Unitary Time Changes of Stationary Processes Yield Oscillatory Processes

by Stephen Crowley

September 16, 2025

Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are square-integrable over compact sets. Applying U_θ to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\ e^{i\lambda\theta(t)}$ and evolutionary spectrum $d\,F_t(\lambda) = \dot{\theta}(t) d\,F(\lambda)$. It is proved that sample paths of any non-degenerate second-order stationary process almost surely lie in $L^2_{\text{loc}}(\mathbb{R})$, making the operator applicable to typical realizations. A zero-localization measure $\mathrm{d}\mu(t) = \delta(Z(t))\,|\dot{Z}(t)|\ d\,t$ induces a Hilbert space $L^2(\mu)$ on the zero set of an oscillatory process Z, and the multiplication operator $(L\,f)(t) = t\,f(t)$ has simple pure point spectrum equal to the zero crossing set of Z. This produces a concrete operator scaffold consistent with a Hilbert–Pólya-type viewpoint.

Table of contents

1 Function Spaces

1.1 σ-compact sets and locally square-integrable functions

Definition 1. ***[σ-compact sets]*** *A subset $U \subseteq \mathbb{R}$ is σ-compact if*

$$U = \bigcup_{n=1}^{\infty} K_n \tag{1}$$

with each K_n compact.

Definition 2. ***[Locally square-integrable functions]*** *Define*

$$L^2_{\mathrm{loc}}(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} : \int_K |f(t)|^2 \, dt < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \tag{2}$$

Remark 3. Every bounded measurable set in $\mathbb{R}$ is compact or contained in a compact set; hence $L^2_{\mathrm{loc}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

2 Gaussian Processes

A Gaussian process is a ...

2.1 Stationary processes

Definition 4. ***[Cramér representation]*** *A zero-mean stationary process X with spectral measure F admits the sample path representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t} \, \Phi(d\lambda) \tag{3}$$

which has covariance

$$R_X(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)} \, dF(\lambda) \tag{4}$$

2.2 Oscillatory Processes

A particularly tractable class of non-stationary Gaussian processes is that of the oscillatory processes as defined by M.B. Priestley in 1965[1].

Definition 5. ***[Oscillatory process]*** *link to Priestley 1965 Let F be a finite nonnegative Borel measure on* $\mathbb{R}$*. Let*

$$A_t \in L^2(F) \forall t \in \mathbb{R} \tag{5}$$

be the gain function and

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{6}$$

be the corresponding oscillatory function then an oscillatory process *is a stochastic process which can be represented as*

$$\begin{aligned} Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda)\, \mathrm{d}\,\Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, \mathrm{d}\,\Phi(\lambda) \end{aligned} \tag{7}$$

where Φ *is a complex orthogonal random measure with spectral measure F which satisfies the relation*

$$\mathrm{d}\,\mathbb{E}[\Phi(\lambda)\overline{\Phi(\mu)}] = \delta\,(\lambda - \mu)\; d\,F(\lambda) \tag{8}$$

and has the corresponding covariance kernel

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \int_{\mathbb{R}} A_t(\lambda)\, \overline{A_s(\lambda)}\, e^{i\lambda(t-s)}\; d\,F(\lambda) \\ &= \int_{\mathbb{R}} \phi_t(\lambda)\overline{\phi_s(\lambda)} \mathrm{d}\,F(\lambda) \end{aligned} \tag{9}$$

Theorem 6. ***[Real-valuedness criterion for oscillatory processes]****Let Z be an oscillatory process with oscillatory function*

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{10}$$

and spectral measure F. Then Z is real-valued if and only if

$$A_t\,(-\lambda) = \overline{A_t(\lambda)} \tag{11}$$

for F-almost every $\lambda \in \mathbb{R}$*, equivalently*

$$\varphi_t\,(-\lambda) = \overline{\varphi_t(\lambda)} \tag{12}$$

for F-almost every $\lambda \in \mathbb{R}$*.*

Proof. Assume Z is real-valued, i.e.

$$Z(t) = \overline{Z(t)} \quad \forall t \in \mathbb{R} \tag{13}$$

Writing its oscillatory representation,

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, \mathrm{d}\Phi(\lambda) \tag{14}$$

and taking the complex conjugate gives

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t} \mathrm{d}\overline{\Phi\,(\lambda)} \tag{15}$$

For a real-valued process, the orthogonal random measure Φ must satisfy

$$\mathrm{d}\overline{\Phi\,(\lambda)} = -\mathrm{d}\,\Phi\,(\lambda) \tag{16}$$

which ensures that the spectral representation produces real values. Substituting this identity and using the substitution

$$\mu = -\lambda \tag{17}$$

it is shown that

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t\,(-\mu)} e^{i\mu t}\, \mathrm{d}\,\Phi\,(\mu) \tag{18}$$

Since $Z(t) = \overline{Z(t)}$, comparison of the integrands (which are unique elements of $L^2(F)$) yields

$$A_t(\lambda) = \overline{A_t\,(-\lambda)} \quad \text{for } F\text{-a.e. } \lambda \tag{19}$$

Equivalently, because the oscillatory function (6) is given by

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{20}$$

we have

$$\varphi_t\,(-\lambda) = \overline{\varphi_t(\lambda)} \quad \text{for } F\text{-a.e. } \lambda \tag{21}$$

Conversely, if

$$A_t\,(-\lambda) = \overline{A_t(\lambda)} \tag{22}$$

for F-a.e. λ, then the same substitution shows that

$$\overline{Z(t)} = Z(t) \quad \forall t \in \mathbb{R} \tag{23}$$

so Z is real-valued. This completes the proof. □

Theorem 7. ***[Existence]*** *Let F be an absolutely continuous spectral measure and the gain function*

$$A_t(\lambda) \in L^2(F) \forall \mathbb{R} \ni t < \infty \tag{24}$$

be measurable in both time and frequency then the time-dependent spectral density is defined by

$$\begin{aligned} S_t(\lambda) \ &= \int_{\mathbb{R}} |A_t(\lambda)|^2 \, dF(\lambda) < \infty \\ &= \int_{\mathbb{R}} |A_t(\lambda)|^2 \, S(\lambda) d\lambda \end{aligned} \tag{25}$$

and there exists a complex orthogonal random measure Φ with spectral measure F such that for each sample path $\varpi \in \Theta$ in the space of sample paths having given covariance constituting the ensemble denoted Θ

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda) \, e^{i\lambda t} \mathrm{d}\Phi(\lambda) \tag{26}$$

is well-defined in $L^2(\Omega)$ and has covariance R_Z as in (9) above.

Proof. The proof proceeds by constructing the stochastic integral using the standard extension procedure. First, the integral is defined for simple functions of the form

$$g(\lambda) = \lim_{n \to \infty} \sum_{j=1}^{n} c_j \mathbf{1}_{E_j}(\lambda) \tag{27}$$

where $\{E_j\}$ are disjoint Borel sets with $F(E_j) < \infty$ and $c_j \in \mathbb{C}$:

$$\int_{\mathbb{R}} g(\lambda) \, \mathrm{d}\Phi(\lambda) = \lim_{n \to \infty} \sum_{j=1}^{n} c_j \, \Phi(E_j) \tag{28}$$

For simple functions such as this, the isometry property holds:

$$\begin{aligned} \mathbb{E}\left[\left| \int_{\mathbb{R}} g(\lambda) \ \Phi \mathrm{d}(\lambda) \right|^2 \right] \ &= \mathbb{E}\left[\lim_{n \to \infty} \left| \sum_{j=1}^{n} c_j \, \Phi(E_j) \right|^2 \right] \\ &= \lim_{n \to \infty} \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \bar{c_k} \mathbb{E}[\Phi(E_j) \overline{\Phi(E_k)}] \\ &= \lim_{n \to \infty} \sum_{j=1}^{n} |c_j|^2 \, F(E_j) \\ &= \int_{\mathbb{R}} |g(\lambda)|^2 \, dF(\lambda) \end{aligned} \tag{29}$$

Since simple functions are dense in $L^2(F)$, the integral is extended by continuity $\forall g \in L^2(F)$ since the oscillatory function (6) is defined by

$$\varphi_t(\lambda) = A_t(\lambda) \, e^{i\lambda t} \in L^2(F) \forall t \in \mathbb{R} \tag{30}$$

and $A_t \in$. Therefore

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda)\, \mathrm{d}\Phi\,(\lambda) \tag{31}$$

is well-defined in $L^2(\Omega)$. The covariance is computed as:

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\Big[\int_{\mathbb{R}} \varphi_t(\lambda) \mathrm{d}\Phi\,(\lambda) \int_{\mathbb{R}} \overline{\varphi_s(\mu)} \mathrm{d}\overline{\Phi\,(\mu)}\Big] \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\mu)} \mathrm{d}\mathbb{E}\,[\Phi(\lambda)\overline{\Phi(\mu)}] \\ &= \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)}\, d\,F(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)} e^{i\lambda(t-s)}\, d\,F(\lambda) \end{aligned} \tag{32}$$

□

3 Unitarily Time-Changed Stationary Processes

3.1 Unitary time-change operator $U_\theta f$

Definition 8. ***[Unitary time-change]*** *Let the time-scaling function $\theta: \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with $\dot{\theta}(t) > 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero. The function θ maps σ-compact sets to σ-compact sets. Define, for f measurable,*

$$(U_\theta\, f)(t) = \sqrt{\dot{\theta}(t)}\; f(\theta(t)) \tag{33}$$

Proposition 9. ***[Inversion of Unitary time-change]*** *The inverse of the unitary time-change operator U in Equation (33) is given by*

$$(U_\theta^{-1}\, g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{34}$$

which is well-defined almost everywhere on every σ-compact set.

Proof. Since $\dot{\theta}(t) = 0$ only on sets of measure zero, and θ^{-1} maps sets of measure zero to sets of measure zero because of the fact that absolutely continuous bijective functions preserve measure-zero sets, the denominator $\sqrt{\dot{\theta}(\theta^{-1}(s))}$ is positive almost everywhere. The expression is therefore well-defined almost everywhere on every σ-compact set, which suffices for defining an element of $L^2_{\mathrm{loc}}(\mathbb{R})$. □

Theorem 10. ***[Local unitarity on σ-compact sets]*** *For every σ-compact set $C \subseteq \mathbb{R}$ and $f \in L^2_{\text{loc}}(\mathbb{R})$,*

$$\int_C |(U_\theta f)(t)|^2\, dt = \int_{\theta(C)} |f(s)|^2\, ds \tag{35}$$

Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\text{loc}}(\mathbb{R})$.

Proof. Let $f \in L^2_{\text{loc}}(\mathbb{R})$ and let C be any σ-compact set. The local L^2-norm of $U_\theta f$ over C is:

$$\begin{aligned} \int_C |(U_\theta f)(t)|^2\, dt &= \int_C \left|\sqrt{\dot{\theta}(t)}\, f(\theta(t))\right|^2 dt \\ &= \int_C \dot{\theta}(t) |f(\theta(t))|^2\, dt \end{aligned} \tag{36}$$

Since θ is absolutely continuous and strictly increasing, applying the change of variables $s = \theta(t)$ gives

$$ds = \dot{\theta}(t)\, dt \tag{37}$$

almost everywhere. Since θ maps σ-compact sets to σ-compact sets, as t ranges over C, $s = \theta(t)$ ranges over $\theta(C)$, which is σ-compact. Therefore:

$$\int_C \dot{\theta}(t)\, |f(\theta(t))|^2\, dt = \int_{\theta(C)} |f(s)|^2\, ds \tag{38}$$

To verify that U_θ^{-1} is indeed the inverse, it is seen that:

$$\begin{aligned} (U_\theta^{-1} U_\theta f)(s) &= \left(U_\theta^{-1} \sqrt{\dot{\theta}(s)}\, f(\theta(s))\right)(s) \\ &= \frac{\sqrt{\dot{\theta}(\theta^{-1}(s))}}{\sqrt{\dot{\theta}(\theta^{-1}(s))}}\, f(\theta(\theta^{-1}(s))) \qquad \forall f \in L^2_{\text{loc}}(\mathbb{R}) \\ &= f(s) \end{aligned} \tag{39}$$

since

$$\theta(\theta^{-1}(s)) = s \tag{40}$$

and similarly, its also plain to see that:

$$\begin{aligned} (U_\theta U_\theta^{-1} g)(t) &= \sqrt{\dot{\theta}(t)}\, (U_\theta^{-1} g)(\theta(t)) \\ &= \frac{\sqrt{\dot{\theta}(t)}}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}}\, g(\theta^{-1}(\theta(t))) \qquad \forall g \in L^2_{\text{loc}}(\mathbb{R}) \\ &= \frac{\sqrt{\dot{\theta}(t)}}{\sqrt{\dot{\theta}(t)}}\, g(t) \\ &= g(t) \end{aligned} \tag{41}$$

since

$$\theta^{-1}(\theta(t)) = t \tag{42}$$

Therefore

$$\begin{aligned}(U_\theta U_\theta^{-1} f)(t) &= (U_\theta^{-1} U_\theta f)(t) \\ &= f(t)\end{aligned} \tag{43}$$

on $L^2_{\mathrm{loc}}(\mathbb{R})$. □

3.1.1 Inverse Filter for Unitary Time Transformations

Theorem 11. *[Inverse Filter for Unitary Time Transformations]Let $\theta: \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\theta'(t) > 0$ almost everywhere. Let $Y(u)$ be a stationary process with unit variance, and define*

$$Z(t) = \sqrt{\dot{\theta}(t)}\; Y(\theta(t)) \tag{44}$$

as the oscillatory process obtained by the unitary time transformation. Then:

1. *The forward filter kernel is*

$$h(t, u) = \sqrt{\dot{\theta}(t)}\; \delta(u - \theta(t)) \tag{45}$$

2. *The inverse filter kernel is*

$$g(t, s) = \frac{\delta(s - \theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \tag{46}$$

3. *The composition $(g \circ h)$ recovers the identity:*

$$\begin{aligned}Y(t) &= \int_{\mathbb{R}} g(t, s)\, Z(s)\; ds \\ &= \frac{Z(\theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}}\end{aligned} \tag{47}$$

Proof. Parts 1–3 are established in sequence.

Let

$$v = \theta(s) \text{ and } s = \theta^{-1}(v) \tag{48}$$

so that

$$ds = \frac{1}{\dot{\theta}(\theta^{-1}(v))}\, dv \tag{49}$$

and

$$\delta\left(\theta^{-1}(v)-\theta^{-1}(t)\right)=\dot{\theta}(\theta^{-1}(t))\ \delta\left(v-t\right) \tag{50}$$

then substitute the inverse filter in Equation (45)

$$g(t,s)=\frac{\delta\left(s-\theta^{-1}(t)\right)}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \tag{51}$$

and unitarily time-changed stationary process operator representation (44)

$$\begin{aligned} Z(t)\ &=(U_\theta Y)(t)\\ &=\sqrt{\dot{\theta}(t)}\ Y(\theta(t)) \end{aligned} \tag{52}$$

to verify that each of the equivalent expressions

$$\begin{aligned} Y(t)\ &=\int_{\mathbb{R}} g(t,s)\,Z(s)\ d\,s\\ &=\int_{\mathbb{R}} \frac{\delta\left(s-\theta^{-1}(t)\right)}{\sqrt{\dot{\theta}(\theta^{-1}(t))}}\ Z(s)\ d\,s\\ &=\int_{\mathbb{R}} \frac{\delta\left(s-\theta^{-1}(t)\right)}{\sqrt{\dot{\theta}(\theta^{-1}(t))}}\ (U_\theta Y)(s)\,d\,s\\ &=\int_{\mathbb{R}} \frac{\delta\left(s-\theta^{-1}(t)\right)}{\sqrt{\dot{\theta}(\theta^{-1}(t))}}\ \sqrt{\dot{\theta}(s)}\ Y(\theta(s))\ d\,s\\ &=\int_{\mathbb{R}} \frac{\delta\left(\theta^{-1}(v)-\theta^{-1}(t)\right)}{\sqrt{\dot{\theta}(\theta^{-1}(t))}}\ \frac{\sqrt{\dot{\theta}(\theta^{-1}(v))}}{\dot{\theta}(\theta^{-1}(v))}Y(v)d\,v\\ &=\int_{\mathbb{R}} \frac{\delta\left(\theta^{-1}(v)-\theta^{-1}(t)\right)}{\sqrt{\dot{\theta}(\theta^{-1}(t))}\ \sqrt{\dot{\theta}(\theta^{-1}(v))}}\ Y(v)\ d\,v\\ &=\int_{\mathbb{R}} \frac{\dot{\theta}(\theta^{-1}(t))\ \delta\left(v-t\right)}{\sqrt{\dot{\theta}(\theta^{-1}(t))}\ \sqrt{\dot{\theta}(\theta^{-1}(v))}}\ Y(v)\ d\,v\\ &=\frac{\dot{\theta}(\theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}\ \sqrt{\dot{\theta}(\theta^{-1}(t))}}\ Y(t)\\ &=\frac{\dot{\theta}(\theta^{-1}(t))}{\left(\sqrt{\dot{\theta}(\theta^{-1}(t))}\right)^2}\,Y(t)\\ &=\frac{\dot{\theta}(\theta^{-1}(t))}{\dot{\theta}(\theta^{-1}(t))}\,Y(t)\\ &=Y(t) \end{aligned} \tag{53}$$

□

3.2 Transformation of Stationary → Oscillatory Processes via U_θ

Theorem 12. ***[Unitary time change yields oscillatory process]*** *Let X be zero-mean stationary as in Definition 4. For scaling function θ as in Definition 8, define*

$$\begin{aligned} Z(t) &= (U_\theta X)(t) \\ &= \sqrt{\dot{\theta}(t)}\; X(\theta(t)) \end{aligned} \tag{54}$$

Then Z is a realization of an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\; e^{i\lambda\theta(t)} \tag{55}$$

gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\; e^{i\lambda(\theta(t)-t)} \tag{56}$$

and covariance

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\left[\sqrt{\dot{\theta}(t)}\, X(\theta(t))\overline{\sqrt{\dot{\theta}(t)}\, X(\theta(t))}\right] \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\; \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\; R_X\left(\theta(t)-\theta(s)\right) \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\, dF(\lambda) \end{aligned} \tag{57}$$

Proof. Applying the unitary time change operator to the spectral representation of $X(t)$:

$$\begin{aligned} Z(t) &= (U_\theta X)(t) \\ &= \sqrt{\dot{\theta}(t)}\; X(\theta(t)) \\ &= \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, \mathrm{d}\,\Phi(\lambda) \\ &= \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}\, \mathrm{d}\,\Phi(\lambda) \\ &= \int_{\mathbb{R}} \varphi_t(\lambda)\, \mathrm{d}\,\Phi(\lambda) \end{aligned} \tag{58}$$

where

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \tag{59}$$

To verify this constitutes an oscillatory representation according to Definition 5, $\varphi_t(\lambda)$ has the form $A_t(\lambda)\, e^{i\lambda t}$:

$$\begin{aligned}\varphi_t(\lambda) &= \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \\ &= \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)}\, e^{i\lambda t} \\ &= A_t(\lambda)\, e^{i\lambda t}\end{aligned} \tag{60}$$

where

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)} \tag{61}$$

Since $\dot{\theta}(t) \geq 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of measure zero, $A_t(\lambda)$ is well defined almost everywhere. Moreover, $A_t \in L^2(F)$ for each t since:

$$\begin{aligned}\int_{\mathbb{R}} |A_t(\lambda)|^2\, d\,F(\lambda) &= \int_{\mathbb{R}} \left|\sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)}\right|^2 d\,F(\lambda) \\ &= \int_{\mathbb{R}} \dot{\theta}(t) |e^{i\lambda(\theta(t)-t)}|^2\, d\,F(\lambda) \\ &= \dot{\theta}(t) \int_{\mathbb{R}} d\,F(\lambda) \\ &= \dot{\theta}(t)\, F(\mathbb{R}) < \infty\end{aligned} \tag{62}$$

where $|e^{i\alpha}| = 1$ for all real α is used. The covariance (57) is computed by substituting the spectral representation and applying Fubuni's theorem to interchange the order of operations.

$$\tag{63}$$ □

Corollary 13. ***[Evolutionary spectrum of unitarily time-changed stationary process]****[1] Link to The evolutionary spectrum, also called the time-varying spectral density, is*

$$\begin{aligned}d\,F_t(\lambda) &= |A_t(\lambda)|^2\, d\,F(\lambda) \\ &= \dot{\theta}(t) d\,F(\lambda)\end{aligned} \tag{64}$$

Proof. By definition of the evolutionary spectrum and using the gain function from Theorem 12:

$$\begin{aligned}d\,F_t(\lambda) &= |A_t(\lambda)|^2\, d\,F(\lambda) \\ &= \left|\sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)}\right|^2 d\,F(\lambda) \\ &= \dot{\theta}(t) |e^{i\lambda(\theta(t)-t)}|^2\, d\,F(\lambda) \\ &= \dot{\theta}(t)\; d\,F(\lambda)\end{aligned} \tag{65}$$

since

$$|e^{i\alpha}| = 1 \forall a \in \mathbb{R} \tag{66}$$ □

3.3 Covariance operator conjugation

Proposition 14. ***[Operator conjugation]*** *Let*

$$(T_K f)(t) := \int_{\mathbb{R}} K(|t-s|)\, f(s)\, d\, s \tag{67}$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h}\, d\, F(\lambda) \tag{68}$$

Define the transformed kernel

$$K_\theta(s,t) := \sqrt{\dot{\theta}(t)\, \dot{\theta}(s)}\;\; K(|\theta(t) - \theta(s)|) \tag{69}$$

then the corresponding integral covariance operator is conjugated $\forall f \in L^2_{\text{loc}}(\mathbb{R})$ *by*

$$\begin{aligned}(T_{K_\theta} f)(t) \;\; &= \int_{\mathbb{R}} K_\theta(s,t)\, f(s)\, d\, s \\ &= (U_\theta\; T_K\; U_\theta^{-1}\; f)(t)\end{aligned} \tag{70}$$

Proof. For any $g \in L^2_{\text{loc}}(\mathbb{R})$, compute:

$$\begin{aligned}((U_\theta T_K U_\theta^{-1})\, g)(t) \;\; &= (U_\theta\, (T_K U_\theta^{-1}\, g))(t) \\ &= \sqrt{\dot{\theta}(t)}\; (T_K U_\theta^{-1}\, g)(\theta(t)) \\ &= \sqrt{\dot{\theta}(t)}\; \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|)\; (U_\theta^{-1}\, g)(\theta(s)) \dot{\theta}(s) d s \\ &= \sqrt{\dot{\theta}(t)}\; \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|) \frac{g(s)}{\sqrt{\dot{\theta}(s)}}\, \dot{\theta}(s)\;\, d\, s \\ &= \sqrt{\dot{\theta}(t)}\; \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|)\; g(s) \sqrt{\dot{\theta}(s)}\;\, d\, s \\ &= \int_{\mathbb{R}} \sqrt{\dot{\theta}(t) \dot{\theta}(s)}\; K(|\theta(t) - \theta(s)|)\; g(s)\;\, d\, s \\ &= \int_{\mathbb{R}} K_\theta(t,s)\; g(s)\;\, d\, s \\ &= (T_{K_\theta}\, g)(t)\end{aligned} \tag{71}$$

$\square$

4 The Ensemble of Sample Path Realizations

Question: is this called local integrability? state this more eloquently

Theorem 15. ***[Sample paths in*** $\boldsymbol{L^2_{\text{loc}}(\mathbb{R})}$***]*** *Let* $\{X(t)\}_{t \in \mathbb{R}}$ *be a second-order stationary process with*

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \tag{72}$$

then, almost surely, every sample path $t \mapsto X(\omega, t) \in L^2_{\text{loc}}(\mathbb{R})$.

Proof. Fix any bounded interval $[a,b]$ and consider the random variable

$$Y_{[a,b]} := \int_a^b X(t)^2 \, dt \tag{73}$$

By stationarity and Fubini's theorem:

$$\begin{aligned}\mathbb{E}[Y_{[a,b]}] = \mathbb{E}\left[\int_a^b X(t)^2 \, dt\right] &= \int_a^b \mathbb{E}[X(t)^2] \, dt \\ &= \int_a^b \sigma^2 \, dt \\ &= \sigma^2 (b-a) < \infty\end{aligned} \tag{74}$$

By Markov's inequality, for any $M > 0$:

$$P\left(Y_{[a,b]} > M\right) \leq \frac{\mathbb{E}[Y_{[a,b]}]}{M} = \frac{\sigma^2 (b-a)}{M} \tag{75}$$

Taking $M \to \infty$, the conclusion is

$$P\left(Y_{[a,b]} < \infty\right) = 1 \tag{76}$$

i.e., almost surely the sample path is square-integrable on $[a,b]$. Since $\mathbb{R}$ is the countable union of bounded intervals:

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n] \tag{77}$$

by countable subadditivity of probability:

$$P\left(\bigcap_{n=1}^{\infty} \left\{\int_{-n}^{n} X(t)^2 \, dt < \infty\right\}\right) = 1 \tag{78}$$

Now let K be any compact set. Then K is bounded, so

$$K \subseteq [-N, N] \tag{79}$$

for some N. Therefore:

$$\int_K X(t)^2 \, dt \leq \int_{-N}^{N} X(t)^2 \, dt < \infty \tag{80}$$

almost surely. This holds for every compact set K, so almost surely every sample path lies in $L^2_{\mathrm{loc}}(\mathbb{R})$. □

5 Zero Localization

The construction

$$\text{stationary } X \xrightarrow{U_\theta} \text{oscillatory } Z \xrightarrow{\mu = \delta(Z)|\dot{Z}|\, dt} L^2(\mu) \xrightarrow{L: t f(t)} (L, \sigma(L)) \tag{81}$$

produces a self-adjoint operator whose eigenvalues equal the zero set of the realization sample path realization $Z(t)$ from the ensemble of possible sample path functions having the given covariance structure and whose spectrum equals the closure of the zero set, determined by the choice of time-change $\theta(t)$, spectral measure $F(\lambda)$, and complex orthogonal random measure $\Phi(\lambda)$ which uniquely corresponds to a given sample path from the ensemble.

5.1 Zero localization measure

Definition 16. ***[Zero localization measure]*** *Let Z be real-valued with $Z \in C^1(\mathbb{R})$ having only simple zeros*

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \tag{82}$$

Define, for Borel $B \subset \mathbb{R}$,

$$\mu(B) = \int_{\mathbb{R}} \boldsymbol{1}_B(t)\ \delta(Z(t))\,|\dot{Z}(t)|\ dt \tag{83}$$

Theorem 17. ***[Atomicity on the zero set]*** *For every $\phi \in C_c^\infty(\mathbb{R})$,*

$$\int_{\mathbb{R}} \phi(t)\ \delta(Z(t))\,|\dot{Z}(t)|\ dt = \sum_{t_0: Z(t_0)=0} \phi(t_0) \tag{84}$$

hence

$$\mu(t) = \sum_{t_0: Z(t_0)=0} \delta_{t_0}(t) \tag{85}$$

Proof. Since all zeros of Z are simple and $Z \in C^1(\mathbb{R})$, by the inverse function theorem each zero t_0 is isolated. Near each zero t_0, Z is locally monotonic, so the one-dimensional change of variables formula for the Dirac delta can be applied. Specifically, near t_0 where $Z(t_0) = 0$ and $\dot{Z}(t_0) \neq 0$, locally

$$Z(t) = (t - t_0)\,\dot{Z}(t_0) + O((t - t_0)^2) \tag{86}$$

holds. The distributional identity for the Dirac delta under smooth changes of variables gives:

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|} \tag{87}$$

Therefore:

$$\begin{aligned} \int_{\mathbb{R}} \phi(t)\ \delta(Z(t))\,|\dot{Z}(t)|\ dt \ &= \int_{-\infty}^{\infty} \phi(t)\,|\dot{Z}(t)| \sum_{t_0: Z(t_0)=0} \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|}\ dt \\ &= \sum_{t_0: Z(t_0)=0} \int_{\mathbb{R}} \phi(t) \frac{|\dot{Z}(t)|\,\delta\,(t - t_0)}{|\dot{Z}(t_0)|}\ dt \\ &= \sum_{t_0: Z(t_0)=0} \frac{|\dot{Z}(t_0)|}{|\dot{Z}(t_0)|}\,\phi(t_0) \\ &= \sum_{t_0: Z(t_0)=0} \phi(t_0) \end{aligned} \tag{88}$$

This shows that μ is the discrete measure

$$\mu(t) = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0}(t) \tag{89}$$

assigning unit mass to each zero. □

5.2 Hilbert space on zeros and multiplication operator

Definition 18. ***[Hilbert space on the zero set]*** *Let $\mathcal{H} = L^2(\mu)$ with inner product*

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} \, \mathrm{d}\mu\,(t) \tag{90}$$

Proposition 19. ***[Atomic structure]*** *Let*

$$\mu = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0} \tag{91}$$

then

$$\mathcal{H} \cong \left\{ f : \{t_0 : Z(t_0) = 0\} \to \mathbb{C} : \sum_{t_0 : Z(t_0) = 0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \tag{92}$$

with orthonormal basis $\{e_{t_0}\}_{t_0 : Z(t_0) = 0}$ where

$$e_{t_0}(t_1) = \delta_{t_0}(t_1) \tag{93}$$

Proof. By the atomic form of μ, for any $f \in L^2(\mu)$:

$$\begin{aligned} \|f\|_{\mathcal{H}}^2 &= \int |f(t)|^2 \, \mathrm{d}\mu\,(t) && (94) \\ &= \int |f(t)|^2 \sum_{t_0 : Z(t_0) = 0} \delta_{t_0}\,(t) && (95) \\ &= \sum_{t_0 : Z(t_0) = 0} |f(t_0)|^2 && (96) \end{aligned}$$

This shows the isomorphism with ℓ^2 where the functions e_{t_0} defined by

$$e_{t_0}(t_1) = \delta_{t_0}(t_1) \tag{97}$$

satisfy the relations

$$\begin{aligned} \langle e_{t_0}, e_{t_1} \rangle &= \int e_{t_0}(t) \overline{e_{t_1}(t)} \mathrm{d}\mu\,(t) \\ &= \sum_{t : Z(t) = 0} \delta_{t_0}(t)\, \delta_{t_1}(t) \\ &= \delta_{t_0}(t_1) \\ &= \delta t_1(t_0) \end{aligned} \tag{98}$$

thus forming an orthonormal set. Thus, any $f(t) \in \mathcal{H}$ can be written as

$$f(t) = \sum_{t_0 : Z(t_0) = 0} f(t_0)\, e_{t_0}(t) \tag{99}$$

proving they form a basis. □

Definition 20. ***[Multiplication operator]*** *Define the linear operator*

$$L\colon \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H} \tag{100}$$

by

$$(L\,f)(t) = t\,f(t) \tag{101}$$

on the support of μ with domain

$$\mathcal{D}(L) := \left\{ f \in \mathcal{H} \colon \int |t\,f(t)|^2\,\mathrm{d}\mu(t) < \infty \right\} \tag{102}$$

Theorem 21. ***[Self-adjointness and spectrum]*** *L is self-adjoint on $\mathcal{H}$ and has pure point, simple spectrum*

$$\sigma(L) = \overline{\{t \in \mathbb{R} \colon Z(t) = 0\}} \tag{103}$$

with eigenvalues $\lambda = t_0$ for each zero t_0 and corresponding eigenvectors e_{t_0}.

Proof. First, self-adjointness is verified. For $f, g \in \mathcal{D}(L)$:

$$\begin{aligned} \langle L\,f, g\rangle &= \int (L\,f)(t)\overline{g(t)}\mathrm{d}\mu(t) \\ &= \int t\,f(t)\overline{g(t)}\,\mathrm{d}\mu(t) \\ &= \int f(t)\overline{t\,g(t)}\,\mathrm{d}\mu(t) \\ &= \int f(t)\overline{(L\,g)(t)}\mathrm{d}\mu(t) \\ &= \langle f, L\,g\rangle \end{aligned} \tag{104}$$

Thus L is symmetric and acts as

$$(L\,f)(t_0) = t_0\,f(t_0) \tag{105}$$

for each t_0 in the atomic representation where

$$Z(t_0) = 0 \tag{106}$$

This is unitarily equivalent to the diagonal operator on ℓ^2 with diagonal entries

$$\{t_0 \colon Z(t_0) = 0\} \tag{107}$$

Such diagonal operators are self-adjoint. For the spectrum calculation:

$$L\,e_{t_0} = t_0\,e_{t_0} \forall \{t_0 \colon Z(t_0) = 0\} \tag{108}$$

holds, so each t_0 is an eigenvalue of L with eigenvector e_{t_0} and since $\{e_{t_0}\}$ forms an orthonormal basis, L has pure point spectrum. The spectrum of a diagonal operator equals the closure of the set of diagonal entries, hence

$$\sigma(L) = \overline{\{t_0 \colon Z(t_0) = 0\}} \tag{109}$$

The eigenvalues are simple. □

5.3 Regularity and Simplicity of Sample Path Zero Crossings

TODO: insert the fundamental theorem on the non-tangency of zero crossings so that it doesnt have to be assumed but is in fact a fundamental theorem of non-degenerate Gaussian processes

Definition 22. ***[Regularity and simplicity]*** *Assume $Z \in C^1(\mathbb{R})$ and every zero is simple:*

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \tag{110}$$

Lemma 23. ***[Local finiteness and delta decomposition]*** *Under Definition 22, zeros are locally finite and*

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0) = 0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \tag{111}$$

whence

$$\mu = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0} \tag{112}$$

Proof. Since $Z \in C^1(\mathbb{R})$ and $\dot{Z}(t_0) \neq 0$ at each zero t_0, the inverse function theorem implies that Z is locally invertible near each zero. Specifically, there exists a neighborhood U_{t_0} of t_0 such that $Z|_{U_{t_0}}$ is strictly monotonic and invertible.

This implies zeros are isolated: if $Z(t_0) = 0$ and $\dot{Z}(t_0) \neq 0$, then there exists $\epsilon > 0$ such that $Z(t) \neq 0$ for $0 < |t - t_0| < \epsilon$. Therefore zeros are locally finite (finitely many in any bounded interval).

For the distributional identity, the one-dimensional change of variables formula for the Dirac delta is considered. If $g: I \rightarrow \mathbb{R}$ is C^1 on interval I with $\dot{g}(x) \neq 0$ for all $x \in I$, then

$$\delta(g(x)) = \sum_{x_0 : g(x_0) = 0} \frac{\delta(x - x_0)}{|\dot{g}(x_0)|} \tag{113}$$

Applying this locally around each zero t_0 of Z, and since zeros are isolated, the local results can be patched together to obtain the global identity:

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0) = 0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \tag{114}$$

Consequently:

$$\begin{aligned} \mathrm{d}\mu(t) &= \delta(Z(t)) |\dot{Z}(t)| \, dt \\ &= \sum_{t_0 : Z(t_0) = 0} \frac{|\dot{Z}(t)|}{|\dot{Z}(t_0)|} \delta(t - t_0) \, dt \\ &= \sum_{t_0 : Z(t_0) = 0} \delta_{t_0}(dt) \end{aligned} \tag{115}$$

where the last equality uses the fact that

$$\frac{|\dot{Z}(t_0)|}{|\dot{Z}(t_0)|} = 1 \tag{116}$$

when evaluating at $t = t_0$. □

5.4 The Kac-Rice Formula For The Expected Zero Counting Function

Theorem 24. (Kac-Rice Formula for Zero Crossings) *Let $Z(t)$ be a centered Gaussian process on $[a, b]$ with covariance $K(s, t) = \mathbb{E}[Z(s)\, Z(t)]$ then the expected number of zeros in $[a, b]$ is*

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{\frac{2}{\pi}}\, \frac{\sqrt{K(t,t)\, K_{\dot{Z}}(t,t) - K_{Z,\dot{Z}}(t,t)^2}}{K(t,t)}\, dt \tag{117}$$

where

$$K(t, t) = \mathbb{E}[Z(t)^2] \tag{118}$$

$$K_{\dot{Z}}(t, t) = -\partial_s^2\, \partial_t\, K(s, t)|_{s=t} \tag{119}$$

and

$$K_{Z,\dot{Z}}(t, t) = \partial_s\, K(s, t)|_{s=t} \tag{120}$$

Proof.

The exact zero counting function is

$$N_{[a,b]} = \int_a^b \delta(Z(t))|\dot{Z}(t)|\ dt \tag{121}$$

so

$$\begin{aligned} \mathbb{E}[N_{[a,b]}] &= \int_a^b \mathbb{E}[\delta(Z(t))|\dot{Z}(t)|]\ dt \\ &= \int_a^b \int_{-\infty}^{\infty} |v|\ p_{Z,\dot{Z}}(0, v)\ dv\ dt \end{aligned} \tag{122}$$

The vector $(Z(t), \dot{Z}(t))$ is bivariate Gaussian with covariance matrix

$$\Sigma = \begin{pmatrix} K(t,t) & K_{Z,\dot{Z}}(t,t) \\ K_{Z,\dot{Z}}(t,t) & K_{\dot{Z}}(t,t) \end{pmatrix} \tag{123}$$

whose determinant is given by

$$\det \Sigma = K(t,t)\, K_{\dot{Z}}(t,t) - K_{Z,\dot{Z}}(t,t)^2 \tag{124}$$

the inverse of which satisfies

$$\Sigma_{22}^{-1} = \frac{K(t,t)}{\det \Sigma} \tag{125}$$

yielding

$$p_{Z,\dot{Z}}(0,v)=\frac{1}{\sqrt{2\,\pi\,K(t,t)}}\cdot\frac{e^{-\frac{K(t,t)\,v^2}{2\det\Sigma}}}{\sqrt{2\,\pi\det\Sigma/K(t,t)}} \tag{126}$$

which factorizes as $p_Z(0)\cdot p_{\dot{Z}|Z}(v|0)$ where

$$p_Z(0)=\frac{1}{\sqrt{2\,\pi\,K(t,t)}} \tag{127}$$

and

$$\dot{Z}|Z=0\sim\mathcal{N}(0,\det\Sigma/K(t,t)) \tag{128}$$

For zero-mean Gaussian $Y\sim\mathcal{N}(0,\sigma^2)$, direct integration gives

$$\begin{aligned}\mathbb{E}[|Y|] &=2\int_0^\infty\frac{y}{\sqrt{2\,\pi\,\sigma^2}}\,e^{-y^2/(2\sigma^2)}\,d\,y\\ &=\frac{2\,\sigma}{\sqrt{2\,\pi}}\int_0^\infty e^{-u}\,d\,u\\ &=\sqrt{\frac{2}{\pi}}\,\sigma\end{aligned} \tag{129}$$

so that combining results yields

$$\begin{aligned}\int_{-\infty}^\infty|v|\;p_{Z,\dot{Z}}(0,v)\;d\,v\; &=\frac{\sqrt{\frac{2}{\pi}}\sqrt{\frac{\det\Sigma}{K(t,t)}}}{\sqrt{2\,\pi\,K(t,t)}}\\ &=\sqrt{\frac{2}{\pi}}\,\frac{\sqrt{\det\Sigma}}{K(t,t)}\end{aligned} \tag{130}$$ □

Theorem 25. ***[Expected Zero-Counting Function]*** *Let $\theta\in\mathcal{F}$ and let*

$$K(t,s)=\operatorname{cov}(Z(t),Z(s)) \tag{131}$$

be twice differentiable at $s=0$ and $t=0$ then expected number of zeros of the process $Z(t)$ in $[a,b]$ is

$$\mathbb{E}[N_{[a,b]}]=\sqrt{-K(0)}\;(\theta(b)-\theta(a)) \tag{132}$$

Proof. The covariance function of the time-changed process is

$$\begin{aligned}K_\theta(s,t)\; &=\operatorname{cov}(Z(t),Z(s))\\ &=\sqrt{\dot{\theta}(s)\dot{\theta}(t)}\,K(|\theta(t)-\theta(s)|)\end{aligned} \tag{133}$$

For the zero-crossing analysis, consider the normalized process. By the Kac-Rice formula:

$$\mathbb{E}[N_{[a,b]}]=\int_a^b\sqrt{-\lim_{s\to t}\frac{\partial^2}{\partial\,s\,\partial\,t}\,K_\theta(s,t)}\;d\,t \tag{134}$$

Computing the mixed partial derivative:

$$\frac{\partial}{\partial t} K_\theta(s,t) = \frac{1}{2}\frac{\dot{\theta}(t)}{\sqrt{\theta(t)}}\sqrt{\dot{\theta}(s)}K(|\theta(t)-\theta(s)|) + \sqrt{\theta(s)}\sqrt{\theta(t)}\dot{K}(|\theta(t)-\theta(s)|)\mathrm{sgn}(\theta(t)-\theta(s))\,\dot{\theta}(t) \tag{135}$$

Taking the limit as $s \to t$ and using the fact that $\dot{K}(0)=0$ for stationary processes:

$$\begin{aligned}\lim_{s\to t}\frac{\partial^2}{\partial s\,\partial t}K_\theta(s,t) &= \lim_{s\to t}\dot{\theta}(s)\,\dot{\theta}(t)\;\ddot{K}(0)\\ &= \dot{\theta}(t)^2\,\ddot{K}(0)\end{aligned} \tag{136}$$

Substituting into the Kac-Rice formula we have

$$\begin{aligned}\mathbb{E}[N_{[a,b]}] &= \int_a^b \sqrt{-\dot{\theta}(t)^2\;\ddot{K}(0)}\;d\,t\\ &= \sqrt{-\ddot{K}(0)}\int_a^b \dot{\theta}(t)\;d\,t\\ &= \sqrt{-\ddot{K}(0)}\;(\theta(b)-\theta(a))\end{aligned} \tag{137}$$

since $\dot{\theta}(t) \geq 0$ almost everywhere. □

Bibliography

[1] Maurice B Priestley. Evolutionary spectra and non-stationary processes. *Journal of the Royal Statistical Society: Series B (Methodological)*, 27(2):204–229, 1965.