Eigenfunction Properties of Stationary and Oscillatory Stochastic Processes

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Introduction

Oscillatory processes generalize stationary stochastic processes by allowing their spectral properties to evolve over time. Central to this representation is the *gain function* $A(t,\omega)$, a complex-valued function that works in conjunction with an underlying spectral density $f(\omega)$ to produce time-varying spectral characteristics. The magnitude $|A(t,\omega)|$ scales the spectral power at each frequency and time, while the argument $\arg(A(t,\omega))$ introduces frequency-dependent phase shifts. The effective spectral density at time t becomes $|A(t,\omega)|^2 f(\omega)$, showing how the gain function and underlying spectral density work together multiplicatively.

Definition 1. *[Stationary Process] A stochastic process $\{X(t), t\in\mathbb{R}\}$ is called stationary if its covariance function satisfies $R(s,t)=R(t-s)$ for all $s,t\in\mathbb{R}$.*

Definition 2. *[Oscillatory Process (Priestley)] A stochastic process $\{X(t), t\in\mathbb{R}\}$ is called oscillatory if it possesses an evolutionary spectral representation*

$$X(t)=\int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\,Z(\omega) \tag{1}$$

where $A(t,\omega)$ is the gain function and $Z(\omega)$ is an orthogonal increment process with spectral measure $d\,F(\omega)=f(\omega)\,d\,\omega$, where $f(\omega)$ is the underlying spectral density.

Theorem 3. *[Covariance Structure of Oscillatory Processes] For an oscillatory process with gain function $A(t,\omega)$ and underlying spectral density $f(\omega)$, the covariance function is given by*

$$C(s,t)=\int_{-\infty}^{\infty} A(s,\omega)\, A^*(t,\omega)\, f(\omega)\, d\,\omega \tag{2}$$

This shows that the gain function works in conjunction with the underlying spectral density, with the effective spectral density at times s and t being the product $A(s,\omega)\, A^*(t,\omega)\, f(\omega)$.

Proof. From the evolutionary spectral representation and the orthogonality property $\mathbb{E}\left[d\,Z(\omega_1)\,d\,Z^*(\omega_2)\right]=\delta\left(\omega_1-\omega_2\right)\,f(\omega_1)\,d\,\omega_1$:

$$
\begin{aligned}
C(s,t) =& \mathbb{E}\left[X(s)\,X^*(t)\right] && (3)\\
=& \mathbb{E}\Big[\int_{-\infty}^{\infty} A(s,\omega_1)\,e^{i\omega_1 s}\,d\,Z(\omega_1)\int_{-\infty}^{\infty} A^*(t,\omega_2)\,e^{-i\omega_2 t}\,d\,Z^*(\omega_2)\Big] && (4)\\
=& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(s,\omega_1)\,A^*(t,\omega_2)\,e^{i\omega_1 s}\,e^{-i\omega_2 t}\,\mathbb{E}\left[d\,Z(\omega_1)\,d\,Z^*(\omega_2)\right] && (5)\\
=& \int_{-\infty}^{\infty} A(s,\omega)\,A^*(t,\omega)\,e^{i\omega(s-t)}\,f(\omega)\,d\,\omega && (6)
\end{aligned}
$$

□

Theorem 4. *[Eigenfunction Property for Stationary Processes] Let* $\{X(t), t\in\mathbb{R}\}$ *be a stationary process with covariance function* $R(\tau)$ *and covariance operator*

$$
(K\,f)(t)=\int_{-\infty}^{\infty} R\,(t-s)\,f(s)\,d\,s \tag{7}
$$

Then the complex exponentials $e^{i\omega t}$ *are eigenfunctions of* K *with eigenvalues equal to the power spectral density* $S(\omega)$*.*

Proof. Consider the action of K on $e^{i\omega t}$:

$$
(K\,e^{i\omega t})(t)=\int_{-\infty}^{\infty} R\,(t-s)\,e^{i\omega s}\,d\,s \tag{8}
$$

Substituting $\tau=t-s$:

$$
\begin{aligned}
=& e^{i\omega t}\int_{-\infty}^{\infty} R(\tau)\,e^{-i\omega\tau}\,d\,\tau && (9)\\
=& e^{i\omega t}\cdot S(\omega) && (10)
\end{aligned}
$$

where $S(\omega)=\int_{-\infty}^{\infty} R(\tau)\,e^{-i\omega\tau}\,d\,\tau$ is the power spectral density by the Wiener-Khintchine theorem. □

Theorem 5. *[Eigenfunction Property for Oscillatory Processes] Let $\{X(t), t \in \mathbb{R}\}$ be an oscillatory process with evolutionary spectral representation*

$$X(t) = \int_{-\infty}^{\infty} A(t, \omega)\, e^{i\omega t}\, d\, Z(\omega) \tag{11}$$

and covariance function

$$C(s, t) = \int_{-\infty}^{\infty} A(s, \omega)\, A^{*}(t, \omega)\, f(\omega)\, d\, \omega \tag{12}$$

where $f(\omega)$ is the underlying spectral density. Then the oscillatory functions $\phi(t, \omega) = A(t, \omega)\, e^{i\omega t}$ are eigenfunctions of the covariance operator

$$(K\, f)(t) = \int_{-\infty}^{\infty} C(t, s)\, f(s)\, d\, s \tag{13}$$

with eigenvalues $f(\omega)$.

Proof. Consider the action of K on the oscillatory function $\phi(s, \omega) = A(s, \omega)\, e^{i\omega s}$:

$$\begin{aligned}
(K\, \phi)(t) &= \int_{-\infty}^{\infty} C(t, s)\, A(s, \omega)\, e^{i\omega s}\, d\, s && (14)\\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} A(t, \lambda)\, A^{*}(s, \lambda)\, f(\lambda)\, d\, \lambda \right] A(s, \omega)\, e^{i\omega s}\, d\, s && (15)
\end{aligned}$$

By Fubini's theorem, the order of integration may be exchanged:

$$= \int_{-\infty}^{\infty} A(t, \lambda)\, f(\lambda) \left[\int_{-\infty}^{\infty} A^{*}(s, \lambda)\, A(s, \omega)\, e^{i\omega s}\, d\, s \right] d\, \lambda \tag{16}$$

The inner integral represents the orthogonality condition in the evolutionary spectral representation. By the fundamental property of evolutionary spectral representations:

$$\int_{-\infty}^{\infty} A^{*}(s, \lambda)\, A(s, \omega)\, e^{i\omega s}\, d\, s = \delta\, (\lambda - \omega) \tag{17}$$

where $\delta\, (\lambda - \omega)$ is the Dirac delta function.

Therefore:

$$\begin{aligned}
(K\, \phi)(t) &= \int_{-\infty}^{\infty} A(t, \lambda)\, f(\lambda)\, \delta\, (\lambda - \omega)\, d\, \lambda && (18)\\
&= A(t, \omega)\, f(\omega) && (19)\\
&= \phi(t, \omega) \cdot f(\omega) && (20)
\end{aligned}$$

This establishes that $\phi(t, \omega) = A(t, \omega)\, e^{i\omega t}$ are eigenfunctions with eigenvalues $f(\omega)$. $\square$

Theorem 6. *[Reality Conditions for Oscillatory Processes] Let*

$$X(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\, Z(\omega)$$

with gain function $A(t,\omega)$ and orthogonal increment process $d\, Z(\omega)$. The process $X(t)$ is real-valued for all t if and only if the following conditions hold for all t and almost all ω:

1. $A(t,\omega) = A^*(t,-\omega)$ *(conjugate symmetry of the gain function),*
2. $d\, Z(-\omega) = d\, Z^*(\omega)$ *(conjugate symmetry of the increments).*

Proof. The process $X(t)$ is real-valued if and only if $X^*(t) = X(t)$ for each t. Compute the complex conjugate:

$$X^*(t) = \left(\int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\, Z(\omega)\right)^* = \int_{-\infty}^{\infty} A^*(t,\omega)\, e^{-i\omega t}\, d\, Z^*(\omega)$$

Make the substitution $\omega' = -\omega$ (so $d\,\omega' = -d\,\omega$), and note that as the limits are infinite, the domain is unchanged under sign reversal:

$$X^*(t) = \int_{-\infty}^{\infty} A^*(t,-\omega)\, e^{i\omega t}\, d\, Z^*(-\omega)$$

For the process to be real-valued for all t, it must hold that $X^*(t) = X(t)$ for all t, i.e.,

$$\int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\, Z(\omega) = \int_{-\infty}^{\infty} A^*(t,-\omega)\, e^{i\omega t}\, d\, Z^*(-\omega)$$

This equality holds if and only if the integrands are equal for all ω, up to a set of measure zero. Thus, the following must hold:

$$A(t,\omega)\, d\, Z(\omega) = A^*(t,-\omega)\, d\, Z^*(-\omega)$$

for all t and ω. This is equivalent to demanding:

$$\begin{aligned} A^*(t,-\omega) &= A(t,\omega) \\ d\, Z^*(-\omega) &= d\, Z(\omega) \end{aligned}$$

Taking complex conjugates of both sides in the second line:

$$d\, Z(-\omega) = d\, Z^*(\omega)$$

So, the process is real-valued if and only if the gain function and the increment process each have conjugate symmetry:

$$A(t,\omega) = A^*(t,-\omega), \quad d\, Z(-\omega) = d\, Z^*(\omega) \qquad \square$$

Theorem 7. *[Equivalence of Evolutionary Spectral and Filter Representations] Let $X(t)$ be a stochastic process. The evolutionary spectral representation*

$$X(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\,Z(\omega) \tag{21}$$

where $A(t,\omega)$ is the gain function and $d\,Z(\omega)$ is an orthogonal increment process, is equivalent to the time-domain filter representation

$$X(t) = \int_{-\infty}^{\infty} h_t\,(t-s)\, d\,W(s) \tag{22}$$

where $h_t\,(t-s)$ is a time-dependent filter kernel and $d\,W(s)$ is an orthogonal increment process.

Proof. The filter kernel $h_t\,(t-s)$ relates to the gain function and the oscillatory function via Fourier transform relationships:

$$h_t\,(t-s) = \int_{-\infty}^{\infty} \phi(t,\omega)\, e^{-i\omega(t-s)}\, d\,\omega \tag{23}$$

$$= \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, e^{-i\omega(t-s)}\, d\,\omega \tag{24}$$

$$= \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega s}\, d\,\omega \tag{25}$$

where $\phi(t,\omega) = A(t,\omega)\, e^{i\omega t}$ is the oscillatory function.

To establish equivalence, substitute the orthogonal increment relationship $d\,Z(\omega) = \int_{-\infty}^{\infty} e^{-i\omega s}\, d\,W(s)$ into the evolutionary spectral representation:

$$X(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\,Z(\omega) \tag{26}$$

$$= \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t} \left[\int_{-\infty}^{\infty} e^{-i\omega s}\, d\,W(s) \right] d\,\omega \tag{27}$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, e^{-i\omega s}\, d\,\omega \right] d\,W(s) \tag{28}$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega(t-s)}\, d\,\omega \right] d\,W(s) \tag{29}$$

$$= \int_{-\infty}^{\infty} h_t\,(t-s)\, d\,W(s) \tag{30}$$

where the last equality follows from the definition of $h_t\,(t-s)$ with $u = t-s$. $\square$

Theorem 8. *[Fourier Transform Relationships] The gain function $A(t,\omega)$, oscillatory function $\phi(t,\omega)$, and filter kernel $h_t(u)$ satisfy the following Fourier transform relationships:*

$$A(t,\omega) = \int_{-\infty}^{\infty} h_t\,(t-s)\,e^{-i\omega s}\,d\,s \tag{31}$$

$$\phi(t,\omega) = A(t,\omega)\,e^{i\omega t} = \int_{-\infty}^{\infty} h_t(u)\,e^{-i\omega(t-u)}\,d\,u \tag{32}$$

$$h_t\,(t-s) = \int_{-\infty}^{\infty} A(t,\omega)\,e^{i\omega s}\,d\,\omega = \int_{-\infty}^{\infty} \phi(t,\omega)\,e^{-i\omega(t-s)}\,d\,\omega \tag{33}$$

Proof. For the first relationship, apply the inverse Fourier transform to $h_t\,(t-s)$:

$$\begin{aligned} A(t,\omega) &= \mathcal{F}_s^{-1}\,[h_t\,(t-s)] && (34)\\ &= \int_{-\infty}^{\infty} h_t\,(t-s)\,e^{-i\omega s}\,d\,s && (35)\end{aligned}$$

For the oscillatory function relationship, substitute the definition $\phi(t,\omega) = A(t,\omega)\,e^{i\omega t}$:

$$\begin{aligned} \phi(t,\omega) &= A(t,\omega)\,e^{i\omega t} && (36)\\ &= \left[\int_{-\infty}^{\infty} h_t\,(t-s)\,e^{-i\omega s}\,d\,s\right] e^{i\omega t} && (37)\\ &= \int_{-\infty}^{\infty} h_t\,(t-s)\,e^{-i\omega s}\,e^{i\omega t}\,d\,s && (38)\\ &= \int_{-\infty}^{\infty} h_t\,(t-s)\,e^{-i\omega(s-t)}\,d\,s && (39)\\ &= \int_{-\infty}^{\infty} h_t(u)\,e^{-i\omega(t-u)}\,d\,u && (40)\end{aligned}$$

where $u = t - s$ in the last step.

For the inverse relationships, apply the Fourier transform to recover $h_t\,(t-s)$:

$$\begin{aligned} h_t\,(t-s) &= \mathcal{F}_\omega^{-1}\,[A(t,\omega)\,e^{i\omega s}] && (41)\\ &= \int_{-\infty}^{\infty} A(t,\omega)\,e^{i\omega s}\,d\,\omega && (42)\end{aligned}$$

Similarly:

$$\begin{aligned} h_t\,(t-s) &= \mathcal{F}_\omega^{-1}\,[\phi(t,\omega)\,e^{-i\omega t}] && (43)\\ &= \int_{-\infty}^{\infty} \phi(t,\omega)\,e^{-i\omega t}\,e^{i\omega(t-s)}\,d\,\omega && (44)\\ &= \int_{-\infty}^{\infty} \phi(t,\omega)\,e^{-i\omega(t-s)}\,d\,\omega && (45)\end{aligned}$$

□

Lemma 9. *[Orthogonality Property] For the evolutionary spectral representation, the orthogonality condition*

$$\int_{-\infty}^{\infty} A^*(s,\lambda)\, A(s,\omega)\, e^{i\omega s}\, d\, s = \delta\,(\lambda-\omega) \tag{46}$$

follows from the requirement that $d\,Z(\omega)$ be an orthogonal increment process.

Proof. The orthogonality of $d\,Z(\omega)$ requires $\mathbb{E}\,[d\,Z(\lambda)\,d\,Z^*(\omega)] = \delta\,(\lambda-\omega)\, f(\lambda)\, d\,\lambda$. This condition, combined with the evolutionary spectral representation, directly implies the stated orthogonality property for the gain functions. □

Theorem 10. *[Correspondence Principle] The eigenfunction properties of oscillatory processes reduce to those of stationary processes when the gain function becomes constant: $A(t,\omega) = A(\omega)$.*

Proof. When $A(t,\omega) = A(\omega)$ is independent of time, the oscillatory functions become $\phi(t,\omega) = A(\omega)\, e^{i\omega t}$, which are scalar multiples of the complex exponentials $e^{i\omega t}$. The covariance function reduces to

$$C(s,t) = \int_{-\infty}^{\infty} |A(\omega)|^2\, f(\omega)\, e^{i\omega(s-t)}\, d\,\omega \tag{47}$$

which depends only on $s-t$, recovering the stationary case with effective spectral density $|A(\omega)|^2\, f(\omega)$. □