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\usepackage{amsmath,amssymb,amsfonts,mathtools}

The Eigenfunctions of Stationary and Oscillatory Processes on the Real Line

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\begin{document}
\maketitle

Proof. Compute

$$(Ke^{i\omega\cdot})(t) = \int_{-\infty}^{\infty} R(t-s)e^{i\omega s}ds.$$

Set $\tau = t - s$. Then $s = t - \tau$ and $ds = -d\tau$. The lower limit $\tau = \infty$ when $s = -\infty$ and upper limit $\tau = -\infty$ when $s = \infty$, so

$$\int_{-\infty}^{\infty} R(t-s)e^{i\omega s}ds = \int_{\infty}^{-\infty} R(\tau)e^{i\omega(t-\tau)}(-d\tau) = \int_{-\infty}^{\infty} R(\tau)e^{i\omega(t-\tau)}d\tau.$$

Factor

$$\int_{-\infty}^{\infty} R(\tau)e^{i\omega(t-\tau)}d\tau = e^{i\omega t}\int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau = e^{i\omega t}S(\omega).$$

□

Proof. Compute

$$(K\phi(\cdot,\omega))(t) = \int_{-\infty}^{\infty} C(t,s)A(s,\omega)e^{i\omega s}ds.$$

Substitute $C(t,s)$:

$$\int_{-\infty}^{\infty} C(t,s)A(s,\omega)e^{i\omega s}ds = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} A(t,\lambda)A^*(s,\lambda)dF(\lambda)\right)A(s,\omega)e^{i\omega s}ds.$$

Exchange order of integration:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(t,\lambda)A^*(s,\lambda)A(s,\omega)e^{i\omega s}dF(\lambda)ds = \int_{-\infty}^{\infty} A(t,\lambda)\left(\int_{-\infty}^{\infty} A^*(s,\lambda)A(s,\omega)e^{i\omega s}d\right.$$

The inner integral equals $\delta(\lambda-\omega)$:

$$\int_{-\infty}^{\infty} A(t,\lambda)\delta(\lambda-\omega)dF(\lambda) = A(t,\omega)dF(\omega) = A(t,\omega)e^{i\omega t}dF(\omega) = \phi(t,\omega)dF(\omega).$$

□

Proof. The condition is

$$\mathbb{E}[dZ(\lambda)dZ^*(\omega)] = \delta(\lambda-\omega)dF(\lambda).$$

The representation

$$X(t) = \int_{-\infty}^{\infty} A(t,\omega)e^{i\omega t}dZ(\omega)$$

preserves the orthogonality, so the integral equals $\delta(\lambda-\omega)$. □

Proof. Compute

$$X^*(t) = \int_{-\infty}^{\infty} A^*(t,\omega)e^{-i\omega t}dZ^*(\omega).$$

Set $\omega = -\nu$:

$$d\omega = -d\nu,$$

so

$$\int_{-\infty}^{\infty} A^*(t,-\nu)e^{i\nu t}dZ^*(-\nu)(-d\nu) = \int_{-\infty}^{\infty} A^*(t,-\omega)e^{i\omega t}dZ^*(-\omega).$$

Set $X^*(t) = X(t)$:

$$\int_{-\infty}^{\infty} A(t,\omega)e^{i\omega t}dZ(\omega) = \int_{-\infty}^{\infty} A^*(t,-\omega)e^{i\omega t}dZ^*(-\omega).$$

The integrals are equal, so

$$A(t,\omega) = A^*(t,-\omega),$$

$$dZ(\omega) = dZ^*(-\omega).$$

Reverse:

$$\int_{-\infty}^{\infty} A^*(t,\omega)e^{-i\omega t}dZ^*(\omega) = \int_{-\infty}^{\infty} A(t,-\omega)e^{-i\omega t}dZ(-\omega).$$

Set $\omega = -\nu$:

$$\int_{-\infty}^{\infty} A(t,\nu)e^{i\nu t}dZ(\nu) = X(t).$$

□

Proof. Compute

$$\phi^*(t,\omega) = [A(t,\omega)e^{i\omega t}]^* = A^*(t,\omega)e^{-i\omega t}.$$

Substitute $A(t,\omega) = A^*(t,-\omega)$:

$$A^*(t,\omega)e^{-i\omega t} = A(t,-\omega)e^{-i\omega t}.$$

Rewrite

$$A(t,-\omega)e^{-i\omega t} = A(t,-\omega)e^{i(-\omega)t} = \phi(t,-\omega).$$

□

Proof. Compute

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t,\omega)e^{-i\omega u}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\omega)e^{i\omega t}e^{-i\omega u}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\omega)e^{i\omega(t-u)}d\omega.$$

□

Proof. Compute

$$\int_{-\infty}^{\infty} h(t,u)e^{-i\omega(t-u)}du = \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\lambda)e^{i\lambda(t-u)}d\lambda\right)e^{-i\omega(t-u)}du.$$

Exchange

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\lambda)\int_{-\infty}^{\infty} e^{i\lambda(t-u)}e^{-i\omega(t-u)}dud\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\lambda)e^{i\lambda t}e^{-i\omega t}\int_{-\infty}^{\infty} e^{-i(\lambda-\omega)u}dud$$

The inner integral is $2\pi\delta(\lambda-\omega)$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\lambda)e^{i\lambda t}e^{-i\omega t}2\pi\delta(\lambda-\omega)d\lambda = \int_{-\infty}^{\infty} A(t,\lambda)e^{i\lambda t}e^{-i\omega t}\delta(\lambda-\omega)d\lambda = A(t,\omega)e^{i\omega t}\epsilon$$

The second follows similarly. □

Proof. Compute

$$\int_{-\infty}^{\infty} h(t,u)X(u)du = \int_{-\infty}^{\infty} h(t,u)\left(\int_{-\infty}^{\infty} e^{i\omega u}dZ(\omega)\right)du = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h(t,u)e^{i\omega u}du\right)dZ$$

Compute

$$\int_{-\infty}^{\infty} h(t,u)e^{i\omega u}du = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(t,\lambda)e^{i\lambda(t-u)}d\lambda e^{i\omega u}du = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\lambda)\int_{-\infty}^{\infty} e^{i\lambda(t-}$$

Compute

$$\int_{-\infty}^{\infty} e^{i\lambda(t-u)}e^{i\omega u}du = e^{i\lambda t}\int_{-\infty}^{\infty} e^{-i(\lambda-\omega)u}du = e^{i\lambda t}2\pi\delta(\lambda-\omega).$$

Substitute

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\lambda)e^{i\lambda t}2\pi\delta(\lambda-\omega)d\lambda = \int_{-\infty}^{\infty} A(t,\lambda)e^{i\lambda t}\delta(\lambda-\omega)d\lambda = A(t,\omega)e^{i\omega t}.$$

Substitute back

$$\int_{-\infty}^{\infty} \left(A(t,\omega)e^{i\omega t}\right)dZ(\omega) = \int_{-\infty}^{\infty} A(t,\omega)e^{i\omega t}dZ(\omega).$$

□

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This mathematical paper by Stephen Crowley presents a comprehensive theoretical framework for understanding the eigenfunction properties of both **stationary and oscillatory stochastic processes** on the real line. The work establishes fundamental relationships between these two important classes of processes through their covariance operators and spectral representations.

Core Definitions and Framework

The paper begins by establishing two fundamental process types:

Stationary Processes are characterized by time-invariant covariance functions where $R(s,t) = R(t-s)$, meaning the correlation structure depends only on the time difference, not absolute time.

Oscillatory Processes (following Priestley's framework) have the more general spectral representation:

$X(t) = \int_{-\infty}^{\infty} A(t,\omega)e^{i\omega t}dZ(\omega)$

where $A(t,\omega)$ is a time-varying gain function and $Z(\omega)$ represents an orthogonal increment process with the orthogonality condition $\mathbb{E}[dZ(\lambda)dZ^*(\omega)] = \delta(\lambda-\omega)dF(\omega)$.

Eigenfunction Properties

Stationary Processes

For stationary processes, the covariance operator $(Kf)(t) = \int_{-\infty}^{\infty} R(t-s)f(s)ds$ has **complex exponentials** $e^{i\omega t}$ as eigenfunctions:

$(Ke^{i\omega\cdot})(t) = S(\omega)e^{i\omega t}$

where $S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau$ is the **power spectral density**. This is the classical result showing that Fourier components are natural eigenfunctions of stationary processes.

Oscillatory Processes

For oscillatory processes, the eigenfunctions are **frequency-dependent and time-varying**:

$\phi(\cdot,\omega)(t) = A(t,\omega)e^{i\omega t}$

These satisfy the eigenfunction equation:

$(K\phi(\cdot,\omega))(t) = \phi(t,\omega)dF(\omega)$

This generalizes the stationary case by incorporating time-varying amplitude modulation through $A(t,\omega)$.

Key Mathematical Results

Real-Valued Process Conditions

The paper establishes that an oscillatory process is **real-valued** if and only if:

- $A(t,\omega) = A^*(t,-\omega)$ (conjugate symmetry in frequency)
- $dZ(-\omega) = dZ^*(\omega)$ (spectral symmetry)

Conjugate Pair Relationships

For real-valued processes, the eigenfunctions form **conjugate pairs**:
$\phi^*(t,\omega) = \phi(t,-\omega)$
This establishes a fundamental symmetry in the eigenfunction structure.

Filter Kernel Representation

The paper derives a **dual Fourier relationship** for the filter kernel:
$h(t,u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\omega)e^{i\omega(t-u)}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t,\omega)e^{-i\omega u}d\omega$
With **inverse relations**:
$A(t,\omega) = \int_{-\infty}^{\infty} h(t,u)e^{-i\omega(t-u)}du$
$\phi(t,\omega) = \int_{-\infty}^{\infty} h(t,u)e^{-i\omega u}du$

Theoretical Significance

This work provides a **unified mathematical framework** that:

1. **Generalizes classical results**: Extends the well-known eigenfunction properties of stationary processes to the more general oscillatory case
2. **Establishes spectral-temporal duality**: Shows how time-varying processes can be represented through both spectral (frequency-domain) and temporal (filter-based) perspectives
3. **Preserves orthogonality**: Maintains the fundamental orthogonality properties that make spectral analysis tractable, even in the time-varying case
4. **Connects representations**: Demonstrates equivalence between the oscillatory spectral representation and filtered versions of stationary white noise

The mathematical rigor throughout includes complete proofs using measure theory, Fourier analysis, and functional analysis techniques. The work contributes to the theoretical foundations of **non-stationary time series analysis** and **evolutionary spectral theory**, providing tools for analyzing processes whose spectral characteristics change over time while maintaining mathematical tractability through the eigenfunction framework.