New Uniformly Convergent Series for the Bessel Functions of the First Kind of Integer Orders

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June 26, 2024

Definition 1. *Let $j_n(x)$ is the spherical Bessel function of the first kind,*

$$\begin{aligned} j_n(z) &= \sqrt{\frac{\pi}{2\,z}}\, J_{n+\frac{1}{2}}(x) \\ &= \frac{1}{\sqrt{z}} \Big(\sin(z)\, R_{n,\frac{1}{2}}(z) - \cos(z)\, R_{n-1,\frac{3}{2}}(z) \Big) \end{aligned} \tag{1}$$

where $R_{n,v}(z)$ are the (misnamed) Lommel polynomials [2]

$$R_{n,v}(z) = \frac{\Gamma(n+v)}{\Gamma(v)} \left(\frac{2}{z}\right)^n {}_2F_3\left(\left[-\frac{n}{2}, \frac{1}{2} - \frac{n}{2} \right]; [v, -n, 1-v-n]; -z^2 \right) \tag{2}$$

where ${}_2F_3$ is a generalized hypergeometric function. The "Lommel polynomials" are actually rational functions of z, not polynomial; but rather "polynomial in $\frac{1}{z}$".

Conjecture 2. *The series*

$$\begin{aligned} J_0(t) &= \sum_{k=0}^{\infty} \lambda_k \psi_k(t) \\ &= \sum_{k=0}^{\infty} \sqrt{\frac{4\,n+1}{\pi}}\, \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, j_{2n}(t) \\ &= \sum_{k=0}^{\infty} \frac{4\,n+1}{\pi}\, \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} (-1)^n\, j_{2n}(t) \end{aligned} \tag{3}$$

converges uniformly for all comlex t except the origin where it has a regular singular point where $\lim_{t\to 0} J_0(t) = 1$.

Conjecture 3. *The eigenfunctions of the stationary integral covariance operator*

$$[T\psi_n](x) = \int_0^\infty J_0\,(x-y)\;\psi_n(x)\mathrm{d}x = \lambda_n \psi_n(x) \tag{4}$$

are given by

$$\psi_n(y) = (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\; j_{2n}(y) \tag{5}$$

and the eigenvalues are given by

$$\begin{aligned} \lambda_n \;&= \int_{-\infty}^{\infty} J_0(x)\;\psi_n(x)\,\mathrm{d}x \\ &= \sqrt{\frac{4\,n+1}{\pi}}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\,(n+1)^2} \\ &= \sqrt{\frac{4\,n+1}{\pi}}\;(n+1)_{\frac{1}{2}}^2 \end{aligned} \tag{6}$$

where $(n+1)_{-\frac{1}{2}}^2$ *is the Pochhammer symbol(ascending/rising factorial).*

Definition 4. *The spectral density of a stationary process is the Fourier tranform of the covariance kernel due to Wiener-Khinchine theorem.*

Definition 5. *Let* $S_n(x)$ *be the orthogonal polynomials whose orthogonality measure is equal to the spectral density of the process. These polynomials shall be called the spectral polynomials corresponding to the process.*

Example 6. Let the kernel function be given by $K(t,s) = J_0(t-s)$ then identify the orthogonal polynomial sequence associated with the spectral density

$$S(\omega) = \int_{-\infty}^{\infty} J_0(x)\,e^{ix\omega}\,\mathrm{d}x = \begin{cases} \dfrac{2}{\sqrt{1-\omega^2}} & \omega \in (-1,1) \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

as being twice the orthogonality measure of the Type-I Chebyshev polynomials $T_n(x)$ so that the orthogonal polynomial sequence is identified as

$$S_n(x) = \sqrt{2}T_n(x) \tag{8}$$

so that

$$\int_{-1}^{1} S_n(\omega)\, S_m(\omega)\, S(\omega)\, \mathrm{d}\omega = \begin{cases} 0 & n \neq m \\ 2\pi & n = m = 0 \\ \pi & n = m \neq 0 \end{cases} \tag{9}$$

Remark 7. If the spectral density does not equal the orthogonality measure of a known set of orthogonal polynomials then such a set can always be generated by applying the Gram-Schmidt process to the monomials so that they are transformed into a set that is orthogonal with respect any given spectral density (of a stationary process).

Definition 8. *The sequence $\hat{S}_n(y)$ of Fourier transforms of the spectral polynomials $S_n(x)$ is given by*

$$\hat{S}_n(y) = \int_{-1}^{1} S_n(x) e^{ixy} \mathrm{d}x \tag{10}$$

Example 9. The Fourier transforms of the Chebyshev polynomials are just the usual infinite Fourier transforms with the integration restricted to the range $-1 \ldots 1$ since $T_n(x) = 0 \forall x \notin [-1, 1]$). Equivalently, the spectral density function can be extended to take the value 0 outside the interval $[-1, 1]$. The derivation of

$$\begin{aligned} \hat{T}_n(y) &= \int_{-\infty}^{\infty} e^{-ixy}\, T_n(x) d\,y = \int_{-1}^{1} e^{-ixy}\, T_n(x) d\,x \\ &= \int_{-\infty}^{\infty} e^{-ixy}\, {}_2F_1\!\left(\left. \begin{matrix} n, & -n \\ & \frac{1}{2} \end{matrix} \right| \frac{1}{2} - \frac{x}{2} \right) d\,x \\ &= \frac{i}{y}\, (e^{-iy} F_n^{+}(y) - e^{i(\pi n + y)} F_n^{-}(y)) \end{aligned} \tag{11}$$

where

$$F_n^{\pm}(y) = {}_3F_1\!\left(\left. \begin{matrix} 1, & n, & -n \\ & & \frac{1}{2} \end{matrix} \right| \frac{\pm i y}{2} \right) \tag{12}$$

can be found in [1].

Definition 10. *Let $Y_n(y)$ be the normalized spectral polynomials $S_n(x)$*

Example 11. When $K = J_0$ the spectral polynomials are given by

$$S_n(x) = \sqrt{2} T_n(x) \tag{13}$$

so that

$$
\begin{aligned}
Y_n(y) &= \frac{\hat{T}_n(y)}{|\hat{T}_n|} \\
&= \frac{i}{y}\left(\frac{e^{-iy}F_n^+(y) - e^{i(\pi n + y)}F_n^-(y)}{\sqrt{\frac{4(-1)^n\pi - (2n^2-1)}{4n^2-1}}}\right)
\end{aligned}
\tag{14}
$$

where the L^2 norm of $\hat{T}_n(y)$ is given by

$$
\begin{aligned}
|\hat{T}_n| &= \sqrt{\int_{-\infty}^{\infty} \hat{T}_n(y)^2 \mathrm{d}y} \\
&= \sqrt{\frac{4(-1)^n\pi - (2n^2-1)}{4n^2-1}}
\end{aligned}
\tag{15}
$$

Conjecture 12. *The eigenfunctions of the integral covariance operator (4) are given by the orthogonal complement of the normalized Fourier transforms $Y_n(y)$ of the spectral polynomials (via the Gram-Schmidt process)*

$$
\psi_n(y) = Y_n^{\perp}(y) = Y_n(y) - \sum_{m=1}^{n-1} \frac{\langle Y_n(y), Y_m^{\perp}(y)\rangle}{\langle Y_m^{\perp}(y), Y_m^{\perp}(y)\rangle} Y_m^{\perp}(y)
\tag{16}
$$

can be equivalently expressed as

$$
\begin{aligned}
\psi_n(y) &= (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, j_{2n}(y) \\
&= (-1)^n \sqrt{\frac{4\,n+1}{\pi}} \sqrt{\frac{\pi}{2\,y}}\, J_{2n+\frac{1}{2}}(y) \\
&= (-1)^n \sqrt{\frac{4\,n+1}{2\,y}}\, J_{2n+\frac{1}{2}}(y) \\
&= \sqrt{\frac{4\,n+1}{4\,\pi}} \int_{-1}^{1} P_{2n}(x)\, e^{ixy}\, \mathrm{d}x
\end{aligned}
\tag{17}
$$

Remark 13. Since T is compact due to its self-adjointness and convergence of the eigenvalues to 0 it converges uniformly since compactness implies uniform convergence of the eigenfunctions. TODO: cite/theorems from [3, 3. Reproducing Kernel Hilbert Space of a Gaussian Process]

1 Simplifying The Convolution

Apply the addition theorem

$$J_0\,(x-y) = \sum_{k=-\infty}^{\infty} J_k(x)\, J_k\,(-y)$$

to the integral covariance operator

$$\begin{aligned}
[T\psi_n](x) \;&= \int_0^{\infty} J_0\,(x-y)\,\psi_n(y)\; d\,y \\
&= \int_0^{\infty} \sum_{k=-\infty}^{\infty} J_k(x)\, J_k\,(-y)\,\psi_n(y)\; d\,y \\
&= \sum_{k=-\infty}^{\infty} J_k(x) \int_0^{\infty} J_k\,(-y)\,\psi_n(y)\; d\,y
\end{aligned}$$

Where $\psi_n(y)$ is:

$$\psi_n(y) = (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, j_{2n}(y) = = (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\,\sqrt{\frac{\pi}{2\,y}}\, J_{2n+\frac{1}{2}}(y)$$

Substituting

$$\int_0^{\infty} J_k\,(-y)\,\psi_n(y)\; d\,y = \frac{\sqrt{4\,n+1}\,(-1)^n \sqrt{\pi}\,\Gamma\left(\frac{k}{2}+n+\frac{1}{2}\right)}{2\,\Gamma\left(-n+\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{k}{2}+n+1\right)\Gamma\left(n+1-\frac{k}{2}\right)}$$

Now, putting it all back into the expansion for $[T\psi_n](x)$:

$$[T\psi_n](x) = \sum_{k=-\infty}^{\infty} J_k(x) \frac{\sqrt{4\,n+1}\,(-1)^n \sqrt{\pi}\,\Gamma\left(\frac{k}{2}+n+\frac{1}{2}\right)}{2\,\Gamma\left(-n+\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{k}{2}+n+1\right)\Gamma\left(n+1-\frac{k}{2}\right)}$$

Bibliography

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[3] Steven P. Lalley. Introduction to gaussian processes. https://galton.uchicago.edu/∼elalley/Courses/386/GaussianProcesses.pdf, 2013.