Metric Entropy and Compactness Properties of Gaussian Processes

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For a Gaussian process $\{X(t): t \in T\}$, the **canonical metric** d measures the square root of the variance of the length of the interval spanning any given pair of points and it is defined as:

$$d(s,t) := \sqrt{\mathbb{E}[(X(s) - X(t))^2]} \tag{1}$$

Definition 1

*The **spectral radius** R of the covariance operator K associated with a Gaussian process is defined as the square of the largest eigenvalue λ_1 of K:*

$$R := \lambda_1^2 \tag{2}$$

The spectral radius indicates the maximum variance contributed by the process in the direction of the first eigenfunction.

Definition 2

*The **covering number** $N(T,d,\varepsilon)$ is the minimum number of points needed to cover the space T within distance ε using the canonical metric d. An upper bound for the covering number is given by:*

$$N(T,d,\varepsilon) \leq \min\{n \in \mathbb{N}: \lambda_{n+1}^2 \leq \varepsilon\} = \sum_{k=1}^{\infty} \theta(\lambda_k^2 - \varepsilon) \tag{3}$$

where $\{\lambda_k\}$ are the eigenvalues of the covariance operator, ordered in decreasing order and θ is the Heaviside step function.

Definition 3

*The **metric entropy** is the logarithm of the covering number:*

$$\log N(T,d,\varepsilon) \tag{4}$$

which measures the complexity of the set T in the canonical metric d at scale $\varepsilon > 0$.

Definition 4

The ***metric entropy integral*** *is defined as:*

$$\int_0^R \log N(T, d, \varepsilon)\ d\varepsilon \tag{5}$$

where $R := \lambda_1^2$ is the spectral radius. This integral quantifies the total complexity of covering the metric space (T, d) as ε varies from R to 0.

Theorem

Theorem 5

Let $\{X(t): t \in T\}$ be a Gaussian process with covariance operator K having eigenvalues $\{\lambda_k\}$. If the eigenvalues satisfy $\lambda_k \to 0$ as $k \to \infty$, then the metric entropy integral

$$\int_0^R \log N(T, d, \varepsilon)\ d\varepsilon < \infty \tag{6}$$

is finite, indicating that the space is relatively compact in the canonical metric d.

Proof

Given $\lambda_k \to 0$, for any $\varepsilon > 0$, there exists a finite set of indices such that $\lambda_k^2 > \varepsilon$. Thus, the covering number $N(T, d, \varepsilon)$ is finite for any $\varepsilon > 0$.

The metric entropy integral is:

$$\begin{aligned}
\int_0^R \log N(T, d, \varepsilon)\ d\varepsilon &= \sum_{k=1}^{N(T,d,R)} \log(k)\,(\lambda_k^2 - \lambda_{k+1}^2) \\
&= \sum_{\lambda_{k+1}^2 \leq \varepsilon} \log(k)\,(\lambda_k^2 - \lambda_{k+1}^2) \\
&= \sum_{\lambda_{k+1} \leq \sqrt{\varepsilon}} \log(k)\,(\lambda_k^2 - \lambda_{k+1}^2)
\end{aligned} \tag{7}$$

Both $\log k$ and $(\lambda_k^2 - \lambda_{k+1}^2)$ are finite, thus the sum is finite, ensuring that the metric entropy integral is finite, implying relative compactness.