Uniformly Convergent Series Expansions for the Bessel Functions of the First Kind of Integer Orders That Are Conjectured To Be The Eigenfunctions of $\int_0^\infty J_0(x-y) f(x) \mathrm{d}x$

by Stephen Crowley <stephencrowley214@gmail.com>

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Abstract

Uniformly convergent eigenfunction expansions for the even-indexed Type -1 Bessel functions are all but proven.

Definition 1. *Let $j_n(x)$ is the spherical Bessel function of the first kind,*

$$\begin{aligned} j_n(z) &= \sqrt{\frac{\pi}{2z}} J_{n+\frac{1}{2}}(x) \\ &= \frac{1}{\sqrt{z}} \left(\sin(z) R_{n,\frac{1}{2}}(z) - \cos(z) R_{n-1,\frac{3}{2}}(z) \right) \end{aligned} \tag{1}$$

where $R_{n,v}(z)$ are the (misnamed) Lommel polynomials [3]

$$R_{n,v}(z) = \frac{\Gamma(n+v)}{\Gamma(v)} \left(\frac{2}{z}\right)^n {}_2F_3\left(\left[-\frac{n}{2}, \frac{1}{2} - \frac{n}{2} \right]; [v, -n, 1-v-n]; -z^2 \right) \tag{2}$$

where ${}_2F_3$ is a generalized hypergeometric function. The "Lommel polynomials" are actually rational functions of z, not polynomial; but rather "polynomial in $\frac{1}{z}$".

Conjecture 2. *The series*

$$\begin{aligned}J_0(t) &= \sum_{k=0}^{\infty} \lambda_k \psi_k(t) \\ &= \sum_{k=0}^{\infty} \sqrt{\frac{4n+1}{\pi}} \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} (-1)^n \sqrt{\frac{4n+1}{\pi}}\, j_{2n}(t) \\ &= \sum_{k=0}^{\infty} \frac{4n+1}{\pi} \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} (-1)^n\, j_{2n}(t)\end{aligned} \tag{3}$$

converges uniformly for all complex t except the origin where it has a regular singular point where $\lim_{t\to 0} J_0(t) = 1$.

Conjecture 3. *The eigenfunctions of the stationary integral covariance operator*

$$[T\psi_n](x) = \int_0^{\infty} J_0(x-y)\, \psi_n(x) \mathrm{d}x = \lambda_n \psi_n(x) \tag{4}$$

are given by

$$\psi_n(y) = (-1)^n \sqrt{\frac{4n+1}{\pi}}\, j_{2n}(y) \tag{5}$$

and the eigenvalues are given by

$$\begin{aligned}\lambda_n &= \int_{-\infty}^{\infty} J_0(x)\, \psi_n(x)\, \mathrm{d}x \\ &= \sqrt{\frac{4n+1}{\pi}} \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} \\ &= \sqrt{\frac{4n+1}{\pi}}\, (n+1)^2_{-\frac{1}{2}}\end{aligned} \tag{6}$$

where $(n+1)^2_{-\frac{1}{2}}$ is the Pochhammer symbol(ascending/rising factorial).

Note 4. *TODO: change $J_0(x)$ to $J_0(|x|)$ and $\psi_n(y)$ to $\psi_n(|y|)$, by doing so the range of integration can be extended from $(0,\infty)$ to $(-\infty,\infty)$ without the integral vanishing due to the fact that the odd-idexed Bessel functions are odd and therefore integration over the full interval vanishes unless the absolute value is used. The absolute value is the natural way to utilize the functions sice they are symmetric or anti-symmetric depending upon the parity of the index and thus no information is lost in making this change and indeed doing so will make all the integer-indexed Type-1 Bessel functions representable via this expansion rather than only the even indexed ones as it is now. The primary aim is to conclusively prove that the proposed functions are the eigenfunctions and eigenvalues of $\int_0^{\infty} J_0(x-y)\, \psi_n(x)\mathrm{d}x$ but solving the operator for each $\int_0^{\infty} J_m(x-y)\, \psi_n(x)\mathrm{d}x$ seems inevitable since the same cofactor appears in*

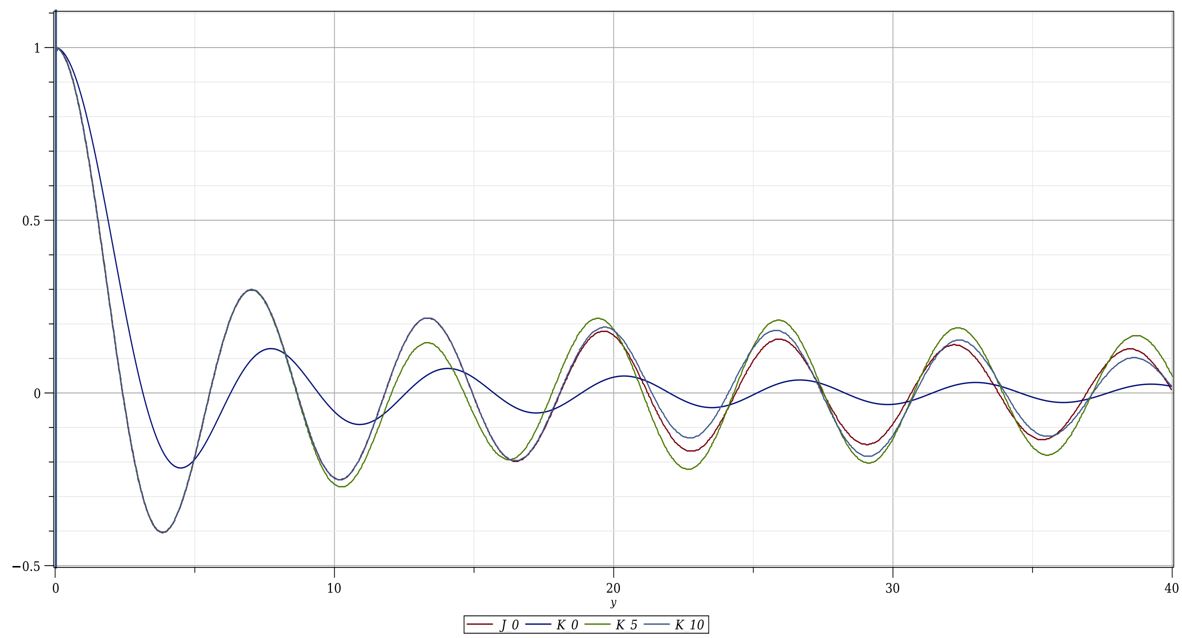

Figure 1. J_0 compared to the finite-rank approximations for rank $0, 5,$ and 10. The figure shows the excellent convergence properties of the proposed eigenfunction expansion $J_0(x) = \lim_{n\to\infty}\sum_{k=0}^{n}\lambda_k\psi_k(x)$.

Remark 5. There is no doubt that the expansion $J_0(x)=\lim_{n\to\infty}\sum_{k=0}^{n}\lambda_k\psi_k(x)$ is correct, but only finding the eigenfunctions of the finite-rank operators and confirming via Rellich's theorem that the finite-rank subspace operators have eigenfunctions which converge to the full-rank operator eigenfunctions $\psi_n(x)$ whose eigenvalues are $\lambda_n=\int_{-\infty}^{\infty}J_0(|x|)\psi_n(x)\mathrm{d}x$

Theorem 6. *The operator defined by Equation (4) is compact relative to the canonical metric induced by the covariance kernel $J_0(|x-y|)$.*

Proof. Due to the spectral theorem for compact operators, if an operator is self-adjoint and has eigenvalues converging to 0, then it is necessarily compact. It is easy to see that

$$\lim_{n\to\infty}\lambda_n = 0 \tag{7}$$

and due to the symmetry of $J_0(|t|)$ it is self-adjoint and therefore the operator T defined in Equation (4) is compact. [1] □

Definition 7. *The spectral density of a stationary process is the Fourier tranform of the covariance kernel due to Wiener-Khinchine theorem.*

Definition 8. *Let $S_n(x)$ be the orthogonal polynomials whose orthogonality measure is equal to the spectral density of the process. These polynomials shall be called the spectral polynomials corresponding to the process.*

Example 9. Let the kernel function be given by $K(t,s) = J_0(t-s)$ then identify the orthogonal polynomial sequence associated with the spectral density

$$S(\omega) = \int_{-\infty}^{\infty} J_0(x)\, e^{ix\omega}\, \mathrm{d}x = \begin{cases} \dfrac{2}{\sqrt{1-\omega^2}} & \omega \in (-1,1) \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

as being twice the orthogonality measure of the Type-I Chebyshev polynomials $T_n(x)$ so that the orthogonal polynomial sequence is identified as

$$S_n(x) = \sqrt{2} T_n(x) \tag{9}$$

so that

$$\int_{-1}^{1} S_n(\omega)\, S_m(\omega)\, S(\omega)\, \mathrm{d}\omega = \begin{cases} 0 & n \neq m \\ 2\pi & n = m = 0 \\ \pi & n = m \neq 0 \end{cases} \tag{10}$$

Remark 10. If the spectral density does not equal the orthogonality measure of a known set of orthogonal polynomials then such a set can always be generated by applying the Gram-Schmidt process to the monomials so that they are transformed into a set that is orthogonal with respect any given spectral density (of a stationary process).

Definition 11. *The sequence $\hat{S}_n(y)$ of Fourier transforms of the spectral polynomials $S_n(x)$ is given by*

$$\hat{S}_n(y) = \int_{-1}^{1} S_n(x) e^{ixy} \mathrm{d}x \tag{11}$$

Example 12. The Fourier transforms of the Chebyshev polynomials are just the usual infinite Fourier transforms with the integration restricted to the range $-1 \ldots 1$ since $T_n(x) = 0 \forall x \notin [-1,1]$). Equivalently, the spectral density function can be extended to take the value 0 outside the interval $[-1,1]$. The derivation of

$$\begin{aligned} \hat{T}_n(y) &= \int_{-\infty}^{\infty} e^{-ixy}\, T_n(x) d\, y = \int_{-1}^{1} e^{-ixy}\, T_n(x) d\, x \\ &= \int_{-\infty}^{\infty} e^{-ixy}\, {}_2F_1\left(\begin{matrix} n, & -n \\ & \frac{1}{2} \end{matrix} \middle| \frac{1}{2} - \frac{x}{2} \right) d\, x \\ &= \frac{i}{y} \left(e^{-iy} F_n^{+}(y) - e^{i(\pi n + y)} F_n^{-}(y) \right) \end{aligned} \tag{12}$$

where

$$F_n^{\pm}(y) = {}_3F_1\left(\begin{matrix} 1, & n, & -n \\ & & \frac{1}{2} \end{matrix} \middle| \frac{\pm iy}{2} \right) \tag{13}$$

can be found in [2].

Definition 13. *Let $Y_n(y)$ be the normalized spectral polynomials $S_n(x)$*

Example 14. When $K = J_0$ the spectral polynomials are given by

$$S_n(x) = \sqrt{2} T_n(x) \tag{14}$$

so that

$$\begin{aligned} Y_n(y) &= \frac{\hat{T}_n(y)}{|\hat{T}_n|} \\ &= \frac{i}{y}\left(\frac{e^{-iy} F_n^+(y) - e^{i(\pi n + y)} F_n^-(y)}{\sqrt{\frac{4(-1)^n \pi - (2n^2 - 1)}{4n^2 - 1}}} \right) \end{aligned} \tag{15}$$

where the L^2 norm of $\hat{T}_n(y)$ is given by

$$\begin{aligned} |\hat{T}_n| &= \sqrt{\int_{-\infty}^{\infty} \hat{T}_n(y)^2 \mathrm{d}y} \\ &= \sqrt{\frac{4(-1)^n \pi - (2n^2 - 1)}{4n^2 - 1}} \end{aligned} \tag{16}$$

Conjecture 15. *The eigenfunctions of the integral covariance operator (4) are given by the orthogonal complement of the normalized Fourier transforms $Y_n(y)$ of the spectral polynomials (via the Gram-Schmidt process)*

$$\psi_n(y) = Y_n^{\perp}(y) = Y_n(y) - \sum_{m=1}^{n-1} \frac{\langle Y_n(y), Y_m^{\perp}(y) \rangle}{\langle Y_m^{\perp}(y), Y_m^{\perp}(y) \rangle} Y_m^{\perp}(y) \tag{17}$$

can be equivalently expressed as

$$\begin{aligned} \psi_n(y) &= (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, j_{2n}(y) \\ &= (-1)^n \sqrt{\frac{4\,n+1}{\pi}} \sqrt{\frac{\pi}{2\,y}}\, J_{2n+\frac{1}{2}}(y) \\ &= (-1)^n \sqrt{\frac{4\,n+1}{2\,y}}\, J_{2n+\frac{1}{2}}(y) \\ &= \sqrt{\frac{4\,n+1}{4\,\pi}} \int_{-1}^{1} P_{2n}(x)\, e^{ixy}\, \mathrm{d}x \end{aligned} \tag{18}$$

Remark 16. Since T is compact due to its self-adjointness and convergence of the eigenvalues to 0 it converges uniformly since compactness implies uniform convergence of the eigenfunctions. TODO: cite/theorems from [4, 3. Reproducing Kernel Hilbert Space of a Gaussian Process]

Theorem 17. *The Bessel function identity*

$$J_0(x-y) = \sum_{k=-\infty}^{\infty} J_k(x)\, J_k(y) \tag{19}$$

can be used to expression Equation (4) as a series

$$\begin{aligned} [T f](y) &= \int_0^{\infty} J_0(x-y)\, f(x)\, d\,x \\ &= \int_0^{\infty} \sum_{k=-\infty}^{\infty} J_k(x)\, J_k(y)\, f(x)\, d\,x \\ &= \sum_{k=-\infty}^{\infty} J_k(y) \int_0^{\infty} J_k(x)\, f(x)\, d\,x \end{aligned}$$

by applying Fubini's theorem to exchange the sum with the integral when f is absolutely integrable.

Proof. TODO: demonstrate the identity is well-known and the interchange is justified when $f \in L^1_{0,\infty}$, e.g., f is Lebesgue absolutely integrable over $[0, \infty]$ □

Example 18. Simplifying The Convolution

Apply the addition theorem

$$J_0(x-y) = \sum_{k=-\infty}^{\infty} J_k(x)\, J_k(y) \tag{20}$$

to the integral covariance operator from Conjecture 17

$$
\begin{aligned}
[T\psi_n](x) &= \int_0^\infty J_0(x-y)\,\psi_n(y)\;d\,y \\
&= \int_0^\infty \sum_{k=-\infty}^{\infty} J_k(x)\,J_k(y)\,\psi_n(y)\;d\,y \\
&= \sum_{k=-\infty}^{\infty} J_k(x)\int_0^\infty J_k(y)\,\psi_n(y)\;d\,y \\
&= \sum_{k=-\infty}^{\infty} J_k(x)\int_0^\infty J_k(y)\,(-1)^n\sqrt{\frac{4\,n+1}{\pi}}\;j_{2n}(y)\;d\,y
\end{aligned} \tag{21}
$$

Where $\psi_n(y)$ is:

$$
\psi_n(y) = (-1)^n\sqrt{\frac{4\,n+1}{\pi}}\;j_{2n}(y) = (-1)^n\sqrt{\frac{4\,n+1}{\pi}}\;\sqrt{\frac{\pi}{2\,y}}\;J_{2n+\frac{1}{2}}(y) \tag{22}
$$

Substituting

$$
\begin{aligned}
\int_0^\infty J_k(y)\,\psi_n(y)\;d\,y &= \int_0^\infty J_k(y)\,(-1)^n\sqrt{\frac{4\,n+1}{\pi}}\;j_{2n}(y)\;d\,y \\
&= \frac{\sqrt{4\,n+1}\,(-1)^n\sqrt{\pi}\,\Gamma\left(\frac{k}{2}+n+\frac{1}{2}\right)}{2\,\Gamma\left(-n+\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{k}{2}+n+1\right)\Gamma\left(n+1-\frac{k}{2}\right)}
\end{aligned} \tag{23}
$$

Now, putting it all back into the expansion for $[T\psi_n](x)$:

$$
[T\psi_n](x) = \sum_{k=-\infty}^{\infty} J_k(x)\frac{\sqrt{4\,n+1}\,(-1)^n\sqrt{\pi}\,\Gamma\left(\frac{k}{2}+n+\frac{1}{2}\right)}{2\,\Gamma\left(-n+\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{k}{2}+n+1\right)\Gamma\left(n+1-\frac{k}{2}\right)} \tag{24}
$$

Remark 19. I checked a few points of Equation 24 and found it to only be correct for $x=0$; I think the calculations need to be redone with the kernel $J_0(|x|)$ instead of $J_0(x)$ because as is, the reproduction of the odd-indexed Bessel functions does not hold, only for the even functions J_{2n} is it true that

$$
J_{2n}(x) = \sum_{k=0}^{\infty} \psi_k(x)\int_0^\infty J_{2n}(y)\psi_k(y)\mathrm{d}k \tag{25}
$$

which is the same as Equation 23 after making the change-of-variables $2n\to n$. Now that I think about that as I write it, I realize that's probably why Equation 24 doesn't quite work as written…hmm.

Conjecture 20.

$$\sum_{k=0}^{\infty} \psi_k(x)^2 = \frac{1}{\pi} \tag{26}$$

Bibliography

[1] Alen Alexanderian. On compact operators. *Journal of Mathematical Analysis and Applications*, 18(2):5–36, 2013.

[2] Atul Dixit, Lin Jiu, Victor H Moll, and Christophe Vignat. The finite fourier transform of classical polynomials. *Journal of the Australian Mathematical Society*, 98:145–160, 2015.

[3] R. Wong K.F. Lee. Asymptotic expansion of the modified lommel polynomials $h_{n,\nu}(x)$ and their zeros. *Proceedings of the American Mathematical Society*, 142(11):3953–3964, 2014.

[4] Steven P. Lalley. Introduction to gaussian processes. https://galton.uchicago.edu/∼elalley/Courses/386/GaussianProcesses.pdf, 2013.