The Operational Matrix of the Random Wave Process

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February 15, 2025

Abstract

An expression for the convolution of a pair of spherical Bessel functions is determined.

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1 Lemmas

Lemma 1. (Terminating Hypergeometric Series) *For any $p \in \mathbb{Z}_{\geq 0}$, the Gauss hypergeometric function terminates:*

$$ {}_2F_1(-p, b; c; z) = \sum_{k=0}^{p} \frac{(-p)_k (b)_k}{(c)_k \, k!} z^k \tag{1} $$

where $(a)_k = \prod_{i=0}^{k-1} (a+i)$

Proof. By definition, the Gauss hypergeometric series is:

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k \, k!} z^k \tag{2} $$

Setting $a = -p$ with $p \in \mathbb{Z}_{\geq 0}$, the Pochhammer symbol $(-p)_k$ becomes zero for all $k > p$. Explicitly:

$$ (-p)_k = \prod_{i=0}^{k-1} (-p+i) = \begin{cases} (-p)(-p+1)\cdots(-p+k-1), & k \leq p \\ 0 & k > p \end{cases} \tag{3} $$

Thus, the series terminates at $k=p$, yielding:

$$
{}_2F_1(-p,b;c;z)=\sum_{k=0}^{p}\frac{(-p)_k(b)_k}{(c)_k\,k!}\,z^k \tag{4}
$$
$\square$

Lemma 2. **(Integral with Incomplete Gamma Function)** *For $j\geq 0$,*

$$
\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{j}e^{ixy}\,dx=\frac{e^{iy}}{2^j}\frac{\gamma(j+1,2\,i\,y)}{(i\,y)^{j+1}} \tag{5}
$$

where $\gamma(s,x)$ denotes the lower incomplete gamma function.

Proof. Substitute $t=\frac{1-x}{2}\Longrightarrow x=1-2\,t$, $d\,x=-2\,d\,t$, adjusting limits:

$$
\int_{1}^{0}t^j\,e^{i(1-2t)\,y}\,(-2\,d\,t)=2\,e^{iy}\int_{0}^{1}t^j\,e^{-2iyt}\,d\,t \tag{6}
$$

Let $u=2\,i\,y\,t\Longrightarrow t=\frac{u}{2\,i\,y}$, $d\,t=\frac{d\,u}{2\,i\,y}$:

$$
\frac{2\,e^{iy}}{(2\,i\,y)^{j+1}}\int_{0}^{2iy}u^j\,e^{-u}\,d\,u=\frac{e^{iy}}{2^j}\frac{\gamma(j+1,2\,i\,y)}{(i\,y)^{j+1}}
$$
$\square$

Lemma 3. **(Legendre Polynomial Representation)** *The Legendre polynomials are hypergeometric functions*

$$
P_m(x)={}_2F_1(-m,m+1;1;\tfrac{1-x}{2}) \tag{7}
$$

Proof. From the Rodrigues formula $P_m(x)=\frac{1}{2^m\,m!}\frac{d^m}{d\,x^m}(x^2-1)^m$ expand using the binomial theorem:

$$
(x^2-1)^m=\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}x^{2k} \tag{8}
$$

Differentiating m times yields terms proportional to x^k, matching the hypergeometric series:

$$
P_m(x)={}_2F_1\left(-m,m+1;1;\tfrac{1-x}{2}\right)
$$
$\square$

2 Main Theorem

Theorem 4. (Fourier Transform of Legendre Polynomial Products) *The Fourier transform of the product of a pair of Legendre polynomials is the convolution of a pair of spherical Bessel functions of the first kind which can be expressed as*

$$\begin{aligned} I_{m,n}(y) &= \int_{-1}^{1} P_m(x)\, P_n(x)\, e^{ixy}\, dx \\ &= e^{iy} \sum_{j=0}^{m+n} \Psi_j(m,n) \frac{\gamma(j+1, 2\,i\,y)}{2^j\,(i\,y)^{j+1}} \end{aligned} \tag{9}$$

where

$$\begin{aligned} \Psi_j(m,n) &= \frac{{}_4F_3\!\left(\begin{matrix} -m,\ m+1,\ -n,\ n+1 \\ 1,\ 1,\ j+1 \end{matrix};1\right)}{j!} \\ &= \sum_{k=\max(0,j-n)}^{\min(j,m)} \frac{(-m)_k\,(m+1)_k\,(-n)_{j-k}\,(n+1)_{j-k}}{(1)_k(1)_{j-k}\,k!(j-k)!} \end{aligned} \tag{10}$$

Proof. Expand both polynomials using their hypergeometric representations:

$$P_m(x) = \sum_{k=0}^{m} \frac{(-m)_k\,(m+1)_k}{(1)_k\,k!}\left(\frac{1-x}{2}\right)^k \tag{11}$$

$$P_n(x) = \sum_{\ell=0}^{n} \frac{(-n)_\ell\,(n+1)_\ell}{(1)_\ell\,\ell!}\left(\frac{1-x}{2}\right)^\ell \tag{12}$$

Their product becomes:

$$\begin{aligned} P_m(x)\,P_n(x) &= \sum_{k=0}^{m}\sum_{\ell=0}^{n} \frac{(-m)_k\,(m+1)_k\,(-n)_\ell\,(n+1)_\ell}{(1)_k(1)_\ell\,k!\ell!}\left(\frac{1-x}{2}\right)^{k+\ell} \\ &= \sum_{k=0}^{m}\sum_{\ell=0}^{n} \frac{(-m)_k\,(m+1)_k\,(-n)_\ell\,(n+1)_\ell}{k!^2\,\ell!^2}\left(\frac{1-x}{2}\right)^{k+\ell} \end{aligned} \tag{13}$$

Reorganize using $j = k + \ell$, with explicit summation bounds:

$$P_m(x)\,P_n(x) = \sum_{j=0}^{m+n} \underbrace{\sum_{k=\max(0,j-n)}^{\min(j,m)} \frac{(-m)_k\,(m+1)_k\,(-n)_{j-k}\,(n+1)_{j-k}}{(1)_k(1)_{j-k}\,k!(j-k)!}}_{\Psi_j(m,n)}\left(\frac{1-x}{2}\right)^j \tag{14}$$

The inner sum coefficients are:

$$\Psi_j(m,n) = \sum_{k=\max(0,j-n)}^{\min(j,m)} \frac{(-m)_k\,(m+1)_k\,(-n)_{j-k}\,(n+1)_{j-k}}{(1)_k(1)_{j-k}\,k!(j-k)!} \tag{15}$$

Apply the identities:

$$(-n)_{j-k} = (-1)^{j-k}\frac{\Gamma(n+1)}{\Gamma(n-j+k+1)} \tag{16}$$

$$(n+1)_{j-k} = \frac{\Gamma(n+j-k+1)}{\Gamma(n+1)} \tag{17}$$

$$(-a)_n\,(a+1)_n = (-1)^n\frac{\Gamma(a+1)}{\Gamma(a-n+1)} \tag{18}$$

so that the substitutions

$$(-n)_{j-k}\,(n+1)_{j-k} = \frac{(-1)^{j-k}\,\Gamma(n+j-k+1)}{\Gamma(n-j+k+1)} \tag{19}$$

and $h = j-k$ can be made so that $\Psi_j(m,n)$ can be expressed as

$$\Psi_j(m,n) = (-1)^j \sum_{h=0}^{\min(n,j)-\max(0,j-m)} \frac{(-m)_{j-h}\,(m+1)_{j-h}\,\Gamma(n+h+1)}{(j-h)!^2\,\Gamma(n-j+h+1)\,h!} \tag{20}$$

to reveal the hypergeometric form by reversing the index transformation:

$$\begin{aligned}\Psi_j(m,n) &= \frac{1}{j!}\sum_{k=0}^{\min(j,m,n)} \frac{(-m)_k\,(m+1)_k\,(-n)_k\,(n+1)_k}{(1)_k(1)_k\,(j+1)_k\,k!}\\ &= \frac{{}_4F_3\left(\begin{matrix}-m, m+1, -n, n+1\\ 1, 1, j+1\end{matrix};1\right)}{j!}\end{aligned} \tag{21}$$

To complete the proof, observe that the hypergeometric form has an extended summation range $k=0$ to $\min(j,m,n)$ compared to the original bounds $k=\max(0,j-n)$ to $\min(j,m)$. However, terms outside the original bounds vanish identically. For $k>\min(j,m)$, the Pochhammer symbol $(-m)_k=0$ by Lemma 1. For $k<\max(0,j-n)$, the term $(-n)_{j-k}$ in the original sum vanishes since $j-k>n$. Thus the extended summation yields the same result, establishing the equality of both representations of $\Psi_j(m,n)$. □