Measure-Preserving Bijective Time Changes of Stationary Gaussian Processes Generates A Subclass of Oscillatory Processes

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August 1, 2025

Abstract

This article establishes that Gaussian processes obtained through measure-preserving bijective unitary time transformations of stationary processes constitute a subclass of oscillatory processes in the sense of Priestley[1]. The transformation $Z(t) = \sqrt{\dot{\theta}(t)} X(\theta(t))$, where $X(t)$ is a realization of stationary Gaussian process and θ is a strictly increasing C^1 differentiable monotonic function, yields an oscillatory process with evolutionary power spectrum $dF_t(\omega) = \dot{\theta}(t) d\mu(\omega)$. An explicit unitary transformation between the input stationary process and the transformed oscillatory process is established, preserving the L^2-norm and providing a complete spectral characterization.

Contents

1 Scaling Functions

Definition 1 (Scaling Functions) *Let $\mathcal{F}$ denote the set of functions $\theta : \mathbb{R} \to \mathbb{R}$ satisfying*

1. *θ is absolutely continuous with*

$$\dot{\theta}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\theta(t) \geq 0 \tag{1}$$

almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero

2. *θ is strictly increasing and bijective.*

Remark 1 The conditions in Definition 1 ensure that θ^{-1} exists and is absolutely continuous. By the inverse function theorem for absolutely continuous functions,

$$\frac{\mathrm{d}}{\mathrm{d}s}(\theta^{-1})(s) = \frac{1}{\dot{\theta}(\theta^{-1}(s))} \tag{2}$$

for almost all s in the range of θ. The condition that $\dot{\theta}(t) = 0$ only on sets of measure zero ensures that $\frac{1}{\dot{\theta}(\theta^{-1}(s))}$ is well-defined almost everywhere.

2 Oscillatory Processes

Definition 2 (Oscillatory Process) *A complex-valued, second-order process $\{X(t)\}_{t\in\mathbb{R}}$ is called* oscillatory *if there exist*

1. *a family of oscillatory basis functions $\{\phi_t(\omega)\}_{t\in\mathbb{R}}$ with*

$$\phi_t(\omega) = A_t(\omega)e^{i\omega t} \tag{3}$$

and a given gain function

$$A_t(\cdot) \in L^2(\mu) \tag{4}$$

2. *and a complex orthogonal random measure $\Phi(\omega)$ with*

$$E|d\Phi(\omega)|^2 = d\mu(\omega) = S(\omega) \tag{5}$$

such that

$$\begin{aligned} Z(t) &= \int_{-\infty}^{\infty} \phi_t(\omega)\, d\Phi(\omega) \\ &= \int_{-\infty}^{\infty} A_t(\omega)e^{i\omega t} d\Phi(\omega) \end{aligned} \tag{6}$$

All stationary processes are oscillatory with $A_t(\omega) = 1$

3 Stationary Reference Process

Let $\{X(t)\}_{t\in\mathbb{R}}$ be a stationary Gaussian process with continuous spectral representation

$$X(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, d\Phi(\omega) \tag{7}$$

where $\Phi(\omega)$ is an orthogonal-increment process with spectral density

$$E|d\Phi(\omega)|^2 = d\mu(\omega) = S(\omega) = \langle \text{fourier transform of } K_X \rangle \tag{8}$$

and μ is a finite measure on $\mathbb{R}$.

4 Time-Changed Process

4.1 Definition and Unitary Operator

Definition 3 (Unitary Time-Change Operator) *For $\theta \in \mathcal{F}$, define the operator $M_\theta : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by*

$$(M_\theta f)(t) = \sqrt{\dot{\theta}(t)}\, f(\theta(t)) \tag{9}$$

Definition 4 (Unitarily Time-Changed Stationary Process) *For $\theta \in \mathcal{F}$, apply the unitary time change operator M_θ from Definition 3 to a realization of a stationary process $X(t)$ from the ensemble $\{X(t)\}$ to define a realization of the unitarily time-changed process*

$$Z(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) \forall t \in \mathbb{R} \tag{10}$$

Definition 5 (Inverse Unitary Time-Change Operator) *The inverse operator $M_\theta^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ corresponding to the unitary time-change operator $(M_\theta f)(t)$ defined in Equation 9 is given by*

$$(M_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{11}$$

Lemma 1 (Well-Definedness of Inverse Operator) *The operator M_θ^{-1} in Definition 5 is well-defined $\forall \theta \in \mathcal{F}$.*

Proof Since $\dot{\theta}(t) = 0$ only on sets of measure zero by Definition 1, and θ^{-1} maps sets of measure zero to sets of measure zero (as it preserves absolute continuity), the denominator $\sqrt{\dot{\theta}(\theta^{-1}(s))}$ is positive almost everywhere. The expression in equation (11) is therefore well-defined almost everywhere, which is sufficient for defining an element of $L^2(\mathbb{R})$. □

Theorem 1 (Unitarity of Transformation Operator) *The operator M_θ defined in equation* (9) *is unitary, i.e.,*

$$\int_{\mathbb{R}} |(M_\theta f)(t)|^2 \, dt = \int_{\mathbb{R}} |f(s)|^2 \, ds \forall f \in L^2(\mathbb{R}) \tag{12}$$

Proof Let $f \in L^2(\mathbb{R})$. The L^2-norm of $M_\theta f$ is computed as follows:

$$\int_{\mathbb{R}} |(M_\theta f)(t)|^2 \, dt = \int_{\mathbb{R}} \left| \sqrt{\dot{\theta}(t)}\, f(\theta(t)) \right|^2 \, dt \tag{13}$$

$$= \int_{\mathbb{R}} \dot{\theta}(t) \, |f(\theta(t))|^2 \, dt \tag{14}$$

Apply the change of variables $s = \theta(t)$. Since θ is absolutely continuous and strictly increasing, its Jacobian is given by

$$ds = \dot{\theta}(t) \, dt \tag{15}$$

almost everywhere. As t ranges over $\mathbb{R}$, $s = \theta(t)$ ranges over $\mathbb{R}$ due to the bijectivity of θ. Therefore:

$$\int_{\mathbb{R}} \dot{\theta}(t) \, |f(\theta(t))|^2 \, dt = \int_{\mathbb{R}} |f(s)|^2 \, ds \tag{16}$$

This establishes equation (12). To complete the proof of unitarity, it remains to show that M_θ^{-1} is indeed the inverse of M_θ. For any $f \in L^2(\mathbb{R})$:

$$\begin{aligned}
(M_\theta^{-1} M_\theta f)(s) &= (M_\theta^{-1}) \left[\sqrt{\dot{\theta}(\cdot)} \, f(\theta(\cdot)) \right] (s) && (17)\\
&= \frac{\sqrt{\dot{\theta}(\theta^{-1}(s))} \, f(\theta(\theta^{-1}(s)))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} && (18)\\
&= f(s) && (19)
\end{aligned}$$

where the last equality uses $\theta(\theta^{-1}(s)) = s$. Similarly, for any $g \in L^2(\mathbb{R})$:

$$\begin{aligned}
(M_\theta M_\theta^{-1} g)(t) &= \sqrt{\dot{\theta}(t)} \, (M_\theta^{-1} g)(\theta(t)) && (20)\\
&= \sqrt{\dot{\theta}(t)} \, \frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} && (21)\\
&= \sqrt{\dot{\theta}(t)} \, \frac{g(t)}{\sqrt{\dot{\theta}(t)}} && (22)\\
&= g(t) && (23)
\end{aligned}$$

Therefore

$$M_\theta M_\theta^{-1} = M_\theta^{-1} M_\theta = I \tag{24}$$

proving that M_θ is unitary. □

Corollary 1 (Measure Preservation) *The transformation M_θ preserves the L^2-measure in the sense that for any measurable set $A \subseteq \mathbb{R}$*

$$\int_A |(M_\theta f)(t)|^2 \, dt = \int_{\theta(A)} |f(s)|^2 \, ds \tag{25}$$

Proof The proof follows the same change of variables argument as in Theorem 1, applied to the characteristic function of the set A. □

4.2 L^2-Norm Preservation

Theorem 2 (Measure Preservation) *The transformation defined in equation* (10) *preserves the L^2-norm in the sense that*

$$\int_I \operatorname{var}(Z(t)) \, dt = \int_{\theta(I)} \operatorname{var}(X(s)) \, ds \tag{26}$$

for any measurable set $I \subseteq \mathbb{R}$.

Proof Using the change of variables $s = \theta(t)$ with $ds = \dot{\theta}(t)\,dt$:

$$\int_I \operatorname{var}(Z(t))\,dt = \int_I \operatorname{var}\left(\sqrt{\dot{\theta}(t)}\,X(\theta(t))\right)\,dt \tag{27}$$

$$= \int_I \dot{\theta}(t)\operatorname{var}(X(\theta(t)))\,dt \tag{28}$$

$$= \int_{\theta(I)} \operatorname{var}(X(s))\,ds \tag{29}$$

□

4.3 Oscillatory Representation

Theorem 3 (Oscillatory Form) *The process* $\{Z(t)\}$ *defined in equation* (10) *is oscillatory with oscillatory functions*

$$\phi_t(\omega) = A_t(\omega)e^{i\omega t} = \sqrt{\dot{\theta}(t)}\,e^{i\omega\theta(t)} \tag{30}$$

and gain functions

$$A_t(\omega) = \sqrt{\dot{\theta}(t)}\,e^{i\omega(\theta(t)-t)} \tag{31}$$

Proof From the spectral representation (7) of the stationary process $X(t)$:

$$Z(t) = \sqrt{\dot{\theta}(t)}X(\theta(t)) \tag{32}$$

$$= \sqrt{\dot{\theta}(t)}\int_{-\infty}^{\infty} e^{i\omega\theta(t)}\,d\Phi(\omega) \tag{33}$$

$$= \int_{-\infty}^{\infty} \sqrt{\dot{\theta}(t)}\,e^{i\omega\theta(t)}\,d\Phi(\omega) \tag{34}$$

$$= \int_{-\infty}^{\infty} \phi_t(\omega)\,d\Phi(\omega) \tag{35}$$

where

$$\phi_t(\omega) = \sqrt{\dot{\theta}(t)}\,e^{i\omega\theta(t)} \tag{36}$$

To verify this is an oscillatory representation according to Definition 2, express $\phi_t(\omega)$ in the form of a function of the time-dependent gain $A_t(\omega)$ as required

$$\begin{aligned}\phi_t(\omega) &= A_t(\omega)e^{i\omega t}\\ &= \sqrt{\dot{\theta}(t)}\,e^{i\omega(\theta(t)-t)}\,e^{i\omega t}\\ &= \sqrt{\dot{\theta}(t)}e^{i\omega(\theta(t)-t+t)}\\ &= \sqrt{\dot{\theta}(t)}\,e^{i\omega\theta(t)}\end{aligned} \tag{37}$$

where

$$A_t(\omega) = \sqrt{\dot{\theta}(t)}\,e^{i\omega(\theta(t)-t)} \tag{38}$$

Since $\dot{\theta}(t) \geq 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of measure zero, the function $A_t(\omega)$ is well-defined almost everywhere. Moreover, $A_t(\cdot) \in L^2(\mu)$ for each t since:

$$\int_{-\infty}^{\infty} |A_t(\omega)|^2 \, d\mu(\omega) = \int_{-\infty}^{\infty} \dot{\theta}(t) \, d\mu(\omega) \tag{39}$$

$$= \dot{\theta}(t) \int_{-\infty}^{\infty} d\mu(\omega) \tag{40}$$

$$= \dot{\theta}(t)\mu(\mathbb{R}) < \infty \tag{41}$$

where the finiteness follows from μ being a finite measure and $\dot{\theta}(t)$ being finite almost everywhere. □

4.4 Envelope and Evolutionary Spectrum

Corollary 2 (Evolutionary Spectrum) *The evolutionary power spectrum is*

$$\begin{aligned} dF_t(\omega) &= |A_t(\omega)|^2 \, d\mu(\omega) \\ &= \dot{\theta}(t) \, d\mu(\omega) \end{aligned} \tag{42}$$

Proof By Definition 2 and the envelope from Equation 4, the evolutionary power spectrum is:

$$dF_t(\omega) = |A_t(\omega)|^2 \, d\mu(\omega) \tag{43}$$

$$= \left| \sqrt{\dot{\theta}(t)} \, e^{i\omega(\theta(t)-t)} \right|^2 \, d\mu(\omega) \tag{44}$$

$$= \dot{\theta}(t) \, |e^{i\omega(\theta(t)-t)}|^2 \, d\mu(\omega) \tag{45}$$

$$= \dot{\theta}(t) \, d\mu(\omega) \tag{46}$$

since

$$|e^{i\alpha}| = 1 \forall \alpha \in \mathbb{R} \tag{47}$$

□

5 Operator Conjugation

Theorem 4 (Operator Conjugation) *Let T_K be the integral covariance operator defined by*

$$(T_K f)(t) = \int_{-\infty}^{\infty} K(|t-s|) f(s) \, ds \tag{48}$$

where $K(h)$ is the stationary kernel

$$K(h) = \int_{-\infty}^{\infty} S(\lambda) e^{i\lambda h} \mathrm{d}\lambda \tag{49}$$

, and let T_{K_θ} be the integral covariance operator defined by

$$\begin{aligned} (T_{K_\theta} f)(t) &= \int_{-\infty}^{\infty} K_\theta(s,t) f(s) \, ds \\ &= \int_{-\infty}^{\infty} K(|\theta(t) - \theta(s)|) \sqrt{\dot{\theta}(t)\dot{\theta}(s)} f(s) \, ds \end{aligned} \tag{50}$$

for the unitarily time-changed kernel

$$K_\theta(s,t) = K(|\theta(t) - \theta(s)|)\sqrt{\dot\theta(t)\dot\theta(s)} \tag{51}$$

. Then

$$T_{K_\theta} = M_\theta T_K M_\theta^{-1} \tag{52}$$

Proof For any $g \in L^2(\mathbb{R})$, compute $(M_\theta T_K M_\theta^{-1} g)(t)$:

$$(M_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot\theta(\theta^{-1}(s))}}, \tag{53}$$

$$(T_K M_\theta^{-1} g)(t) = \int_{-\infty}^{\infty} K(|t-s|)\frac{g(\theta^{-1}(s))}{\sqrt{\dot\theta(\theta^{-1}(s))}} ds. \tag{54}$$

Apply the change of variables $u = \theta^{-1}(s)$, so $s = \theta(u)$ and $ds = \dot\theta(u)du$:

$$(T_K M_\theta^{-1} g)(t) = \int_{-\infty}^{\infty} K(|t-\theta(u)|)\frac{g(u)}{\sqrt{\dot\theta(u)}}\dot\theta(u)du \tag{55}$$

$$= \int_{-\infty}^{\infty} K(|t-\theta(u)|)g(u)\sqrt{\dot\theta(u)}du. \tag{56}$$

Now apply M_θ:

$$(M_\theta T_K M_\theta^{-1} g)(t) = \sqrt{\dot\theta(t)}(T_K M_\theta^{-1} g)(\theta(t)) \tag{57}$$

$$= \sqrt{\dot\theta(t)}\int_{-\infty}^{\infty} K(|\theta(t)-\theta(u)|)g(u)\sqrt{\dot\theta(u)}du \tag{58}$$

$$= \int_{-\infty}^{\infty} K(|\theta(t)-\theta(u)|)\sqrt{\dot\theta(t)\dot\theta(u)}g(u)du. \tag{59}$$

Finally, apply the change of variables $s = u$ (since the integration variable appears as u in the transformed expression):

$$(M_\theta T_K M_\theta^{-1} g)(t) = \int_{-\infty}^{\infty} K(|\theta(t)-\theta(s)|)\sqrt{\dot\theta(t)\dot\theta(s)}g(s)ds \tag{60}$$

$$= (T_{K_\theta} g)(t) \tag{61}$$

This establishes the conjugation relation (52). □

6 Expected Zero Count

Theorem 5 (Expected Zero-Counting Function) *Let $\theta \in \mathcal{F}$ and let*

$$K(\tau) = \operatorname{cov}(X(t), X(t+\tau)) \tag{62}$$

be twice differentiable at $\tau = 0$. The expected number of zeros of the process Z_t in $[a,b]$ is

$$\mathbb{E}[N_{[a,b]}] = \sqrt{-\ddot K(0)}\,(\theta(b) - \theta(a)) \tag{63}$$

Proof The covariance function of the time-changed process is

$$K_\theta(s,t) = \operatorname{cov}(Z_s, Z_t) = \sqrt{\dot\theta(s)\dot\theta(t)}\, K(|\theta(t) - \theta(s)|) \tag{64}$$

For the zero-crossing analysis, consider the normalized process. By the Kac-Rice formula:

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{-\lim_{s\to t} \frac{\partial^2}{\partial s \partial t} K_\theta(s,t)\, dt} \tag{65}$$

Computing the mixed partial derivative:

$$\frac{\partial}{\partial t} K_\theta(s,t) = \frac{1}{2} \frac{\ddot\theta(t)}{\sqrt{\dot\theta(t)}} \sqrt{\dot\theta(s)} K(|\theta(t) - \theta(s)|) \tag{66}$$

$$+ \sqrt{\dot\theta(s)\dot\theta(t)} \dot K(|\theta(t) - \theta(s)|) \operatorname{sgn}(\theta(t) - \theta(s)) \dot\theta(t). \tag{67}$$

Taking the limit as $s \to t$ and using the fact that $\dot K(0) = 0$ for stationary processes:

$$\lim_{s\to t} \frac{\partial^2}{\partial s \partial t} K_\theta(s,t) = \dot\theta(s)\dot\theta(t) \ddot K(0) \tag{68}$$

$$= \dot\theta(t)^2 \ddot K(0) \tag{69}$$

Substituting into the Kac-Rice formula:

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{-\dot\theta(t)^2 \ddot K(0)}\, dt \tag{70}$$

$$= \sqrt{-\ddot K(0)} \int_a^b \dot\theta(t)\, dt \tag{71}$$

$$= \sqrt{-\ddot K(0)}\, (\theta(b) - \theta(a)) \tag{72}$$

Here the second equality uses $\dot\theta(t) \geq 0$ almost everywhere. □

7 Conclusion

This analysis establishes that Gaussian processes generated by measure-preserving bijective time changes of stationary processes form a well-defined subclass of oscillatory processes. The key contributions include:

1. The rigorous construction of the unitary operator M_θ and its inverse, with proper treatment of the case where $\dot\theta(t) = 0$ on sets of measure zero.
2. The explicit oscillatory representation with envelope function $A_t(\omega) = \sqrt{\dot\theta(t)} e^{i\omega(\theta(t)-t)}$.
3. The evolutionary power spectrum formula $dF_t(\omega) = \dot\theta(t) d\mu(\omega)$.
4. The operator conjugation relationship $T_{K_\theta} = M_\theta T_K M_\theta^{-1}$.
5. A closed-form expression for the expected zero count in terms of the range of the time transformation.

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