Theorem 1

(Stone [?] and von Neumann [?]) *Let $U(t)$ and $V(t)$ be (continuous) unitary representations of $\mathbb{R}^d$ on $\mathcal{H}$ satisfying the Weyl relations (3). Then there is a Hilbert space $\mathbb{K}$ and a unitary map $T\colon \mathcal{H} \to L^2(\mathbb{R}^d, \mathbb{K})$ that transforms $U(t)$ and $V(t)$ to the Schrödinger representation. The representation is irreducible if and only if $\mathbb{K}$ is one dimensional.*

For systems with infinitely many degrees of freedom, the analog of the Schrödinger representation is called the Fock representation (see, for instance, Streater and Wightman [?]). However, there are infinitely many other inequivalent irreducible representations as well (Gårding and Wightman [?]) and according to a theorem of Haag (see Streater and Wightman [?]) these cannot be avoided in nontrivial field theories. As mentioned earlier, the maps T implementing other representations of the Weyl relations are related to Fourier integral operators.

Mackey [?] has given an important reformulation of the Stone–von Neumann theorem. One represents the position observables by orthogonal projections P_E in Hilbert space $\mathcal{H}$ for any (Borel) set $E \subseteq Q$, where Q represents position space. One requires $E \mapsto P_E$ to be a (projection-valued) measure. (For $Q = \mathbb{R}^3$, an example of these are the spectral projections associated with the usual position operators, i.e., with $\mathcal{H} = L^2(\mathbb{R}^3)$,

$$P_E \psi = \chi_E \psi \tag{1}$$

where χ_E is the characteristic function of $E \subseteq \mathbb{R}^3$.) If a group G acts on Q, the momentum observables will arise as a representation $U(g)$ of G on $\mathcal{H}$. (For example, if $G = \mathbb{R}^3 = Q$, we obtain $U(g)$ as described earlier.) The position and momentum are linked by

$$U(g)\, P_E U(g)^{-1} = P_{g \cdot E} \tag{2}$$

where $g \cdot E$ is the translate of E under g in the given action. Equations (5) are an abstract form of the Weyl relations (3) [or the Heisenberg relations (1)]. One calls a projection-valued measure and a representation satisfying (5) a system of imprimitivity. Mackey then proves a general result of which the Stone–von Neumann theorem is a special case.

Besides $G = \mathbb{R}^d$, one wishes to take the Euclidean group for G and still impose (5). This leads to what is referred to as the Mackey–Wightman system.analysis. Since one should only work with expectation values, one should only require $U(g)$ to be a projective representation. As Bargmann has shown, we can then adjust things so that we have a true representation of the covering group $\tilde{G} = \mathbb{R}^3 \times SU(2)$. Mackey and Wightman then use the generalized Stone–von Neumann theorem to show that if we have a system of imprimitivity based on $\mathbb{R}^3$ for $\tilde{G}$, then it is unitarily equivalent to the system,

$$P_E f = \chi_E f \tag{3}$$

and

$$(U_{(a,A)} f)(x) = D_A f\,(A^{-1}(x - a)) \tag{4}$$

on $L^2(\mathbb{R}^3, \mathcal{K})$, where $a \in \mathbb{R}^3$, $A \in SU(2)$ (which by projection to $SO(3)$, acts on $\mathbb{R}^3$), and D_A is a unitary representation of $SU(2)$ on $\mathcal{K}$.

Thus the unitary representations of $SU(2)$ classify Euclidean invariant systems. In quantum mechanics texts, the irreducible unitary representations of $SU(2)$ are shown to be of dimension n, $n = 1, 2, 3, \ldots$ and correspond to particles of spin $s = n/2$.

By analogy with the classical case, one can show that a quantum dynamical system with Hamiltonian operator H_{op} is Euclidean invariant on $\mathbb{R}^3$ when H_{op} is a function of the Laplacian; the relevant fact from operator theory is that every translation and rotational invariant operator on $\mathbb{R}^n$ is a function of the Laplacian.

We can go to the Galilei group and the Lorentz group as in the classical case. For the Galilei case we are again forced into $H_{\mathrm{op}} = -(1/2m)\,\Delta$ acting on spin wave functions. For the case of the Lorentz group things are more interesting. Here H_{op} depends on the spin and one recovers, for example, the Klein–Gordon and Dirac operators, as Bargmann and Wigner have shown. Any such H_{op} satisfies

$$H_{\mathrm{op}}^2 = m^2 c^4 - c^2 \Delta \tag{5}$$

the mass-energy relation, independent of spin. (Mass-zero particles, e.g., the photon and neutrino are exceptional in that they are not localizable in the sense that their position operators have the form previously described, so this case is dealt with separately.) We refer the reader to Varadarajan [?] for details of the aforementioned results and the appropriate references.