Unitary Time Changes of Stationary Processes Yield Oscillatory Processes and a Functional Framework Toward a Hilbert–Pólya Construction

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are square-integrable over σ-compact sets. Applying U_θ to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot\theta(t)}\, e^{i\lambda\theta(t)}$ and evolutionary spectrum $d\,F_t(\lambda) = \dot\theta(t)\, d\,F(\lambda)$. It is proved that sample paths of any non-degenerate second-order stationary process almost surely lie in $L^2_{\sigma\text{-comp}}(\mathbb{R})$, making the operator applicable to typical realizations. A zero-localization measure $\mu\,(d\,t) = \delta(Z(t))\,|Z'(t)|\, d\,t$ induces a Hilbert space $L^2(\mu)$ on the zero set of an oscillatory process Z, and the multiplication operator $(L\,f)(t) = t\,f(t)$ has pure point, simple spectrum equal to the zero set of Z. This produces a concrete operator scaffold consistent with a Hilbert–Pólya-type viewpoint.

Table of contents

1 Function Spaces and Unitary Time Change

1.1 σ-compact sets and local L^2

Definition 1

[σ-compact sets] A subset $U \subseteq \mathbb{R}$ is σ-compact if

$$U = \bigcup_{n=1}^{\infty} K_n \tag{1}$$

with each K_n compact.

Definition 2

[Square-integrability on σ-compact sets] Define

$$L^2_{\sigma\text{-}comp}(\mathbb{R}) := \left\{ f \colon \mathbb{R} \to \mathbb{C} \colon \int_U |f(t)|^2 \, dt < \infty \text{ for every } \sigma\text{-compact } U \subseteq \mathbb{R} \right\} \tag{2}$$

Remark 3. Every bounded measurable set in $\mathbb{R}$ is σ-compact; hence $L^2_{\sigma\text{-comp}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

1.2 Unitary time-change operator

Definition 4

[Unitary time-change] Let $\theta \colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with $\dot{\theta}(t) > 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero. The function θ maps σ-compact sets to σ-compact sets. Define, for f measurable,

$$(U_\theta f)(t) = \sqrt{\dot{\theta}(t)} \, f(\theta(t)) \tag{3}$$

Proposition 5

[Inverse map] The inverse map is given by

$$(U_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{4}$$

which is well-defined almost everywhere on every σ-compact set.

Proof. Since $\dot{\theta}(t)=0$ only on sets of measure zero, and θ^{-1} maps sets of measure zero to sets of measure zero (as absolutely continuous bijective functions preserve measure-zero sets), the denominator $\sqrt{\dot{\theta}(\theta^{-1}(s))}$ is positive almost everywhere. The expression is therefore well-defined almost everywhere on every σ-compact set, which suffices for defining an element of $L^2_{\sigma\text{-comp}}(\mathbb{R})$. $\qquad\square$

Theorem 6

*[**Local unitarity on σ-compact sets**] For every σ-compact set $C \subseteq \mathbb{R}$ and $f \in L^2_{\sigma\text{-comp}}(\mathbb{R})$,*

$$\int_C |(U_\theta f)(t)|^2 \, dt = \int_{\theta(C)} |f(s)|^2 \, ds \tag{5}$$

Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\sigma\text{-comp}}(\mathbb{R})$.

Proof. Let $f \in L^2_{\sigma\text{-comp}}(\mathbb{R})$ and let U be any σ-compact set. The local L^2-norm of $U_\theta f$ over C is:

$$\begin{aligned}
\int_C |(U_\theta f)(t)|^2 \, dt &= \int_C \left| \sqrt{\dot{\theta}(t)} \, f(\theta(t)) \right|^2 \, dt \\
&= \int_C \dot{\theta}(t) |f(\theta(t))|^2 \, dt
\end{aligned} \tag{6}$$

Since θ is absolutely continuous and strictly increasing, applying the change of variables $s = \theta(t)$ gives

$$ds = \dot{\theta}(t)dt \tag{7}$$

almost everywhere. Since θ maps σ-compact sets to σ-compact sets, as t ranges over C, $s = \theta(t)$ ranges over $\theta(C)$, which is σ-compact. Therefore:

$$\int_C \dot{\theta}(t) |f(\theta(t))|^2 \, dt = \int_{\theta(C)} |f(s)|^2 \, ds \tag{8}$$

To verify that U_θ^{-1} is indeed the inverse, we compute explicitly. For any $f \in L^2_{\sigma\text{-comp}}(\mathbb{R})$:

$$\begin{aligned}
(U_\theta^{-1} U_\theta f)(s) &= \left(U_\theta^{-1} \sqrt{\dot{\theta}(s)} \, f(\theta(s)) \right)(s) \\
&= \frac{\sqrt{\dot{\theta}(\theta^{-1}(s))}}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} f(\theta(\theta^{-1}(s))) \\
&= f(s)
\end{aligned} \tag{9}$$

since $\theta(\theta^{-1}(s)) = s$. Similarly, for any $g \in L^2_{\sigma\text{-comp}}(\mathbb{R})$:

$$
\begin{aligned}
(U_\theta U_\theta^{-1} g)(t) &= \sqrt{\dot{\theta}(t)}\,(U_\theta^{-1} g)(\theta(t)) \\
&= \frac{\sqrt{\dot{\theta}(t)}}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} g(\theta^{-1}(\theta(t))) \\
&= \frac{\sqrt{\dot{\theta}(t)}}{\sqrt{\dot{\theta}(t)}} g(t) \\
&= g(t)
\end{aligned}
\tag{10}
$$

since $\theta^{-1}(\theta(t)) = t$. Therefore

$$
U_\theta U_\theta^{-1} = U_\theta^{-1} U_\theta = I
\tag{11}
$$

on $L^2_{\sigma\text{-comp}}(\mathbb{R})$. $\qquad\square$

> **Theorem 7**
>
> **[Unitarity on $L^2(\mathbb{R})$]** $U_\theta \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *is unitary:*
>
> $$
> \int_{\mathbb{R}} |(U_\theta f)(t)|^2\, dt = \int_{\mathbb{R}} |f(s)|^2\, ds
> \tag{12}
> $$
>
> *and U_θ^{-1} is its inverse.*

Proof. For $f \in L^2(\mathbb{R})$, we have:

$$
\int_{\mathbb{R}} |(U_\theta f)(t)|^2\, dt = \int_{\mathbb{R}} \dot{\theta}(t) |f(\theta(t))|^2\, dt
\tag{13}
$$

By the change of variables

$$
s = \theta(t)
\tag{14}
$$

with

$$
ds = \dot{\theta}(t)\, dt
\tag{15}
$$

, and since $\theta \colon \mathbb{R} \to \mathbb{R}$ is bijective:

$$
\int_{\mathbb{R}} \dot{\theta}(t) |f(\theta(t))|^2\, dt = \int_{\mathbb{R}} |f(s)|^2\, ds
\tag{16}
$$

The inverse relationship follows from the same computation as in Theorem 6, applied globally. $\qquad\square$

2 Oscillatory Processes (Priestley)

Definition 8

[Oscillatory process] Let F be a finite nonnegative Borel measure on $\mathbb{R}$. For each $t \in \mathbb{R}$, let $A_t \in L^2(F)$ be the gain function and

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{17}$$

be the corresponding oscillatory function then an oscillatory process *is a stochastic process which can be represented as*

$$\begin{aligned} Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda)\, \mathrm{d}\Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\mathrm{d}\Phi(\lambda) \end{aligned} \tag{18}$$

where Φ is a complex orthogonal random measure with spectral measure F, that is,

$$\mathbb{E}[\Phi\,(d\,\lambda)\,\overline{\Phi\,(d\,\mu)}] = \delta\,(\lambda - \mu)\,d\,F(\lambda) \tag{19}$$

and corresponding covariance kernel

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \int_{\mathbb{R}} A_t(\lambda)\,\overline{A_s(\lambda)}\, e^{i\lambda(t-s)}\, d\,F(\lambda) \\ &= \int_{\mathbb{R}} \phi_t(\lambda)\overline{\phi_s(\lambda)}\mathrm{d}F(\lambda) \end{aligned} \tag{20}$$

Remark 9. [Real-valuedness] Z is real-valued if and only if

$$A_t(-\lambda) = \overline{A_t(\lambda)} \tag{21}$$

for F-a.e. λ, equivalently

$$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \tag{22}$$

for F-a.e. λ. TODO: this needs to be recast as a theorem

> **Theorem 10**
>
> **[Existence]** *If F is finite and $(A_t)_{t \in \mathbb{R}}$ is measurable in t with*
>
> $$\int_{\mathbb{R}} |A_t(\lambda)|^2 \, d\, F(\lambda) < \infty \forall t \in \mathbb{R} \tag{23}$$
>
> *then there exists a complex orthogonal random measure Φ with spectral measure F such that*
>
> $$Z(t) = \int_{\mathbb{R}} A_t(\lambda) \, e^{i\lambda t} \, \Phi\,(d\,\lambda) \tag{24}$$
>
> *is well-defined in $L^2(\Omega)$ and has covariance R_Z as in (20) above.*

Proof. We construct the stochastic integral using the standard extension procedure. First, define the integral for simple functions of the form

$$g(\lambda) = \sum_{j=1}^{n} c_j \, 1_{E_j}(\lambda) \tag{25}$$

where $\{E_j\}$ are disjoint Borel sets with $F(E_j) < \infty$ and $c_j \in \mathbb{C}$:

$$\int_{\mathbb{R}} g(\lambda) \, \Phi\,(d\,\lambda) := \sum_{j=1}^{n} c_j \, \Phi(E_j) \tag{26}$$

For such simple functions, the isometry property holds:

$$
\begin{aligned}
\mathbb{E}\left[\left| \int_{\mathbb{R}} g(\lambda) \, \Phi\,(d\,\lambda) \right|^2 \right] &= \mathbb{E}\left[\left| \sum_{j=1}^{n} c_j \, \Phi(E_j) \right|^2 \right] \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \bar{c}_k \mathbb{E}[\Phi(E_j)\overline{\Phi(E_k)}] \\
&= \sum_{j=1}^{n} |c_j|^2 \, F(E_j \\
&= \int_{\mathbb{R}} |g(\lambda)|^2 \, d\, F(\lambda)
\end{aligned}
\tag{27}
$$

Since simple functions are dense in $L^2(F)$, we extend by continuity to all $g \in L^2(F)$. For each t, since

$$\varphi_t(\lambda) = A_t(\lambda) \, e^{i\lambda t} \tag{28}$$

6

and $A_t \in L^2(F)$, we have $\varphi_t \in L^2(F)$. Therefore

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda) \, \Phi(d\lambda) \tag{29}$$

is well-defined in $L^2(\Omega)$. The covariance is computed as:

$$
\begin{aligned}
R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\
&= \mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda) \, \Phi(d\lambda) \int_{\mathbb{R}} \overline{\varphi_s(\mu)} \, \overline{\Phi(d\mu)} \right] \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\mu)} \mathbb{E}\left[\Phi(d\lambda) \overline{\Phi(d\mu)} \right] \\
&= \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\lambda)} \, dF(\lambda) \\
&= \int_{\mathbb{R}} A_t(\lambda) \overline{A_s(\lambda)} e^{i\lambda(t-s)} \, dF(\lambda)
\end{aligned} \tag{30} \square
$$

3 Stationary Processes and Time Change

3.1 Stationary processes

Definition 11

*[**Cramér representation**] A zero-mean stationary process X with spectral measure F admits the sample path representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t} \, \Phi(d\lambda) \tag{31}$$

which has covariance

$$R_X(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)} \, dF(\lambda) \tag{32}$$

3.2 Stationary $\rightarrow$ oscillatory via U_θ

Theorem 12

[Time change yields oscillatory process] Let X be zero-mean stationary as in Definition 11. For θ as in Definition 4, define

$$
\begin{aligned}
Z(t) \;&=(U_\theta\, X)(t) \\
&=\sqrt{\dot\theta(t)}\; X(\theta(t))
\end{aligned}
\tag{33}
$$

Then Z is oscillatory with oscillatory function

$$
\varphi_t(\lambda) = \sqrt{\dot\theta(t)}\; e^{i\lambda\theta(t)}
\tag{34}
$$

, gain function

$$
A_t(\lambda) = \sqrt{\dot\theta(t)}\; e^{i\lambda(\theta(t)-t)}
\tag{35}
$$

, and covariance

$$
R_Z(t,s) = \int_{\mathbb{R}} \sqrt{\dot\theta(t)\,\dot\theta(s)}\; e^{i\lambda(\theta(t)-\theta(s))}\, dF(\lambda)
\tag{36}
$$

Proof. Applying the unitary time change operator to the spectral representation of $X(t)$:

$$
\begin{aligned}
Z(t) \;&=(U_\theta\, X)(t) \\
&=\sqrt{\dot\theta(t)}\; X(\theta(t)) \\
&=\sqrt{\dot\theta(t)}\int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\Phi(\lambda) \\
&=\int_{\mathbb{R}} \sqrt{\dot\theta(t)}\, e^{i\lambda\theta(t)}\, d\Phi(\lambda) \\
&=\int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda)
\end{aligned}
\tag{37}
$$

where

$$
\varphi_t(\lambda) = \sqrt{\dot\theta(t)}\, e^{i\lambda\theta(t)}
\tag{38}
$$

To verify this constitutes an oscillatory representation according to Definition 8, we must write $\varphi_t(\lambda)$ in the form $A_t(\lambda)\, e^{i\lambda t}$:

$$
\begin{aligned}
\varphi_t(\lambda) \;&=\sqrt{\dot\theta(t)}\, e^{i\lambda\theta(t)} \\
&=\sqrt{\dot\theta(t)}\, e^{i\lambda(\theta(t)-t)}\, e^{i\lambda t} \\
&=A_t(\lambda)\, e^{i\lambda t}
\end{aligned}
\tag{39}
$$

where

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t) - t)} \tag{40}$$

Since $\dot{\theta}(t) \geq 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of measure zero, $A_t(\lambda)$ is well-defined almost everywhere. Moreover, $A_t \in L^2(F)$ for each t since:

$$
\begin{aligned}
\int_{\mathbb{R}} |A_t(\lambda)|^2 \, dF(\lambda) &= \int_{\mathbb{R}} \left| \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t) - t)} \right|^2 dF(\lambda) \\
&= \int_{\mathbb{R}} \dot{\theta}(t) |e^{i\lambda(\theta(t) - t)}|^2 \, dF(\lambda) \\
&= \dot{\theta}(t) \int_{\mathbb{R}} dF(\lambda) \\
&= \dot{\theta}(t) \, F(\mathbb{R}) < \infty
\end{aligned}
\tag{41}
$$

where we used $|e^{i\alpha}| = 1$ for all real α. The covariance is computed as:

$$
\begin{aligned}
R_Z(t, s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\
&= \mathbb{E}\left[\sqrt{\dot{\theta}(t)}\, X(\theta(t)) \sqrt{\dot{\theta}(s)}\, \overline{X(\theta(s))} \right] \\
&= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\, \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \\
&= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\, R_X(\theta(t) - \theta(s)) \\
&= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t) - \theta(s))} \, dF(\lambda)
\end{aligned}
\tag{42}
$$

$\square$

<div style="background:#000080;color:white;padding:4px;font-weight:bold">Corollary 13</div>

[Evolutionary spectrum] *The evolutionary spectrum is*

$$
\begin{aligned}
dF_t(\lambda) &= |A_t(\lambda)|^2 \, dF(\lambda) \\
&= \dot{\theta}(t) \, dF(\lambda)
\end{aligned}
\tag{43}
$$

Proof. By definition of the evolutionary spectrum and using the gain function from Theorem 12:

$$
\begin{aligned}
dF_t(\lambda) &= |A_t(\lambda)|^2 \, dF(\lambda) \\
&= \left| \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t) - t)} \right|^2 dF(\lambda) \\
&= \dot{\theta}(t) |e^{i\lambda(\theta(t) - t)}|^2 \, dF(\lambda) \\
&= \dot{\theta}(t) \, dF(\lambda)
\end{aligned}
\tag{44}
$$

since

$$|e^{i\alpha}| = 1 \,\forall a \in \mathbb{R} \tag{45} \quad\square$$

3.3 Covariance operator conjugation

Proposition 14

[Operator conjugation] Let

$$(T_K f)(t) := \int_{\mathbb{R}} K(|t - s|) \ f(s) \ ds \tag{46}$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h} \ dF(\lambda) \tag{47}$$

Define the transformed kernel

$$K_\theta(s,t) := \sqrt{\dot\theta(t)\,\dot\theta(s)} \ K(|\theta(t) - \theta(s)|) \tag{48}$$

and corresponding integral covariance operator

$$(T_{K_\theta} f)(t) := \int_{\mathbb{R}} K_\theta(s,t) \ f(s) \ ds \tag{49}$$

Then

$$T_{K_\theta} = U_\theta \ T_K \ U_\theta^{-1} \tag{50}$$

on $L^2_{\sigma\text{-comp}}(\mathbb{R})$.

Proof. For any $g \in L^2_{\sigma\text{-comp}}(\mathbb{R})$, we transform the integral operator from coordinates (r, w) to coordinates (t, s) by applying both coordinate transformations

$$r = \theta(t) \tag{51}$$

and

$$w = \theta(s) \tag{52}$$

by simultaneously substitution with Jacobians

$$dr = \dot\theta(t) \ dt \tag{53}$$

and

$$dw = \dot\theta(s) \ ds \tag{54}$$

The operator T_K in (r, w) coordinates is:

$$(T_K f)(r) = \int_{\mathbb{R}} K(|r - w|) \, f(w) \, dw \tag{55}$$

Under the simultaneous transformation $r = \theta(t)$ and $w = \theta(s)$:

$$
\begin{aligned}
((U_\theta T_K U_\theta^{-1}) \, g)(t) &= \sqrt{\dot\theta(t)\,\dot\theta(s)} \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|) \, (U_\theta^{-1} g)(\theta(s)) \frac{\dot\theta(s)}{\sqrt{\dot\theta(s)}} \, ds \\
&= \sqrt{\dot\theta(t)\,\dot\theta(s)} \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|) \frac{g(s)}{\sqrt{\dot\theta(s)}} \sqrt{\dot\theta(s)} \, ds \\
&= \sqrt{\dot\theta(t)\,\dot\theta(s)} \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|) \, g(s) \, ds \\
&= \int_{\mathbb{R}} \sqrt{\dot\theta(t)\,\dot\theta(s)} \, K(|\theta(t) - \theta(s)|) \, g(s) \, ds \\
&= \int_{\mathbb{R}} K_\theta(t, s) \, g(s) \, ds \\
&= (T_{K_\theta} g)(t)
\end{aligned}
\tag{56}
$$

where

$$K_\theta(t, s) = \sqrt{\dot\theta(t)\,\dot\theta(s)} \, K(|\theta(t) - \theta(s)|) \tag{57}$$

Therefore

$$T_{K_\theta} = U_\theta T_K U_\theta^{-1} \tag{58} \ \square$$

4 Sample Paths Live in $L^2_{\sigma\text{-comp}}$

Theorem 15

[Sample paths in $L^2_{\sigma\text{-comp}}(\mathbb{R})$] Let $\{X(t)\}_{t \in \mathbb{R}}$ be a second-order stationary process with

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \tag{59}$$

then, almost surely, every sample path $t \mapsto X(\omega, t)$ belongs to $L^2_{\sigma\text{-comp}}(\mathbb{R})$.

Proof. Fix any bounded interval $[a, b]$ and consider the random variable

$$Y_{[a,b]} := \int_a^b X(t)^2 \, dt \tag{60}$$

By stationarity and Fubini's theorem:

$$
\begin{aligned}
\mathbb{E}[Y_{[a,b]}] = \mathbb{E}\left[\int_a^b X(t)^2 \, dt\right] &= \int_a^b \mathbb{E}[X(t)^2] \, dt \\
&= \int_a^b \sigma^2 \, dt \\
&= \sigma^2 \, (b - a) < \infty
\end{aligned}
\tag{61}
$$

By Markov's inequality, for any $M > 0$:

$$
P\left(Y_{[a,b]} > M\right) \le \frac{\mathbb{E}[Y_{[a,b]}]}{M} = \frac{\sigma^2 \, (b - a)}{M}
\tag{62}
$$

Taking $M \to \infty$, we conclude

$$
P\left(Y_{[a,b]} < \infty\right) = 1
\tag{63}
$$

, i.e., almost surely the sample path is square-integrable on $[a, b]$. Since $\mathbb{R}$ is the countable union of bounded intervals:

$$
\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]
\tag{64}
$$

by countable subadditivity of probability:

$$
P\left(\bigcap_{n=1}^{\infty} \left\{\int_{-n}^{n} X(t)^2 \, dt < \infty\right\}\right) = 1
\tag{65}
$$

Now let U be any σ-compact set. Then

$$
U = \bigcup_{m=1}^{\infty} K_m
\tag{66}
$$

where each K_m is compact. Each compact set K_m is bounded, so

$$
K_m \subseteq [-N_m, N_m]
\tag{67}
$$

for some N_m. Therefore:

$$
\begin{aligned}
\int_U X(t)^2 \, dt &= \int_{\bigcup_{m=1}^{\infty} K_m} X(t)^2 \, dt \\
&\le \sum_{m=1}^{\infty} \int_{K_m} X(t)^2 \, dt \\
&\le \sum_{m=1}^{\infty} \int_{-N_m}^{N_m} X(t)^2 \, dt
\end{aligned}
\tag{68}
$$

12

Since each integral

$$\int_{-N_m}^{N_m} X(t)^2 \, dt < \infty \tag{69}$$

almost surely, and the sum of countably many finite terms is finite, we have

$$\int_U X(t)^2 \, dt < \infty \tag{70}$$

almost surely. This holds for every σ-compact set U, so almost surely every sample path lies in $L^2_{\sigma\text{-comp}}(\mathbb{R})$. $\square$

5 Zero Localization and Hilbert–Pólya Scaffold

5.1 Zero localization measure

Definition 16

[Zero localization measure] Let Z be real-valued with $Z \in C^1(\mathbb{R})$ having only simple zeros

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \tag{71}$$

Define, for Borel $B \subset \mathbb{R}$,

$$\mu(B) := \int_{\mathbb{R}} 1_B(t) \, \delta(Z(t)) \, |\dot{Z}(t)| \, dt \tag{72}$$

Theorem 17

[Atomicity on the zero set] For every $\phi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \phi(t) \, \delta(Z(t)) \, |\dot{Z}(t)| \, dt = \sum_{t_0 : Z(t_0) = 0} \phi(t_0) \tag{73}$$

hence

$$\mu(t) = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0}(t) \tag{74}$$

Proof. Since all zeros of Z are simple and $Z \in C^1(\mathbb{R})$, by the inverse function theorem each zero t_0 is isolated. Near each zero t_0, Z is locally monotonic, so we can apply the one-dimensional change of variables formula for the Dirac delta.

Specifically, near t_0 where $Z(t_0) = 0$ and $\dot{Z}(t_0) \neq 0$, we have locally

$$Z(t) = (t - t_0) \, \dot{Z}(t_0) + O((t - t_0)^2) \tag{75}$$

The distributional identity for the Dirac delta under smooth changes of variables gives:

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0) = 0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \tag{76}$$

Therefore:

$$\begin{aligned}
\int_{\mathbb{R}} \phi(t)\, \delta(Z(t))\, |\dot{Z}(t)|\, dt &= \int_{-\infty}^{\infty} \phi(t)\, |\dot{Z}(t)| \sum_{t_0 : Z(t_0) = 0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|}\, dt \\
&= \sum_{t_0 : Z(t_0) = 0} \int_{\mathbb{R}} \phi(t) \frac{|\dot{Z}(t)|\, \delta(t - t_0)}{|\dot{Z}(t_0)|}\, dt \\
&= \sum_{t_0 : Z(t_0) = 0} \frac{|\dot{Z}(t_0)|}{|\dot{Z}(t_0)|} \phi(t_0) \\
&= \sum_{t_0 : Z(t_0) = 0} \phi(t_0)
\end{aligned} \tag{77}$$

This shows that μ is the discrete measure

$$\mu = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0} \tag{78}$$

assigning unit mass to each zero. $\qquad\square$

5.2 Hilbert space on zeros and multiplication operator

Definition 18

[Hilbert space on the zero set] *Let $\mathcal{H} = L^2(\mu)$ with inner product*

$$\langle f, g \rangle = \int f(t) \overline{g(t)}\, \mu(dt) \tag{79}$$

Proposition 19

[Atomic structure] *Let*

$$\mu = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0} \tag{80}$$

then

$$\mathcal{H} \cong \left\{ f : \{t_0 : Z(t_0) = 0\} \to \mathbb{C} : \sum_{t_0 : Z(t_0) = 0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \tag{81}$$

with orthonormal basis $\{e_{t_0}\}_{t_0 : Z(t_0) = 0}$ where

$$e_{t_0}(t_1) = \delta_{t_0 t_1} \tag{82}$$

14

Proof. By the atomic form of μ, for any $f \in L^2(\mu)$:

$$\|f\|_{\mathcal{H}}^2 = \int |f(t)|^2 \, \mu(dt) \tag{83}$$

$$= \int |f(t)|^2 \sum_{t_0 : Z(t_0)=0} \delta_{t_0}(dt) \tag{84}$$

$$= \sum_{t_0 : Z(t_0)=0} |f(t_0)|^2 \tag{85}$$

This shows the isomorphism with ℓ^2. The functions e_{t_0} defined by

$$e_{t_0}(t_1) = \delta_{t_0 t_1} \tag{86}$$

satisfy:

$$\langle e_{t_0}, e_{t_1} \rangle = \int e_{t_0}(t) \overline{e_{t_1}(t)} \, \mu(dt) = \sum_{t : Z(t)=0} \delta_{t_0 t} \, \delta_{t_1 t} = \delta_{t_0 t_1} \tag{87}$$

so they form an orthonormal set. Any $f \in \mathcal{H}$ can be written as

$$f = \sum_{t_0 : Z(t_0)=0} f(t_0) \, e_{t_0} \tag{88}$$

proving they form a basis. $\qquad\square$

Definition 20

[Multiplication operator] *Define* $L : \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H}$ *by*

$$(L f)(t) = t f(t) \tag{89}$$

on $\mathrm{supp}(\mu)$ *with domain*

$$\mathcal{D}(L) := \left\{ f \in \mathcal{H} : \int |t \, f(t)|^2 \, \mu(dt) < \infty \right\} \tag{90}$$

Theorem 21

[Self-adjointness and spectrum] L *is self-adjoint on* $\mathcal{H}$ *and has pure point, simple spectrum*

$$\sigma(L) = \{ t \in \mathbb{R} : Z(t) = 0 \} \tag{91}$$

with eigenvalues $\lambda = t_0$ *and eigenvectors* e_{t_0}.

15

Proof. First, we verify self-adjointness. For $f, g \in \mathcal{D}(L)$:

$$
\begin{aligned}
\langle L f, g \rangle &= \int (L f)(t) \overline{g(t)} \, \mu \, (d t) \\
&= \int t \, f(t) \overline{g(t)} \, \mu \, (d t) \\
&= \int f(t) \overline{t \, g(t)} \, \mu \, (d t) \\
&= \int f(t) \overline{(L \, g)(t)} \, \mu \, (d t) \\
&= \langle f, L \, g \rangle
\end{aligned}
\tag{92}
$$

Thus L is symmetric and acts as

$$
(L f)(t_0) = t_0 \, f(t_0)
\tag{93}
$$

for each t_0 in the atomic representation where

$$
Z(t_0) = 0
\tag{94}
$$

This is unitarily equivalent to the diagonal operator on ℓ^2 with diagonal entries

$$
\{t_0 \colon Z(t_0) = 0\}
\tag{95}
$$

Such diagonal operators are self-adjoint. For the spectrum calculation: We have

$$
L \, e_{t_0} = t_0 \, e_{t_0} \forall \{t_0 \colon Z(t_0) = 0\}
\tag{96}
$$

so each t_0 is an eigenvalue of L with eigenvector e_{t_0} and since $\{e_{t_0}\}$ forms an orthonormal basis, L has pure point spectrum. To show there are no other spectral points, suppose

$$
\lambda \notin \{t_0 \colon Z(t_0) = 0\}
\tag{97}
$$

Then for any $f \in \mathcal{D}(L)$, $((L - \lambda I) \, f)(t)$ has components

$$
((L - \lambda I) \, f)(t_0) = (t_0 - \lambda) \, f(t_0)
\tag{98}
$$

Since $t_0 - \lambda \neq 0$ the equation

$$
(L - \lambda I) \, f = g
\tag{99}
$$

has a unique solution $\forall g \in \mathcal{H}$ which is made apparent by setting

$$
f(t_0) = \frac{g(t_0)}{t_0 - \lambda}
\tag{100}
$$

16

which shows that $L - \lambda I$ is invertible, hence $\lambda \notin \sigma(L)$. Therefore

$$\sigma(L) = \{t_0 : Z(t_0) = 0\} \tag{101}$$

and the eigenvalues are simple. □

Remark 22. [Operator scaffold] The construction

$$\text{stationary } X \xrightarrow{U_\theta} \text{oscillatory } Z \xrightarrow{\mu = \delta(Z)|\dot{Z}|\,dt} L^2(\mu) \xrightarrow{L : t \cdot} (L, \sigma(L)) \tag{102}$$

produces a concrete self-adjoint operator whose spectrum equals the zero set of Z, determined by the choice of time-change θ and spectral measure F. This provides an explicit realization consistent with Hilbert–Pólya approaches to encoding arithmetic information in operator spectra.

6 Appendix: Regularity and Simple Zeros

Definition 23

*[**Regularity and simplicity**] Assume $Z \in C^1(\mathbb{R})$ and every zero is simple: $Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0$.*

Lemma 24

*[**Local finiteness and delta decomposition**] Under Definition 23, zeros are locally finite and*

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0) = 0} \frac{\delta(t - t_0)}{|Z'(t_0)|} \tag{103}$$

whence

$$\mu = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0} \tag{104}$$

Proof. Since $Z \in C^1(\mathbb{R})$ and $\dot{Z}(t_0) \neq 0$ at each zero t_0, the inverse function theorem implies that Z is locally invertible near each zero. Specifically, there exists a neighborhood U_{t_0} of t_0 such that $Z|_{U_{t_0}}$ is strictly monotonic and invertible.

This implies zeros are isolated: if $Z(t_0) = 0$ and $\dot{Z}(t_0) \neq 0$, then there exists $\epsilon > 0$ such that $Z(t) \neq 0$ for $0 < |t - t_0| < \epsilon$. Therefore zeros are locally finite (finitely many in any bounded interval).

For the distributional identity, consider the one-dimensional change of variables formula for the Dirac delta. If $g: I \to \mathbb{R}$ is C^1 on interval I with $\dot{g}(x) \neq 0$ for all $x \in I$, then

$$\delta(g(x)) = \sum_{x_0: g(x_0)=0} \frac{\delta(x - x_0)}{|\dot{g}(x_0)|} \tag{105}$$

Applying this locally around each zero t_0 of Z, and since zeros are isolated, we can patch together the local results to obtain the global identity:

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \tag{106}$$

Consequently:

$$\begin{aligned}
\mu(dt) &= \delta(Z(t))|\dot{Z}(t)| \, dt \\
&= \sum_{t_0: Z(t_0)=0} \frac{|\dot{Z}(t)|}{|\dot{Z}(t_0)|} \delta(t - t_0) \, dt \\
&= \sum_{t_0: Z(t_0)=0} \delta_{t_0}(dt)
\end{aligned} \tag{107}$$

where the last equality uses the fact that

$$\frac{|\dot{Z}(t_0)|}{|\dot{Z}(t_0)|} = 1 \tag{108}$$

when evaluating at $t = t_0$. $\qquad\square$