New Uniformly Convergent Series for the Bessel Functions of the First Kind of Integer Orders

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Definition 1

Let $j_n(x)$ is the spherical Bessel function of the first kind,

$$\begin{aligned} j_n(z) &= \sqrt{\frac{\pi}{2\,z}}\,J_{n+\frac{1}{2}}(x) \\ &= \frac{1}{\sqrt{z}}\left(\sin(z)\,R_{n,\frac{1}{2}}(z) - \cos(z)\,R_{n-1,\frac{3}{2}}(z)\right) \end{aligned} \tag{1}$$

where $R_{n,v}(z)$ are the (misnamed) Lommel polynomials [2]

$$R_{n,v}(z) = \frac{\Gamma(n+v)}{\Gamma(v)}\left(\frac{2}{z}\right)^n {}_2F_3\left(\left[-\frac{n}{2},\frac{1}{2}-\frac{n}{2}\right];[v,-n,1-v-n];-z^2\right) \tag{2}$$

where ${}_2F_3$ is a generalized hypergeometric function. The "Lommel polynomials" are actually rational functions of z, not polynomial; but rather "polynomial in $\frac{1}{z}$".

Conjecture 2

The series

$$\begin{aligned} J_0(t) &= \sum_{k=0}^{\infty} \lambda_k \psi_k(t) \\ &= \sum_{k=0}^{\infty} \sqrt{\frac{4\,n+1}{\pi}}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2}(-1)^n\sqrt{\frac{4\,n+1}{\pi}}\,j_{2n}(t) \\ &= \sum_{k=0}^{\infty} \frac{4\,n+1}{\pi}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2}(-1)^n\,j_{2n}(t) \end{aligned} \tag{3}$$

converges uniformly for all comlex t except the origin where it has a regular singular point where $\lim_{t\to 0} J_0(t) = 1$.

Conjecture 3

The eigenfunctions of the stationary integral covariance operator

$$[T\psi_n](x) = \int_0^\infty J_0\left(x-y\right)\psi_n(x)\mathrm{d}x = \lambda_n\psi_n(x) \tag{4}$$

are given by

$$\psi_n(y) = (-1)^n\sqrt{\frac{4\,n+1}{\pi}}\,j_{2n}(y) \tag{5}$$

and the eigenvalues are given by

$$\begin{aligned}\lambda_n &= \int_{-\infty}^\infty J_0(x)\,\psi_n(x)\,\mathrm{d}x \\ &= \sqrt{\frac{4\,n+1}{\pi}}\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\left(n+1\right)^2} \\ &= \sqrt{\frac{4\,n+1}{\pi}}\,(n+1)_{\frac{1}{2}}^2\end{aligned} \tag{6}$$

where $(n+1)_{-\frac{1}{2}}^2$ is the Pochhammer symbol(ascending/rising factorial).

Definition 4

The spectral density of a stationary process is the Fourier tranform of the covariance kernel due to Wiener-Khinchine theorem.

Definition 5

Let $S_n(x)$ be the orthogonal polynomials whose orthogonality measure is equal to the spectral density of the process. These polynomials shall be called the spectral polynomials corresponding to the process.

Example 6. Let the kernel function be given by $K(t,s) = J_0(t-s)$ then identify the orthogonal polynomial sequence associated with the spectral density

$$S(\omega) = \int_{-\infty}^\infty J_0(x)\,e^{ix\omega}\,\mathrm{d}x = \begin{cases}\dfrac{2}{\sqrt{1-\omega^2}} & \omega\in(-1,1)\\ 0 & \text{otherwise}\end{cases} \tag{7}$$

as being twice the orthogonality measure of the Type-I Chebyshev polynomials $T_n(x)$ so that the orthogonal polynomial sequence is identified as

$$S_n(x) = \sqrt{2}T_n(x) \tag{8}$$

so that

$$\int_{-1}^{1} S_n(\omega)\, S_m(\omega)\, S(\omega)\, \mathrm{d}\omega = \begin{cases} 0 & n \neq m \\ 2\pi & n = m = 0 \\ \pi & n = m \neq 0 \end{cases} \tag{9}$$

Remark 7. If the spectral density does not equal the orthogonality measure of a known set of orthogonal polynomials then such a set can always be generated by applying the Gram-Schmidt process to the monomials so that they are transformed into a set that is orthogonal with respect any given spectral density (of a stationary process).

Definition 8

The sequence $\hat{S}_n(y)$ of Fourier transforms of the spectral polynomials $S_n(x)$ is given by

$$\hat{S}_n(y) = \int_{-1}^{1} S_n(x) e^{ixy} \mathrm{d}x \tag{10}$$

Example 9. The Fourier transforms of the Chebyshev polynomials are just the usual infinite Fourier transforms with the integration restricted to the range $-1 \ldots 1$ since $T_n(x) = 0 \forall x \notin [-1, 1]$). Equivalently, the spectral density function can be extended to take the value 0 outside the interval $[-1, 1]$. The derivation of

$$\begin{aligned} \hat{T}_n(y) &= \int_{-\infty}^{\infty} e^{-ixy}\, T_n(x) d\, y = \int_{-1}^{1} e^{-ixy}\, T_n(x) d\, x \\ &= \int_{-\infty}^{\infty} e^{-ixy}\, {}_2F_1\left(\begin{matrix} n, & -n \\ & \frac{1}{2} \end{matrix} \middle| \frac{1}{2} - \frac{x}{2} \right) d\, x \\ &= \frac{i}{y} \left(e^{-iy} F_n^{+}(y) - e^{i(\pi n + y)} F_n^{-}(y)\right) \end{aligned} \tag{11}$$

where

$$F_n^{\pm}(y) = {}_3F_1\left(\begin{matrix} 1, & n, & -n \\ & & \frac{1}{2} \end{matrix} \middle| \frac{\pm i y}{2} \right) \tag{12}$$

can be found in [1].

Definition 10

Let $Y_n(y)$ be the normalized spectral polynomials $S_n(x)$

Example 11. When $K = J_0$ the spectral polynomials are given by

$$S_n(x) = \sqrt{2} T_n(x) \tag{13}$$

so that

$$
\begin{aligned}
Y_n(y) \; =& \frac{\hat{T}_n(y)}{|\hat{T}_n|} \\
=& \frac{i}{y}\left(\frac{e^{-iy}F_n^{+}(y) - e^{i(\pi n + y)}F_n^{-}(y)}{\sqrt{\frac{4(-1)^n\pi - (2n^2-1)}{4n^2-1}}}\right)
\end{aligned} \tag{14}
$$

where the L^2 norm of $\hat{T}_n(y)$ is given by

$$
\begin{aligned}
|\hat{T}_n| \; =& \sqrt{\int_{-\infty}^{\infty} \hat{T}_n(y)^2 \mathrm{d}y} \\
=& \sqrt{\frac{4(-1)^n\pi - (2n^2-1)}{4n^2-1}}
\end{aligned} \tag{15}
$$

Conjecture 12

The eigenfunctions of the integral covariance operator (4) are given by the orthogonal complement of the normalized Fourier transforms $Y_n(y)$ of the spectral polynomials (via the Gram-Schmidt process)

$$
\psi_n(y) = Y_n^{\perp}(y) = Y_n(y) - \sum_{m=1}^{n-1} \frac{\langle Y_n(y), Y_m^{\perp}(y)\rangle}{\langle Y_m^{\perp}(y), Y_m^{\perp}(y)\rangle} Y_m^{\perp}(y) \tag{16}
$$

can be equivalently expressed as

$$
\begin{aligned}
\psi_n(y) \; =& (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, j_{2n}(y) \\
=& (-1)^n \sqrt{\frac{4\,n+1}{\pi}} \sqrt{\frac{\pi}{2\,y}}\, J_{2n+\frac{1}{2}}(y) \\
=& (-1)^n \sqrt{\frac{4\,n+1}{2\,y}}\, J_{2n+\frac{1}{2}}(y) \\
=& \sqrt{\frac{4\,n+1}{4\,\pi}} \int_{-1}^{1} P_{2n}(x)\, e^{ixy}\, \mathrm{d}x
\end{aligned} \tag{17}
$$

Remark 13. Since T is compact due to its self-adjointness and convergence of the eigenvalues to 0 it converges uniformly since compactness implies uniform convergence of the eigenfunctions. TODO: cite/theorems from [3, 3. Reproducing Kernel Hilbert Space of a Gaussian Process]

Bibliography

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