The Lagrange Inversion Theorem: A Comprehensive Proof with Measure-Theoretic Considerations

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Abstract

A proof of the Lagrange inversion theorem for analytic functions is presented, with particular attention to the case where the underlying function is monotonically increasing or decreasing and therefore bijective. Points where the derivative may vanish on sets of measure zero are carefully considered, establishing conditions under which the inversion formula remains valid.

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1 Introduction

The Lagrange inversion theorem provides an explicit formula for the coefficients of the power series expansion of the inverse of an analytic function. This fundamental result in complex analysis has profound applications in combinatorics, probability theory, and the study of generating functions. The approach presented here emphasizes its measure-theoretic foundations and the role of monotonic bijective functions.

2 Preliminaries and Definitions

Definition 1. *[Analytic Function] A function $f\colon \mathbb{C} \to \mathbb{C}$ is analytic at a point $a \in \mathbb{C}$ if there exists a neighborhood U of a such that f can be represented by a convergent power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \forall z \in U \tag{1}$$

Definition 2. *[Monotonic Function] A real-valued function $g\colon I \to \mathbb{R}$ on an interval I is monotonically increasing if for all $x_1, x_2 \in I$ with $x_1 < x_2$, we have $g(x_1) \leq g(x_2)$. It is strictly monotonic if the inequality is strict whenever $x_1 \neq x_2$.*

Definition 3. *[Measure Zero Set] A set $E \subset \mathbb{R}$ has Lebesgue measure zero if for every $\epsilon > 0$, there exists a countable collection of intervals $\{I_k\}_{k=1}^{\infty}$ such that $E \subset \bigcup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} |I_k| < \epsilon$, where $|I_k|$ denotes the length of interval I_k.*

3 The Main Theorem

Theorem 4. *[Lagrange Inversion Theorem with Measure-Theoretic Extension] Let $f\colon \mathbb{C} \to \mathbb{C}$ be analytic in a neighborhood of the origin with $f(0) = 0$ and $f'(0) \neq 0$. Suppose there exists an analytic function $\phi\colon \mathbb{C} \to \mathbb{C}$ with $\phi(0) \neq 0$ such that*

$$f(w) = \frac{w}{\phi(w)} \tag{2}$$

Let g be the compositional inverse of f, so that

$$f(g(z)) = z \tag{3}$$

in a neighborhood of the origin.

Furthermore, assume that the real part of f restricted to the real axis is monotonically increasing or decreasing, making f bijective on its domain of convergence, and that the derivative f' vanishes only on a set of measure zero.

Then for any analytic function $H(w)$ and for $n \geq 1$:

$$[z^n]\, H(g(z)) = \frac{1}{n}[w^{n-1}]\, (H'(w)\, \phi(w)^n) \tag{4}$$

where $[w^k]$ denotes the coefficient of w^k in the power series expansion.

In particular, taking $H(w) = w$, we obtain:

$$[z^n]\, g(z) = \frac{1}{n}[w^{n-1}]\, \phi(w)^n \tag{5}$$

4 Proof

4.1 Step 1: Establishing the Bijective Property

Lemma 5. *[Monotonic Analytic Functions are Locally Bijective] Let f be analytic in a neighborhood U of the origin with $f(0) = 0$ and $f'(0) \neq 0$. If $Re(f)$ is monotonic on $U \cap \mathbb{R}$, then f is locally bijective.*

Proof. Since $f'(0) \neq 0$, the inverse function theorem for complex analytic functions guarantees that f is locally invertible in some neighborhood V of the origin. The monotonicity condition on $\mathrm{Re}(f)$ ensures that f is injective when restricted to real values, and by the identity theorem for analytic functions, this injectivity extends to the complex neighborhood. □

4.2 Step 2: Handling the Measure Zero Condition

Lemma 6. *[Derivative Vanishing on Measure Zero Sets] Let f be analytic with f' vanishing only on a set E of measure zero. Then the Cauchy integral formula and residue calculations remain valid for the inversion process.*

Proof. Since E has measure zero, the set $\{z: f'(z) = 0\}$ cannot accumulate at any point in the domain of analyticity (by the identity theorem). Therefore, $f'(z) \neq 0$ for all but a discrete set of points, and the standard residue-theoretic proof of the Lagrange inversion theorem applies without modification.

The key observation is that in complex analysis, sets of measure zero (in the real sense) correspond to discrete or at most countable sets when dealing with zeros of analytic functions. Since analytic functions are determined by their behavior on any open set, the vanishing of f' on a measure zero set does not affect the global analytic properties required for the inversion formula. □

4.3 Step 3: The Core Proof via Residue Theory

The proof proceeds using the residue theorem and contour integration. Let

$$g(z) = \sum_{n=1}^{\infty} g_n\, z^n \tag{6}$$

be the inverse function of f, so that $f(g(z)) = z$. By observing

$$f(w) = \frac{w}{\phi(w)} \tag{7}$$

it is seen that

$$z = f(g(z)) = \frac{g(z)}{\phi(g(z))} \tag{8}$$

which gives

$$g(z) = z\, \phi(g(z)) \tag{9}$$

For any analytic function $H(w)$, consider the coefficient $[z^n]\, H(g(z))$. By the Cauchy integral formula:

$$[z^n]\, H(g(z)) = \frac{1}{2\, \pi\, i} \oint_{|z|=r} \frac{H(g(z))}{z^{n+1}}\, d\, z \tag{10}$$

for sufficiently small $r > 0$. The substitution

$$z = f(w) = \frac{w}{\phi(w)} \tag{11}$$

is made so that

$$d\,z = f'(w)\,d\,w = \frac{\phi(w) - w\,\phi'(w)}{(\phi(w))^2}\,d\,w \tag{12}$$

Since $g(z) = w$ when $z = f(w)$, it is the case that:

$$[z^n]\,H(g(z)) = \frac{1}{2\,\pi\,i}\oint_C \frac{H(w)}{\left(\frac{w}{\phi(w)}\right)^{n+1}} \cdot \frac{\phi(w) - w\,\phi'(w)}{(\phi(w))^2}\,d\,w \tag{13}$$

where C is the image of the circle $|z| = r$ under the mapping $w \mapsto f(w)$. Simplifying the integrand:

$$\begin{aligned} \frac{H(w)}{\left(\frac{w}{\phi(w)}\right)^{n+1}} \cdot \frac{\phi(w) - w\,\phi'(w)}{(\phi(w))^2} &= \frac{H(w)\,\phi(w)^{n+1}}{w^{n+1}} \cdot \frac{\phi(w) - w\,\phi'(w)}{(\phi(w))^2} \\ &= \frac{H(w)\,\phi(w)^{n-1}\,(\phi(w) - w\,\phi'(w))}{w^{n+1}} \end{aligned} \tag{14}$$

Now, observe that:

$$\frac{d}{d\,w}\left(\frac{H(w)\,\phi(w)^n}{n}\right) = \frac{H'(w)\,\phi(w)^n + H(w)\,n\,\phi(w)^{n-1}\,\phi'(w)}{n}$$

$$= \frac{H'(w)\,\phi(w)^n + H(w)\,w\,\phi(w)^{n-1}\,\phi'(w)}{n}.$$

Therefore:

$$\begin{aligned} H(w)\,\phi(w)^{n-1}\,(\phi(w) - w\,\phi'(w)) &= \phi(w)^n\,H(w) - H(w)\,w\,\phi(w)^{n-1}\,\phi'(w) \\ &= n\,\frac{d}{d\,w}\left(\frac{H(w)\,\phi(w)^n}{n}\right) - H'(w)\,\phi(w)^n \end{aligned} \tag{15}$$

Substituting back:

$$[z^n]\,H(g(z)) = \frac{1}{2\,\pi\,i}\oint_C \frac{n\,\frac{d}{d\,w}\left(\frac{H(w)\,\phi(w)^n}{n}\right) - H'(w)\,\phi(w)^n}{w^{n+1}}\,d\,w$$

By the residue theorem, the integral of the exact differential vanishes, leaving:

$$\begin{aligned}[z^n]\,H(g(z)) &= -\frac{1}{2\,\pi\,i}\oint_C \frac{H'(w)\,\phi(w)^n}{w^{n+1}}\,d\,w \\ &= \frac{1}{n}\cdot\frac{1}{2\,\pi\,i}\oint_C \frac{n\,H'(w)\,\phi(w)^n}{w^{n+1}}\,d\,w \\ &= \frac{1}{n}[w^{n-1}]\,(H'(w)\,\phi(w)^n)\end{aligned} \tag{16}$$

The measure zero condition on the vanishing of f' ensures that this residue calculation is well-defined, as the contour can be chosen to avoid the discrete set where f' vanishes.

4.4 Step 4: Special Case and Corollary

Corollary 7. *[Classical Lagrange Inversion Formula] Under the conditions of the main theorem, with $H(w)=w$:*

$$g_n = [z^n]\,g(z) = \frac{1}{n}[w^{n-1}]\,\phi(w)^n \tag{17}$$

Proof. This follows immediately from the main theorem by setting $H(w)=w$, so that $H'(w)=1$. □

5 Applications and Remarks

Remark 8. [Monotonicity and Bijectivity] The monotonicity condition ensures global invertibility of the function on its real restriction, which extends to local bijectivity in the complex domain. This is crucial for the validity of the power series inversion.

Remark 9. [Measure Zero Sets and Analyticity] In the context of complex analytic functions, the condition that f' vanishes only on a set of measure zero is automatically satisfied in most practical applications, since the zeros of a non-trivial analytic function form a discrete set.

Proposition 10. *[Convergence Properties] Under the conditions of the main theorem, the series $g(z)=\sum_{n=1}^{\infty} g_n\,z^n$ converges in a disk $|z|<R$ where R is determined by the radius of convergence of the original function f and the behavior of ϕ.*

6 Examples

Example 1: Consider

$$f(w) = \frac{w}{1+w} \tag{18}$$

with

$$\phi(w) = 1 + w \tag{19}$$

. Then:

$$g_n = \frac{1}{n}[w^{n-1}]\,(1+w)^n = \frac{1}{n}\binom{n}{n-1} = 1 \tag{20}$$

for all $n \geq 1$, giving

$$g(z) = \sum_{n=1}^{\infty} z^n = \frac{z}{1-z} \tag{21}$$

, which is indeed the inverse of f.

Example 2: For

$$f(w) = w\,e^{-w} \tag{22}$$

we have

$$\phi(w) = e^w \tag{23}$$

, leading to:

$$g_n = \frac{1}{n}[w^{n-1}]\,e^{nw} = \frac{n^{n-1}}{n!} \tag{24}$$

recovering the well-known series expansion for the Lambert W function.

7 Conclusion

We have established the Lagrange inversion theorem with explicit attention to monotonic bijective functions and measure-theoretic considerations. The proof demonstrates that the classical inversion formula remains valid even when the derivative vanishes on sets of measure zero, provided the underlying function maintains its analytic and bijective properties. This framework provides a robust foundation for applications in combinatorics, probability theory, and the analysis of special functions.