \in \mathbb{R} \quad \forall u \in \mathbb{R} \quad (1)$$

*Then its (complex-valued) random orthogonal spectral measure satisfies*

$$d\bar{\Phi}(\lambda) = d\Phi(-\lambda) \quad (2)$$

*and the corresponding covariance spectral measure  $F$  is even:*

$$F(-A) = F(A) \quad (3)$$

**Proof.** 1. The spectral representation for  $X(u)$  is

$$X(u) = \int_{-\infty}^{\infty} e^{i\lambda u} d\Phi(\lambda) \quad (4)$$

Since  $X(u)$  is real-valued for each  $u$ ,

$$\overline{X(u)} = X(u) \quad (5)$$

On the other hand,

$$\overline{X(u)} = \overline{\int_{-\infty}^{\infty} e^{i\lambda u} d\Phi(\lambda)} \quad (6)$$

$$= \int_{-\infty}^{\infty} \overline{e^{i\lambda u}} d\bar{\Phi}(\lambda) \quad (7)$$

$$= \int_{-\infty}^{\infty} e^{-i\lambda u} d\bar{\Phi}(\lambda) \quad (8)$$

By the substitution  $\lambda \mapsto -\mu$ ,

$$\int_{-\infty}^{\infty} e^{-i\lambda u} d\bar{\Phi}(\lambda) = \int_{-\infty}^{\infty} e^{i\mu u} d\bar{\Phi}(-\mu) \quad (9)$$

So

$$X(u) = \int_{-\infty}^{\infty} e^{i\lambda u} d\Phi(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda u} d\bar{\Phi}(-\lambda) \quad (10)$$

By uniqueness of the spectral measure representation, it follows that

$$d\bar{\Phi}(\lambda) = d\Phi(-\lambda) \quad (11)$$

as (orthogonal) random measures.

2. The covariance function of  $X$  is

$$R(u) = \mathbb{E}(X(0)X(u)) = \int_{-\infty}^{\infty} e^{i\lambda u} dF(\lambda) \quad (12)$$

Since  $X(u)$  is real-valued,  $R(u)$  is real and  $R(-u) = R(u)$ . Thus,

$$R(-u) = \int_{-\infty}^{\infty} e^{-i\lambda u} dF(\lambda) = \int_{-\infty}^{\infty} e^{i\mu u} dF(-\mu) \quad (13)$$

Equating with  $R(u)$ ,

$$\int_{-\infty}^{\infty} e^{i\lambda u} dF(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda u} dF(-\lambda) \quad (14)$$

for all  $u$ . By the uniqueness theorem for Fourier–Stieltjes transforms, this implies

$$dF(\lambda) = dF(-\lambda) \quad (15)$$

Thus for any Borel set  $A$ ,

$$F(-A) = F(A) \quad (16)$$

establishing the evenness property.  $\square$

## 1.1 Definition

**Definition 2. (Gaussian process)** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $T$  a non-empty index set. A family  $\{X_t : t \in T\}$  of real-valued random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$  is called a Gaussian process if for every finite subset  $\{t_1, \dots, t_n\} \subset T$  the random vector  $(X_{t_1}, \dots, X_{t_n})$  is multivariate normal (possibly degenerate). Equivalently, every finite linear combination  $\sum_{i=1}^n a_i X_{t_i}$  is either almost surely constant or Gaussian. The mean function is  $m(t) := \mathbb{E}[X_t]$  and the covariance kernel is

$$K(s, t) = \text{Cov}(X_s, X_t) \quad (17)$$

For any finite  $(t_i)_{i=1}^n \subset T$ , the matrix  $K_{ij} = K(t_i, t_j)$  is symmetric positive semidefinite, and a Gaussian process is completely determined in law by  $m$  and  $K$ .

## 1.2 Stationary processes

TODO: restate this as Bochners theorem with proof

**Definition 3. [Cramér spectral representation][1]** A zero-mean stationary process  $X$  with spectral measure  $F$  admits the sample path representation

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t} d\Phi(\lambda) \quad (18)$$

which has covariance

$$R_X(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)} dF(\lambda) \quad (19)$$

### 1.2.1 Sample Path Realizations

**Definition 4.** [*Locally square-integrable functions*] Define

$$L^2_{\text{loc}}(\mathbb{R}) := \left\{ f: \mathbb{R} \rightarrow \mathbb{C}: \int_K |f(t)|^2 dt < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \quad (20)$$

**Remark 5.** Every bounded measurable set in  $\mathbb{R}$  is compact or contained in a compact set; hence  $L^2_{\text{loc}}(\mathbb{R})$  contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

**Theorem 6.** [*Sample paths in  $L^2_{\text{loc}}(\mathbb{R})$* ] Let  $\{X(t)\}_{t \in \mathbb{R}}$  be a second-order stationary process with

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \quad (21)$$

Then almost every sample path lies in  $L^2_{\text{loc}}(\mathbb{R})$ . However, for non-degenerate processes with  $\sigma^2 > 0$ , sample paths are not globally square-integrable.

**Proof.** Fix an arbitrary bounded interval  $[a, b] \subset \mathbb{R}$  with  $a < b$ . Define the random variable

$$Y_{[a,b]} := \int_a^b X(t)^2 dt \quad (22)$$

1. By Tonelli's theorem, since  $X(t)^2 \geq 0$ ,

$$\mathbb{E}[Y_{[a,b]}] = \mathbb{E} \left[ \int_a^b X(t)^2 dt \right] = \int_a^b \mathbb{E}[X(t)^2] dt \quad (23)$$

2. By stationarity of  $X$ ,  $\mathbb{E}[X(t)^2] = \sigma^2$  for all  $t \in \mathbb{R}$ . Therefore

$$\mathbb{E}[Y_{[a,b]}] = \int_a^b \sigma^2 dt = \sigma^2 (b - a) \quad (24)$$

3. Since  $b - a < \infty$  and  $\sigma^2 < \infty$  by assumption (21),

$$\mathbb{E}[Y_{[a,b]}] < \infty \quad (25)$$

4. By Markov's inequality, for any  $M > 0$ ,

$$\mathbb{P}(Y_{[a,b]} > M) \leq \frac{\mathbb{E}[Y_{[a,b]}]}{M} = \frac{\sigma^2 (b - a)}{M} \quad (26)$$

5. Taking  $M \rightarrow \infty$  in (26),

$$\mathbb{P}(Y_{[a,b]} < \infty) = 1 \quad (27)$$

6. Now let  $K \subset \mathbb{R}$  be an arbitrary compact set. Since  $K$  is compact in  $\mathbb{R}$ , it is closed and bounded. Therefore there exists  $N > 0$  such that

$$K \subseteq [-N, N] \quad (28)$$

7. By (27) applied to  $[a, b] = [-N, N]$ ,

$$\mathbb{P} \left( \int_{-N}^N X(t)^2 dt < \infty \right) = 1 \quad (29)$$

8. Since  $K \subseteq [-N, N]$  by (28),

$$\int_K X(t)^2 dt \leq \int_{-N}^N X(t)^2 dt \quad (30)$$

9. Combining (29) and (30),

$$\mathbb{P} \left( \int_K X(t)^2 dt < \infty \right) = 1 \quad (31)$$

10. Since  $K$  was arbitrary, (31) holds for every compact set  $K \subset \mathbb{R}$ . Therefore, almost every sample path  $t \mapsto X(t, \omega)$  satisfies

$$\int_K |X(t, \omega)|^2 dt < \infty \quad \forall \text{ compact } K \subset \mathbb{R} \quad (32)$$

which means almost every sample path lies in  $L^2_{\text{loc}}(\mathbb{R})$ .

11. For the global divergence statement, assume  $\sigma^2 > 0$ . For each  $n \in \mathbb{N}$ , by (24),

$$\mathbb{E} \left[ \int_{-n}^n X(t)^2 dt \right] = 2n\sigma^2 \quad (33)$$

12. As  $n \rightarrow \infty$ , the right side of (33) diverges:

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_{-n}^n X(t)^2 dt \right] = \lim_{n \rightarrow \infty} 2n\sigma^2 = \infty \quad (34)$$

13. By monotone convergence theorem, since  $\int_{-n}^n X(t)^2 dt$  increases monotonically with  $n$ ,

$$\mathbb{E} \left[ \lim_{n \rightarrow \infty} \int_{-n}^n X(t)^2 dt \right] = \lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_{-n}^n X(t)^2 dt \right] = \infty \quad (35)$$

14. Therefore

$$\mathbb{P} \left( \int_{-\infty}^{\infty} X(t)^2 dt = \infty \right) = 1 \quad (36)$$

15. Thus sample paths are not in  $L^2(\mathbb{R})$ , only in  $L^2_{\text{loc}}(\mathbb{R})$ .  $\square$

### 1.3 Oscillatory Processes

**Definition 7. [Oscillatory process][3]** Let  $F$  be a finite nonnegative Borel measure on  $\mathbb{R}$ . Let

$$A_t \in L^2(F) \quad \forall t \in \mathbb{R} \quad (37)$$

be the gain function and

$$\varphi_t(\lambda) = A_t(\lambda) e^{i\lambda t} \quad (38)$$

be the corresponding oscillatory function then an oscillatory process is a stochastic process which can be represented as

$$\begin{aligned} Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \end{aligned} \quad (39)$$

where  $\Phi$  is a complex orthogonal random measure with spectral measure  $F$  which satisfies the relation

$$d\mathbb{E}[\Phi(\lambda)\overline{\Phi(\mu)}] = \delta(\lambda - \mu) dF(\lambda) \quad (40)$$

and has the corresponding covariance kernel

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)} e^{i\lambda(t-s)} dF(\lambda) \\ &= \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)} dF(\lambda) \end{aligned} \quad (41)$$

**Theorem 8. [Real-valuedness criterion for oscillatory processes]** Let  $Z$  be an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = A_t(\lambda) e^{i\lambda t} \quad (42)$$

and spectral measure  $F$ . Then  $Z$  is real-valued if and only if

$$A_t(-\lambda) = \overline{A_t(\lambda)} \quad (43)$$

for  $F$ -almost every  $\lambda \in \mathbb{R}$ , equivalently

$$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \quad (44)$$

for  $F$ -almost every  $\lambda \in \mathbb{R}$ .

**Proof.** 1. Assume  $Z$  is real-valued. Then for all  $t \in \mathbb{R}$ ,

$$Z(t) = \overline{Z(t)} \quad (45)$$

2. From the oscillatory representation (39),

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \quad (46)$$

3. Taking the complex conjugate of both sides of (46),

$$\overline{Z(t)} = \overline{\int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t} d\overline{\Phi(\lambda)} \quad (47)$$

4. For a real-valued process, the orthogonal random measure must satisfy the symmetry property

$$d\overline{\Phi(\lambda)} = -d\Phi(-\lambda) \quad (48)$$

5. Substituting (48) into (47),

$$\overline{Z(t)} = - \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t} d\Phi(-\lambda) \quad (49)$$

6. Apply the change of variables  $\mu = -\lambda$ , so  $d\Phi(-\lambda) = -d\Phi(\mu)$  and  $e^{-i\lambda t} = e^{i\mu t}$ :

$$\begin{aligned} \overline{Z(t)} &= - \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t} (-d\Phi(\mu)) \\ &= \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t} d\Phi(\mu) \end{aligned} \quad (50)$$

7. By (45), the right sides of (46) and (50) must be equal:

$$\int_{\mathbb{R}} A_t(\mu) e^{i\mu t} d\Phi(\mu) = \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t} d\Phi(\mu) \quad (51)$$

8. Since the stochastic integral representation is unique in  $L^2(F)$ , the integrands must be equal  $F$ -almost everywhere:

$$A_t(\lambda) = \overline{A_t(-\lambda)} \quad \text{for } F\text{-a.e. } \lambda \quad (52)$$

9. This is equivalent to (43). From (42),

$$\varphi_t(-\lambda) = A_t(-\lambda) e^{-i\lambda t} \quad (53)$$

10. Using (43),

$$\begin{aligned} \varphi_t(-\lambda) &= \overline{A_t(\lambda)} e^{-i\lambda t} \\ &= \overline{A_t(\lambda)} e^{i\lambda t} \\ &= \overline{\varphi_t(\lambda)} \end{aligned} \quad (54)$$

establishing (44).

11. Conversely, assume (43) holds. Reversing the steps from (50) to (45) shows that  $\overline{Z(t)} = Z(t)$  for all  $t$ , so  $Z$  is real-valued.  $\square$

**Theorem 9. [Existence of Oscillatory Processes]** Let  $F$  be an absolutely continuous spectral measure and the gain function

$$A_t(\lambda) \in L^2(F) \quad \forall t \in \mathbb{R} \quad (55)$$

be measurable in both time and frequency then the time-dependent spectral density is defined by

$$\begin{aligned} S_t(\lambda) &= \int_{\mathbb{R}} |A_t(\lambda)|^2 dF(\lambda) < \infty \\ &= \int_{\mathbb{R}} |A_t(\lambda)|^2 S(\lambda) d\lambda \end{aligned} \quad (56)$$

and there exists a complex orthogonal random measure  $\Phi$  with spectral measure  $F$  such that for each sample path  $\omega_0 \in \Omega$

$$Z(t, \omega_0) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda, \omega_0) \quad (57)$$

is well-defined in  $L^2(\Omega)$  and has covariance  $R_Z$  as in (41).

**Proof.**

1. Define the space of simple functions on  $\mathbb{R}$ : for disjoint Borel sets  $\{E_j\}_{j=1}^n$  with  $F(E_j) < \infty$  and coefficients  $\{c_j\}_{j=1}^n \subset \mathbb{C}$ ,

$$g(\lambda) = \sum_{j=1}^n c_j 1_{E_j}(\lambda) \quad (58)$$

2. For simple functions, define the stochastic integral

$$\int_{\mathbb{R}} g(\lambda) d\Phi(\lambda) := \sum_{j=1}^n c_j \Phi(E_j) \quad (59)$$

3. Compute the second moment:

$$\begin{aligned} \mathbb{E} \left[ \left| \int_{\mathbb{R}} g(\lambda) d\Phi(\lambda) \right|^2 \right] &= \mathbb{E} \left[ \left| \sum_{j=1}^n c_j \Phi(E_j) \right|^2 \right] \\ &= \mathbb{E} \left[ \sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \Phi(E_j) \overline{\Phi(E_k)} \right] \end{aligned} \quad (60)$$

4. By linearity of expectation,

$$\mathbb{E} \left[ \sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \Phi(E_j) \overline{\Phi(E_k)} \right] = \sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \mathbb{E} [\Phi(E_j) \overline{\Phi(E_k)}] \quad (61)$$



5. By the orthogonality relation (40), since  $E_j \cap E_k = \emptyset$  for  $j \neq k$ ,

$$\mathbb{E}[\Phi(E_j)\overline{\Phi(E_k)}] = \begin{cases} F(E_j) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \quad (62)$$

6. Substituting (62) into (61),

$$\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \mathbb{E}[\Phi(E_j)\overline{\Phi(E_k)}] = \sum_{j=1}^n |c_j|^2 F(E_j) \quad (63)$$

7. The right side of (63) equals

$$\sum_{j=1}^n |c_j|^2 F(E_j) = \int_{\mathbb{R}} |g(\lambda)|^2 dF(\lambda) \quad (64)$$

8. Therefore the isometry property holds for simple functions:

$$\mathbb{E}\left[\left|\int_{\mathbb{R}} g(\lambda) d\Phi(\lambda)\right|^2\right] = \int_{\mathbb{R}} |g(\lambda)|^2 dF(\lambda) \quad (65)$$

9. The space of simple functions is dense in  $L^2(F)$ . For any  $h(\lambda) \in L^2(F)$  and  $\epsilon > 0$ , there exists a simple function  $g(\lambda)$  such that

$$\int_{\mathbb{R}} |h(\lambda) - g(\lambda)|^2 dF(\lambda) < \epsilon \quad (66)$$

10. By the isometry (65) and completeness of  $L^2(\Omega)$ , the integral extends uniquely by continuity to all  $h(\lambda) \in L^2(F)$ .

11. Since  $A_t \in L^2(F)$  by assumption (55), and  $|e^{i\lambda t}| = 1$ ,

$$\int_{\mathbb{R}} |\varphi_t(\lambda)|^2 dF(\lambda) = \int_{\mathbb{R}} |A_t(\lambda)|^2 dF(\lambda) < \infty \quad (67)$$

so  $\varphi_t \in L^2(F)$ .

12. Therefore

$$\begin{aligned} Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \end{aligned} \quad (68)$$

is well-defined in  $L^2(\Omega)$ .

13. To compute the covariance, use the sesquilinearity of the stochastic integral:

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \overline{\int_{\mathbb{R}} \varphi_s(\mu) d\Phi(\mu)}\right] \end{aligned} \quad (69)$$

14. By Fubini's theorem for stochastic integrals,

$$\mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \overline{\int_{\mathbb{R}} \varphi_s(\mu) d\Phi(\mu)}\right] = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\mu)} \mathbb{E}[d\Phi(\lambda) \overline{d\Phi(\mu)}] \quad (70)$$

15. Using the orthogonality relation (40),

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\mu)} \delta(\lambda - \mu) dF(\lambda) dF(\mu) = \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\lambda)} dF(\lambda) \quad (71)$$

16. Substituting the definition (38),

$$R_Z(t, s) = \int_{\mathbb{R}} A_t(\lambda) \overline{A_s(\lambda)} e^{i\lambda(t-s)} dF(\lambda) \quad (72)$$

as claimed in (41).  $\square$

### 1.3.1 Integral Representations of Oscillatory Process Covariance Kernels

[2]

## 2 Unitarily Time-Changed Stationary Processes

### 2.1 Unitary Time-Change Operator $U_\theta f$

**Theorem 10.** *[Unitary time-change operator  $U_\theta$  and its inverse  $U_\theta^{-1}$ ] Let the time-change function  $\theta: \mathbb{R} \rightarrow \mathbb{R}$  be absolutely continuous, strictly increasing, and bijective, with*

$$\dot{\theta}(t) > 0 \quad (73)$$

*almost everywhere and  $\dot{\theta}(t) = 0$  only on sets of Lebesgue measure zero. For  $f$  measurable, define*

$$(U_\theta f)(t) = \sqrt{\dot{\theta}(t)} f(\theta(t)) \quad (74)$$

*Its inverse is given by*

$$(U_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \quad (75)$$

*For every compact set  $K \subseteq \mathbb{R}$  and  $f \in L^2_{\text{loc}}(\mathbb{R})$ ,*

$$\int_K |(U_\theta f)(t)|^2 dt = \int_{\theta(K)} |f(s)|^2 ds \quad (76)$$

*Moreover,  $U_\theta^{-1}$  is the inverse of  $U_\theta$  on  $L^2_{\text{loc}}(\mathbb{R})$ .*

**Proof.** 1. Let  $f \in L^2_{\text{loc}}(\mathbb{R})$  and let  $K \subset \mathbb{R}$  be compact. From the definition (74),

$$\int_K |(U_\theta f)(t)|^2 dt = \int_K \left| \sqrt{\dot{\theta}(t)} f(\theta(t)) \right|^2 dt \quad (77)$$

2. Expanding the square,

$$\int_K \left| \sqrt{\dot{\theta}(t)} f(\theta(t)) \right|^2 dt = \int_K \dot{\theta}(t) |f(\theta(t))|^2 dt \quad (78)$$

3. Since  $\theta$  is absolutely continuous and strictly increasing,  $\theta' = \dot{\theta}$  exists almost everywhere and  $\dot{\theta}(t) > 0$  a.e.
4. Apply the change of variables  $s = \theta(t)$ . Then

$$ds = \dot{\theta}(t) dt \quad (79)$$

5. The inverse function  $t = \theta^{-1}(s)$  exists since  $\theta$  is strictly increasing and bijective.
6. As  $t$  ranges over  $K$ , the variable  $s = \theta(t)$  ranges over  $\theta(K)$ .
7. Since  $\theta$  is continuous and  $K$  is compact,  $\theta(K)$  is compact.
8. Substituting (79) into (78),

$$\int_K \dot{\theta}(t) |f(\theta(t))|^2 dt = \int_{\theta(K)} |f(s)|^2 ds \quad (80)$$

9. This establishes the local isometry (76).
10. To verify  $U_\theta^{-1}$  is the inverse, compute:

$$(U_\theta^{-1} U_\theta f)(s) = U_\theta^{-1} (U_\theta f)(s) \quad (81)$$

11. By definition (75),

$$U_\theta^{-1} (U_\theta f)(s) = \frac{(U_\theta f)(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \quad (82)$$

12. By definition (74),

$$(U_\theta f)(\theta^{-1}(s)) = \sqrt{\dot{\theta}(\theta^{-1}(s))} f(\theta(\theta^{-1}(s))) \quad (83)$$

13. Since  $\theta \circ \theta^{-1} = \text{id}$ ,

$$f(\theta(\theta^{-1}(s))) = f(s) \quad (84)$$

14. Substituting (83) and (84) into (82),

$$\frac{\sqrt{\dot{\theta}(\theta^{-1}(s))} f(s)}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} = f(s) \quad (85)$$

15. Therefore

$$U_\theta^{-1} U_\theta = \text{id} \quad (86)$$

16. Similarly, compute:

$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)} (U_\theta^{-1} g)(\theta(t)) \quad (87)$$

17. By definition (75),

$$(U_\theta^{-1} g)(\theta(t)) = \frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} \quad (88)$$

18. Since  $\theta^{-1} \circ \theta = \text{id}$ ,

$$g(\theta^{-1}(\theta(t))) = g(t), \quad \theta^{-1}(\theta(t)) = t \quad (89)$$

19. Substituting (89) into (88),

$$\frac{g(t)}{\sqrt{\dot{\theta}(t)}} \quad (90)$$

20. Therefore from (87),

$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)} \cdot \frac{g(t)}{\sqrt{\dot{\theta}(t)}} = g(t) \quad (91)$$

21. Thus

$$U_\theta U_\theta^{-1} = \text{id} \quad (92)$$

22. Combining (86) and (92),  $U_\theta^{-1}$  is the two-sided inverse of  $U_\theta$  on  $L_{\text{loc}}^2(\mathbb{R})$ .  $\square$

## 2.2 Transformation of Stationary $\rightarrow$ Oscillatory Processes via $U_\theta$

**Theorem 11.** *[Unitary time changes of stationary processes produce oscillatory process] Let  $X$  be zero-mean stationary as in Definition 3. For scaling function  $\theta$  as in Theorem 10, define*

$$\begin{aligned} Z(t) &= (U_\theta X)(t) \\ &= \sqrt{\dot{\theta}(t)} X(\theta(t)) \end{aligned} \quad (93)$$

*Then  $Z$  is a realization of an oscillatory process with oscillatory function*

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \quad (94)$$

*gain function*

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \quad (95)$$

*and covariance kernel*

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\left[\sqrt{\dot{\theta}(t)} X(\theta(t)) \overline{\sqrt{\dot{\theta}(s)} X(\theta(s))}\right] \\ &= \sqrt{\dot{\theta}(t)\dot{\theta}(s)} \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \\ &= \sqrt{\dot{\theta}(t)\dot{\theta}(s)} R_X(\theta(t) - \theta(s)) \\ &= \sqrt{\dot{\theta}(t)\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} dF(\lambda) \end{aligned} \quad (96)$$

**Proof.** 1. From the Cramér representation (18),

$$X(u) = \int_{\mathbb{R}} e^{i\lambda u} d\Phi(\lambda) \quad (97)$$

2. Substituting  $u = \theta(t)$  into (97),

$$X(\theta(t)) = \int_{\mathbb{R}} e^{i\lambda\theta(t)} d\Phi(\lambda) \quad (98)$$

3. From the definition (93),

$$Z(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)} d\Phi(\lambda) \quad (99)$$

4. By linearity of the stochastic integral,

$$Z(t) = \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} d\Phi(\lambda) \quad (100)$$

5. Define

$$\varphi_t(\lambda) := \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \quad (101)$$

6. Then (100) becomes

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \quad (102)$$

which is the oscillatory representation (39).

7. To express this in terms of the standard oscillatory function form, Define the gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \quad (103)$$

8. Then verify the oscillatory function form (38) factorizes

$$\begin{aligned} \varphi_t(\lambda) &= A_t(\lambda) e^{i\lambda t} \\ &= \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} e^{i\lambda t} \\ &= \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t+t)} \\ &= \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \end{aligned} \quad (104)$$

9. To compute the covariance, use (41):

$$R_Z(t, s) = \mathbb{E}[Z(t)\overline{Z(s)}] \quad (105)$$

10. Substituting (93),

$$R_Z(t, s) = \mathbb{E}\left[\sqrt{\dot{\theta}(t)} X(\theta(t)) \overline{\sqrt{\dot{\theta}(s)} X(\theta(s))}\right] \quad (106)$$

11. Since  $\dot{\theta}$  is deterministic,

$$R_Z(t, s) = \sqrt{\dot{\theta}(t)} \sqrt{\dot{\theta}(s)} \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \quad (107)$$

12. By stationarity of  $X$ , using (19),

$$\mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] = R_X(\theta(t) - \theta(s)) = \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} dF(\lambda) \quad (108)$$

13. Substituting (108) into (107),

$$R_Z(t, s) = \sqrt{\dot{\theta}(t)\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} dF(\lambda) \quad (109)$$

establishing (96). □

### 2.2.1 Time-Varying Filter Representations

**Theorem 12.** *[Forward and Inverse Time-Dependent Convolution Representations for Unitary Time Transformations] Let  $\theta: \mathbb{R} \rightarrow \mathbb{R}$  be absolutely continuous, strictly increasing, and bijective with  $\theta'(t) > 0$  almost everywhere. Let  $X(u)$  be a stationary process, and define the oscillatory process obtained by the forward unitary time transformation  $U_\theta$*

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) = Z(t) = \int_{\mathbb{R}} h(t, u) X(u) du \quad (110)$$

where the (forward) impulse response function is given by

$$h(t, u) = \sqrt{\dot{\theta}(t)} \delta(u - \theta(t)) \quad (111)$$

then likewise the transformation can be reversed by expression the stationary process as

$$X(u) = (U_\theta^{-1} Z)(u) = \frac{Z(\theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} = \int_{\mathbb{R}} g(u, t) Z(t) dt \quad (112)$$

where the inverse impulse response function is

$$g(u, t) = \frac{\delta(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \quad (113)$$

**Proof.** TODO: insert proof here □

**Corollary 13.** *[Evolutionary spectrum of unitarily time-changed stationary process] The evolutionary spectrum is*

$$dF_t(\lambda) = \dot{\theta}(t) dF(\lambda) \quad (114)$$

**Proof.** 1. The evolutionary spectrum is defined by

$$dF_t(\lambda) = |A_t(\lambda)|^2 dF(\lambda) \quad (115)$$

2. From (95),

$$|A_t(\lambda)|^2 = \left| \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \right|^2 \quad (116)$$

3. Since  $|e^{i\alpha}| = 1$  for all real  $\alpha$ ,

$$|e^{i\lambda(\theta(t)-t)}|^2 = 1 \quad (117)$$

4. Therefore

$$|A_t(\lambda)|^2 = \left( \sqrt{\dot{\theta}(t)} \right)^2 \cdot 1 = \dot{\theta}(t) \quad (118)$$

5. Substituting (118) into (115),

$$dF_t(\lambda) = \dot{\theta}(t) dF(\lambda) \quad (119) \quad \square$$

## 2.3 Covariance operator conjugation

**Proposition 14. [Operator conjugation]** *Let*

$$(T_K f)(t) := \int_{\mathbb{R}} K(|t-s|) f(s) ds \quad (120)$$

*with stationary kernel*

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h} dF(\lambda) \quad (121)$$

*Define the transformed kernel*

$$K_{\theta}(s, t) := \sqrt{\dot{\theta}(t) \dot{\theta}(s)} K(|\theta(t) - \theta(s)|) \quad (122)$$

*then the corresponding integral covariance operator is conjugated for all  $f \in L^2_{\text{loc}}(\mathbb{R})$  by*

$$(T_{K_{\theta}} f)(t) = (U_{\theta} T_K U_{\theta}^{-1} f)(t) \quad (123)$$

**Proof.** 1. From (123), expand the right side:

$$(U_{\theta} T_K U_{\theta}^{-1} f)(t) = \sqrt{\dot{\theta}(t)} (T_K U_{\theta}^{-1} f)(\theta(t)) \quad (124)$$

2. By definition (120),

$$(T_K U_{\theta}^{-1} f)(\theta(t)) = \int_{\mathbb{R}} K(|\theta(t) - s|) (U_{\theta}^{-1} f)(s) ds \quad (125)$$

3. By definition (75),

$$(U_{\theta}^{-1} f)(s) = \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \quad (126)$$

4. Substituting (126) into (125),

$$\int_{\mathbb{R}} K(|\theta(t) - s|) \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} ds \quad (127)$$

5. Apply the change of variables  $s = \theta(u)$ , so  $ds = \dot{\theta}(u) du$  and  $\theta^{-1}(s) = u$ :

$$\int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \frac{f(u)}{\sqrt{\dot{\theta}(u)}} \dot{\theta}(u) du \quad (128)$$

6. Simplify:

$$\int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \frac{\dot{\theta}(u)}{\sqrt{\dot{\theta}(u)}} f(u) du = \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \sqrt{\dot{\theta}(u)} f(u) du \quad (129)$$

7. Substituting (129) into (124),

$$\sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \sqrt{\dot{\theta}(u)} f(u) du \quad (130)$$

8. Bring the constant inside the integral:

$$\int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} \sqrt{\dot{\theta}(u)} K(|\theta(t) - \theta(u)|) f(u) du \quad (131)$$

9. By definition (122),

$$\sqrt{\dot{\theta}(t)} \sqrt{\dot{\theta}(u)} K(|\theta(t) - \theta(u)|) = K_{\theta}(u, t) \quad (132)$$

10. Therefore

$$\int_{\mathbb{R}} K_{\theta}(u, t) f(u) du = (T_{K_{\theta}} f)(t) \quad (133)$$

establishing (123).  $\square$

### 3 Zero Localization

**Definition 15. [Zero localization measure]** Let  $Z$  be real-valued with  $Z \in C^1(\mathbb{R})$  having only simple zeros

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \quad (134)$$

Define, for Borel  $B \subset \mathbb{R}$ ,

$$\mu(B) = \int_{\mathbb{R}} 1_B(t) \delta(Z(t)) |\dot{Z}(t)| dt \quad (135)$$

**Theorem 16. [Atomicity and local finiteness of zeros and delta decomposition]** Under the assumptions of Definition 15, zeros are locally finite and one has

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \quad (136)$$

whence

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (137)$$

**Proof.** 1. For any smooth test function  $\phi$  with compact support, apply the standard change of variables formula for the delta function. Let  $\{t_0^{(1)}, t_0^{(2)}, \dots\}$  denote the zeros of  $Z$ .



2. By the change of variables formula for distributions,

$$\int_{\mathbb{R}} \phi(t) \delta(Z(t)) dt = \sum_{t_0: Z(t_0)=0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} \quad (138)$$

3. The right side of (138) equals

$$\sum_{t_0: Z(t_0)=0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} = \sum_{t_0: Z(t_0)=0} \int_{\mathbb{R}} \phi(t) \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} dt \quad (139)$$

4. By Fubini's theorem (justified since the sum has locally finite terms due to  $C^1$  regularity and simple zeros),

$$\sum_{t_0: Z(t_0)=0} \int_{\mathbb{R}} \phi(t) \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} dt = \int_{\mathbb{R}} \phi(t) \sum_{t_0: Z(t_0)=0} \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} dt \quad (140)$$

5. Comparing (138) and (140),

$$\int_{\mathbb{R}} \phi(t) \delta(Z(t)) dt = \int_{\mathbb{R}} \phi(t) \sum_{t_0: Z(t_0)=0} \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} dt \quad (141)$$

6. Since  $\phi$  is arbitrary,

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} \quad (142)$$

establishing (136).

7. Substituting (142) into the definition (135),

$$\mu(B) = \int_{\mathbb{R}} 1_B(t) \sum_{t_0: Z(t_0)=0} \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} |\dot{Z}(t)| dt \quad (143)$$

8. By the sifting property of the delta function,  $|\dot{Z}(t)|$  evaluated at  $t=t_0$  gives  $|\dot{Z}(t_0)|$ :

$$\int_{\mathbb{R}} 1_B(t) \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} |\dot{Z}(t)| dt = \frac{1_B(t_0) |\dot{Z}(t_0)|}{|\dot{Z}(t_0)|} = 1_B(t_0) \quad (144)$$

9. Summing over all zeros,

$$\mu(B) = \sum_{t_0: Z(t_0)=0} 1_B(t_0) = \sum_{t_0 \in B: Z(t_0)=0} 1 \quad (145)$$

10. This is precisely the atomic measure

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (146)$$

establishing (137). □

**Definition 17. [Hilbert space on the zero set]** Let  $\mathcal{H} = L^2(\mu)$  with inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} d\mu(t) \quad (147)$$

**Proposition 18. [Atomic structure]** Let

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (148)$$

then

$$\mathcal{H} \cong \left\{ f: \{t_0: Z(t_0)=0\} \rightarrow \mathbb{C}: \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \quad (149)$$

with orthonormal basis  $\{e_{t_0}\}_{t_0: Z(t_0)=0}$  where

$$e_{t_0}(t_1) = \delta_{t_0, t_1} \quad (150)$$

**Proof.** 1. By (148),  $\mu$  is a purely atomic measure with atoms at the zero set.

2. For any  $f \in L^2(\mu)$ , the  $L^2$  norm is

$$\|f\|_{L^2(\mu)}^2 = \int_{\mathbb{R}} |f(t)|^2 d\mu(t) \quad (151)$$

3. Substituting (148),

$$\int_{\mathbb{R}} |f(t)|^2 d\mu(t) = \int_{\mathbb{R}} |f(t)|^2 \sum_{t_0: Z(t_0)=0} \delta_{t_0}(dt) = \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 \quad (152)$$

4. Therefore

$$\|f\|_{L^2(\mu)}^2 = \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 \quad (153)$$

5. This is precisely the  $\ell^2$  norm on the zero set.

6. Define the map  $\Psi: L^2(\mu) \rightarrow \ell^2$  by

$$\Psi(f) = (f(t_0))_{t_0: Z(t_0)=0} \quad (154)$$

7. From (153),  $\Psi$  is an isometry:

$$\|\Psi(f)\|_{\ell^2}^2 = \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 = \|f\|_{L^2(\mu)}^2 \quad (155)$$

8.  $\Psi$  is surjective: for any sequence  $(c_{t_0}) \in \ell^2$ , define  $f(t) = \sum_{t_0} c_{t_0} \delta_{t_0}(t)$ , which is in  $L^2(\mu)$ .

9. Therefore  $\Psi$  is a Hilbert space isomorphism, establishing (149).

10. For the orthonormal basis, define  $e_{t_0}$  by (150).

11. Then

$$\langle e_{t_0}, e_{t_1} \rangle = \int_{\mathbb{R}} e_{t_0}(t) \overline{e_{t_1}(t)} d\mu(t) = \sum_{s: Z(s)=0} \delta_{t_0,s} \delta_{t_1,s} = \delta_{t_0,t_1} \quad (156)$$

12. Therefore  $\{e_{t_0}\}$  is an orthonormal set.

13. Since every  $f \in L^2(\mu)$  can be written as

$$f = \sum_{t_0: Z(t_0)=0} f(t_0) e_{t_0} \quad (157)$$

the set  $\{e_{t_0}\}$  is complete, hence an orthonormal basis.  $\square$

**Definition 19. [Multiplication operator]** Define the linear operator

$$L: \mathcal{D}(L) \subset \mathcal{H} \rightarrow \mathcal{H} \quad (158)$$

by

$$(L f)(t) = t f(t) \quad (159)$$

on the support of  $\mu$  with domain

$$\mathcal{D}(L) := \left\{ f \in \mathcal{H}: \int |t f(t)|^2 d\mu(t) < \infty \right\} \quad (160)$$

**Theorem 20. [Self-adjointness and spectrum]**  $L$  is self-adjoint on  $\mathcal{H}$  and has pure point, simple spectrum

$$\sigma(L) = \overline{\{t \in \mathbb{R}: Z(t) = 0\}} \quad (161)$$

with eigenvalues  $\lambda = t_0$  for each zero  $t_0$  and corresponding eigenvectors  $e_{t_0}$ .

**Proof.** 1. For  $f, g \in \mathcal{D}(L)$ , compute the inner product:

$$\langle L f, g \rangle = \int_{\mathbb{R}} (L f)(t) \overline{g(t)} d\mu(t) \quad (162)$$

2. By definition (159),

$$\int_{\mathbb{R}} t f(t) \overline{g(t)} d\mu(t) \quad (163)$$

3. Since  $t$  is real-valued,  $\bar{t} = t$ , so

$$\int_{\mathbb{R}} t f(t) \overline{g(t)} d\mu(t) = \int_{\mathbb{R}} f(t) \overline{t g(t)} d\mu(t) \quad (164)$$

4. The right side of (164) is

$$\int_{\mathbb{R}} f(t) \overline{(L g)(t)} d\mu(t) = \langle f, L g \rangle \quad (165)$$

5. Therefore

$$\langle L f, g \rangle = \langle f, L g \rangle \quad (166)$$

for all  $f, g \in \mathcal{D}(L)$ , establishing that  $L$  is symmetric.

6. Since  $L$  is a multiplication operator on  $L^2(\mu)$ , it is self-adjoint (by standard functional analysis).

7. To determine the spectrum, compute the action on basis vectors. From (159) and (150),

$$(L e_{t_0})(t) = t e_{t_0}(t) = t \delta_{t_0}(t) \quad (167)$$

8. By the sifting property,

$$t \delta_{t_0}(t) = t_0 \delta_{t_0}(t) = t_0 e_{t_0}(t) \quad (168)$$

9. Therefore

$$L e_{t_0} = t_0 e_{t_0} \quad (169)$$

10. This shows that each  $t_0$  is an eigenvalue with eigenvector  $e_{t_0}$ .

11. Since the  $\{e_{t_0}\}$  form a complete orthonormal basis (Proposition 18), the spectrum is pure point.

12. Each eigenspace is one-dimensional (spanned by  $e_{t_0}$ ), so the spectrum is simple and given by the closure of the zero set

$$\sigma(L) = \{t_0: Z(t_0) = 0\} = \overline{\{t \in \mathbb{R}: Z(t) = 0\}} \quad (170) \quad \square$$

### 3.1 The Kac-Rice Formula For The Expected Zero Counting Function

**Theorem 21. [Kac-Rice Formula for Zero Crossings]** Let  $Z(t)$  be a centered Gaussian process on  $[a, b]$  with covariance  $K(s, t) = \mathbb{E}[Z(s) Z(t)]$  then the expected number of zeros in  $[a, b]$  is

$$\mathbb{E}[N_{[a, b]}] = \int_a^b \sqrt{\frac{2}{\pi}} \frac{\sqrt{K(t, t) K_{\dot{Z}}(t, t) - K_{Z, \dot{Z}}(t, t)^2}}{K(t, t)} dt \quad (171)$$

where

$$K(t, t) = \mathbb{E}[Z(t)^2] \quad (172)$$

$$K_{\dot{Z}}(t, t) = -\partial_s^2 \partial_t K(s, t)|_{s=t} \quad (173)$$

and

$$K_{Z, \dot{Z}}(t, t) = \partial_s K(s, t)|_{s=t} \quad (174)$$

**Proof.** 1. The expected number of zeros is given by the Kac-Rice formula for the level crossing density at level zero.

2. For a Gaussian process  $Z(t)$ , the joint distribution of  $(Z(t), \dot{Z}(t))$  is bivariate normal with covariance matrix

$$\Sigma(t) = \begin{pmatrix} K(t, t) & K_{Z, \dot{Z}}(t, t) \\ K_{Z, \dot{Z}}(t, t) & K_{\dot{Z}}(t, t) \end{pmatrix} \quad (175)$$

3. The determinant of  $\Sigma(t)$  is

$$\det \Sigma(t) = K(t, t) K_{\dot{Z}}(t, t) - K_{Z, \dot{Z}}(t, t)^2 \quad (176)$$

4. The Kac-Rice formula states that the expected number of zeros in  $[a, b]$  is

$$\mathbb{E}[N_{[a, b]}] = \int_a^b p_{Z(t), \dot{Z}(t)}(0, v) |v| dv dt \quad (177)$$

where  $p_{Z(t), \dot{Z}(t)}$  is the joint density of  $(Z(t), \dot{Z}(t))$ .

5. For a centered bivariate normal distribution with covariance  $\Sigma(t)$ ,

$$p_{Z(t), \dot{Z}(t)}(z, v) = \frac{1}{2\pi \sqrt{\det \Sigma(t)}} \exp \left( -\frac{1}{2} \begin{pmatrix} z \\ v \end{pmatrix}^T \Sigma(t)^{-1} \begin{pmatrix} z \\ v \end{pmatrix} \right) \quad (178)$$

6. At  $z=0$ ,

$$p_{Z(t), \dot{Z}(t)}(0, v) = \frac{1}{2\pi \sqrt{\det \Sigma(t)}} \exp \left( -\frac{v^2 K(t, t)}{2 \det \Sigma(t)} \right) \quad (179)$$

7. Integrating (179) against  $|v|$ ,

$$\int_{-\infty}^{\infty} p_{Z(t), \dot{Z}(t)}(0, v) |v| dv = \frac{1}{2\pi \sqrt{\det \Sigma(t)}} \int_{-\infty}^{\infty} |v| \exp \left( -\frac{v^2 K(t, t)}{2 \det \Sigma(t)} \right) dv \quad (180)$$

8. Using symmetry,  $\int_{-\infty}^{\infty} |v| e^{-av^2} dv = 2 \int_0^{\infty} v e^{-av^2} dv = \frac{1}{a} \sqrt{\frac{\pi}{a}}$ :

$$\int_{-\infty}^{\infty} |v| \exp \left( -\frac{v^2 K(t, t)}{2 \det \Sigma(t)} \right) dv = \sqrt{\frac{2\pi \det \Sigma(t)}{K(t, t)}} \quad (181)$$

9. Substituting (181) into (180),

$$\frac{1}{2\pi \sqrt{\det \Sigma(t)}} \cdot \sqrt{\frac{2\pi \det \Sigma(t)}{K(t, t)}} = \sqrt{\frac{2}{\pi}} \frac{\sqrt{\det \Sigma(t)}}{K(t, t)} \quad (182)$$

10. Using (176),

$$\sqrt{\frac{2}{\pi}} \frac{\sqrt{K(t, t) K_{\dot{Z}}(t, t) - K_{Z, \dot{Z}}(t, t)^2}}{K(t, t)} \quad (183)$$

11. Integrating (183) over  $[a, b]$  yields (171).  $\square$

### 3.1.1 The Expected Zero Counting Function for Unitarily Time-Changed Stationary Processes

**Theorem 22. (Expected Zero-Counting Function Of The Oscillatory Process Subclass of Unitarily Time-Changed Stationary Processes)** *Let  $\theta: \mathbb{R} \rightarrow \mathbb{R}$  be absolutely continuous, strictly increasing, and bijective with  $\dot{\theta}(t) > 0$  almost everywhere. Let  $X$  be a centered stationary Gaussian process with spectral measure  $F$  and covariance function*

$$K(h) = \int_{\mathbb{R}} e^{i\omega h} dF(\omega) \quad (184)$$

*twice differentiable at  $h=0$ . Define the unitarily time-changed process*

$$Z(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) \quad (185)$$

*Then  $Z$  is a centered Gaussian process with covariance*

$$K_Z(t, s) = \sqrt{\dot{\theta}(t)\dot{\theta}(s)} K(\theta(t) - \theta(s)) \quad (186)$$

*and the expected number of zeros in  $[0, T]$  is*

$$\mathbb{E}[N_{[0, T]}] = \frac{1}{\sqrt{2\pi K(0)}} \sqrt{-\ddot{K}(0)} [\theta(T) - \theta(0)] \quad (187)$$

**Proof.** 1. By the Kac-Rice formula:

$$\mathbb{E}[N_{[0, T]}] = \int_0^T \sqrt{\frac{2}{\pi}} \frac{\sqrt{-\lim_{s \rightarrow t} \frac{\partial^2}{\partial s \partial t} K_Z(s, t)}}{K_Z(t, t)} dt \quad (188)$$

2. Differentiate (186) with respect to  $s$ :

$$\frac{\partial}{\partial s} K_Z(s, t) = \frac{\ddot{\theta}(s)}{2\sqrt{\dot{\theta}(s)\dot{\theta}(t)}} K(\theta(t) - \theta(s)) - \dot{\theta}(s) \sqrt{\dot{\theta}(s)\dot{\theta}(t)} \dot{K}(\theta(t) - \theta(s)) \quad (189)$$

3. Differentiate (189) with respect to  $t$  and take  $s \rightarrow t$ . Since  $\dot{K}(0) = 0$  by stationarity:

$$\lim_{s \rightarrow t} \frac{\partial^2}{\partial s \partial t} K_Z(s, t) = -\dot{\theta}(t)^2 \ddot{K}(0) \quad (190)$$

4. From (186) with  $s = t$ :

$$K_Z(t, t) = \dot{\theta}(t) K(0) \quad (191)$$

5. Substitute (190) and (191) into (188):

$$\mathbb{E}[N_{[0, T]}] = \int_0^T \sqrt{\frac{2}{\pi}} \frac{\sqrt{\dot{\theta}(t)^2 (-\ddot{K}(0))}}{\dot{\theta}(t) K(0)} dt \quad (192)$$

6. Since  $\dot{\theta}(t) > 0$  and  $\ddot{K}(0) < 0$ :

$$\mathbb{E}[N_{[0,T]}] = \int_0^T \sqrt{\frac{2}{\pi}} \frac{\dot{\theta}(t) \sqrt{-\ddot{K}(0)}}{\dot{\theta}(t) K(0)} dt = \frac{\sqrt{-\ddot{K}(0)}}{\sqrt{2\pi} K(0)} \int_0^T \dot{\theta}(t) dt \quad (193)$$

7. Evaluate the integral:

$$\mathbb{E}[N_{[0,T]}] = \sqrt{-2\pi \frac{\ddot{K}(0)}{K(0)}} [\theta(T) - \theta(0)] \quad (194) \quad \square$$

**Theorem 23.** [*Deterministic zero-crossing at vanishing derivative*] Let  $X$  be a zero-mean stationary process with spectral measure  $F$  as in Definition 3 and finite variance  $\sigma^2 = \mathbb{E}[X(t)^2] < \infty$ . Let  $\theta: \mathbb{R} \rightarrow \mathbb{R}$  be the time-change function from Theorem 10, which is absolutely continuous (has derivative  $\dot{\theta}$  that exists almost everywhere and is Lebesgue integrable), strictly increasing (so  $\theta(t_1) < \theta(t_2)$  whenever  $t_1 < t_2$ ), and bijective (one-to-one and onto). The derivative  $\dot{\theta}(t)$  is strictly positive almost everywhere, meaning  $\dot{\theta}(t) > 0$  for all  $t$  except possibly on a set of Lebesgue measure zero. Define the transformed process

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) \quad (195)$$

as in equation (74). Consider a point  $t_0 \in \mathbb{R}$  where the derivative vanishes:  $\dot{\theta}(t_0) = 0$ . Then **every sample path of  $Z$  passes through zero at  $t_0$** : for all  $\omega \in \Omega$ ,

$$Z(t_0, \omega) = 0 \quad (196)$$

This is a **deterministic zero-crossing**: unlike the random zero-crossings of the stationary process  $X$ , which occur probabilistically according to Bulinskaya's statistics, the zero at  $t_0$  occurs with certainty in every realization of  $Z$ . The randomness of  $X$  is completely suppressed at  $t_0$  by the vanishing amplitude  $\sqrt{\dot{\theta}(t_0)} = 0$ .

**Proof.** 1. Consider a point  $t_0 \in \mathbb{R}$  where  $\dot{\theta}(t_0) = 0$ .

2. From the definition (195), the value of  $Z$  at  $t_0$  for any sample path  $\omega \in \Omega$  is

$$Z(t_0, \omega) = \sqrt{\dot{\theta}(t_0)} \cdot X(\theta(t_0), \omega) \quad (197)$$

3. Since  $\dot{\theta}(t_0) = 0$  by hypothesis,

$$\sqrt{\dot{\theta}(t_0)} = \sqrt{0} = 0 \quad (198)$$

4. Substituting (198) into (197),

$$Z(t_0, \omega) = 0 \cdot X(\theta(t_0), \omega) = 0 \quad (199)$$

regardless of the value of  $X(\theta(t_0), \omega)$ .

5. Since  $\omega \in \Omega$  was arbitrary, equation (199) holds for every sample path:

$$Z(t_0, \omega) = 0 \quad \forall \omega \in \Omega \quad (200)$$

6. Therefore  $t_0$  is a deterministic zero-crossing: the process  $Z$  reaches zero at  $t_0$  in every realization, not probabilistically.

7. As a direct consequence, the variance of  $Z$  at  $t_0$  is zero:

$$\text{Var}[Z(t_0)] = \mathbb{E}[(Z(t_0) - \mathbb{E}[Z(t_0)])^2] = \mathbb{E}[0^2] = 0 \quad (201)$$

8. By Corollary 13, the evolutionary spectrum at  $t_0$  vanishes:

$$dF_{t_0}(\lambda) = \dot{\theta}(t_0) dF(\lambda) = 0 \cdot dF(\lambda) = 0 \quad (202)$$

meaning there is no spectral energy at  $t_0$ .

9. The point  $t_0$  belongs to the zero set  $\{t \in \mathbb{R}: Z(t, \omega) = 0\}$  for every  $\omega \in \Omega$ . By Definition 4, this deterministic zero-crossing differs fundamentally from the random zero-crossings governed by the statistics of the stationary process  $X$ : it occurs because the amplitude factor  $\sqrt{\dot{\theta}(t_0)}$  vanishes, completely eliminating the influence of the random process  $X$  at that instant.  $\square$

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