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Motion of a Vortex Filament and its Relation to Elastica

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Steady rotation of a very thin plane vortex filament with uniform angular velocity $-\Omega$ in the direction opposite to the circulatory rotation around it is discussed on the basis of the localized induction equation. It is shown that the possible form of the filament is that of the plane elastic filament of flexural rigidity B under the action of thrust F applied at its ends provided that

$$G/\Omega=B/F ,$$

where G is the coefficient of local induction proportional to the circulation of the filament.

The motion of a very thin vortex filament in an unbounded perfect fluid has been discussed by several authors[1-4] on the basis of the so-called localized induction equation which is mentioned by Hama[1] as initially suggested by Arms and is valid for very small values of the radius a of the filament compared with its radius of curvature R.

In this note, it is shown that an analytic solution of this equation corresponding to the simple rotation of a plane filament exists and is perfectly equivalent to the finite deformation of a plane elastic filament, *i.e.* the *elastica*.

The localized-induction equation states that the local velocity of the filament $\boldsymbol{v}$ is G/R in the direction of its binormal $\boldsymbol{i}_b$:

$$\boldsymbol{v}=(G/R)\boldsymbol{i}_b , \tag{1}$$

where G is the coefficient of local induction proportional to the circulation of the filament and may be assumed to be constant for very small values of a/R.

Let y be the distance from the x-axis about which our filament rotates with uniform angular velocity $-\Omega$, s be the distance measured along the filament, and θ be the angle between the tangent to the filament and the x-axis. Then, by noting $\boldsymbol{v}=-\Omega y\boldsymbol{i}_b$ and $1/R=\mathrm{d}\theta/\mathrm{d}s$, eq. (1) reduces to $G\,\mathrm{d}\theta/\mathrm{d}s=-\Omega y$, or by diffentiation with respect to s

$$G\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2}=-\Omega\frac{\mathrm{d}y}{\mathrm{d}s}=-\Omega\sin\theta . \tag{2}$$

This equation is nothing but the equation for Euler's elastica[5] *i.e.* the plane elastic filament of flexural rigidity B under the action of thrust F applied at its ends, provided that

$$G/\Omega=B/F . \tag{3}$$

The kinetic analogue to a simple pendulum in which s is replaced by the time t is also valid.

In this manner, the classical solution for elastica in terms of Jacobian elliptic function is easily transferable to our problem:

$$\left.\begin{aligned} x&=\left(\frac{G}{\Omega}\right)^{1/2}\left[-\xi+2\int_0^{\xi}\mathrm{dn}^2(\xi,k)\,\mathrm{d}\xi\right] ,\\ y&=A\,\mathrm{cn}(\xi,k) , \end{aligned}\right\} \tag{4}$$

where A is the maximum distance from the axis and

$$\xi=\left(\frac{\Omega}{G}\right)^{1/2}s , \quad k=\frac{1}{2}\left(\frac{\Omega}{G}\right)^{1/2}A . \tag{5}$$

For small values of k eqs. (4)–(5) yield

$$\left.\begin{aligned} \frac{y}{A}&=\left(1-\frac{k^2}{16}\right)\cos v+\frac{k^2}{16}\cos 3v+0(k^4) ,\\ v&=\left(\frac{\Omega}{G}\right)^{1/2}\left(1+\frac{3}{4}k^2\right)x , \end{aligned}\right\} \tag{6}$$

which is in accordance with the form calculated by Kambe in relation to the experiment on the stability of a vortex ring[6] and corresponds to Kelvin's sinusoidal filament[2,4] for $k=0$.

The distance Δx between nodal points ($\xi=\pm\mathrm{K}$), *i.e.*

$$\Delta x=2\left(\frac{G}{\Omega}\right)^{1/2}(2\mathrm{E}-\mathrm{K}) , \tag{7}$$

is shown in Fig. 1 as a function of k, where K and

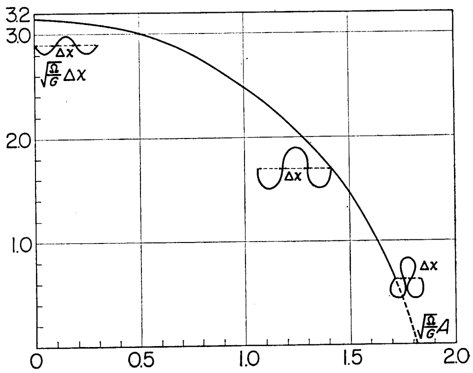

Fig. 1.

E are the complete elliptic integrals of the first and the second kind respectively.

As k increases Δx decreases and the "bay" formed by the curve bulges, so that the filament cuts the axis perpendicularly at $k^2=1/2$, and at $k \fallingdotseq 0.8551$ neighbouring bays start to cross; thereafter our solution is not acceptable since the interaction between two widely separated values of s is neglected in our approximation.

The other solution represented by $y=A\,\mathrm{dn}$ is unacceptable, since it yields a series of *loops* lying altogether on one side of the axis.

The general feature of our acceptable case is partly in accordance with that found by numerical experiments by Hama[2] and Takaki and Yoshizawa[7] on initially sinusoidal filaments, except for three-dimensional deformation for large k.

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