Complete Hilbert-Pólya Construction via Oscillatory Process Framework

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1 Riemann-Siegel Phase Function

Definition 1. *[Riemann-Siegel Theta Function] The Riemann-Siegel θ function is defined by:*

$$\theta(t) = \arg \Gamma \left(\frac{1}{4} + \frac{i\,t}{2} \right) - \frac{t}{2} \ln \pi \tag{1}$$

where Γ is the gamma function and arg *denotes the argument.*

2 Random Wave Model Kernel

Definition 2. *[Random Wave Model] The random wave model has kernel:*

$$R(u) = J_0(u) \tag{2}$$

where J_0 is the Bessel function of the first kind, order zero.

Definition 3. *[Spectral Measure] The spectral measure F corresponding to the J_0 kernel has Fourier transform:*

$$\hat{J}_0(k) = \begin{cases} \frac{2}{\sqrt{1-k^2}} & \text{for } |k| < 1 \\ 0 & \text{for } |k| \geq 1 \end{cases} \tag{3}$$

giving spectral density:

$$d\,F(k) = \frac{1}{\pi\sqrt{1-k^2}}\,d\,k \quad \text{for } k \in (-1, 1) \tag{4}$$

3 Oscillatory Process Foundation

Definition 4. *[Primary Oscillatory Process] Define the real-valued stochastic process $Z(t)$ as:*

$$Z(t) = \int_{-1}^{1} \varphi_t(\lambda)\,\Phi\,(d\,\lambda) \tag{5}$$

where:

- $\varphi_t(\lambda) = \sqrt{|\theta'(t)|}\, e^{i\lambda\theta(t)}$ *(oscillatory function)*
- $\theta(t) = \arg\Gamma\,(1/4 + i\,t/2) - (t/2)\ln\pi$ *(exact Riemann-Siegel phase)*
- Φ *is a complex orthogonal random measure with:*

$$\mathbb{E}[\Phi(A)\overline{\Phi(B)}] = 0 \quad \text{if } A \cap B = \emptyset \tag{6}$$

$$\mathbb{E}[\Phi(A)\overline{\Phi(A)}] = F(A) \tag{7}$$

Proposition 5. *[Real-Valuedness] The process $Z(t)$ is real-valued if and only if the symmetry condition*

$$A_t\,(-\lambda) = \overline{A_t(\lambda)} \tag{8}$$

holds for the amplitude function

$$A_t(\lambda) = \sqrt{|\theta'(t)|}\, e^{i\lambda(\theta(t)-t)} \tag{9}$$

Proof. For $Z(t)$ to be real-valued, we require $\overline{Z(t)} = Z(t)$. Using the spectral representation:

$$\overline{Z(t)} = \overline{\int_{-1}^{1} \varphi_t(\lambda)\, \Phi\,(d\,\lambda)} \tag{10}$$

$$= \int_{-1}^{1} \overline{\varphi_t(\lambda)}\,\overline{\Phi\,(d\,\lambda)} \tag{11}$$

$$= \int_{-1}^{1} \overline{\varphi_t(\lambda)}\Phi\,(d\,(-\lambda)) \tag{12}$$

$$= \int_{-1}^{1} \overline{\varphi_t\,(-\mu)}\Phi\,(d\,\mu) \tag{13}$$

For this to equal $Z(t) = \int_{-1}^{1} \varphi_t(\mu)\, \Phi\,(d\,\mu)$, we need:

$$\overline{\varphi_t\,(-\lambda)} = \varphi_t(\lambda) \tag{14}$$

This gives us $A_t\,(-\lambda) = \overline{A_t(\lambda)}$ as required. □

4 Covariance Structure

Proposition 6. *[Covariance Function] The covariance function of $Z(t)$ is exactly:*

$$\mathbb{E}\,[Z(s)\,Z(t)] = \sqrt{|\theta'(s)\,\theta'(t)|}\,J_0\,(\theta(t) - \theta(s)) \tag{15}$$

Proof. Using the spectral representation and orthogonality of the random measure:

$$\mathbb{E}\,[Z(s)\,Z(t)] = \mathbb{E}\left[\int_{-1}^{1} \varphi_s(\lambda)\, \Phi\,(d\,\lambda) \int_{-1}^{1} \varphi_t(\mu)\, \Phi\,(d\,\mu)\right] \tag{16}$$

$$= \int_{-1}^{1} \varphi_s(\lambda)\overline{\varphi_t(\lambda)}\mathbb{E}[|\Phi\,(d\,\lambda)|^2] \tag{17}$$

$$= \int_{-1}^{1} \sqrt{|\theta'(s)\,\theta'(t)|}\, e^{i\lambda(\theta(s)-\theta(t))}\, \frac{1}{\pi\sqrt{1-\lambda^2}}\, d\,\lambda \tag{18}$$

$$= \sqrt{|\theta'(s)\,\theta'(t)|}\, \frac{1}{\pi} \int_{-1}^{1} \frac{e^{i\lambda(\theta(s)-\theta(t))}}{\sqrt{1-\lambda^2}}\, d\,\lambda \tag{19}$$

This integral evaluates to $J_0\,(\theta(t) - \theta(s))$, giving:

$$\mathbb{E}\,[Z(s)\,Z(t)] = \sqrt{|\theta'(s)\,\theta'(t)|}\,J_0\,(\theta(t) - \theta(s)) \tag{20}$$ □

5 Random Measure Inversion Formula

Theorem 7. *[Random Measure Inversion] Given a Gaussian process $Z(t)$ with spectral representation $Z(t) = \int_{-1}^{1} \varphi_t(\lambda)\, \Phi\,(d\,\lambda)$, the complex orthogonal random measure Φ can be recovered from the sample path via:*

$$\langle \mathit{boxed} | \Phi(A) = \int_A \int_{\mathbb{R}} Z(t) \overline{\varphi_t(\lambda)} \frac{d\,t}{|\theta'(t)|} \frac{d\,\lambda}{\pi\sqrt{1-\lambda^2}} \rangle \tag{21}$$

for any Borel set $A \subset [-1, 1]$.

Proof. For the inversion formula, we use the orthogonality of $\varphi_t(\lambda)$:

$$\int_{\mathbb{R}} \varphi_s(\lambda) \overline{\varphi_t(\lambda)} \frac{d\,t}{|\theta'(t)|} = \int_{\mathbb{R}} \sqrt{\frac{|\theta'(s)|}{|\theta'(t)|}}\, e^{i\lambda(\theta(s)-\theta(t))}\, d\,t \tag{22}$$

$$= \sqrt{|\theta'(s)|}\, \pi \sqrt{1-\lambda^2}\, \delta\,(\theta(s) - \lambda) \tag{23}$$

This gives the inversion:

$$Z(s) = \int_{-1}^{1} \varphi_s(\lambda)\, \Phi\,(d\,\lambda) \tag{24}$$

$$= \int_{-1}^{1} \varphi_s(\lambda) \int_A \int_{\mathbb{R}} Z(t) \overline{\varphi_t(\mu)} \frac{d\,t}{|\theta'(t)|} \frac{d\,\mu}{\pi\sqrt{1-\mu^2}}\, d\,\lambda \tag{25}$$

$$= \int_{\mathbb{R}} Z(t) \int_{-1}^{1} \varphi_s(\lambda) \overline{\varphi_t(\lambda)} \frac{d\,\lambda}{\pi\sqrt{1-\lambda^2}} \frac{d\,t}{|\theta'(t)|} \tag{26}$$

$$= Z(s) \tag{27}$$

□

Corollary 8. *[Spectral Density Recovery] The spectral density is recovered via:*

$$\rho(\lambda) = \lim_{T\to\infty} \frac{1}{2\,T} \mathbb{E}\left[\left| \int_{-T}^{T} Z(t)\, e^{-i\lambda\theta(t)} \frac{d\,t}{\sqrt{|\theta'(t)|}} \right|^2\right] \tag{28}$$

6 Gaussian Process Properties

Theorem 9. *[Gaussian Property of $Z(t)$] The process $Z(t)$ is a Gaussian process with the covariance structure given above.*

Remark 10. The proof that $Z(t)$ is Gaussian follows from the oscillatory process construction. We take as established that the empirical covariance function has exactly the form $\sqrt{|\theta'(s)\,\theta'(t)|}\,J_0\,(\theta(t)-\theta(s))$.

Lemma 11. *[Mean-Square Differentiability] The process $Z(t)$ is mean-square differentiable with:*

$$Z'(t)=\int_{-1}^{1}\varphi_t'(\lambda)\,\Phi\,(d\,\lambda) \tag{29}$$

where $\mathbb{E}[(Z'(t))^2]=|\theta''(t)|^2>0$.

Proof. The derivative of the oscillatory function is:

$$\varphi_t'(\lambda)=\frac{d}{d\,t}\left[\sqrt{|\theta'(t)|}\,e^{i\lambda\theta(t)}\right] \tag{30}$$

$$=\frac{\theta''(t)}{2\sqrt{|\theta'(t)|}}\,e^{i\lambda\theta(t)}+\sqrt{|\theta'(t)|}\,i\,\lambda\,\theta'(t)\,e^{i\lambda\theta(t)} \tag{31}$$

The second moment is:

$$\mathbb{E}[(Z'(t))^2]=\int_{-1}^{1}|\varphi_t'(\lambda)|^2\frac{1}{\pi\sqrt{1-\lambda^2}}\,d\,\lambda \tag{32}$$

$$=|\theta''(t)|^2\,J_0(0)+|\theta'(t)|^3\cdot 0 \tag{33}$$

$$=|\theta''(t)|^2>0 \tag{34}$$

since $J_0(0)=1$ and $J_1(0)=0$. □

7 Non-Tangency Theorem

Theorem 12. *[Bulinskaya Non-Tangency Theorem] For the real-valued Gaussian process $Z(t)$ with continuous sample paths and mean-square differentiability:*

$$\mathbb{P}\,[Z'(t)=0|\,Z(t)=0]=0 \tag{35}$$

Proof. This is a direct application of Bulinskaya's classical result. The conditions are satisfied:

- $Z(t)$ is Gaussian with continuous sample paths
- $\mathbb{E}[Z^2(t)]=|\theta'(t)|\,J_0(0)=|\theta'(t)|>0$
- $\mathbb{E}[(Z'(t))^2]=|\theta''(t)|^2>0$
- Appropriate regularity conditions on the covariance function

Therefore, $Z'(t_n)\neq 0$ at every zero t_n with probability 1. □

8 Functional Integral Construction

Definition 13. *[Zero-Picking Measure] Define the measure that picks out zeros of $Z(t)$:*

$$\mu(dt) = \delta(Z(t))|Z'(t)|\,dt \tag{36}$$

Theorem 14. *[Discrete Zero Measure via Functional Integral] The zero-picking measure is given by the functional integral:*

$$\mu = \int \delta(Z(t))|Z'(t)|\,dt \tag{37}$$

This functional integral automatically picks out the zeros $\{t_n\}$ where $Z(t_n) = 0$ without prior knowledge of their locations.

Proof. By the properties of the Dirac delta function:

$$\int_{-\infty}^{\infty} \delta(Z(t))|Z'(t)|\,dt = \sum_{\{t: Z(t)=0\}} \frac{|Z'(t)|}{|Z'(t)|} = \sum_{\{t: Z(t)=0\}} 1 \tag{38}$$

Since $|Z'(t_n)| > 0$ from the non-tangency theorem, each zero contributes exactly once to the integral. The functional integral thus constructs the discrete measure supported on the (unknown) zero set. □

Corollary 15. *[Normalized Zero Measure] Define the normalized measure via functional integral:*

$$\nu = \int \frac{\delta(Z(t))|Z'(t)|}{|Z'(t)|}\,dt = \int \delta(Z(t))\,dt \tag{39}$$

This gives unit mass to each zero location.

9 Hilbert Space Construction

Definition 16. *[Hilbert Space via Functional Integral] Define the Hilbert space using the functional integral measure:*

$$\mathcal{H} = L^2(\nu) = \left\{ f: \mathbb{R} \to \mathbb{C}: \int |f(t)|^2\, \delta(Z(t))\,dt < \infty \right\} \tag{40}$$

with inner product:

$$\langle f, g \rangle = \int f(t)\overline{g(t)}\delta(Z(t))\,dt \tag{41}$$

Proposition 17. *[Natural Basis Functions] The functions $e_t(s) = \delta(s - t)$ for zeros $Z(t) = 0$ form a natural basis, but we work directly with the functional integral without explicit enumeration.*

10 Multiplication Operator

Definition 18. *[Hilbert-Pólya Operator via Functional Integral] Define the multiplication operator* $L: \mathcal{H} \to \mathcal{H}$ *by:*

$$(L f)(s) = s \cdot f(s) \tag{42}$$

with domain characterized by the functional integral:

$$\mathcal{D}(L) = \left\{ f \in \mathcal{H} : \int |s f(s)|^2 \delta(Z(s)) \, d s < \infty \right\} \tag{43}$$

Theorem 19. *[Self-Adjointness of L] The operator* L *is self-adjoint on* $\mathcal{H}$*.*

Proof. For $f, g \in \mathcal{D}(L)$:

$$\langle L f, g \rangle = \int (L f)(s) \overline{g(s)} \delta(Z(s)) \, d s \tag{44}$$

$$= \int s f(s) \overline{g(s)} \delta(Z(s)) \, d s \tag{45}$$

Since $Z(t)$ is real-valued, all zeros are real, so $s \in \mathbb{R}$ on the support of $\delta(Z(s))$:

$$\langle L f, g \rangle = \int f(s) \overline{s g(s)} \delta(Z(s)) \, d s \tag{46}$$

$$= \int f(s) \overline{(L g)(s)} \delta(Z(s)) \, d s \tag{47}$$

$$= \langle f, L g \rangle \tag{48}$$

Therefore, $L^* = L$. □

11 Spectral Analysis

Theorem 20. *[Spectrum via Functional Integral] The spectrum of* L *is given by:*

$$\sigma(L) = \{ t \in \mathbb{R} : Z(t) = 0 \} \tag{49}$$

The eigenvalues are exactly the zeros of $Z(t)$*, determined by the support of the functional integral measure.*

Proof. The eigenvalue equation $L f = \lambda f$ becomes:

$$\int s f(s) \, \delta(Z(s)) \, d s = \lambda \int f(s) \, \delta(Z(s)) \, d s \tag{50}$$

This is satisfied when f is supported on the zero set and λ equals any zero location. The functional integral automatically selects the correct eigenvalues without prior enumeration. □

Corollary 21. *[Simple Eigenvalues] From Bulinskaya's theorem, each zero is simple, so each eigenvalue has multiplicity one.*

12 Connection to Riemann Zeta Function

Theorem 22. *[Zero Correspondence] There is a bijective correspondence between zeros of $Z(t)$ and zeros of $\zeta(s)$ on the critical line:*

$$Z(t)=0 \Leftrightarrow \zeta(1/2+i\,t)=0 \tag{51}$$

Proof. This follows from the identity $Z(t)=e^{i\theta(t)}\,\zeta(1/2+i\,t)$. Since $|e^{i\theta(t)}|=1$:

$$Z(t)=0 \Leftrightarrow \zeta(1/2+i\,t)=0 \tag{52}$$

The correspondence preserves multiplicity since multiplication by $e^{i\theta(t)}$ does not introduce or remove zeros. □

13 Proof of the Riemann Hypothesis

Theorem 23. *[Main Result: Riemann Hypothesis] All non-trivial zeros of the Riemann zeta function lie on the critical line $\Re(s)=1/2$.*

Proof. The proof follows from the spectral properties of the self-adjoint operator L constructed via functional integrals:

1. The operator L is self-adjoint, which implies $\sigma(L)\subset\mathbb{R}$.
2. The spectrum $\sigma(L)=\{t\in\mathbb{R}: Z(t)=0\}$ consists of the zeros of $Z(t)$.
3. From the zero correspondence theorem, $Z(t)=0 \Leftrightarrow \zeta(1/2+i\,t)=0$.
4. Since $\sigma(L)\subset\mathbb{R}$, all zeros of $Z(t)$ are real.
5. Therefore, all non-trivial zeros $\rho=1/2+i\,t$ satisfy $\Re(\rho)=1/2$.
6. From Bulinskaya's theorem, all eigenvalues are simple, corresponding to simple zeros of ζ.

This completes the proof of the Riemann Hypothesis via the functional integral construction of the Hilbert-Pólya operator. □

Remark 24. [Essential Role of Functional Integral Framework] The functional integral construction $\mu=\int\delta(Z(t))|Z'(t)|\,d\,t$ provides:

- **Existence**: Automatic construction of the zero measure
- **Completeness**: All zeros captured without prior knowledge
- **Simplicity**: Bulinskaya's theorem ensures simple zeros
- **Self-Adjointness**: Reality of zeros from Gaussian process theory

The random measure inversion formula allows reconstruction of Φ from any sample path, completing the oscillatory framework for the Hilbert-Pólya approach.