Unitary Time Changes of Stationary Processes Yield Oscillatory Processes

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are locally square-integrable (meaning over compact sets). Applying U_θ to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}$, evolutionary spectrum $d\,F_t(\lambda) = \dot{\theta}(t)\, d\,F(\lambda)$, and expected zero-counting function $\mathbb{E}[N_{[0,T]}] = \frac{[\theta(T) - \theta(0)]}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}}$. The sample paths of any non-degenerate second-order stationary process are locally square integrable, making the unitary time-change operator U_θ applicable to typical realizations. A zero-localization measure $d\,\mu(t) = \delta(Z(t))|\dot{Z}(t)|\, d\,t$ induces a Hilbert space $L^2(\mu)$ on the zero set of each oscillatory process realization $Z(t)$, and the multiplication operator $(L\,f)(t) = t\,f(t)$ has simple pure point spectrum equal to the zero crossing set of Z.

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1 Gaussian Processes

Unless otherwise stated, all processes considered will be real-valued.

Theorem 1. *Let $X(u)$ be a real-valued process:*

$$X(u) \in \mathbb{R} \quad \forall u \in \mathbb{R} \tag{1}$$

Then its (complex-valued) random orthogonal spectral measure satisfies

$$d\,\bar{\Phi}(\lambda) = d\,\Phi\,(-\lambda) \tag{2}$$

and the corresponding covariance spectral measure F is even:

$$F\,(-A) = F(A) \tag{3}$$

Proof. 1. The spectral representation for $X(u)$ is

$$X(u) = \int_{-\infty}^{\infty} e^{i\lambda u}\ d\,\Phi(\lambda) \tag{4}$$

Since $X(u)$ is real-valued for each u,

$$\overline{X(u)} = X(u) \tag{5}$$

On the other hand,

$$\overline{X(u)} = \overline{\int_{-\infty}^{\infty} e^{i\lambda u}\ d\,\Phi(\lambda)} \tag{6}$$

$$= \int_{-\infty}^{\infty} \overline{e^{i\lambda u}}\, d\,\bar{\Phi}(\lambda) \tag{7}$$

$$= \int_{-\infty}^{\infty} e^{-i\lambda u}\ d\,\bar{\Phi}(\lambda) \tag{8}$$

By the substitution $\lambda \mapsto -\mu$,

$$\int_{-\infty}^{\infty} e^{-i\lambda u}\ d\,\bar{\Phi}(\lambda) = \int_{-\infty}^{\infty} e^{i\mu u}\ d\,\bar{\Phi}(-\mu) \tag{9}$$

So

$$X(u) = \int_{-\infty}^{\infty} e^{i\lambda u}\ d\,\Phi(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda u}\ d\,\bar{\Phi}(-\lambda) \tag{10}$$

By uniqueness of the spectral measure representation, it follows that

$$d\,\bar{\Phi}(\lambda) = d\,\Phi\,(-\lambda) \tag{11}$$

as (orthogonal) random measures.

2. The covariance function of X is

$$R(u) = \mathbb{E}\left(X(0)\, X(u)\right) = \int_{-\infty}^{\infty} e^{i\lambda u}\, d\,F(\lambda) \tag{12}$$

Since $X(u)$ is real-valued, $R(u)$ is real and $R\,(-u) = R(u)$. Thus,

$$R\,(-u) = \int_{-\infty}^{\infty} e^{-i\lambda u}\, d\,F(\lambda) = \int_{-\infty}^{\infty} e^{i\mu u}\, d\,F\,(-\mu) \tag{13}$$

Equating with $R(u)$,

$$\int_{-\infty}^{\infty} e^{i\lambda u}\, d\,F(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda u}\, d\,F\,(-\lambda) \tag{14}$$

for all u. By the uniqueness theorem for Fourier–Stieltjes transforms, this implies

$$d\,F(\lambda) = d\,F\,(-\lambda) \tag{15}$$

Thus for any Borel set A,

$$F\,(-A) = F(A) \tag{16}$$

establishing the evenness property. □

1.1 Definition

Definition 2. ***(Gaussian process)*** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T a non-empty index set. A family $\{X_t \colon t \in T\}$ of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Gaussian process if for every finite subset $\{t_1, \ldots, t_n\} \subset T$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ is multivariate normal (possibly degenerate). Equivalently, every finite linear combination $\sum_{i=1}^{n} a_i\, X_{t_i}$ is either almost surely constant or Gaussian. The mean function is $m(t) := \mathbb{E}[X_t]$ and the covariance kernel is*

$$K(s, t) = \operatorname{Cov}(X_s, X_t) \tag{17}$$

For any finite $(t_i)_{i=1}^{n} \subset T$, the matrix $K_{ij} = K(t_i, t_j)$ is symmetric positive semidefinite, and a Gaussian process is completely determined in law by m and K.

1.2 Stationary processes

Definition 3. ***[Cramér spectral representation]*** *[1] A zero-mean stationary process X with spectral measure F admits the sample path representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, d\,\Phi(\lambda) \tag{18}$$

which has covariance

$$R_X\,(t - s) = \int_{\mathbb{R}} e^{i\lambda(t-s)}\, d\,F(\lambda) \tag{19}$$

1.2.1 Sample Path Realizations

Definition 4. ***[Locally square-integrable functions]*** *Define*

$$L^2_{\mathrm{loc}}(\mathbb{R}) := \left\{ f: \mathbb{R} \to \mathbb{C}: \int_K |f(t)|^2 \, dt < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \tag{20}$$

Remark 5. Every bounded measurable set in $\mathbb{R}$ is compact or contained in a compact set; hence $L^2_{\mathrm{loc}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

Theorem 6. ***[Sample paths in $L^2_{\mathrm{loc}}(\mathbb{R})$]*** *Let $\{X(t)\}_{t \in \mathbb{R}}$ be a second-order stationary process with*

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \tag{21}$$

Then almost every sample path lies in $L^2_{\mathrm{loc}}(\mathbb{R})$.

Proof. Fix an arbitrary bounded interval $[a, b] \subset \mathbb{R}$ with $a < b$. Define the random variable

$$Y_{[a,b]} := \int_a^b X(t)^2 \, dt \tag{22}$$

By Fubini's theorem,

$$\mathbb{E}[Y_{[a,b]}] = \int_a^b \mathbb{E}[X(t)^2] \, dt = (b - a) \, \sigma^2 < \infty \tag{23}$$

By Markov's inequality, $\mathbb{P}\,(Y_{[a,b]} = \infty) = 0$. Thus $Y_{[a,b]} < \infty$ almost surely. Covering compacts by countably many dyadic intervals yields the result: for every compact $K \subset \mathbb{R}$, almost surely $\int_K X(t)^2 \, d\, t < \infty$. □

2 Oscillatory Processes

Definition 7. ***[Oscillatory process]*** *[2] Let F be a finite nonnegative Borel measure on $\mathbb{R}$. Let*

$$A_t \in L^2(F) \quad \forall t \in \mathbb{R} \tag{24}$$

be the gain function and

$$\varphi_t(\lambda) = A_t(\lambda) \, e^{i\lambda t} \tag{25}$$

be the corresponding oscillatory function; then an oscillatory process is a stochastic process which can be represented as

$$\begin{aligned} Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda) \, d\, \Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda) \, e^{i\lambda t} \, d\, \Phi(\lambda) \end{aligned} \tag{26}$$

where Φ is a complex orthogonal random measure with spectral measure F which satisfies the relation

$$d\, \mathbb{E}[\Phi(\lambda)\overline{\Phi(\mu)}] = \delta\,(\lambda - \mu)\, d\, F(\lambda) \tag{27}$$

and has the corresponding covariance kernel

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)} e^{i\lambda(t-s)} \, d\, F(\lambda) \\ &= \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)} d\, F(\lambda) \end{aligned} \tag{28}$$

Theorem 8. ***[Real-valuedness criterion for oscillatory processes]*** *Let Z be an oscillatory process with oscillatory function*

$$\varphi_t(\lambda) = A_t(\lambda) \, e^{i\lambda t} \tag{29}$$

and spectral measure F. Then Z is real-valued if and only if

$$A_t\,(-\lambda) = \overline{A_t(\lambda)} \tag{30}$$

for F-almost every $\lambda \in \mathbb{R}$, equivalently

$$\varphi_t\,(-\lambda) = \overline{\varphi_t(\lambda)} \tag{31}$$

for F-almost every $\lambda \in \mathbb{R}$.

Proof. 1. Assume Z is real-valued. Then for all $t \in \mathbb{R}$,

$$Z(t) = \overline{Z(t)} \tag{32}$$

2. From the oscillatory representation (26),

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda) \tag{33}$$

3. Taking the complex conjugate of both sides of (33),

$$\overline{Z(t)} = \overline{\int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t}\, d\overline{\Phi(\lambda)} \tag{34}$$

4. For a real-valued process, the orthogonal random measure must satisfy the symmetry property from Theorem 1:

$$d\overline{\Phi(\lambda)} = d\,\Phi\,(-\lambda) \tag{35}$$

5. Substituting (35) into (34),

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t}\, d\,\Phi\,(-\lambda) \tag{36}$$

6. Apply the change of variables $\mu = -\lambda$, so $d\,\Phi\,(-\lambda) = d\,\Phi(\mu)$ and $e^{-i\lambda t} = e^{i\mu t}$:

$$\overline{Z(t)} \;= \int_{\mathbb{R}} \overline{A_t\,(-\mu)} e^{i\mu t}\, d\,\Phi(\mu) \tag{37}$$

7. By (32), the right sides of (33) and (37) must be equal:

$$\int_{\mathbb{R}} A_t(\mu)\, e^{i\mu t}\, d\,\Phi(\mu) = \int_{\mathbb{R}} \overline{A_t\,(-\mu)} e^{i\mu t}\, d\,\Phi(\mu) \tag{38}$$

8. Since the stochastic integral representation is unique in $L^2(F)$, the integrands must be equal F-almost everywhere:

$$A_t(\lambda) = \overline{A_t\,(-\lambda)} \quad \text{for } F\text{-a.e. } \lambda \tag{39}$$

9. This is equivalent to (30). From (29),

$$\varphi_t\,(-\lambda) = A_t\,(-\lambda)\, e^{-i\lambda t} \tag{40}$$

10. Using (30),

$$\begin{aligned} \varphi_t\,(-\lambda) \;&= \overline{A_t(\lambda)} e^{-i\lambda t} \\ &= \overline{A_t(\lambda)\, e^{i\lambda t}} \\ &= \overline{\varphi_t(\lambda)} \end{aligned} \tag{41}$$

establishing (31).

11. Conversely, assume (30) holds. Reversing the steps from (37) to (32) shows that $\overline{Z(t)} = Z(t)$ for all t, so Z is real-valued. □

Theorem 9. ***[Existence of Oscillatory Processes]*** *Let F be an absolutely continuous spectral measure and the gain function*

$$A_t(\lambda) \in L^2(F) \quad \forall t \in \mathbb{R} \tag{42}$$

be measurable in both time and frequency; then the time-dependent spectral density is defined by

$$\begin{aligned} S_t(\lambda) &= \int_{\mathbb{R}} |A_t(\lambda)|^2 \, d\, F(\lambda) < \infty \\ &= \int_{\mathbb{R}} |A_t(\lambda)|^2 \, S(\lambda) \, d\, \lambda \end{aligned} \tag{43}$$

and there exists a complex orthogonal random measure Φ with spectral measure F such that for each sample path $\omega_0 \in \Omega$

$$Z(t, \omega_0) = \int_{\mathbb{R}} A_t(\lambda) \, e^{i\lambda t} \, d\, \Phi(\lambda, \omega_0) \tag{44}$$

is well-defined in $L^2(\Omega)$ and has covariance R_Z as in (28).

Proof. 1. Define the space of simple functions on $\mathbb{R}$: for disjoint Borel sets $\{E_j\}_{j=1}^n$ with $F(E_j) < \infty$ and coefficients $\{c_j\}_{j=1}^n \subset \mathbb{C}$,

$$g(\lambda) = \sum_{j=1}^n c_j \, 1_{E_j}(\lambda) \tag{45}$$

2. For simple functions, define the stochastic integral

$$\int_{\mathbb{R}} g(\lambda) \, d\, \Phi(\lambda) := \sum_{j=1}^n c_j \, \Phi(E_j) \tag{46}$$

3. Compute the second moment:

$$\begin{aligned} \mathbb{E}\left[\left|\int_{\mathbb{R}} g(\lambda) \, d\, \Phi(\lambda)\right|^2\right] &= \mathbb{E}\left[\left|\sum_{j=1}^n c_j \, \Phi(E_j)\right|^2\right] \\ &= \mathbb{E}\left[\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c_k} \Phi(E_j) \overline{\Phi(E_k)}\right] \end{aligned} \tag{47}$$

4. By linearity of expectation,

$$\mathbb{E}\left[\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c_k} \Phi(E_j) \overline{\Phi(E_k)}\right] = \sum_{j=1}^n \sum_{k=1}^n c_j \bar{c_k} \mathbb{E}[\Phi(E_j) \overline{\Phi(E_k)}] \tag{48}$$

5. By the orthogonality relation (27), since $E_j \cap E_k = \emptyset$ for $j \neq k$,

$$\mathbb{E}[\Phi(E_j) \overline{\Phi(E_k)}] = \begin{cases} F(E_j) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \tag{49}$$

6. Substituting (49) into (48),

$$\sum_{j=1}^{n}\sum_{k=1}^{n} c_j\bar{c_k}\mathbb{E}[\Phi(E_j)\overline{\Phi(E_k)}]=\sum_{j=1}^{n} |c_j|^2 F(E_j) \tag{50}$$

7. The right side of (50) equals

$$\sum_{j=1}^{n} |c_j|^2 F(E_j)=\int_{\mathbb{R}}|g(\lambda)|^2 d F(\lambda) \tag{51}$$

8. Therefore the isometry property holds for simple functions:

$$\mathbb{E}\left[\left|\int_{\mathbb{R}} g(\lambda)\, d\,\Phi(\lambda)\right|^2\right]=\int_{\mathbb{R}}|g(\lambda)|^2 d F(\lambda) \tag{52}$$

9. The space of simple functions is dense in $L^2(F)$. For any $h(\lambda)\in L^2(F)$ and $\epsilon>0$, there exists a simple function $g(\lambda)$ such that

$$\int_{\mathbb{R}}|h(\lambda)-g(\lambda)|^2 d F(\lambda)<\epsilon \tag{53}$$

10. By the isometry (52) and completeness of $L^2(\Omega)$, the integral extends uniquely by continuity to all $h(\lambda)\in L^2(F)$.

11. Since $A_t\in L^2(F)$ by assumption (42), and $|e^{i\lambda t}|=1$,

$$\int_{\mathbb{R}}|\varphi_t(\lambda)|^2 d F(\lambda)=\int_{\mathbb{R}}|A_t(\lambda)|^2 d F(\lambda)<\infty \tag{54}$$

so $\varphi_t\in L^2(F)$.

12. Therefore

$$\begin{aligned} Z(t) &=\int_{\mathbb{R}}\varphi_t(\lambda)\, d\,\Phi(\lambda)\\ &=\int_{\mathbb{R}}A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda)\end{aligned} \tag{55}$$

is well-defined in $L^2(\Omega)$.

13. To compute the covariance, use the sesquilinearity of the stochastic integral:

$$\begin{aligned} R_Z(t,s) &=\mathbb{E}[Z(t)\overline{Z(s)}]\\ &=\mathbb{E}\left[\int_{\mathbb{R}}\varphi_t(\lambda)\, d\,\Phi(\lambda)\,\overline{\int_{\mathbb{R}}\varphi_s(\mu)\, d\,\Phi(\mu)}\right]\end{aligned} \tag{56}$$

14. By Fubini's theorem for stochastic integrals,

$$\mathbb{E}\left[\int_{\mathbb{R}}\varphi_t(\lambda)\, d\,\Phi(\lambda)\,\overline{\int_{\mathbb{R}}\varphi_s(\mu)\, d\,\Phi(\mu)}\right]=\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi_t(\lambda)\overline{\varphi_s(\mu)}\mathbb{E}\,[d\,\Phi(\lambda)\overline{d\,\Phi(\mu)}] \tag{57}$$

15. Using the orthogonality relation (27),

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi_t(\lambda)\overline{\varphi_s(\mu)}\delta(\lambda-\mu)\,dF(\lambda)\,dF(\mu)=\int_{\mathbb{R}}\varphi_t(\lambda)\overline{\varphi_s(\lambda)}dF(\lambda) \tag{58}$$

16. Substituting the definition (25),

$$R_Z(t,s)=\int_{\mathbb{R}}A_t(\lambda)\overline{A_s(\lambda)}e^{i\lambda(t-s)}\,dF(\lambda) \tag{59}$$

as claimed in (28). □

3 Unitarily Time-Changed Stationary Processes

3.1 Unitary Time-Change Operator $U_\theta f$

Theorem 10. ***[Unitary time-change operator U_θ and its inverse U_θ^{-1}]*** *Let the time-change function $\theta:\mathbb{R}\to\mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with*

$$\dot{\theta}(t)>0 \tag{60}$$

almost everywhere and $\dot{\theta}(t)=0$ only on sets of Lebesgue measure zero. For f measurable, define

$$(U_\theta f)(t)=\sqrt{\dot{\theta}(t)}\;f(\theta(t)) \tag{61}$$

Its inverse is given by

$$(U_\theta^{-1}g)(s)=\frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{62}$$

For every compact set $K\subseteq\mathbb{R}$ and $f\in L^2_{\mathrm{loc}}(\mathbb{R})$,

$$\int_K|(U_\theta f)(t)|^2dt=\int_{\theta(K)}|f(s)|^2ds \tag{63}$$

Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\mathrm{loc}}(\mathbb{R})$.

Proof. 1. Let $f\in L^2_{\mathrm{loc}}(\mathbb{R})$ and let $K\subset\mathbb{R}$ be compact. From the definition (61),

$$\int_K|(U_\theta f)(t)|^2\,dt=\int_K\left|\sqrt{\dot{\theta}(t)}\;f(\theta(t))\right|^2dt \tag{64}$$

2. Expanding the square,

$$\int_K\left|\sqrt{\dot{\theta}(t)}\;f(\theta(t))\right|^2dt=\int_K\dot{\theta}(t)|f(\theta(t))|^2\,dt \tag{65}$$

3. Since θ is absolutely continuous and strictly increasing, $\theta' = \dot{\theta}$ exists almost everywhere and $\dot{\theta}(t) > 0$ a.e.
4. Apply the change of variables $s = \theta(t)$. Then

$$d\,s = \dot{\theta}(t)\,d\,t \tag{66}$$

5. The inverse function $t = \theta^{-1}(s)$ exists since θ is strictly increasing and bijective.
6. As t ranges over K, the variable $s = \theta(t)$ ranges over $\theta(K)$.
7. Since θ is continuous and K is compact, $\theta(K)$ is compact.
8. Substituting (66) into (65),

$$\int_K \dot{\theta}(t)\,|\,f(\theta(t))\,|^2 d\,t = \int_{\theta(K)} |\,f(s)|^2\,d\,s \tag{67}$$

9. This establishes the local isometry (63).
10. To verify U_θ^{-1} is the inverse, compute:

$$(U_\theta^{-1}\,U_\theta\,f)(s) = U_\theta^{-1}\,(U_\theta\,f)(s) \tag{68}$$

11. By definition (62),

$$U_\theta^{-1}\,(U_\theta\,f)(s) = \frac{(U_\theta\,f)(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{69}$$

12. By definition (61),

$$(U_\theta\,f)(\theta^{-1}(s)) = \sqrt{\dot{\theta}(\theta^{-1}(s))}\;f(\theta(\theta^{-1}(s))) \tag{70}$$

13. Since $\theta \circ \theta^{-1} = \mathrm{id}$,

$$f(\theta(\theta^{-1}(s))) = f(s) \tag{71}$$

14. Substituting (70) and (71) into (69),

$$\frac{\sqrt{\dot{\theta}(\theta^{-1}(s))}\;f(s)}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} = f(s) \tag{72}$$

15. Therefore

$$U_\theta^{-1}\,U_\theta = \mathrm{id} \tag{73}$$

16. Similarly, compute:

$$(U_\theta\,U_\theta^{-1}\,g)(t) = \sqrt{\dot{\theta}(t)}\;(U_\theta^{-1}\,g)(\theta(t)) \tag{74}$$

17. By definition (62),

$$(U_\theta^{-1}\,g)(\theta(t)) = \frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} \tag{75}$$

18. Since $\theta^{-1} \circ \theta = \mathrm{id}$,

$$g(\theta^{-1}(\theta(t))) = g(t), \quad \theta^{-1}(\theta(t)) = t \tag{76}$$

19. Substituting (76) into (75),

$$\frac{g(t)}{\sqrt{\dot{\theta}(t)}} \tag{77}$$

20. Therefore from (74),

$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)} \cdot \frac{g(t)}{\sqrt{\dot{\theta}(t)}} = g(t) \tag{78}$$

21. Thus

$$U_\theta U_\theta^{-1} = \mathrm{id} \tag{79}$$

22. Combining (73) and (79), U_θ^{-1} is the two-sided inverse of U_θ on $L^2_{\mathrm{loc}}(\mathbb{R})$. □

3.2 Transformation of Stationary → Oscillatory Processes via U_θ

Theorem 11. ***[Unitary time changes of stationary processes produce oscillatory process]*** *Let X be zero-mean stationary as in Definition 3. For scaling function θ as in Theorem 10, define*

$$\begin{aligned} Z(t) &= (U_\theta X)(t) \\ &= \sqrt{\dot{\theta}(t)}\, X(\theta(t)) \end{aligned} \tag{80}$$

Then Z is a realization of an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \tag{81}$$

gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)} \tag{82}$$

and covariance kernel

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\left[\sqrt{\dot{\theta}(t)}\, X(\theta(t)) \overline{\sqrt{\dot{\theta}(s)}\, X(\theta(s))}\right] \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\; \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\; R_X\left(\theta(t)-\theta(s)\right) \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\, dF(\lambda) \end{aligned} \tag{83}$$

Proof. 1. From the Cramér representation (18),

$$X(u) = \int_{\mathbb{R}} e^{i\lambda u}\, d\Phi(\lambda) \tag{84}$$

2. Substituting $u = \theta(t)$ into (84),

$$X(\theta(t)) = \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\Phi(\lambda) \tag{85}$$

3. From the definition (80),

$$Z(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\,\Phi(\lambda) \tag{86}$$

4. By linearity of the stochastic integral,

$$Z(t) = \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}\, d\,\Phi(\lambda) \tag{87}$$

5. Define

$$\varphi_t(\lambda) := \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \tag{88}$$

6. Then (87) becomes

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda)\, d\,\Phi(\lambda) \tag{89}$$

   which is the oscillatory representation (26).

7. To express this in terms of the standard oscillatory function form, define the gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)} \tag{90}$$

8. Then verify the oscillatory function form (25) factorizes

$$\begin{aligned} \varphi_t(\lambda) &= A_t(\lambda)\, e^{i\lambda t} \\ &= \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)}\, e^{i\lambda t} \\ &= \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t+t)} \\ &= \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \end{aligned} \tag{91}$$

9. To compute the covariance, use (28):

$$R_Z(t,s) = \mathbb{E}[Z(t)\overline{Z(s)}] \tag{92}$$

10. Substituting (80),

$$R_Z(t,s) = \mathbb{E}\left[\sqrt{\dot{\theta}(t)}\, X(\theta(t))\overline{\sqrt{\dot{\theta}(s)}\, X(\theta(s))}\right] \tag{93}$$

11. Since $\dot{\theta}$ is deterministic,

$$R_Z(t,s) = \sqrt{\dot{\theta}(t)}\, \sqrt{\dot{\theta}(s)}\, \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \tag{94}$$

12. By stationarity of X, using (19),

$$\mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] = R_X\,(\theta(t)-\theta(s)) = \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\, d\,F(\lambda) \tag{95}$$

13. Substituting (95) into (94),

$$R_Z(t,s) = \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\, d\,F(\lambda) \tag{96}$$

establishing (83). □

3.2.1 Time-Varying Filter Representations

Theorem 12. *Let $\theta: \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\theta'(t) > 0$ almost everywhere. Let $X(u)$ be a stationary process, and define the oscillatory process obtained by the forward unitary time transformation U_θ*

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) = \int_{\mathbb{R}} h(t, u)\, X(u)\, d u \tag{97}$$

where the (forward) impulse response function is given by

$$h(t, u) = \sqrt{\dot{\theta}(t)}\, \delta\,(u - \theta(t)) \tag{98}$$

Then likewise the transformation can be reversed by expressing the stationary process as

$$X(u) = (U_\theta^{-1} Z)(u) = \frac{Z(\theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} = \int_{\mathbb{R}} g(u, t)\, Z(t)\, d t \tag{99}$$

where the inverse impulse response function is

$$g(u, t) = \frac{\delta\,(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \tag{100}$$

Proof.

1. Recall the forward unitary transformation from Theorem 10:

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) \tag{101}$$

2. To express this as a convolution integral, note that the Dirac delta function satisfies the sifting property: for any continuous function $f: \mathbb{R} \to \mathbb{R}$,

$$\int_{\mathbb{R}} f(u)\, \delta\,(u - a)\, d u = f(a) \tag{102}$$

for any $a \in \mathbb{R}$.

3. Substituting $f(u) = X(u)$ and $a = \theta(t)$, which is well-defined since θ is bijective and continuous,

$$X(\theta(t)) = \int_{\mathbb{R}} X(u)\, \delta\,(u - \theta(t))\, d u \tag{103}$$

4. Multiplying both sides by $\sqrt{\dot{\theta}(t)}$ and substituting into (101),

$$Z(t) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} X(u)\, \delta\,(u - \theta(t))\, d u = \int_{\mathbb{R}} \left[\sqrt{\dot{\theta}(t)}\, \delta\,(u - \theta(t)) \right] X(u)\, d u \tag{104}$$

5. Thus, the forward impulse response function is

$$h(t, u) = \sqrt{\dot{\theta}(t)}\, \delta\,(u - \theta(t)) \tag{105}$$

establishing (97).

6. For the inverse transformation, recall from Theorem 10 that

$$X(u) = (U_\theta^{-1} Z)(u) = \frac{Z(\theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \tag{106}$$

7. Let $s = \theta^{-1}(u)$, so $u = \theta(s)$ and $Z(\theta^{-1}(u)) = Z(s)$. The sifting property applied to $Z(t)$ with point $\theta^{-1}(u)$ gives

$$Z(\theta^{-1}(u)) = \int_{\mathbb{R}} Z(t)\, \delta\,(t - \theta^{-1}(u))\, d\,t \tag{107}$$

8. Substituting into (106),

$$X(u) = \frac{1}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \int_{\mathbb{R}} Z(t)\, \delta\,(t - \theta^{-1}(u))\, d\,t = \int_{\mathbb{R}} \left[\frac{\delta\,(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \right] Z(t)\, d\,t \tag{108}$$

9. Thus, the inverse impulse response function is

$$g(u,t) = \frac{\delta\,(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \tag{109}$$

establishing (99) and (100).

10. To confirm invertibility, substitute (104) into (108). The integral becomes

$$X(u) = \int_{\mathbb{R}} g(u,t) \left[\int_{\mathbb{R}} h(t,v)\, X(v)\, d\,v \right] d\,t \tag{110}$$

11. By Fubini's theorem, since all measures are positive and the delta functions ensure finite support,

$$X(u) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(u,t)\, h(t,v)\, X(v)\, d\,v\;\, d\,t \tag{111}$$

12. Integrating the kernel

$$g(u,t)\, h(t,v) = \frac{\delta\,(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \cdot \sqrt{\dot{\theta}(t)}\, \delta\,(v - \theta(t)) \tag{112}$$

over t results in $t = \theta^{-1}(u)$, so

$$\sqrt{\dot{\theta}(t)} = \sqrt{\dot{\theta}(\theta^{-1}(u))} \tag{113}$$

and

$$\delta\,(v - \theta(t)) = \delta\,(v - u) \tag{114}$$

yielding

$$\int_{\mathbb{R}} g(u,t)\, h(t,v)\, d\,t = \delta\,(v - u) \tag{115}$$

13. Thus, (111) simplifies to

$$\int_{\mathbb{R}} \delta\,(v-u)\,X(v)\,d\,v = X(u) \tag{116}$$

confirming the transformations are inverses. □

Corollary 13. *The evolutionary spectrum is*

$$d\,F_t(\lambda) = \dot{\theta}(t)\,d\,F(\lambda) \tag{117}$$

Proof. 1. The evolutionary spectrum is defined by

$$d\,F_t(\lambda) = |A_t(\lambda)|^2\,d\,F(\lambda) \tag{118}$$

2. From (82),

$$|A_t(\lambda)|^2 = \left|\sqrt{\dot{\theta}(t)}\,e^{i\lambda(\theta(t)-t)}\right|^2 \tag{119}$$

3. Since $|e^{i\alpha}| = 1$ for all real α,

$$|e^{i\lambda(\theta(t)-t)}|^2 = 1 \tag{120}$$

4. Therefore

$$|A_t(\lambda)|^2 = \left(\sqrt{\dot{\theta}(t)}\right)^2 \cdot 1 = \dot{\theta}(t) \tag{121}$$

5. Substituting (121) into (118),

$$d\,F_t(\lambda) = \dot{\theta}(t)\,d\,F(\lambda) \tag{122}$$ □

3.3 Covariance operator conjugation

Proposition 14. *Let*

$$(T_K\,f)(t) := \int_{\mathbb{R}} K(|t-s|)\,f(s)\,d\,s \tag{123}$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h}\,d\,F(\lambda) \tag{124}$$

Define the transformed kernel

$$K_\theta(s,t) := \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\,K(|\theta(t)-\theta(s)|) \tag{125}$$

then the corresponding integral covariance operator is conjugated for all $f \in L^2_{\mathrm{loc}}(\mathbb{R})$ by

$$(T_{K_\theta}\,f)(t) = (U_\theta\,T_K\,U_\theta^{-1}\,f)(t) \tag{126}$$

Proof. 1. From (126), expand the right side:

$$(U_\theta\,T_K\,U_\theta^{-1}\,f)(t) = \sqrt{\dot{\theta}(t)}\,(T_K\,U_\theta^{-1}\,f)(\theta(t)) \tag{127}$$

2. By definition (123),

$$(T_K U_\theta^{-1} f)(\theta(t)) = \int_{\mathbb{R}} K(|\theta(t) - s|)\, (U_\theta^{-1} f)(s)\, d s \tag{128}$$

3. By definition (62),

$$(U_\theta^{-1} f)(s) = \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{129}$$

4. Substituting (129) into (128),

$$\int_{\mathbb{R}} K(|\theta(t) - s|) \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}}\, d s \tag{130}$$

5. Apply the change of variables $s = \theta(u)$, so $d s = \dot{\theta}(u)\, d u$ and $\theta^{-1}(s) = u$:

$$\int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \frac{f(u)}{\sqrt{\dot{\theta}(u)}}\, \dot{\theta}(u)\, d u \tag{131}$$

6. Simplify:

$$\int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \frac{\dot{\theta}(u)}{\sqrt{\dot{\theta}(u)}}\, f(u)\, d u = \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \sqrt{\dot{\theta}(u)}\, f(u)\, d u \tag{132}$$

7. Substituting (132) into (127),

$$\sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \sqrt{\dot{\theta}(u)}\, f(u)\, d u \tag{133}$$

8. Bring the constant inside the integral:

$$\int_{\mathbb{R}} \sqrt{\dot{\theta}(t)}\, \sqrt{\dot{\theta}(u)}\, K(|\theta(t) - \theta(u)|)\, f(u)\, d u \tag{134}$$

9. By definition (125),

$$\sqrt{\dot{\theta}(t)}\, \sqrt{\dot{\theta}(u)}\, K(|\theta(t) - \theta(u)|) = K_\theta(u, t) \tag{135}$$

10. Therefore

$$\int_{\mathbb{R}} K_\theta(u, t)\, f(u)\, d u = (T_{K_\theta} f)(t) \tag{136}$$

establishing (126). □

4 Zero Localization

Definition 15. *Let Z be real-valued with $Z \in C^1(\mathbb{R})$ having only simple zeros*

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \tag{137}$$

Define, for Borel $B \subset \mathbb{R}$,

$$\mu(B) = \int_{\mathbb{R}} 1_B(t)\, \delta(Z(t)) |\dot{Z}(t)|\, d t \tag{138}$$

Theorem 16. *Under the assumptions of Definition 15, zeros are locally finite and one has*

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0) = 0} \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|} \tag{139}$$

whence

$$\mu = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0} \tag{140}$$

Proof.

1. For any smooth test function ϕ with compact support, apply the standard change of variables formula for the delta function. Let $\{t_0^{(1)}, t_0^{(2)}, \ldots\}$ denote the zeros of Z.
2. By the change of variables formula for distributions,

$$\int_{\mathbb{R}} \phi(t)\, \delta(Z(t))\, d t = \sum_{t_0 : Z(t_0) = 0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} \tag{141}$$

3. The right side of (141) equals

$$\sum_{t_0 : Z(t_0) = 0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} = \sum_{t_0 : Z(t_0) = 0} \int_{\mathbb{R}} \phi(t) \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|}\, d t \tag{142}$$

4. By Fubini's theorem (justified since the sum has locally finite terms due to C^1 regularity and simple zeros),

$$\sum_{t_0 : Z(t_0) = 0} \int_{\mathbb{R}} \phi(t) \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|}\, d t = \int_{\mathbb{R}} \phi(t) \sum_{t_0 : Z(t_0) = 0} \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|}\, d t \tag{143}$$

5. Comparing (141) and (143),

$$\int_{\mathbb{R}} \phi(t)\, \delta(Z(t))\, d t = \int_{\mathbb{R}} \phi(t) \sum_{t_0 : Z(t_0) = 0} \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|}\, d t \tag{144}$$

6. Since ϕ is arbitrary,

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0) = 0} \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|} \tag{145}$$

establishing (139).

7. Substituting (145) into the definition (138),

$$\mu(B) = \int_{\mathbb{R}} 1_B(t) \sum_{t_0: Z(t_0)=0} \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} |\dot{Z}(t)|\, d\,t \tag{146}$$

8. By the sifting property of the delta function, $|\dot{Z}(t)|$ evaluated at $t = t_0$ gives $|\dot{Z}(t_0)|$:

$$\int_{\mathbb{R}} 1_B(t) \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} |\dot{Z}(t)|\, d\,t = \frac{1_B(t_0)|\dot{Z}(t_0)|}{|\dot{Z}(t_0)|} = 1_B(t_0) \tag{147}$$

9. Summing over all zeros,

$$\mu(B) = \sum_{t_0: Z(t_0)=0} 1_B(t_0) = \sum_{t_0 \in B: Z(t_0)=0} 1 \tag{148}$$

10. This is precisely the atomic measure

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \tag{149}$$

establishing (140). □

Definition 17. *Let $\mathcal{H} = L^2(\mu)$ be the Hilbert space with inner product*

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} d\,\mu(t) \tag{150}$$

Proposition 18. ***[Atomic structure]*** *Let*

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \tag{151}$$

then

$$\mathcal{H} \cong \left\{ f: \{t_0: Z(t_0) = 0\} \to \mathbb{C}: \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \tag{152}$$

with orthonormal basis $\{e_{t_0}\}_{t_0: Z(t_0)=0}$ where

$$e_{t_0}(t_1) = \delta_{t_0, t_1} \tag{153}$$

Proof. 1. By (151), μ is a purely atomic measure with atoms at the zero set.

2. For any $f \in L^2(\mu)$, the L^2 norm is

$$\|f\|_{L^2(\mu)}^2 = \int_{\mathbb{R}} |f(t)|^2\, d\,\mu(t) \tag{154}$$

3. Substituting (151),

$$\int_{\mathbb{R}} |f(t)|^2 \, d\mu(t) = \int_{\mathbb{R}} |f(t)|^2 \sum_{t_0 : Z(t_0) = 0} \delta_{t_0}(dt) = \sum_{t_0 : Z(t_0) = 0} |f(t_0)|^2 \tag{155}$$

4. Therefore

$$\|f\|_{L^2(\mu)}^2 = \sum_{t_0 : Z(t_0) = 0} |f(t_0)|^2 \tag{156}$$

5. This is precisely the ℓ^2 norm on the zero set.
6. Define the map $\Psi: L^2(\mu) \to \ell^2$ by

$$\Psi(f) = (f(t_0))_{t_0 : Z(t_0) = 0} \tag{157}$$

7. From (156), Ψ is an isometry:

$$\|\Psi(f)\|_{\ell^2}^2 = \sum_{t_0 : Z(t_0) = 0} |f(t_0)|^2 = \|f\|_{L^2(\mu)}^2 \tag{158}$$

8. Ψ is surjective: for any sequence $(c_{t_0}) \in \ell^2$, define $f(t) = \sum_{t_0} c_{t_0} \delta_{t_0}(t)$, which is in $L^2(\mu)$.
9. Therefore Ψ is a Hilbert space isomorphism, establishing (152).
10. For the orthonormal basis, define e_{t_0} by (153).
11. Then

$$\langle e_{t_0}, e_{t_1} \rangle = \int_{\mathbb{R}} e_{t_0}(t) \overline{e_{t_1}(t)} d\mu(t) = \sum_{s : Z(s) = 0} \delta_{t_0, s} \delta_{t_1, s} = \delta_{t_0, t_1} \tag{159}$$

12. Therefore $\{e_{t_0}\}$ is an orthonormal set.
13. Since every $f \in L^2(\mu)$ can be written as

$$f = \sum_{t_0 : Z(t_0) = 0} f(t_0) \, e_{t_0} \tag{160}$$

   the set $\{e_{t_0}\}$ is complete, hence an orthonormal basis. ☐

Definition 19. ***[Multiplication operator]*** *Define the linear operator*

$$L: \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H} \tag{161}$$

by

$$(Lf)(t) = t f(t) \tag{162}$$

on the support of μ with domain

$$\mathcal{D}(L) := \left\{ f \in \mathcal{H}: \int |t f(t)|^2 \, d\mu(t) < \infty \right\} \tag{163}$$

Theorem 20. ***[Self-adjointness and spectrum]*** *L is self-adjoint on $\mathcal{H}$ and has pure point, simple spectrum*

$$\sigma(L) = \overline{\{t \in \mathbb{R} : Z(t) = 0\}} \tag{164}$$

with eigenvalues $\lambda = t_0$ for each zero t_0 and corresponding eigenvectors e_{t_0}.

Proof.

1. For $f, g \in \mathcal{D}(L)$, compute the inner product:

$$\langle L f, g \rangle = \int_{\mathbb{R}} (L f)(t)\overline{g(t)} d\mu(t) \tag{165}$$

2. By definition (162),

$$\int_{\mathbb{R}} t f(t)\overline{g(t)} d\mu(t) \tag{166}$$

3. Since t is real-valued, $\bar{t} = t$, so

$$\int_{\mathbb{R}} t f(t)\overline{g(t)} d\mu(t) = \int_{\mathbb{R}} f(t)\overline{t g(t)} d\mu(t) \tag{167}$$

4. The right side of (167) is

$$\int_{\mathbb{R}} f(t)\overline{(L g)(t)} d\mu(t) = \langle f, L g \rangle \tag{168}$$

5. Therefore

$$\langle L f, g \rangle = \langle f, L g \rangle \tag{169}$$

   for all $f, g \in \mathcal{D}(L)$, establishing that L is symmetric.

6. Since L is a multiplication operator on $L^2(\mu)$, it is self-adjoint (by standard functional analysis).

7. To determine the spectrum, compute the action on basis vectors. From (162) and (153),

$$(L e_{t_0})(t) = t e_{t_0}(t) = t \delta_{t_0}(t) \tag{170}$$

8. By the sifting property,

$$t \delta_{t_0}(t) = t_0 \delta_{t_0}(t) = t_0 e_{t_0}(t) \tag{171}$$

9. Therefore

$$L e_{t_0} = t_0 e_{t_0} \tag{172}$$

10. This shows that each t_0 is an eigenvalue with eigenvector e_{t_0}.

11. Since the $\{e_{t_0}\}$ form a complete orthonormal basis (Proposition 18), the spectrum is pure point.

12. Each eigenspace is one-dimensional (spanned by e_{t_0}), so the spectrum is simple and given by the closure of the zero set

$$\sigma(L) = \{t_0 \colon Z(t_0) = 0\} = \overline{\{t \in \mathbb{R} \colon Z(t) = 0\}} \tag{173}$$ □

4.1 Simplicity of Zeros and Their Expected Counting Function

Theorem 21. (Bulinskaya) *Let $X(t)$ be a centered stationary Gaussian process with covariance function $K(h) = \mathbb{E}[X(t)\, X(t+h)]$ that is twice differentiable at $h = 0$ with $K(0) > 0$ and $\ddot{K}(0) < 0$. Then the zero set of X has no accumulation points almost surely. In particular, on any compact interval $[a, b]$, the number of zeros is almost surely finite.*

Proof.

1. The twice-differentiability of K at $h = 0$ implies that the spectral measure F has finite second moment:

$$\lambda_2 = \int_{\mathbb{R}} \omega^2\, d\, F(\omega) = -\ddot{K}(0) < \infty. \tag{174}$$

2. This finite second moment implies that X has mean-square continuous derivative $\dot{X}(t)$, with

$$\mathbb{E}[\dot{X}(t)^2] = -\ddot{K}(0) = \lambda_2 > 0. \tag{175}$$

3. Since $\dot{X}(t)$ is a non-degenerate centered Gaussian process, it is continuous almost surely and does not vanish identically on any interval.

4. For any zero t_0 of X (i.e., $X(t_0) = 0$), if $\dot{X}(t_0) \neq 0$, then X crosses transversally through zero at t_0, making t_0 an isolated zero.

5. The joint distribution of $(X(t_0), \dot{X}(t_0))$ is bivariate Gaussian with covariance matrix

$$\begin{pmatrix} K(0) & K'(0) \\ K'(0) & -\ddot{K}(0) \end{pmatrix} = \begin{pmatrix} K(0) & 0 \\ 0 & -\ddot{K}(0) \end{pmatrix}, \tag{176}$$

   where $K'(0) = 0$ by evenness of K. Since X and $\dot{X}$ are uncorrelated Gaussians, they are independent.

6. At any zero t_0 of X, the derivative $\dot{X}(t_0)$ is Gaussian with mean zero and variance $-\ddot{K}(0) > 0$, hence

$$\mathbb{P}\,[\dot{X}(t_0) = 0 |\, X(t_0) = 0] = \mathbb{P}\,[\dot{X}(t_0) = 0] = 0. \tag{177}$$

   Thus almost surely $\dot{X}(t_0) \neq 0$, making t_0 an isolated zero.

7. Since every zero of X is isolated almost surely, the zero set can have no accumulation points almost surely.

8. On a compact interval $[a, b]$, a set with no accumulation points is finite, completing the proof. □

Theorem 22. (Expected Zero-Counting Function with Deterministic Atoms) *Let $\theta: \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\dot{\theta}(t) \geq 0$ for all t and $\dot{\theta}(t) > 0$ almost everywhere. Define the zero-derivative set*

$$T_0 := \{t \in \mathbb{R}: \dot{\theta}(t) = 0\} \tag{178}$$

Let X be a centered stationary Gaussian process with spectral measure F and covariance function

$$K(h) = \int_{\mathbb{R}} e^{i\omega h}\, dF(\omega) \tag{179}$$

twice differentiable at $h = 0$ with $\ddot{K}(0) < 0$ and $K(0) > 0$. Define the unitarily time-changed process

$$Z(t) = \sqrt{\dot{\theta}(t)}\; X(\theta(t)) \tag{180}$$

Then Z is a centered Gaussian process with covariance

$$K_Z(t, s) = \sqrt{\dot{\theta}(t)\dot{\theta}(s)}\; K\left(\theta(t) - \theta(s)\right) \tag{181}$$

For any compact interval $[0, T]$, define

$$N_{\det}([0, T]) := \#(T_0 \cap [0, T]) \tag{182}$$

By Theorem 21, $N_{\det}([0, T])$ is finite almost surely. The expected number of zeros of Z in $[0, T]$ decomposes as

$$\mathbb{E}[N_{[0,T]}(Z)] = N_{\det}([0, T]) + \frac{\theta(T) - \theta(0)}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}} \tag{183}$$

Proof.

1. Define

$$T_0 := \{t \in \mathbb{R}: \dot{\theta}(t) = 0\}, \qquad I_T := [0, T] \setminus T_0 \tag{184}$$

   For any $t_0 \in T_0$,

$$Z(t_0) = \sqrt{\dot{\theta}(t_0)}\; X(\theta(t_0)) = 0 \cdot X(\theta(t_0)) = \tag{185}$$

   so each $t_0 \in T_0$ is a deterministic zero of Z on every sample path.

2. By Theorem 21, the zero set of X has no accumulation points almost surely. Since θ is strictly increasing and continuous, the image $\theta\left(T_0 \cap [0, T]\right)$ inherits this property: points in $T_0 \cap [0, T]$ cannot accumulate because their images under θ would then be accumulation points of zeros of X. Therefore $T_0 \cap [0, T]$ is at most countable with no accumulation points, hence finite on the compact interval $[0, T]$. Thus

$$N_{\det}([0, T]) = \#(T_0 \cap [0, T]) < \infty \tag{186}$$

3. On the complement $I_T = [0, T] \setminus T_0$, the derivative satisfies $\dot{\theta}(t) > 0$, hence

$$Z(t) = 0 \quad \Longleftrightarrow \quad X(\theta(t)) = 0, \qquad t \in I_T \tag{187}$$

Define

$$Y(t) := X(\theta(t)). \tag{188}$$

The random zero set of Z on $[0, T]$ coincides with the zero set of Y on I_T, and the total zero count decomposes as

$$N_{[0,T]}(Z) = N_{\text{det}}([0, T]) + N_{\text{rand}}([0, T]) \tag{189}$$

where $N_{\text{rand}}([0, T])$ counts zeros of Y in I_T.

4. For $t, s \in \mathbb{R}$,

$$K_Y(t, s) = \mathbb{E}\left[X(\theta(t))\, X(\theta(s))\right] = K\left(\theta(t) - \theta(s)\right) \tag{190}$$

Differentiate with respect to s:

$$\frac{\partial}{\partial s} K_Y(s, t) = -\dot{\theta}(s)\ \dot{K}\left(\theta(t) - \theta(s)\right) \tag{191}$$

Since $K(h)$ is even, $\dot{K}(0) = 0$. Taking $s \to t$,

$$\lim_{s \to t} \frac{\partial}{\partial s} K_Y(s, t) = 0 \tag{192}$$

5. The mixed partial derivative is

$$\frac{\partial^2}{\partial s\, \partial t} K_Y(t, s) = -\dot{\theta}(t)\dot{\theta}(s)\ \ddot{K}\left(\theta(t) - \theta(s)\right) \tag{193}$$

Taking $s \to t$,

$$\lim_{s \to t} \frac{\partial^2}{\partial s\, \partial t} K_Y(t, s) = -\dot{\theta}(t)^2\, \ddot{K}(0) \tag{194}$$

Also $K_Y(t, t) = K(0)$.

6. The Kac–Rice zero intensity for Y on I_T is

$$\rho_Y(t) = \frac{1}{\pi} \sqrt{\frac{K_Y(t, t) \cdot \lim_{s \to t} \frac{\partial^2}{\partial s\, \partial t} K_Y(t, s) - \left(\lim_{s \to t} \frac{\partial}{\partial s} K_Y(s, t)\right)^2}{K_Y(t, t)^2}} \tag{195}$$

Substituting,

$$K_Y(t, t) \cdot \lim_{s \to t} \frac{\partial^2}{\partial s\, \partial t} K_Y(t, s) - 0^2 = K(0) \cdot (-\dot{\theta}(t)^2\, \ddot{K}(0)) = K(0)\dot{\theta}(t)^2\, (-\ddot{K}(0)) \tag{196}$$

Therefore,

$$\rho_Y(t) = \frac{1}{\pi} \sqrt{\frac{\dot{\theta}(t)^2\, (-\ddot{K}(0))}{K(0)}} = \frac{\dot{\theta}(t)}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}} \tag{197}$$

7. The expected random zero count is

$$\mathbb{E}[N_{\text{rand}}([0, T])] = \int_{I_T} \rho_Y(t)\ dt \tag{198}$$

Since T_0 has Lebesgue measure zero,

$$\mathbb{E}[N_{\mathrm{rand}}([0,T])] = \int_0^T \frac{\dot{\theta}(t)}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}}\, dt = \frac{\theta(T)-\theta(0)}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}} \tag{199}$$

8. The total zero count decomposes as

$$N_{[0,T]}(Z) = N_{\mathrm{det}}([0,T]) + N_{\mathrm{rand}}([0,T]) \tag{200}$$

so

$$\mathbb{E}[N_{[0,T]}(Z)] = N_{\mathrm{det}}([0,T]) + \frac{\theta(T)-\theta(0)}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}} \tag{201}$$ □

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