Gaussian Processes Generated By Monotonically Modulated Stationary Gaussian Process Kernels

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Definition 1

Let $\mathcal{F}$ denote the class of functions $f\colon \mathbb{R} \to \mathbb{R}$ which are:

1. *piecewise continuous with piecewise continuous first derivative,*
2. *monotonically increasing*

$$f(t) < f(s) \forall -\infty \leqslant t < s \leqslant \infty \tag{1}$$

3. *and have a finite limiting derivative at infinity*

$$\lim_{t \to \infty} \dot{f}(t) < \infty \tag{2}$$

Theorem 2

(Eigenfunctions) *For any stationary kernel $K(t, s) = K(|t - s|)$, the eigenfunctions of the modulated kernel*

$$K_f(s, t) = K(|f(t) - f(s)|) \tag{3}$$

take the form:

$$\phi_n(t) = \psi_n(f(t)) \sqrt{\dot{f}(t)} \tag{4}$$

where where $f \in \mathcal{F}$ and ψ_n are the normalized eigenfunctions of the original unmodulated kernel $K(|t - s|)$.

Proof. The eigenfunction equation for the modulated kernel is:

$$\int_{-\infty}^{\infty} K(|f(t)-f(s)|)\,\phi_n(s)\,ds = \lambda_n\,\phi_n(t) \tag{5}$$

Under the change of variables $u = f(s)$, $v = f(t)$:

$$\int_{-\infty}^{\infty} K(|v-u|)\frac{\phi_n(f^{-1}(u))}{\dot{f}(f^{-1}(u))}\,du = \lambda_n\,\phi_n(f^{-1}(v)) \tag{6}$$

Let

$$\psi_n(u) = \frac{\phi_n(f^{-1}(u))}{\sqrt{\dot{f}(f^{-1}(u))}} \tag{7}$$

Then:

$$\int_{-\infty}^{\infty} K(|v-u|)\,\psi_n(u)\,du = \lambda_n\,\psi_n(v) \tag{8}$$

This is precisely the eigenfunction equation for the original kernel $K(|t-s|)$. Therefore, if ψ_n are the eigenfunctions of the original kernel, then

$$\phi_n(t) = \psi_n(f(t))\sqrt{\dot{f}(t)} \tag{9}$$

are the eigenfunctions of the modulated kernel. □

Theorem 3

(Normalization) *If ψ_n are normalized eigenfunctions of the original kernel, then $\phi_n(t) = \psi_n(f(t))\sqrt{\dot{f}(t)}$ are automatically normalized eigenfunctions of the modulated kernel, requiring no additional normalization constants.*

Proof. For normalized ψ_n:

$$\int_{-\infty}^{\infty} |\phi_n(t)|^2 dt = \int_{-\infty}^{\infty} |\psi_n(f(t))|^2 \dot{f}(t)\,dt \tag{10}$$

Under the change of variables $u = f(t)$:

$$\int_{-\infty}^{\infty} |\psi_n(u)|^2 \, d\,u = 1 \tag{11}$$

Therefore the ϕ_n are already normalized without additional constants. □

Corollary 4

(Eigenvalue Invariance) *The eigenvalues $\{\lambda_n\}$ of the modulated kernel K_f are identical to those of the original kernel K.*

Remark 5. This result demonstrates that monotonic modulation preserves the spectral structure of any stationary kernel through composition with the modulation function. The transformation operator

$$(T\phi)(t) = \sqrt{\dot{f}(t)} \, \phi(f(t)) \tag{12}$$

provides an explicit isometry between the original and modulated kernel Hilbert spaces, explaining why no additional normalization constants are needed.