> $X := unapply(int(\mathrm{ChebyshevT}(n, x)\cdot\exp(-I\cdot x\cdot y), x=-1..1), n, y);$

$$X := (n, y) \mapsto \int_{-1}^{1} \mathrm{ChebyshevT}(n, x)\cdot \mathrm{e}^{-\mathrm{I}\cdot x\cdot y}\, \mathrm{d}x \tag{1}$$

> $V := (m, y) \rightarrow \left\{ \begin{array}{l} piecewise\left(y=0, 2, \frac{-\mathrm{I}\cdot(\mathrm{e}^{\mathrm{I}\cdot y}-\mathrm{e}^{-\mathrm{I}\cdot y})}{y}\right) \\ V0(m) \\ simplify\left(\left(\frac{\mathrm{I}\cdot\left(\mathrm{hypergeom}\left([1, m, -m], \left[\frac{1}{2}\right], \frac{\frac{\mathrm{I}}{2}}{y}\right)\cdot\mathrm{e}^{2\cdot\mathrm{I}\cdot\pi\cdot m-\mathrm{I}\cdot y}-\mathrm{hypergeom}\left([1, m, -m\right.\right.}{y}\right.\right. \end{array} \right.$

:

>

> $[seq(limit(V(m, y), y=0), m=1..10)]$

$$\left[0, -\frac{2}{3}, 0, -\frac{2}{15}, 0, -\frac{2}{35}, 0, -\frac{2}{63}, 0, -\frac{2}{99}\right] \tag{2}$$

> $recurrence := LREtools[GuessRecurrence](\mathbf{(2)}, a(n));$

$$recurrence := (n+4)\, a(n+2) - n\, a(n) = 0 \tag{3}$$

> $lresol := convert\left(rsolve\left(\left\{a(1) = -\frac{2}{3}, a(2) = 0, recurrence\right\}, a(n)\right), \exp\right);$

$$lresol := \frac{-\mathrm{e}^{\mathrm{I}\,\pi\,(-1+n)}-1}{(n+2)\,n} \tag{4}$$

> $V0 := unapply(piecewise(n=0, limit(lresol, n=0), n=-2, limit(lresol, n=-2), lresol), n)$

$$V0 := n \mapsto \begin{cases} \frac{\mathrm{I}}{2}\cdot\pi & n=0 \\ -\frac{\mathrm{I}}{2}\cdot\pi & n=-2 \\ \frac{-\mathrm{e}^{\mathrm{I}\cdot\pi\cdot(n-1)}-1}{(n+2)\cdot n} & otherwise \end{cases} \tag{5}$$

> $lprint(V(n, y))$

```
piecewise(n = 0,piecewise(y = 0,2,-I*(exp(I*y)-exp(-I*y))/y),y = 0,
piecewise(n
= 0,1/2*I*Pi,n = -2,-1/2*I*Pi,1/(n+2)/n*(-exp(I*Pi*(-1+n))-1)),I*
(hypergeom([1,
n, -n],[1/2],1/2*I/y)*exp(I*(2*Pi*n-y))-hypergeom([1, n, -n],[1/2],
-1/2*I/y)*
exp(I*(Pi*n+y)))/y)
```

>

> $Vector([seq(V(n, y), n=0..5)])$ assuming $y > 0$

$$\left[\begin{array}{c}\frac{-\mathrm{I}\,(\mathrm{e}^{\mathrm{I}\,y}-\mathrm{e}^{-\mathrm{I}\,y})}{y}\\ \frac{2\,\mathrm{I}\,(y\cos(y)-\sin(y))}{y^2}\\ \frac{2\sin(y)\,y^2+8\,y\cos(y)-8\sin(y)}{y^3}\\ \frac{2\,\mathrm{I}\,(\cos(y)\,y^3-9\sin(y)\,y^2-24\,y\cos(y)+24\sin(y))}{y^4}\\ \frac{(2\,y^4-160\,y^2+384)\sin(y)+32\,y\,(y^2-12)\cos(y)}{y^5}\\ \frac{2\,\mathrm{I}\left((y^4-200\,y^2+1920)\,y\cos(y)-25\left(y^4-\frac{168}{5}\,y^2+\frac{384}{5}\right)\sin(y)\right)}{y^6}\end{array}\right] \tag{6}$$

> $[seq(limit(V(m,y), y=0), m=0..10)]$

$$\left[2, 0, -\frac{2}{3}, 0, -\frac{2}{15}, 0, -\frac{2}{35}, 0, -\frac{2}{63}, 0, -\frac{2}{99}\right] \tag{7}$$

>

>

> $Vnorms := [seq(int(V(m,y)^2, y=0..\mathrm{infinity}), m=0..20)]$

$$Vnorms := \left[2\,\pi, -\frac{2\,\pi}{3}, \frac{14\,\pi}{15}, -\frac{34\,\pi}{35}, \frac{62\,\pi}{63}, -\frac{98\,\pi}{99}, \frac{142\,\pi}{143}, -\frac{194\,\pi}{195}, \frac{254\,\pi}{255}, -\frac{322\,\pi}{323}, \frac{398\,\pi}{399}, -\frac{482\,\pi}{483}, \frac{574\,\pi}{575}, -\frac{674\,\pi}{675}, \frac{782\,\pi}{783}, -\frac{898\,\pi}{899}, \frac{1022\,\pi}{1023}, -\frac{1154\,\pi}{1155}, \frac{1294\,\pi}{1295}, -\frac{1442\,\pi}{1443}, \frac{1598\,\pi}{1599}\right] \tag{8}$$

> $lresol := LREtools[GuessRecurrence](\mathbf{(8)}, a(n));$

$$lresol := (2\,n+3)\,(2\,n^2-1)\,a(1+n)+(2\,n-1)\,(2\,n^2+4\,n+1)\,a(n)=0 \tag{9}$$

> $Vnorm := unapply(convert(simplify(rsolve(\{a(0)=Vnorms[1], a(1) = Vnorms[2], a(2)=Vnorms[3], a(3)=Vnorms[4], a(4)=Vnorms[5], lresol\}, a)), \exp), n);$

$$Vnorm := n \mapsto \frac{(4\cdot\pi\cdot n^2-2\cdot\pi)\cdot\mathrm{e}^{\mathrm{I}\cdot n\cdot\pi}}{4\cdot n^2-1} \tag{10}$$

>

>

> $Y := unapply\left(simplify\left(\frac{V(m,y)}{Vnorm(m)}\right), m, y\right):$
    *# Normalized Fourier transformed Type-I Chebyshev Polynomials*

> $simplify(Y(1,y))$

$$\frac{\begin{cases} 1 & y=0 \\ \frac{-3\,\mathrm{I}\,(y\cos(y)-\sin(y))}{y^2} & \textit{otherwise} \end{cases}}{\pi} \qquad \textbf{(11)}$$

> $Matrix(map(simplify, [seq([(X(m, y))-(V(m, y))], m = 0..4)]))$ assuming $y \neq 0$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \textbf{(12)}$$

>

>

> $f := unapply(int(\mathrm{BesselJ}(0, y)\cdot\exp(-I\cdot x\cdot y), y = 0..\mathrm{infinity}), x)$ assuming $x \neq 0$

$$f := x \mapsto \frac{1}{\sqrt{-x^2+1}} \qquad \textbf{(13)}$$

```
> # Gram-Schmidt process
GramSchmidt := proc(vecList, a, b)
local n, m, inner, projection, u, v, orthoList;
orthoList := [ ];
for n from 1 to numelems(vecList) do
   v := vecList[n];
   projection := 0;
   for m from 1 to n−1 do
      u := orthoList[m];
      inner := int(u * v, y = a .. b);
      projection := projection + inner * u;
   end do;
   u := simplify(v − projection);
   u := simplify(u / sqrt(int(u * conjugate(u), y = a .. b)));
   orthoList := [op(orthoList), u];
end do;
return orthoList;
 end proc:
```

```
> # Generate the functions
 phi := map(simplify, [seq(Y(n, y), n = 0 .. 20)]) :
```

>

```
>  # Apply Gram-Schmidt process
psi := GramSchmidt(phi,−infinity, infinity) :
```

> $Vector([seq(int(\mathrm{phi}[m]\cdot \mathrm{BesselJ}(n, y), y=-\mathrm{infinity}..\mathrm{infinity}), m = 1..6)])$

$$\left[\begin{array}{c} \frac{(-1)^n \sin\left(\frac{\pi n}{2}\right)}{\pi n} + \frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n} \\ \frac{3\,\mathrm{I}\,(-1)^n \cos\left(\frac{\pi n}{2}\right) n}{\pi\,(-1+n)\,(1+n)} - \frac{3\,\mathrm{I}\cos\left(\frac{\pi n}{2}\right) n}{\pi\,(-1+n)\,(1+n)} \\ \frac{15\,n\,(-1)^n \sin\left(\frac{\pi n}{2}\right)}{7\,(-2+n)\,(n+2)\,\pi} + \frac{15\,n \sin\left(\frac{\pi n}{2}\right)}{7\,(-2+n)\,(n+2)\,\pi} \\ \frac{35\,\mathrm{I}\,n\,(-1)^n \cos\left(\frac{\pi n}{2}\right)}{17\,(n-3)\,(3+n)\,\pi} - \frac{35\,\mathrm{I}\,n \cos\left(\frac{\pi n}{2}\right)}{17\,(n-3)\,(3+n)\,\pi} \\ \frac{63\,n\,(-1)^n \sin\left(\frac{\pi n}{2}\right)}{31\,(n-4)\,(n+4)\,\pi} + \frac{63\,n \sin\left(\frac{\pi n}{2}\right)}{31\,(n-4)\,(n+4)\,\pi} \\ \frac{99\,\mathrm{I}\,(-1)^n \cos\left(\frac{\pi n}{2}\right) n}{49\,(n-5)\,(5+n)\,\pi} - \frac{99\,\mathrm{I}\cos\left(\frac{\pi n}{2}\right) n}{49\,(n-5)\,(5+n)\,\pi} \end{array}\right] \quad (14)$$

> $LREtools[GuessRecurrence](\mathbf{(14)}, a(n))$

$$FAIL \quad (15)$$

>

>

> $\mathrm{eta} := k \rightarrow \sqrt{\frac{4\cdot k+1}{3+2\cdot k}} \cdot \left(\sin(\mathrm{Pi}*n/2) * product((n+2*i-1)*(n-(2*i-1)), i = 1..k)\cdot n\cdot \mathrm{sqrt}\left(2\cdot\left(k+\frac{3}{2}\right)\right)\right) / (product((n+2*j-2)*(n-(2*j-2)), j = 1..(k+1)) * \mathrm{sqrt}(\mathrm{Pi}));$

$$\eta := k \mapsto \frac{\sqrt{\frac{4\cdot k+1}{3+2\cdot k}} \cdot \sin\left(\frac{\pi\cdot n}{2}\right) \cdot \left(\prod_{i=1}^{k} (n+2\cdot i-1)\cdot(n-2\cdot i+1)\right) \cdot n \cdot \sqrt{3+2\cdot k}}{\left(\prod_{j=1}^{k+1} (n+2\cdot j-2)\cdot(n-2\cdot j+2)\right)\cdot\sqrt{\pi}} \quad (16)$$

> $wtf := k \rightarrow \mathrm{sqrt}\left(\frac{4\,k-3}{2\,k+1}\right)$

$$wtf := k \mapsto \sqrt{\frac{4\cdot k-3}{2\cdot k+1}} \quad (17)$$

> $ftw := unapply(wtf(k+1), k);$

$$ftw := k \mapsto \sqrt{\frac{4 \cdot k + 1}{3 + 2 \cdot k}} \tag{18}$$

> $Matrix([seq([\mathrm{eta}(k), int(\mathrm{psi}[2k+1] \cdot \mathrm{BesselJ}(0, y), y = 0..\mathrm{infinity})], k = 0..3)])$

$$\left[\left[\frac{\sin\left(\frac{\pi n}{2}\right)}{\sqrt{\pi}\, n}, \frac{\sqrt{\pi}}{2}\right], \left[\frac{\sin\left(\frac{\pi n}{2}\right)(1+n)(-1+n)\sqrt{5}}{n(n+2)(-2+n)\sqrt{\pi}}, \frac{\sqrt{\pi}\sqrt{5}}{8}\right], \left[\frac{3\sin\left(\frac{\pi n}{2}\right)(-1+n)(1+n)(3+n)(n-3)}{n(n+2)(-2+n)(n+4)(n-4)\sqrt{\pi}}, \frac{27\sqrt{\pi}}{128}\right], \left[\frac{\sqrt{13}\sin\left(\frac{\pi n}{2}\right)(-1+n)(1+n)(3+n)(n-3)(5+n)(n-5)}{n(n+2)(-2+n)(n+4)(n-4)(n+6)(n-6)\sqrt{\pi}}, \frac{25\sqrt{\pi}\sqrt{13}}{512}\right]\right] \tag{19}$$

> $map(simplify, map(z \to limit(z, n = 0), \mathbf{(19)}))$

$$\begin{bmatrix} \frac{\sqrt{\pi}}{2} & \frac{\sqrt{\pi}}{2} \\ \frac{\sqrt{\pi}\sqrt{5}}{8} & \frac{\sqrt{\pi}\sqrt{5}}{8} \\ \frac{27\sqrt{\pi}}{128} & \frac{27\sqrt{\pi}}{128} \\ \frac{25\sqrt{\pi}\sqrt{13}}{512} & \frac{25\sqrt{\pi}\sqrt{13}}{512} \end{bmatrix} \tag{20}$$

> $evalf(\mathbf{(20)})$

$$\begin{bmatrix} 0.8862269255 & 0.8862269255 \\ 0.4954159121 & 0.4954159121 \\ 0.3738769842 & 0.3738769842 \\ 0.3120445919 & 0.3120445919 \end{bmatrix} \tag{21}$$

>

> $int(\mathrm{psi}[1] \cdot \mathrm{BesselJ}(0, y), y = 0..\mathrm{infinity})$

$$\frac{\sqrt{\pi}}{2} \tag{22}$$

> $simplify(\mathrm{eta}(k))$

$$\frac{\sqrt{3+2k}\,\cos\left(\frac{\pi n}{2}\right)\Gamma\left(k+\frac{1}{2}-\frac{n}{2}\right)\Gamma\left(k+\frac{1}{2}+\frac{n}{2}\right)\sqrt{\frac{4k+1}{3+2k}}}{2\sqrt{\pi}\,\Gamma\left(k+1-\frac{n}{2}\right)\Gamma\left(k+1+\frac{n}{2}\right)} \tag{23}$$

>

> $eta0 := unapply(simplify(limit(\mathrm{eta}(k), n=0)), k)$ assuming $k :: posint$

$$\eta 0 := k \mapsto \frac{\Gamma\left(k+\frac{1}{2}\right)^2 \cdot \sqrt{4 \cdot k+1}}{2 \cdot \Gamma(k+1)^2 \cdot \sqrt{\pi}} \tag{24}$$

>

> $Vector([seq(eta0(k), k = 0..8)])$

$$\begin{bmatrix} \frac{\sqrt{\pi}}{2} \\ \frac{\sqrt{\pi}\sqrt{5}}{8} \\ \frac{9\sqrt{\pi}\sqrt{9}}{128} \\ \frac{25\sqrt{\pi}\sqrt{13}}{512} \\ \frac{1225\sqrt{\pi}\sqrt{17}}{32768} \\ \frac{3969\sqrt{\pi}\sqrt{21}}{131072} \\ \frac{53361\sqrt{\pi}\sqrt{25}}{2097152} \\ \frac{184041\sqrt{\pi}\sqrt{29}}{8388608} \\ \frac{41409225\sqrt{\pi}\sqrt{33}}{2147483648} \end{bmatrix} \tag{25}$$

> $[seq(int(\mathrm{psi}[k]^2, y = 0..\mathrm{infinity}), k = 1..10)]$

(26)

$$\left[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] \quad (26)$$

>

>

> $sinfact := unapply\left(\frac{y^i \cdot \mathrm{sqrt}(\mathrm{Pi})}{\sin(y) \cdot \mathrm{sqrt}(2 \cdot i - 1)}, i, y\right);$

$$sinfact := (i, y) \mapsto \frac{y^i \cdot \sqrt{\pi}}{\sin(y) \cdot \sqrt{2 \cdot i - 1}} \quad (27)$$

> $cosfact := unapply\left(\frac{y^{i-1} \cdot \sqrt{\pi}}{\cos(y) \cdot \sqrt{2 \cdot i - 1}}, i, y\right);$

$$cosfact := (i, y) \mapsto \frac{y^{i-1} \cdot \sqrt{\pi}}{\cos(y) \cdot \sqrt{2 \cdot i - 1}} \quad (28)$$

> $hmmsin := map(z \to (op(1, z)), \mathrm{psi}) : coolsin := Vector([seq(simplify(hmmsin[i] \cdot sinfact(i, y)), i = 1..op(1, hmmsin))])$

Error, range bounds in seq must be numeric or character

>

> $hmmcos := map(z \to (op(2, z)), \mathrm{psi}) : coolcos := Vector([seq(simplify(hmmcos[i] \cdot cosfact(i, y)), i = 1..op(1, hmmsin))])$

>

> $cosMatrix := convert(convert(map(z \to ListTools{:}\text{-}Reverse(convert(MTM{:}\text{-}coeffs(z, y), list)), coolcos[2..op(1, coolcos)]), list), Matrix)$

>

> $sinMatrix := convert(convert(map(z \to ListTools{:}\text{-}Reverse(convert(MTM{:}\text{-}coeffs(z, y), list)), coolsin[2..op(1, coolsin)]), list), Matrix)$

> $B := unapply(convert(\mathrm{pochhammer}(n+1, k) * \mathrm{binomial}(n, k) / 2\verb|^|k, \mathrm{binomial}), n, k)$

>

> $C := unapply((simplify((-1)^{n+1} \cdot (-1)\verb|^|\mathrm{binomial}(k, 2) \cdot B(k+1, k-(2n-1)))), n, k)$ assuming $n :: nonnegint, k :: nonnegint;$#column row

> $Cgen := unapply\left(simplify\left(sum\left(\frac{C(n, m) \cdot x^n}{n!}, n = 0..\mathrm{infinity}\right)\right), x, m\right);$

>

> $Vector([seq(simplify(Cgen(x, m)), m = 0..10)])$

>

>

> $Matrix\left(\left[seq\left(\left[seq\left(C(n, k), n = 0..\mathrm{floor}\left(\frac{k+1}{2}\right)\right)\right], k = 0..10\right)\right]\right)$

>

> $LS := unapply(simplify(((-1)\verb|^|(m + \mathrm{binomial}(n, 2)) * 2\verb|^|(n-2*m) * \mathrm{factorial}(n-m) * \mathrm{binomial}(1/2-m+n, n-2*m) / \mathrm{factorial}(m))), m, n);$

```
>
> coolcos := unapply(simplify(sum(LS(n, m)·x^n, n = 0..infinity) ), m, x)
>
>
> psisin := unapply( sum(C(m, n-2)·x^(2m), m = 0..n-2)·sqrt(2n-1)·sin(x) / (x^n·sqrt(Pi)),
      n, x);
> seq(is(op(1, collect(psi[n], sin(y))) = psisin(n, y)), n = 1..op(1, psi))
>
>
> psicos :=
      unapply(simplify(convert( 1/(x^n·sqrt(Pi)) (sum(LS(m, n-2)·x^(2m+1), m
       = 0..n-2)·sqrt(2n-1)·cos(x)), exp)), n, x) assuming n :: posint, x
       :: real
>
>
> [seq( int(yay(2n-1, y), y = 0..infinity) , n = 1..10)]
> map(z→z^2, ??)
> LREtools[GuessRecurrence](??, a(n))
> simplify(psicos(n, y))
> map(z→denom(z)/4, ??); # 4^(number of 1's in binary expansion of 2 n.)  https://oeis·org
       /A056982
> map(z→numer(z)/Pi, ??);
      #(4n+1) Binomial[2n, n]^2 / 4^DigitCount[n, 2, 1], https://oeis.
      org/A110257
>
```