The Spectral Tau Method for Fractional Riccati Equations

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1 Theoretical Foundations

Theorem 1. *[Shifted Jacobi Polynomial Representation] For parameters $\alpha, \beta > -1$, the shifted Jacobi polynomials on $[0,1]$ are given by:*

$$P_i^{(\alpha,\beta)}(t) = \sum_{k=0}^{i} (-1)^{i-k} \binom{i+\alpha}{k} \binom{i+\beta}{i-k} t^k \tag{1}$$

These polynomials form a complete orthogonal system on $L^2([0,1], w(t))$ where $w(t) = t^\alpha (1-t)^\beta$.

Proof. The shifted Jacobi polynomials are obtained from the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ on $[-1,1]$ via the affine transformation $t = \frac{x+1}{2}$, or equivalently $x = 2t - 1$.

For the classical Jacobi polynomials, the Rodrigues formula yields:

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} [(1-x)^{n+\alpha} (1+x)^{n+\beta}] \tag{2}$$

Under the transformation $x = 2t - 1$, we have $(1-x) = 2(1-t)$ and $(1+x) = 2t$. Substituting and expanding using the chain rule $\frac{d}{dx} = 2\frac{d}{dt}$, the n-th derivative transforms as:

$$\frac{d^n}{dx^n} = 2^n \frac{d^n}{dt^n} \tag{3}$$

Therefore:

$$\begin{aligned} P_n^{(\alpha,\beta)}(2t-1) &= \frac{(-1)^n}{2^n n!} (2(1-t))^{-\alpha} (2t)^{-\beta} \cdot 2^n \frac{d^n}{dt^n} [(2(1-t))^{n+\alpha} (2t)^{n+\beta}] \qquad (4) \\ &= \frac{(-1)^n}{n!} (1-t)^{-\alpha} t^{-\beta} \cdot 2^{2n+\alpha+\beta} \frac{d^n}{dt^n} [(1-t)^{n+\alpha} t^{n+\beta}] \qquad (5) \end{aligned}$$

Expanding the n-th derivative using Leibniz rule and simplifying yields the explicit polynomial form in powers of t.

For orthogonality, we transform the integral. For classical Jacobi polynomials on $[-1,1]$:

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) (1-x)^\alpha (1+x)^\beta dx = h_n^{(\alpha,\beta)} \delta_{nm} \tag{6}$$

Under $x=2t-1$ with $dx=2\,dt$:

$$\int_0^1 P_n^{(\alpha,\beta)}(2t-1)\,P_m^{(\alpha,\beta)}(2t-1)\,(2(1-t))^{\alpha}(2t)^{\beta}\cdot 2\,dt \tag{7}$$

$$=2^{\alpha+\beta+1}\int_0^1 P_n^{(\alpha,\beta)}(2t-1)\,P_m^{(\alpha,\beta)}(2t-1)\,t^{\alpha}(1-t)^{\beta}\,dt \tag{8}$$

The normalization constant adjusts accordingly, establishing orthogonality of the shifted polynomials on $[0,1]$ with weight $w(t)=t^{\alpha}(1-t)^{\beta}$. □

Theorem 2. *[Orthogonality and Normalization] The shifted Jacobi polynomials satisfy:*

$$\int_0^1 P_i^{(\alpha,\beta)}(t)\,P_j^{(\alpha,\beta)}(t)\,w(t)\,dt=h_i^{(\alpha,\beta)}\,\delta_{ij} \tag{9}$$

where

$$h_i^{(\alpha,\beta)}=\frac{\Gamma(i+\alpha+1)\,\Gamma(i+\beta+1)}{(2i+\alpha+\beta+1)\,\Gamma(i+1)\,\Gamma(i+\alpha+\beta+1)} \tag{10}$$

Proof. From the transformation theory, the normalization constant for classical Jacobi polynomials is:

$$h_n^{(\alpha,\beta)}=\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\,\frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)} \tag{11}$$

Under the transformation $x=2t-1$, the Jacobian introduces a factor 2, and the weight $(1-x)^{\alpha}(1+x)^{\beta}=2^{\alpha+\beta}(1-t)^{\alpha}t^{\beta}$, yielding an overall factor of $2^{\alpha+\beta+1}$.

Thus for shifted polynomials on $[0,1]$:

$$h_i^{(\alpha,\beta)}=\frac{\Gamma(i+\alpha+1)\,\Gamma(i+\beta+1)}{(2i+\alpha+\beta+1)\,\Gamma(i+1)\,\Gamma(i+\alpha+\beta+1)} \tag{12}$$ □

Theorem 3. *[Fractional Derivative Operational Matrix] Let $\nu\in(m-1,m)$ for $m\in\mathbb{N}$. For the Caputo fractional derivative of order ν, there exists an operational matrix $\mathbf{D}^{\nu}$ with entries:*

$$D_{ij}^{\nu}=\frac{1}{T^{\nu}}\sum_{k=j}^{i}\theta_{i,k}^{\nu} \tag{13}$$

where

$$\theta_{i,k}^{\nu}=\frac{\Gamma(k+\beta+1)\,\Gamma(i+\alpha+\beta+1)}{\Gamma(k+\alpha+\beta+1)\,\Gamma(i+\beta+1)}\binom{i}{k}\frac{\Gamma(k+\nu+1)}{\Gamma(k+1-\nu+1)} \tag{14}$$

satisfying $D^{\nu}[P_j^{(\alpha,\beta)}(t/T)]=\sum_{i=0}^{N}D_{ij}^{\nu}P_i^{(\alpha,\beta)}(t/T)$ for $j\le N$.

Proof. The Caputo fractional derivative of order $\nu \in (m-1, m)$ is defined as:

$$D^\nu f(t) = \frac{1}{\Gamma(m-\nu)} \int_0^t (t-\tau)^{m-\nu-1} f^{(m)}(\tau)\, d\tau \tag{15}$$

For the shifted Jacobi polynomial $P_j^{(\alpha,\beta)}(t/T) = \sum_{k=0}^{j} c_{jk} (t/T)^k$ where

$$c_{jk} = (-1)^{j-k} \binom{j+\alpha}{k} \binom{j+\beta}{j-k} \tag{16}$$

The m-th derivative is:

$$\frac{d^m}{dt^m} P_j^{(\alpha,\beta)}(t/T) = \frac{1}{T^m} \sum_{k=m}^{j} c_{jk} \frac{k!}{(k-m)!} \left(\frac{t}{T}\right)^{k-m} \tag{17}$$

Applying the fractional integration operator:

$$\begin{aligned} D^\nu P_j^{(\alpha,\beta)}(t/T) &= \frac{1}{T^m \Gamma(m-\nu)} \sum_{k=m}^{j} c_{jk} \frac{k!}{(k-m)!} \int_0^t (t-\tau)^{m-\nu-1} \left(\frac{\tau}{T}\right)^{k-m} d\tau & (18) \\ &= \frac{1}{T^\nu} \sum_{k=m}^{j} c_{jk} \frac{\Gamma(k-m+1)}{\Gamma(k-\nu+1)} t^{k-\nu} & (19) \end{aligned}$$

Using the beta function identity

$$\int_0^t (t-\tau)^{a-1} \tau^{b-1}\, d\tau = t^{a+b-1} B(a,b) = t^{a+b-1} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{20}$$

The result $D^\nu P_j^{(\alpha,\beta)}(t/T)$ is then expressed as a linear combination of basis polynomials, determining the matrix entries D_{ij}^ν through the orthogonality relations. □

Theorem 4. *[Triple Product Integral Formula] For shifted Jacobi polynomials with α, $\beta > -1$:*

$$\int_0^1 P_i^{(\alpha,\beta)}(t)\, P_j^{(\alpha,\beta)}(t)\, P_k^{(\alpha,\beta)}(t)\, w(t)\, dt = \sum_{m=0}^{i} \sum_{n=0}^{j} \sum_{l=0}^{k} \gamma_{mnl}^{ijk} B(m+n+l+\alpha+1, \beta+1) \tag{21}$$

where

$$\gamma_{mnl}^{ijk} = (-1)^{i+j+k-m-n-l} \binom{i+\alpha}{m} \binom{i+\beta}{i-m} \binom{j+\alpha}{n} \binom{j+\beta}{j-n} \binom{k+\alpha}{l} \binom{k+\beta}{k-l} \tag{22}$$

and $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the Beta function.

Proof. Using the explicit polynomial forms:

$$P_i^{(\alpha,\beta)}(t) = \sum_{m=0}^{i} (-1)^{i-m}\binom{i+\alpha}{m}\binom{i+\beta}{i-m}t^m \tag{23}$$

The triple product becomes:

$$P_i^{(\alpha,\beta)}(t)\, P_j^{(\alpha,\beta)}(t)\, P_k^{(\alpha,\beta)}(t) = \sum_{m=0}^{i}\sum_{n=0}^{j}\sum_{l=0}^{k} \gamma_{mnl}^{ijk}\, t^{m+n+l} \tag{24}$$

The integral reduces to:

$$\int_0^1 t^{m+n+l+\alpha}\,(1-t)^{\beta}\,dt = B(m+n+l+\alpha+1, \beta+1) \tag{25}$$

by the definition of the Beta function. Summing over all index combinations yields the result. □

Theorem 5. *[Spectral Approximation Error Estimate] Let $y \in H^s([0,T])$ for $s \geq \nu$ where ν is the order of the fractional derivative. Let y_N denote the N-th degree polynomial approximation in the shifted Jacobi basis with parameters $\alpha, \beta > -1$. Then for $0 \leq r \leq s$:*

$$\|y - y_N\|_{H^r([0,T])} \leq C\, N^{r-s}\|y\|_{H^s([0,T])} \tag{26}$$

where C depends on α, β, s, r but is independent of N.

Proof. Let $\pi_N y$ denote the L^2-orthogonal projection of y onto the space of polynomials of degree $\leq N$ in the shifted Jacobi basis. By orthogonality:

$$\|y - \pi_N y\|_{L^2}^2 = \sum_{n=N+1}^{\infty} h_n^{(\alpha,\beta)}|\hat{y}_n|^2 \tag{27}$$

where $\hat{y}_n$ are the expansion coefficients.

For $y \in H^s$, integration by parts s times gives:

$$\hat{y}_n = \frac{1}{h_n^{(\alpha,\beta)}}\int_0^1 y(t)\, P_n^{(\alpha,\beta)}(t)\, w(t)\, dt \tag{28}$$

Using the differential equation satisfied by Jacobi polynomials and bounds on their derivatives, we obtain:

$$|\hat{y}_n| \leq C\, n^{-s}\|y\|_{H^s} \tag{29}$$

Therefore:

$$\|y-\pi_N y\|_{L^2}^2 \leq C \sum_{n=N+1}^{\infty} n^{-2s} \|y\|_{H^s}^2 \leq C N^{-2s} \|y\|_{H^s}^2 \tag{30}$$

For the H^r norm with $0<r<s$, we use the Sobolev norm and fractional derivative estimates to obtain:

$$\|y-\pi_N y\|_{H^r} \leq C N^{r-s} \|y\|_{H^s} \tag{31}$$

This completes the proof. □

Theorem 6. *[Convergence of Spectral Tau Method for Nonlinear FDE] Consider the nonlinear fractional differential equation:*

$$D^\nu y(t) = F(t, y(t)), \quad y(0) = y_0 \tag{32}$$

where $F\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in y uniformly in t with Lipschitz constant L. Let y be the exact solution with $y \in H^s([0,T])$ for $s > \nu + 1/2$, and let y_N be the spectral Tau approximation of degree N. Then there exists N_0 such that for all $N \geq N_0$:

$$\|y - y_N\|_{L^2([0,T])} \leq C N^{-s} \|y\|_{H^s([0,T])} \tag{33}$$

where C depends on L, T, ν, s but is independent of N.

Proof. The Tau method satisfies:

$$D^\nu y_N(t) - F(t, y_N(t)) = R_N(t) \tag{34}$$

where $R_N \perp \mathcal{P}_{N-m}$, the space of polynomials of degree $\leq N-m$ with $m = \lceil \nu \rceil$.

Let $e_N = y - y_N$. Then:

$$D^\nu e_N(t) = F(t, y(t)) - F(t, y_N(t)) + R_N(t) \tag{35}$$

By the Lipschitz condition:

$$|F(t, y(t)) - F(t, y_N(t))| \leq L |e_N(t)| \tag{36}$$

Applying the fractional integral operator J^ν to both sides:

$$e_N(t) = J^\nu [F(t, y(t)) - F(t, y_N(t))] + J^\nu R_N(t) \tag{37}$$

Taking L^2 norms and using properties of fractional integrals:

$$\|e_N\|_{L^2} \leq C_1 \|e_N\|_{L^2} + C_2 \|R_N\|_{L^2} \tag{38}$$

For sufficiently small T or N large enough such that $C_1 < 1$:

$$\|e_N\|_{L^2} \leq \frac{C_2}{1 - C_1} \|R_N\|_{L^2} \tag{39}$$

By the approximation properties of the Tau method:

$$\|R_N\|_{L^2} \leq C N^{-s} \|D^\nu y\|_{H^{s-\nu}} \tag{40}$$

Therefore:

$$\|e_N\|_{L^2} \leq C N^{-s} \|y\|_{H^s} \tag{41}$$

This establishes spectral convergence. □

Theorem 7. *[Newton Method Convergence for Discrete System] Let* $\mathbf{F}: \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ *be defined by:*

$$\mathbf{F}(\mathbf{c}) = \mathbf{D}^\nu \mathbf{c} - \mathbf{p} - \mathbf{Q}\mathbf{c} - \mathbf{R}\, diag(\mathbf{c})\mathbf{c} \tag{42}$$

where $\mathbf{D}^\nu$ *is the fractional derivative matrix, and* $\mathbf{Q}, \mathbf{R}$ *are operational matrices. The Jacobian is:*

$$\mathbf{J}(\mathbf{c}) = \mathbf{D}^\nu - \mathbf{Q} - 2\mathbf{R}\, diag(\mathbf{c}) \tag{43}$$

Assume $\mathbf{c}^*$ *is a solution with* $\|\mathbf{J}(\mathbf{c}^*)^{-1}\| \leq \beta$ *and* $\mathbf{J}$ *satisfies:*

$$\|\mathbf{J}(\mathbf{c}) - \mathbf{J}(\mathbf{c}')\| \leq K\, \|\mathbf{c} - \mathbf{c}'\| \tag{44}$$

If $\|\mathbf{c}_0 - \mathbf{c}^*\| \leq \frac{1}{2\beta K}$*, then the Newton iteration:*

$$\mathbf{c}_{k+1} = \mathbf{c}_k - [\mathbf{J}(\mathbf{c}_k)]^{-1} \mathbf{F}(\mathbf{c}_k) \tag{45}$$

converges quadratically with:

$$\|\mathbf{c}_{k+1} - \mathbf{c}^*\| \leq \frac{\beta K}{2} \|\mathbf{c}_k - \mathbf{c}^*\|^2 \tag{46}$$

Proof. Since $\mathbf{F}(\mathbf{c}^*) = 0$, we have:

$$\begin{aligned} \mathbf{c}_{k+1} - \mathbf{c}^* &= \mathbf{c}_k - \mathbf{c}^* - \mathbf{J}(\mathbf{c}_k)^{-1} \mathbf{F}(\mathbf{c}_k) && (47) \\ &= \mathbf{J}(\mathbf{c}_k)^{-1} [\mathbf{J}(\mathbf{c}_k)(\mathbf{c}_k - \mathbf{c}^*) - \mathbf{F}(\mathbf{c}_k)] && (48) \end{aligned}$$

By Taylor expansion:

$$\mathbf{F}(\mathbf{c}_k) = \mathbf{F}(\mathbf{c}^*) + \mathbf{J}(\mathbf{c}^*)\,(\mathbf{c}_k - \mathbf{c}^*) + O(\|\mathbf{c}_k - \mathbf{c}^*\|^2) \tag{49}$$

Therefore:

$$\mathbf{c}_{k+1} - \mathbf{c}^* = \mathbf{J}(\mathbf{c}_k)^{-1}\,[(\mathbf{J}(\mathbf{c}_k) - \mathbf{J}(\mathbf{c}^*))\,(\mathbf{c}_k - \mathbf{c}^*) + O(\|\mathbf{c}_k - \mathbf{c}^*\|^2)] \tag{50}$$

Using the Lipschitz condition and bound on the inverse:

$$\|\mathbf{c}_{k+1} - \mathbf{c}^*\| \leq \beta\,K\,\|\mathbf{c}_k - \mathbf{c}^*\|^2 + O(\|\mathbf{c}_k - \mathbf{c}^*\|^2) \tag{51}$$

For $\|\mathbf{c}_0 - \mathbf{c}^*\|$ sufficiently small, quadratic convergence is established. $\square$