Oscillatory Processes

by Stephen Crowley

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Definition 1. (Stationary Process) *A stochastic process* $\{X(t), t \in \mathbb{R}\}$ *is stationary when* $R(s,t) = R(t-s)$ *for all* $s, t \in \mathbb{R}$.

Theorem 2. (Filter Representation of Nonstationary Process) *Oscillatory processes* $Z(t)$ *satisfy*

$$Z(t) = \int_{-\infty}^{\infty} A_t(\omega)\, e^{i\omega t}\, d\,\Phi(\omega) = \int_{-\infty}^{\infty} h(t,u)\, X(u)\, d\,u \tag{1}$$

where $A_t(\omega)$ *is a gain function satisfying*

$$A_t(\omega) = A_t^*(-\omega) \tag{2}$$

and $\Phi(\omega)$ *is an orthogonal increment process.*

$$X(u) = \int_{-\infty}^{\infty} e^{i\omega u}\, d\,\Phi(\omega) \tag{3}$$

Proof.

$$\begin{aligned}
Z(t) &= \int_{-\infty}^{\infty} h(t,u)\, X(u)\, d\,u \\
&= \int_{-\infty}^{\infty} h(t,u) \int_{-\infty}^{\infty} e^{i\omega u}\, d\,\Phi(\omega) d\,u \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t,u)\, e^{i\omega u}\, d\,u\, d\,\Phi(\omega) \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_t(\lambda)\, e^{i\lambda(t-u)}\, d\,\lambda\ \ e^{i\omega u}\, d\,u\, d\,\Phi(\omega) \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} A_t(\lambda) \int_{-\infty}^{\infty} e^{i\lambda(t-u)}\, e^{i\omega u}\, d\,u d\,\lambda\, d\,\Phi(\omega) \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} A_t(\lambda)\, e^{i\lambda t} \int_{-\infty}^{\infty} e^{i(\omega-\lambda)u}\, d\,u\, d\,\lambda\, d\,\Phi(\omega) \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} A_t(\lambda)\, e^{i\lambda t}\, 2\pi\, \delta(\omega-\lambda)\, d\,\lambda\, d\,\Phi(\omega) \\
&= \int_{-\infty}^{\infty} A_t(\omega)\, e^{i\omega t}\, d\,\Phi(\omega)
\end{aligned} \tag{4}$$

where the interchanges are justified by quadratic integrability of the time-dependent gain functions $A_t(\lambda)$ with respect to the spectral measure $S(\lambda) = dF(\lambda) \forall t \in \mathbb{R}$ $\square$

Theorem 3. (Eigenfunction Property for Stationary Processes) *Let $R(\tau)$ be a stationary covariance function. Let the corresponding integral coariance operator be defined*

$$(K f)(t) = \int_{-\infty}^{\infty} R(t-s)\, f(s)\, ds \tag{5}$$

then

$$K e^{i\omega t} = S(\omega)\, e^{i\omega t} \tag{6}$$

where the eigenvalue is the correponding element of the spectral density

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau \tag{7}$$

Proof.

$$\begin{aligned}(K f)(t) &= \int_{-\infty}^{\infty} R(t-s)\, e^{i\omega s}\, ds \\ &= \int_{-\infty}^{\infty} R(\tau)\, e^{i\omega(t-\tau)}\, d\tau \\ &= e^{i\omega t} \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau \\ &= S(\omega)\, e^{i\omega t}\end{aligned} \tag{8}$$

$\square$

Theorem 4. (Eigenfunction Property for Oscillatory Processes) *Let*

$$C(s,t) = \int_{-\infty}^{\infty} A_s(\omega)\, A_t^*(\omega)\, d\,F(\omega) \tag{9}$$

and

$$(K\,f)(t) = \int_{-\infty}^{\infty} C(t,s)\; f(s)\, d\,s \tag{10}$$

then the oscillatory functions

$$\phi(t,\omega) = A_t(\omega)\, e^{i\omega t} \tag{11}$$

are eigenfunction of K with eigenvalues $S(\lambda) = d\,F(\omega) \forall \omega$

$$(K\,\phi(\cdot,\omega))(t) = \phi_t(\lambda) S(\lambda) \tag{12}$$

Proof.

$$\begin{aligned}
K\,\phi(\cdot,\omega)(t) \;\; &= \int_{-\infty}^{\infty} C(t,s)\; \phi(s,\omega)\, d\,s \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} A_t(\lambda)\, A_s^*(\lambda)\, d\,F(\lambda) \right) A_s(\omega)\, e^{i\omega s}\, d\,s \\
&= \int_{-\infty}^{\infty} A_t(\lambda) \left[\int_{-\infty}^{\infty} A_s^*(\lambda)\, A_s(\omega)\, e^{i\omega s}\, d\,s \right] d\,F(\lambda) \\
&= \int_{-\infty}^{\infty} A_t(\lambda)\, \delta\,(\lambda - \omega)\, d\,F(\lambda) \\
&= A_t(\omega)\, d\,F(\omega) \\
&= \phi(t,\omega)\, d\,F(\omega)
\end{aligned} \tag{13}$$

□

Lemma 5. (Orthogonality Property)

$$\int_{-\infty}^{\infty} A_s^*(\lambda)\, A_s(\omega)\, e^{i\omega s}\, d\,s = \delta\,(\lambda - \omega)$$

Proof. The orthogonality of $\Phi(\omega)$ is

$$\mathbb{E}\,[d\,\Phi(\lambda)\, d\,\Phi^*(\omega)] = \delta\,(\lambda - \omega)\, d\,F(\lambda).$$

The representation

$$Z(t) = \int_{-\infty}^{\infty} A_t(\omega)\, e^{i\omega t}\, d\,\Phi(\omega)$$

with this covariance property, forces the stated orthogonality among the time-varying modulating amplitudes. □

Theorem 6. (Real-Valued Oscillatory Processes) *The process $Z(t)$ is real-valued if and only if*

$$A_t(\omega) = A_t^*(-\omega) \tag{14}$$

and

$$d\,\Phi(-\omega) = d\,\Phi^*(\omega) \tag{15}$$

Proof. Compute

$$Z^*(t) = \int_{-\infty}^{\infty} A_t^*(\omega)\, e^{-i\omega t}\, d\,\Phi^*(\omega).$$

Set $\omega = -\nu$, so $d\,\omega = -d\,\nu$,

$$Z^*(t) = \int_{+\infty}^{-\infty} A_t^*(-\nu)\, e^{i\nu t}\, d\,\Phi^*(-\nu)\,(-d\,\nu) = \int_{-\infty}^{\infty} A_t^*(-\omega)\, e^{i\omega t}\, d\,\Phi^*(-\omega).$$

For $Z(t)$ to be real-valued,

$$Z(t) = Z^*(t)$$

for all t, so it is necessary that for all ω,

$$A_t(\omega) = A_t^*(-\omega), \qquad d\,\Phi(\omega) = d\,\Phi^*(-\omega).$$

If these hold, then

$$Z^*(t) = \int_{-\infty}^{\infty} A_t^*(-\omega)\, e^{i\omega t}\, d\,\Phi^*(-\omega) = \int_{-\infty}^{\infty} A_t(\omega)\, e^{i\omega t}\, d\,\Phi(\omega) = Z(t).$$ □

Theorem 7. (Eigenfunction Conjugate Pairs) $\phi^*(t,\omega) = \phi(t,-\omega)$.

Proof.

$$\phi^*(t,\omega) = [A_t(\omega)\, e^{i\omega t}]^* = A_t^*(\omega)\, e^{-i\omega t}$$

By the conjugate symmetry property,

$$A_t^*(\omega)\, e^{-i\omega t} = A_t(-\omega)\, e^{-i\omega t} = A_t(-\omega)\, e^{i(-\omega)t} = \phi(t,-\omega)$$ □

Theorem 8. (Filter Kernel: Dual Fourier Formula)

$$h(t,u)=\frac{1}{2\,\pi}\int_{-\infty}^{\infty}A_t(\omega)\,e^{i\omega\,(t-u)}\,d\,\omega=\frac{1}{2\,\pi}\int_{-\infty}^{\infty}\phi(t,\omega)\,e^{-i\omega u}\,d\,\omega$$

Proof. $\frac{1}{2\,\pi}\int_{-\infty}^{\infty}\phi(t,\ \omega)\ e^{-i\omega u}\ d\ \omega\ =\ \frac{1}{2\,\pi}\int_{-\infty}^{\infty}[A_t(\omega)\ e^{i\omega t}]\ e^{-i\omega u}\ d\ \omega\ =\ \frac{1}{2\,\pi}\int_{-\infty}^{\infty}A_t(\omega)\,e^{i\omega\,(t-u)}\,d\,\omega$ □

Theorem 9. (Inverse Relations)

$$A_t(\omega)=\int_{-\infty}^{\infty}h(t,u)\,e^{-i\omega\,(t-u)}\,d\,u \tag{16}$$

$$\phi(t,\omega)=\int_{-\infty}^{\infty}h(t,u)\,e^{-i\omega u}\,d\,u \tag{17}$$

Proof.
$$\int_{-\infty}^{\infty}h(t,u)\,e^{-i\omega\,(t-u)}\,d\,u=\int_{-\infty}^{\infty}\frac{1}{2\,\pi}\int_{-\infty}^{\infty}A_t(\lambda)\,e^{i\lambda\,(t-u)}\,d\,\lambda\ \ e^{-i\omega\,(t-u)}\,d\,u$$

$$=\frac{1}{2\,\pi}\int_{-\infty}^{\infty}A_t(\lambda)\left[\int_{-\infty}^{\infty}e^{i\lambda\,(t-u)}\,e^{-i\omega\,(t-u)}\,d\,u\right]d\,\lambda$$

$$=\frac{1}{2\,\pi}\int_{-\infty}^{\infty}A_t(\lambda)\,e^{i\lambda t}\,e^{-i\omega t}\left[\int_{-\infty}^{\infty}e^{-i\,(\lambda-\omega)\,u}\,d\,u\right]d\,\lambda=\frac{1}{2\,\pi}\int_{-\infty}^{\infty}A_t(\lambda)\,e^{i\lambda t}\,e^{-i\omega t}\,2\,\pi\,\delta\,(\lambda-\omega)\,d\,\lambda$$

$$=\int_{-\infty}^{\infty}A_t(\lambda)\,e^{i\lambda t}\,e^{-i\omega t}\,\delta\,(\lambda-\omega)\,d\,\lambda=A_t(\omega)\,e^{i\omega t}\,e^{-i\omega t}=A_t(\omega)$$

The formula for $\phi(t,\omega)$ is found similarly. □