Unitary Time Changes of Stationary Processes Yield Oscillatory Processes

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are locally square-integrable. Applying U_θ to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}$, evolutionary spectrum $dF_t(\lambda) = \dot{\theta}(t)\, dF(\lambda)$, and expected zero-counting function $\mathbb{E}[N_{[0,T]}] = N_{\det}([0,T]) + \frac{[\theta(T) - \theta(0)]}{\pi}\sqrt{-\frac{\ddot{K}(0)}{K(0)}}$, where $N_{\det}([0, T])$ counts deterministic zeros from critical points of the time-change. The sample paths of any non-degenerate second-order stationary process are locally square integrable, making the unitary time-change operator U_θ applicable to typical realizations. A zero-localization measure $d\,\mu(t) = \delta(Z(t))|\dot{Z}(t)|\, d\,t$ induces a Hilbert space $L^2(\mu)$ on the zero set of each oscillatory process realization $Z(t)$, and the multiplication operator $(L\,f)(t) = t\,f(t)$ has simple pure point spectrum equal to the zero crossing set of Z.

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1 Gaussian Processes

Unless otherwise stated, all processes considered will be real-valued.

Theorem 1. *Let $X(u)$ be a real-valued process:*

$$X(u) \in \mathbb{R} \quad \forall u \in \mathbb{R} \tag{1}$$

Then its (complex-valued) random orthogonal spectral measure satisfies

$$d\,\bar{\Phi}(\lambda) = d\,\Phi(-\lambda) \tag{2}$$

and the corresponding covariance spectral measure F is even:

$$F(-A) = F(A) \tag{3}$$

Proof.   1. The spectral representation for $X(u)$ is

$$X(u) = \int_{-\infty}^{\infty} e^{i\lambda u}\, d\,\Phi(\lambda) \tag{4}$$

Since $X(u)$ is real-valued for each u,

$$\overline{X(u)} = X(u) \tag{5}$$

On the other hand,

$$\overline{X(u)} = \overline{\int_{-\infty}^{\infty} e^{i\lambda u}\, d\,\Phi(\lambda)} \tag{6}$$

$$= \int_{-\infty}^{\infty} \overline{e^{i\lambda u}}\, d\,\bar{\Phi}(\lambda) \tag{7}$$

$$= \int_{-\infty}^{\infty} e^{-i\lambda u}\, d\,\bar{\Phi}(\lambda) \tag{8}$$

By the substitution $\lambda \mapsto -\mu$,

$$\int_{-\infty}^{\infty} e^{-i\lambda u}\, d\,\bar{\Phi}(\lambda) = \int_{-\infty}^{\infty} e^{i\mu u}\, d\,\bar{\Phi}(-\mu) \tag{9}$$

So

$$X(u) = \int_{-\infty}^{\infty} e^{i\lambda u}\, d\,\Phi(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda u}\, d\,\bar{\Phi}(-\lambda) \tag{10}$$

By uniqueness of the spectral measure representation, it follows that

$$d\,\bar{\Phi}(\lambda) = d\,\Phi(-\lambda) \tag{11}$$

as (orthogonal) random measures.

2. The covariance function of X is

$$R(u) = \mathbb{E}\left(X(0)\,X(u)\right) = \int_{-\infty}^{\infty} e^{i\lambda u}\, dF(\lambda) \tag{12}$$

Since $X(u)$ is real-valued, $R(u)$ is real and $R(-u) = R(u)$. Thus,

$$R(-u) = \int_{-\infty}^{\infty} e^{-i\lambda u}\, dF(\lambda) = \int_{-\infty}^{\infty} e^{i\mu u}\, dF(-\mu) \tag{13}$$

Equating with $R(u)$,

$$\int_{-\infty}^{\infty} e^{i\lambda u}\, dF(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda u}\, dF(-\lambda) \tag{14}$$

for all u. By the uniqueness theorem for Fourier–Stieltjes transforms, this implies

$$dF(\lambda) = dF(-\lambda) \tag{15}$$

Thus for any Borel set A,

$$F(-A) = F(A) \tag{16}$$

establishing the evenness property. $\qquad\square$

1.1 Definition

Definition 2. (Gaussian process) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T a nonempty index set. A family $\{X_t : t \in T\}$ of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Gaussian process if for every finite subset $\{t_1, \ldots, t_n\} \subset T$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ is multivariate normal (possibly degenerate). Equivalently, every finite linear combination $\sum_{i=1}^{n} a_i X_{t_i}$ is either almost surely constant or Gaussian. The mean function is $m(t) := \mathbb{E}[X_t]$ and the covariance kernel is*

$$K(s, t) = \mathrm{Cov}(X_s, X_t) \tag{17}$$

For any finite $(t_i)_{i=1}^{n} \subset T$, the matrix $K_{ij} = K(t_i, t_j)$ is symmetric positive semidefinite, and a Gaussian process is completely determined in law by m and K.

1.2 Stationary processes

Definition 3. [Cramér spectral representation] *[1] A zero-mean stationary process X with spectral measure F admits the sample path representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, d\Phi(\lambda) \tag{18}$$

which has covariance

$$R_X(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)}\, dF(\lambda) \tag{19}$$

1.2.1 Fourier Transform Conventions

Definition 4. *[Fourier transform conventions] The forward and inverse Fourier transforms on $L^2(\mathbb{R})$ are defined by*

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, d\,u \tag{20}$$

and

$$f(u) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda)\, e^{i\lambda u}\, d\,\lambda \tag{21}$$

where the factor $\frac{1}{2\,\pi}$ is incorporated into the definition of the inverse transform.

1.2.2 Sample Path Realizations

Definition 5. *[Locally square-integrable functions] Define*

$$L^2_{\mathrm{loc}}(\mathbb{R}) := \left\{ f\colon \mathbb{R} \to \mathbb{C} \colon \int_K |f(t)|^2\, d\,t < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \tag{22}$$

Remark 6. Every bounded measurable set in $\mathbb{R}$ is contained in a compact set; hence $L^2_{\mathrm{loc}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

Theorem 7. *[Sample paths in $\mathbf{L^2_{loc}(\mathbb{R})}$] Let $\{X(t)\}_{t\in\mathbb{R}}$ be a second-order stationary process with*

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \tag{23}$$

Then almost every sample path lies in $L^2_{\mathrm{loc}}(\mathbb{R})$.

Proof. Fix an arbitrary bounded interval $[a,b] \subset \mathbb{R}$ with $a < b$. Define the random variable

$$Y_{[a,b]} := \int_a^b X(t)^2\, d\,t \tag{24}$$

By Fubini's theorem,

$$\mathbb{E}[Y_{[a,b]}] = \int_a^b \mathbb{E}[X(t)^2]\, d\,t = (b-a)\,\sigma^2 < \infty \tag{25}$$

By Markov's inequality, $\mathbb{P}\left(Y_{[a,b]} = \infty\right) = 0$. Thus $Y_{[a,b]} < \infty$ almost surely. Covering compacts by countably many dyadic intervals yields the result: for every compact $K \subset \mathbb{R}$, almost surely $\int_K X(t)^2\, d\,t < \infty$. $\square$

2 Oscillatory Processes

Definition 8. *[Oscillatory process] [2] Let F be a finite nonnegative Borel measure on $\mathbb{R}$. Let*

$$A_t \in L^2(F) \quad \forall t \in \mathbb{R} \tag{26}$$

be the gain function and

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{27}$$

be the corresponding oscillatory function; then an oscillatory process is a stochastic process which can be represented as

$$
\begin{aligned}
Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda)\, d\,\Phi(\lambda) \\
&= \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda)
\end{aligned} \tag{28}
$$

where Φ is a complex orthogonal random measure with spectral measure F which satisfies the relation

$$d\,\mathbb{E}[\Phi(\lambda)\overline{\Phi(\mu)}] = \delta\,(\lambda - \mu)\, d\,F(\lambda) \tag{29}$$

and has the corresponding covariance kernel

$$
\begin{aligned}
R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\
&= \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)} e^{i\lambda(t-s)}\, d\,F(\lambda) \\
&= \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)} d\,F(\lambda)
\end{aligned} \tag{30}
$$

Theorem 9. *[Real-valuedness criterion for oscillatory processes]* *Let Z be an oscillatory process with oscillatory function*

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{31}$$

and spectral measure F. Then Z is real-valued if and only if

$$A_t(-\lambda) = \overline{A_t(\lambda)} \tag{32}$$

for F-almost every $\lambda \in \mathbb{R}$, equivalently

$$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \tag{33}$$

for F-almost every $\lambda \in \mathbb{R}$.

Proof.   1. Assume Z is real-valued. Then for all $t \in \mathbb{R}$,

$$Z(t) = \overline{Z(t)} \tag{34}$$

2. From the oscillatory representation (28),

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda) \tag{35}$$

3. Taking the complex conjugate of both sides of (35),

$$\overline{Z(t)} = \overline{\int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t}\, d\overline{\Phi(\lambda)} \tag{36}$$

4. For a real-valued process, the orthogonal random measure must satisfy the symmetry property from Theorem 1:

$$d\overline{\Phi(\lambda)} = d\,\Phi\,(-\lambda) \tag{37}$$

5. Substituting (37) into (36),

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t}\, d\,\Phi\,(-\lambda) \tag{38}$$

6. Apply the change of variables $\mu = -\lambda$, so $d\,\Phi\,(-\lambda) = d\,\Phi(\mu)$ and $e^{-i\lambda t} = e^{i\mu t}$:

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t}\, d\,\Phi(\mu) \tag{39}$$

7. By (34), the right sides of (35) and (39) must be equal:

$$\int_{\mathbb{R}} A_t(\mu)\, e^{i\mu t}\, d\,\Phi(\mu) = \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t}\, d\,\Phi(\mu) \tag{40}$$

8. Since the stochastic integral representation is unique in $L^2(F)$, the integrands must be equal F-almost everywhere:

$$A_t(\lambda) = \overline{A_t(-\lambda)} \quad \text{for } F\text{-a.e. } \lambda \tag{41}$$

9. This is equivalent to (32). From (31),

$$\varphi_t(-\lambda) = A_t(-\lambda)\, e^{-i\lambda t} \tag{42}$$

10. Using (32),

$$\begin{aligned} \varphi_t(-\lambda) &= \overline{A_t(\lambda)} e^{-i\lambda t} \\ &= \overline{A_t(\lambda)\, e^{i\lambda t}} \\ &= \overline{\varphi_t(\lambda)} \end{aligned} \tag{43}$$

establishing (33).

11. Conversely, assume (32) holds. Reversing the steps from (39) to (34) shows that $\overline{Z(t)} = Z(t)$ for all t, so Z is real-valued. $\qquad\square$

Theorem 10. *[**Existence of Oscillatory Processes**] Let F be an absolutely continuous spectral measure and the gain function*

$$A_t(\lambda) \in L^2(F) \quad \forall t \in \mathbb{R} \tag{44}$$

be measurable in both time and frequency; then the time-dependent spectral density is defined by

$$\begin{aligned} S_t(\lambda) &= \int_{\mathbb{R}} |A_t(\lambda)|^2\, d\,F(\lambda) < \infty \\ &= \int_{\mathbb{R}} |A_t(\lambda)|^2\, S(\lambda)\, d\,\lambda \end{aligned} \tag{45}$$

and there exists a complex orthogonal random measure Φ with spectral measure F such that for each sample path $\omega_0 \in \Omega$

$$Z(t, \omega_0) = \int_{\mathbb{R}} A_t(\lambda) \, e^{i\lambda t} \, d\, \Phi(\lambda, \omega_0) \tag{46}$$

is well-defined in $L^2(\Omega)$ and has covariance R_Z as in (30).

Proof. 1. Define the space of simple functions on $\mathbb{R}$: for disjoint Borel sets $\{E_j\}_{j=1}^n$ with $F(E_j) < \infty$ and coefficients $\{c_j\}_{j=1}^n \subset \mathbb{C}$,

$$g(\lambda) = \sum_{j=1}^n c_j \mathbf{1}_{E_j}(\lambda) \tag{47}$$

2. For simple functions, define the stochastic integral

$$\int_{\mathbb{R}} g(\lambda) \, d\, \Phi(\lambda) := \sum_{j=1}^n c_j \, \Phi(E_j) \tag{48}$$

3. Compute the second moment:

$$\mathbb{E}\left[\left|\int_{\mathbb{R}} g(\lambda) \, d\, \Phi(\lambda)\right|^2\right] = \mathbb{E}\left[\left|\sum_{j=1}^n c_j \, \Phi(E_j)\right|^2\right]$$
$$= \mathbb{E}\left[\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \Phi(E_j) \overline{\Phi(E_k)}\right] \tag{49}$$

4. By linearity of expectation,

$$\mathbb{E}\left[\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \Phi(E_j) \overline{\Phi(E_k)}\right] = \sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \mathbb{E}[\Phi(E_j) \overline{\Phi(E_k)}] \tag{50}$$

5. By the orthogonality relation (29), since $E_j \cap E_k = \emptyset$ for $j \neq k$,

$$\mathbb{E}[\Phi(E_j) \overline{\Phi(E_k)}] = \begin{cases} F(E_j) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \tag{51}$$

6. Substituting (51) into (50),

$$\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \mathbb{E}[\Phi(E_j) \overline{\Phi(E_k)}] = \sum_{j=1}^n |c_j|^2 \, F(E_j) \tag{52}$$

7. The right side of (52) equals

$$\sum_{j=1}^n |c_j|^2 \, F(E_j) = \int_{\mathbb{R}} |g(\lambda)|^2 \, d\, F(\lambda) \tag{53}$$

8. Therefore the isometry property holds for simple functions:

$$\mathbb{E}\left[\left|\int_{\mathbb{R}} g(\lambda)\, d\,\Phi(\lambda)\right|^2\right] = \int_{\mathbb{R}} |g(\lambda)|^2\, d\,F(\lambda) \tag{54}$$

9. The space of simple functions is dense in $L^2(F)$. For any $h(\lambda) \in L^2(F)$ and $\epsilon > 0$, there exists a simple function $g(\lambda)$ such that

$$\int_{\mathbb{R}} |h(\lambda) - g(\lambda)|^2\, d\,F(\lambda) < \epsilon \tag{55}$$

10. By the isometry (54) and completeness of $L^2(\Omega)$, the integral extends uniquely by continuity to all $h(\lambda) \in L^2(F)$.

11. Since $A_t \in L^2(F)$ by assumption (44), and $|e^{i\lambda t}| = 1$,

$$\int_{\mathbb{R}} |\varphi_t(\lambda)|^2\, d\,F(\lambda) = \int_{\mathbb{R}} |A_t(\lambda)|^2\, d\,F(\lambda) < \infty \tag{56}$$

so $\varphi_t \in L^2(F)$.

12. Therefore

$$\begin{aligned} Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda)\, d\,\Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda) \end{aligned} \tag{57}$$

is well-defined in $L^2(\Omega)$.

13. To compute the covariance, use the sesquilinearity of the stochastic integral:

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda)\, d\,\Phi(\lambda)\, \overline{\int_{\mathbb{R}} \varphi_s(\mu)\, d\,\Phi(\mu)}\right] \end{aligned} \tag{58}$$

14. By Fubini's theorem for stochastic integrals,

$$\mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda)\, d\,\Phi(\lambda)\, \overline{\int_{\mathbb{R}} \varphi_s(\mu)\, d\,\Phi(\mu)}\right] = \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\mu)}\, \mathbb{E}\left[d\,\Phi(\lambda)\overline{d\,\Phi(\mu)}\right] \tag{59}$$

15. Using the orthogonality relation (29),

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\mu)}\delta\,(\lambda - \mu)\, d\,F(\lambda)\, d\,F(\mu) = \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)}\, d\,F(\lambda) \tag{60}$$

16. Substituting the definition (27),

$$R_Z(t,s) = \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)}e^{i\lambda(t-s)}\, d\,F(\lambda) \tag{61}$$

as claimed in (30). □

Definition 11. *[Forward impulse response]* *For an oscillatory process* $Z(t) = \int \varphi_t(\lambda)\, d\,\Phi(\lambda)$ *with oscillatory function* $\varphi_t(\lambda)$, *define the forward impulse response function*

$$h(t, u) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \varphi_t(\lambda)\, e^{-i\lambda u}\, d\,\lambda \tag{62}$$

That is, $h(t, \cdot)$ *is the inverse Fourier transform (in the variable* u*) of the oscillatory function* $\varphi_t(\cdot)$ *at time* t, *with the* $\frac{1}{2\,\pi}$ *factor as in Definition 4.*

Theorem 12. *[Filter representation via impulse response]* *Let* X *be a zero-mean stationary process with Cramér representation* $X(u) = \int e^{i\lambda u}\, d\,\Phi(\lambda)$ *and spectral measure* F, *and let* Z *be an oscillatory process with oscillatory function* $\varphi_t(\lambda)$ *and the same orthogonal random measure* Φ. *Then*

$$Z(t) = \int_{-\infty}^{\infty} h(t, u)\, X(u)\, d\,u \tag{63}$$

where $h(t, u)$ *is the forward impulse response of Definition 11.*

Proof.　1. Substitute the definitions of $h(t, u)$ and $X(u)$:

$$\int_{-\infty}^{\infty} h(t, u)\, X(u)\, d\,u = \int_{-\infty}^{\infty} \left[\frac{1}{2\,\pi} \int_{-\infty}^{\infty} \varphi_t(\lambda)\, e^{-i\lambda u}\, d\,\lambda \right] \left[\int_{-\infty}^{\infty} e^{i\mu u}\, d\,\Phi(\mu) \right] d\,u \tag{64}$$

2. Apply Fubini's theorem:

$$= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_t(\lambda) \left[\int_{-\infty}^{\infty} e^{i(\mu - \lambda)u}\, d\,u \right] d\,\lambda\, d\,\Phi(\mu) \tag{65}$$

3. The inner integral over u is the Fourier representation of the Dirac delta:

$$\int_{-\infty}^{\infty} e^{i(\mu - \lambda)u}\, d\,u = 2\,\pi\,\delta\,(\mu - \lambda) \tag{66}$$

4. Substitute:

$$= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_t(\lambda) \cdot 2\,\pi\,\delta\,(\mu - \lambda)\, d\,\lambda\, d\,\Phi(\mu) \tag{67}$$

5. Simplify:

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_t(\lambda)\,\delta\,(\mu - \lambda)\, d\,\lambda\, d\,\Phi(\mu) \tag{68}$$

6. Apply the sifting property of the delta function:

$$= \int_{-\infty}^{\infty} \varphi_t(\mu)\, d\,\Phi(\mu) = Z(t) \tag{69} \quad □$$

3   Unitarily Time-Changed Stationary Processes

3.1   Unitary Time-Change Operator $U_\theta f$

Theorem 13. *[**Unitary time-change operator U_θ and its inverse U_θ^{-1}**] Let the time-change function $\theta\colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with*

$$\dot{\theta}(t) > 0 \tag{70}$$

almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero. For f measurable, define

$$(U_\theta f)(t) = \sqrt{\dot{\theta}(t)}\, f(\theta(t)) \tag{71}$$

Its inverse is given by

$$(U_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{72}$$

For every compact set $K \subseteq \mathbb{R}$ and $f \in L^2_{\mathrm{loc}}(\mathbb{R})$,

$$\int_K |(U_\theta f)(t)|^2 dt = \int_{\theta(K)} |f(s)|^2 \, ds \tag{73}$$

Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\mathrm{loc}}(\mathbb{R})$.

> **Proof.**   1. Let $f \in L^2_{\mathrm{loc}}(\mathbb{R})$ and let $K \subset \mathbb{R}$ be compact. From the definition (71),
>
> $$\int_K |(U_\theta f)(t)|^2 \, dt = \int_K \left|\sqrt{\dot{\theta}(t)}\, f(\theta(t))\right|^2 dt \tag{74}$$
>
> 2. Expanding the square,
>
> $$\int_K \left|\sqrt{\dot{\theta}(t)}\, f(\theta(t))\right|^2 dt = \int_K \dot{\theta}(t)|f(\theta(t))|^2 \, dt \tag{75}$$
>
> 3. Since θ is absolutely continuous and strictly increasing, $\theta' = \dot{\theta}$ exists almost everywhere and $\dot{\theta}(t) > 0$ a.e.
>
> 4. Apply the change of variables $s = \theta(t)$. Then
>
> $$ds = \dot{\theta}(t)\, dt \tag{76}$$
>
> 5. The inverse function $t = \theta^{-1}(s)$ exists since θ is strictly increasing and bijective.
>
> 6. As t ranges over K, the variable $s = \theta(t)$ ranges over $\theta(K)$.
>
> 7. Since θ is continuous and K is compact, $\theta(K)$ is compact.
>
> 8. Substituting (76) into (75),
>
> $$\int_K \dot{\theta}(t)\, |f(\theta(t))|^2 dt = \int_{\theta(K)} |f(s)|^2 \, ds \tag{77}$$

9. This establishes the local isometry (73).

10. To verify U_θ^{-1} is the inverse, compute:
$$(U_\theta^{-1} U_\theta f)(s) = U_\theta^{-1} (U_\theta f)(s) \tag{78}$$

11. By definition (72),
$$U_\theta^{-1} (U_\theta f)(s) = \frac{(U_\theta f)(\theta^{-1}(s))}{\sqrt{\dot\theta(\theta^{-1}(s))}} \tag{79}$$

12. By definition (71),
$$(U_\theta f)(\theta^{-1}(s)) = \sqrt{\dot\theta(\theta^{-1}(s))}\, f(\theta(\theta^{-1}(s))) \tag{80}$$

13. Since $\theta \circ \theta^{-1} = \mathrm{id}$,
$$f(\theta(\theta^{-1}(s))) = f(s) \tag{81}$$

14. Substituting (80) and (81) into (79),
$$\frac{\sqrt{\dot\theta(\theta^{-1}(s))}\, f(s)}{\sqrt{\dot\theta(\theta^{-1}(s))}} = f(s) \tag{82}$$

15. Therefore
$$U_\theta^{-1} U_\theta = \mathrm{id} \tag{83}$$

16. Similarly, compute:
$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\dot\theta(t)}\, (U_\theta^{-1} g)(\theta(t)) \tag{84}$$

17. By definition (72),
$$(U_\theta^{-1} g)(\theta(t)) = \frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\dot\theta(\theta^{-1}(\theta(t)))}} \tag{85}$$

18. Since $\theta^{-1} \circ \theta = \mathrm{id}$,
$$g(\theta^{-1}(\theta(t))) = g(t), \quad \theta^{-1}(\theta(t)) = t \tag{86}$$

19. Substituting (86) into (85),
$$\frac{g(t)}{\sqrt{\dot\theta(t)}} \tag{87}$$

20. Therefore from (84),
$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\dot\theta(t)} \cdot \frac{g(t)}{\sqrt{\dot\theta(t)}} = g(t) \tag{88}$$

21. Thus
$$U_\theta U_\theta^{-1} = \mathrm{id} \tag{89}$$

22. Combining (83) and (89), U_θ^{-1} is the two-sided inverse of U_θ on $L^2_{\mathrm{loc}}(\mathbb{R})$. □

3.2   Transformation of Stationary $\rightarrow$ Oscillatory Processes via U_θ

Theorem 14. *[Unitary time changes of stationary processes produce oscillatory process] Let X be zero-mean stationary as in Definition 3. For scaling function θ as in Theorem 13, define*

$$
\begin{aligned}
Z(t) \ &= (U_\theta X)(t) \\
&= \sqrt{\dot\theta(t)}\, X(\theta(t))
\end{aligned}
\tag{90}
$$

Then Z is a realization of an oscillatory process with oscillatory function

$$
\varphi_t(\lambda) = \sqrt{\dot\theta(t)}\, e^{i\lambda\theta(t)}
\tag{91}
$$

gain function

$$
A_t(\lambda) = \sqrt{\dot\theta(t)}\, e^{i\lambda(\theta(t)-t)}
\tag{92}
$$

and covariance kernel

$$
\begin{aligned}
R_Z(t,s) \ &= \mathbb{E}[Z(t)\overline{Z(s)}] \\
&= \mathbb{E}\left[\sqrt{\dot\theta(t)}\, X(\theta(t)) \overline{\sqrt{\dot\theta(s)}\, X(\theta(s))} \right] \\
&= \sqrt{\dot\theta(t)\,\dot\theta(s)}\; \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \\
&= \sqrt{\dot\theta(t)\,\dot\theta(s)}\; R_X\left(\theta(t)-\theta(s)\right) \\
&= \sqrt{\dot\theta(t)\,\dot\theta(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\, dF(\lambda)
\end{aligned}
\tag{93}
$$

Proof.   1. From the Cramér representation (18),

$$
X(u) = \int_{\mathbb{R}} e^{i\lambda u}\, d\Phi(\lambda)
\tag{94}
$$

2. Substituting $u = \theta(t)$ into (94),

$$
X(\theta(t)) = \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\Phi(\lambda)
\tag{95}
$$

3. From the definition (90),

$$
Z(t) = \sqrt{\dot\theta(t)}\, X(\theta(t)) = \sqrt{\dot\theta(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\Phi(\lambda)
\tag{96}
$$

4. By linearity of the stochastic integral,

$$
Z(t) = \int_{\mathbb{R}} \sqrt{\dot\theta(t)}\, e^{i\lambda\theta(t)}\, d\Phi(\lambda)
\tag{97}
$$

5. Define

$$
\varphi_t(\lambda) := \sqrt{\dot\theta(t)}\, e^{i\lambda\theta(t)}
\tag{98}
$$

6. Then (97) becomes

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda) \, d\,\Phi(\lambda) \tag{99}$$

   which is the oscillatory representation (28).

7. To express this in terms of the standard oscillatory function form, define the gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)} \, e^{i\lambda(\theta(t)-t)} \tag{100}$$

8. Then verify the oscillatory function form (27) factorizes

$$\begin{aligned} \varphi_t(\lambda) \ &= A_t(\lambda) \, e^{i\lambda t} \\ &= \sqrt{\dot{\theta}(t)} \, e^{i\lambda(\theta(t)-t)} \, e^{i\lambda t} \\ &= \sqrt{\dot{\theta}(t)} \, e^{i\lambda(\theta(t)-t+t)} \\ &= \sqrt{\dot{\theta}(t)} \, e^{i\lambda\theta(t)} \end{aligned} \tag{101}$$

9. To compute the covariance, use (30):

$$R_Z(t,s) = \mathbb{E}[Z(t)\overline{Z(s)}] \tag{102}$$

10. Substituting (90),

$$R_Z(t,s) = \mathbb{E}\left[\sqrt{\dot{\theta}(t)} \, X(\theta(t)) \overline{\sqrt{\dot{\theta}(s)} \, X(\theta(s))} \right] \tag{103}$$

11. Since $\dot{\theta}$ is deterministic,

$$R_Z(t,s) = \sqrt{\dot{\theta}(t)} \, \sqrt{\dot{\theta}(s)} \, \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \tag{104}$$

12. By stationarity of X, using (19),

$$\mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] = R_X\left(\theta(t)-\theta(s)\right) = \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} \, d\,F(\lambda) \tag{105}$$

13. Substituting (105) into (104),

$$R_Z(t,s) = \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} \, d\,F(\lambda) \tag{106}$$

   establishing (93). □

3.2.1 Time-Varying Filter Representations

Theorem 15. *Let $\theta \colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\dot{\theta}(t) > 0$ almost everywhere. Let $X(u)$ be a stationary process, and define the oscillatory process obtained by the forward unitary time transformation U_θ*

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)} \, X(\theta(t)) = \int_{\mathbb{R}} h(t,u) \, X(u) \, d\,u \tag{107}$$

where the forward impulse response function is given by

$$h(t, u) = \sqrt{\dot{\theta}(t)}\, \delta\left(u - \theta(t)\right) \tag{108}$$

For the unitary time-change case, this impulse response is a specialization of the general impulse response from Definition 11. Specifically, with $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}$, the Fourier-based impulse response

$$h(t, u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}\, e^{-i\lambda u}\, d\lambda = \sqrt{\dot{\theta}(t)}\, \delta\left(u - \theta(t)\right) \tag{109}$$

simplifies to the Dirac form via the inverse Fourier transform.

Then likewise the transformation can be reversed by expressing the stationary process as

$$X(u) = (U_\theta^{-1} Z)(u) = \frac{Z(\theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} = \int_{\mathbb{R}} g(u, t)\, Z(t)\, dt \tag{110}$$

where the inverse impulse response function is

$$g(u, t) = \frac{\delta\left(t - \theta^{-1}(u)\right)}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \tag{111}$$

Proof.   1. Recall the forward unitary transformation from Theorem 13:

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) \tag{112}$$

2. To express this as a convolution integral, note that the Dirac delta function satisfies the sifting property: for any continuous function $f\colon \mathbb{R} \to \mathbb{R}$,

$$\int_{\mathbb{R}} f(u)\, \delta\left(u - a\right)\, du = f(a) \tag{113}$$

for any $a \in \mathbb{R}$.

3. Substituting $f(u) = X(u)$ and $a = \theta(t)$, which is well-defined since θ is bijective and continuous,

$$X(\theta(t)) = \int_{\mathbb{R}} X(u)\, \delta\left(u - \theta(t)\right)\, du \tag{114}$$

4. Multiplying both sides by $\sqrt{\dot{\theta}(t)}$ and substituting into (112),

$$Z(t) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} X(u)\, \delta\left(u - \theta(t)\right)\, du = \int_{\mathbb{R}} \left[\sqrt{\dot{\theta}(t)}\, \delta\left(u - \theta(t)\right)\right] X(u)\, du \tag{115}$$

5. Thus, the forward impulse response function is

$$h(t, u) = \sqrt{\dot{\theta}(t)}\, \delta\left(u - \theta(t)\right) \tag{116}$$

establishing (107).

6. To verify this as a specialization of the Fourier-based impulse response from Definition 11, substitute $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}$:

$$h(t,u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}\, e^{-i\lambda u}\, d\lambda = \sqrt{\dot{\theta}(t)}\, \frac{1}{2\pi}\int e^{i\lambda(\theta(t)-u)}\, d\lambda = \sqrt{\dot{\theta}(t)}\,\delta\,(u - \theta(t)) \tag{117}$$

by the standard Fourier representation of the Dirac delta.

7. For the inverse transformation, recall from Theorem 13 that

$$X(u) = (U_\theta^{-1}Z)(u) = \frac{Z(\theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \tag{118}$$

8. Let $s = \theta^{-1}(u)$, so $u = \theta(s)$ and $Z(\theta^{-1}(u)) = Z(s)$. The sifting property applied to $Z(t)$ with point $\theta^{-1}(u)$ gives

$$Z(\theta^{-1}(u)) = \int_{\mathbb{R}} Z(t)\,\delta\,(t - \theta^{-1}(u))\, dt \tag{119}$$

9. Substituting into (118),

$$X(u) = \frac{1}{\sqrt{\dot{\theta}(\theta^{-1}(u))}}\int_{\mathbb{R}} Z(t)\,\delta\,(t - \theta^{-1}(u))\, dt = \int_{\mathbb{R}}\left[\frac{\delta\,(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}}\right]Z(t)\, dt \tag{120}$$

10. Thus, the inverse impulse response function is

$$g(u,t) = \frac{\delta\,(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \tag{121}$$

establishing (110) and (111).

11. To confirm invertibility, substitute (115) into (120). The integral becomes

$$X(u) = \int_{\mathbb{R}} g(u,t)\left[\int_{\mathbb{R}} h(t,v)\,X(v)\, dv\right]dt \tag{122}$$

12. By Fubini's theorem, since all measures are positive and the delta functions ensure finite support,

$$X(u) = \int_{\mathbb{R}}\int_{\mathbb{R}} g(u,t)\,h(t,v)\,X(v)\, dv\, dt \tag{123}$$

13. Integrating the impulse response composition

$$g(u,t)\,h(t,v) = \frac{\delta\,(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}}\cdot\sqrt{\dot{\theta}(t)}\,\delta\,(v - \theta(t)) \tag{124}$$

over t results in $t = \theta^{-1}(u)$, so

$$\sqrt{\dot{\theta}(t)} = \sqrt{\dot{\theta}(\theta^{-1}(u))} \tag{125}$$

and

$$\delta\left(v - \theta(t)\right) = \delta\left(v - u\right) \tag{126}$$

yielding

$$\int_{\mathbb{R}} g(u,t)\, h(t,v)\, d\, t = \delta\left(v - u\right) \tag{127}$$

14. Thus, (123) simplifies to

$$\int_{\mathbb{R}} \delta\left(v - u\right) X(v)\, d\, v = X(u) \tag{128}$$

confirming the transformations are inverses.   □

Corollary 16. *The evolutionary spectrum is*

$$d\, F_t(\lambda) = \dot{\theta}(t)\, d\, F(\lambda) \tag{129}$$

Proof.   1. The evolutionary spectrum is defined by

$$d\, F_t(\lambda) = |A_t(\lambda)|^2\, d\, F(\lambda) \tag{130}$$

2. From (92),

$$|A_t(\lambda)|^2 = \left| \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t) - t)} \right|^2 \tag{131}$$

3. Since $|e^{i\alpha}| = 1$ for all real α,

$$|e^{i\lambda(\theta(t) - t)}|^2 = 1 \tag{132}$$

4. Therefore

$$|A_t(\lambda)|^2 = \left(\sqrt{\dot{\theta}(t)} \right)^2 \cdot 1 = \dot{\theta}(t) \tag{133}$$

5. Substituting (133) into (130),

$$d\, F_t(\lambda) = \dot{\theta}(t)\, d\, F(\lambda) \tag{134}$$ □

3.3   Covariance operator conjugation

Proposition 17. *Let*

$$(T_K f)(t) := \int_{\mathbb{R}} K(|t - s|)\, f(s)\, d\, s \tag{135}$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h}\, d\, F(\lambda) \tag{136}$$

Define the transformed kernel

$$K_\theta(s, t) := \sqrt{\dot{\theta}(t)\, \dot{\theta}(s)}\, K(|\theta(t) - \theta(s)|) \tag{137}$$

then the corresponding integral covariance operator is conjugated for all $f \in L^2_{\mathrm{loc}}(\mathbb{R})$ by

$$(T_{K_\theta} f)(t) = (U_\theta T_K U_\theta^{-1} f)(t) \tag{138}$$

Proof.   1. From (138), expand the right side:

$$(U_\theta T_K U_\theta^{-1} f)(t) = \sqrt{\dot{\theta}(t)} \, (T_K U_\theta^{-1} f)(\theta(t)) \tag{139}$$

2. By definition (135),

$$(T_K U_\theta^{-1} f)(\theta(t)) = \int_{\mathbb{R}} K(|\theta(t) - s|) \, (U_\theta^{-1} f)(s) \, ds \tag{140}$$

3. By definition (72),

$$(U_\theta^{-1} f)(s) = \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{141}$$

4. Substituting (141) into (140),

$$\int_{\mathbb{R}} K(|\theta(t) - s|) \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \, ds \tag{142}$$

5. Apply the change of variables $s = \theta(u)$, so $ds = \dot{\theta}(u) \, du$ and $\theta^{-1}(s) = u$:

$$\int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \frac{f(u)}{\sqrt{\dot{\theta}(u)}} \dot{\theta}(u) \, du \tag{143}$$

6. Simplify:

$$\int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \frac{\dot{\theta}(u)}{\sqrt{\dot{\theta}(u)}} f(u) \, du = \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \sqrt{\dot{\theta}(u)} \, f(u) \, du \tag{144}$$

7. Substituting (144) into (139),

$$\sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \sqrt{\dot{\theta}(u)} \, f(u) \, du \tag{145}$$

8. Bring the constant inside the integral:

$$\int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} \sqrt{\dot{\theta}(u)} \, K(|\theta(t) - \theta(u)|) \, f(u) \, du \tag{146}$$

9. By definition (137),

$$\sqrt{\dot{\theta}(t)} \sqrt{\dot{\theta}(u)} \, K(|\theta(t) - \theta(u)|) = K_\theta(u, t) \tag{147}$$

10. Therefore

$$\int_{\mathbb{R}} K_\theta(u, t) \, f(u) \, du = (T_{K_\theta} f)(t) \tag{148}$$

establishing (138).   □

4 Zero Localization

Definition 18. *Let Z be real-valued with $Z \in C^1(\mathbb{R})$ having only simple zeros*

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \tag{149}$$

Define, for Borel $B \subset \mathbb{R}$,

$$\mu(B) = \int_{\mathbb{R}} \mathbf{1}_B(t)\, \delta(Z(t)) |\dot{Z}(t)|\, dt \tag{150}$$

Theorem 19. *Under the assumptions of Definition 18, zeros are locally finite and one has*

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0) = 0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \tag{151}$$

whence

$$\mu = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0} \tag{152}$$

Proof.   1. For any smooth test function ϕ with compact support, apply the standard change of variables formula for the delta function. Let $\{t_0^{(1)}, t_0^{(2)}, \ldots\}$ denote the zeros of Z.

2. By the change of variables formula for distributions,

$$\int_{\mathbb{R}} \phi(t)\, \delta(Z(t))\, dt = \sum_{t_0 : Z(t_0) = 0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} \tag{153}$$

3. The right side of (153) equals

$$\sum_{t_0 : Z(t_0) = 0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} = \sum_{t_0 : Z(t_0) = 0} \int_{\mathbb{R}} \phi(t) \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|}\, dt \tag{154}$$

4. By Fubini's theorem (justified since the sum has locally finite terms due to C^1 regularity and simple zeros),

$$\sum_{t_0 : Z(t_0) = 0} \int_{\mathbb{R}} \phi(t) \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|}\, dt = \int_{\mathbb{R}} \phi(t) \sum_{t_0 : Z(t_0) = 0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|}\, dt \tag{155}$$

5. Comparing (153) and (155),

$$\int_{\mathbb{R}} \phi(t)\, \delta(Z(t))\, dt = \int_{\mathbb{R}} \phi(t) \sum_{t_0 : Z(t_0) = 0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|}\, dt \tag{156}$$

6. Since ϕ is arbitrary,

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0) = 0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \tag{157}$$

establishing (151).

7. Substituting (157) into the definition (150),

$$\mu(B) = \int_{\mathbb{R}} \mathbf{1}_B(t) \sum_{t_0 : Z(t_0)=0} \frac{\delta\left(t - t_0\right)}{|\dot{Z}(t_0)|} |\dot{Z}(t)| \, dt \tag{158}$$

8. By the sifting property of the delta function, $|\dot{Z}(t)|$ evaluated at $t = t_0$ gives $|\dot{Z}(t_0)|$:

$$\int_{\mathbb{R}} \mathbf{1}_B(t) \frac{\delta\left(t - t_0\right)}{|\dot{Z}(t_0)|} |\dot{Z}(t)| \, dt = \frac{\mathbf{1}_B(t_0)|\dot{Z}(t_0)|}{|\dot{Z}(t_0)|} = \mathbf{1}_B(t_0) \tag{159}$$

9. Summing over all zeros,

$$\mu(B) = \sum_{t_0 : Z(t_0)=0} \mathbf{1}_B(t_0) = \sum_{t_0 \in B : Z(t_0)=0} 1 \tag{160}$$

10. This is precisely the atomic measure

$$\mu = \sum_{t_0 : Z(t_0)=0} \delta_{t_0} \tag{161}$$

establishing (152). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 20. *Let $\mathcal{H} = L^2(\mu)$ be the Hilbert space with inner product*

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} d\mu(t) \tag{162}$$

Proposition 21. [Atomic structure] *Let*

$$\mu = \sum_{t_0 : Z(t_0)=0} \delta_{t_0} \tag{163}$$

then

$$\mathcal{H} \cong \left\{ f : \{t_0 : Z(t_0) = 0\} \to \mathbb{C} : \sum_{t_0 : Z(t_0)=0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \tag{164}$$

with orthonormal basis $\{e_{t_0}\}_{t_0 : Z(t_0)=0}$ where

$$e_{t_0}(t_1) = \delta_{t_0, t_1} \tag{165}$$

Proof.   1. By (163), μ is a purely atomic measure with atoms at the zero set.

2. For any $f \in L^2(\mu)$, the L^2 norm is

$$\|f\|_{L^2(\mu)}^2 = \int_{\mathbb{R}} |f(t)|^2 \, d\mu(t) \tag{166}$$

3. Substituting (163),

$$\int_{\mathbb{R}} |f(t)|^2 \, d\mu(t) = \int_{\mathbb{R}} |f(t)|^2 \sum_{t_0 : Z(t_0)=0} \delta_{t_0}(dt) = \sum_{t_0 : Z(t_0)=0} |f(t_0)|^2 \tag{167}$$

4. Therefore

$$\|f\|_{L^2(\mu)}^2 = \sum_{t_0 : Z(t_0)=0} |f(t_0)|^2 \tag{168}$$

5. This is precisely the ℓ^2 norm on the zero set.

6. Define the map $\Psi \colon L^2(\mu) \to \ell^2$ by

$$\Psi(f) = (f(t_0))_{t_0 : Z(t_0)=0} \tag{169}$$

7. From (168), Ψ is an isometry:

$$\|\Psi(f)\|_{\ell^2}^2 = \sum_{t_0 : Z(t_0)=0} |f(t_0)|^2 = \|f\|_{L^2(\mu)}^2 \tag{170}$$

8. Ψ is surjective: for any sequence $(c_{t_0}) \in \ell^2$, define $f(t) = \sum_{t_0} c_{t_0} \delta_{t_0}(t)$, which is in $L^2(\mu)$.

9. Therefore Ψ is a Hilbert space isomorphism, establishing (164).

10. For the orthonormal basis, define e_{t_0} by (165).

11. Then

$$\langle e_{t_0}, e_{t_1} \rangle = \int_{\mathbb{R}} e_{t_0}(t) \overline{e_{t_1}(t)} d\mu(t) = \sum_{s : Z(s)=0} \delta_{t_0,s} \delta_{t_1,s} = \delta_{t_0,t_1} \tag{171}$$

12. Therefore $\{e_{t_0}\}$ is an orthonormal set.

13. Since every $f \in L^2(\mu)$ can be written as

$$f = \sum_{t_0 : Z(t_0)=0} f(t_0) e_{t_0} \tag{172}$$

the set $\{e_{t_0}\}$ is complete, hence an orthonormal basis. $\square$

Definition 22. *[Multiplication operator] Define the linear operator*

$$L \colon \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H} \tag{173}$$

by

$$(Lf)(t) = t f(t) \tag{174}$$

on the support of μ with domain

$$\mathcal{D}(L) := \left\{ f \in \mathcal{H} \colon \int |t f(t)|^2 d\mu(t) < \infty \right\} \tag{175}$$

Theorem 23. *[Self-adjointness and spectrum] L is self-adjoint on $\mathcal{H}$ and has pure point, simple spectrum*

$$\sigma(L) = \overline{\{t \in \mathbb{R} \colon Z(t) = 0\}} \tag{176}$$

with eigenvalues $\lambda = t_0$ for each zero t_0 and corresponding eigenvectors e_{t_0}.

   Proof.   1. For $f, g \in \mathcal{D}(L)$, compute the inner product:

$$\langle Lf, g \rangle = \int_{\mathbb{R}} (Lf)(t) \overline{g(t)} d\mu(t) \tag{177}$$

2. By definition (174),

$$\int_{\mathbb{R}} t f(t) \overline{g(t)} d\mu(t) \tag{178}$$

3. Since t is real-valued, $\bar{t} = t$, so

$$\int_{\mathbb{R}} t\, f(t)\overline{g(t)} d\mu(t) = \int_{\mathbb{R}} f(t)\overline{t\, g(t)} d\mu(t) \tag{179}$$

4. The right side of (179) is

$$\int_{\mathbb{R}} f(t)\overline{(L\,g)(t)} d\mu(t) = \langle f, L\,g \rangle \tag{180}$$

5. Therefore

$$\langle L\,f, g \rangle = \langle f, L\,g \rangle \tag{181}$$

for all $f, g \in \mathcal{D}(L)$, establishing that L is symmetric.

6. Since L is a multiplication operator on $L^2(\mu)$, it is self-adjoint (by standard functional analysis).

7. To determine the spectrum, compute the action on basis vectors. From (174) and (165),

$$(L\, e_{t_0})(t) = t\, e_{t_0}(t) = t\, \delta_{t_0}(t) \tag{182}$$

8. By the sifting property,

$$t\, \delta_{t_0}(t) = t_0\, \delta_{t_0}(t) = t_0\, e_{t_0}(t) \tag{183}$$

9. Therefore

$$L\, e_{t_0} = t_0\, e_{t_0} \tag{184}$$

10. This shows that each t_0 is an eigenvalue with eigenvector e_{t_0}.

11. Since the $\{e_{t_0}\}$ form a complete orthonormal basis (Proposition 21), the spectrum is pure point.

12. Each eigenspace is one-dimensional (spanned by e_{t_0}), so the spectrum is simple and given by the closure of the zero set

$$\sigma(L) = \{t_0 : Z(t_0) = 0\} = \overline{\{t \in \mathbb{R} : Z(t) = 0\}} \tag{185} \quad \square$$

4.1 Simplicity of Zeros and Their Expected Counting Function

Theorem 24. (Bulinskaya) *Let $X(t)$ be a centered stationary Gaussian process with covariance function $K(h) = \mathbb{E}\left[X(t)\,X(t+h)\right]$ that is twice differentiable at $h = 0$ with $K(0) > 0$ and $\ddot{K}(0) < 0$. Then the zero set of X has no accumulation points almost surely. In particular, on any compact interval $[a, b]$, the number of zeros is almost surely finite.*

Proof.   1. The twice-differentiability of K at $h = 0$ implies that the spectral measure F has finite second moment:

$$\lambda_2 = \int_{\mathbb{R}} \omega^2 \, dF(\omega) = -\ddot{K}(0) < \infty. \tag{186}$$

2. This finite second moment implies that X has mean-square continuous derivative $\dot{X}(t)$, with

$$\mathbb{E}[\dot{X}(t)^2] = -\ddot{K}(0) = \lambda_2 > 0. \tag{187}$$

3. Since $\dot{X}(t)$ is a non-degenerate centered Gaussian process, it is continuous almost surely and does not vanish identically on any interval.

4. For any zero t_0 of X (i.e., $X(t_0) = 0$), if $\dot{X}(t_0) \neq 0$, then X crosses transversally through zero at t_0, making t_0 an isolated zero.

5. The joint distribution of $(X(t_0), \dot{X}(t_0))$ is bivariate Gaussian with covariance matrix

$$\begin{pmatrix} K(0) & K'(0) \\ K'(0) & -\ddot{K}(0) \end{pmatrix} = \begin{pmatrix} K(0) & 0 \\ 0 & -\ddot{K}(0) \end{pmatrix}, \tag{188}$$

where $K'(0) = 0$ by evenness of K. Since X and $\dot{X}$ are uncorrelated Gaussians, they are independent.

6. At any zero t_0 of X, the derivative $\dot{X}(t_0)$ is Gaussian with mean zero and variance $-\ddot{K}(0) > 0$, hence

$$\mathbb{P}\left[\dot{X}(t_0) = 0 \mid X(t_0) = 0\right] = \mathbb{P}\left[\dot{X}(t_0) = 0\right] = 0. \tag{189}$$

Thus almost surely $\dot{X}(t_0) \neq 0$, making t_0 an isolated zero.

7. Since every zero of X is isolated almost surely, the zero set can have no accumulation points almost surely.

8. On a compact interval $[a, b]$, a set with no accumulation points is finite, completing the proof. $\square$

Theorem 25. (Expected Zero-Counting Function with Deterministic Atoms)
Let $\theta: \mathbb{R} \to \mathbb{R}$ be C^1 (continuously differentiable), strictly increasing, and bijective with $\dot{\theta}(t) \geq 0$ for all t and $\dot{\theta}(t) > 0$ almost everywhere. Define the zero-derivative set

$$T_0 := \{t \in \mathbb{R} : \dot{\theta}(t) = 0\} \tag{190}$$

and assume that T_0 is at most countable with no accumulation points on any compact interval. Let X be a centered stationary Gaussian process with spectral measure F and covariance function

$$K(h) = \int_{\mathbb{R}} e^{i\omega h} \, dF(\omega) \tag{191}$$

twice differentiable at $h = 0$ with $\ddot{K}(0) < 0$ and $K(0) > 0$. Define the unitarily time-changed process

$$Z(t) = \sqrt{\dot{\theta}(t)} \, X(\theta(t)) \tag{192}$$

Then Z is a centered Gaussian process with covariance

$$K_Z(t, s) = \sqrt{\dot{\theta}(t) \, \dot{\theta}(s)} \, K\left(\theta(t) - \theta(s)\right) \tag{193}$$

For any compact interval $[0, T]$, define

$$N_{\det}([0, T]) := \#(T_0 \cap [0, T]) \tag{194}$$

By the assumption on T_0, $N_{\det}([0, T])$ is finite. The expected number of zeros of Z in $[0, T]$ decomposes as

$$\mathbb{E}[N_{[0,T]}(Z)] = N_{\det}([0, T]) + \frac{\theta(T) - \theta(0)}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}} \tag{195}$$

Proof. 1. Define

$$T_0 := \{t \in \mathbb{R} : \dot{\theta}(t) = 0\}, \qquad I_T := [0, T] \setminus T_0 \tag{196}$$

For any $t_0 \in T_0$,

$$Z(t_0) = \sqrt{\dot{\theta}(t_0)}\, X(\theta(t_0)) = 0 \cdot X(\theta(t_0)) = 0 \tag{197}$$

so each $t_0 \in T_0$ is a deterministic zero of Z on every sample path.

2. By the assumption that T_0 has no accumulation points on compact intervals, $T_0 \cap [0, T]$ is finite. Thus

$$N_{\det}([0, T]) = \#(T_0 \cap [0, T]) < \infty. \tag{198}$$

3. On the complement $I_T = [0, T] \setminus T_0$, the derivative satisfies $\dot{\theta}(t) > 0$, hence

$$Z(t) = 0 \iff X(\theta(t)) = 0, \qquad t \in I_T \tag{199}$$

Define

$$Y(t) := X(\theta(t)). \tag{200}$$

The random zero set of Z on $[0, T]$ coincides with the zero set of Y on I_T, and the total zero count decomposes as

$$N_{[0,T]}(Z) = N_{\det}([0, T]) + N_{\mathrm{rand}}([0, T]) \tag{201}$$

where $N_{\mathrm{rand}}([0, T])$ counts zeros of Y in I_T.

4. For $t, s \in \mathbb{R}$,

$$K_Y(t, s) = \mathbb{E}\left[X(\theta(t))\, X(\theta(s))\right] = K\left(\theta(t) - \theta(s)\right) \tag{202}$$

Differentiate with respect to s:

$$\frac{\partial}{\partial s} K_Y(s, t) = -\dot{\theta}(s)\, \dot{K}\left(\theta(t) - \theta(s)\right) \tag{203}$$

Since $K(h)$ is even, $\dot{K}(0) = 0$. Taking $s \to t$,

$$\lim_{s \to t} \frac{\partial}{\partial s} K_Y(s, t) = 0 \tag{204}$$

5. The mixed partial derivative is

$$\frac{\partial^2}{\partial s\,\partial t}\,K_Y(t,s) = -\dot{\theta}(t)\,\dot{\theta}(s)\;\ddot{K}\left(\theta(t)-\theta(s)\right) \tag{205}$$

Taking $s \to t$,

$$\lim_{s\to t}\frac{\partial^2}{\partial s\,\partial t}\,K_Y(t,s) = -\dot{\theta}(t)^2\,\ddot{K}(0) \tag{206}$$

Also $K_Y(t,t) = K(0)$.

6. The Kac–Rice zero intensity for Y on I_T is

$$\rho_Y(t) = \frac{1}{\pi}\sqrt{\frac{K_Y(t,t)\cdot\lim_{s\to t}\frac{\partial^2}{\partial s\,\partial t}K_Y(t,s) - \left(\lim_{s\to t}\frac{\partial}{\partial s}K_Y(s,t)\right)^2}{K_Y(t,t)^2}} \tag{207}$$

Substituting,

$$K_Y(t,t)\cdot\lim_{s\to t}\frac{\partial^2}{\partial s\,\partial t}K_Y(t,s) - 0^2 = K(0)\cdot(-\dot{\theta}(t)^2\,\ddot{K}(0)) = K(0)\,\dot{\theta}(t)^2\,(-\ddot{K}(0)) \tag{208}$$

Therefore,

$$\rho_Y(t) = \frac{1}{\pi}\sqrt{\frac{\dot{\theta}(t)^2\,(-\ddot{K}(0))}{K(0)}} = \frac{\dot{\theta}(t)}{\pi}\sqrt{-\frac{\ddot{K}(0)}{K(0)}} \tag{209}$$

7. The expected random zero count is

$$\mathbb{E}[N_{\mathrm{rand}}([0,T])] = \int_{I_T}\rho_Y(t)\,dt \tag{210}$$

Since T_0 is countable (hence Lebesgue measure zero),

$$\mathbb{E}[N_{\mathrm{rand}}([0,T])] = \int_0^T\frac{\dot{\theta}(t)}{\pi}\sqrt{-\frac{\ddot{K}(0)}{K(0)}}\,dt = \frac{\theta(T)-\theta(0)}{\pi}\sqrt{-\frac{\ddot{K}(0)}{K(0)}} \tag{211}$$

8. The total zero count factors as

$$N_{[0,T]}(Z) = N_{\det}([0,T]) + N_{\mathrm{rand}}([0,T]) \tag{212}$$

so

$$\mathbb{E}[N_{[0,T]}(Z)] = N_{\det}([0,T]) + \frac{\theta(T)-\theta(0)}{\pi}\sqrt{-\frac{\ddot{K}(0)}{K(0)}} \tag{213} \;\square$$

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