The Szegő-Jacobi Parameters Are A Map From The Moments of a Measure to The Corresponding Set of Orthogonal Polynomials Which Have That Measure as Their Orthogonalizing Measure

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Szegő-Jacobi parameters translate the moments of measures into parameters that define the orthogonal polynomials relative to that measure. Two notable routes to solution are the Golub-Welsch algorithm and the Stieltjes procedure.

Golub-Welsch Algorithm: Finding the Eigenspaces of the Jacobi Matrix Constructed From the Szegő-Jacobi Parameters(Equivalently the 3-term Recurrence)

This algorithm efficiently computes the eigenvalues and eigenvectors of Jacobi matrices, which correspond to the zeros of orthogonal polynomials and their weights in the Gaussian quadrature formula, respectively. It involves constructing a tridiagonal matrix from the Szegő-Jacobi parameters, then using eigenvalue solvers to find its eigenvalues and eigenvectors.

1. **Construction**: Construct a symmetric tridiagonal matrix J using the recurrence coefficients (Szegő-Jacobi parameters) a_n and b_n as the entries, where a_n are the off-diagonal elements and b_n the diagonal elements.

2. **Eigenvalue Problem**: Solve the eigenvalue problem

$$J v = \lambda v$$

   where v are the eigenvectors and λ the eigenvalues.

3. **Orthogonal Polynomials Zeros and Weights**: The eigenvalues λ represent the zeros of the orthogonal polynomials. The first element of the normalized eigenvectors, squared and multiplied by the weight of the corresponding orthogonal polynomial, gives the weights for the Gaussian quadrature formula.

Stieltjes Procedure: Orthogonalizing Monomial Sequences with the Gram-Schmidt Process

The Stieltjes procedure is a method for generating orthogonal polynomials and their recurrence coefficients from a given measure. It relies on the Gram-Schmidt orthogonalization process applied to the sequence of monomials with respect to the inner product defined by the measure.

1. **Orthogonalization**: Starting with a sequence of monomials $1, x, x^2, \ldots$, apply the Gram-Schmidt process using the inner product

$$\langle f, g \rangle = \int f(x)\, g(x)\, d\,\mu(x) \tag{1}$$

   where $d\,\mu(x)$ is the measure with respect to which the polynomials are orthogonal.

2. **Recurrence Coefficients**: This process inherently produces the recurrence coefficients a_n and b_n, which are used to construct orthogonal polynomials through the three-term recurrence relation

$$P_{n+1}(x) = (x - a_n)\, P_n(x) - b_n\, P_{n-1}(x) \tag{2}$$