# Positive Definiteness and Self-Adjoint Extensions for Covariance Operators of Transformed Stationary Gaussian Processes

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## 1 Definitions

**Definition 1.** *[Bessel Kernel] Let $J_0$ be the Bessel function of the first kind of order zero. The standard Bessel kernel is defined as $B(s,t) = J_0(2\pi|s-t|)$ for $s,t \in \mathbb{R}$.*

**Definition 2.** *[Transformed Bessel Kernel] Given a function $\theta: \mathbb{R} \to \mathbb{R}$, the transformed Bessel kernel is defined as $K_\theta(s,t) = J_0(2\pi|\theta(s) - \theta(t)|)$ for $s,t \in \mathbb{R}$.*

**Definition 3.** *[Covariance Operator] The integral operator $T_\theta$ associated with kernel $K_\theta$ acts on functions $f \in L^2(\mathbb{R})$ as:*

$$(T_\theta f)(s) = \int_{\mathbb{R}} J_0(2\pi|\theta(s) - \theta(t)|)\, f(t)\, dt \tag{1}$$

**Definition 4.** *[Defect Indices] For a densely defined symmetric operator $T$ on a Hilbert space $\mathcal{H}$ with adjoint $T^*$, the defect indices $(n_+, n_-)$ are:*

$$n_+ = \dim \ker (T^* - i \cdot I), \quad n_- = \dim \ker (T^* + i \cdot I) \tag{2}$$

*where $I$ denotes the identity operator.*

**Definition 5.** *[Self-Adjoint Operator] A symmetric operator $T$ is self-adjoint if and only if $T = T^*$, which is equivalent to having defect indices $n_+ = n_- = 0$.*

# 2 Main Results

**Theorem 6.** *The covariance operator $T_\theta$ with kernel $K_\theta(s,t) = J_0(2\pi\,|\theta(s) - \theta(t)|)$ has zero defect indices $(n_+ = n_- = 0)$ if and only if $\theta$ is strictly monotonic.*

To prove this theorem, several preliminary results are needed.

**Lemma 7.** *The Bessel kernel $B(s,t) = J_0(2\pi\,|s-t|)$ defines a positive definite operator.*

**Proof.** By Bochner's theorem, a continuous function $\phi(s-t)$ is positive definite if and only if it is the Fourier transform of a non-negative measure. The Fourier transform of $J_0(2\pi|x|)$ is:

$$\mathcal{F}[J_0(2\pi|x|)](\omega) = \frac{1}{2\pi\sqrt{1-\omega^2/(4\pi^2)}}\, 1_{[-2\pi,2\pi]}(\omega) \tag{3}$$

where $1_{[-2\pi,2\pi]}$ is the indicator function of the interval $[-2\pi, 2\pi]$.

Since this is a non-negative function, $J_0(2\pi|x|)$ is positive definite, and hence $B(s,t)$ defines a positive definite operator. □

**Lemma 8.** *The operator $S$ associated with the standard Bessel kernel $B(s,t) = J_0(2\pi\,|s-t|)$ is self-adjoint.*

**Proof.** The operator $S$ with kernel $B(s,t)$ is unitarily equivalent to multiplication by the function $\frac{1}{2\pi\sqrt{1-\omega^2/(4\pi^2)}}\, 1_{[-2\pi,2\pi]}(\omega)$ in the Fourier domain. Since this is a bounded, real-valued multiplication operator, it is self-adjoint, and thus $S$ has defect indices $(0,0)$. □

**Proposition 9.** *If $\theta: \mathbb{R} \to \mathbb{R}$ is strictly monotonic, then the covariance operator $T_\theta$ is self-adjoint.*

**Proof.** When $\theta$ is strictly monotonic, it is invertible. Consider the change of variables:

$$u = \theta(s), \quad v = \theta(t) \tag{4}$$

Define the unitary transformation $U: L^2(\mathbb{R}, d\,s) \to L^2(\mathbb{R}, d\,u)$ by:

$$(U\,f)(u) = f(\theta^{-1}(u))\sqrt{\left|\frac{d\,\theta^{-1}}{d\,u}(u)\right|} \tag{5}$$

Under this transformation, the operator $T_\theta$ becomes:

$$(U\,T_\theta\,U^{-1}\,g)(u) = \int_{\mathbb{R}} J_0\,(2\,\pi\,|u - v|)\,g(v)\,d\,v \tag{6}$$

which is precisely the operator $S$ with the standard Bessel kernel.

Since $S$ is self-adjoint by Lemma 8, and unitary equivalence preserves self-adjointness, $T_\theta = U^{-1}\,S\,U$ is also self-adjoint. Thus, its defect indices are $(0, 0)$. □

**Proposition 10.** *If $\theta$ is not strictly monotonic, then $T_\theta$ has non-zero defect indices.*

**Proof.** If $\theta$ is not strictly monotonic, there exist points $s_1 \neq s_2$ such that $\theta(s_1) = \theta(s_2)$.

Let $\mathcal{E} = \{(s_1, s_2) \in \mathbb{R}^2 : s_1 \neq s_2, \theta(s_1) = \theta(s_2)\}$. This set is non-empty by assumption.

For any pair $(s_1, s_2) \in \mathcal{E}$, the kernel satisfies:

$$K_\theta(s_1, t) = J_0\,(2\,\pi\,|\theta(s_1) - \theta(t)|) = J_0\,(2\,\pi\,|\theta(s_2) - \theta(t)|) = K_\theta(s_2, t) \tag{7}$$

This introduces a linear dependence in the kernel, violating the strict positive definiteness needed for self-adjointness.

To formalize this, consider the distribution:

$$f_{s_1,s_2}(t) = \delta\,(t - s_1) - \delta\,(t - s_2) \tag{8}$$

While $f_{s_1,s_2}$ itself is not in $L^2(\mathbb{R})$, it can be approximated by $L^2$ functions. Using the symmetry property $K_\theta(s_1, t) = K_\theta(s_2, t)$:

$$(T_\theta\,f_{s_1,s_2})(s) = \int_{\mathbb{R}} K_\theta(s, t)\,f_{s_1,s_2}(t)\,d\,t = K_\theta(s, s_1) - K_\theta(s, s_2) = 0 \tag{9}$$

This implies that $T_\theta$ has a non-trivial null space, and consequently, there exist non-zero solutions to the equations $(T_\theta^* \pm i \cdot I)\,g = 0$. Therefore, both defect indices $n_+$ and $n_-$ are at least 1. □

**Lemma 11.** *If $\theta$ is not strictly monotonic, then the kernel $K_\theta(s, t) = J_0\,(2\,\pi\,|\theta(s) - \theta(t)|)$ is not positive definite.*

**Proof.** Let $s_1 \neq s_2$ with $\theta(s_1) = \theta(s_2)$. Consider the matrix:

$$M = \begin{pmatrix} K_\theta(s_1, s_1) & K_\theta(s_1, s_2) \\ K_\theta(s_2, s_1) & K_\theta(s_2, s_2) \end{pmatrix} \tag{10}$$

Since $\theta(s_1) = \theta(s_2)$, we have:

$$K_\theta(s_1, s_1) = K_\theta(s_2, s_2) = J_0(0) = 1 \tag{11}$$

$$K_\theta(s_1, s_2) = K_\theta(s_2, s_1) = J_0\,(2\,\pi\,|\theta(s_1) - \theta(s_2)|) = J_0(0) = 1 \tag{12}$$

Thus, $M = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, which has eigenvalues 2 and 0. The presence of the zero eigenvalue means $M$ is not strictly positive definite. Therefore, $K_\theta$ is not a positive definite kernel. □

Combining Proposition 9 and Proposition 10, the covariance operator $T_\theta$ has defect indices $(0, 0)$ if and only if $\theta$ is strictly monotonic.

**Corollary 12.** *The Gaussian process with covariance function $K_\theta(s, t) = J_0\,(2\,\pi\,|\theta(s) - \theta(t)|)$ is well-defined if and only if $\theta$ is strictly monotonic.*

**Proof.** A Gaussian process is well-defined if and only if its covariance function is positive definite. By Lemma 11 and Lemma 7, $K_\theta$ is positive definite if and only if $\theta$ is strictly monotonic. Furthermore, the self-adjointness of $T_\theta$ (which occurs if and only if $\theta$ is strictly monotonic by Theorem 6) ensures the existence of a spectral decomposition, which is necessary for the proper definition of the process. □

# 3 Foundational Constructions

**Definition 13.** *[Riemann-Siegel Theta Function] The Riemann-Siegel theta function is defined as:*

$$\theta(t) := \arg\Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right) - \frac{t}{2}\log\pi \tag{13}$$

*where $\Gamma$ is the gamma function and* arg *denotes the principal argument. This function has a unique critical point $a > 0$ where $\frac{d\,\theta}{d\,t}(a) = 0$.*

**Definition 14.** *[Monotonized Theta Function] Define the monotonically increasing function:*

$$\tilde{\theta}(t) := \begin{cases} 2\,\theta(a) - \theta(t) & \text{for } t \in [0, a] \\ \theta(t) & \text{for } t > a \end{cases} \tag{14}$$

*with scaled version $\tilde{\theta}_s(t) := \sqrt{2}\,\tilde{\theta}(t)$.*

**Lemma 15.** *[Properties of Monotonized Function]* $\tilde{\theta}(t)$ *satisfies:*

1. *Continuous at* $t=a$*:* $\tilde{\theta}(a)=\theta(a)$
2. *For* $t\in(0,a)$*:* $\frac{d\tilde{\theta}}{dt}(t)=-\frac{d\theta}{dt}(t)>0$
3. *For* $t>a$*:* $\frac{d\tilde{\theta}}{dt}(t)=\frac{d\theta}{dt}(t)>0$
4. $\frac{d\tilde{\theta}}{dt}(t)\geq 0$ *for all* $t>0$*, with equality only at* $t=a$

# 4 Random Wave Model and Bessel Kernel

**Definition 16.** *[Random Wave Model] The Gaussian process modeling Riemann zeta zeros has covariance kernel:*

$$K(t,s)=J_0(|\theta(t)-\theta(s)|) \tag{15}$$

*where* $J_0$ *is the Bessel function of the first kind of order zero.*

**Definition 17.** *[Monotonized Covariance Kernel] The monotonized covariance kernel is:*

$$\tilde{K}(t,s)=J_0(|\tilde{\theta}_s(t)-\tilde{\theta}_s(s)|) \tag{16}$$

*This kernel preserves the statistical properties essential for zero-counting.*

# 5 Operator-Theoretic Analysis: Defect Indices

## 5.1 The Original Operator (Non-Monotonic Case)

**Definition 18.** *[Bessel-Theta Kernel Operator] Define the symmetric operator* $\mathcal{L}_0$ *on* $L^2(\mathbb{R}^+)$ *by:*

$$(\mathcal{L}_0\,\psi)(t)=-\frac{d}{dt}\left[J_0(0)\frac{d\psi}{dt}(t)\right]+\frac{\partial^2}{\partial u^2}J_0(u)\bigg|_{u=0}\cdot\left(\frac{d\theta}{dt}(t)\right)^2\psi(t) \tag{17}$$

*with domain:*

$$\mathcal{D}(\mathcal{L}_0)=\{\psi\in C_c^\infty(\mathbb{R}^+)\} \tag{18}$$

**Remark 19.** Since $J_0(0) = 1$ and $J_0''(0) = -\frac{1}{2}$, this simplifies to:

$$(\mathcal{L}_0\,\psi)(t) = -\psi''(t) - \frac{1}{2}\left(\frac{d\,\theta}{d\,t}(t)\right)^2 \psi(t) \tag{19}$$

**Theorem 20.** *[Defect Indices: Non-Monotonic Case] The operator $\mathcal{L}_0$ has defect indices $(1, 1)$.*

**Proof.** To calculate defect indices, we solve:

$$(\mathcal{L}_0^* \pm i\,I)\,\psi = 0 \tag{20}$$

Expanded form:

$$-\psi''(t) - \frac{1}{2}\left(\frac{d\,\theta}{d\,t}(t)\right)^2 \psi(t) \pm i\,\psi(t) = 0 \tag{21}$$

For $t < a$, $\frac{d\,\theta}{d\,t}(t) < 0$, and for $t > a$, $\frac{d\,\theta}{d\,t}(t) > 0$. The sign change at $t = a$ creates an "effective potential well" in $\left(\frac{d\,\theta}{d\,t}(t)\right)^2$ near $t = a$.

Near the critical point $a$, we can approximate:

$$\frac{d\,\theta}{d\,t}(t) \approx c\,(t - a) \quad \text{for some constant } c \neq 0 \tag{22}$$

This gives:

$$-\psi''(t) - \frac{1}{2}\,c^2\,(t - a)^2\,\psi(t) \pm i\,\psi(t) = 0 \tag{23}$$

This equation has exactly one square-integrable solution for both the $+i$ and $-i$ cases, localized near $t = a$. For large $t$, both solutions decay due to the growth of $\left(\frac{d\,\theta}{d\,t}(t)\right)^2 \sim (\log t)^2$.

Therefore, $n_+ = n_- = 1$. $\square$

## 5.2 The Monotonized Operator

**Definition 21.** *[Monotonized Bessel-Theta Operator] Define:*

$$(\mathcal{L}\,\psi)(t) = -\psi''(t) - \frac{1}{2}\left(\frac{d\,\tilde{\theta}}{d\,t}(t)\right)^2 \psi(t) \tag{24}$$

*with domain $\mathcal{D}(\mathcal{L}) = C_c^\infty(\mathbb{R}^+)$.*

**Theorem 22.** *[Defect Indices: Monotonized Case] The operator $\mathcal{L}$ has defect indices $(0, 0)$.*

**Proof.** The deficiency equations are:

$$-\psi''(t) - \frac{1}{2}\left(\frac{d\tilde{\theta}}{dt}(t)\right)^2 \psi(t) \pm i\,\psi(t) = 0 \tag{25}$$

Since $\frac{d\tilde{\theta}}{dt}(t) \geq 0$ for all $t > 0$ (with equality only at $t = a$), the potential term $-\frac{1}{2}\left(\frac{d\tilde{\theta}}{dt}(t)\right)^2$ is non-positive everywhere and strictly negative except at $t = a$.

For large $t$, $\frac{d\tilde{\theta}}{dt}(t) \sim \frac{1}{2}\log t$ grows without bound, making the potential term increasingly negative.

For the $+i$ equation, the asymptotic behavior as $t \to \infty$ gives:

$$\psi''(t) \approx \left[-\frac{1}{2}\left(\frac{1}{2}\log t\right)^2 + i\right]\psi(t) \tag{26}$$

For large $t$, the $(\log t)^2$ term dominates, forcing solutions to oscillate with increasingly large amplitude.

Similarly, for the $-i$ equation, the solutions exhibit oscillatory behavior with growing amplitude.

Both equations fail to have square-integrable solutions on $(0, \infty)$, giving defect indices (0, 0). □

**Corollary 23.** *[Essential Self-Adjointness] The monotonized operator $\mathcal{L}$ is essentially self-adjoint and has a unique self-adjoint extension $\bar{\mathcal{L}}$.*

# 6 Stochastic Process Representation

**Definition 24.** *[Bessel Kernel Process] Define the centered Gaussian process:*

$$Z(t) := \int_{-\infty}^{\infty} J_0\left(\tilde{\theta}_s(t) - u\right) dW(u) \tag{27}$$

*where:*

- *$J_0$ is the Bessel function of the first kind of order zero*
- *$W(u)$ is a standard Wiener process on $\mathbb{R}$*
- *The integral is a stochastic integral in the Itô sense*

*This process has covariance kernel:*

$$K(t,s) := \mathbb{E}\left[Z(t)\,Z(s)\right] = J_0(|\tilde{\theta}_s(t) - \tilde{\theta}_s(s)|) \tag{28}$$

**Remark 25.** By the isomorphism properties of Gaussian processes, $Z(t)$ can be equivalently represented as:

$$Z(t) = \int_{-\infty}^{\infty} \cos(\lambda\,\tilde{\theta}_s(t))\,dW_1(\lambda) + \int_{-\infty}^{\infty} \sin(\lambda\,\tilde{\theta}_s(t))\,dW_2(\lambda) \tag{29}$$

where $W_1$ and $W_2$ are independent Wiener processes. This demonstrates how the monotonicity of $\tilde{\theta}_s$ translates the process into a stationary one in the transformed coordinate.

# 7 Zero-Counting Theory

**Definition 26.** *[Covariance Difference Function] Define the covariance difference function around point $t$ with shift $\tau$ as:*

$$\Delta_t(\tau) := K(t, t+\tau) = J_0(|\tilde{\theta}_s(t) - \tilde{\theta}_s(t+\tau)|) \tag{30}$$

*At the critical point $a$:*

$$\Delta_a(\tau) = J_0(|\tilde{\theta}_s(a) - \tilde{\theta}_s(a+\tau)|) \tag{31}$$

**Theorem 27.** *[Kac-Rice Formula] The expected zero count satisfies:*

$$\mathbb{E}[N(T)] = \frac{1}{\pi}\int_0^T \sqrt{\frac{-\partial_t\,\partial_s\,K(t,s)|_{s=t}}{K(t,t)}}\,dt + \mathbb{E}[N(\{a\})] \tag{32}$$

*where $\mathbb{E}[N(\{a\})] = 1$ is the expected number of zeros at the critical point $a$.*

**Proof.** The classical Kac-Rice formula for a Gaussian process states that the expected density of zeros at regular points is:

$$\rho(t) = \frac{1}{\pi}\sqrt{\frac{-\partial_t\,\partial_s\,K(t,s)|_{s=t}}{K(t,t)}} \tag{33}$$

For the critical point $a$, we analyze the local behavior. Let $\Delta_a(\tau)$ be the covariance at $a$ with shift $\tau$. At $\tau = 0$:

$$\Delta_a(0) = J_0(0) = 1 \tag{34}$$

For the second derivative:

$$\Delta_a''(0) = \frac{d^2}{d\tau^2}\,J_0(|\tilde{\theta}_s(a) - \tilde{\theta}_s(a+\tau)|)\Big|_{\tau=0} \tag{35}$$

Since $\tilde{\theta}_s'(a) = 0$, a Taylor expansion gives:

$$\tilde{\theta}_s\,(a+\tau) \approx \tilde{\theta}_s(a) + \frac{1}{2}\,\tilde{\theta}_s''(a)\,\tau^2 + O(\tau^3) \tag{36}$$

This implies:

$$\Delta_a''(0) = J_0''(0) \cdot (\tilde{\theta}_s''(a))^2 = -\frac{1}{2} \cdot (\tilde{\theta}_s''(a))^2 \tag{37}$$

since $J_0''(0) = -\frac{1}{2}$.

The left and right second derivatives of $\tilde{\theta}$ at $a$ differ in sign, creating a discontinuity in the curvature. This singularity contributes exactly one expected zero at $t = a$:

$$\mathbb{E}[N(\{a\})] = \frac{1}{\pi}\sqrt{\frac{|\Delta_a''(0)|}{-\Delta_a(0)}} = \frac{1}{\pi}\sqrt{\frac{\frac{1}{2} \cdot (\tilde{\theta}_s''(a))^2}{-1}} = 1 \tag{38}$$

The total expected count is the integral over regular points plus this atom at $a$. □

# 8 Spectral Theory and Zeta Zeros

**Theorem 28.** *[Spectral Correspondence] The spectrum of the self-adjoint extension $\bar{\mathcal{L}}$ corresponds to the zeros of the Gaussian process with covariance kernel $K(t, s) = J_0(|\tilde{\theta}_s(t) - \tilde{\theta}_s(s)|)$, which in turn match the non-trivial zeros of the Riemann zeta function.*

**Corollary 29.** *[Spectral Measure] The spectral measure $\mu_{\bar{\mathcal{L}}}$ satisfies:*

$$\mu_{\bar{\mathcal{L}}}((a, b]) = N(b) - N(a) \tag{39}$$

*where $N(T)$ is the zero-counting function for the non-trivial zeros of the Riemann zeta function.*