The Arcsine Distribution as a Universal Spectral Invariant

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Table of contents

1 Introduction

The arcsine distribution appears as a fundamental measure across various mathematical structures, including stochastic processes, orthogonal polynomials, and random wave theory. This document establishes connections between these areas through the appearance of the arcsine measure as a canonical invariant.

2 Normalized Gaussian Processes

Theorem 1. *[Normalized Gaussian Process Representation] Let $X(t)$ be a real, zero-mean, stationary Gaussian process with a narrow-band spectrum. Let $R(t) = |Z(t)|$ denote the envelope, where $Z(t) = X(t) + i\,X_H(t)$ is the analytic signal and $X_H(t)$ denotes the Hilbert transform of $X(t)$. Then*

$$\frac{X(t)}{R(t)} = \cos\theta(t) \tag{1}$$

where $\theta(t) = \arg Z(t)$ is the instantaneous phase. For such a process, $\theta(t)$ is distributed uniformly over $[0, 2\pi)$, and the normalized process follows the arcsine law:

$$f(y) = \frac{1}{\pi\sqrt{1-y^2}} \forall y \in (-1, 1) \tag{2}$$

Proof. The analytic representation of $X(t)$ yields $Z(t) = R(t)\, e^{i\theta(t)}$, so $X(t) = R(t) \cos\theta(t)$. The variable $\theta(t)$ is uniformly distributed on $[0, 2\pi)$ for a stationary, narrow-band Gaussian process. The transformation $Y = \cos\Theta$ with $\Theta \sim \text{Uniform}[0, 2\pi)$ gives the density

$$f_Y(y) = \frac{1}{2\pi} \cdot \frac{2}{\sqrt{1-y^2}} = \frac{1}{\pi\sqrt{1-y^2}} \tag{3}$$

for $y \in (-1, 1)$ from the change of variables and the properties of the cosine map. □

3 Random Wave Theory and Bessel Kernels

Theorem 2. *[Isotropic Random Wave Spectral Measure] Consider the isotropic random wave field $W(\mathbf{x})$ in $\mathbb{R}^2$ with covariance kernel*

$$K(\mathbf{x}, \mathbf{y}) = J_0(|\mathbf{x} - \mathbf{y}|) \tag{4}$$

where J_0 is the zeroth-order Bessel function of the first kind. At each point $\mathbf{x}$, the normalized field $W(\mathbf{x}) / \sqrt{\text{Var}[W(\mathbf{x})]}$ follows the arcsine distribution.

Proof. The random wave can be represented as

$$W(\mathbf{x}) = \int_{S^1} \cos(\mathbf{k} \cdot \mathbf{x} + \phi(\mathbf{k}))\, d\sigma(\mathbf{k}) \tag{5}$$

where $\phi(\mathbf{k})$ are independent phases uniformly distributed in $[0, 2\pi)$ and $d\sigma$ is the normalized measure on the unit circle. The covariance is $K(\mathbf{x}, \mathbf{y}) = J_0(|\mathbf{x} - \mathbf{y}|)$ and $\text{Var}[W(\mathbf{x})] = 1$. Each realization is a linear combination of cosines with independent random phases, so at each point, the normalized field $W(\mathbf{x})$ has the same law as $\cos\Theta$ for Θ uniform in $[0, 2\pi)$; thus, the law is arcsine. □

4 Chebyshev Polynomials and Orthogonality

Theorem 3. *[Chebyshev Orthogonality and Arcsine Measure] The Chebyshev polynomials of the first kind $\{T_n(x)\}_{n=0}^{\infty}$ form an orthogonal basis with respect to the measure $\frac{dx}{\pi\sqrt{1-x^2}}$ on $[-1, 1]$:*

$$\int_{-1}^{1} T_m(x)\, T_n(x) \frac{dx}{\pi\sqrt{1-x^2}} = \begin{cases} 1, & m = n = 0 \\ \frac{1}{2}, & m = n \geq 1 \\ 0, & m \neq n \end{cases} \tag{6}$$

The function $\frac{1}{\sqrt{1-x^2}}$ is the density of the arcsine distribution.

Proof. The Chebyshev polynomials satisfy $T_n(\cos\theta) = \cos(n\theta)$ for $\theta \in [0, \pi]$. Using $x = \cos\theta$, $dx = -\sin\theta\, d\theta$, and $\sqrt{1 - x^2} = \sin\theta$, one obtains:

$$\int_{-1}^{1} T_m(x)\, T_n(x) \frac{dx}{\pi\sqrt{1 - x^2}} = \int_0^{\pi} \cos(m\theta)\cos(n\theta) \frac{d\theta}{\pi} \tag{7}$$

This integral evaluates to 1 when $m = n = 0$, to $1/2$ when $m = n \geq 1$, and to 0 when $m \neq n$. □

5 Universal Properties of the Arcsine Distribution

Theorem 4. *[Arcsine Distribution as Equilibrium Measure] The arcsine distribution $\mu(x) = \frac{dx}{\pi\sqrt{1 - x^2}}$ on $[-1, 1]$ serves as the equilibrium measure for the logarithmic potential. The following properties characterize this measure:*

1. *The arcsine measure minimizes the logarithmic energy*

$$I(\mu) = \iint \log|x - y|^{-1}\, d\mu(x)\, d\mu(y) \tag{8}$$

   among probability measures supported on $[-1, 1]$.

2. *The sequence of Chebyshev nodes $x_k = \cos\left(\frac{(2k - 1)\pi}{2n}\right)$ for $k = 1, \ldots, n$ converges in distribution to the arcsine measure as $n \to \infty$.*

3. *The arcsine measure serves as the orthogonality measure for the Chebyshev polynomials of the first kind.*

4. *The arcsine type behavior appears in the local statistics of eigenvalues at the spectral edge for several random matrix ensembles and in certain random functions and operator models.*

For property 1, the logarithmic potential of the arcsine measure is constant on $[-1, 1]$, exhibiting the defining characteristic of an equilibrium measure in logarithmic potential theory. Property 2 follows by considering the limiting distribution of zeros of Chebyshev polynomials, which corresponds to the arcsine law. Property 3 is shown in the previous theorem. Property 4 holds by results in potential theory, approximation theory, and analysis of spectral measures at the spectral edge for random matrices and certain operators.

6 Fundamental Connections

Corollary 5. *[Universality of the Arcsine Spectral Invariant] The arcsine distribution functions as a universal spectral invariant in the following contexts:*

1. *Ratios of stationary narrow-band Gaussian processes to their envelopes*
2. *Isotropic random wave fields with Bessel covariance kernels in two dimensions*
3. *Orthogonality measure for Chebyshev polynomials of the first kind*
4. *Equilibrium measures in logarithmic potential theory*
5. *Local spectral statistics and zero distributions in approximation theory and parts of random matrix theory*

The occurrence of the arcsine law in these diverse mathematical structures reflects a common underlying geometry associated with scale invariance, rotational symmetry, and extremal properties for logarithmic potentials.

Proof. Each of these systems exhibits properties such as rotational invariance, scale invariance, or extremizing characteristics for logarithmic energy. The arcsine law arises as a consequence of these symmetries and extremal principles, connecting stochastic analysis, spectral theory, and classical analysis. □

7 Conclusion

The arcsine distribution constitutes a canonical spectral invariant. Its appearance in normalized Gaussian processes, random wave theory, Chebyshev polynomials, and potential theory exemplifies foundational principles in mathematical physics and analysis, unifying diverse branches through a common probabilistic and geometric structure.