

# Unitary Time Changes of Stationary Processes Yield Oscillatory Processes and a Functional Framework Toward a Hilbert– Pólya Construction

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## Abstract

A unitary time-change operator  $U_\theta$  is constructed for absolutely continuous, strictly increasing time reparametrizations  $\theta$ , acting on functions that are square-integrable over  $\sigma$ -compact sets. Applying  $U_\theta$  to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function  $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)}$  and evolutionary spectrum  $dF_t(\lambda) = \dot{\theta}(t) dF(\lambda)$ . It is proved that sample paths of any non-degenerate second-order stationary process almost surely lie in  $L^2_{\sigma\text{-comp}}(\mathbb{R})$ , making the operator applicable to typical realizations. A zero-localization measure  $\mu(dt) = \delta(Z(t)) |Z'(t)| dt$  induces a Hilbert space  $L^2(\mu)$  on the zero set of an oscillatory process  $Z$ , and the multiplication operator  $(Lf)(t) = t f(t)$  has pure point, simple spectrum equal to the zero set of  $Z$ . This produces a concrete operator scaffold consistent with a Hilbert–Pólya-type viewpoint.

## Table of contents

<b>1</b>	<b>Function Spaces and Unitary Time Change</b>	<b>2</b>
1.1	$\sigma$ -compact sets and local $L^2$	2
1.2	Unitary time-change operator	2
<b>2</b>	<b>Oscillatory Processes (Priestley)</b>	<b>5</b>
<b>3</b>	<b>Stationary Processes and Time Change</b>	<b>7</b>
3.1	Stationary processes	7
3.2	Stationary $\rightarrow$ oscillatory via $U_\theta$	8
3.3	Covariance operator conjugation	10
<b>4</b>	<b>Sample Paths Live in <math>L^2_{\sigma\text{-comp}}</math></b>	<b>11</b>
<b>5</b>	<b>Zero Localization and Hilbert–Pólya Scaffold</b>	<b>13</b>
5.1	Zero localization measure	13
5.2	Hilbert space on zeros and multiplication operator	14
<b>6</b>	<b>Appendix: Regularity and Simple Zeros</b>	<b>17</b>

# 1 Function Spaces and Unitary Time Change

## 1.1 $\sigma$ -compact sets and local $L^2$

### Definition 1

*[ $\sigma$ -compact sets] A subset  $U \subseteq \mathbb{R}$  is  $\sigma$ -compact if*

$$U = \bigcup_{n=1}^{\infty} K_n \quad (1)$$

*with each  $K_n$  compact.*

### Definition 2

*[Square-integrability on  $\sigma$ -compact sets] Define*

$$L^2_{\sigma\text{-comp}}(\mathbb{R}) := \left\{ f: \mathbb{R} \rightarrow \mathbb{C}: \int_U |f(t)|^2 dt < \infty \text{ for every } \sigma\text{-compact } U \subseteq \mathbb{R} \right\} \quad (2)$$

**Remark 3.** Every bounded measurable set in  $\mathbb{R}$  is  $\sigma$ -compact; hence  $L^2_{\sigma\text{-comp}}(\mathbb{R})$  contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

## 1.2 Unitary time-change operator

### Definition 4

*[Unitary time-change] Let  $\theta: \mathbb{R} \rightarrow \mathbb{R}$  be absolutely continuous, strictly increasing, and bijective, with  $\dot{\theta}(t) > 0$  almost everywhere and  $\dot{\theta}(t) = 0$  only on sets of Lebesgue measure zero. The function  $\theta$  maps  $\sigma$ -compact sets to  $\sigma$ -compact sets. Define, for  $f$  measurable,*

$$(U_{\theta} f)(t) = \sqrt{\dot{\theta}(t)} f(\theta(t)) \quad (3)$$

### Proposition 5

*[Inverse map] The inverse map is given by*

$$(U_{\theta}^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\theta'(\theta^{-1}(s))}} \quad (4)$$

*which is well-defined almost everywhere on every  $\sigma$ -compact set.*

**Proof.** Since  $\dot{\theta}(t) = 0$  only on sets of measure zero, and  $\theta^{-1}$  maps sets of measure zero to sets of measure zero (as absolutely continuous bijective functions preserve measure-zero sets), the denominator  $\sqrt{\dot{\theta}(\theta^{-1}(s))}$  is positive almost everywhere. The expression is therefore well-defined almost everywhere on every  $\sigma$ -compact set, which suffices for defining an element of  $L^2_{\sigma\text{-comp}}(\mathbb{R})$ .  $\square$

### Theorem 6

**[Local unitarity on  $\sigma$ -compact sets]** For every  $\sigma$ -compact set  $U \subseteq \mathbb{R}$  and  $f \in L^2_{\sigma\text{-comp}}(\mathbb{R})$ ,

$$\int_U |(U_\theta f)(t)|^2 dt = \int_{\theta(U)} |f(s)|^2 ds \quad (5)$$

Moreover,  $U_\theta^{-1}$  is the inverse of  $U_\theta$  on  $L^2_{\sigma\text{-comp}}(\mathbb{R})$ .

**Proof.** Let  $f \in L^2_{\sigma\text{-comp}}(\mathbb{R})$  and let  $U$  be any  $\sigma$ -compact set. The local  $L^2$ -norm of  $U_\theta f$  over  $U$  is:

$$\begin{aligned} \int_U |(U_\theta f)(t)|^2 dt &= \int_U \left| \sqrt{\dot{\theta}(t)} f(\theta(t)) \right|^2 dt \\ &= \int_U \dot{\theta}(t) |f(\theta(t))|^2 dt \end{aligned} \quad (6)$$

Since  $\theta$  is absolutely continuous and strictly increasing, applying the change of variables  $s = \theta(t)$  gives

$$ds = \dot{\theta}(t) dt \quad (7)$$

almost everywhere. Since  $\theta$  maps  $\sigma$ -compact sets to  $\sigma$ -compact sets, as  $t$  ranges over  $U$ ,  $s = \theta(t)$  ranges over  $\theta(U)$ , which is  $\sigma$ -compact. Therefore:

$$\int_U \dot{\theta}(t) |f(\theta(t))|^2 dt = \int_{\theta(U)} |f(s)|^2 ds \quad (8)$$

To verify that  $U_\theta^{-1}$  is indeed the inverse, we compute explicitly. For any  $f \in L^2_{\sigma\text{-comp}}(\mathbb{R})$ :

$$\begin{aligned} (U_\theta^{-1} U_\theta f)(s) &= \left( U_\theta^{-1} \sqrt{\dot{\theta}(\cdot)} f(\theta(\cdot)) \right)(s) \\ &= \frac{\sqrt{\dot{\theta}(\theta^{-1}(s))}}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} f(\theta(\theta^{-1}(s))) \\ &= f(s) \end{aligned} \quad (9)$$

since  $\theta(\theta^{-1}(s)) = s$ . Similarly, for any  $g \in L^2_{\sigma\text{-comp}}(\mathbb{R})$ :

$$\begin{aligned}
(U_\theta U_\theta^{-1} g)(t) &= \sqrt{\dot{\theta}(t)} (U_\theta^{-1} g)(\theta(t)) \\
&= \frac{\sqrt{\dot{\theta}(t)}}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} g(\theta^{-1}(\theta(t))) \\
&= \frac{\sqrt{\dot{\theta}(t)}}{\sqrt{\dot{\theta}(t)}} g(t) \\
&= g(t)
\end{aligned} \tag{10}$$

since  $\theta^{-1}(\theta(t)) = t$ . Therefore

$$U_\theta U_\theta^{-1} = U_\theta^{-1} U_\theta = I \tag{11}$$

on  $L^2_{\sigma\text{-comp}}(\mathbb{R})$ . □

### Theorem 7

**[Unitarity on  $L^2(\mathbb{R})$ ]**  $U_\theta: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  is unitary:

$$\int_{\mathbb{R}} |(U_\theta f)(t)|^2 dt = \int_{\mathbb{R}} |f(s)|^2 ds \tag{12}$$

and  $U_\theta^{-1}$  is its inverse.

**Proof.** For  $f \in L^2(\mathbb{R})$ , we have:

$$\int_{\mathbb{R}} |(U_\theta f)(t)|^2 dt = \int_{\mathbb{R}} \dot{\theta}(t) |f(\theta(t))|^2 dt \tag{13}$$

By the change of variables

$$s = \theta(t) \tag{14}$$

with

$$ds = \dot{\theta}(t) dt \tag{15}$$

, and since  $\theta: \mathbb{R} \rightarrow \mathbb{R}$  is bijective:

$$\int_{\mathbb{R}} \dot{\theta}(t) |f(\theta(t))|^2 dt = \int_{\mathbb{R}} |f(s)|^2 ds \tag{16}$$

The inverse relationship follows from the same computation as in Theorem 6, applied globally.  $\square$

## 2 Oscillatory Processes (Priestley)

### Definition 8

**[Oscillatory process]** Let  $F$  be a finite nonnegative Borel measure on  $\mathbb{R}$ . For each  $t \in \mathbb{R}$ , let  $A_t \in L^2(F)$  be the gain function and

$$\varphi_t(\lambda) := A_t(\lambda) e^{i\lambda t} \quad (17)$$

be the corresponding oscillatory function then an oscillatory process is a stochastic process which can be represented as

$$\begin{aligned} Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \end{aligned} \quad (18)$$

where  $\Phi$  is a complex orthogonal random measure with spectral measure  $F$ , that is,

$$\mathbb{E}[\Phi(d\lambda) \overline{\Phi(d\mu)}] = \delta(\lambda - \mu) dF(\lambda) \quad (19)$$

and corresponding covariance kernel

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t) \overline{Z(s)}] \\ &= \int_{\mathbb{R}} A_t(\lambda) \overline{A_s(\lambda)} e^{i\lambda(t-s)} dF(\lambda) \\ &= \int_{\mathbb{R}} \phi_t(\lambda) \overline{\phi_s(\lambda)} dF(\lambda) \end{aligned} \quad (20)$$

**Remark 9.** [Real-valuedness]  $Z$  is real-valued if and only if

$$A_t(-\lambda) = \overline{A_t(\lambda)} \quad (21)$$

for  $F$ -a.e.  $\lambda$ , equivalently

$$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \quad (22)$$

for  $F$ -a.e.  $\lambda$ . TODO: this needs to be recast as a theorem

### Theorem 10

**[Existence]** If  $F$  is finite and  $(A_t)_{t \in \mathbb{R}}$  is measurable in  $t$  with

$$\int_{\mathbb{R}} |A_t(\lambda)|^2 dF(\lambda) < \infty \forall t \in \mathbb{R} \quad (23)$$

then there exists a complex orthogonal random measure  $\Phi$  with spectral measure  $F$  such that

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} \Phi(d\lambda) \quad (24)$$

is well-defined in  $L^2(\Omega)$  and has covariance  $R_Z$  as in (20) above.

**Proof.** We construct the stochastic integral using the standard extension procedure. First, define the integral for simple functions of the form

$$g(\lambda) = \sum_{j=1}^n c_j 1_{E_j}(\lambda) \quad (25)$$

where  $\{E_j\}$  are disjoint Borel sets with  $F(E_j) < \infty$  and  $c_j \in \mathbb{C}$ :

$$\int_{\mathbb{R}} g(\lambda) \Phi(d\lambda) := \sum_{j=1}^n c_j \Phi(E_j) \quad (26)$$

For such simple functions, the isometry property holds:

$$\begin{aligned} \mathbb{E} \left[ \left| \int_{\mathbb{R}} g(\lambda) \Phi(d\lambda) \right|^2 \right] &= \mathbb{E} \left[ \left| \sum_{j=1}^n c_j \Phi(E_j) \right|^2 \right] \\ &= \sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \mathbb{E}[\Phi(E_j) \overline{\Phi(E_k)}] \\ &= \sum_{j=1}^n |c_j|^2 F(E_j) \\ &= \int_{\mathbb{R}} |g(\lambda)|^2 dF(\lambda) \end{aligned} \quad (27)$$

Since simple functions are dense in  $L^2(F)$ , we extend by continuity to all  $g \in L^2(F)$ . For each  $t$ , since

$$\varphi_t(\lambda) = A_t(\lambda) e^{i\lambda t} \quad (28)$$

and  $A_t \in L^2(F)$ , we have  $\varphi_t \in L^2(F)$ . Therefore

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda) \Phi(d\lambda) \quad (29)$$

is well-defined in  $L^2(\Omega)$ . The covariance is computed as:

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t) \overline{Z(s)}] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda) \Phi(d\lambda) \int_{\mathbb{R}} \overline{\varphi_s(\mu)} \overline{\Phi(d\mu)}\right] \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\mu)} \mathbb{E}[\Phi(d\lambda) \overline{\Phi(d\mu)}] \\ &= \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\lambda)} dF(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda) \overline{A_s(\lambda)} e^{i\lambda(t-s)} dF(\lambda) \end{aligned} \quad (30) \quad \square$$

### 3 Stationary Processes and Time Change

#### 3.1 Stationary processes

##### Definition 11

**[Cramér representation]** A zero-mean stationary process  $X$  with spectral measure  $F$  admits the sample path representation

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t} \Phi(d\lambda) \quad (31)$$

which has covariance

$$R_X(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)} dF(\lambda) \quad (32)$$

### 3.2 Stationary $\rightarrow$ oscillatory via $U_\theta$

#### Theorem 12

**[Time change yields oscillatory process]** Let  $X$  be zero-mean stationary as in Definition 11. For  $\theta$  as in Definition 4, define

$$\begin{aligned} Z(t) &= (U_\theta X)(t) \\ &= \sqrt{\dot{\theta}(t)} X(\theta(t)) \end{aligned} \quad (33)$$

Then  $Z$  is oscillatory with oscillatory function

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \quad (34)$$

, gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \quad (35)$$

, and covariance

$$R_Z(t, s) = \int_{\mathbb{R}} \sqrt{\dot{\theta}(t) \dot{\theta}(s)} e^{i\lambda(\theta(t)-\theta(s))} dF(\lambda) \quad (36)$$

**Proof.** Applying the unitary time change operator to the spectral representation of  $X(t)$ :

$$\begin{aligned} Z(t) &= (U_\theta X)(t) \\ &= \sqrt{\dot{\theta}(t)} X(\theta(t)) \\ &= \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)} d\Phi(\lambda) \\ &= \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} d\Phi(\lambda) \\ &= \int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \end{aligned} \quad (37)$$

where

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \quad (38)$$

To verify this constitutes an oscillatory representation according to Definition 8, we must write  $\varphi_t(\lambda)$  in the form  $A_t(\lambda) e^{i\lambda t}$ :

$$\begin{aligned} \varphi_t(\lambda) &= \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \\ &= \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} e^{i\lambda t} \\ &= A_t(\lambda) e^{i\lambda t} \end{aligned} \quad (39)$$



where

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \quad (40)$$

Since  $\dot{\theta}(t) \geq 0$  almost everywhere and  $\dot{\theta}(t) = 0$  only on sets of measure zero,  $A_t(\lambda)$  is well-defined almost everywhere. Moreover,  $A_t \in L^2(F)$  for each  $t$  since:

$$\begin{aligned} \int_{\mathbb{R}} |A_t(\lambda)|^2 dF(\lambda) &= \int_{\mathbb{R}} \left| \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \right|^2 dF(\lambda) \\ &= \int_{\mathbb{R}} \dot{\theta}(t) |e^{i\lambda(\theta(t)-t)}|^2 dF(\lambda) \\ &= \dot{\theta}(t) \int_{\mathbb{R}} dF(\lambda) \\ &= \dot{\theta}(t) F(\mathbb{R}) < \infty \end{aligned} \quad (41)$$

where we used  $|e^{i\alpha}| = 1$  for all real  $\alpha$ . The covariance is computed as:

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\left[\sqrt{\dot{\theta}(t)} X(\theta(t)) \sqrt{\dot{\theta}(s)} \overline{X(\theta(s))}\right] \\ &= \sqrt{\dot{\theta}(t)\dot{\theta}(s)} \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \\ &= \sqrt{\dot{\theta}(t)\dot{\theta}(s)} R_X(\theta(t) - \theta(s)) \\ &= \sqrt{\dot{\theta}(t)\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} dF(\lambda) \end{aligned} \quad (42)$$

□

### Corollary 13

**[Evolutionary spectrum]** The evolutionary spectrum is

$$\begin{aligned} dF_t(\lambda) &= |A_t(\lambda)|^2 dF(\lambda) \\ &= \dot{\theta}(t) dF(\lambda) \end{aligned} \quad (43)$$

**Proof.** By definition of the evolutionary spectrum and using the gain function from Theorem 12:

$$\begin{aligned} dF_t(\lambda) &= |A_t(\lambda)|^2 dF(\lambda) \\ &= \left| \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \right|^2 dF(\lambda) \\ &= \dot{\theta}(t) |e^{i\lambda(\theta(t)-t)}|^2 dF(\lambda) \\ &= \dot{\theta}(t) dF(\lambda) \end{aligned} \quad (44)$$

since

$$|e^{i\alpha}| = 1 \forall \alpha \in \mathbb{R} \quad (45) \quad \square$$

### 3.3 Covariance operator conjugation

#### Proposition 14

*[Operator conjugation] Let*

$$(T_K f)(t) := \int_{\mathbb{R}} K(|t-s|) f(s) \, ds \quad (46)$$

*with stationary kernel*

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h} \, dF(\lambda) \quad (47)$$

*Define the transformed kernel*

$$K_{\theta}(s, t) := \sqrt{\dot{\theta}(t) \dot{\theta}(s)} \, K(|\theta(t) - \theta(s)|) \quad (48)$$

*and corresponding integral covariance operator*

$$(T_{K_{\theta}} f)(t) := \int_{\mathbb{R}} K_{\theta}(s, t) f(s) \, ds \quad (49)$$

*Then*

$$T_{K_{\theta}} = U_{\theta} \, T_K \, U_{\theta}^{-1} \quad (50)$$

*on  $L^2_{\sigma\text{-comp}}(\mathbb{R})$ .*

**Proof.** For any  $g \in L^2_{\sigma\text{-comp}}(\mathbb{R})$ , we transform the integral operator from coordinates  $(r, w)$  to coordinates  $(t, s)$  by applying both coordinate transformations

$$r = \theta(t) \quad (51)$$

and

$$w = \theta(s) \quad (52)$$

by simultaneously substitution with Jacobians

$$dr = \dot{\theta}(t) \, dt \quad (53)$$

and

$$dw = \dot{\theta}(s) \, ds \quad (54)$$

The operator  $T_K$  in  $(r, w)$  coordinates is:

$$(T_K f)(r) = \int_{\mathbb{R}} K(|r - w|) f(w) dw \quad (55)$$

Under the simultaneous transformation  $r = \theta(t)$  and  $w = \theta(s)$ :

$$\begin{aligned} ((U_\theta T_K U_\theta^{-1}) g)(t) &= \sqrt{\dot{\theta}(t) \dot{\theta}(s)} \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|) (U_\theta^{-1} g)(\theta(s)) \frac{\dot{\theta}(s)}{\sqrt{\dot{\theta}(s)}} ds \\ &= \sqrt{\dot{\theta}(t) \dot{\theta}(s)} \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|) \frac{g(s)}{\sqrt{\dot{\theta}(s)}} \sqrt{\dot{\theta}(s)} ds \\ &= \sqrt{\dot{\theta}(t) \dot{\theta}(s)} \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|) g(s) ds \\ &= \int_{\mathbb{R}} \sqrt{\dot{\theta}(t) \dot{\theta}(s)} K(|\theta(t) - \theta(s)|) g(s) ds \\ &= \int_{\mathbb{R}} K_\theta(t, s) g(s) ds \\ &= (T_{K_\theta} g)(t) \end{aligned} \quad (56)$$

where

$$K_\theta(t, s) = \sqrt{\dot{\theta}(t) \dot{\theta}(s)} K(|\theta(t) - \theta(s)|) \quad (57)$$

Therefore

$$T_{K_\theta} = U_\theta T_K U_\theta^{-1} \quad (58) \quad \square$$

## 4 Sample Paths Live in $L^2_{\sigma\text{-comp}}$

### Theorem 15

*[Sample paths in  $L^2_{\sigma\text{-comp}}(\mathbb{R})$ ] Let  $\{X(t)\}_{t \in \mathbb{R}}$  be a second-order stationary process with*

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \quad (59)$$

*then, almost surely, every sample path  $t \mapsto X(\omega, t)$  belongs to  $L^2_{\sigma\text{-comp}}(\mathbb{R})$ .*

**Proof.** Fix any bounded interval  $[a, b]$  and consider the random variable

$$Y_{[a, b]} := \int_a^b X(t)^2 dt \quad (60)$$

By stationarity and Fubini's theorem:

$$\begin{aligned}\mathbb{E}[Y_{[a,b]}] &= \mathbb{E}\left[\int_a^b X(t)^2 dt\right] = \int_a^b \mathbb{E}[X(t)^2] dt \\ &= \int_a^b \sigma^2 dt \\ &= \sigma^2(b-a) < \infty\end{aligned}\tag{61}$$

By Markov's inequality, for any  $M > 0$ :

$$P(Y_{[a,b]} > M) \leq \frac{\mathbb{E}[Y_{[a,b]}]}{M} = \frac{\sigma^2(b-a)}{M}\tag{62}$$

Taking  $M \rightarrow \infty$ , we conclude

$$P(Y_{[a,b]} < \infty) = 1\tag{63}$$

, i.e., almost surely the sample path is square-integrable on  $[a, b]$ . Since  $\mathbb{R}$  is the countable union of bounded intervals:

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]\tag{64}$$

by countable subadditivity of probability:

$$P\left(\bigcap_{n=1}^{\infty} \left\{\int_{-n}^n X(t)^2 dt < \infty\right\}\right) = 1\tag{65}$$

Now let  $U$  be any  $\sigma$ -compact set. Then

$$U = \bigcup_{m=1}^{\infty} K_m\tag{66}$$

where each  $K_m$  is compact. Each compact set  $K_m$  is bounded, so

$$K_m \subseteq [-N_m, N_m]\tag{67}$$

for some  $N_m$ . Therefore:

$$\begin{aligned}\int_U X(t)^2 dt &= \int_{\bigcup_{m=1}^{\infty} K_m} X(t)^2 dt \\ &\leq \sum_{m=1}^{\infty} \int_{K_m} X(t)^2 dt \\ &\leq \sum_{m=1}^{\infty} \int_{-N_m}^{N_m} X(t)^2 dt\end{aligned}\tag{68}$$

Since each integral

$$\int_{-N_m}^{N_m} X(t)^2 dt < \infty \quad (69)$$

almost surely, and the sum of countably many finite terms is finite, we have

$$\int_U X(t)^2 dt < \infty \quad (70)$$

almost surely. This holds for every  $\sigma$ -compact set  $U$ , so almost surely every sample path lies in  $L^2_{\sigma\text{-comp}}(\mathbb{R})$ .  $\square$

## 5 Zero Localization and Hilbert–Pólya Scaffold

### 5.1 Zero localization measure

#### Definition 16

*[Zero localization measure] Let  $Z$  be real-valued with  $Z \in C^1(\mathbb{R})$  having only simple zeros*

$$Z(t_0) = 0 \Rightarrow Z'(t_0) \neq 0 \quad (71)$$

*Define, for Borel  $B \subset \mathbb{R}$ ,*

$$\mu(B) := \int_{\mathbb{R}} 1_B(t) \delta(Z(t)) |Z'(t)| dt \quad (72)$$

#### Theorem 17

*[Atomicity on the zero set] For every  $\phi \in C_c^\infty(\mathbb{R})$ ,*

$$\int_{\mathbb{R}} \phi(t) \delta(Z(t)) |Z'(t)| dt = \sum_{t_0: Z(t_0)=0} \phi(t_0) \quad (73)$$

*hence*

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (74)$$

**Proof.** Since all zeros of  $Z$  are simple and  $Z \in C^1(\mathbb{R})$ , by the inverse function theorem each zero  $t_0$  is isolated. Near each zero  $t_0$ ,  $Z$  is locally monotonic, so we can apply the one-dimensional change of variables formula for the Dirac delta.

Specifically, near  $t_0$  where  $Z(t_0) = 0$  and  $Z'(t_0) \neq 0$ , we have locally

$$Z(t) = (t - t_0) Z'(t_0) + O((t - t_0)^2) \quad (75)$$

The distributional identity for the Dirac delta under smooth changes of variables gives:

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta(t-t_0)}{|Z'(t_0)|} \quad (76)$$

Therefore:

$$\begin{aligned} \int_{\mathbb{R}} \phi(t) \delta(Z(t)) |Z'(t)| dt &= \int_{\mathbb{R}} \phi(t) |Z'(t)| \sum_{t_0: Z(t_0)=0} \frac{\delta(t-t_0)}{|Z'(t_0)|} dt \\ &= \sum_{t_0: Z(t_0)=0} \int_{\mathbb{R}} \phi(t) \frac{|Z'(t)| \delta(t-t_0)}{|Z'(t_0)|} dt \\ &= \sum_{t_0: Z(t_0)=0} \frac{|Z'(t_0)| \phi(t_0)}{|Z'(t_0)|} \\ &= \sum_{t_0: Z(t_0)=0} \phi(t_0) \end{aligned} \quad (77)$$

This shows that  $\mu$  is the discrete measure

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (78)$$

assigning unit mass to each zero. □

## 5.2 Hilbert space on zeros and multiplication operator

### Definition 18

*[Hilbert space on the zero set] Let  $\mathcal{H} := L^2(\mu)$  with inner product*

$$\langle f, g \rangle = \int f(t) \overline{g(t)} \mu(dt) \quad (79)$$

### Proposition 19

*[Atomic structure] Let*

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (80)$$

*then*

$$\mathcal{H} \cong \left\{ f: \{t_0: Z(t_0)=0\} \rightarrow \mathbb{C}: \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \quad (81)$$

*with orthonormal basis  $\{e_{t_0}\}_{t_0: Z(t_0)=0}$  where*

$$e_{t_0}(t_1) = \delta_{t_0 t_1} \quad (82)$$

**Proof.** By the atomic form of  $\mu$ , for any  $f \in L^2(\mu)$ :

$$\|f\|_{\mathcal{H}}^2 = \int |f(t)|^2 \mu(dt) \quad (83)$$

$$= \int |f(t)|^2 \sum_{t_0: Z(t_0)=0} \delta_{t_0}(dt) \quad (84)$$

$$= \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 \quad (85)$$

This shows the isomorphism with  $\ell^2$ . The functions  $e_{t_0}$  defined by

$$e_{t_0}(t_1) = \delta_{t_0 t_1} \quad (86)$$

satisfy:

$$\langle e_{t_0}, e_{t_1} \rangle = \int e_{t_0}(t) \overline{e_{t_1}(t)} \mu(dt) = \sum_{t: Z(t)=0} \delta_{t_0 t} \delta_{t_1 t} = \delta_{t_0 t_1} \quad (87)$$

so they form an orthonormal set. Any  $f \in \mathcal{H}$  can be written as

$$f = \sum_{t_0: Z(t_0)=0} f(t_0) e_{t_0} \quad (88)$$

proving they form a basis. □

### Definition 20

*[Multiplication operator] Define  $L: \mathcal{D}(L) \subset \mathcal{H} \rightarrow \mathcal{H}$  by*

$$(L f)(t) = t f(t) \quad (89)$$

*on  $\text{supp}(\mu)$  with domain*

$$\mathcal{D}(L) := \left\{ f \in \mathcal{H}: \int |t f(t)|^2 \mu(dt) < \infty \right\} \quad (90)$$

### Theorem 21

*[Self-adjointness and spectrum]  $L$  is self-adjoint on  $\mathcal{H}$  and has pure point, simple spectrum*

$$\sigma(L) = \{t \in \mathbb{R}: Z(t) = 0\} \quad (91)$$

*with eigenvalues  $\lambda = t_0$  and eigenvectors  $e_{t_0}$ .*

**Proof.** First, we verify self-adjointness. For  $f, g \in \mathcal{D}(L)$ :

$$\begin{aligned}
\langle Lf, g \rangle &= \int (Lf)(t) \overline{g(t)} \mu(dt) \\
&= \int t f(t) \overline{g(t)} \mu(dt) \\
&= \int f(t) \overline{t g(t)} \mu(dt) \\
&= \int f(t) \overline{(Lg)(t)} \mu(dt) \\
&= \langle f, Lg \rangle
\end{aligned} \tag{92}$$

Thus  $L$  is symmetric and acts as

$$(Lf)(t_0) = t_0 f(t_0) \tag{93}$$

for each  $t_0$  in the atomic representation where  $Z(t_0) = 0$ . This is unitarily equivalent to the diagonal operator on  $\ell^2$  with diagonal entries  $\{t_0: Z(t_0) = 0\}$ . Such diagonal operators are self-adjoint. For the spectrum calculation: We have

$$L e_{t_0} = t_0 e_{t_0} \tag{94}$$

so each  $t_0$  where  $Z(t_0) = 0$  is an eigenvalue of  $L$  with eigenvector  $e_{t_0}$ . Since  $\{e_{t_0}\}$  forms an orthonormal basis,  $L$  has pure point spectrum. To show there are no other spectral points, suppose  $\lambda \notin \{t_0: Z(t_0) = 0\}$ . Then for any  $f \in \mathcal{D}(L)$ ,  $(L - \lambda I)f$  has components

$$((L - \lambda I)f)(t_0) = (t_0 - \lambda) f(t_0) \tag{95}$$

Since  $t_0 - \lambda \neq 0$  for all zeros  $t_0$ , we can solve

$$(L - \lambda I)f = g \tag{96}$$

uniquely for any  $g \in \mathcal{H}$  by setting

$$f(t_0) = \frac{g(t_0)}{t_0 - \lambda} \tag{97}$$

This shows  $L - \lambda I$  is invertible, so  $\lambda \notin \sigma(L)$ . Therefore

$$\sigma(L) = \{t_0: Z(t_0) = 0\} \tag{98}$$

and the eigenvalues are simple. □



**Remark 22. [Operator scaffold]** The construction

$$\text{stationary } X \xrightarrow{U_\theta} \text{oscillatory } Z \xrightarrow{\mu=\delta(Z)|Z'|dt} L^2(\mu) \xrightarrow{L:t\cdot} (L, \sigma(L)) \quad (99)$$

produces a concrete self-adjoint operator whose spectrum equals the zero set of  $Z$ , determined by the choice of time-change  $\theta$  and spectral measure  $F$ . This provides an explicit realization consistent with Hilbert–Pólya approaches to encoding arithmetic information in operator spectra.

## 6 Appendix: Regularity and Simple Zeros

### Definition 23

**[Regularity and simplicity]** Assume  $Z \in C^1(\mathbb{R})$  and every zero is simple:  $Z(t_0) = 0 \Rightarrow Z'(t_0) \neq 0$ .

### Lemma 24

**[Local finiteness and delta decomposition]** Under Definition 23, zeros are locally finite and

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta(t-t_0)}{|Z'(t_0)|} \quad (100)$$

whence

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (101)$$

**Proof.** Since  $Z \in C^1(\mathbb{R})$  and  $Z'(t_0) \neq 0$  at each zero  $t_0$ , the inverse function theorem implies that  $Z$  is locally invertible near each zero. Specifically, there exists a neighborhood  $U_{t_0}$  of  $t_0$  such that  $Z|_{U_{t_0}}$  is strictly monotonic and invertible.

This implies zeros are isolated: if  $Z(t_0) = 0$  and  $Z'(t_0) \neq 0$ , then there exists  $\epsilon > 0$  such that  $Z(t) \neq 0$  for  $0 < |t - t_0| < \epsilon$ . Therefore zeros are locally finite (finitely many in any bounded interval).

For the distributional identity, consider the one-dimensional change of variables formula for the Dirac delta. If  $g: I \rightarrow \mathbb{R}$  is  $C^1$  on interval  $I$  with  $g'(x) \neq 0$  for all  $x \in I$ , then

$$\delta(g(x)) = \sum_{x_0: g(x_0)=0} \frac{\delta(x-x_0)}{|g'(x_0)|} \quad (102)$$

Applying this locally around each zero  $t_0$  of  $Z$ , and since zeros are isolated, we can patch together the local results to obtain the global identity:

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta(t-t_0)}{|Z'(t_0)|} \quad (103)$$

Consequently:

$$\begin{aligned} \mu(dt) &= \delta(Z(t)) |Z'(t)| dt \\ &= \sum_{t_0: Z(t_0)=0} \frac{|Z'(t)|}{|Z'(t_0)|} \delta(t-t_0) dt \\ &= \sum_{t_0: Z(t_0)=0} \delta_{t_0}(dt) \end{aligned} \quad (104)$$

where the last equality uses the fact that

$$\frac{|Z'(t)|}{|Z'(t_0)|} = 1 \quad (105)$$

when evaluating at  $t = t_0$ . □