STRONG LAW OF LARGE NUMBERS FOR WEAKLY HARMONIZABLE PROCESSES

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Abstract

If $X\colon \mathbb{R} \to L^2_{\mathbb{C}}(\Omega, \mathscr{A}, P)$ is a weakly harmonizable process with spectral stochastic measure $\mu\colon \mathscr{B}_{\mathbf{R}} \to L^2_{\mathbb{C}}(\Omega, \mathscr{A}, P)$, we first prove that

$$\lim_{t \to +\infty} \frac{1}{2t} \int_{-t}^{t} X(s) \mathrm{d}s = \mu(0) \quad \text{a.s.} \tag{1}$$

if and only if there exists some integer $p \geqslant 2$ such that

$$\lim_{n \to +\infty} \mu(|u| < p^{-n}) = \mu(0) \quad \text{a.s.} \tag{2}$$

As a consequence we then get criteria for the strong law of large numbers for the process X to hold, i.e.

$$\lim_{t \to +\infty} \frac{1}{2t} \int_{-t}^{t} X(s) \mathrm{d}s = 0 \quad \text{a.s.} \tag{3}$$

These are extensions to the weakly harmonizable case of results previously obtained by several authors and specially by Gaposhkin in the strongly harmonizable case.

*harmonizable processes * stochastic measures * bimeasures * strong laws of large numbers*

1 Introduction

1.1

Let $(\Omega, \mathscr{A}, P)$ be a probability space. A weakly harmonizable process $X\colon \mathbb{R} \to L^2_{\mathbb{C}}(\Omega, \mathscr{A}, P)$ is the Fourier transform of a stochastic measure i.e. a σ-additive set function $\mu\colon \mathscr{B}_{\mathbb{R}} \to L^2_{\mathbb{C}}(\Omega, \mathscr{A}, P)$, which is called its spectral stochastic measure.

The spectral bimeasure of X is the complex function defined on $\mathscr{B}_{\mathbb{R}} \times \mathscr{B}_{\mathbb{R}}$ by

$$M\,(A \times B) = E\,(\mu(A) \cdot \overline{\mu(B)}), \quad A, B \in \mathscr{B}_{\mathbb{R}} \tag{4}$$

The process X is called strongly harmonizable if its spectral bimeasure M is extendable to a measure (known as its spectral measure) on $\mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}}$. More particularly, if M concentrates on the diagonal Δ of $\mathbb{R} \times \mathbb{R}$, i.e. if

$$M(B) = M\,(B \cap \Delta), \quad B \in \mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbf{R}}, \tag{5}$$

X is a continuous (in q.m.) (wide sense) stationary process, and conversely.

It is well known that there exist non extendable spectral bimeasures (e.g., [6, Example 1]). We will overcome the technical problems generated by this difficulty with the help of the following Miamee and Salehi's domination lemma [7]: for every spectral bimeasure $M\colon \mathscr{B}_{\mathbb{R}} \times \mathscr{B}_{\mathbb{R}} \to \mathbb{C}$ there exists a bounded non-negative measure m on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ such that for any bounded measurable function $f\colon \mathbb{R} \to \mathbb{C}$, one has

$$0 \leqslant \iint f(t) \cdot \overline{f(s)} M(\,dt,\, ds) \leqslant \int |f(t)|^2\, m(\,dt) \tag{6}$$

where we use the concept of integration w.r.t. the bimeasure M as introduced by Moché [8, Chapter IV] (see also [11]).

1.2 The mean process

Let X be a weakly harmonizable process. Since it is continuous, we can suppose that it is measurable and has locally integrable sample paths (if X is separable, it actually has such a measurable modification). Therefore we can define a new second order process $\sigma_X\colon]\,0, +\infty\,[\to L^2_{\mathbb{C}}\,(\Omega, \mathscr{A}, P)$ called the (time averaged) mean process of X such that

$$\sigma_X(t) = \frac{1}{2\,t} \int_{-t}^{t} X(s) \mathrm{d}s, \quad t > 0\,(\text{strong } L^2\text{-integral}), \tag{7}$$

$$\sigma_X(t, \omega) = \frac{1}{2\,t} \int_{-t}^{t} X(s, \omega) \mathrm{d}s, \quad t > 0, \omega \in \Omega \tag{8}$$

1.3 Convergence of the mean process-previous results

We recall that one has [11, inversion formulae]

$$\sigma_X(t) = \int_{\mathbb{R}} \frac{\sin\,(t\,u)}{t\,u} \cdot \mu(\mathrm{d}u) \underset{t \to +\infty}{\longrightarrow} \mu(0) \quad (\text{in q.m.}). \tag{9}$$

Does this result remain true for the a.s. convergence? If we put

$$\sigma_X(t) = \Psi_X(t) + \mu\left(|u| < 2^{-n}\right), \quad 2^n + 1 < t \leqslant 2^{n+1} + 1, \tag{10}$$

Gaposhkin proved [5] that in the strongly harmonizable case, one has

$$\Psi_X(t) \underset{t \to +\infty}{\longrightarrow} 0 \quad \text{a.s} \tag{11}$$

So he obtained that X obeys the strong law of large numbers (SLLN) if and only if one has

$$\mu\left(|u| < 2^{-n}\right) \underset{n \to +\infty}{\longrightarrow} 0 \quad \text{a.s.} \tag{12}$$

He also deduced [4] that in the continuous stationary case, if M denotes the spectral bimeasure of X as well as its trace on Δ, X obeys the SLLN if the following conditions are both fulfilled:

$$M(0) = 0 \, \text{and there exists a real number } u_0 > 0 \text{such that} \tag{13}$$

$$\int_{(0 < |u| < u_0)} \left(\log\log \frac{1}{|u|}\right)^2 M(\,du) < +\infty. \tag{14}$$

Various other criteria for the SLLN in the strongly harmonizable case had been previously settled [1,2,10]. All of them use the total variation measure of M and consequently are not applicable if M is not extendable to a spectral measure [9, 2. Harmonizability].

1.4 The new results

In Section 2 we extend (2) and (3) to the weakly harmonizable case while Section 3 is devoted to the SLLN. More particularly, theorem 3.2 is an extension of the criterion (13) from the continuous stationary case to the weakly harmonizable one.

2 Asymptotic behaviour of the mean process

2.1

Let X be a weakly harmonizable process, p an integer $\geqslant$2, and let us put

$$\begin{aligned} \sigma_X(t) \; &= \Psi_X(p,t) + \mu\left(|u| < p^{-q}\right) \\ & \quad t > p+1, q \in \mathbb{N} \setminus (0), p^q + 1 < t \leqslant p^{q+1} + 1, \end{aligned} \tag{15}$$

$$\begin{aligned} \Psi_X(p,t) \; &= (\sigma_X(t) - \sigma_X(n)) + (\sigma_X(n) - \sigma_X(p^q)) + (\sigma_X(p^q) - \mu\left(|u| < p^{-q}\right)) \\ & \quad n, q \in \mathbb{N} \setminus (0), n < t \leqslant n+1, \quad p^q < n \leqslant p^{q+1}. \end{aligned} \tag{16}$$

We are going to prove that each term of the right-hand side of (16) converges almost surely to 0 as t tends to infinity. This is already done for the first term since Rousseau has proved [10, Prop. 1] that

$$\mathrm{Sup}(|\sigma_X(t)-\sigma_X(n)|; n<t\leqslant n+1)\underset{n\to+\infty}{\longrightarrow}0 \quad \text{a.s.} \tag{17}$$

2.2

For the second term, we have

Proposition 1. $\mathrm{Lim}_{q\to+\infty}\mathrm{Max}(|\sigma_X(n)-\sigma_X(p^q)|; p^q<n\leqslant p^{q+1})=0$ *a.s.*

Proof. In order to simplify the proof, let us suppose that $p=2$ (for $p>2$, see [3, Chapter 3]).

For every integer $q\geqslant 1, k$ such that $1\leqslant k\leqslant q$ and every $\boldsymbol{e}\in E(k)=\{0,1\}^k$, we put

$$\begin{aligned} a(q,k,e) &= 2^q+1+\sum_{j=1}^{k} e_j\, 2^{q-j} \\ b(q,k,e) &= \begin{cases} 2^q+1+\sum_{j=1}^{k-1} e_j\, 2^{q-j} & \text{if } k\geqslant 2 \\ 2^q & \text{if } k=1 \end{cases} \end{aligned} \tag{18}$$

and let $(\alpha_k, k\geqslant 1)$ be a sequence of strictly positive numbers. Utilizing Rousseau's majorization lemma [10], we can deduce that

$$\begin{aligned} &\mathrm{Max}(|\sigma_X(n)-\sigma_X(2^q)|^2; 2^q<n\leqslant 2^{q+1}) \\ &\leqslant \left(\sum_{k=1}^{q}\alpha_k^{-1}\right)\left(\sum_{k=1}^{q}\alpha_k\left(\sum_{e\in E(k)}\left|\sum_{j=b(q,q,ke)+1}^{a(q,k,e)}(\sigma_X(j)-\sigma_X\,(j-1))\right|^2\right)\right) \\ &= \left(\sum_{k=1}^{q}\alpha_k^{-1}\right)\left(\sum_{k=1}^{q}\alpha_k\left(\sum_{e\in E(k)}|\,\sigma_X\,(a(q,k,e)-\sigma_X(b(q,k,e))|^2)\right)\right) \end{aligned} \tag{19}$$

Then we obtain, by integration,

$$\begin{aligned} E\ &(\mathrm{Max}(|\sigma_X(n)-\sigma_X(2^q)|^2; 2^q<n\leqslant 2^{q+1})) \\ &\leqslant\left(\sum_{k=1}^{q}\alpha_k^{-1}\right)\left(\sum_{k=1}^{q}2^k\,\alpha_k\cdot\mathrm{Max}(E(|\sigma_X(a(q,k,e))-\sigma_X(b(q,k,e))|^2); e\in E(k))\right) \\ &=\left(\sum_{k=1}^{q}\alpha_k^{-1}\right)\left(\sum_{k=1}^{q}2^k\,\alpha_k\mathrm{Max}\left(\iint f_{q,k,e}(u)\cdot f_{q,k,e}(v)\,M(\,du,\,dv); e\in E(k)\right)\right) \end{aligned} \tag{20}$$

from (4), where

$$f_{q,k,e}(u) = \frac{\sin(a(q,k,e)\cdot u)}{a(q,k,e)\cdot u} - \frac{\sin(b(q,k,e)\cdot u)}{b(q,k,e)\cdot u}, \quad u \in \mathbb{R}. \tag{21}$$

Now we use the key idea of the proof i.e. we reduce the problem to the classical stationary case through the domination lemma: there exists a bounded non-negative measure m on $(\mathbb{R}, \mathscr{B}_{\mathbf{R}})$ associated to the spectral bimeasure M such that

$$0 \iint f_{q,k,e}(u)\cdot f_{q,k,e}(v)\, M(\,du,\, dv) \leqslant \int f^2_{q,k,e}(u)\, m(\,du) \tag{22}$$

So we have

$$\begin{aligned} &\sum_{q=1}^{+\infty} E(\mathrm{Max}|\sigma_X(n) - \sigma_X(2^q)|^2; 2^q < n \leqslant 2^{q+1}) \\ &\quad \leqslant \sum_{q=1}^{+\infty} \left(\sum_{k=1}^{q} \alpha_k^{-1}\right)\left(\sum_{k=1}^{q} 2^k\, \alpha_k \mathrm{Max}\left(\int f^2_{q,k,e}(u)\, m(\,du); e \in E(k)\right)\right) \end{aligned} \tag{23}$$

The end of the proof is not new: if we divide the integration domain of the last integral into the following four parts:

$$(|u| < 2^{-q-1}), \quad (2^{-q-1} \leqslant |u| < 2^{-q+k}), \quad (2^{-q+k} \leqslant |u| < 1), \quad (1 \leqslant |u|), \tag{24}$$

and if $\alpha_k = \alpha^k, k \geqslant 1, 1 < \alpha < 2$, it appears four convergent series [4, Theorem 1; 10, Prop. 4] so that the series (23) is also convergent. We can obviously conclude that

$$\max(|\sigma_X(n) - \sigma_X(2^q)|; 2^q < n \leqslant 2^{q+1}) \underset{q\to+\infty}{\longrightarrow} 0 \text{a.s.} \tag{25}$$ $\square$

2.3

It is easy to prove that

$$\sigma_X(p^q) - \mu(|u| < p^{-q}) \underset{q\to+\infty}{\longrightarrow} 0 \quad \text{a.s.} \tag{26}$$

but one will need once again the domination lemma:

$$\begin{gathered} \sigma_X(p^q) - \mu(|u| < p^{-q}) \\ = \\ \int_{(p^{-q} \leqslant |u|)} \frac{\sin(p^q u)}{p^q u}\, \mu(\mathrm{d}u) + \int_{(|u| < p^{-q})} \left(\frac{\sin(p^q u)}{p^q u} - 1\right) \mu(\mathrm{d}u) \end{gathered} \tag{27}$$

$$
\begin{aligned}
&E(|\sigma_X(p^q) - \mu\,(|u| < p^{-q})|^2) \\
&\quad \leqslant 2 \iint_{(p^{-q} \leqslant |u|, |v|)} \frac{\sin\,(p^q u)}{p^q u} \cdot \frac{\sin\,(p^q v)}{p^q v} M(\,du,\,dv) \\
&\quad + 2 \iint_{(|u|, |v| < p^{-q})} \left(\frac{\sin\,(p^q u)}{p^q u} - 1\right)\left(\frac{\sin\,(p^q v)}{p^q v} - 1\right) M(\,du,\,dv) \\
&\quad \leqslant 2 \int_{(p^{-q} \leqslant |u|)} \left(\frac{\sin\,(p^q u)}{p^q u}\right)^2 m(\,du) + 2 \int_{(|u| < p^{-q})} \left(\frac{\sin\,(p^q u)}{p^q u} - 1\right)^2 m(\,du).
\end{aligned} \tag{28}
$$

So we are now in the stationary case from which [4, Theorem 1] we can prove that

$$
\sum_{q=1}^{+\infty} E(|\sigma_X(p^q) - \mu\,(|u| < p^{-q})|^2) < +\infty \tag{29}
$$

so that (26) is true. At last we can summarize (15), (16), (17), (25) and (26) by the following theorem.

2.4

Theorem 2. *For every weakly harmonizable process X and every integer $p \geqslant 2$, one has*

$$
\Psi_X(p,t) \underset{t \to +\infty}{\longrightarrow} 0 \quad a.s. \tag{30}
$$

so that the following two conditions are equivalent:

(i) $\sigma_X(t)$ converges a.s. as tends to infinity.

(ii) there exists an integer $p \geqslant 2$ such that $\mu\,(|u| < p^{-q})$ converges a.s. when q tends to infinity.

Moreover, one then has, for every integer $p \geqslant 2$,

$$
\lim_{t \to +\infty} \sigma_X(t) = \lim_{q \to +\infty} \mu\,(|u| < p^{-q}) = \mu(0) \quad a.s. \tag{31}
$$

3 Criteria for the SLLN

3.1

The next statement is an obvious corollary of the theorem in 2.4.

Theorem 3. *Let X be a weakly harmonizable process: it obeys the SLLN if and only if there exists an integer $p \geqslant 2$ such that:*

$$\lim_{q \to +\infty} \mu\left(|u| < p^{-q}\right) = 0 \quad a.s. \tag{32}$$

3.2

We can now give an extension of Gaposhkin's criterion (13):

Theorem 4. *Let X be a weakly harmonizable process. If there exists a bounded nonnegative measure M_0 on $(\mathbb{R}^2, \mathscr{B}_{\mathbb{R}^2})$ such that*

(i) for every event A of the ring generated by the intervals, one has

$$M\left(A \times A\right) \leqslant M_0\left(A \times A\right) \tag{33}$$

(ii) there exists a real number $u_0 > 0$ such that

$$\iint_{(0<|u|,|v|<u_0)} \left(\log\log\frac{1}{|u|}\right)\left(\log\log\frac{1}{|v|}\right) M_0(\,du,\,dv) < +\infty \tag{34}$$

then one has

$$\sigma_X(t) \underset{t \to +\infty}{\longrightarrow} \mu(0) \quad a.s., \tag{35}$$

and X obeys the SLLN if and only if $M(0,0) = 0$.

Proof. (a) The theorem in 2.4 shows that we have only to prove that

$$\lim_{n \to +\infty} \mu\left(|u| < 2^{-n}\right) = \mu(0) \quad \text{a.s.} \tag{36}$$

More particularly, since we have

$$\mu\left(|u| < 2^{-n}\right) = \mu(0) + \mu\left(0 < |u| < 2^{-2^q}\right) - \mu\left(2^{-n} \leqslant |u| < 2^{-2^q}\right) \forall 2^q < n \leqslant 2^{q+1} \tag{37}$$

we have only to prove that the last two terms converge to 0 a.s. as n tends to infinity.

(b) Let q_0 be an integer such that $2^{-2^{q_0}} < u_0$. Putting

$$B_q = \left(0 < |u| < 2^{-2^q}\right) \tag{38}$$

and utilizing (i) and (ii), we obtain

$$\begin{aligned}
\sum_{q=q_0}^{+\infty} E(|\mu(B_q)|^2) &\leqslant \sum_{q=q_0}^{+\infty} M_0\,(B_q \times B_q) \\
&\leqslant \sum_{q=q_0}^{+\infty} q^{-2} \iint_{B_q \times B_q} \left(\log_2 \log_2 \frac{1}{|u|}\right)\left(\log_2 \log_2 \frac{1}{|v|}\right) M_0(\,du,\,dv) \\
&\leqslant \left(\sum_{q=q_0}^{+\infty} q^{-2}\right) \iint_{(0<|u|,|v|<u_0)} \left(\log_2 \log_2 \frac{1}{|u|}\right)\left(\log_2 \log_2 \frac{1}{|v|}\right) M_0(\,du,\,dv) \\
&< +\infty
\end{aligned} \tag{39}$$

where $\log_2$ is the log function to the base 2. Therefore we have

$$\mu\,(0<|u|<2^{-2^q}) \underset{q\to+\infty}{\longrightarrow} 0 \text{a.s.} \tag{40}$$

(c) Using the previous notations, we put

$$A\,(q,k,e) = (2^{-a(q,k,e)} \leqslant |u| < 2^{-b(q,k,e)}) \tag{41}$$

and

$$C_q = (2^{-2^{q+1}} \leqslant |u| < 2^{-2^q}). \tag{42}$$

By means of (i), (ii) and the Rousseau's majorization lemma, one has

$$\begin{aligned}
&\sum_{q=q_0}^{+\infty} E(\mathrm{Max}(|\mu\,(2^{-n} \leqslant |u| < 2^{-2^q})|^2; 2^q < n \leqslant 2^{q+1})) \\
&\quad \leqslant \sum_{q=q_0}^{+\infty} q\left(\sum_{k=1}^{q}\left(\sum_{e\in E(k)} E(|\mu\,(A\,(q,k,e))|^2)\right)\right) \\
&\quad \leqslant \sum_{q=q_0}^{+\infty} q\left(\sum_{k=1}^{q}\left(\sum_{e\in E(k)} M_0\,(A\,(q,k,e) \times A\,(q,k,e))\right)\right) \\
&\quad \leqslant \sum_{q=q_0}^{+\infty} q\left(\sum_{k=1}^{q} M_0\,(C_q \times C_q)\right) = \sum_{q=q_0}^{+\infty} q^2\, M_0\,(C_q \times C_q) \\
&\quad \leqslant \sum_{q=q_0}^{+\infty} \iint_{C_q \times C_q} \left(\log_2 \log_2 \frac{1}{|u|}\right)\left(\log_2 \log_2 \frac{1}{|v|}\right) M_0(\,du,\,dv) \\
&\quad \leqslant \iint_{(0<|u|,|v|<u_0)} \left(\log_2 \log_2 \frac{1}{|u|}\right)\left(\log_2 \log_2 \frac{1}{|v|}\right) M_0(\,du,\,dv) \\
&\quad < +\infty
\end{aligned} \tag{43}$$

Therefore we have

$$\mathrm{Max}(|\mu\,(2^{-n} \leqslant |u| < 2^{-2^q})|; 2^q < n \leqslant 2^{q+1}) \underset{q\to+\infty}{\longrightarrow} 0 \quad \text{a.s.,} \tag{44}$$

and this completes the proof. □

3.3 Remarks

(a) In the strongly harmonizable case, the both conditions (i) and (ii) of the theorem in 3.2 can be replaced by the following single one: there exists a real number u_0 such that

$$\iint_{(0<|u|,|v|<u_0)} \left(\log\log\frac{1}{|u|}\right)\left(\log\log\frac{1}{|v|}\right)|M|(\mathrm{d}u,\, dv) < +\infty \tag{45}$$

where $|M|$ is the total variation measure of M. Moreover, if X is continuous and stationary, it reduces exactly to the Gaposhkin's criterion (13).

In the general weakly harmonizable case, the condition (ii) can be replaced by more practical ones [3, Lemma 4.2.6 and Remark 4.2.7] which are also extensions of the corresponding result of Gaposhkin [4, Corollary 3].

(b) At last, once again as Gaposhkin [4] we have got some information about the rate of convergence of $\sigma_X(t)$ towards $\mu(0)$:

Theorem 5. *Let X be a weakly harmonizable process. Suppose that there exists a non-decreasing function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that*

(i) there exist integers $p \geqslant 2, q_0$ and a real $A, 1 < A < \sqrt{p}$ for which we have

$$g^2(p^{p^{q+1}}) \leqslant A\, g^2(p^{p^q}) \quad q\mathit{integer}, q \geqslant q_0 \tag{46}$$

(ii) there exist a bounded non-negative measure m on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ which dominates M (in the sense of the domination lemma) and a real number u_0 such that

$$\int_{(0<|u|<u_0)} \left(\left(\log\log\frac{1}{|u|}\right) g\left(\frac{1}{|u|}\right)\right)^2 m(\,du) < +\infty \tag{47}$$

Then one has

$$\lim_{t\to+\infty} g(t)\cdot(\sigma_X(t) - \mu(0)) = 0 \quad a.s. \tag{48}$$

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