Spectral Theory of Oscillatory Non-Stationary Processes and RKHS Framework

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1 Introduction

This document synthesizes the spectral analysis of oscillatory non-stationary processes within the framework established by Priestley [Priestley1981] and the theory of reproducing kernel Hilbert spaces (RKHS). The deterministic and stochastic duality of spectral measures, their representation via integral operators, and the foundational operators that connect paths and measures is explored. The focus is on continuous-time processes, their spectral representations, and the convergence of the limits involved in their definitions.

2 Preliminaries and Definitions

2.1 Oscillatory Non-Stationary Processes

Definition 1. (Oscillatory Non-Stationary Process) *A real-valued stochastic process* $\{X(t): t \in \mathbb{R}\}$ *is called* oscillatory non-stationary *in the sense of Priestley [Priestley1981] if there exists a family of amplitude functions* $A_t(\omega)$*, slowly varying in* t*, and a family of phase functions* $\theta(\omega)$ *such that*

$$X(t) = \int_{-\infty}^{\infty} A_t(\omega)\, e^{i(\theta(\omega)t + \phi(\omega,t))}\, d\,Z(\omega) \tag{1}$$

where $d\,Z(\omega)$ *is an orthogonal increment process, and the spectral representation extends the classical stationary case to a time-varying spectral density.*

2.2 Evolutionary Spectrum

Definition 2. (Evolutionary Spectrum) *The* evolutionary spectrum $h_t(\omega)$ *of an oscillatory non-stationary process is defined as*

$$h_t(\omega) = |A_t(\omega)|^2\; d\,\mu(\omega) \tag{2}$$

where μ *is a spectral measure, and* $A_t(\omega)$ *encodes the time-varying spectral gain.*

3 Spectral Representation and Measures

Definition 3. *A process is said to be analytic when its sample paths are infinitely differentiable and belong to a suitable RKHS associated with the kernel corresponding to the processes spectral measure.*

3.1 Spectral Measure and Path Correspondence

Theorem 4. (Path-Measure Isomorphism) *Let* $\{X(t)\}$ *be an analytic oscillatory process with spectral measure* $d\,Z(\omega)$*. Then, conditioned on the sample path* $X(\cdot)$*, the associated spectral measure (or random measure) is a deterministic functional of* X*, given explicitly by*

$$d\,Z(\omega) = \lim_{N\to\infty} \sum_{k=1}^{N} \left\langle X(\,\cdot), e^{i\omega_k \cdot} \right\rangle \delta\,(\omega - \omega_k) \tag{3}$$

where the limit is in the appropriate Hilbert space norm, and the measure is uniquely determined by the path.

Proof. The proof follows from the spectral theorem for self-adjoint operators and the isometric isomorphism between the path space and the spectral measure space. Since the process is analytic, the spectral measure can be reconstructed exactly via the integral representation:

$$X(t) = \int_{-\infty}^{\infty} A_t(\omega)\, e^{i\omega t}\, d\, Z(\omega)$$

Conditioning on $X(\cdot)$ fixes the spectral measure as a functional of the path, making it deterministic. ☐

3.2 Limit and Approximation Lemmas

Lemma 5. (Discretization Limit) *Let $\{\omega_k\}$ be a sequence of frequency bins with mesh size tending to zero. Then, for a fixed analytic path $X(t)$,*

$$\lim_{|\Delta\omega| \to 0} \sum_k \left\langle X(\,\cdot\,), e^{i\omega_k \cdot} \right\rangle \delta\,(\omega - \omega_k) = d\, Z(\omega) \tag{4}$$

where the convergence is in the sense of the weak topology of measures.

Proof. This follows from the Riemann sum approximation of the integral representation of $X(t)$ and the properties of the spectral measure as a finite measure in the continuous spectrum. ☐

Lemma 6. (Spectral Approximation of Analytic Paths) *For an analytic path $X(t)$, the spectral measure can be approximated arbitrarily well by band-limited filters via the sinc kernel:*

$$X(t) \approx \int_{-\Omega}^{\Omega} A_t(\omega)\, e^{i\omega t}\, d\, Z(\omega) \tag{5}$$

with the approximation improving as the cutoff frequency $\Omega \to \infty$.

Proof. This is a consequence of the Paley–Wiener theorem and the fact that the Fourier transform of an analytic function with entire extension is supported on the entire real line, allowing the approximation by band-limited projections. ☐

4 Reproducing Kernel Hilbert Space Framework

4.1 Kernel and RKHS

Definition 7. (Reproducing Kernel) *A kernel* $K\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *is called a* reproducing kernel *if it is positive definite and for the Hilbert space*

$$\mathcal{H}_K = \overline{span}\{K(\cdot, t)\colon t\in\mathbb{R}\} \tag{6}$$

the reproducing property holds:

$$f(t) = \langle f, K(\cdot, t)\rangle_{\mathcal{H}_K} \quad \forall f\in\mathcal{H}_K \tag{7}$$

4.2 Analysis and Synthesis Operators

Theorem 8. (Operators in RKHS) *Let* $\{f_i\}$ *be a frame in* $\mathcal{H}_K$*, with analysis operator* $T\colon \mathcal{H}_K\to\ell^2$

$$T(f) = (\langle f, f_i\rangle)_i \tag{8}$$

and synthesis operator $T^*\colon \ell^2\to\mathcal{H}_K$

$$T^*(a_i) = \sum_i a_i f_i \tag{9}$$

Then, the composition $S = T^* T$ *is the frame operator, which is bounded, invertible, and satisfies*

$$f = S^{-1}\left(\sum_i \langle f, f_i\rangle f_i\right) \tag{10}$$

Proof. Standard frame theory results [Christensen2003] apply, noting that the kernel functions $K(\cdot, t)$ form a continuous frame for the RKHS. □

5 Limits, Corollaries, and Remarks

Corollary 9. (Determinism of Path-Conditioned Measure) *For a fixed analytic sample path* $X(t)$*, the spectral measure* $d\,Z(\omega)$ *is a deterministic function, explicitly given by the spectral projection of* $X(t)$ *onto the basis functions* $e^{i\omega t}$*. The randomness collapses upon conditioning, and the measure is uniquely determined by the path.*

Remark 10. The duality between the path and measure representations is a manifestation of the isomorphism in the RKHS framework, where the spectral measure encodes all frequency content and the path is reconstructed via the inverse spectral integral. This is analogous to the synthesis and analysis operators in frame theory, providing a rigorous foundation for spectral synthesis.

6 Conclusion

This synthesis demonstrates that the spectral measure associated with an analytic, oscillatory non-stationary process is a deterministic functional once conditioned on the sample path. The integral operators, spectral representations, and RKHS structures provide a rigorous mathematical framework connecting the path and measure, with limits and approximations justified via classical analysis. The duality and isomorphism underpin many modern approaches in spectral analysis, kernel methods, and stochastic process theory.

Bibliography

[Priestley1981] M. B. Priestley, *Spectral Analysis and Time Series*, Academic Press, 1981.

[Christensen2003] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, 2003.