# Unitary Time Changes of Stationary Processes Yield Oscillatory Processes and a Functional Framework Toward a Hilbert–Pólya Construction

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# 1 Unitary Time Change on $L^2(\mathbb{R})$

**Definition 1.** *[Unitary time change operator on $L^2(\mathbb{R})$] Let $\theta: \mathbb{R} \to \mathbb{R}$ be absolutely continuous with $\theta'(t) \neq 0$ almost everywhere. Define $U_\theta: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by*

$$(U_\theta f)(t) := \sqrt{|\theta'(t)|}\ f(\theta(t)) \qquad \forall f \in L^2(\mathbb{R}) \tag{1}$$

**Theorem 2.** *[Unitarity of $U_\theta$] $U_\theta$ is unitary on $L^2(\mathbb{R})$.*

**Proof.** By absolute continuity and $\theta'(t) \neq 0$ a.e., the change-of-variables formula gives

$$\int_{\mathbb{R}} |(U_\theta f)(t)|^2 dt = \int_{\mathbb{R}} |\theta'(t)||f(\theta(t))|^2 dt = \int_{\mathbb{R}} |f(u)|^2 du \tag{2}$$

so $U_\theta$ is an isometry. Since $\theta$ admits an a.e. inverse $\theta^{-1}$ with the same regularity and non-vanishing derivative a.e., one has $U_{\theta^{-1}} U_\theta = \mathrm{Id}$ and $U_\theta U_{\theta^{-1}} = \mathrm{Id}$ a.e., hence $U_\theta$ is unitary. $\square$

# 2 Oscillatory Processes in the Sense of Priestley

**Definition 3.** *[Oscillatory process, gain and oscillatory function] Let $F$ be a finite non-negative Borel measure on $\mathbb{R}$. For each $t \in \mathbb{R}$ let $A_t: \mathbb{R} \to \mathbb{C}$ be measurable and square-integrable with respect to $F$. Define*

$$\varphi_t(\lambda) := A_t(\lambda)\ e^{i\lambda t} \tag{3}$$

*An* oscillatory process *$Z$ is a stochastic process with spectral representation*

$$Z(t) := \int_{\mathbb{R}} \varphi_t(\lambda)\ \Phi\,(d\,\lambda) = \int_{\mathbb{R}} A_t(\lambda)\ e^{i\lambda t}\ \Phi\,(d\,\lambda) \tag{4}$$

*where $\Phi$ is a complex orthogonal random measure with spectral measure $F$ satisfying the orthogonality of infinitesimal increments*

$$\mathbb{E}[\Phi\,(d\,\lambda)\,\overline{\Phi\,(d\,\mu)}] = \delta\,(\lambda - \mu)\ d\,F(\lambda) \tag{5}$$

*The covariance kernel is*

$$R_Z(t,s) := \mathbb{E}[Z(t)\,\overline{Z(s)}] = \int_{\mathbb{R}} A_t(\lambda)\,\overline{A_s(\lambda)}\,e^{i\lambda(t-s)}\ d\,F(\lambda) \tag{6}$$

**Remark 4.** [Real-valuedness] $Z$ is real-valued if and only if, for each fixed $t$, $A_t\,(-\lambda) = \overline{A_t(\lambda)}$ for $F$-a.e. $\lambda$, equivalently $\varphi_t\,(-\lambda) = \overline{\varphi_t(\lambda)}$ for $F$-a.e. $\lambda$.

**Theorem 5.** *[Existence of oscillatory processes with prescribed $(A_t)_t$] Let $F$ be finite and $(A_t)_t$ measurable with $\int |A_t(\lambda)|^2\, d\,F(\lambda) < \infty$ for each $t$. There exists a complex orthogonal random measure $\Phi$ on $\mathbb{R}$ with spectral measure $F$ such that $Z(t) = \int \varphi_t(\lambda)\ \Phi\,(d\,\lambda)$ is well-defined in $L^2(\Omega)$ and has covariance*

$$R_Z(t,s) = \int_{\mathbb{R}} \varphi_t(\lambda)\,\overline{\varphi_s(\lambda)}\,d\,F(\lambda) = \int_{\mathbb{R}} A_t(\lambda)\,\overline{A_s(\lambda)}\,e^{i\lambda(t-s)}\ d\,F(\lambda) \tag{7}$$

**Proof.** Construct the stochastic integral first for simple functions in $L^2\,(\mathbb{R}, F)$ and extend by isometry using

$$\mathbb{E}\left[\left|\int g(\lambda)\ \Phi\,(d\,\lambda)\right|^2\right] = \int |g(\lambda)|^2\ d\,F(\lambda) \tag{8}$$

. Apply with $g = \varphi_t$ to obtain $Z(t)$ and the stated covariance. □

# 3 Unitary Time Changes Map Stationary to Oscillatory

**Definition 6.** *[Stationary process via Cramér representation] A zero-mean stationary process $X$ with spectral measure $F$ admits*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\ \Phi\,(d\,\lambda) \tag{9}$$

*with covariance*

$$R_X\,(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)}\ d\,F(\lambda) \tag{10}$$

**Theorem 7.** *[Unitary time change yields an oscillatory process] Let $X$ be zero-mean stationary with*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\ \Phi\,(d\,\lambda) \tag{11}$$

*Let $\theta$ satisfy the hypotheses of the unitary time change and set*

$$Z(t) := (U_\theta\, X)(t) = \sqrt{|\theta'(t)|}\ X(\theta(t)) \tag{12}$$

*Then $Z$ is an oscillatory process with oscillatory function*

$$\varphi_t(\lambda) = \sqrt{|\theta'(t)|}\ e^{i\lambda\theta(t)} \tag{13}$$

*and gain*

$$A_t(\lambda) = \sqrt{|\theta'(t)|}\ e^{i\lambda(\theta(t)-t)} \tag{14}$$

*The covariance is*

$$R_Z(t,s) = \int_{\mathbb{R}} A_t(\lambda)\,\overline{A_s(\lambda)}\,e^{i\lambda(t-s)}\ d\,F(\lambda) = \int_{\mathbb{R}} \sqrt{|\theta'(t)\,\theta'(s)|}\ e^{i\lambda(\theta(t)-\theta(s))}\ d\,F(\lambda \tag{15}$$

**Proof.** Compute

$$Z(t) = \sqrt{|\theta'(t)|}\ X(\theta(t)) = \sqrt{|\theta'(t)|}\int_{\mathbb{R}} e^{i\lambda\theta(t)}\ \Phi\,(d\,\lambda) = \int_{\mathbb{R}} \sqrt{|\theta'(t)|}\ e^{i\lambda\theta(t)}\ \Phi\,(d\,\lambda) \tag{16}$$

Thus

$$\varphi_t(\lambda) = \sqrt{|\theta'(t)|}\ e^{i\lambda\theta(t)} \tag{17}$$

and

$$A_t(\lambda) = \varphi_t(\lambda)\,e^{-i\lambda t} \tag{18}$$

The covariance follows from orthogonality of $\Phi$. □

**Remark 8.** [Real-valuedness under time change] If $X$ is real-valued and $\theta$ is real with $\theta'(t) > 0$ a.e., then $Z$ is real-valued by the Hermitian symmetry of $A_t$.

# 4 Zero Localization by a Functional Measure

**Definition 9.** *[Zero localization measure] Let $Z$ be real-valued, with sample paths in $C^1(\mathbb{R})$, and such that every zero of $Z$ is simple (i.e. $Z(t_0) = 0 \Longrightarrow Z'(t_0) \neq 0$). Define the measure on Borel $B \subset \mathbb{R}$ by*

$$\mu(B) := \int_{\mathbb{R}} 1_B(t)\ \delta(Z(t))\, |Z'(t)|\ dt \tag{19}$$

**Theorem 10.** *[Support and mass on the zero set] For any test function $\phi \in C_c^\infty(\mathbb{R})$,*

$$\int_{\mathbb{R}} \phi(t)\ \delta(Z(t))\, |Z'(t)|\ dt = \sum_{t_0 : Z(t_0) = 0} \phi(t_0), \tag{20}$$

*and hence*

$$\mu = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0} \tag{21}$$

*is a discrete measure assigning unit mass to each simple zero of $Z$.*

**Proof.** At a simple zero $t_0$, the distributional identity holds:

$$\delta(Z(t)) = \frac{\delta\,(t - t_0)}{|Z'(t_0)|} + \sum_{t_1 \neq t_0 : Z(t_1) = 0} \frac{\delta\,(t - t_1)}{|Z'(t_1)|} \tag{22}$$

Multiplying by $|Z'(t)|$ and integrating against $\phi$ yields the stated identity and the atomic form of $\mu$. □

# 5 Hilbert Space on the Zero Set and Multiplication Operator

**Definition 11.** *[Hilbert space on the zero set via $\mu$] Define*

$$\mathcal{H} := L^2(\mu) = \left\{ f : \mathbb{R} \to \mathbb{C} : \|f\|_{\mathcal{H}}^2 = \int |f(t)|^2\ \delta(Z(t))\, |Z'(t)|\ dt < \infty \right\} \tag{23}$$

*The inner product is*

$$\langle f, g \rangle = \int f(t) \overline{g(t)}\, \delta(Z(t))\, |Z'(t)|\ dt \tag{24}$$

**Proposition 12.** *[Atomic structure] With* $\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0}$, *one has*

$$\mathcal{H} = \left\{ f: \{t_0: Z(t_0)=0\} \to \mathbb{C}: \sum_{Z(t_0)=0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \tag{25}$$

*and the functions* $e_{t_0}$ *defined by*

$$e_{t_0}(t_1) = \delta_{t_0 t_1} \tag{26}$$

*form an orthonormal basis.*

**Proof.** Substitute the atomic form of $\mu$ into the $L^2$-definition to obtain the $\ell^2$-structure; the canonical coordinate functions form an ONB. □

**Definition 13.** *[Multiplication operator] Define* $L: \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H}$ *by*

$$(L f)(t) = t\ f(t) \tag{27}$$

*on* $\sup(\mu)$, *with*

$$\mathcal{D}(L) = \left\{ f \in \mathcal{H}: \int |t\ f(t)|^2\ \delta(Z(t))\,|Z'(t)|\ dt < \infty \right\} \tag{28}$$

**Theorem 14.** *[Self-adjointness and spectrum]* $L$ *is self-adjoint on* $\mathcal{H}$, *and its spectrum is exactly*

$$\sigma(L) = \{\, t \in \mathbb{R}: Z(t) = 0 \,\} \tag{29}$$

*with pure point spectrum consisting of simple eigenvalues* $\lambda = t_0$ *(for each zero* $t_0$*) and eigenvectors* $e_{t_0}$.

**Proof.** For $f, g \in \mathcal{D}(L)$,

$$\langle L f, g \rangle = \int t\ f(t)\,\overline{g(t)}\,\delta(Z(t))\,|Z'(t)|\ dt = \int f(t)\,\overline{t\ g(t)}\,\delta(Z(t))\,|Z'(t)|\ dt = \langle f, L g \rangle \tag{30}$$

so $L$ is symmetric. On the atomic space, $L$ is unitarily equivalent to the diagonal operator $(c_{t_0}) \mapsto (t_0\, c_{t_0})$ on $\ell^2$, which is self-adjoint with spectrum equal to the set of diagonal entries $\{t_0: Z(t_0) = 0\}$, each simple, with eigenvectors the coordinate basis identified with $e_{t_0}$. □

# 6 Time-Changed Stationary Processes and a Hilbert–Pólya Scaffold

**Definition 15.** *[Arithmetic phase time change] Let* $\theta: \mathbb{R} \to \mathbb{R}$ *be an absolutely continuous phase with* $\theta'(t) > 0$ *a.e. encoding the target arithmetic structure (e.g. a Riemann–Siegel-type phase). Let* $X$ *be zero-mean stationary with spectral measure* $F$ *and orthogonal random measure* $\Phi$. *Define the time-changed oscillatory process*

$$Z(t) = \int_{\mathbb{R}} \sqrt{|\theta'(t)|}\ e^{i\lambda\theta(t)}\ \Phi(d\lambda) \tag{31}$$

**Definition 16.** *[Zero-localized Hilbert space and operator] With the zero localization measure $\mu(dt) = \delta(Z(t))\,|Z'(t)|\,dt$, define $\mathcal{H} = L^2(\mu)$ and $L$ as multiplication by $t$ on $\mathcal{H}$.*

**Theorem 17.** *[Spectral encoding of zero set] The spectrum of $L$ is the zero set of $Z$:*

$$\sigma(L) = \{t: Z(t) = 0\},$$

*and $L$ has simple pure point spectrum with eigenvectors supported at individual zeros.*

**Proof.** Follows from the established atomic structure of $\mu$ and the diagonal form of $L$ on $L^2(\mu)$. $\square$

**Remark 18.** [Operator scaffold] The sequence

$$\text{stationary } X \xrightarrow{U_\theta} \text{oscillatory } Z \xrightarrow{\delta(Z)\,|Z'|\,dt} \mu \xrightarrow{L^2(\mu)} \mathcal{H} \xrightarrow{t\cdot} L \tag{32}$$

produces a concrete self-adjoint operator whose spectrum equals the (constructed) zero set governed by the choice of $\theta$ and $F$. Aligning $\theta$ and $F$ to a prescribed arithmetic target sets the stage for a Hilbert–Pólya-type identification.

# 7 Appendix: Regularity and Simple Zeros

**Definition 19.** *[Regularity and simplicity] Assume $Z \in C^1(\mathbb{R})$ and every zero of $Z$ is simple: $Z(t_0) = 0 \Longrightarrow Z'(t_0) \neq 0$.*

**Lemma 20.** *[Local finiteness and decomposition] Under the above condition, zeros are locally finite and the distributional identity*

$$\delta(Z(t)) = \sum_{t_0: Z(t_0) = 0} \frac{\delta(t - t_0)}{|Z'(t_0)|} \tag{33}$$

*holds, yielding $\mu = \sum_{t_0} \delta_{t_0}$.*

**Proof.** Continuity and $Z'(t_0) \neq 0$ imply isolated zeros by the inverse function theorem; the distributional identity is standard from the one-dimensional change-of-variables formula for the Dirac delta under monotone $C^1$ maps near each zero. $\square$