

# Injective Measure-Preserving Time-Changes of Stationary Processes are Oscillatory

BY STEPHEN CROWLEY

July 24, 2025

## Oscillatory Processes and Normalized Injective Time-Changes

**Definition 1.** *[Oscillatory Process] A complex-valued second-order stochastic process  $\{X_t\}_{t \in I}$  is said to be oscillatory if there exists a family of functions  $\phi_t(\omega)$  and a complex orthogonal increment process  $Z(\omega)$  with  $E |dZ(\omega)|^2 = d\mu(\omega)$  such that*

$$X_t = \int_{-\infty}^{\infty} \phi_t(\omega) dZ(\omega) \quad (1)$$

where  $\phi_t(\omega) = A_t(\omega) e^{i\omega t}$  and  $A_t(\omega)$  is quadratically integrable with respect to  $d\mu$ .

**Definition 2.** *[Stationary Process] A second-order process  $\{S_t\}_{t \in J}$  is stationary if it admits the spectral representation*

$$S_t = \int_{-\infty}^{\infty} e^{i\omega t} dZ(\omega) \quad (2)$$

for some orthogonal increment process  $Z(\omega)$  with  $E |dZ(\omega)|^2 = d\mu(\omega)$ .

**Theorem 3.** *[Time-Varying Filter for Injective Time-Change] Let  $S_t$  be a stationary process and  $\theta: \mathbb{R} \rightarrow \mathbb{R}$  be smooth and strictly increasing with  $\theta'(t) > 0$ . To achieve the transformation*

$$X_t = \sqrt{\theta'(t)} S_{\theta(t)} \quad (3)$$

via convolution  $X_t = \int_{-\infty}^{\infty} S_{t-u} h_t(u) du$ , the time-varying impulse response must be

$$h_t(u) = \sqrt{\theta'(t)} \delta(u - (t - \theta(t))) \quad (4)$$

**Proof.** For the convolution to yield  $X_t = \sqrt{\theta'(t)} S_{\theta(t)}$ , the argument of  $S$  in the integrand must equal  $\theta(t)$  when the delta function is activated. This requires:

$$t - u = \theta(t)$$

Solving for  $u$ :

$$u = t - \theta(t) \quad (5)$$

Therefore:

$$h_t(u) = \sqrt{\theta'(t)} \delta(u - (t - \theta(t))) \quad (6)$$

Verification by direct computation:

$$X_t = \int_{-\infty}^{\infty} S_{t-u} h_t(u) du \quad (7)$$

$$= \int_{-\infty}^{\infty} S_{t-u} \sqrt{\theta'(t)} \delta(u - (t - \theta(t))) du \quad (8)$$

$$= \sqrt{\theta'(t)} S_{t-(t-\theta(t))} \quad (\text{by sifting property}) \quad (9)$$

$$= \sqrt{\theta'(t)} S_{\theta(t)} \quad (10)$$

□

**Theorem 4.** *[Spectral Envelope for Injective Time-Change] For the time-varying filter  $h_t(u) = \sqrt{\theta'(t)} \delta(u - (t - \theta(t)))$ , the spectral envelope is*

$$A_t(\omega) = \sqrt{\theta'(t)} e^{i\omega(t-\theta(t))}$$

**Proof.** The spectral envelope is the Fourier transform of  $h_t(u)$ :

$$A_t(\omega) = \int_{-\infty}^{\infty} e^{i\omega u} h_t(u) du \quad (11)$$

$$= \int_{-\infty}^{\infty} e^{i\omega u} \sqrt{\theta'(t)} \delta(u - (t - \theta(t))) du \quad (12)$$

$$= \sqrt{\theta'(t)} e^{i\omega(t-\theta(t))} \quad (\text{by sifting property}) \quad (13)$$

□

**Theorem 5.** *[Oscillatory Representation of Injective Time-Change] The process  $X_t = \sqrt{\theta'(t)} S_{\theta(t)}$  admits the oscillatory representation*

$$X_t = \int_{-\infty}^{\infty} \phi_t(\omega) dZ(\omega)$$

where

$$\phi_t(\omega) = \sqrt{\theta'(t)} e^{i\omega\theta(t)}$$

**Proof.** Starting from the spectral representation of  $S_t$ :

$$X_t = \sqrt{\theta'(t)} S_{\theta(t)} \quad (14)$$

$$= \sqrt{\theta'(t)} \int_{-\infty}^{\infty} e^{i\omega\theta(t)} dZ(\omega) \quad (15)$$

$$= \int_{-\infty}^{\infty} \sqrt{\theta'(t)} e^{i\omega\theta(t)} dZ(\omega) \quad (16)$$

Thus  $\phi_t(\omega) = \sqrt{\theta'(t)} e^{i\omega\theta(t)}$ . □

**Corollary 6.** *[Envelope in Standard Form] The oscillatory functions can be written in the standard form  $\phi_t(\omega) = A_t(\omega) e^{i\omega t}$  where*

$$A_t(\omega) = \sqrt{\theta'(t)} e^{i\omega(\theta(t)-t)}$$

**Proof.** Factor the exponential:

$$\phi_t(\omega) = \sqrt{\theta'(t)} e^{i\omega\theta(t)} \quad (17)$$

$$= \sqrt{\theta'(t)} e^{i\omega(\theta(t)-t)} e^{i\omega t} \quad (18)$$

$$= A_t(\omega) e^{i\omega t} \quad (19)$$

where  $A_t(\omega) = \sqrt{\theta'(t)} e^{i\omega(\theta(t)-t)}$ . □

**Theorem 7.** *[Verification of Filter-Envelope Consistency] The spectral envelope*

$$A_t(\omega) = \sqrt{\theta'(t)} e^{i\omega(\theta(t)-t)} \quad (20)$$

*is the Fourier transform of the time-varying impulse response*

$$h_t(u) = \sqrt{\theta'(t)} \delta(u - (t - \theta(t))) \quad (21)$$

**Proof.** Compute the Fourier transform:

$$\mathcal{F}[h_t](\omega) = \int_{-\infty}^{\infty} e^{i\omega u} \sqrt{\theta'(t)} \delta(u - (t - \theta(t))) du \quad (22)$$

$$= \sqrt{\theta'(t)} e^{i\omega(t - \theta(t))} \quad (23)$$

$$= A_t(\omega) \quad (24)$$

Compute the inverse Fourier transform:

$$\mathcal{F}^{-1}[A_t](u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega u} \sqrt{\theta'(t)} e^{i\omega(\theta(t)-t)} d\omega \quad (25)$$

$$= \frac{\sqrt{\theta'(t)}}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(\theta(t)-t-u)} d\omega \quad (26)$$

$$= \sqrt{\theta'(t)} \delta(\theta(t) - t - u) \quad (27)$$

$$= \sqrt{\theta'(t)} \delta(u - (t - \theta(t))) \quad (28)$$

$$= h_t(u) \quad (29)$$

□