The Borg-Levinson Theorem: $\boldsymbol{\lambda_n(V_1) = \lambda_n(V_2) \forall n}$ implies $\boldsymbol{V_1(x) = V_2(x)}$ When $\boldsymbol{V_1(x)}$ and $\boldsymbol{V_2(x)}$ are Even Functions

In exploring the structure of spectral problems and the inference theory of stochastic processes, a pivotal aspect emerges from the study of Bessel polynomials by Krall. The investigation into real continuous orthogonalizing functions of bounded variation unveiled the indeterminacy of the associated moment problem. The revelation of an infinite set of possible orthogonalizing measures, initially considered problematic, is indeed foundational for the continuous transformation of a potential function while maintaining unchanged eigenvalues. This characteristic underlines the mathematical framework where the non-uniqueness of measures becomes a prerequisite for the invariant nature of eigenvalues as the eigenfunction evolves under the potential's influence.

Given two potential functions $V_1(x)$ and $V_2(x)$, it is critical to understand under what conditions their spectral data, specifically the eigenvalues $\lambda_n(V_1) = \lambda_n(V_2)$ for all $n \geq 1$, imply the equivalence of $V_1(x)$ and $V_2(x)$. Borg and Levinson's theorems provide insights into this matter by establishing the uniqueness of the potential function from spectral data under certain symmetries. Their first theorem posits that if $V_1(x)$ and $V_2(x)$ are even, then $\lambda_n(V_1) = \lambda_n(V_2)$ for all $n > 1$ necessarily implies $V_1(x) = V_2(x)$.

Theorem 1 (Borg-Levinson):

$$\text{If } V_1(x) \text{ and } V_2(x) \text{ are even, then } \lambda_n(V_1) = \lambda_n(V_2)\, \forall n > 1 \text{ implies that } V_1(x) = V_2(x). \tag{1}$$

This theorem is complemented by a second theorem addressing non-even potentials, indicating that additional information, such as the derivatives of eigenfunctions at the boundary, is necessary for establishing uniqueness.

Theorem 2 (Borg-Levinson): Let $y(x;\lambda)$ satisfy the differential equation with boundary conditions

$$y(0;\lambda) = 0, \tag{2}$$
$$\dot{y}(0;\lambda) = 1, \tag{3}$$
$$k_n(V) = \dot{y}(1;\lambda_n(V)). \tag{4}$$

The mathematical parallel between the spectral analysis of differential operators and the distinguishability of hypotheses in statistical decision theory extends beyond mere analogy to share a foundational structure. Both domains hinge on the extraction of underlying parameters from observations — in spectral theory, these are the potential functions from eigenvalues, and in statistical inference, the determination of which hypothesis is supported by the data.

The critical insight from Krall's examination of Bessel polynomials lies in recognizing the non-uniqueness of orthogonalizing measures not as a limitation but as a necessary condition for the flexibility of spectral characteristics. This observation directly translates to the inference process in statistical decision theory, where the ability to distinguish between hypotheses depends on the underlying assumptions and the information gleaned from the observations.

1 Reiteration

The discussion revolves around an interesting facet of spectral theory, particularly in the context of quantum mechanics or mathematical physics, where the notion of the potential function $V(x)$ plays a central role in determining the behavior of a quantum system. The indeterminacy of the moment problem, as mentioned in relation to the Bessel polynomials and orthogonalizing measures, indeed points to a deeper mathematical principle: the non-uniqueness of solutions to certain spectral problems.

The core question here pertains to whether two different potential functions, $V_1(x)$ and $V_2(x)$, can have the same set of eigenvalues ($\lambda_n(V_1) = \lambda_n(V_2)$ for all $n \geq 1$). This question touches upon the uniqueness or lack thereof in the inversion of spectral data to determine a potential. The example provided, where $V_2(x) = V_1(1-x) \neq V_1(x)$, demonstrates a counterexample to the naive expectation of uniqueness. This non-uniqueness illustrates the subtlety in the relationship between a potential function and its spectral properties.

However, Borg and Levinson's theorem provides a significant breakthrough in this context by establishing conditions under which uniqueness can be assured, at least for a subclass of potential functions. The theorem states:

Theorem 1 (Borg-Levinson): If $V_1(x)$ and $V_2(x)$ are even functions, then the condition

$$\lambda_n(V_1) = \lambda_n(V_2)\, \forall n > 1$$

implies that $V_1(x) = V_2(x)$.

This result is monumental because it offers a criterion (evenness of the potential) under which the spectral data (eigenvalues) uniquely determine the potential, up to the specified symmetry.

For potentials that are not even, Borg and Levinson further showed that additional information—specifically, *the values of the derivatives of the eigenfunctions at the boundary*—can provide the necessary conditions for uniqueness. This is encapsulated in their second theorem, which asserts that knowledge of the boundary behavior of the system's wave functions, alongside the spectral data, can uniquely determine the potential.

The mathematical exploration of these theorems and their implications illuminates the intricate dance between a system's potential landscape and its quantum or vibrational spectra. It also underlines a profound truth in mathematical physics: the geometry and boundary conditions of a system are deeply entwined with its spectral characteristics, and subtle symmetries or conditions can dramatically influence the interpretability and uniqueness of solutions to spectral problems.