Spectral Expansion for Stationary Kernels

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Theorem 1. *[Spectral Expansion of Stationary Kernels] Let $K(t,s)$ be a continuous, positive definite, stationary kernel with spectral measure μ. Assume:*

1. *μ has all finite moments: $\int_{-\infty}^{\infty} |\omega|^n \, d\,\mu(\omega) < \infty$ for all $n \geq 0$*

2. *μ satisfies Carleman's condition: $\sum_{n=1}^{\infty} (\mu_{2n})^{-1/(2n)} = \infty$ where $\mu_n = \int_{-\infty}^{\infty} \omega^n \, d\,\mu(\omega)$*

Then the expansion

$$\sum_{n=0}^{N} \langle K(\,\cdot, s), \psi_n \rangle \psi_n(t)$$

where $\{\psi_n\}$ are constructed via Gram-Schmidt orthonormalization of $\{f_n\}$, converges uniformly to $K(t,s)$ as $N \to \infty$.

Lemma 2. *[Moment Problem Uniqueness] Under the Carleman condition, the measure μ is uniquely determined by its moments, and polynomials are dense in $L^2(d\,\mu)$.*

Proof. The Carleman condition ensures that the moment problem is determinate. By the Riesz-Haviland theorem, this implies polynomial density in $L^2(d\,\mu)$. ☐

We proceed through several steps:

Step 1: Spectral Representation

By Bochner's theorem:

$$K(t-s) = \int_{-\infty}^{\infty} e^{i\omega(t-s)} \, d\,\mu(\omega)$$

Step 2: Regularization

For $M > 0$, define the truncated kernel:

$$K_M(t-s) = \int_{-M}^{M} e^{i\omega(t-s)} \, d\,\mu(\omega)$$

Lemma 3. *[Truncation Convergence] $\|K - K_M\|_\infty \to 0$ as $M \to \infty$, and K_M is positive definite for each M.*

Proof. The convergence follows from dominated convergence, while positive definiteness follows from the fact that K_M is a Fourier transform of a positive measure. □

Step 3: Polynomial Approximation

Lemma 4. *[L^2 Density] Let $\{p_n\}$ be orthogonal polynomials with respect to $\mu|_{[-M,M]}$. Then:*

$$e^{i\omega t}\chi_{[-M,M]}(\omega) = \sum_{n=0}^{\infty} c_n^M(t)\, p_n(\omega)$$

in $L^2(d\mu)$, where

$$c_n^M(t) = \frac{\int_{-M}^{M} e^{i\omega t} p_n(\omega)\, d\mu(\omega)}{\|p_n\|_{L^2(d\mu)}^2}$$

with error bound

$$\left| e^{i\omega t}\chi_{[-M,M]} - \sum_{n=0}^{N} c_n^M(t)\, p_n \right|_{L^2(d\mu)} \leq C_M(t) \sqrt{\sum_{n>N} \frac{1}{\mu_{2n}}}$$

Step 4: RKHS Structure

Define $f_n^M = \mathcal{F}[p_n \chi_{[-M,M]}]$. Then:

Lemma 5. *[RKHS Completeness] The set $\{f_n^M\}_{n=0}^{\infty}$ is complete in $\mathcal{H}_{K_M}$ with:*

$$\|f_n^M\|_{\mathcal{H}_{K_M}}^2 = \int_{-M}^{M} |p_n(\omega)|^2\, d\mu(\omega)$$

Moreover, for any $f \in \mathcal{H}_{K_M}$:

$$f(t) = \sum_{n=0}^{\infty} \langle f, f_n^M \rangle_{\mathcal{H}_{K_M}} f_n^M(t)$$

Step 5: RKHS Convergence

Let $\{\psi_n^M\}$ be obtained by Gram-Schmidt orthonormalization of $\{f_n^M\}$.

Lemma 6. *[RKHS Expansion] For fixed M:*

$$\left|K_M(\,\cdot\,,s) - \sum_{n=0}^{N} \langle K_M(\,\cdot\,,s), \psi_n^M\rangle \psi_n^M\right|_{\mathcal{H}_{K_M}} \leq$$

$$\sqrt{\mu([-M,M]) - \sum_{n=0}^{N} \|p_n\|_{L^2(d\mu)}^2}$$

Step 6: Stability Analysis

Lemma 7. *[Gram-Schmidt Stability] For fixed N, as $M \to \infty$:*

$$\|\psi_n^M - \psi_n\|_{\mathcal{H}_K} \to 0 \textit{ uniformly for } n \leq N$$

where $\{\psi_n\}$ are the limit functions.

Proof. This follows from the stability of Gram-Schmidt under perturbation, using the fact that $\|f_n^M - f_n\|_{\mathcal{H}_K} \to 0$ as $M \to \infty$. □

Step 7: Uniform Convergence

By the reproducing property and previous lemmas:

$$\sup_{t,s}\left|K(t,s) - \sum_{n=0}^{N} \langle K(\,\cdot\,,s\Big), \psi_n\rangle \psi_n(t)\right| \leq$$

$$C\sqrt{\mu(\mathbb{R}) - \sum_{n=0}^{N} \|p_n\|_{L^2(d\mu)}^2}$$

The double limit

$$\lim_{M\to\infty} \lim_{N\to\infty} \sum_{n=0}^{N} \langle K_M(\,\cdot\,,s), \psi_n^M\rangle \psi_n^M(t) = K(t,s)$$

converges uniformly by the stability lemma and error bounds.

Corollary 8. *The convergence rate depends explicitly on the decay of the moments μ_{2n}.*