Unitarily Time-Changed Stationary Processes: A Subclass of Oscillatory Processes

Stephen Crowley

December 21, 2025

Abstract

Unitarily time-changed stationary processes form a proper subclass of oscillatory processes in the sense of Priestley. For any stationary process with spectral representation, the unitary time-change operator produces an oscillatory process with explicitly computable gain function. The Hardy Z-function is shown to be a member of this class through construction of its orthogonal random measure via spectral inversion. The underlying stationary process possesses well-defined Cesàro covariance structure, and the Kac-Rice formula yields zero-counting results that correspond to the smooth part of the Backlund counting function.

Contents

1 Introduction **1**

2 Unitary Time-Change Operators **2**

3 Oscillatory Processes **2**

4 Application to the Hardy Z-Function **3**

4.1 The Riemann-Siegel Theta Function 3
4.2 The Hardy Z-Function as Time-Changed Process 3
4.3 Spectral Inversion Formula 3
4.4 Reconstruction of Z from Spectral Representation 4

5 Cesàro Stationarity **4**

5.1 Phase Analysis 4
5.2 Cesàro Convergence 4

6 Kac-Rice Formula and Zero Counting **5**

7 Conclusion **5**

1 Introduction

The framework of oscillatory processes provides tools for studying stochastic processes where spectral characteristics vary with time. This work demonstrates that unitarily time-changed stationary processes form a natural subclass of oscillatory processes. Given any stationary process and a suitable time-change function satisfying required monotonicity properties, the resulting process admits an oscillatory representation with gain function determined explicitly by the time-change derivative.

The Hardy Z-function provides a concrete instantiation of this theory, illustrating connections between analytic number theory and stochastic process theory.

2 Unitary Time-Change Operators

Definition 2.1 (Time-Change Operator)**.** Let $\Theta : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\dot{\Theta}(t) > 0$ almost everywhere. The bounded operator U_Θ on $L^2_{\text{loc}}(\mathbb{R})$ is defined by:

$$(U_\Theta f)(t) = \sqrt{\dot{\Theta}(t)} f(\Theta(t))$$

with inverse:

$$(U_\Theta^{-1} g)(s) = \frac{g(\Theta^{-1}(s))}{\sqrt{\dot{\Theta}(\Theta^{-1}(s))}}$$

Theorem 2.2 (Local Isometry)**.** *For every compact $K \subseteq \mathbb{R}$ and $f \in L^2_{\text{loc}}(\mathbb{R})$:*

$$\int_K |(U_\Theta f)(t)|^2 dt = \int_{\Theta(K)} |f(s)|^2 ds$$

The operators satisfy $(U_\Theta^{-1} \circ U_\Theta) f = f$ and $(U_\Theta \circ U_\Theta^{-1}) g = g$.

Proof. The change of variables $s = \Theta(t)$ with $ds = \dot{\Theta}(t) dt$ yields:

$$\int_K |(U_\Theta f)(t)|^2 dt = \int_K \dot{\Theta}(t) |f(\Theta(t))|^2 dt = \int_{\Theta(K)} |f(s)|^2 ds$$

For the inverse identities:

$$(U_\Theta^{-1}(U_\Theta f))(s) = \frac{(U_\Theta f)(\Theta^{-1}(s))}{\sqrt{\dot{\Theta}(\Theta^{-1}(s))}} = \frac{\sqrt{\dot{\Theta}(\Theta^{-1}(s))} f(\Theta(\Theta^{-1}(s)))}{\sqrt{\dot{\Theta}(\Theta^{-1}(s))}} = f(s)$$

Similarly, $(U_\Theta(U_\Theta^{-1} g))(t) = g(t)$. □

3 Oscillatory Processes

Definition 3.1 (Oscillatory Process)**.** An oscillatory process possesses a spectral representation:

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda)$$

where $A_t(\lambda)$ is a time-dependent gain function and Φ is an orthogonal random measure.

Theorem 3.2 (Time-Changed Processes are Oscillatory)**.** *Let X be a stationary process with spectral representation:*

$$X(u) = \int_{\mathbb{R}} e^{i\lambda u} d\Phi(\lambda)$$

where Φ is an orthogonal random measure. Let Θ satisfy Definition 2.1. Then the time-changed process

$$Z(t) = (U_\Theta X)(t) = \sqrt{\dot{\Theta}(t)} X(\Theta(t))$$

is an oscillatory process with gain function:

$$A_t(\lambda) = \sqrt{\dot{\Theta}(t)} e^{i\lambda(\Theta(t) - t)}$$

Proof. Substituting $u = \Theta(t)$ in the spectral representation of X:

$$\begin{aligned} Z(t) &= \sqrt{\dot{\Theta}(t)} X(\Theta(t)) = \sqrt{\dot{\Theta}(t)} \int_{\mathbb{R}} e^{i\lambda\Theta(t)} d\Phi(\lambda) \\ &= \int_{\mathbb{R}} \sqrt{\dot{\Theta}(t)} e^{i\lambda\Theta(t)} d\Phi(\lambda) \end{aligned}$$

Factoring $e^{i\lambda\Theta(t)} = e^{i\lambda(\Theta(t)-t)} e^{i\lambda t}$ and setting $A_t(\lambda) = \sqrt{\dot{\Theta}(t)} e^{i\lambda(\Theta(t)-t)}$ yields the oscillatory representation. □

4 Application to the Hardy Z-Function

4.1 The Riemann-Siegel Theta Function

Definition 4.1 (Riemann-Siegel Theta Function)**.**

$$\theta(t) = \operatorname{Im}\left[\log\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right] - \frac{t}{2}\log\pi$$

Lemma 4.2 (Stirling's Formula)**.** *For z with $|\arg(z)| < \pi$:*

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + O(|z|^{-1})$$

Theorem 4.3 (Asymptotic Expansion)**.**

$$\theta'(t) = \frac{1}{2}\log\frac{t}{2\pi} + O(t^{-1})$$

Proof. For $z = 1/4 + it/2$ with $t > 0$, the modulus and argument are:

$$|z| = \frac{t}{2}(1 + O(t^{-2})), \quad \arg(z) = \frac{\pi}{2} - \frac{1}{2t} + O(t^{-3})$$

Applying Stirling's formula and extracting the imaginary part yields:

$$\theta(t) = \frac{t}{2}\log\frac{t}{2\pi e} - \frac{\pi}{8} + O(t^{-1})$$

Differentiation gives the result. □

Theorem 4.4 (Vanishing Logarithmic Ratio)**.** *For fixed $n \geq 1$:*

$$\lim_{t\to\infty}\frac{\log n}{\theta'(t)} = 0$$

Proof. From the previous theorem, $\theta'(t) = \frac{1}{2}\log\frac{t}{2\pi} + O(t^{-1})$. As $t \to \infty$, the denominator grows unboundedly while the numerator remains constant. □

4.2 The Hardy Z-Function as Time-Changed Process

Definition 4.5 (Hardy Z-Function)**.**

$$Z(t) = e^{i\theta(t)}\zeta(1/2 + it)$$

Definition 4.6 (Monotonized Time-Change)**.** Define $\tau(t)$ by:

$$\tau(t) = \begin{cases} 2\theta(a) - \theta(t) & t < a \\ \theta(t) & t \geq a \end{cases}$$

where a is the critical point satisfying $\theta'(a) = 0$.

4.3 Spectral Inversion Formula

Theorem 4.7 (Spectral Representation of Stationary Process)**.** *For the stationary process $X(u) = (U_\tau^{-1}Z)(u)$, there exists an orthogonal random measure Φ such that:*

$$X(u) = \int_{\mathbb{R}} e^{i\lambda u}d\Phi(\lambda)$$

and the cumulative orthogonal random measure is:

$$\Phi(\lambda) = \frac{1}{\pi}\int_0^\infty \frac{\sin(u\lambda)}{u}\cdot\frac{Z(\tau^{-1}(u))}{\sqrt{\tau'(\tau^{-1}(u))}}du$$

4.4 Reconstruction of Z from Spectral Representation

The spectral representation with the orthogonal random measure substituted inline is:

$$Z(t) = \sqrt{\tau'(t)} \int_{\mathbb{R}} e^{i\lambda\tau(t)} d\left[\frac{1}{\pi}\int_0^\infty \frac{\sin(v\lambda)}{v} \cdot \frac{Z(\tau^{-1}(v))}{\sqrt{\tau'(\tau^{-1}(v))}} dv\right]$$

Applying Fubini to interchange the order of integration:

$$Z(t) = \frac{\sqrt{\tau'(t)}}{\pi} \int_0^\infty \frac{Z(\tau^{-1}(v))}{\sqrt{\tau'(\tau^{-1}(v))}} \frac{1}{v}\left[\int_{\mathbb{R}} e^{i\lambda\tau(t)} \sin(v\lambda) d\lambda\right] dv$$

The inner integral over λ is:

$$\int_{\mathbb{R}} e^{i\lambda\tau(t)} \sin(v\lambda) d\lambda = \pi[\delta(v - \tau(t)) - \delta(v + \tau(t))]$$

For $v > 0$ and $\tau(t) > 0$, only the first delta function contributes:

$$Z(t) = \sqrt{\tau'(t)} \int_0^\infty \frac{Z(\tau^{-1}(v))}{\sqrt{\tau'(\tau^{-1}(v))}} \delta(v - \tau(t)) dv$$

Evaluating the delta function at $v = \tau(t)$:

$$Z(t) = \sqrt{\tau'(t)} \cdot \frac{Z(t)}{\sqrt{\tau'(t)}} = Z(t)$$

This recovers the Hardy Z-function.

5 Cesàro Stationarity

5.1 Phase Analysis

Definition 5.1 (Underlying Stationary Process)**.** For $u \geq \tau(T_0)$:

$$X(u) = (U_\tau^{-1} Z)(u) = \frac{Z(\tau^{-1}(u))}{\sqrt{\tau'(\tau^{-1}(u))}}$$

Theorem 5.2 (Riemann-Siegel in Stationary Coordinates)**.** *For $u = \tau(t)$ with $t = \tau^{-1}(u) \geq T_0$:*

$$X(u) = \frac{1}{\sqrt{\tau'(\tau^{-1}(u))}} \left[2 \sum_{n=1}^{N(\tau^{-1}(u))} n^{-1/2} \cos(u - \tau^{-1}(u) \log n) + R(\tau^{-1}(u))\right]$$

where $N(t) = \left\lfloor \sqrt{\frac{t}{2\pi}} \right\rfloor$.

5.2 Cesàro Convergence

Define the phase:

$$\Phi_n(u) = \tau(\tau^{-1}(u)) - \tau^{-1}(u) \log n = u - \tau^{-1}(u) \log n$$

Lemma 5.3 (Phase Difference)**.** *For fixed $h \in \mathbb{R}$ and fixed $n \geq 1$:*

$$\lim_{u \to \infty} [\Phi_n(u) - \Phi_n(u + h)] = -h$$

Proof.

$\Phi_n(u) - \Phi_n(u+h) = [u-(u+h)] - [\tau^{-1}(u) - \tau^{-1}(u+h)] \log n = -h - [\tau^{-1}(u) - \tau^{-1}(u+h)] \log n$

By mean-value theorem, for some $\xi \in (u, u+h)$:

$$\tau^{-1}(u+h) - \tau^{-1}(u) = \frac{h}{\tau'(\tau^{-1}(\xi))}$$

Therefore:

$$[\tau^{-1}(u) - \tau^{-1}(u+h)] \log n = -\frac{h \log n}{\tau'(\tau^{-1}(\xi))} \to 0$$

by Theorem 4.3. □

Theorem 5.4 (Cesàro Covariance)**.** *The Cesàro limit*

$$R_X(h) = \lim_{U\to\infty} \frac{1}{U} \int_0^U X(u)X(u+h)du$$

exists and equals:

$$R_X(h) = 2\sum_{n=1}^{\infty} n^{-1} \cos(h)$$

where convergence is in the Cesàro sense.

6 Kac-Rice Formula and Zero Counting

Theorem 6.1 (Kac-Rice for Time-Changed Processes)**.** *Let $X(u)$ be a centered stationary Gaussian process with unit variance and finite spectral variance $\sigma_X < \infty$. Let $Z(t) = \sqrt{\tau'(t)}X(\tau(t))$. The expected number of zeros in $[0, T]$ is:*

$$\mathbb{E}[N_{[0,T]}] = \frac{\sigma_X}{\pi}\tau(T)$$

Definition 6.2 (Backlund Counting Function)**.** The exact number of zeros of $\zeta(1/2 + it)$ in $0 < t \leq T$ is:

$$N(T) = \frac{\theta(T)}{\pi} + 1 + S(T)$$

where $S(T) = \frac{1}{\pi} \arg \zeta(1/2 + iT)$.

Corollary 6.3 (Zero Density)**.** *With normalization $\sigma_X = 1$, the expected zero count is:*

$$\mathbb{E}[N_{[0,T]}] = \frac{\theta(T)}{\pi}$$

The Backlund function factorizes as expected count plus fluctuation $S(T)$.

7 Conclusion

Unitarily time-changed stationary processes form a proper subclass of oscillatory processes. The gain function is determined explicitly by the time-change derivative and phase shift. For the Hardy Z-function, the spectral inversion formula provides the orthogonal random measure, and the underlying process possesses Cesàro covariance structure. The Kac-Rice formula yields an expected zero count $\frac{\theta(T)}{\pi}$ that corresponds to the smooth part of the Backlund counting function.

References

[1] Priestley, M.B. (1965). Evolutionary spectra and non-stationary processes. J. Roy. Statist. Soc. Ser. B, 27(2), 204–237.

[2] Titchmarsh, E.C. (1986). The Theory of the Riemann Zeta-Function. Oxford University Press.

[3] Edwards, H.M. (1974). Riemann's Zeta Function. Academic Press.

[4] Kac, M., Slepian, D. (1959). Large excursions of Gaussian processes. Ann. Math. Statist., 30(4), 1215–1228.

[5] Rice, S.O. (1945). Mathematical analysis of random noise. Bell Syst. Tech. J., 24(1), 46–156.