Spectral Recovery and Pre-Envelope Theory: Formal Theorems

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1 Spectral Recovery Theory

Theorem 1. *[Spectral Recovery for Real Stationary Processes]Let $X(t)$ be a real-valued, zero-mean, stationary Gaussian process with spectral representation:*

$$X(t) = \int_0^{\infty} \cos(\lambda t)\, dU(\lambda) + \sin(\lambda t)\, dV(\lambda) \tag{1}$$

Then the orthogonal random measures $U(\lambda)$ and $V(\lambda)$ are recovered from the sample path by:

$$U(\lambda) = \frac{2}{\pi} \int_0^{\infty} X(t) \cos(\lambda t)\, dt \tag{2}$$

$$V(\lambda) = \frac{2}{\pi} \int_0^{\infty} X(t) \sin(\lambda t)\, dt \tag{3}$$

Proof. Apply the orthogonality relations of the trigonometric system. For $U(\lambda)$:

$$\int_0^{\infty} X(t) \cos(\mu t)\, dt = \int_0^{\infty} \left[\int_0^{\infty} \cos(\lambda t)\, dU(\lambda) + \sin(\lambda t)\, dV(\lambda) \right] \cos(\mu t)\, dt \tag{4}$$

Interchanging integration order:

$$= \int_0^{\infty} dU(\lambda) \int_0^{\infty} \cos(\lambda t) \cos(\mu t)\, dt + \int_0^{\infty} dV(\lambda) \int_0^{\infty} \sin(\lambda t) \cos(\mu t)\, dt \tag{5}$$

Using orthogonality:

$$\int_0^\infty \cos(\lambda t)\cos(\mu t)\,dt = \frac{\pi}{2}\delta(\lambda-\mu) \tag{6}$$

$$\int_0^\infty \sin(\lambda t)\cos(\mu t)\,dt = 0 \tag{7}$$

Therefore:

$$\int_0^\infty X(t)\cos(\mu t)\,dt = \frac{\pi}{2}U(\mu) \tag{8}$$

Hence: $U(\mu) = \frac{2}{\pi}\int_0^\infty X(t)\cos(\mu t)\,dt$. The proof for $V(\lambda)$ is analogous. □

Theorem 2. *[Analytic Signal Representation]For any real-valued stationary process $X(t)$, the analytic signal $Z(t)$ is defined as:*

$$Z(t) = X(t) + i\,\hat{X}(t) \tag{9}$$

where the quadrature process is:

$$\hat{X}(t) = \int_0^\infty \sin(\lambda t)\;dU(\lambda) - \cos(\lambda t)\;dV(\lambda) \tag{10}$$

Then $Z(t)$ admits the complex exponential representation:

$$Z(t) = \int_0^\infty e^{i\lambda t}\,d\zeta(\lambda) \tag{11}$$

where $d\zeta(\lambda) = dU(\lambda) - i\;dV(\lambda)$ is the pre-envelope spectral measure.

Proof.

$$Z(t) = X(t) + i\,\hat{X}(t) \tag{12}$$

$$\begin{aligned} &= \int_0^\infty [\cos(\lambda t)\;dU(\lambda) + \sin(\lambda t)\;dV(\lambda)] \\ &\quad + i\int_0^\infty [\sin(\lambda t)\;dU(\lambda) - \cos(\lambda t)\;dV(\lambda)] \end{aligned} \tag{13}$$

Collecting terms:

$$\begin{aligned} &= \int_0^\infty [(\cos(\lambda t) + i\sin(\lambda t))\;dU(\lambda) + (\sin(\lambda t) - i\cos(\lambda t))\;dV(\lambda)] \\ &= \int_0^\infty [e^{i\lambda t}\;dU(\lambda) - i\,e^{i\lambda t}\;dV(\lambda)] \\ &= \int_0^\infty e^{i\lambda t}[dU(\lambda) - i\;dV(\lambda)] \end{aligned} \tag{14}$$

Setting $d\zeta(\lambda) = dU(\lambda) - i\ dV(\lambda)$ completes the proof. $\square$

Theorem 3. *[Pre-Envelope Spectral Measure Properties]The pre-envelope spectral measure $\zeta(\lambda)$ satisfies:*

1. $E[d\zeta(\lambda)] = 0$
2. $E[|d\zeta(\lambda)|^2] = dG(\lambda)$ *where $G(\lambda)$ is the spectral distribution function*
3. $E[d\zeta(\lambda_1)\overline{d\zeta(\lambda_2)}] = 0$ *for* $\lambda_1 \neq \lambda_2$

Proof. 1.

$$E[d\zeta(\lambda)] = E[dU(\lambda) - i\ dV(\lambda)] = 0 - i \cdot 0 = 0 \tag{15}$$

2.

$$\begin{aligned} E[|d\zeta(\lambda)|^2] =& E[(dU(\lambda) - i\ dV(\lambda))(\overline{dU(\lambda) - i\ dV(\lambda)})] \\ =& E[(dU(\lambda) - i\ dV(\lambda))(dU(\lambda) + i\ dV(\lambda))] \\ =& E[|dU(\lambda)|^2] + E[|dV(\lambda)|^2] = dG(\lambda) \end{aligned} \tag{16}$$

Since

$$E[|dU(\lambda)|^2] = E[|dV(\lambda)|^2] = \frac{dG(\lambda)}{2} \tag{17}$$

by symmetry.

3. Orthogonality follows from the orthogonality of U and V increments.

$\square$

2 Envelope Theory

Theorem 4. *[Envelope as Absolute Value of Analytic Signal]The envelope $R(t)$ of the real process $X(t)$ is given by:*

$$R(t) = |Z(t)| = \sqrt{X^2(t) + \hat{X}^2(t)} \tag{18}$$

where $Z(t)$ is the analytic signal.

Proof. By definition:

$$R(t) = |Z(t)| = |X(t) + i\hat{X}(t)| = \sqrt{X^2(t) + \hat{X}^2(t)} \tag{19}$$

This establishes that the envelope is the modulus of the pre-envelope process $Z(t)$. □

Theorem 5. *[Polar Representation of Complex Process]The analytic signal $Z(t)$ admits the polar representation:*

$$Z(t) = R(t)\, e^{i\Theta(t)} \tag{20}$$

where:

1. $R(t) = |Z(t)| = \sqrt{X^2(t) + \hat{X}^2(t)}$ *is the envelope*

2. $\Theta(t) = \arctan\left(\frac{\hat{X}(t)}{X(t)}\right)$ *is the instantaneous phase*

Proof. For any complex number $z = a + i\, b$, the polar form is

$$z = |z|\, e^{i \arg(z)} \tag{21}$$

where:

$$|z| = \sqrt{a^2 + b^2} \tag{22}$$

$$\arg(z) = \arctan\left(\frac{b}{a}\right) \tag{23}$$

Applying this to $Z(t) = X(t) + i\, \hat{X}(t)$:

$$R(t) = |Z(t)| = \sqrt{X^2(t) + \hat{X}^2(t)} \tag{24}$$

$$\Theta(t) = \arg(Z(t)) = \arctan\left(\frac{\hat{X}(t)}{X(t)}\right) \tag{25}$$

Therefore:

$$Z(t) = R(t)\, e^{i\Theta(t)} \tag{26}$$

□

Theorem 6. *[Instantaneous Frequency]The instantaneous frequency $\omega(t)$ of the process $X(t)$ is defined as:*

$$\omega(t) = \frac{d\,\Theta(t)}{d\,t} \tag{27}$$

where $\Theta(t)$ is the instantaneous phase from Theorem 5.

Proof. By definition of instantaneous frequency as the time derivative of phase:

$$\omega(t) = \frac{d}{dt}\left[\arctan\left(\frac{\hat{X}(t)}{X(t)}\right)\right] \tag{28}$$

Using the chain rule:

$$\omega(t) = \frac{\frac{d}{dt}\left(\frac{\hat{X}(t)}{X(t)}\right)}{1 + \left(\frac{\hat{X}(t)}{X(t)}\right)^2} \tag{29}$$

$$= \frac{X^2(t)}{X^2(t) + \hat{X}^2(t)} \cdot \frac{\hat{X}'(t)\,X(t) - \hat{X}(t)\,X'(t)}{X^2(t)} \tag{30}$$

$$= \frac{\hat{X}'(t)\,X(t) - \hat{X}(t)\,X'(t)}{X^2(t) + \hat{X}^2(t)} = \frac{\hat{X}'(t)\,X(t) - \hat{X}(t)\,X'(t)}{R^2(t)} \tag{31}$$

This expresses the instantaneous frequency in terms of the original process and its quadrature. ☐

3 Spectral Inversion

Corollary 7. *[Spectral Recovery from Analytic Signal]The pre-envelope spectral measure can be recovered from the analytic signal by:*

$$\zeta(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(t)\,e^{-i\lambda t}\,dt \tag{32}$$

Proof. This follows directly from the inverse Fourier transform applied to the complex exponential representation in Theorem 2. ☐