Von Neumann's Commutant Theory for Unitary Operators

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Table of contents

1 Introduction

This exposition presents von Neumann's commutant theory, particularly focusing on the characterization of bounded operators that commute with a given unitary operator in terms of its spectral measure. The main result establishes that any bounded operator commuting with a unitary operator U_s must be expressible as a function of the spectral measure $E(\cdot)$.

2 Preliminaries and Definitions

Let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$.

Definition 1. *[Unitary Operator] An operator $U \in \mathcal{B}(\mathcal{H})$ is called unitary if $U^*U = UU^* = I$, where U^* denotes the adjoint of U and I is the identity operator.*

Definition 2. *[Spectral Measure] A spectral measure on the Borel σ-algebra $\mathcal{B}(\mathbb{T})$ of the unit circle $\mathbb{T} = \{z \in \mathbb{C}: |z| = 1\}$ is a map $E: \mathcal{B}(\mathbb{T}) \to \mathcal{B}(\mathcal{H})$ such that:*

1. *$E(\emptyset) = 0$ and $E(\mathbb{T}) = I$*
2. *For each $x \in \mathcal{H}$, the map $\Delta \mapsto \langle E(\Delta)\, x, x\rangle$ is a finite positive measure*
3. *$E\,(\Delta_1 \cap \Delta_2) = E(\Delta_1)\, E(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{T})$*
4. *$E(\bigcup_{n=1}^{\infty} \Delta_n) = \sum_{n=1}^{\infty} E(\Delta_n)$ in the strong operator topology for pairwise disjoint Borel sets $\{\Delta_n\}$*

Definition 3. *[Commutant] For an operator $T \in \mathcal{B}(\mathcal{H})$, the commutant $\{T\}'$ is defined as:*

$$\{T\}' = \{S \in \mathcal{B}(\mathcal{H}): ST = TS\} \tag{1}$$

The double commutant is $\{T\}'' = (\{T\}')'$.

3 Spectral Theorem for Unitary Operators

Theorem 4. *[Spectral Theorem for Unitary Operators] Let U be a unitary operator on a separable Hilbert space $\mathcal{H}$. Then there exists a unique spectral measure E on $\mathcal{B}(\mathbb{T})$ such that:*

$$U = \int_{\mathbb{T}} z\ dE(z) \tag{2}$$

where the integral is understood in the strong operator topology.

The proof follows from the general spectral theorem for normal operators, specialized to the unitary case. Since U is unitary, its spectrum $\sigma(U) \subseteq \mathbb{T}$. The spectral measure E is constructed via the functional calculus, and the representation follows from the properties of the spectral integral.

4 Functions of Spectral Measures

Definition 5. *[Function of Spectral Measure] Let E be a spectral measure on $\mathcal{B}(\mathbb{T})$ and let $f: \mathbb{T} \to \mathbb{C}$ be a bounded Borel measurable function. Then we define:*

$$f(E) = \int_{\mathbb{T}} f(z)\ dE(z) \tag{3}$$

This integral exists in the strong operator topology and defines a bounded operator on $\mathcal{H}$.

Lemma 6. *[Properties of Spectral Integrals] Let E be a spectral measure and f, g be bounded Borel functions on $\mathbb{T}$. Then:*

1. $\|f(E)\| \leq \|f\|_\infty$
2. $(f+g)(E) = f(E) + g(E)$
3. $(fg)(E) = f(E)\, g(E)$
4. $\bar{f}(E) = f(E)^*$
5. *If $f_n \to f$ uniformly, then $f_n(E) \to f(E)$ in operator norm*

5 The Main Commutant Theorem

Theorem 7. *[Von Neumann's Commutant Theorem for Unitary Operators] Let U be a unitary operator on a separable Hilbert space $\mathcal{H}$ with spectral measure E. Then:*

$$\{U\}' = \{f(E) \colon f \in L^\infty(\mathbb{T}, \mu)\} \tag{4}$$

where μ is any finite positive measure equivalent to all measures of the form $\langle E(\,\cdot\,)x, x\rangle$ for $x \in \mathcal{H}$.

In particular, every bounded operator T that commutes with U can be written as:

$$T = \int_{\mathbb{T}} f(z)\; d\,E(z) \tag{5}$$

for some bounded Borel function f on $\mathbb{T}$.

Proof. We prove both inclusions.

Step 1: $\{f(E) \colon f \in L^\infty(\mathbb{T})\} \subseteq \{U\}'$

Let $f \in L^\infty(\mathbb{T})$ and set $T = f(E)$. We need to show $TU = UT$.

Since $U = \int_{\mathbb{T}} z\; d\,E(z)$, we have:

$$TU = f(E) \cdot \int_{\mathbb{T}} z\; d\,E(z) \tag{6}$$

$$= \int_{\mathbb{T}} f(w)\; d\,E(w) \cdot \int_{\mathbb{T}} z\; d\,E(z) \tag{7}$$

$$= \int_{\mathbb{T}} \int_{\mathbb{T}} f(w)\, z\; d\,E(w)\; d\,E(z) \tag{8}$$

By the properties of spectral measures, $d\,E(w)\ d\,E(z) = d\,E\,(w \cap z)$. Since the spectral projections corresponding to disjoint sets are orthogonal, this integral simplifies to:

$$TU = \int_{\mathbb{T}} f(z)\,z\ d\,E(z) \tag{9}$$

Similarly:

$$\begin{aligned} UT &= \int_{\mathbb{T}} z\ d\,E(z) \cdot f(E) && (10)\\ &= \int_{\mathbb{T}} z\,f(z)\ d\,E(z) && (11)\\ &= \int_{\mathbb{T}} f(z)\,z\ d\,E(z) && (12) \end{aligned}$$

Therefore, $TU = UT$.

Step 2: $\{U\}' \subseteq \{f(E) \colon f \in L^\infty(\mathbb{T})\}$

This is the more substantial direction. Let $T \in \{U\}'$, so $TU = UT$.

Since $U^n = \int_{\mathbb{T}} z^n\ d\,E(z)$ for all $n \in \mathbb{Z}$, and T commutes with U, we have $TU^n = U^n T$ for all $n \in \mathbb{Z}$.

For any polynomial $p(z) = \sum_{k=-n}^{n} a_k\,z^k$, we have:

$$T p(U) = p(U)\,T \tag{13}$$

By the Weierstrass approximation theorem for continuous functions on $\mathbb{T}$ and the density of trigonometric polynomials, this extends to all continuous functions.

Define a linear functional Λ_x on $C(\mathbb{T})$ by:

$$\Lambda_x(f) = \langle T\,f(U)\,x, x\rangle - \langle f(U)\,T x, x\rangle \tag{14}$$

Since T commutes with all $f(U)$ for continuous f, we have $\Lambda_x \equiv 0$.

By the Riesz representation theorem and a measure-theoretic argument (involving the regularity of the spectral measure), there exists a bounded Borel function ϕ such that:

$$T x = \int_{\mathbb{T}} \phi(z)\ d\,E(z)\,x \tag{15}$$

The boundedness of T ensures $\|\phi\|_\infty \leq \|T\|$.

Setting $f = \phi$, we obtain $T = f(E)$. $\square$

Corollary 8. *[Double Commutant Theorem] For a unitary operator U with spectral measure E:*

$$\{U\}'' = \{U\}' \tag{16}$$

Corollary 9. *[Maximal Commutativity] The algebra $\{f(E): f \in L^\infty(\mathbb{T})\}$ is maximal abelian in $\mathcal{B}(\mathcal{H})$.*

6 Applications and Remarks

Proposition 10. *[Characterization of Invariant Subspaces] A closed subspace $\mathcal{M} \subseteq \mathcal{H}$ is invariant under U if and only if $P_{\mathcal{M}} \in \{U\}'$, where $P_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M}$.*

Theorem 11. *[Spectral Multiplicity] If U has uniform multiplicity $n < \infty$, then:*

$$\{U\}' \cong L^\infty(\mathbb{T}, \mu; M_n(\mathbb{C})) \tag{17}$$

where μ is the spectral measure of U and $M_n(\mathbb{C})$ denotes $n \times n$ complex matrices.

This completes the exposition of von Neumann's commutant theory for unitary operators, establishing the fundamental result that bounded operators commuting with U are precisely the functions of its spectral measure $E(\cdot)$.