Proof Using Stone's Theorem and von Neumann's Commutant Theory for Oscillatory Processes

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Theorem 1. ***(Characterization of Oscillatory Processes).*** *A curve $x(t)$ in a Hilbert space H is oscillatory if and only if it is a deformed stationary curve $x(t) = \mathbf{A}_t[y(t)]$, where $y(t)$ is stationary and $\mathbf{A}_t$ commutes with the shift group $\{U_s\}$.*

Proof. We will proceed in steps, invoking Stone's spectral theorem and von Neumann's theory of commutants.

Step 1: Stationary Process Representation via Stone's Theorem

Let $y(t)$ be a continuous stationary curve in H. By Stone's theorem, there exists a unique projection-valued measure $E(\cdot)$ on $\mathbb{R}$ such that

$$y(t) = \int_{-\infty}^{\infty} e^{i\lambda t} E(d\lambda)\, y_0 \tag{1}$$

where

$$y_0 = y(0) \tag{2}$$

The shift group $\{U_s\}_{s\in\mathbb{R}}$ acts by time translation:

$$U_s[y(t)] = y(t+s) = \int_{-\infty}^{\infty} e^{i\lambda(t+s)} E(d\lambda)\, y_0 \tag{3}$$

The group can also be represented spectrally:

$$U_s = \int_{-\infty}^{\infty} e^{i\lambda s} E(d\lambda) \tag{4}$$

which is unitary and strongly continuous.

Step 2: Oscillatory Processes as Deformed Stationary Curves

Let us consider Priestley's oscillatory process:

$$x(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\, a_t(\lambda)\, E\,(d\,\lambda)\, y_0 \tag{5}$$

where each $a_t(\lambda)$ is a measurable function, and $E(\cdot)$ is the same projection-valued measure as above.

Define the operator

$$\mathbf{A}_t = \int_{-\infty}^{\infty} a_t(\lambda)\, E\,(d\,\lambda) \tag{6}$$

Then $x(t)$ can be written as

$$x(t) = \mathbf{A}_t[y(t)] \tag{7}$$

Here, $y(t)$ is the stationary process from Step 1, and $x(t)$ is its deformation via $\mathbf{A}_t$.

Step 3: Commutation with the Shift Group

For $x(t)$ to be a deformed stationary curve in this sense, the operators $\mathbf{A}_t$ must commute with the shift group:

$$\mathbf{A}_t[U_s[h]] = U_s[\mathbf{A}_t[h]] \quad \forall h \in \mathrm{Dom}(\mathbf{A}_t), s \in \mathbb{R} \tag{8}$$

Because $\mathbf{A}_t$ is defined through $E(\cdot)$, spectral calculus ensures

$$\mathbf{A}_t U_s = U_s \mathbf{A}_t, \tag{9}$$

since multiplication by $a_t(\lambda)$ and $e^{i\lambda s}$ commute under the spectral integral.

Step 4: Converse via von Neumann's Commutant Theory

Now suppose $x(t) = \mathbf{A}_t[y(t)]$ with $\mathbf{A}_t$ commuting with U_s. By von Neumann's theory of commutants, the set of all (possibly unbounded) closed operators on H which commute with the group $\{U_s\}$ are precisely the spectral integrals:

$$\mathbf{A}_t = \int_{-\infty}^{\infty} a_t(\lambda)\, E\,(d\,\lambda) \tag{10}$$

for some measurable $a_t(\cdot)$. Thus, any such deformation is an oscillatory process of the desired form.

Step 5: Nondeterminism and Moving Average Representation

Suppose $y(t)$ is purely nondeterministic, admitting a moving average representation ([Karhunen]):

$$y(t) = \int_{-\infty}^{t} f\,(t-u)\ \xi\,(d\,u) \tag{11}$$

where ξ is a suitable random measure. Then

$$x(t) = \mathbf{A}_t[y(t)] = \int_{-\infty}^{t} h(t,u)\ \xi\,(d\,u) \tag{12}$$

$$h(t,u) := \mathbf{A}_t(f\,(t-u)) \tag{13}$$

Thus, any oscillatory process can be represented as a moving average of a purely nondeterministic process. □

Conclusion

By Stone's theorem and von Neumann's commutant theorem, oscillatory processes are precisely deformed stationary curves whose deformation operators commute with the shift group. This yields their spectral structure and moving average representations.

Bibliography

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