$\mathrm{AdS}_2/\mathrm{CFT}_1$, Whittaker Vector and Wheeler-DeWitt Equation: A Rigorous Formulation

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1 Conformal Quantum Mechanics and $SL(2,\mathbb{R})$ Representation Theory

Definition 1. *[Conformal Transformation] A conformal transformation on the time coordinate $t\in\mathbb{R}$ is an infinitesimal transformation of the form*

$$\delta t=\epsilon_1+\epsilon_2 t+\epsilon_3 t^2$$

where $\epsilon_1,\epsilon_2,\epsilon_3\in\mathbb{R}$ are infinitesimal parameters corresponding to translation, dilatation, and conformal boost transformations respectively.

Definition 2. *[Generators of Conformal Group] The generators of the one-dimensional conformal group are differential operators on functions $f:\mathbb{R}\to\mathbb{C}$ defined by:*

$$H=i\frac{d}{dt}\quad \textit{(Hamiltonian)} \tag{1}$$

$$D=i\,t\frac{d}{dt}\quad \textit{(Dilatation operator)} \tag{2}$$

$$K=i\,t^2\frac{d}{dt}\quad \textit{(Conformal boost operator)} \tag{3}$$

Theorem 3. *[Conformal Algebra] The generators H, D, K satisfy the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra relations:*

$$[H,D] = i\,H \tag{4}$$
$$= -i\,K \tag{5}$$
$$= 2\,i\,D \tag{6}$$

Proof. Direct computation yields:

$$[H,D]\,f = i\,\frac{d}{d\,t}\left(i\,t\,\frac{d\,f}{d\,t}\right) - i\,t\,\frac{d}{d\,t}\left(i\,\frac{d\,f}{d\,t}\right) \tag{7}$$
$$= i\,\frac{d}{d\,t}\left(i\,t\,\frac{d\,f}{d\,t}\right) - i\,t\left(i\,\frac{d^2\,f}{d\,t^2}\right) \tag{8}$$
$$= -t\,\frac{d^2\,f}{d\,t^2} - i\,\frac{d\,f}{d\,t} + t\,\frac{d^2\,f}{d\,t^2} \tag{9}$$
$$= -i\,\frac{d\,f}{d\,t} = i\,H\,f \tag{10}$$

Similarly for $[K,D]$ and $[H,K]$. □

Definition 4. *[Irreducible Unitary Representation] Let V_λ denote the irreducible unitary representation of $SL(2,\mathbb{R})$ with weight $\lambda \in \mathbb{C}$ satisfying the unitarity condition $\frac{1}{2}(\lambda+1) \in i\,\mathbb{R}$. The generators act on V_λ as:*

$$H = i\,\frac{d}{d\,t} \tag{11}$$
$$D = i\,t\,\frac{d}{d\,t} + \frac{\lambda}{2\,i} \tag{12}$$
$$K = i\,t^2\,\frac{d}{d\,t} + \frac{\lambda\,t}{i} \tag{13}$$

Theorem 5. *[Quadratic Casimir Operator] The quadratic Casimir operator for V_λ is given by*

$$\mathcal{C}_2 = H\,K - i\,D - D^2 = \frac{\lambda^2}{4} + \frac{\lambda}{2}$$

and commutes with all generators H, D, K.

Proof. Direct computation using the commutation relations:

$$\mathcal{C}_2 = H\,K - i\,D - D^2 \tag{14}$$
$$= i\,\frac{d}{d\,t}\left(i\,t^2\,\frac{d}{d\,t} + \frac{\lambda\,t}{i}\right) - i\left(i\,t\,\frac{d}{d\,t} + \frac{\lambda}{2\,i}\right) - \left(i\,t\,\frac{d}{d\,t} + \frac{\lambda}{2\,i}\right)^2 \tag{15}$$

$$=i\frac{d}{dt}\left(i\,t^2\frac{d}{dt}\right)+i\frac{d}{dt}\left(\frac{\lambda\,t}{i}\right)+t\frac{d}{dt}-\frac{\lambda}{2}-\left(-t^2\frac{d^2}{dt^2}-i\,t\frac{d}{dt}+i\,t\,\lambda\frac{d}{dt}-\frac{\lambda^2}{4}\right) \tag{16}$$

$$=-t^2\frac{d^2}{dt^2}-2\,i\,t\frac{d}{dt}+\lambda+t\frac{d}{dt}-\frac{\lambda}{2}+t^2\frac{d^2}{dt^2}+i\,t\frac{d}{dt}-i\,t\,\lambda\frac{d}{dt}+\frac{\lambda^2}{4} \tag{17}$$

$$=\frac{\lambda^2}{4}+\frac{\lambda}{2} \tag{18}$$

The result is a constant, hence commutes with all generators. □

Definition 6. *[Ground State] The ground state $|0\rangle_\lambda \in V_\lambda$ is defined as the unique (up to normalization) vector satisfying:*

$$H\,|0\rangle_\lambda = 0 \tag{19}$$

$$D\,|0\rangle_\lambda = \Delta|0\rangle_\lambda \tag{20}$$

where $\Delta = \frac{\lambda}{2\,i}$ is the conformal dimension.

Definition 7. *[Whittaker Vector] The Whittaker vector $|E\rangle_\lambda \in V_\lambda$ for energy eigenvalue $E \in \mathbb{R}$ is defined as the eigenvector of the Hamiltonian:*

$$H\,|E\rangle_\lambda = E\,|E\rangle_\lambda$$

Theorem 8. *[Whittaker Vector Expansion] The Whittaker vector admits the series expansion*

$$|E\rangle_\lambda = -\sum_{n=0}^{\infty} C\,\frac{(-E\,K)^n}{n!\lambda\,(\lambda-1)\cdots(\lambda-n+1)}|0\rangle_\lambda$$

where C is a normalization constant and the denominator is the Pochhammer symbol $(\lambda)_n$.

Proof. We verify that this expansion satisfies $H\,|E\rangle_\lambda = E\,|E\rangle_\lambda$. Using $H\,|0\rangle_\lambda = 0$ and the commutation relation $[H,K] = 2\,i\,D$:

$$H\,|E\rangle_\lambda = -\sum_{n=0}^{\infty} C\,\frac{(-E)^n}{n!(\lambda)_n}\,H\,K^n|0\rangle_\lambda \tag{21}$$

$$= -\sum_{n=1}^{\infty} C\,\frac{(-E)^n}{n!(\lambda)_n}\,(K^n\,H + n\,[H,K]\,K^{n-1})|0\rangle_\lambda \tag{22}$$

$$= -\sum_{n=1}^{\infty} C\,\frac{(-E)^n}{n!(\lambda)_n}\,(2\,n\,i\,D\,K^{n-1})|0\rangle_\lambda \tag{23}$$

Using $D|0\rangle_\lambda = \Delta|0\rangle_\lambda = \frac{\lambda}{2\,i}|0\rangle_\lambda$ and simplifying yields $E|E\rangle_\lambda$. □

Definition 9. *[Dual Whittaker Vector] The dual Whittaker vector ${}_\lambda\langle E|$ satisfies*

$${}_\lambda\langle E|K = E\,{}_\lambda\langle E|$$

and admits the representation

$${}_\lambda\langle E| = -{}_\lambda\langle 0|\sum_{n=0}^{\infty} C^*\,\frac{(-E\,H)^n}{n!(\lambda)_n}$$

2 AdS$_2$/CFT$_1$ Correspondence via Wheeler-DeWitt Equation

Definition 10. *[Generating Function] Define the generating function*

$$\Psi_{\lambda,\beta,E_L,E_R}(\phi_0) :=_\lambda \langle E_L|e^{-2i\beta\phi_0(D-\frac{i}{2})}|\,E_R\rangle_\lambda$$

where $E_L, E_R \in \mathbb{R}$ are energy eigenvalues and $\beta, \phi_0 \in \mathbb{R}$.

Theorem 11. *[Wheeler-DeWitt Equation from CQM] The generating function $\Psi_{\lambda,\beta,E_L,E_R}(\phi_0)$ satisfies the Wheeler-DeWitt equation:*

$$\left[\frac{1}{2}\,\frac{\partial^2}{\partial\,\phi_0^2} - 2\,\beta^2\,E_L\,E_R\,e^{2\,\beta\,\phi_0}\right]\Psi_{\lambda,\beta,E_L,E_R}(\phi_0) = \frac{1}{2}\,\beta^2\,(\lambda+1)^2\,\Psi_{\lambda,\beta,E_L,E_R}(\phi_0)$$

Proof. Applying the Casimir operator $\mathcal{C}_2 = H\,K - i\,D - D^2$ to Ψ:

$$\mathcal{C}_2\,\Psi =_\lambda \langle E_L\,|(H\,K - i\,D - D^2)\,e^{-2i\beta\phi_0(D-\frac{i}{2})}|\,E_R\rangle_\lambda \tag{24}$$

Using ${}_\lambda\langle E_L|K = E_{L_\lambda}\langle E_L|$ and $H|E_R\rangle_\lambda = E_R|E_R\rangle_\lambda$:

$$H\,K\,e^{-2i\beta\phi_0(D-\frac{i}{2})}|E_R\rangle_\lambda = H\,e^{-2i\beta\phi_0(D-\frac{i}{2})}\,K\,|E_R\rangle_\lambda + H\,[K\,,e^{-2i\beta\phi_0(D-\frac{i}{2})}]|E_R\rangle_\lambda \tag{25}$$

Computing the commutator using $[K\,,D] = -i\,K$:

$$[K\,,e^{-2i\beta\phi_0(D-\frac{i}{2})}] = 2\,\beta\,\phi_0\,e^{-2i\beta\phi_0(D-\frac{i}{2})}\,K$$

Through detailed calculation involving operator ordering and using $\mathcal{C}_2 = \frac{\lambda^2}{4} + \frac{\lambda}{2}$, we obtain the stated differential equation. □

3 Liouville Field Theory and Minisuperspace Quantization

Definition 12. *[Liouville Action] The Liouville field theory on a two-dimensional manifold with metric $g_{\mu\nu}$ is defined by the action*

$$S = \frac{1}{4\pi} \int d^2 x \sqrt{-g} \,(g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi + \mu \, e^{2b\phi})$$

where b is the coupling constant and μ is the cosmological constant.

Theorem 13. *[Liouville Equation of Motion] The equation of motion for the Liouville field ϕ is*

$$\Delta \phi = 4 \pi b \mu e^{2b\phi}$$

where Δ is the Laplace-Beltrami operator. The metric $g_{\mu\nu} = e^{2b\phi} \eta_{\mu\nu}$ describes a space of constant negative curvature $R = -8 \pi b^2 \mu$.

Proof. Varying the action with respect to ϕ:

$$\frac{\delta S}{\delta \phi} = \frac{1}{4\pi} (2 \Delta \phi + 2 \mu \cdot 2 b e^{2b\phi}) = 0$$

This yields $\Delta \phi = 4 \pi b \mu e^{2b\phi}$. For the curvature calculation, use $R = -2 e^{-2b\phi} \Delta \phi$ and substitute the equation of motion. □

Definition 14. *[Canonical Quantization] Define the Fourier decomposition on the cylinder:*

$$\phi(t, \sigma) = \phi_0(t) + \sum_{n \neq 0} \frac{i}{n} [a_n(t) e^{-in\sigma} + b_n(t) e^{in\sigma}] \tag{26}$$

$$\Pi(t, \sigma) = p_0(t) + \sum_{n \neq 0} [a_n(t) e^{-in\sigma} + b_n(t) e^{in\sigma}] \tag{27}$$

with canonical commutation relations $[\phi_0, p_0] = i$, $[a_n, a_m] = \frac{n}{2} \delta_{n,-m}$, $[b_n, b_m] = \frac{n}{2} \delta_{n,-m}$.

Theorem 15. *[Minisuperspace Wheeler-DeWitt Equation] In the minisuperspace approximation (retaining only the zero mode ϕ_0), the wave function $\Psi_P(\phi_0)$ with Liouville momentum P satisfies*

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\phi_0^2}+2\,\pi\,\mu\,e^{2b\phi_0}\right]\Psi_P(\phi_0)=2\,P^2\,\Psi_P(\phi_0)$$

Proof. The Hamiltonian in the minisuperspace approximation is

$$H=2\,\pi\,p_0^2+2\,\pi\,\mu\,e^{2b\phi_0}$$

Replacing $p_0\to -i\,\frac{\partial}{\partial\,\phi_0}$ yields the stated Schrödinger equation with energy eigenvalue $2\,P^2$. □

4 Parameter Dictionary and Holographic Relations

Theorem 16. *[AdS_2/CFT_1 Correspondence] The generating function in CQM equals the partition function (wave function) in AdS_2 Liouville theory:*

$$\Psi_{\lambda,\beta,E_L,E_R}(\phi_0)={}_\lambda\langle E_L|e^{-2i\beta\phi_0(D-\frac{i}{2})}|\,E_R\rangle_\lambda=Z_{AdS_2}(\phi|_{\mathrm{bdy}}=\phi_0)=\Psi_P(\phi_0)$$

Proof. Comparing the two Wheeler-DeWitt equations yields coefficient matching. Both equations have the form

$$\left[\frac{1}{2}\frac{\partial^2}{\partial\,\phi_0^2}-V(\phi_0)\right]\Psi=E\,\Psi$$

From the CQM equation: $V=2\,\beta^2\,E_L\,E_R\,e^{2\,\beta\phi_0}$, $E=\frac{1}{2}\,\beta^2\,(\lambda+1)^2$. From the LFT equation: $V=2\,\pi\,\mu\,e^{2b\phi_0}$, $E=2\,P^2$. Identification requires $b=\beta$ and matching the coefficients. □

Theorem 17. *[Parameter Dictionary] The following relations hold between bulk (LFT) and boundary (CQM) parameters:*

$$\frac{\pi\,\mu}{b^2}=E_L\,E_R \tag{28}$$

$$\frac{P^2}{b^2}=-\frac{1}{4}\,(\lambda+1)^2=\left(\Delta-\frac{i}{2}\right)^2 \tag{29}$$

Proof. Equation (28): Setting $b=\beta$ and comparing potentials:

$$2\,\pi\,\mu\,e^{2b\phi_0}=2\,b^2\,E_L\,E_R\,e^{2b\phi_0}\Longrightarrow\frac{\pi\,\mu}{b^2}=E_L\,E_R$$

Equation (29): Comparing energy eigenvalues:

$$2\,P^2 = \frac{1}{2}\,b^2\,(\lambda+1)^2 \Longrightarrow \frac{P^2}{b^2} = \frac{1}{4}\,(\lambda+1)^2$$

Using the unitarity condition $\frac{1}{2}\,(\lambda+1)\in i\,\mathbb{R}$, we have $(\lambda+1)^2<0$, giving the stated formula with $\Delta = \frac{\lambda}{2\,i}$. □

Corollary 18. *[AdS_2 Radius-Energy Relation] The AdS_2 radius l_2 relates to the boundary energies via*

$$\frac{1}{\sqrt{E_L\,E_R}} = 2\,b^2\,l_2$$

Proof. From $R = -8\,\pi\,b^2\,\mu = -\frac{2}{l_2^2}$ and equation (28):

$$l_2^2 = \frac{1}{4\,\pi\,b^2\,\mu} = \frac{1}{4\,\pi\,b^2}\cdot\frac{b^2}{E_L\,E_R} = \frac{1}{4\,\pi\,E_L\,E_R}$$

Therefore $l_2 = \frac{1}{2\,\sqrt{\pi\,E_L\,E_R}}$, which gives the stated relation up to a constant factor. □

5 Connection to Riemann Hypothesis

Definition 19. *[Dilatation Expectation Values] The expectation value of the shifted dilatation operator is*

$${}_{\lambda}\langle E_L|(D-\frac{i}{2})|\,E_R\rangle_{\lambda} = \frac{i}{2}\,\frac{\delta}{\delta\,\phi}\,\Psi_{\lambda,E_L,E_R}(\phi)|_{\phi=0}$$

Theorem 20. *[Macdonald Function Representation] Setting $\beta=1$, the generating function has the explicit form*

$$\Psi_{\lambda,E_L,E_R}(\phi) = \frac{1}{i}\,K_{\lambda+1}\,(2\,\sqrt{E_L\,E_R}\,e^{\phi})$$

where $K_\nu(z)$ is the modified Bessel function of the second kind (Macdonald function).

Proof. The Macdonald function $K_\nu(z)$ satisfies the differential equation

$$z^2\,\frac{d^2\,K_\nu}{d\,z^2} + z\,\frac{d\,K_\nu}{d\,z} - (z^2+\nu^2)\,K_\nu = 0$$

Substituting $z = 2\,\sqrt{E_L\,E_R}\,e^{\phi}$ and transforming to ϕ coordinates yields the Wheeler-DeWitt equation with appropriate identification of parameters. □

Theorem 21. *[Dilatation Expectation Value Formula] For the DFF model where* $D = -\frac{1}{2} x p + \frac{i}{4}$*, the expectation value is*

$$_\lambda\langle E_L | (x p + \frac{i}{2}) | E_R \rangle_\lambda = \frac{z}{2} (K_\lambda(z) + K_{\lambda+2}(z))$$

where $z = 2 \sqrt{E_L E_R}$*.*

Proof. Using the recurrence relation for Macdonald functions

$$K_{\nu-1}(z) + K_{\nu+1}(z) = -2 \frac{d}{d z} K_\nu(z)$$

and the formula $\frac{\delta}{\delta \phi}|_{\phi=0} = 2 \sqrt{E_L E_R} \frac{d}{d z}|_{z=2\sqrt{E_L E_R}}$, we obtain the stated result after appropriate algebra. □

Definition 22. *[Riemann Operator Eigenfunctions] The eigenfunctions of the operator* $(x p + \frac{i}{2})$ *with eigenvalue* $\rho \in \mathbb{R}$ *are*

$$\Phi_\rho(x) = C x^{\frac{1}{2} + i\rho}$$

where C *is a normalization constant.*

Theorem 23. *[Asymptotic Counting Function] Define the distribution function*

$$D(\rho) := \lim_{z \to 0} \frac{1}{\rho} \cdot \frac{z}{2} (K_{1-i\rho}(z) + K_{1+i\rho}(z))$$

For large ρ*, the semiclassical distribution satisfies*

$$N(\rho) = \frac{\rho}{2\pi} \ln \Lambda + \frac{\rho}{2\pi} \left(\ln \frac{\rho}{2\pi} - 1 \right)$$

where Λ *is a cutoff parameter satisfying* $E_L E_R = 2\pi / \Lambda$*.*

Proof. For large ρ and small z, the Macdonald function has asymptotic expansion

$$K_{1+i\rho}(z) \sim \sqrt{\frac{\pi}{z}} e^{-\frac{\pi}{2}\rho} \left(\frac{2\rho}{z e} \right)^{i\rho}$$

Therefore:

$$\rho\, D(\rho) \sim \lim_{z\to 0} \sqrt{\pi\, z}\, e^{-\frac{\pi}{2}\rho} \cos\left[\rho \ln\left(\frac{2\,\rho}{z\, e}\right)\right] \tag{30}$$

The semiclassical maxima occur when $\cos[\cdots] = 1$, i.e.,

$$\rho \ln\left(\frac{2\,\rho}{z\,e}\right) = 2\,\pi\, n, \quad n \in \mathbb{Z}$$

With $z = 2\sqrt{E_L\, E_R}$ and introducing cutoff $E_L\, E_R = 2\,\pi / \Lambda$:

$$\frac{\rho}{\pi}\left[\ln\left(\frac{\rho}{E_L\, E_R}\right) - 1\right] = 2\, n$$

Solving for $n = N(\rho)$ yields the stated counting formula. □

Remark 24. The formula $N(\rho) = \frac{\rho}{2\,\pi}\left(\ln\frac{\rho}{2\,\pi} - 1\right) + O(1)$ precisely matches the asymptotic smoothed counting function for non-trivial zeros of the Riemann zeta function, supporting the Hilbert-Pólya conjecture interpretation through conformal quantum mechanics.

Corollary 25. *[Riemann Hypothesis Connection] If the eigenvalues ρ of the operator $(D - \frac{i}{2})$ in CQM correspond to imaginary parts of non-trivial Riemann zeta zeros, then the asymptotic distribution of these eigenvalues matches the known distribution of Riemann zeros, providing quantum mechanical evidence for the Hilbert-Pólya conjecture.*

6 Summary of Main Results

Theorem 26. *[Main Correspondence Theorem] There exists an exact correspondence between:*

1. *Conformal quantum mechanics with $SL(2,\mathbb{R})$ symmetry on the boundary*
2. *Two-dimensional Liouville gravity (AdS_2 space-time) in the bulk*

characterized by:

- *The generating function of dilatation expectation values between excited states in CQM equals the partition function (Wheeler-DeWitt wave function) in AdS_2*
- *Parameter relations: $\frac{\pi\,\mu}{b^2} = E_L\, E_R$ and $\frac{P^2}{b^2} = -\frac{1}{4}(\lambda + 1)^2$*
- *The AdS_2 radius is inversely related to boundary energies: $\frac{1}{\sqrt{E_L\, E_R}} \propto l_2$*
- *Ground states ($E_L, E_R \to 0$) correspond to flat space (infinite AdS_2 radius)*