Proof of Orthonormality of a Certain Sequence of Spherical Bessel Functions over $[0, \infty]$

by Stephen Crowley

Lemma 1

The functions

$$\psi_n(y) = \sqrt{\frac{4\,n+1}{y}}\,(-1)^n\,J_{2n+\frac{1}{2}}(y) \tag{1}$$

are orthonormal over the interval 0 *to* ∞*, i.e.,*

$$\int_0^\infty \psi_j(y)\,\psi_k(y)\;d\,y = \delta_{jk} \tag{2}$$

where δ_{jk} *is the Kronecker delta.*

Proof. Consider the integral

$$I = \int_0^\infty \psi_j(y)\,\psi_k(y)\;d\,y \tag{3}$$

which can be expressed as

$$I = \int_0^\infty \sqrt{\frac{4\,j+1}{y}}\,(-1)^j\,J_{2j+\frac{1}{2}}(y)\sqrt{\frac{4\,k+1}{y}}\,(-1)^k\,J_{2k+\frac{1}{2}}(y)\;d\,y \tag{4}$$

This simplifies to

$$I = \sqrt{(4\,j+1)\,(4\,k+1)}\,(-1)^{j+k}\int_0^\infty \frac{J_{2j+\frac{1}{2}}(y)\,J_{2k+\frac{1}{2}}(y)}{y}\;d\,y \tag{5}$$

Using the orthogonality relation for Bessel functions [1],

$$\int_0^\infty \frac{J_{v+2m+1}(x)\,J_{v+2n+1}(x)}{x}\,dx = \frac{\sin(\pi\,m-\pi\,n)}{2\,\pi\,(v+m+n+1)\,(m-n)} = \frac{\delta_{m,n}}{2(2n+v+1)} \tag{6}$$

where $\delta_{m,n} = \begin{cases} 1 & m=n \\ 0 & \text{otherwise} \end{cases}$ is the Kronecker delta, it can be seen that the limit when $m=n$ is given by

$$\lim_{m\to n}\frac{\sin(\pi\,m-\pi\,n)}{2\,\pi\,(v+m+n+1)\,(m-n)} = \frac{1}{2(2n+v+1)} \tag{7}$$

and the value of Equation (6) when $m \neq n$ is zero due to the numerator being the Kronecker delta implemented as a sine function in this instance.

Letting $v = -\frac{1}{2}$ and substituting this result back, we have

$$\begin{aligned} I &= \sqrt{(4j+1)(4k+1)}\,(-1)^{j+k} \frac{\delta_{j,k}}{2\left(2k+\frac{1}{2}\right)} \\ &= \sqrt{(4j+1)(4k+1)}\,(-1)^{j+k} \frac{\delta_{j,k}}{4k+1} \end{aligned} \tag{8}$$

For $j \neq k$, $\delta_{jk} = 0$, yielding $I = 0$. For $j = k$, $\delta_{jk} = 1$ and $(-1)^{2k} = 1 \forall k \in \mathbb{Z}$ giving

$$I = \frac{\sqrt{(4j+1)(4j+1)}}{4j+1} = 1 \tag{9}$$

Hence, $\psi_j(y)$ and $\psi_k(y)$ are orthonormal. □