Gaussian Processes Generated By Monotonically Modulated Stationary Gaussian Process Kernels

by Stephen Crowley

January 3, 2025

Definition 1

Let $\mathcal{F}$ denote the class of functions $f: \mathbb{R} \rightarrow \mathbb{R}$ which are:

1. *piecewise continuous with piecewise continuous first derivative,*
2. *strictly monotonically increasing*

$$f(t) < f(s) \forall -\infty \leqslant t < s \leqslant \infty \tag{1}$$

3. *and have a finite limiting derivative at infinity*

$$\lim_{t \rightarrow \infty} \dot{f}(t) < \infty \tag{2}$$

Theorem 2

(Eigenfunctions) *For any stationary kernel $K(t,s) = K(|t-s|)$, the eigenfunctions of the modulated kernel*

$$K_f(s,t) = K(|f(t) - f(s)|) \tag{3}$$

take the form:

$$\phi_n(t) = \psi_n(f(t)) \sqrt{\dot{f}(t)} \tag{4}$$

where where $f \in \mathcal{F}$ and ψ_n are the normalized eigenfunctions of the original unmodulated kernel $K(|t-s|)$.

Proof. The eigenfunction equation for the modulated kernel is:

$$\int_{-\infty}^{\infty} K(|f(t)-f(s)|)\,\phi_n(s)\,ds=\lambda_n\,\phi_n(t) \tag{5}$$

The variables can be changed by substituting $u=f(s)$, $v=f(t)$:

$$\int_{-\infty}^{\infty} K(|v-u|)\frac{\phi_n(f^{-1}(u))}{\dot{f}(f^{-1}(u))}\,du=\lambda_n\,\phi_n(f^{-1}(v)) \tag{6}$$

which is valid due to the strict monotonicity of f which assures its invertability. Let

$$\psi_n(u)=\frac{\phi_n(f^{-1}(u))}{\sqrt{\dot{f}(f^{-1}(u))}} \tag{7}$$

Then:

$$\int_{-\infty}^{\infty} K(|v-u|)\,\psi_n(u)\,du=\lambda_n\,\psi_n(v) \tag{8}$$

This is precisely the eigenfunction equation for the original kernel $K(|t-s|)$. Therefore, if ψ_n are the eigenfunctions of the original kernel, then

$$\phi_n(t)=\psi_n(f(t))\sqrt{\dot{f}(t)} \tag{9}$$

are the eigenfunctions of the modulated kernel. □

Theorem 3

(Normalization) *If ψ_n are normalized eigenfunctions of the original kernel, then $\phi_n(t)=\psi_n(f(t))\sqrt{\dot{f}(t)}$ are automatically normalized eigenfunctions of the modulated kernel, requiring no additional normalization constants.*

Proof. For normalized ψ_n:

$$\int_{-\infty}^{\infty} |\phi_n(t)|^2 dt=\int_{-\infty}^{\infty} |\psi_n(f(t))|^2 \dot{f}(t)\,dt \tag{10}$$

Under the change of variables $u = f(t)$:

$$\int_{-\infty}^{\infty} |\psi_n(u)|^2 \, du = 1 \tag{11}$$

Therefore the ϕ_n are already normalized without additional constants. $\square$

Corollary 4

(Eigenvalue Invariance) *The eigenvalues $\{\lambda_n\}$ of the modulated kernel K_f are identical to those of the original kernel K.*

Remark 5. This result demonstrates that monotonic modulation preserves the spectral structure of any stationary kernel through composition with the modulation function. The transformation operator

$$(T\phi)(t) = \sqrt{\dot{f}(t)}\, \phi(f(t)) \tag{12}$$

provides an explicit isometry between the original and modulated kernel Hilbert spaces, explaining why no additional normalization constants are needed.

Theorem 6

(Mean Zero-Counting Function) *Let $f \in \mathcal{F}$ and let $K(\cdot)$ be any positive-definite, stationary covariance function, twice differentiable at 0. Consider the centered Gaussian process with covariance*

$$K_f(s,t) = K(|f(t) - f(s)|) \tag{13}$$

Then the expected number of zeros in $[0, T]$ is

$$\mathbb{E}[N([0,T])] = \sqrt{-K''(0)}\, \left(f(T) - f(0)\right) \tag{14}$$

Proof. By the Kac-Rice formula:

$$\mathbb{E}[N([0,T])] = \int_0^T \sqrt{-\lim_{s \to t} \frac{\partial^2}{\partial t \, \partial s} K_f(s,t)} \, dt \tag{15}$$

Computing the mixed partial derivative and taking the limit as $s \to t$:

$$\lim_{s \to t} \frac{\partial^2}{\partial t \, \partial s} K_f(s,t) = -\ddot{K}(0) \, \dot{f}(t)^2 \tag{16}$$

Therefore

$$\mathbb{E}[N([0,T])] = \sqrt{-\ddot{K}(0)} \int_0^T \dot{f}(t) \, dt = \sqrt{-\ddot{K}(0)} \, (f(T) - f(0)) \tag{17}$$

so that

$$\begin{aligned} \sqrt{-\ddot{K}(0)} \, (f(T) - f(0)) &= \sqrt{-\ddot{K}(0)} \int_0^T \dot{f}(t) \, dt \\ &= \int_0^T \sqrt{-\ddot{K}(0) \dot{f}(t)^2} \, dt \\ &= \int_0^T \sqrt{-\lim_{s \to t} \frac{\partial^2}{\partial t \, \partial s} K(|f(t) - f(s)|)} \, dt \end{aligned} \tag{18}$$

which is precisely the Kac-Rice formula for the expected zero-count. $\square$