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Testing for Harmonizability: A Mathematical Framework for Stochastic Process Analysis

Introduction

The concept of harmonizability represents a fundamental characterization of stochastic processes that admits frequency domain analysis through spectral decomposition[1]. A stochastic process X(t) is harmonizable when it possesses a quadratic mean representation involving complex exponentials, enabling the application of linear filtering theory with standard frequency interpretations[1].

The mathematical foundation rests on the representation:

$$X(t, \omega) = \int_{-\infty}^{\infty} \exp(itx)\, dZ(x, \omega)$$

where Z(x, ω) constitutes a process whose covariance exhibits bounded variation in the plane[1]. This decomposition proves particularly valuable in engineering applications, as it facilitates the analysis of linear time-invariant systems through frequency response functions[1].

Theoretical Framework

Core Characterization Theorem

The central theoretical contribution establishes that any function R(s,t) serving simultaneously as a covariance function and a Fourier-Stieltjes transform with respect to some bounded variation function G(x,y) necessarily renders G(x,y) itself a covariance[1]. This result provides the theoretical foundation for determining harmonizability through Fourier-Stieltjes transform characterizations.

The proof methodology involves constructing the normalized function:

$$G_a(x, y) = G(x, y) - G(a, y) - G(x, a) + G(a, a)$$

and demonstrating non-negative definiteness through the inversion theorem, ultimately establishing that G(x,y) maintains non-negative definite structure[1].

Bochner-Eberlein Characterization

The Bochner-Eberlein theorem provides the operational criterion for harmonizability[1]. A function R(s,t) admits the representation:

$$R(s, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(isx - ity)\, d^2G(x, y)$$

with bounded variation G(x,y) if and only if for arbitrary sequences $\{(s_j,t_j)\}$ and complex coefficients $\{c_j\}$, the inequality:

$$\left|\sum_{j=1}^{n} c_j R(s_j, t_j)\right| \leq M\left[\sum_{j,k=1}^{n} c_j c_k^* \exp(i(xs_j - yt_j - xs_k + yt_k))\right]^{1/2}$$

holds for some constant M > 0[1].

Applications and Examples

Gaussian Process Powers

For real Gaussian harmonizable processes X(t), the nth power X^n(t) maintains harmonizability for any positive integer n[1]. The proof employs characteristic function analysis, expressing the covariance of X^n(t) as:

$$\mathbb{E}[X^n(s)X^n(t)] = \sum_{p,q,r\geq 0} c_z(p,q,r,n) R^p(s,s) R^q(s,t) R^r(t,s) R^p(t,t)$$

where the summation extends over all non-negative integers satisfying n = 2p + q + r[1]. The harmonizability follows from the closure properties of Fourier-Stieltjes transforms and covariances under products and positive linear combinations[1].

Composition with Fourier-Stieltjes Transforms

When X(t) represents a harmonizable process with spectral support confined to a bounded set A, and g(t) constitutes a Fourier-Stieltjes transform of finite variation G(x), both X(t + g(t)) and X(g(t)) preserve harmonizability[1]. The demonstration involves showing that the mappings t ↦ exp[ix'g(t)] and t ↦ exp[-iy'g(t)] maintain Fourier-Stieltjes transform properties, thereby satisfying Bochner's conditions[1].

Integral Compositions

The framework extends to integral compositions of harmonizable covariances[1]. For harmonizable covariances R_1 and R_2, the integral:

$$R_3(s,t) = \int_T R_1(s,u) R_2(u,t)\, du$$

over any finite Lebesgue measure set T produces a harmonizable covariance R_3[1]. The proof utilizes the Bochner-Eberlein condition with variation bounds $M_1M_2m(T)$, where m(T) denotes the Lebesgue measure of T[1].

Moving Average Processes

Moving average processes of the form:

$$X(t) = \int_{-\infty}^{\infty} f(t-u)\, dZ(u)$$

where f(t-u) constitutes a Fourier-Stieltjes transform and G(u,v) = E{Z(u)Z*(v)} exhibits bounded variation, maintain harmonizability[1]. The covariance function becomes:

$$R(s,t) = \iint \exp[ix(s-u) - iy(t-v)]\, d^2G(u,v)\, dH(x)\, dH^*(y)$$

representing a Fourier-Stieltjes transform with variation bound M_G M_H²[1].

Oscillatory Processes

Priestley's oscillatory processes, characterized by:

$$f(t,u) = \exp(iut) \int_{-\infty}^{\infty} \exp(itx)\, dH_u(x)$$

achieve harmonizability under the condition:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |dH_u(x)|\, |dH_u^*(y)|\, dF(u) < \infty$$

where F(u) represents the orthogonal increment measure[1]. The proof employs repeated applications of Bochner's theorem in conjunction with Fubini's theorem[1].

Significance and Impact

The harmonizability framework provides a rigorous mathematical foundation for frequency domain analysis of non-stationary stochastic processes[1]. The methodology enables the extension of classical spectral analysis techniques to broader classes of processes while maintaining theoretical rigor through Fourier-Stieltjes transform characterizations[1].

The practical implications extend to engineering applications where linear filtering interpretations remain valid despite non-stationarity, provided the underlying processes satisfy harmonizability conditions[1]. This framework thus bridges theoretical stochastic process theory with practical signal processing applications through mathematically sound characterizations[1].

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