On A New Bessel Function Identity For The Fourier Transform of J_0

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Lemma 1

Let J_n be the Bessel function of the first kind of order n, then

$$\sum_{m=1}^{\infty} \left(J_{m-1}^2(m\,v) + J_{m+1}^2(m\,v)\right) = \frac{1}{\sqrt{1-v^2}} \forall 0 \le v < 1 \tag{1}$$

Proof. The case $v=0$ is trivial as the only nonvanishing term on the left-hand side is $J_0^2(0)=1$. We henceforth assume $0<v<1$.

Using the integral representation 10.9.26 in [?],

$$J_n^2(z) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} J_{2n}(2\,z\cos\theta)\,\mathrm{d}\theta \tag{2}$$

we have

$$\begin{aligned}\sum_{m=1}^{\infty} \left(J_{m-1}^2(m\,v) + J_{m+1}^2(m\,v)\right) &= \frac{2}{\pi}\sum_{m=1}^{\infty}\int_0^{\frac{\pi}{2}} (J_{2m-2}(2\,m\,v\cos\theta) + J_{2m+2}(2\,m\,v\cos\theta))\mathrm{d}\theta \\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\mathrm{d}\theta \sum_{m=1}^{\infty} \left(J_{2m-2}(2\,m\,v\cos\theta) + J_{2m+2}(2\,m\,v\cos\theta)\right)\end{aligned} \tag{3}$$

where the interchange of the sum and the integral is justified because the summands fall off exponentially in m, uniformly in θ, as seen from 10.20.4 in [?], recalling that $0 \le v\cos\theta \le v < 1$.

For $0 \le \theta < \frac{\pi}{2}$, we use Bessel function identities to rewrite the summands in (3) and

$$J_{2m-2}(2\,m\,v\cos\theta) + J_{2m+2}(2\,m\,v\cos\theta) = J_{2m}(2\,m\,v\cos\theta)\left(\frac{4}{v^2\cos^2\theta} - 2\right) - \frac{\frac{\dot{J}_{2m}(2\,m\,v\cos\theta)}{2\,m}}{v\cos\theta} \tag{4}$$

and we then evaluate the sum over m by the identities

$$\sum_{m=1}^{\infty} J_{2m}(2\,m\,t) = \frac{t^2}{2\,(1-t^2)} \tag{5}$$

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{\dot{J}_{2m}(2\,m\,t)}{2\,m} &= \frac{1}{2}\left(\sum_{k=1}^{\infty}\frac{\dot{J}_k(k\,t)}{k} - \sum_{k=1}\frac{\dot{J}_k(k\,t)}{k}(-1)^{k-1}\right) \\ &= \frac{1}{2}\left[\frac{1}{2} + \frac{t}{4} - \left(\frac{1}{2} - \frac{t}{4}\right)\right] \\ &= \frac{t}{4}\end{aligned} \tag{6}$$

valid for $0 \leq t < 1$, using 8.517.3, 8.518.1 and 8.518.2 in [?]. Hence

$$\begin{aligned}
\sum_{m=1}^{\infty} \left(J_{m-1}^2(m\,v) + J_{m+1}^2(m\,v)\right) =& \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left[\left(\frac{4}{v^2\cos^2\theta} - 2\right)\frac{v^2\cos^2\theta}{2\,(1 - v^2\cos^2\theta)} - \frac{4}{v\cos\theta}\,\frac{v\cos\theta}{4}\right]\mathrm{d}\theta \\
=& \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{}{1 - v^2\cos^2\theta}\mathrm{d}\theta \\
=& \frac{1}{\sqrt{1 - v^2}}
\end{aligned} \tag{7}$$

where the last integral is elementary.[1] □

Lemma 1 it was proved in [1] that but they made no mention of the fact that this is the (one-sided) Fourier transform

$$\lambda(v) = \int_0^{\infty} J_0(x)e^{ixv}\mathrm{d}x = \frac{1}{\sqrt{1 - v^2}} \tag{8}$$

which is the orthogonality measure of the Chebyshev polynomials of the first kind

$$\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1 - v^2}}\mathrm{d}x = \delta_{n,m}\,\|T_n\| \tag{9}$$

where $\|T_n\| = \int_{-1}^{1} T_n(x)^2\mathrm{d}x$.

Bibliography

[1] Cameron R D Bunney and Jorma Louko. Circular motion analogue unruh effect in a 2+1 thermal bath: robbing from the rich and giving to the poor. *Classical and Quantum Gravity*, 40(15):155001–27, 2023.