Pre-envelopes of Nonstationary Bandpass Processes

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Abstract

Many of the useful properties of pre-envelopes of real waveforms are shown to hold when the notion of pre-envelope is applied to random processes which are not wide-sense stationary. In particular, if $x(t)$ represents a real random process which is not wide-sense stationary and $y(t)$ is its Hilbert transform, then $x(t)$ and $y(t)$ have the same autocovariance function and have zero crosscovariance at the same instant. The autocovariance function of the pre-envelope $z(t)$, given by $z(t) = x(t) + j\,y(t)$, is twice the pre-envelope of the autocovariance function of $x(t)$.

The notion of a time-dependent power density spectrum allows a simple interpretation of a bandpass random process which is not wide-sense stationary. The well-known form of the autocovariance function of a wide-sense stationary bandpass process carries over with simple changes to processes which are not wide-sense stationary.

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1 Introduction

As defined by Arens [arens1957] and Dugundji [dugundji1958], the *pre-envelope* of a real waveform $x(t)$ is the complex-valued function $z(t)$ whose real part is $x(t)$ and whose imaginary part is $y(t)$, the Hilbert transform of $x(t)$. The Hilbert transform is defined as:

$$y(t) = \mathcal{H}[x(t)] = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{x(s)}{t-s} \, d\,s \tag{1}$$

where P.V. denotes the Cauchy principal value of the integral.

The envelope of $x(t)$ is defined as $|z(t)|$. Dugundji has derived a number of interesting properties for the time auto- and cross-correlations of real waveforms. The same properties hold for auto- and crosscovariances of wide-sense stationary random processes by replacing time averages by ensemble averages. In this paper it is shown that similar results hold for nonstationary random processes. The particular interest lies in bandpass processes.

The fundamental motivation for studying pre-envelopes of nonstationary processes stems from their application in communication theory, radar signal processing, and general signal analysis where the assumption of stationarity is often violated in practice.

2 Mathematical Preliminaries

Before proceeding with the main results, it is necessary to establish the mathematical framework and conventions that will be used throughout this work.

Definition 1. *[Hilbert Transform]For a function $f(t) \in L^2(\mathbb{R})$, the Hilbert transform is defined as*

$$\mathcal{H}[f(t)] = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(s)}{t-s} \, d\,s \tag{2}$$

where the integral exists in the principal value sense.

Alternative forms of the Hilbert transform that will prove useful are:

$$y(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x\,(t+s)}{s} \, d\,s = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x\,(t-s)}{s} \, d\,s \tag{3}$$

Definition 2. *[Pre-envelope]The pre-envelope of a real-valued function $x(t)$ is the complex-valued function*

$$z(t) = x(t) + j\,\mathcal{H}[x(t)] = x(t) + j\,y(t) \tag{4}$$

3 Covariance Functions and Their Properties

Definition 3. *[Autocovariance Function]The autocovariance function of a complex-valued random process $z(t)$ is given by*

$$R_z(t,\tau) = \mathbb{E}\left[z(t)\, z^*\left(t-\tau\right)\right] \tag{5}$$

where z^* denotes the complex conjugate and $\mathbb{E}[\cdot]$ denotes the expectation operator.

Remark 4. In the special case of a wide-sense stationary process, the autocovariance function depends only on τ, the time displacement [doob1953].

Definition 5. *[Crosscovariance Function]The crosscovariance function of two complex-valued random processes $z(t)$ and $w(t)$ is given by*

$$R_{zw}(t,\tau) = \mathbb{E}\left[z(t)\, w^*\left(t-\tau\right)\right] \tag{6}$$

Definition 6. *[τ-Hilbert Transform]The τ-Hilbert transform $\hat{R}(t,\tau)$ of a covariance function $R(t,\tau)$ is the Hilbert transform with the integration performed over τ:*

$$\hat{R}(t,\tau) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{R(t,s)}{\tau - s}\, ds = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R(t,\tau+s)}{s}\, ds \tag{7}$$

3.1 Fundamental Theorems

Theorem 7. *[Cross-covariance Hilbert Relationship]The crosscovariance function of $x(t)$ and $y(t)$ is the negative of the τ-Hilbert transform of the autocovariance function of $x(t)$:*

$$R_{xy}(t,\tau) = -\hat{R}_x(t,\tau) \tag{8}$$

Proof. Starting from the definition of the crosscovariance function:

$$R_{xy}(t,\tau) = \mathbb{E}\left[x(t)\, y^*\left(t-\tau\right)\right] \tag{9}$$

$$= \mathbb{E}\left[x(t) \cdot \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x\left(t-\tau-s\right)}{s}\, ds\right] \tag{10}$$

By the linearity of expectation and assuming the interchange of expectation and integration is valid (which holds under suitable regularity conditions):

$$R_{xy}(t,\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathbb{E}\left[x(t)\, x\left(t-\tau-s\right)\right]}{s}\, ds \tag{11}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R_x(t,\tau+s)}{s}\, ds \tag{12}$$

Making the substitution $u = \tau + s$, so $s = u - \tau$ and $d\,s = d\,u$:

$$R_{xy}(t,\tau) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R_x(t,u)}{u-\tau}\,d\,u \tag{13}$$
$$= -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R_x(t,u)}{\tau-u}\,d\,u \tag{14}$$
$$= -\hat{R}_x(t,\tau) \tag{15}$$

□

Theorem 8. *[Hilbert Transform of Cross-covariance]The τ-Hilbert transform of the crosscovariance function satisfies:*

$$\hat{R}_{xy}(t,\tau) = R_x(t,\tau) \tag{16}$$

Proof. From Theorem 7, we have $R_{xy}(t,\tau) = -\hat{R}_x(t,\tau)$. Taking the τ-Hilbert transform of both sides:

$$\hat{R}_{xy}(t,\tau) = -\hat{\hat{R}}_x(t,\tau) \tag{17}$$

Using the fundamental property of Hilbert transforms that $\mathcal{H}[\mathcal{H}[f(t)]] = -f(t)$ for suitable functions f:

$$\hat{R}_{xy}(t,\tau) = -(-R_x(t,\tau)) = R_x(t,\tau) \tag{18}$$

□

Theorem 9. *[Double Hilbert Transform]The τ-Hilbert transform applied twice yields:*

$$\hat{\hat{R}}_x(t,\tau) = -R_x(t,\tau) \tag{19}$$

Proof. This follows directly from the fundamental property of Hilbert transforms. For a function $f(\tau)$ that satisfies appropriate regularity conditions, $\mathcal{H}[\mathcal{H}[f(\tau)]] = -f(\tau)$. Applied to the covariance function $R_x(t,\tau)$ viewed as a function of τ, this gives the desired result. □

Corollary 10. *[Symmetry Properties]The following symmetry relationships hold:*

$$R_{yx}(t,\tau) = -R_{xy}(t,-\tau) \tag{20}$$
$$R_y(t,\tau) = R_x(t,\tau) \tag{21}$$

Proof. For equation (20):

$$R_{yx}(t,\tau) = \mathbb{E}\left[y(t)\, x\,(t-\tau)\right] \tag{22}$$
$$= \mathbb{E}\left[x\,(t-\tau)\, y(t)\right] \tag{23}$$
$$= \mathbb{E}\left[x\,(t-\tau)\, y^*(t)\right]^* \tag{24}$$
$$= R_{xy}^*(t,-\tau) \tag{25}$$

Since $x(t)$ and $y(t)$ are real-valued, $R_{xy}(t,\tau)$ is real, so $R_{xy}^*(t,-\tau) = R_{xy}(t,-\tau)$. However, from Theorem 7, $R_{xy}(t,\tau) = -\hat{R}_x(t,\tau)$, and using properties of the Hilbert transform, one can show that $R_{yx}(t,\tau) = -R_{xy}(t,-\tau)$.

For equation (21), this follows from the fact that $y(t) = \mathcal{H}[x(t)]$ and the isometric property of the Hilbert transform on $L^2(\mathbb{R})$. □

Theorem 11. *[Pre-envelope Autocovariance]The autocovariance function of the pre-envelope $z(t) = x(t) + j\,y(t)$ is given by:*

$$R_z(t,\tau) = 2\left[R_x(t,\tau) + j\,\hat{R}_x(t,\tau)\right] \tag{26}$$

Proof. From the definition of $z(t)$ and the autocovariance function:

$$R_z(t,\tau) = \mathbb{E}\left[z(t)\, z^*\,(t-\tau)\right] \tag{27}$$
$$= \mathbb{E}\left[(x(t) + j\,y(t))\,(x\,(t-\tau) - j\,y\,(t-\tau))\right] \tag{28}$$
$$= \mathbb{E}\left[x(t)\, x\,(t-\tau)\right] + j\,\mathbb{E}\left[y(t)\, x\,(t-\tau)\right] \tag{29}$$
$$- j\,\mathbb{E}\left[x(t)\, y\,(t-\tau)\right] + \mathbb{E}\left[y(t)\, y\,(t-\tau)\right] \tag{30}$$

Using the results from previous theorems:

$$R_z(t,\tau) = R_x(t,\tau) + j\,R_{yx}(t,\tau) - j\,R_{xy}(t,\tau) + R_y(t,\tau) \tag{31}$$
$$= R_x(t,\tau) + j\,(-R_{xy}(t,-\tau)) - j\,R_{xy}(t,\tau) + R_x(t,\tau) \tag{32}$$
$$= 2\,R_x(t,\tau) + j\,(R_{xy}(t,-\tau) - R_{xy}(t,\tau)) \tag{33}$$

From Theorem 7, $R_{xy}(t,\tau) = -\hat{R}_x(t,\tau)$, and using properties of the Hilbert transform:

$$R_z(t,\tau) = 2\,R_x(t,\tau) + j\,(-\hat{R}_x(t,-\tau) - (-\hat{R}_x(t,\tau))) \tag{34}$$
$$= 2\,R_x(t,\tau) + j\,(\hat{R}_x(t,\tau) - \hat{R}_x(t,-\tau)) \tag{35}$$
$$= 2\,R_x(t,\tau) + 2\,j\,\hat{R}_x(t,\tau) \tag{36}$$
$$= 2\left[R_x(t,\tau) + j\,\hat{R}_x(t,\tau)\right] \tag{37}$$

where the last step uses the odd symmetry property of the Hilbert transform. □

4 Time-Dependent Spectral Densities

Following Lampard [lampard1954], the concept of time-dependent power spectral density provides a framework for analyzing nonstationary processes in the frequency domain.

Definition 12. *[Time-Dependent Power Spectral Density]The time-dependent power density spectrum $S_x(\omega, t)$ of the random process $x(t)$ is defined by*

$$S_x(\omega, t) = \int_{-\infty}^{\infty} R_x(t, \tau)\, e^{-j\omega\tau}\, d\,\tau \tag{38}$$

Theorem 13. *[Fourier Transform of τ-Hilbert Transform]The Fourier transform (with respect to τ) of the τ-Hilbert transform of $R_x(t, \tau)$ is:*

$$\mathcal{F}_\tau[\hat{R}_x(t, \tau)] = -j \cdot sgn(\omega) \cdot S_x(\omega, t) \tag{39}$$

where $sgn(\omega)$ is the signum function.

Proof. This follows from the fundamental property of Hilbert transforms in the frequency domain. The Hilbert transform corresponds to multiplication by $-j \cdot \mathrm{sgn}(\omega)$ in the frequency domain. Therefore:

$$\begin{aligned} \mathcal{F}_\tau[\hat{R}_x(t, \tau)] &= (-j \cdot \mathrm{sgn}(\omega)) \cdot \mathcal{F}_\tau[R_x(t, \tau)] && (40) \\ &= -j \cdot \mathrm{sgn}(\omega) \cdot S_x(\omega, t) && (41) \end{aligned}$$

□

Theorem 14. *[Pre-envelope Spectral Density]The time-dependent power density spectrum $S_z(\omega, t)$ of the pre-envelope $z(t)$ is given by:*

$$S_z(\omega, t) = \begin{cases} 4\, S_x(\omega, t), & \omega > 0 \\ 2\, S_x(0, t), & \omega = 0 \\ 0, & \omega < 0 \end{cases} \tag{42}$$

Proof. From Theorem 11 and taking the Fourier transform:

$$\begin{aligned} S_z(\omega, t) &= \mathcal{F}_\tau[R_z(t, \tau)] && (43) \\ &= 2\, \mathcal{F}_\tau\, [R_x(t, \tau) + j\, \hat{R}_x(t, \tau)] && (44) \\ &= 2\, [S_x(\omega, t) + j\, (-j \cdot \mathrm{sgn}(\omega) \cdot S_x(\omega, t))] && (45) \\ &= 2\, S_x(\omega, t)\, [1 + \mathrm{sgn}(\omega)] && (46) \end{aligned}$$

Evaluating the signum function:

$$S_z(\omega,t)=\begin{cases}2\,S_x(\omega,t)\,(1+1)=4\,S_x(\omega,t), & \omega>0\\ 2\,S_x(\omega,t)\,(1+0)=2\,S_x(\omega,t), & \omega=0\\ 2\,S_x(\omega,t)\,(1-1)=0, & \omega<0\end{cases} \tag{47}$$

□

5 Physical Interpretation and Instantaneous Spectra

Page's [page1952, kharkevich1960] concept of the instantaneous spectrum provides a physical interpretation of $S_x(\omega,t)$. For any sample function of the random process, one may conceive of the energy density as being distributed in time and frequency such that the energy expended in the time interval (t_1,t_2) due to frequency components between f_1 and f_2 is given by:

$$E_{f_1,f_2}^{t_1,t_2}=\int_{t_1}^{t_2}\int_{f_1}^{f_2} p(t,f)\;d\,f\;d\,t \tag{48}$$

where $p(t,f)$ is the instantaneous energy spectrum of the function.

Definition 15. *[Ensemble Average Energy Spectrum] The ensemble average of the instantaneous energy spectrum $p(t,f)$ is the time-dependent power density spectrum:*

$$\mathbb{E}[p(t,\omega)]=S_x(\omega,t) \tag{49}$$

where $\omega=2\,\pi\,f$.

6 Bandpass Random Processes

$S_x(\omega,t)$ provides a convenient way of describing bandpass random processes even when the process is not wide-sense stationary.

Definition 16. *[Nonstationary Bandpass Process] A bandpass random process is one whose time-dependent power density spectrum $S_x(\omega,t)$ has bandpass character for all t.*

For analysis, it is convenient to choose a reference frequency f_0 (with $\omega_c=2\,\pi\,f_0$) in the vicinity of the band to enable the representation:

$$S_x(\omega,t)=S_c(\omega-\omega_c,t)+S_c(-\omega-\omega_c,t) \tag{50}$$

indicating that $S_x(\omega, t)$ is an even function of ω. Here, $S_c(\omega, t)$ has low-frequency character when ω_c is in the vicinity of the band.

Remark 17. There is no requirement for symmetry of $S_x(\omega, t)$ about ω_c, and ω_c need not be within the actual signal band.

Theorem 18. *[Bandpass Autocovariance Structure]For a bandpass process with spectrum given by equation (50), the autocovariance function can be written as:*

$$R_x(t,\tau) = R_c(t,\tau)\cos(\omega_c\tau) + R_s(t,\tau)\sin(\omega_c\tau) \tag{51}$$

where:

$$R_c(t,\tau) = \frac{1}{\pi}\int_0^\infty S_x(\omega,t)\cos((\omega-\omega_c)\tau)\,d\omega \tag{52}$$

$$R_s(t,\tau) = \frac{1}{\pi}\int_0^\infty S_x(\omega,t)\sin((\omega-\omega_c)\tau)\,d\omega \tag{53}$$

Proof. Applying the inverse Fourier transform to equation (38):

$$R_x(t,\tau) = \frac{1}{2\pi}\int_{-\infty}^\infty S_x(\omega,t)\,e^{j\omega\tau}\,d\omega \tag{54}$$

Substituting equation (50):

$$R_x(t,\tau) = \frac{1}{2\pi}\int_{-\infty}^\infty [S_c(\omega-\omega_c,t) + S_c(-\omega-\omega_c,t)]\,e^{j\omega\tau}\,d\omega \tag{55}$$

Making appropriate substitutions and using the fact that $S_x(\omega, t)$ is real and even:

$$R_x(t,\tau) = \frac{1}{2\pi}\int_{-\infty}^\infty S_c(\nu,t)\,[e^{j(\nu+\omega_c)\tau} + e^{j(-\nu+\omega_c)\tau}]\,d\nu \tag{56}$$

$$= \frac{e^{j\omega_c\tau}}{2\pi}\int_{-\infty}^\infty S_c(\nu,t)\,e^{j\nu\tau}\,d\nu + \frac{e^{-j\omega_c\tau}}{2\pi}\int_{-\infty}^\infty S_c(\nu,t)\,e^{-j\nu\tau}\,d\nu \tag{57}$$

Since $S_c(\nu, t)$ is real and has appropriate symmetry properties, this reduces to:

$$R_x(t,\tau) = \mathrm{Re}\left[\frac{e^{j\omega_c\tau}}{\pi}\int_0^\infty S_c(\nu,t)\,e^{j\nu\tau}\,d\nu\right] \tag{58}$$

$$= R_c(t,\tau)\cos(\omega_c\tau) + R_s(t,\tau)\sin(\omega_c\tau) \tag{59}$$

□

6.1 Quadrature Representation

An alternative approach to deriving the autocovariance function is to represent $x(t)$ in quadrature form:

$$x(t) = x_c(t)\cos(\omega_c t) + x_s(t)\sin(\omega_c t) \tag{60}$$

Theorem 19. *[Quadrature Component Properties]For the representation in equation (60) to yield the same autocovariance as equation (51), the following conditions must hold:*

$$R_c(t,\tau) = \mathbb{E}[x_c(t)\,x_c(t-\tau)] = \mathbb{E}[x_s(t)\,x_s(t-\tau)] \tag{61}$$
$$R_s(t,\tau) = \mathbb{E}[x_c(t)\,x_s(t-\tau)] = -\mathbb{E}[x_s(t)\,x_c(t-\tau)] \tag{62}$$

Proof. Computing $R_x(t,\tau) = \mathbb{E}[x(t)\,x(t-\tau)]$ using equation (60):

$$2\,R_x(t,\tau) = \tag{63}$$
$$\mathbb{E}[x_c(t)\,x_c(t-\tau)][\cos(\omega_c\tau) + \cos(\omega_c(2t-\tau))] \tag{64}$$
$$+\mathbb{E}[x_s(t)\,x_s(t-\tau)][\cos(\omega_c\tau) - \cos(\omega_c(2t-\tau))] \tag{65}$$
$$+\mathbb{E}[x_c(t)\,x_s(t-\tau)][\sin(\omega_c\tau) + \sin(\omega_c(2t-\tau))] \tag{66}$$
$$-\mathbb{E}[x_s(t)\,x_c(t-\tau)][\sin(\omega_c\tau) - \sin(\omega_c(2t-\tau))] \tag{67}$$

For this to match equation (51), the coefficients of $\cos(\omega_c(2t-\tau))$ and $\sin(\omega_c(2t-\tau))$ must vanish, leading to the stated conditions. □

7 Hilbert Transform of Bandpass Autocovariance

Theorem 20. *[Hilbert Transform of Bandpass Function]For a function $R_x(t,\tau)$ of the form in equation (51), where R_c and R_s satisfy a mild bandwidth requirement, the τ-Hilbert transform is:*

$$\hat{R}_x(t,\tau) = R_c(t,\tau)\sin(\omega_c\tau) - R_s(t,\tau)\cos(\omega_c\tau) \tag{68}$$

Proof. The bandwidth requirement ensures that when the Fourier transforms of R_c and R_s are translated by ω_c, the results are zero for $\omega < 0$. Under this condition, the Hilbert transform of a bandpass function follows standard results [urkowitz1962].

For a function of the form $f(\tau) = A(\tau)\cos(\omega_c\tau) + B(\tau)\sin(\omega_c\tau)$ where A and B have sufficiently narrow bandwidth, the Hilbert transform is:

$$\mathcal{H}[f(\tau)] = A(\tau)\sin(\omega_c\tau) - B(\tau)\cos(\omega_c\tau) \tag{69}$$

Applying this to equation (51) yields the result. □

Corollary 21. *[Pre-envelope of Bandpass Process]Combining equations (51) and (68), the autocovariance function of the pre-envelope is:*

$$R_z(t,\tau) = 2\,[R_c(t,\tau) - j\,R_s(t,\tau)]\,e^{j\omega_c\tau} \tag{70}$$

Theorem 22. *[Direct Bandpass Pre-envelope]If $x_c(t)$ and $x_s(t)$ satisfy the bandwidth requirement, then:*

$$y(t) = x_c(t)\sin(\omega_c t) - x_s(t)\cos(\omega_c t) \tag{71}$$
$$z(t) = [x_c(t) - j\,x_s(t)]\,e^{j\omega_c t} \tag{72}$$

Proof. This follows directly from applying the Hilbert transform to equation (60) and using the properties established in Theorem 20. □

8 Zero Cross-covariance Property

Theorem 23. *[Zero Cross-covariance at Origin]The real part $x(t)$ and imaginary part $y(t)$ of the pre-envelope have zero cross-covariance at the same time instant:*

$$R_{xy}(t,0) = 0 \tag{73}$$

Proof. From Theorem 7, $R_{xy}(t,0) = -\hat{R}_x(t,0)$.

From equation (68) with $\tau = 0$:

$$\hat{R}_x(t,0) = R_c(t,0)\sin(0) - R_s(t,0)\cos(0) \tag{74}$$
$$= 0 - R_s(t,0) \tag{75}$$
$$= -R_s(t,0) \tag{76}$$

From equation (53), with $\tau = 0$:

$$R_s(t,0) = \frac{1}{\pi}\int_0^\infty S_x(\omega,t)\sin((\omega-\omega_c)\cdot 0)\,d\omega \tag{77}$$
$$= \frac{1}{\pi}\int_0^\infty S_x(\omega,t)\sin(0)\,d\omega \tag{78}$$
$$= 0 \tag{79}$$

Therefore, $R_{xy}(t,0) = -(-0) = 0$. □

Corollary 24. *[Quadrature Component Independence]At the same time instant, the quadrature components are uncorrelated:*

$$\mathbb{E}\left[x_c(t)\,x_s(t)\right]=0 \tag{80}$$

Proof. This follows directly from equation (62) with $\tau=0$ and the result of Theorem 23. □

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