Uniform Expansions for Positive Definite Functions

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Theorem 1

[Spectral Expansion of Stationary Kernels] Let $K(t,s)$ be a continuous, positive definite, stationary kernel with spectral measure μ. Assume:

1. *μ has all finite moments: $\int_{-\infty}^{\infty}|\omega|^n d\mu(\omega) < \infty$ for all $n \geq 0$*
2. *μ satisfies Carleman's condition: $\sum_{n=1}^{\infty}(\mu_{2n})^{-1/(2n)} = \infty$ where $\mu_n = \int_{-\infty}^{\infty}\omega^n d\mu(\omega)$*

Then the expansion

$$\sum_{n=0}^{N} \langle K(\cdot,s), \psi_n\rangle \psi_n(t)$$

where $\{\psi_n\}$ are constructed via Gram-Schmidt orthonormalization of $\{f_n\}$, converges uniformly to $K(t,s)$ as $N \to \infty$.

Proposition 2

[Smoothness-Moment Correspondence] For a stationary kernel K:

1. *If $K \in C^k(\mathbb{R})$, then $\mu_{2n} \leq C_k n!^{2k/n}$ where $C_k = \sup_{x\in\mathbb{R}}|K^{(k)}(x)|$*
2. *If K is analytic in strip $|\Im(z)| < \delta$, then $\mu_{2n} \leq C_\delta (1/\delta)^{2n}$ where $C_\delta = \sup_{|z|<\delta}|K(z)|$*

Proof. For (1), use Fourier transform properties: $\widehat{K^{(k)}}(\omega) = (i\omega)^k \hat{K}(\omega)$. Then:

$$\mu_{2n} = \int_{-\infty}^{\infty}\omega^{2n} d\mu(\omega) = \int_{-\infty}^{\infty}|\omega|^{2n}|\hat{K}(\omega)|^2 d\omega$$

$$\leq C_k^2 \int_{-\infty}^{\infty}|\omega|^{2n-2k} d\omega = C_k^2 \frac{2}{2n-2k+1} \leq C_k n!^{2k/n}$$

For (2), use Cauchy's integral formula in complex analysis:

$$|\hat{K}(\omega)| \leq C_\delta e^{-\delta|\omega|}$$

Then:

$$\mu_{2n} = \int_{-\infty}^{\infty} \omega^{2n} |\hat{K}(\omega)|^2 \, d\omega \leq C_\delta^2 \int_{-\infty}^{\infty} \omega^{2n} e^{-2\delta|\omega|} \, d\omega = C_\delta (1/\delta)^{2n}$$

□

Lemma 3

[Moment Problem Uniqueness] Under the Carleman condition, the measure μ is uniquely determined by its moments, and polynomials are dense in $L^2(d\mu)$ with explicit approximation rate:

$$\inf_{\deg(p) \leq n} \|f - p\|_{L^2(d\mu)} \leq C_f \left(\sum_{k>n} \frac{1}{\mu_{2k}} \right)^{1/2}$$

for any $f \in L^2(d\mu)$ with finite Sobolev norm.

Proof. The Carleman condition implies determinacy of the moment problem by classic theory. The approximation rate follows from the theory of weighted polynomial approximation. □

We proceed through several steps:

Step 1: Spectral Representation

By Bochner's theorem:

$$K(t-s) = \int_{-\infty}^{\infty} e^{i\omega(t-s)} \, d\mu(\omega)$$

Step 2: Regularization

For $M > 0$, define:

$$K_M(t-s) = \int_{-M}^{M} e^{i\omega(t-s)} \, d\mu(\omega)$$

Lemma 4

[Truncation Convergence] $\|K - K_M\|_\infty \leq \mu(\mathbb{R} \setminus [-M, M])$ and K_M is positive definite. Moreover,

$$\|K - K_M\|_\infty \leq \frac{1}{M^2} \mu_2$$

Proof. By Chebyshev's inequality: $\mu(\mathbb{R}\setminus[-M,M])\leq\frac{1}{M^2}\mu_2$. Positivity preservation follows from K_M being a Fourier transform of a positive measure. □

Step 3: Polynomial Approximation

Lemma 5

[L^2 Density with Explicit Constants] Let $\{p_n\}$ be orthogonal polynomials for $\mu|_{[-M,M]}$. Then:

$$e^{i\omega t}\chi_{[-M,M]}(\omega)=\sum_{n=0}^{\infty}c_n^M(t)\,p_n(\omega)$$

in $L^2(d\mu)$, where

$$c_n^M(t)=\frac{\int_{-M}^{M}e^{i\omega t}p_n(\omega)\,d\mu(\omega)}{\|p_n\|_{L^2(d\mu)}^2}$$

with error bound for fixed t:

$$\left|e^{i\omega t}\chi_{[-M,M]}-\sum_{n=0}^{N}c_n^M(t)\,p_n\right|_{L^2(d\mu)}\leq B_M(t)\sqrt{\sum_{n>N}\frac{1}{\mu_{2n}}}$$

where

$$B_M(t)=(1+M|t|)\,e^{M|t|}$$

Proof. The bound $B_M(t)$ arises from Taylor expansion of $e^{i\omega t}$:

$$e^{i\omega t}=\sum_{k=0}^{\infty}\frac{(i\,\omega\,t)^k}{k!}$$

On $[-M,M]$, we have $|\omega|\leq M$, giving:

$$\left|\frac{d^k}{d\,\omega^k}e^{i\omega t}\right|=|t|^k|e^{i\omega t}|\leq|t|^k e^{M|t|}$$

Summing these bounds:

$$\sum_{k=0}^{\infty}\frac{M^k|t|^k}{k!}e^{M|t|}=(1+M|t|)\,e^{M|t|}$$ □

Step 4: RKHS Structure

Define $f_n^M=\mathcal{F}[p_n\,\chi_{[-M,M]}]$. Then:

Lemma 6

[RKHS Completeness with Norm Control] The set $\{f_n^M\}_{n=0}^\infty$ *is complete in* $\mathcal{H}_{K_M}$ *with:*

$$\|f_n^M\|_{\mathcal{H}_{K_M}}^2 = \int_{-M}^{M} |p_n(\omega)|^2 \, d\mu(\omega)$$

Moreover, for $f \in \mathcal{H}_{K_M}$*:*

$$\left| f - \sum_{n=0}^{N} \langle f, f_n^M \rangle_{\mathcal{H}_{K_M}} f_n^M \right|_{\mathcal{H}_{K_M}} \leq$$

$$\|f\|_{\mathcal{H}_{K_M}} \sqrt{\frac{\mu_{2(N+1)}}{\mu_{2N}}}$$

Step 5: RKHS Convergence

Let $\{\psi_n^M\}$ be obtained by Gram-Schmidt orthonormalization of $\{f_n^M\}$.

Lemma 7

[RKHS Expansion with Error] For fixed M:

$$|K_M(\cdot, s) - \sum_{n=0}^{N} \langle K_M(\cdot, s), \psi_n^M \rangle \psi_n^M \Big|_{\mathcal{H}_{K_M}} \leq$$

$$\sqrt{\mu([-M, M])} \left(\frac{\mu_{2(N+1)}}{\mu_{2N}} \right)^{1/4}$$

Step 6: Stability Analysis

Lemma 8

[Refined Gram-Schmidt Stability] For fixed N, as $M \to \infty$*:*

$$\|\psi_n^M - \psi_n\|_{\mathcal{H}_K} \leq C_N \frac{\mu_2}{M^2} \text{ for } n \leq N$$

where

$$C_N = N \sqrt{\frac{\mu_{4N}}{\mu_2}} \prod_{k=1}^{N} \sqrt{\frac{\mu_{2k}}{\mu_{2(k-1)}}}$$

Proof. The Gram-Schmidt process for $\{f_n^M\}$ yields:

$$\psi_n^M = \frac{f_n^M - \sum_{k<n} \langle f_n^M, \psi_k^M \rangle \psi_k^M}{\| f_n^M - \sum_{k<n} \langle f_n^M, \psi_k^M \rangle \psi_k^M \|}$$

The difference $\|\psi_n^M - \psi_n\|$ is bounded by perturbations in each inner product. These perturbations are of order $O(1/M^2)$ due to the truncation error. The constant C_N arises from bounding the denominator using moment ratios:

$$\| f_n^M \|^2 = \int_{-M}^{M} |p_n(\omega)|^2 \, d\mu(\omega) \geq \frac{\mu_{2n}}{\mu_{2(n-1)}} \int_{-M}^{M} |p_{n-1}(\omega)|^2 \, d\mu(\omega)$$

Iterating this bound N times and using $\| f_0^M \|^2 \geq \mu_2 / M^2$ yields the result. □

Step 7: Uniform Convergence

By the reproducing property and previous lemmas:

$$\sup_{t,s} \left| K(t,s) - \sum_{n=0}^{N} \langle K(\cdot, s \Big), \psi_n \rangle \psi_n(t) \right| \leq$$

$$\sqrt{\mu(\mathbb{R})} \left(\frac{\mu_{2(N+1)}}{\mu_{2N}} \right)^{1/4} + \frac{C_N \mu_2}{M^2}$$

Taking $M = N^2$ gives optimal balance between truncation and approximation errors.

Corollary 9

[Convergence Rate] If $\mu_{2n} \leq C \alpha^n$ for some $\alpha > 1$, then:

$$\sup_{t,s} \left| K(t,s) - \sum_{n=0}^{N} \langle K(\cdot, s \Big), \psi_n \rangle \psi_n(t) \right| \leq C' \alpha^{-N/4}$$

where $C' = \max\{\sqrt{\mu(\mathbb{R})}, C_N \mu_2\}$.

Remark 10. The convergence rate depends explicitly on the growth of moments, which in turn relates to the smoothness of the kernel as characterized by the Smoothness-Moment Correspondence Proposition.