An oscilatory process (Priestley 1965)X_t can be represented as:

$$X_t = \int_{-\infty}^{\infty} e^{i\omega t} A_t(\omega)\ d Z(\omega) \tag{1}$$

where $A_t(\omega)$ is the time-varying gain function and $d Z(\omega)$ represents a process with orthogonal increments.

Theorem 1. *[Fourier Domain Relationship] The relationship between the gain function $A_t(\omega)$ and the time-varying filter $h_t(u)$ is given by:*

$$A_t(\omega)\, e^{i\omega t} = \int_{-\infty}^{\infty} h_t(u)\, e^{i\omega u}\ d u \tag{2}$$

Remark 2. Note that in some literature, $h_t(u)$ may be denoted as $a(t,\tau)$, where t is the time parameter and τ or u represents the lag parameter.

Theorem 3. *[Explicit Definition of $h_t(u)$] The time-varying filter $h_t(u)$ is explicitly defined as:*

$$h_t(u) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} A_t(\omega)\, e^{i\omega(t-u)}\ d\,\omega \tag{3}$$

Proof. We start with the Fourier domain relationship:

$$A_t(\omega)\, e^{i\omega t} = \int_{-\infty}^{\infty} h_t(u)\, e^{i\omega u}\ d u \tag{4}$$

To isolate $h_t(u)$, we apply the inverse Fourier transform by multiplying both sides by $e^{-i\omega v}$ and integrating with respect to ω:

$$\int_{-\infty}^{\infty} A_t(\omega)\, e^{i\omega t}\, e^{-i\omega v}\ d\,\omega = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h_t(u)\, e^{i\omega u}\ d\,u \right) e^{-i\omega v}\ d\,\omega \tag{5}$$

$$\int_{-\infty}^{\infty} A_t(\omega)\, e^{i\omega(t-v)}\ d\,\omega = \int_{-\infty}^{\infty} h_t(u) \left(\int_{-\infty}^{\infty} e^{i\omega(u-v)}\ d\,\omega \right) d\,u \tag{6}$$

The inner integral on the right-hand side is:

$$\int_{-\infty}^{\infty} e^{i\omega(u-v)}\ d\,\omega = 2\,\pi\,\delta\,(u-v) \tag{7}$$

where $\delta(\cdot)$ is the Dirac delta function.

Therefore:

$$\int_{-\infty}^{\infty} A_t(\omega)\, e^{i\omega(t-v)}\; d\,\omega = \int_{-\infty}^{\infty} h_t(u) \cdot 2\,\pi\,\delta\,(u-v)\; d\,u \tag{8}$$

$$= 2\,\pi\, h_t(v) \tag{9}$$

Solving for $h_t(v)$ and replacing v with u:

$$h_t(u) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} A_t(\omega)\, e^{i\omega(t-u)}\; d\,\omega \tag{10}$$

This gives the explicit definition of $h_t(u)$. □

Theorem 4. *[Alternative Process Representation] The process X_t can also be represented as:*

$$X_t = \int_{-\infty}^{\infty} h_t(u)\, X_S\,(t-u)\; d\,u \tag{11}$$

where $X_S(t)$ is a stationary process with power spectral density $S_{XX}(\omega)$.

Proof. Starting from the original spectral representation:

$$X_t = \int_{-\infty}^{\infty} e^{i\omega t}\, A_t(\omega)\; d\,Z(\omega) \tag{12}$$

We substitute the Fourier relationship:

$$X_t = \int_{-\infty}^{\infty} \frac{1}{e^{i\omega t}} \left(\int_{-\infty}^{\infty} h_t(u)\, e^{i\omega u}\; d\,u \right) e^{i\omega t}\; d\,Z(\omega) \tag{13}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_t(u)\, e^{i\omega u}\; d\,u\; d\,Z(\omega) \tag{14}$$

Interchanging the order of integration:

$$X_t = \int_{-\infty}^{\infty} h_t(u) \left(\int_{-\infty}^{\infty} e^{i\omega u}\; d\,Z(\omega) \right) d\,u \tag{15}$$

$$= \int_{-\infty}^{\infty} h_t(u) \left(\int_{-\infty}^{\infty} e^{i\omega(t-[t-u])}\; d\,Z(\omega) \right) d\,u \tag{16}$$

Define the stationary process:

$$X_S\,(t-u) = \int_{-\infty}^{\infty} e^{i\omega(t-u)}\; d\,Z(\omega) \tag{17}$$

Therefore:

$$X_t = \int_{-\infty}^{\infty} h_t(u)\, X_S\,(t-u)\; d\,u \tag{18}$$

This demonstrates that X_t can be represented as the output of a time-varying filter $h_t(u)$ applied to a stationary process $X_S(t)$. $\square$

1 Conclusion

The explicit definition of the time-varying filter $h_t(u)$ in terms of the gain function $A_t(\omega)$ is:

$$h_t(u) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} A_t(\omega)\, e^{i\omega\,(t-u)}\; d\,\omega \tag{19}$$

This relationship provides a critical link between the frequency-domain representation using the gain function and the time-domain representation using the time-varying filter. The presence of the factor $e^{i\omega\,(t-u)}$ in the integrand is essential and distinguishes this from a simple inverse Fourier transform.