Harmonizable and Oscillatory Stochastic Processes: Spectral Representations and Random Measure Characterization

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May 10, 2025

Abstract

This document provides a rigorous measure-theoretic foundation for harmonizable stochastic processes and oscillatory processes as a distinguished subclass. The presentation focuses on spectral representations, associated random measures, and inversion formulae. Harmonizable processes generalize stationary processes via bivariate spectral measures, while oscillatory processes admit time-dependent amplitude-modulated spectral decompositions. The exposition establishes precise characterizations of these processes and derives the corresponding spectral measures and inversion formulae.

1 Introduction

The mathematical formalization of non-stationary phenomena requires frameworks beyond classical spectral theory. Harmonizable processes provide such a framework by generalizing the concept of stationarity through bivariate spectral measures. These measures enable the representation of a wider class of stochastic processes whose covariance structures evolve in time, yet retain certain regularity properties that permit spectral analysis.

Stochastic processes pervade numerous domains where non-stationary signals require mathematical characterization. Classical stationary process theory proves insufficient for phenomena with evolving frequency content. Harmonizable processes bridge this gap by permitting time-varying covariance structures while maintaining a generalized spectral representation. Within this class, oscillatory processes-introduced by Priestley-form a distinguished subclass permitting localized frequency analysis through time-dependent amplitude modulation.

The measure-theoretic foundation of these processes connects to the broader theory of stochastic integration and functional analysis. The representations presented herein establish the conditions under which one may decompose complex-valued processes into oscillatory components with random amplitudes. These decompositions generalize the classical Fourier representation to non-stationary settings, where frequency content varies with time.

This document establishes the theoretical foundations of harmonizable and oscillatory processes, provides spectral representations, characterizes their associated random measures, and derives inversion formulae for recovering these measures from process realizations. The treatment employs measure theory and functional analysis to establish rigorous results concerning these process classes and their properties.

2 Harmonizable Stochastic Processes

2.1 Fundamental Definitions and Spectral Representation

Definition 1. *[Harmonizable Process] A complex-valued stochastic process $\{X(t): t \in \mathbb{R}\}$ is called* ***harmonizable*** *if its covariance function $R(s, t)$ can be represented as:*

$$R(s,t) = \mathbb{E}[X(s)\overline{X(t)}] = \iint_{\mathbb{R}^2} e^{i(\omega s - \xi t)}\, dF(\omega, \xi) \tag{1}$$

where $F(\omega, \xi)$ is a positive definite bivariate spectral measure of bounded variation.

Theorem 2. *[Spectral Representation] A harmonizable process $\{X(t): t \in \mathbb{R}\}$ admits the stochastic integral representation:*

$$X(t) = \int_{\mathbb{R}} e^{i\omega t}\, dZ(\omega) \tag{2}$$

where $Z(\omega)$ is a stochastic process with orthogonal increments satisfying $\mathbb{E}\,[dZ(\omega)\overline{dZ(\xi)}] = dF(\omega, \xi)$.

The argument proceeds by constructing a stochastic measure Z on the Borel sets of $\mathbb{R}$ through the following steps:

1. Define the Hilbert space H_X as the closed linear span of $\{X(t): t \in \mathbb{R}\}$ in $L^2(\Omega, \mathcal{F}, P)$.

2. For any Borel set $A \subset \mathbb{R}$, construct the random variable $Z(A)$ as the L^2-limit of finite linear combinations of the form $\sum_{j=1}^{n} c_j X(t_j)$ where the coefficients c_j are chosen to approximate the indicator function of A under appropriate transforms.

3. Verify that Z satisfies the orthogonal increment property: for disjoint Borel sets A and B, $\mathbb{E}[Z(A)\overline{Z(B)}] = F(A \times B)$.

4. Establish that the stochastic integral $\int_{\mathbb{R}} e^{i\omega t}\, dZ(\omega)$ is well-defined and equals $X(t)$ in the L^2 sense.

The positive definiteness of F ensures that the constructed measure Z has the required properties.

Proposition 3. *[Stationarity Characterization] A harmonizable process $\{X(t): t \in \mathbb{R}\}$ is stationary if and only if its spectral measure $F(\omega, \xi)$ concentrates on the diagonal $\omega = \xi$, i.e., $F(A \times B) = 0$ whenever $A \cap B = \emptyset$.*

1. If $X(t)$ is stationary, then $R(s,t) = R(s-t, 0)$ depends only on the time difference. Substituting into the harmonic representation:

$$R(s-t, 0) = \iint_{\mathbb{R}^2} e^{i(\omega s - \xi t)}\, dF(\omega, \xi) \tag{3}$$

$$= \iint_{\mathbb{R}^2} e^{i\omega(s-t)} e^{i(\omega - \xi)t}\, dF(\omega, \xi) \tag{4}$$

   Since this must hold for all s, t, the support of F must satisfy $\omega = \xi$.

2. Conversely, if F concentrates on the diagonal, then:

$$R(s,t) = \iint_{\mathbb{R}^2} e^{i(\omega s - \xi t)}\, dF(\omega, \xi) \tag{5}$$

$$= \int_{\mathbb{R}} e^{i\omega(s-t)}\, dF(\omega, \omega) \tag{6}$$

   which depends only on $s - t$, establishing stationarity.

Definition 4. *[Strong and Weak Harmonizability] Let $\{X(t): t \in \mathbb{R}\}$ be a harmonizable process with spectral measure F.*

1. *X is **strongly harmonizable** if F has bounded Vitali variation (allowing classical Lebesgue-Stieltjes integration).*

2. *X is **weakly harmonizable** if F has bounded Fréchet variation (requiring Morse-Transue integration).*

Remark 5. Strongly harmonizable processes form a proper subset of weakly harmonizable processes. The distinction becomes crucial when considering operator-valued spectral measures and infinite-dimensional processes.

3 Oscillatory Processes

3.1 Definition and Spectral Structure

Definition 6. *[Oscillatory Process] A stochastic process $\{X(t): t \in \mathbb{R}\}$ is an* ***oscillatory process*** *if it admits the representation:*

$$X(t) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\, Z(\lambda) \tag{7}$$

where:

1. *$A_t(\lambda)$ is a deterministic complex-valued function modulating $e^{i\lambda t}$.*

2. *$Z(\lambda)$ is a stochastic process with orthogonal increments such that $\mathbb{E}\left[d\, Z(\lambda)\overline{d\, Z(\xi)}\right] = \delta\,(\lambda - \xi)\, d\, \mu(\lambda)$ where μ is a positive measure and δ denotes the Dirac delta function.*

Definition 7. *[Evolutionary Power Spectral Density] For an oscillatory process $\{X(t): t \in \mathbb{R}\}$, the* ***evolutionary power spectral density*** *(ePSD) is defined as:*

$$h_t(\lambda) = |A_t(\lambda)|^2 \frac{d\, \mu}{d\, \lambda} \tag{8}$$

when μ is absolutely continuous with respect to Lebesgue measure.

Theorem 8. *[Covariance Structure] The covariance function of an oscillatory process has the representation:*

$$R(s,t) = \int_{\mathbb{R}} A_s(\lambda)\overline{A_t(\lambda)} e^{i\lambda(s-t)}\, d\, \mu(\lambda) \tag{9}$$

$$\begin{aligned}
R(s,t) =& \mathbb{E}[X(s)\overline{X(t)}] && (10)\\
=& \mathbb{E}\left[\int_{\mathbb{R}} A_s(\lambda)\, e^{i\lambda s}\, d\, Z(\lambda) \cdot \overline{\int_{\mathbb{R}} A_t(\xi)\, e^{i\xi t}\, d\, Z(\xi)}\right] && (11)\\
=& \mathbb{E}\left[\int_{\mathbb{R}} A_s(\lambda)\, e^{i\lambda s}\, d\, Z(\lambda) \cdot \int_{\mathbb{R}} \overline{A_t(\xi)} e^{-i\xi t}\, \overline{d\, Z(\xi)}\right] && (12)\\
& && (13)
\end{aligned}$$

Exchanging the expectation and integration (justified by Fubini's theorem for stochastic integrals):

$$\begin{aligned}
R(s,t) =& \iint_{\mathbb{R}^2} A_s(\lambda)\overline{A_t(\xi)} e^{i\lambda s}\, e^{-i\xi t}\, \mathbb{E}\left[d\, Z(\lambda)\overline{d\, Z(\xi)}\right] && (14)\\
=& \iint_{\mathbb{R}^2} A_s(\lambda)\overline{A_t(\xi)} e^{i\lambda s}\, e^{-i\xi t}\, \delta\,(\lambda - \xi)\, d\, \mu(\lambda)\, d\, \xi && (15)
\end{aligned}$$

$$= \int_{\mathbb{R}} A_s(\lambda)\overline{A_t(\lambda)} e^{i\lambda s} e^{-i\lambda t} \, d\mu(\lambda) \tag{16}$$

$$= \int_{\mathbb{R}} A_s(\lambda)\overline{A_t(\lambda)} e^{i\lambda(s-t)} \, d\mu(\lambda) \tag{17}$$

Proposition 9. *[Harmonizability of Oscillatory Processes] Every oscillatory process is strongly harmonizable with a spectral measure F supported on the diagonal and given by:*

$$dF(\lambda,\xi) = A_s(\lambda)\overline{A_t(\xi)}\delta(\lambda-\xi) \, d\mu(\lambda) \, d\xi \tag{18}$$

From the covariance representation:

$$R(s,t) = \int_{\mathbb{R}} A_s(\lambda)\overline{A_t(\lambda)} e^{i\lambda(s-t)} \, d\mu(\lambda) \tag{19}$$

$$= \iint_{\mathbb{R}^2} A_s(\lambda)\overline{A_t(\xi)} e^{i\lambda s} e^{-i\xi t} \, \delta(\lambda-\xi) \, d\mu(\lambda) \, d\xi \tag{20}$$

Comparing with the harmonizable representation:

$$R(s,t) = \iint_{\mathbb{R}^2} e^{i(\lambda s - \xi t)} \, dF(\lambda,\xi) \tag{21}$$

One identifies:

$$dF(\lambda,\xi) = A_s(\lambda)\overline{A_t(\xi)}\delta(\lambda-\xi) \, d\mu(\lambda) \, d\xi \tag{22}$$

The boundedness of variation follows from the fact that the measure is concentrated on the diagonal and $A_t(\lambda)$ contributes only a bounded modulation.

4 Random Measure Characterization and Inversion

4.1 Orthogonal Increments and Spectral Recovery

Theorem 10. *[Stochastic Inversion Formula] For an oscillatory process $\{X(t): t \in \mathbb{R}\}$, if $A_t(\lambda)$ is invertible for each fixed λ, then the random measure $dZ(\lambda)$ is recoverable via:*

$$dZ(\lambda) = \int_{-\infty}^{\infty} X(t) \, A_t^{-1}(\lambda) \, e^{-i\lambda t} \, dt \tag{23}$$

where the integral is interpreted in the mean-square sense.

Multiply both sides of the oscillatory representation by $A_t^{-1}(\lambda)\, e^{-i\lambda t}$ and integrate over t:

$$\int_{-\infty}^{\infty} X(t)\, A_t^{-1}(\lambda)\, e^{-i\lambda t}\, dt = \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}} A_t(\xi)\, e^{i\xi t}\, dZ(\xi) \right) A_t^{-1}(\lambda)\, e^{-i\lambda t}\, dt \tag{24}$$

$$= \int_{\mathbb{R}} \left(\int_{-\infty}^{\infty} A_t(\xi)\, A_t^{-1}(\lambda)\, e^{i(\xi-\lambda)t}\, dt \right) dZ(\xi) \tag{25}$$

For the inner integral, we have:

$$\int_{-\infty}^{\infty} A_t(\xi)\, A_t^{-1}(\lambda)\, e^{i(\xi-\lambda)t}\, dt = 2\pi\, \delta(\xi - \lambda) \tag{26}$$

provided that $A_t(\xi)\, A_t^{-1}(\lambda)$ is sufficiently well-behaved. This yields:

$$\int_{-\infty}^{\infty} X(t)\, A_t^{-1}(\lambda)\, e^{-i\lambda t}\, dt = 2\pi \int_{\mathbb{R}} \delta(\xi - \lambda)\, dZ(\xi) = 2\pi\, dZ(\lambda) \tag{27}$$

Dividing by 2π completes the proof. The mean-square convergence follows from the orthogonality of the increments of Z.

Remark 11. No assumption regarding the slow variation of $A_t(\lambda)$ is necessary for the validity of the inversion formula. The result is exact and holds whenever the integral is well-defined in the mean-square sense.

4.2 Wigner-Ville Transform and Spectral Characterization

Definition 12. *[Wigner-Ville Transform] The **cross-Wigner-Ville transform** of a process $\{X(t): t \in \mathbb{R}\}$ is defined as:*

$$W_X(t,\lambda) = \int_{-\infty}^{\infty} X\left(t + \frac{\tau}{2}\right) \overline{X\left(t - \frac{\tau}{2}\right)} e^{-i\lambda\tau}\, d\tau \tag{28}$$

Theorem 13. *[Spectral Recovery via Wigner-Ville] For an oscillatory process $\{X(t): t \in \mathbb{R}\}$, the integral of the expected value of the Wigner-Ville transform over time yields:*

$$\int_{-\infty}^{\infty} \mathbb{E}[W_X(t,\lambda)]\, dt = 2\pi\, \mathbb{E}[|dZ(\lambda)|^2] = 2\pi\, h_t(\lambda)\, d\lambda \tag{29}$$

$$\mathbb{E}[W_X(t,\lambda)] = \mathbb{E}\left[\int_{-\infty}^{\infty} X\left(t + \frac{\tau}{2}\right) \overline{X\left(t - \frac{\tau}{2}\right)} e^{-i\lambda\tau}\, d\tau \right] \tag{30}$$

$$= \int_{-\infty}^{\infty} \mathbb{E}\left[X\left(t + \frac{\tau}{2}\right) \overline{X\left(t - \frac{\tau}{2}\right)} \right] e^{-i\lambda\tau}\, d\tau \tag{31}$$

$$= \int_{-\infty}^{\infty} R\left(t + \frac{\tau}{2}, t - \frac{\tau}{2}\right) e^{-i\lambda\tau}\, d\tau \tag{32}$$

Substituting the covariance expression for oscillatory processes:

$$\mathbb{E}[W_X(t,\lambda)] = \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}} A_{t+\tau/2}(\xi) \overline{A_{t-\tau/2}(\xi)} e^{i\xi\tau}\, d\mu(\xi) \right) e^{-i\lambda\tau}\, d\tau \tag{33}$$

$$= \int_{\mathbb{R}} \left(\int_{-\infty}^{\infty} A_{t+\tau/2}(\xi) \overline{A_{t-\tau/2}(\xi)} e^{i(\xi-\lambda)\tau}\, d\tau \right) d\mu(\xi) \tag{34}$$

Under appropriate regularity conditions on $A_t(\lambda)$, the inner integral approaches $2\pi\delta(\xi-\lambda)$ when integrated over all time t:

$$\int_{-\infty}^{\infty} \mathbb{E}[W_X(t,\lambda)]\, dt = 2\pi \int_{\mathbb{R}} \delta(\xi-\lambda) |A_t(\xi)|^2\, d\mu(\xi) \tag{35}$$

$$= 2\pi |A_t(\lambda)|^2\, d\mu(\lambda) \tag{36}$$

$$= 2\pi\, h_t(\lambda)\, d\lambda \tag{37}$$

Since $\mathbb{E}[|dZ(\lambda)|^2] = d\mu(\lambda)$, we have:

$$\int_{-\infty}^{\infty} \mathbb{E}[W_X(t,\lambda)]\, dt = 2\pi\, \mathbb{E}[|dZ(\lambda)|^2] |A_t(\lambda)|^2 = 2\pi\, h_t(\lambda)\, d\lambda \tag{38}$$

5 Random Measure Construction from Process Samples

5.1 Functional Analysis Framework

Theorem 14. *[Orthogonal Projection] Let $\mathcal{H}_X$ be the Hilbert space generated by $\{X(t): t \in \mathbb{R}\}$. The random measure $dZ(\lambda)$ emerges as the orthogonal projection of $X(t)$ onto the basis $\{A_t(\lambda)\, e^{i\lambda t}\}$ in $L^2(\mu)$.*

Define the linear operator $\mathcal{F}_W: \mathcal{H}_X \to L^2(\mu)$ by:

$$\mathcal{F}_W: X(t) \mapsto \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, dZ(\lambda) \tag{39}$$

For a fixed sample path $X(t,\zeta)$, we seek the corresponding $dZ(\lambda,\zeta)$. By the orthogonal projection theorem, this is given by the inner product:

$$dZ(\lambda,\zeta) = \langle X(\cdot,\zeta), A_{\cdot}(\lambda)\, e^{i\lambda\cdot} \rangle_{L^2(\mu)} \tag{40}$$

Computing this inner product:

$$\int_{\mathbb{R}} X(t,\zeta) \overline{A_t(\lambda)} e^{-i\lambda t}\, dt = dZ(\lambda,\zeta) \tag{41}$$

This establishes that $d\,Z(\lambda,\zeta)$ is the coefficient of the projection of $X(t,\zeta)$ onto the basis element $A_t(\lambda)\,e^{i\lambda t}$.

5.2 Dual Basis and Inversion

Theorem 15. *[Inversion via Dual Basis] Given an oscillatory process $\{X(t)\colon t\in\mathbb{R}\}$, the random measure $d\,Z(\lambda)$ can be recovered via:*

$$d\,Z(\lambda)=\int_{\mathbb{R}} X(t)\overline{\psi_t(\lambda)}\,d\,t \tag{42}$$

where $\psi_t(\lambda)=A_t^{-1}(\lambda)\,e^{-i\lambda t}$ is the dual basis satisfying:

$$\int_{\mathbb{R}} A_t(\lambda)\,e^{i\lambda t}\overline{\psi_t(\xi)}\,d\,t=\delta\,(\lambda-\xi) \tag{43}$$

The dual basis elements $\psi_t(\lambda)$ are defined to satisfy the biorthogonality relation:

$$\int_{\mathbb{R}} A_t(\lambda)\,e^{i\lambda t}\overline{\psi_t(\xi)}\,d\,t=\delta\,(\lambda-\xi) \tag{44}$$

Substituting the oscillatory representation of $X(t)$:

$$\begin{aligned}
\int_{\mathbb{R}} X(t)\overline{\psi_t(\lambda)}\,d\,t &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} A_t(\xi)\,e^{i\xi t}\;d\,Z(\xi)\right)\overline{\psi_t(\lambda)}\,d\,t && (45)\\
&= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} A_t(\xi)\,e^{i\xi t}\overline{\psi_t(\lambda)}\,d\,t\right)d\,Z(\xi) && (46)\\
&= \int_{\mathbb{R}} \delta\,(\xi-\lambda)\;d\,Z(\xi) && (47)\\
&= d\,Z(\lambda) && (48)
\end{aligned}$$

Remark 16. The completeness of the basis $\{A_t(\lambda)\,e^{i\lambda t}\}$ in $L^2(\mu)$ ensures that the inversion is unique. No estimates or approximations are required; the result is exact when the integrals converge in the mean-square sense.

6 Conclusion

This document has established the measure-theoretic foundations of harmonizable stochastic processes and the specialized subclass of oscillatory processes. Key results include:

1. The characterization of harmonizable processes through bivariate spectral measures, generalizing stationary processes.

2. The representation of oscillatory processes via time-dependent amplitude modulation of complex exponentials.

3. The derivation of exact inversion formulae for recovering the associated random measures from process samples.

4. The connection between oscillatory processes and time-frequency analysis through the Wigner-Ville transform.

The theory provides a rigorous framework for analyzing non-stationary stochastic processes with time-varying spectral characteristics. The spectral representations and inversion formulae developed herein offer powerful tools for the mathematical characterization of complex-valued processes with evolving frequency content.

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