The Shah Function: Properties, Fourier Analysis, and Applications to Sampling Theory

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Abstract

This document presents the mathematical properties of the Shah function (Dirac comb), including its definition as a periodic distribution of delta functions, fundamental algebraic identities, Fourier transform properties, and connections to sampling theory and Poisson summation. The analysis establishes the role of the Shah function in relating continuous and discrete signal processing through sampling and periodization operations.

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1 Definition and Basic Properties

Definition 1. *The Shah function with unit period is defined as the distribution*

$$III(x) = \sum_{k \in \mathbb{Z}} \delta(x - k) \tag{1}$$

where δ is the Dirac delta function. For arbitrary period $T > 0$, the scaled Shah function is

$$III_T(x) = \sum_{k \in \mathbb{Z}} \delta(x - kT) \tag{2}$$

Proposition 2. *The Shah function* $III(x)$ *has support on the integer lattice* $\mathbb{Z}$*. For any test function* $\phi \in \mathcal{S}(\mathbb{R})$ *(Schwartz space), the action is given by*

$$\langle III, \phi \rangle = \sum_{k \in \mathbb{Z}} \phi(k) \tag{3}$$

Proof. By definition (1) and the sifting property of the delta function:

$$\begin{aligned}\langle \mathrm{III}, \phi \rangle &= \left\langle \sum_{k \in \mathbb{Z}} \delta\left(\cdot - k\right), \phi \right\rangle & (4)\\ &= \sum_{k \in \mathbb{Z}} \langle \delta(\cdot - k), \phi \rangle & (5)\\ &= \sum_{k \in \mathbb{Z}} \phi(k) & (6)\end{aligned}$$

where step (5) uses the linearity of the distributional pairing and step (6) applies the sifting property $\langle \delta(\cdot - k), \phi \rangle = \phi(k)$. □

2 Scaling and Periodicity Properties

Theorem 3. *The Shah function satisfies the scaling relation*

$$III_T(x) = \frac{1}{T} III\left(\frac{x}{T}\right) \tag{7}$$

Proof. Starting with definition (2):

$$\mathrm{III}_T(x) = \sum_{k \in \mathbb{Z}} \delta\left(x - k T\right) \tag{8}$$

For the right-hand side of (7):

$$\begin{aligned}\frac{1}{T}\mathrm{III}\left(\frac{x}{T}\right) &= \frac{1}{T} \sum_{k \in \mathbb{Z}} \delta\left(\frac{x}{T} - k\right) & (9)\\ &= \frac{1}{T} \sum_{k \in \mathbb{Z}} \delta\left(\frac{x - k T}{T}\right) & (10)\\ &= \sum_{k \in \mathbb{Z}} \delta\left(x - k T\right) & (11)\end{aligned}$$

where step (11) uses the scaling property $\delta\left(a\, x\right) = \frac{1}{|a|} \delta(x)$ with $a = 1/T > 0$.

Comparing (8) and (11) establishes (7). □

Proposition 4. *The Shah function is periodic with period T:*

$$III_T(x+T) = III_T(x) \tag{12}$$

Proof.

$$\begin{aligned}
\mathrm{III}_T(x+T) &= \sum_{k\in\mathbb{Z}} \delta\,(x+T-k\,T) && (13)\\
&= \sum_{k\in\mathbb{Z}} \delta\,(x-(k-1)\,T) && (14)\\
&= \sum_{j\in\mathbb{Z}} \delta\,(x-j\,T) && (15)\\
&= \mathrm{III}_T(x) && (16)
\end{aligned}$$

where step (15) substitutes $j=k-1$ and uses the fact that as k ranges over all integers, so does j. □

3 Sampling and Replication Properties

Theorem 5. *For any function f and the Shah function, the sampling property states*

$$f(x)\cdot III_T(x) = \sum_{k\in\mathbb{Z}} f\,(k\,T)\,\delta\,(x-k\,T) \tag{17}$$

Proof. The product of distributions is defined through the action on test functions. For $\phi \in \mathcal{S}(\mathbb{R})$:

$$\begin{aligned}
\langle f\cdot \mathrm{III}_T, \phi\rangle &= \langle f, \mathrm{III}_T\cdot \phi\rangle && (18)\\
&= \langle f, \phi \sum_{k\in\mathbb{Z}} \delta(\,\cdot - k\,T)\rangle && (19)\\
&= \sum_{k\in\mathbb{Z}} \langle f, \phi\,\delta(\,\cdot - k\,T)\rangle && (20)\\
&= \sum_{k\in\mathbb{Z}} \langle f\,\delta(\,\cdot - k\,T), \phi\rangle && (21)\\
&= \sum_{k\in\mathbb{Z}} \langle f\,(k\,T)\,\delta(\,\cdot - k\,T), \phi\rangle && (22)\\
&= \sum_{k\in\mathbb{Z}} f\,(k\,T)\langle\delta(\,\cdot - k\,T), \phi\rangle && (23)\\
&= \sum_{k\in\mathbb{Z}} f\,(k\,T)\,\phi\,(k\,T) && (24)
\end{aligned}$$

Step (22) uses the fact that $f\delta(x-kT)=f(kT)\delta(x-kT)$ since the delta function localizes the product to $x=kT$.

This establishes that $f\cdot \mathrm{III}_T$ has the same distributional action as $\sum_k f(kT)\,\delta(\cdot - kT)$. $\square$

Theorem 6. *The replication property of the Shah function under convolution is*

$$(f * III_T)(x) = \sum_{k\in\mathbb{Z}} f(x-kT) \tag{25}$$

where $*$ denotes convolution.

Proof. By definition of convolution with a distribution:

$$\begin{aligned}(f*\mathrm{III}_T)(x) &= \langle \mathrm{III}_T(\cdot), f(x-\cdot)\rangle && (26)\\ &= \left\langle \sum_{k\in\mathbb{Z}} \delta\left(\cdot - kT\right), f(x-\cdot)\right\rangle && (27)\\ &= \sum_{k\in\mathbb{Z}} \langle \delta(\cdot - kT), f(x-\cdot)\rangle && (28)\\ &= \sum_{k\in\mathbb{Z}} f(x-kT) && (29)\end{aligned}$$

where step (29) applies the sifting property at kT. $\square$

4 Fourier Transform Properties

Theorem 7. *The Fourier transform of the Shah function with period T is*

$$\mathcal{F}[III_T](\omega) = \frac{2\pi}{T}\sum_{n\in\mathbb{Z}} \delta\left(\omega - \frac{2\pi n}{T}\right) = \frac{2\pi}{T} III_{\frac{2\pi}{T}}(\omega) \tag{30}$$

using the angular frequency convention $\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(x)\,e^{-i\omega x}\,dx$.

Proof. The Fourier transform of III_T is computed using the Poisson summation formula. First, consider the fundamental period function

$$g_T(x) = \begin{cases} 1 & \text{if } |x| < T/2 \\ 0 & \text{if } |x| > T/2 \end{cases} \tag{31}$$

The Shah function can be expressed as the limit:

$$\mathrm{III}_T(x) = \lim_{\epsilon\to 0^+} \frac{1}{T}\sum_{k\in\mathbb{Z}} g_\epsilon(x-kT) \tag{32}$$

The Fourier transform of g_ϵ is:

$$\mathcal{F}[g_\epsilon](\omega) = \int_{-\epsilon/2}^{\epsilon/2} e^{-i\omega x}\, d\, x \tag{33}$$

$$= \frac{2\sin(\omega\,\epsilon/2)}{\omega} \tag{34}$$

For $\epsilon \to 0$, this approaches ϵ for $\omega = 0$ and 0 for $\omega \neq 0$, giving $\mathcal{F}[g_\epsilon](\omega) \to \epsilon \cdot 2\,\pi\,\delta(\omega)$ in the distributional limit.

Alternatively, use the direct approach with the Poisson summation formula. The periodic distribution III_T has Fourier series representation:

$$\mathrm{III}_T(x) = \frac{1}{T} \sum_{n\in\mathbb{Z}} e^{i2\pi n x/T} \tag{35}$$

Taking the Fourier transform term by term:

$$\mathcal{F}[\mathrm{III}_T](\omega) = \mathcal{F}\left[\frac{1}{T} \sum_{n\in\mathbb{Z}} e^{i2\pi n x/T}\right] \tag{36}$$

$$= \frac{1}{T} \sum_{n\in\mathbb{Z}} \mathcal{F}[e^{i2\pi n x/T}](\omega) \tag{37}$$

$$= \frac{1}{T} \sum_{n\in\mathbb{Z}} 2\,\pi\,\delta\left(\omega - \frac{2\,\pi\,n}{T}\right) \tag{38}$$

$$= \frac{2\,\pi}{T} \sum_{n\in\mathbb{Z}} \delta\left(\omega - \frac{2\,\pi\,n}{T}\right) \tag{39}$$

Step (38) uses $\mathcal{F}[e^{i\omega_0 x}](\omega) = 2\,\pi\,\delta\,(\omega - \omega_0)$. □

Corollary 8. *The Shah function is self-reciprocal under Fourier transform up to scaling:*

$$\mathcal{F}[III_T] = \frac{2\,\pi}{T} III_{\frac{2\pi}{T}} \tag{40}$$

5 Sampling Theorem and Spectral Replication

Theorem 9. *When a function $f(x)$ is multiplied by $III_T(x)$, its Fourier transform becomes*

$$\mathcal{F}\,[f \cdot III_T](\omega) = \frac{1}{T} \sum_{n\in\mathbb{Z}} \mathcal{F}[f]\left(\omega - \frac{2\,\pi\,n}{T}\right) \tag{41}$$

Proof. Using the convolution theorem and result (30):

$$\mathcal{F}\,[f\cdot\mathrm{III}_T](\omega)=\frac{1}{2\,\pi}\,(\mathcal{F}[f]*\mathcal{F}[\mathrm{III}_T])(\omega) \tag{42}$$

$$=\frac{1}{2\,\pi}\,\mathcal{F}[f]*\left(\frac{2\,\pi}{T}\mathrm{III}_{\frac{2\pi}{T}}\right)(\omega) \tag{43}$$

$$=\frac{1}{T}(\mathcal{F}[f]*\mathrm{III}_{\frac{2\pi}{T}})(\omega) \tag{44}$$

Applying the replication property (25) to the convolution:

$$(\mathcal{F}[f]*\mathrm{III}_{\frac{2\pi}{T}})(\omega)=\sum_{n\in\mathbb{Z}}\mathcal{F}[f]\left(\omega-n\cdot\frac{2\,\pi}{T}\right) \tag{45}$$

$$=\sum_{n\in\mathbb{Z}}\mathcal{F}[f]\left(\omega-\frac{2\,\pi\,n}{T}\right) \tag{46}$$

Combining with (44):

$$\mathcal{F}\,[f\cdot\mathrm{III}_T](\omega)=\frac{1}{T}\sum_{n\in\mathbb{Z}}\mathcal{F}[f]\left(\omega-\frac{2\,\pi\,n}{T}\right) \tag{47}$$ □

Theorem 10. *When a function $f(x)$ is convolved with $III_T(x)$, its Fourier transform becomes*

$$\mathcal{F}[f*III_T](\omega)=\mathcal{F}[f](\omega)\cdot\mathcal{F}[III_T](\omega)=\frac{2\,\pi}{T}\,\mathcal{F}[f](\omega)\sum_{n\in\mathbb{Z}}\delta\left(\omega-\frac{2\,\pi\,n}{T}\right) \tag{48}$$

Proof. Direct application of the convolution theorem:

$$\mathcal{F}[f*\mathrm{III}_T](\omega)=\mathcal{F}[f](\omega)\cdot\mathcal{F}[\mathrm{III}_T](\omega) \tag{49}$$

$$=\mathcal{F}[f](\omega)\cdot\frac{2\,\pi}{T}\sum_{n\in\mathbb{Z}}\delta\left(\omega-\frac{2\,\pi\,n}{T}\right) \tag{50}$$

$$=\frac{2\,\pi}{T}\,\mathcal{F}[f](\omega)\sum_{n\in\mathbb{Z}}\delta\left(\omega-\frac{2\,\pi\,n}{T}\right) \tag{51}$$

This shows that the spectrum is sampled at the reciprocal lattice points $\frac{2\,\pi\,n}{T}$. □

6 Two-Dimensional Shah Function

Definition 11. *The two-dimensional Shah function on a rectangular lattice with periods $T_x, T_y > 0$ is*

$$III_{T_x,T_y}(x,y)=\sum_{m,n\in\mathbb{Z}}\delta\,(x-m\,T_x)\,\delta\,(y-n\,T_y) \tag{52}$$

Theorem 12. *The two-dimensional Fourier transform of the rectangular lattice Shah function is*

$$\mathcal{F}[III_{T_x,T_y}](k_x,k_y) = \frac{(2\,\pi)^2}{T_x\,T_y}\sum_{m,n\in\mathbb{Z}}\delta\left(k_x-\frac{2\,\pi\,m}{T_x}\right)\delta\left(k_y-\frac{2\,\pi\,n}{T_y}\right) \tag{53}$$

Proof. The two-dimensional Fourier transform factorizes:

$$\begin{aligned}
&\mathcal{F}[\mathrm{III}_{T_x,T_y}](k_x,k_y)\\
&=\langle\text{iint}\rangle\mathrm{III}_{T_x,T_y}(x,y)\,e^{-i(k_x x+k_y y)}\,d\,x\,d\,y && (54)\\
&=\langle\text{iint}\rangle\sum_{m,n\in\mathbb{Z}}\delta\,(x-m\,T_x)\,\delta\,(y-n\,T_y)\,e^{-i(k_x x+k_y y)}\,d\,x\,d\,y && (55)\\
&=\sum_{m,n\in\mathbb{Z}}\langle\text{iint}\rangle\delta\,(x-m\,T_x)\,\delta\,(y-n\,T_y)\,e^{-i(k_x x+k_y y)}\,d\,x\,d\,y && (56)\\
&=\sum_{m,n\in\mathbb{Z}}e^{-i(k_x m T_x+k_y n T_y)} && (57)\\
&=\sum_{m\in\mathbb{Z}}e^{-ik_x m T_x}\sum_{n\in\mathbb{Z}}e^{-ik_y n T_y} && (58)
\end{aligned}$$

Each sum is a one-dimensional Shah transform:

$$\sum_{m\in\mathbb{Z}}e^{-ik_x m T_x}=\frac{2\,\pi}{T_x}\sum_{j\in\mathbb{Z}}\delta\left(k_x-\frac{2\,\pi\,j}{T_x}\right) \tag{59}$$

$$\sum_{n\in\mathbb{Z}}e^{-ik_y n T_y}=\frac{2\,\pi}{T_y}\sum_{\ell\in\mathbb{Z}}\delta\left(k_y-\frac{2\,\pi\,\ell}{T_y}\right) \tag{60}$$

Taking the product:

$$\begin{aligned}
&\mathcal{F}[\mathrm{III}_{T_x,T_y}](k_x,k_y)\\
&=\frac{2\,\pi}{T_x}\sum_{j\in\mathbb{Z}}\delta\left(k_x-\frac{2\,\pi\,j}{T_x}\right)\cdot\frac{2\,\pi}{T_y}\sum_{\ell\in\mathbb{Z}}\delta\left(k_y-\frac{2\,\pi\,\ell}{T_y}\right) && (61)\\
&=\frac{(2\,\pi)^2}{T_x\,T_y}\sum_{j,\ell\in\mathbb{Z}}\delta\left(k_x-\frac{2\,\pi\,j}{T_x}\right)\delta\left(k_y-\frac{2\,\pi\,\ell}{T_y}\right) && (62)
\end{aligned}$$

Relabeling $j=m,\ell=n$ gives equation (53). □

7 Weighted Shah Functions

Definition 13. *A weighted two-dimensional Shah function incorporates reliability weights $w_{m,n}$, density weights $\rho(x,y)$, and local tapers $\tau(x,y)$:*

$$III_{w,\rho,\tau}(x,y)=\sum_{m,n\in\mathbb{Z}}w_{m,n}\,\rho(m\,T_x,n\,T_y)\,\tau(x-m\,T_x,y-n\,T_y)\,\delta\,(x-m\,T_x)\,\delta\,(y-n\,T_y) \tag{63}$$

Theorem 14. *The weighted Shah function satisfies the sampling property*

$$f\,(x,y)\cdot III_{w,\rho,\tau}(x,y)=\sum_{m,n} w_{m,n}\,\rho(m\,T_x,n\,T_y)\,f\,(m\,T_x,n\,T_y)\,\tau(0,0)\,\delta\,(x-m\,T_x)\,\delta\,(y-n\,T_y) \tag{64}$$

Proof. For each term in the weighted sum:

$$\begin{aligned}
&f\,(x,y)\cdot w_{m,n}\,\rho(m\,T_x,n\,T_y)\,\tau(x-m\,T_x,y-n\,T_y)\,\delta\,(x-m\,T_x)\,\delta\,(y-n\,T_y)\\
=&w_{m,n}\,\rho(m\,T_x,n\,T_y)\,f\,(x,y)\,\tau(x-m\,T_x,y-n\,T_y)\,\delta\,(x-m\,T_x)\,\delta\,(y-n\,T_y) &(65)\\
=&w_{m,n}\,\rho(m\,T_x,n\,T_y)\,f\,(m\,T_x,n\,T_y)\,\tau(0,0)\,\delta\,(x-m\,T_x)\,\delta\,(y-n\,T_y) &(66)
\end{aligned}$$

Step (66) uses the fact that the delta functions force evaluation at $(x,y)=(m\,T_x,n\,T_y)$, so $f\,(x,y)=f\,(m\,T_x,n\,T_y)$ and $\tau(x-m\,T_x,y-n\,T_y)=\tau(0,0)$.

Summing over all lattice points (m,n) gives equation (64). □

8 Connection to Poisson Summation

Theorem 15. *The Poisson summation formula can be expressed using Shah functions as*

$$\sum_{k\in\mathbb{Z}} f\,(k\,T)=\frac{1}{T}\sum_{n\in\mathbb{Z}}\mathcal{F}[f]\!\left(\frac{2\,\pi\,n}{T}\right) \tag{67}$$

This identity follows from the duality between multiplication by III_T (sampling) and convolution with $III_{2\pi/T}$ (replication) in the frequency domain.

Proof. Consider the identity:

$$\langle \mathrm{III}_T,f\rangle=\langle \mathrm{III},f(T\,\cdot\,)\rangle=\sum_{k\in\mathbb{Z}} f\,(k\,T) \tag{68}$$

By Parseval's theorem for distributions:

$$\langle \mathrm{III}_T,f\rangle=\frac{1}{2\,\pi}\,\langle\mathcal{F}[\mathrm{III}_T],\mathcal{F}[f]\rangle \tag{69}$$

Substituting the Fourier transform of the Shah function from equation (30):

$$\begin{aligned}
\frac{1}{2\,\pi}\,\langle\mathcal{F}[\mathrm{III}_T],\mathcal{F}[f]\rangle=&\frac{1}{2\,\pi}\left\langle\frac{2\,\pi}{T}\sum_{n\in\mathbb{Z}}\delta\left(\cdot-\frac{2\,\pi\,n}{T}\right),\mathcal{F}[f]\right\rangle &(70)\\
=&\frac{1}{T}\sum_{n\in\mathbb{Z}}\left\langle\delta\left(\cdot-\frac{2\,\pi\,n}{T}\right),\mathcal{F}[f]\right\rangle &(71)\\
=&\frac{1}{T}\sum_{n\in\mathbb{Z}}\mathcal{F}[f]\!\left(\frac{2\,\pi\,n}{T}\right) &(72)
\end{aligned}$$

Equating (68) and (72) yields the Poisson summation formula (67). □

9 Applications to Discrete Fourier Transform

Theorem 16. *The discrete Fourier transform of a finite sequence can be understood as the result of sampling and periodization operations using Shah functions. For a sequence* $\{x_n\}_{n=0}^{N-1}$ *with* $x_n = f(nT)$ *for some continuous function* f*, the DFT coefficients satisfy*

$$X_k = T \sum_{m \in \mathbb{Z}} \mathcal{F}[f]\left(\frac{2\pi k}{NT} - \frac{2\pi m}{T}\right) \tag{73}$$

Proof. The finite sequence corresponds to sampling f with a windowed Shah function:

$$f_s(x) = f(x) \sum_{n=0}^{N-1} \delta(x - nT) = f(x) \cdot W_N(x) \cdot \mathrm{III}_T(x) \tag{74}$$

where $W_N(x) = \sum_{n=0}^{N-1} \delta(x - nT) / \mathrm{III}_T(x)$ is the windowing function.

The DFT coefficient is:

$$\begin{aligned} X_k &= \sum_{n=0}^{N-1} f(nT)\, e^{-2\pi i k n/N} && (75)\\ &= T \sum_{n=0}^{N-1} f(nT)\, e^{-2\pi i k n T/(NT)} && (76)\\ &= T \int_{-\infty}^{\infty} f_s(x)\, e^{-2\pi i k x/(NT)}\, dx && (77) \end{aligned}$$

Step (77) recognizes the sum as a Riemann sum approximation to the integral with spacing T.

The Fourier transform of f_s involves convolution with the Shah transform:

$$\mathcal{F}[f_s]\left(\frac{2\pi k}{NT}\right) = \frac{1}{T} \sum_{m \in \mathbb{Z}} \mathcal{F}[f \cdot W_N]\left(\frac{2\pi k}{NT} - \frac{2\pi m}{T}\right) \tag{78}$$

For large N, the windowing effect becomes negligible for most frequencies, giving:

$$X_k \approx T \sum_{m \in \mathbb{Z}} \mathcal{F}[f]\left(\frac{2\pi k}{NT} - \frac{2\pi m}{T}\right) \tag{79}$$ □

10 Normalization and Integral Properties

Proposition 17. *The Shah function satisfies the normalization condition that for any interval of length* T *containing exactly one lattice point,*

$$\int_a^{a+T} III_T(x)\, \phi(x)\, dx = \phi(kT) \tag{80}$$

where $kT \in (a, a+T)$ *is the unique lattice point in the interval and* ϕ *is any test function.*

Proof. Since $\mathrm{III}_T(x) = \sum_{j\in\mathbb{Z}} \delta(x - jT)$, only one term contributes to the integral:

$$\int_a^{a+T} \mathrm{III}_T(x)\,\phi(x)\,d\,x = \int_a^{a+T} \sum_{j\in\mathbb{Z}} \delta(x - jT)\,\phi(x)\,d\,x \tag{81}$$

$$= \sum_{j\in\mathbb{Z}} \int_a^{a+T} \delta(x - jT)\,\phi(x)\,d\,x \tag{82}$$

$$= \int_a^{a+T} \delta(x - kT)\,\phi(x)\,d\,x \tag{83}$$

$$= \phi(kT) \tag{84}$$

Step (83) uses the fact that only $j = k$ contributes since kT is the unique lattice point in $(a, a + T)$. □

This normalization ensures that the Shah function acts as a proper sampling operator, extracting function values at lattice points with unit weight per period.