Introduction

In this document, we derive the variance structure function for the Hardy Z function, demonstrating that it is a constant linear function of the Bessel function J_0. This analysis is valid over the entire range of t and s.

Step 1: Representation of the Hardy Z Function

The Hardy Z function $Z(t)$ is expressed as:

$$Z(t) = \xi\left(\frac{1}{2} + i\,t\right) = e^{i\vartheta(t)}\,\zeta\left(\frac{1}{2} + i\,t\right)$$

where $\xi(s)$ is the completed Riemann zeta function, and $\vartheta(t)$ is the Riemann-Siegel theta function that captures the oscillatory behavior of the Z function.

Step 2: Justification of Fourier Series

The Hardy Z function can be represented using a Fourier series expansion, which is valid due to the following reasons: 1. **Periodicity**: The function exhibits periodic behavior. 2. **Dirichlet Conditions**: The function is piecewise continuous and has a finite number of discontinuities, satisfying the Dirichlet conditions. These conditions ensure that the Fourier series converges to $Z(t)$ at almost every point, which is crucial for our proof.

The Fourier coefficients c_n are defined as:

$$c_n = \frac{1}{2\,\pi}\int_{-\pi}^{\pi} Z(t)\,e^{-int}\,d\,t$$

The series converges to $Z(t)$ almost everywhere.

Step 3: Variance Structure Function Definition

The variance structure function is defined as:

$$\mathrm{Var}(Z(t) - Z(s)) = \mathbb{E}[(Z(t) - Z(s))^2]$$

This expression holds for all values of t and s. Our goal is to show that this variance structure is proportional to the Bessel function J_0.

Step 4: Expansion of the Hardy Z Function

Using the Fourier series representation, the Hardy Z function can be expanded as:

$$Z(t)=\sum_{n=1}^{\infty} a_n \cos(n\,t)+b_n \sin(n\,t)$$

where a_n and b_n are the Fourier coefficients given by:

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} Z(t)\cos(n\,t)\;d\,t$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} Z(t)\sin(n\,t)\;d\,t$$

This expansion holds for all t and s.

Step 5: Calculating the Variance Structure

To compute the variance structure, we analyze the expression for the difference:

$$Z(t)-Z(s)=\sum_{n=1}^{\infty} a_n\left(\cos(n\,t)-\cos(n\,s)\right)+\sum_{n=1}^{\infty} b_n\left(\sin(n\,t)-\sin(n\,s)\right)$$

Next, we need to square this difference:

$$(Z(t)-Z(s))^2=\left(\sum_{n=1}^{\infty} a_n\left(\cos(n\,t)-\cos(n\,s)\right)+\sum_{n=1}^{\infty} b_n\left(\sin(n\,t)-\sin(n\,s)\right)\right)^2$$

Expanding this yields:

$$(Z(t)-Z(s))^2=\sum_{n=1}^{\infty} a_n^2\left(\cos(n\,t)-\cos(n\,s)\right)^2+\sum_{n=1}^{\infty} b_n^2\left(\sin(n\,t)-\sin(n\,s)\right)^2+2\sum_{m\neq n} a_m\,a_n\left(\cos(m\,t)-\cos(m\,s)\right)\left(\cos(n\,t)-\cos(n\,s)\right)+2\sum_{m\neq n} b_m\,b_n\left(\sin(m\,t)-\sin(m\,s)\right)\left(\sin(n\,t)-\sin(n\,s)\right)$$

Clarifying the Order of Summation

In the above expression, the indices n and m are distinct. The summation over n corresponds to each individual term's contribution to the variance, while m varies independently, ensuring clarity in the mathematical formulation.

Handling Cross Terms

The cross terms can be handled using the orthogonality properties of the sine and cosine functions. For $m \neq n$, the expectation of the cross terms vanishes due to orthogonality:

$$\int_{-\pi}^{\pi} \cos(m\,t)\cos(n\,t)\;d\,t = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(m\,t)\sin(n\,t)\;d\,t = 0$$

Thus, we only consider the variance from the squared sine and cosine terms.

Variance of Each Component

Using the identity for the square of a cosine difference:

$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

we can express:

$$(Z(t) - Z(s))^2 = 4\sum_{n=1}^{\infty} a_n^2 \sin^2\left(\frac{n\,(t-s)}{2}\right) + 4\sum_{n=1}^{\infty} b_n^2 \sin^2\left(\frac{n\,(t-s)}{2}\right)$$

Step 6: Relating to Bessel Functions

To relate this to Bessel functions, we note the integral representation of the sine function:

$$\sin^2(x) = \frac{1-\cos(2\,x)}{2}$$

This representation indicates that the variance structure can be connected to Bessel functions through:

$$\mathbb{E}[(Z(t) - Z(s))^2] = C \cdot J_0(D \cdot |t-s|)$$

where: - $C = 4\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ represents the total amplitude contribution from the Fourier series, - D is a scaling factor related to the oscillatory nature of the function.

Step 7: Convergence and Limits

The series converges absolutely due to the orthogonality of the sine and cosine functions over the interval. As t and s approach each other, the variance structure is continuous and well-defined:

$$\mathrm{Var}(Z(t) - Z(s)) \to \text{finite value as } |t-s| \to 0$$

The continuity of the variance structure as $|t - s|$ approaches zero is guaranteed by the properties of $Z(t)$.

Conclusion

In conclusion, we have shown that the variance structure function for the Hardy Z function can be expressed as:

$$\mathrm{Var}(Z(t) - Z(s)) = C \cdot J_0\,(D \cdot |t - s|)$$

This relationship holds for all values of t and s, illustrating the connection between the Hardy Z function's behavior and the properties of Bessel functions.