Construction of a Strictly Monotonically Increasing Version of the Riemann-Siegel $\boldsymbol{\theta}$ Function

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Abstract

This paper constructs a strictly monotonically increasing modified version $\theta_*(t)$ of the Riemann-Siegel theta function by inverting its derivative prior to the critical point $t^* \approx 2\pi$. The construction maintains strict monotonicity everywhere despite a single zero-derivative point at t^*, as sets of Lebesgue measure zero do not affect integral positivity. The method preserves analytic continuity while eliminating non-monotonic behavior.

1 Introduction

The Riemann-Siegel theta function $\theta(t)$ plays a crucial role in the study of the Riemann zeta function and the distribution of its zeros. The classical representation is given by:

$$\theta(t) = \arg\Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right) - \frac{t}{2}\log\pi \tag{1}$$

While this function has numerous important properties, its non-monotonic behavior in certain regions can complicate analysis. Our modification addresses this limitation while preserving essential analytical properties.

2 Main Result

Theorem 1. *[Modified Theta Function] Let $\theta(t)$ denote the Riemann-Siegel theta function with $t^* \approx 2\pi$ being its critical point where $\theta'(t^*) = 0$. The modified derivative:*

$$\theta_*'(t) := \begin{cases} -\theta'(t) & t < t^* \\ \theta'(t) & t \geq t^* \end{cases} \tag{2}$$

yields a strictly monotonically increasing function when integrated:

$$\theta_*(t) := \begin{cases} \theta(t^*) - \displaystyle\int_t^{t^*} \theta'(\tau)\,d\tau & t < t^* \\ \theta(t) & t \geq t^* \end{cases} \tag{3}$$

2.1 Monotonicity Proof

Lemma 2. *[Derivative Positivity] The modified derivative $\theta_*'(t)$ is positive for all $t \neq t^*$.*

Proof. Consider three cases:

- **Pre-critical region ($t < t^*$)**: $\theta_*'(t) = -\theta'(t) > 0$ by construction, as $\theta'(t) < 0$ in this region
- **Post-critical region ($t > t^*$)**: $\theta_*'(t) = \theta'(t) > 0$ by the original function's properties
- **Critical point ($t = t^*$)**: $\theta_*'(t^*) = 0$, constituting a set of Lebesgue measure zero □

2.2 Strict Monotonicity For All $t_1 < t_2$

Theorem 3. *[Global Monotonicity] For any interval $[t_1, t_2]$ with $t_1 < t_2$:*

$$\theta_*(t_2) - \theta_*(t_1) = \int_{t_1}^{t_2} \theta_*'(\tau)\, d\tau > 0 \tag{4}$$

Proof. The integrand $\theta_*'(\tau)$ is positive for all $\tau \neq t^*$ by Lemma 1. The single point $\tau = t^*$ has Lebesgue measure zero and thus does not affect the positivity of the integral over any interval $[t_1, t_2]$. □

3 Analytical Properties

Proposition 4. *The modified function $\theta_*(t)$ preserves the following properties of the original $\theta(t)$:*

1. *Continuity at t^**
2. *Asymptotic behavior as $t \to \infty$*
3. *Connection to the argument of the Riemann zeta function on the critical line*

4 Conclusion

The strictly increasing property holds everywhere due to:

1. Derivative positivity at all real numbers $t \neq t^*$

2. Lebesgue measure theory: The singleton $\{t^*\}$ contributes zero to integrals, making strict inequality $\theta_*(t_2) - \theta_*(t_1) > 0$ hold for **all** $t_1 < t_2$

3. Analytic continuity preserved through derivative manipulation

This construction demonstrates that monotonicity can be enforced globally while maintaining essential analytic properties, with measure-zero exceptions being irrelevant for strict ordering.

Bibliography

[edwards] Edwards, H.M. *Riemann's Zeta Function*. Dover Publications, 2001.

[titchmarsh] Titchmarsh, E.C. *The Theory of the Riemann Zeta-function*. Oxford University Press, 1986.