ORTHOGONALIZING WEIGHTS OF TCHEBYCHEV SETS OF POLYNOMIALS

KIL H. KWON, SUNG S. KIM AND SUNG S. HAN

ABSTRACT

We characterize distributions with respect to which the members of a Tchebychev set of polynomials are orthogonal when they satisfy differential equations with polynomial coefficients. As an application, we find a real weight of bounded variation with support in $[0, \infty)$ for Bessel polynomials.

1. *Introduction*

After H. L. Krall [**12**] (compare Theorem 2.1) classified all the Tchebychev sets of polynomials $\{p_n(x)\}_0^\infty$ (TSPs) satisfying differential equations with polynomial coefficients

$$L_{2r}y = \sum_{i=0}^{2r} \sum_{j=0}^{i} \ell_{ij} x^j y^{(i)} = \lambda_n y, \tag{1.1}$$

where the ℓ_{ij} are real constants and

$$\lambda_n = \ell_{00} + \ell_{11} n + \ldots + \ell_{2r,2r} n(n-1)\ldots(n-2r+1), \tag{1.2}$$

there were many attempts to find their orthogonalizing weights [**5**, **6**, **9**, **10**, **11**, **14**, **15**]. For example, R. D. Morton and A. M. Krall [**15**] introduced a formal δ-series

$$w(x) = \sum_{0}^{\infty} (-1)^n \mu_n \delta^{(n)}(x)/n! \tag{1.3}$$

which is, at least formally, an orthogonalizing weight, and found classical or distributional weights from (1.3) through Fourier transforms under a suitable restriction on the growth of $\{\mu_n\}_0^\infty$. Here, we note that the moments $\{\mu_n\}_0^\infty$ of any TSP $\{p_n(x)\}_0^\infty$ can be computed without referring to the weight, for example, via the three-term recurrence relation satisfied by them. On the other hand, noting that all known differential equations of the form (1.1) having a TSP as solutions can be made symmetric when multiplied by a suitable function, L. L. Littlejohn [**14**] found r homogeneous differential equations, of which any non-trivial weak solution is an orthogonalizing weight for the TSP (compare Theorem 2.2). These two methods have been used quite successfully to find the weights of all known TSPs arising as solutions of (1.1) [**14**, **15**]. Except for the classical orthogonal polynomial sets such as Jacobi, Laguerre and Hermite polynomials, the weights turn out to be distributions (or even a hyperfunction in the case of Bessel polynomials [**5**, **6**]). However, due to the classical theorem of R. P. Boas [**1**] (see also [**3**]) on the Stieltjes moment problem, any TSP must be orthogonal with respect to a real weight of bounded variation.

Received 4 February 1991; revised 12 August 1991.

1991 *Mathematics Subject Classification* 33C45.

Bull. London Math. Soc. 24 (1992) 361–367

Recently, from the viewpoint of the pure moment problem, A. J. Duran [4] developed a way of constructing the real orthogonalizing weight of any TSP in the Schwartz space S or S^+. In particular, he found a real weight for the Bessel polynomials in the series form using their moments.

In this work, we shall give a distinct way of constructing the orthogonalizing weights for the TSPs satisfying (1.1). By improving the result in [14], we first find a necessary and sufficient condition for a distribution to be an orthogonalizing weight for the TSP satisfying (1.1); it is given by r non-homogeneous differential equations obtained from (1.1). Then as an application, we find a real weight of bounded variation with support in $[0, \infty)$ for the Bessel polynomials. Contrary to the one in [4], the weight is given in the closed form, which is directly related to the differential equation of which the Bessel polynomials are solutions. The same method would also work on other TSPs.

This work is partially supported by KOSEF and GARC. We are very grateful to the kind suggestions from the referee for the improvement of this work.

2. *Main theorem*

Let $\{\mu_n\}_0^\infty$ be a sequence of real numbers satisfying

$$\Delta_n = \det[\mu_{i+j}]_{i,j=0}^n \neq 0, \quad n \geqslant 0. \tag{2.1}$$

The Tchebychev set of polynomials associated to $\{\mu_n\}_0^\infty$ is a sequence $\{p_n(x)\}_0^\infty$ of monic polynomials defined by $p_0(x) = 1$ and

$$p_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \cdots & \mu_n \\ \mu_1 & \cdots & \mu_{n+1} \\ & \vdots & \\ \mu_{n-1} & \cdots & \mu_{2n-1} \\ 1 & \cdots & x^n \end{vmatrix}, \quad n \geqslant 0.$$

In [12], H. L. Krall proved the following classification theorem.

THEOREM 2.1. *Let $\{p_n(x)\}$ be a TSP associated to $\{\mu_n\}$. Then $p_n(x)$ satisfies the differential equations* (1.1) *for $n = 0, 1, \ldots$ if and only if the moments $\{\mu_n\}$ satisfy r recurrence relations*

$$S_k(m) = \sum_{i=2k+1}^{2r} \sum_{j=0}^{i} \binom{i-k-1}{k} P(m-2k-1, i-2k-1)\,\ell_{i,k-j}\,\mu_{m-j} = 0, \tag{2.2}$$

$2k+1 \leqslant 2r$, $m = 2k+1, 2k+2, \ldots,$ *where* $P(n,k) = n(n-1)\ldots(n-k+1)$.

We call the r recurrence relations in (2.2) the moment equations for $\{p_n(x)\}$. From now on, we assume that these r moment equations are uniquely solvable for $\{\mu_n\}$ (up to a constant multiple).

On the other hand, L. L. Littlejohn [13] characterized a symmetric factor of the operator L_{2r}: a function $f(x)$ is a symmetric factor of L_{2r} (that is, $f(x)\,L_{2r}$ is formally symmetric) if and only if $f(x)$ satisfies r homogeneous equations

$$R_k f = \sum_{s=k}^{r} \sum_{j=0}^{2s-2k+1} \binom{2s}{2k-1} \binom{2s-2k+1}{j} \frac{2^{2s-2k+1}}{s-k+1} B_{2s-2k+2}\, \ell_{2s}^{(2s-2k+1-j)}(x) f^{(j)}(x) - \ell_{2k-1}(x) f(x) = 0, \quad (2.3)$$

$k = 1, \dots, r$, where $\ell_i(x) = \sum_{j=1}^{i} \ell_{ij} x^j$ and the B_{2i} are the Bernoulli numbers defined by

$$\frac{x}{\exp(x)-1} = 1 - \frac{x}{2} + \sum_{1}^{\infty} \frac{B_{2i} x^{2i}}{(2i)!}.$$

Littlejohn also proved the following in **[14]**.

THEOREM 2.2. *Let $\{p_n(x)\}$ be the TSP associated to the moments $\{\mu_n\}$. Assume that for each $n = 0, 1, \dots$, $p_n(x)$ satisfies the differential equations (1.1). If $w(x)$ is a distributional solution of the r homogeneous equations in (2.3) such that $\langle w, 1\rangle \neq 0$, then $w(x)$ is an orthogonalizing weight for $\{p_n(x)\}$ provided that $w(x)$ decays very rapidly as $|x|$ tends to ∞ so that it can act on polynomials.*

By replacing w with $\mu_0 \langle w, 1\rangle^{-1} w$, we may even have $\langle w, x^n\rangle = \mu_n$, $n = 0, 1, \dots$. Moreover, we have another set of r moment relations

$$\begin{aligned} T_k(m) &= \langle R_k w, x^m\rangle \\ &= \sum_{s=k}^{r} \sum_{j=0}^{2s} \binom{2s}{2k-1} P(m, 2s-2k+1) \frac{2^{2s-2k+1}}{s-k+1} B_{2s-2k+2}\, \ell_{2j,s}\, \mu_{m+2k-2s+j-1} \\ &\quad + \sum_{j=0}^{2k-1} \ell_{2k-1,j}\, \mu_{m+j} = 0, \qquad k = 1, 2, \dots, \quad m = 0, 1, \dots, \end{aligned} \quad (2.4)$$

which is equivalent to (2.2) **[14]**.

What about the converse of Theorem 2.2? More precisely, let $\{p_n(x)\}$ be a TSP associated to $\{\mu_n\}$ satisfying the differential equations (1.1) for each $n = 0, 1, \dots$. If the $\{p_n(x)\}$ are orthogonal with respect to a distribution $w(x)$, does $w(x)$ satisfy the r homogeneous equations $R_k w = 0$, $k = 1, \dots, r$? This question leads us to the following improvement of Theorem 2.2 which characterizes the orthogonalizing weight for any TSP satisfying (1.1).

THEOREM 2.3. *Let $\{p_n(x)\}$ be the TSP associated to the moments $\{\mu_n\}$. Assume that for each $n = 0, 1, \dots$, $p_n(x)$ satisfies the differential equations (1.1). If the $\{p_n(x)\}$ are orthogonal with respect to a distribution $w(x)$ acting on polynomials, then all $R_k w$, $k = 1, 2, \dots, r$, must have 0-moments, that is,*

$$\langle R_k w, x^m\rangle = 0, \quad k = 1, 2, \dots, r, \ m = 0, 1, \dots. \quad (2.5)$$

Conversely, if a distribution $w(x)$ is such that

(a) w decays very rapidly as $|x|$ tends to ∞,

(b) w is non-trivial in the sense that $\langle w, 1\rangle \neq 0$,

(c) $R_k w$, $k = 1, 2, \dots, r$, have 0-moments,

then $w(x)$ is an orthogonalizing weight for the TSP $\{p_n(x)\}$.

Proof. The necessity follows immediately from the fact that, by Theorem 2.1, the moments $\{\mu_n\}$ of $\{p_n\}$ must satisfy r recurrence relations (2.2) or equivalently (2.4). Conversely, by the assumption (c) the moments $\{\langle w, x^n\rangle\}$ of w must satisfy the r moment equations (2.4) or equivalently (2.2). By replacing w with $\mu_0\langle w, 1\rangle^{-1}w$ if necessary, we may assume that $\langle w, 1\rangle = \mu_0$. Then w generates moments, that is, $\langle w, x^n\rangle = \mu_n$, $n = 0, 1, \ldots$, since the moment equations (2.4) are assumed to be uniquely solvable. Since any TSP is orthogonal with respect to any moment generating linear functional [9], $w(x)$ is an orthogonalizing weight of $\{p_n(x)\}$.

Following [14], we call the r differential operators R_k in (2.3) weight operators for $\{p_n(x)\}$.

Finally, let us note that there are many distributions, or even continuous functions, not identically 0, which have 0-moments. For example, we have

$$\int_0^\infty x^n \exp(-x^{1/4}) \sin x^{1/4}\, dx = 0, \quad n = 0, 1, \ldots,$$

which is due to Stieltjes [16].

3. *Application*

As an application of Theorem 2.3, we shall construct a real weight of bounded variation for the Bessel polynomials, which are polynomial solutions of

$$x^2 y''(x) + (2x+2)\, y'(x) = n(n+1)\, y(x). \tag{3.1}$$

The complex orthogonality of the Bessel polynomials has been well known for a long time. But, despite many attempts [5, 9, 10, 14, 15], the real weight for them has not been found, even though there must be such a weight, by the classical work of R. P. Boas [1] on the Stieltjes moment problem. In [5, 6], it is shown that the formal δ-series (1.3) corresponding to the Bessel polynomials converges only in the space of hyperfunctions to a hyperfunction with support at $\{0\}$, which cannot be a differential of a function of bounded variation. The symmetry factor $f(x)$ of equation (3.1) must be a solution of

$$x^2 f'(x) - 2f(x) = 0, \tag{3.2}$$

of which the point $x = 0$ is an irregular singular point. By the index theorem of H. Komatsu [7], equation (3.2) has exactly three linearly independent hyperfunction solutions: $f_0(x) = [\exp(-2/z)]$, $f_1(x) = \exp(-2/x + i0)$, and $f_2(x) = \exp(-2/x)$ for $x > 0$ and $f_2(x) = 0$ for $x \leqslant 0$ [7, 8]. Only the first, $f_0(x)$, which is a hyperfunction with a defining function $\exp(-2/z)$ and support at $\{0\}$, can be an orthogonalizing weight for the Bessel polynomials; it gives the complex orthogonality of Bessel polynomials [5, 6]. Hence equation (3.2) cannot give a real weight of bounded variation for the Bessel polynomials. Instead, by Theorem 2.3, we consider the following non-homogeneous equation

$$x^2 w'(x) - 2w(x) = g(x), \tag{3.3}$$

where $g(x)$ is any function which generates 0-moments. Let us take $g(x)$ to be

$$g(x) = \begin{cases} 0, & x \leqslant 0, \\ \exp(-x^{1/4}) \sin x^{1/4}, & x > 0. \end{cases} \tag{3.4}$$

For $x \neq 0$, equation (3.3) is satisfied by

$$w(x) = \begin{cases} 0, & x \leqslant 0, \\ -\exp(-2/x) \displaystyle\int_x^\infty t^{-2} \exp\left(-t^{1/4} + \frac{2}{t}\right) \sin t^{1/4}\, dt, & x > 0. \end{cases} \tag{3.5}$$

Note that $w(x)$ is continuous on R (even analytic for $x \neq 0$) and $|w(x)|$ decays to 0 exponentially as $|x|$ tends to ∞, so that $w(x)$ is in $L^1(R)$. Moreover, it is easy to see that $w(x)$ satisfies equation (3.3) on R in the sense of distribution. In fact, if we let w' and $[w']$ be the derivatives of w in the distributional and classical sense respectively, then $[w']$ has an algebraic singularity at $x = 0$ and $w' = \mathrm{v.p.}[w']$, the Cauchy principal value of $[w']$.

On the other hand, we have, for any integer $n \geqslant 0$,

$$\begin{aligned} \langle w, x^{n+1} \rangle = \int_0^\infty w(x)\, x^{n+1}\, dx &= \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty w(x)\, x^{n+1}\, dx \\ &= \frac{-1}{n+2} \lim_{\varepsilon \to 0^+} \left[\int_\varepsilon^\infty (2x^n w(x) + x^n g(x))\, dx \right] \\ &= \frac{-2}{n+2} \langle w, x^n \rangle \end{aligned}$$

by integration by parts and the fact that $g(x)$ has 0-moments.

In other words, the moments $\mu_n = \langle w, x^n \rangle$, $n = 0, 1, \ldots$, of w satisfy

$$(n+2)\, \mu_{n+1} + 2\mu_n = 0, \quad n = 0, 1, \ldots, \tag{3.6}$$

which is exactly the moment equation for the Bessel polynomials. Also, we must have $\langle w, 1 \rangle \neq 0$ (see Remark 3.1) since $\langle w, 1 \rangle = \int_0^\infty w(x)\, dx$ is an alternating series in which the absolute value of each term is strictly decreasing. Therefore, by Theorem 2.3, $w(x)$ is an orthogonalizing weight for the Bessel polynomials. Finally, if we set

$$\mu(x) = \int_0^x w(t)\, dt, \tag{3.7}$$

then $\mu(x)$ is a C^1-function (in fact, analytic for $x \neq 0$) of bounded variation with support in $[0, \infty)$, and the Bessel polynomials are orthogonal with respect to the Stieltjes measure $d\mu(x)$.

REMARK 3.1. Here we shall give more precise numerical evidence for $\langle w, 1 \rangle \neq 0$. Set

$$I = -\langle w, 1 \rangle = \int_0^\infty F(t)\, dt = \int_0^\infty \int_0^t t^{-2} \exp\left(-t^{1/4} + \frac{2}{t} - \frac{2}{x}\right) \sin t^{1/4}\, dx\, dt.$$

We shall show that $I > 0$ by dividing I into

$$I = I_1 + I_2,$$

where

$$I_1 = \int_0^{\pi^4} F(t)\,dt \quad \text{and} \quad I_2 = \int_{\pi^4}^{\infty} F(t)\,dt,$$

and estimating I_1 and I_2 separately. We have

$$\begin{aligned} |I_2| &\leqslant \int_{\pi^4}^{\infty} \int_0^t t^{-2} \exp\left(-t^{1/4} + \frac{2}{t} - \frac{2}{x}\right) dx\,dt \\ &\leqslant \int_{\pi^4}^{\infty} t^{-1} \exp(-t^{1/4})\,dt \\ &\leqslant (4/\pi)\exp(-\pi) \end{aligned}$$

and

$$I_1 \geqslant I_3 + I_4,$$

where

$$\begin{aligned} I_3 &= \int_1^{(\pi/2)^4} \int_{(\sqrt{3}-1)t}^{t} t^{-2} \exp\left(-t^{1/4} + \frac{2}{t} - \frac{2}{x}\right) \sin t^{1/4}\,dx\,dt \\ &\geqslant (2-\sqrt{3})\sin(1)\exp(1-\sqrt{3}) \int_1^{(\pi/2)^4} t^{-1}\exp(-t^{1/4})\,dt \\ &\geqslant (8/\pi)(2-\sqrt{3})\sin(1)\exp(1-\sqrt{3})[\exp(-1)-\exp(-\pi/2)] \end{aligned}$$

and

$$\begin{aligned} I_4 &= \int_{(\pi/2)^4}^{(3\pi/4)^4} \int_{t/2}^{t} t^{-2} \exp\left(-t^{1/4} + \frac{2}{t} - \frac{2}{x}\right) \sin t^{1/4}\,dx\,dt \\ &\geqslant (\sqrt{2}/4)\exp(-32/\pi^4) \int_{(\pi/2)^4}^{(3\pi/4)^4} t^{-1}\exp(-t^{1/4})\,dt \\ &\geqslant (4\sqrt{2}/3\pi)\exp(-32/\pi^4)[\exp(-\pi/2)-\exp(-3\pi/4)]. \end{aligned}$$

Hence we have

$$\begin{aligned} I_1/|I_2| \geqslant\ & 2(2-\sqrt{3})\sin(1)\exp(1-\sqrt{3})[\exp(\pi-1)-\exp(\pi/2)] \\ & + (\sqrt{2}/3)\exp(-32/\pi^4)[\exp(\pi/2)-\exp(\pi/4)], \end{aligned}$$

in which the right-hand side can be shown numerically to be larger than 1.5.

REMARK 3.2. In equation (3.3), we may replace $g(x)$ by another 0-moment generating function

$$h(x) = \begin{cases} 0, & x \leqslant 0, \\ \sin(2\pi \ln x)\exp(-\ln^2 x), & x > 0, \end{cases}$$

which is also due to Stieltjes (compare [**2**]). Then we obtain

$$w(x) = \begin{cases} 0, & x \leqslant 0, \\ -\exp(-2/x)\displaystyle\int_x^{\infty} t^{-2}\exp(2/t)\,h(t)\,dt, & x > 0. \end{cases}$$

For this new $w(x)$, we can repeat the same argument as the one for the old $w(x)$ in (3.5) and obtain another function of bounded variation as in (3.7), with respect to which the Bessel polynomials are orthogonal.

REMARK 3.3. Theorem 2.3 can also be applied to other TSPs. In particular, when $r = 1$, we are led to consider

$$\ell_2(x)\, w'(x) + (\ell_2'(x) - \ell_1(x))\, w(x) = f(x), \tag{3.8}$$

where $f(x)$ is any function having 0-moments. If $\ell_2(x) \neq 0$ for $x < c$, we may take $f(x)$ to be $g(x-c)$, where $g(x)$ is the function given by (3.4). Then equation (3.8) has the continuous function

$$w(x) = \begin{cases} 0, & x \leqslant c, \\ (-1/s(x)) \displaystyle\int_x^\infty s(t)\, g(x-c)/\ell_2(x)\, dx, & x > c \end{cases}$$

as a distributional solution, where

$$s(x) = \exp \int (\ell_2'(x) - \ell_1(x))/\ell_2(x)\, dx.$$

Since $w(x)$ decays to 0 exponentially as $|x|$ tends to ∞, $\mu(x) = \int_0^x w(t)\, dt$ is a function of bounded variation with support in $[c, \infty)$. In this way, we can obtain real weights of bounded variation in the closed form for the generalized Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}$ and the generalized Laguerre polynomials $\{L_n^{(\alpha)}(x)\}$, when $\alpha < -1$ and $\beta < -1$ are not integers.

References

1. R. P. BOAS, 'The Stieltjes moment problem for functions of bounded variation', *Bull. Amer. Math. Soc.* 45 (1949) 399–404.
2. T. S. CHIHARA, *An introduction to orthogonal polynomials* (Gordon and Breach, New York, 1978).
3. A. J. DURAN, 'The Stieltjes moment problem for rapidly decreasing functions', *Proc. Amer. Math. Soc.* 107 (3) (1989) 731–741.
4. A. J. DURAN, 'Functions with given moments and weight functions for orthogonal polynomials', *Rocky Mountain J. Math.*, to appear.
5. S. S. KIM and K. H. KWON, 'Generalized weights for orthogonal polynomials', *Differential Integral Equations* 4 (3) (1991) 601–608.
6. S. S. KIM and K. H. KWON, 'Hyperfunctional weights for orthogonal polynomials', *Resultate Math.* 18 (1990) 273–281.
7. H. KOMATSU, 'On the index of ordinary differential operators', *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 18 (1971) 379–398.
8. H. KOMATSU, *An introduction to the theory of hyperfunctions*, Lecture Notes in Math. 287 (Springer, New York, 1973) 3–40.
9. A. M. KRALL, 'Orthogonal polynomials through moment generating functionals', *SIAM J. Math. Anal.* 9 (1978) 600–603.
10. A. M. KRALL, 'The Bessel polynomial moment problem', *Acta Math. Acad. Sci. Hungar.* 38 (1981) 105–107.
11. A. M. KRALL, R. P. KANWAL and L. L. LITTLEJOHN, 'Distributional solutions of ordinary differential equations', *Proc. Conf. Oscillation, Bifurcation, Chaos* (University of Toronto, 1986).
12. H. L. KRALL, 'Certain differential equations for Tchebychev polynomials', *Duke Math. J.* 4 (1938) 705–718.
13. L. L. LITTLEJOHN, 'Symmetry factors for differential equations', *MAA Monthly* 7 (1983) 462–464.
14. L. L. LITTLEJOHN, 'On the classification of differential equations having orthogonal polynomial solutions', *Ann. Mat. Pura Appl.* 4 (1984) 35–53.
15. R. D. MORTON and A. M. KRALL, 'Distributional weight functions for orthogonal polynomials', *SIAM J. Math. Anal.* 9 (1978) 604–626.
16. E. D. RAINVILLE, *Special functions* (Macmillan, New York, 1960).

Department of Mathematics
KAIST
P.O. Box 150
Cheongryang
Seoul, 130-650
Korea