Delta Functions, Heaviside Steps, and Level Crossing Counts for Differentiable Paths

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Contents

1 Foundations of Distributions on Real Line

Definition 1 (Schwartz Test Function Space) *The Schwartz space $\mathcal{S}(\mathbb{R})$ is the space of all infinitely differentiable functions $\phi : \mathbb{R} \to \mathbb{R}$ such that for every pair of nonnegative integers m, n,*

$$\sup_{x\in\mathbb{R}} |x^m \phi^{(n)}(x)| < \infty \tag{1}$$

Functions in $\mathcal{S}(\mathbb{R})$ are called rapidly decreasing smooth test functions.

Definition 2 (Tempered Distribution) *A tempered distribution is a continuous linear functional*

$$T : \mathcal{S}(\mathbb{R}) \to \mathbb{R} \tag{2}$$

Definition 3 (Dirac Delta Distribution) *The Dirac delta distribution $\delta_a \in \mathcal{S}'(\mathbb{R})$ centered at $a \in \mathbb{R}$ is defined by*

$$\langle \delta_a, \phi \rangle = \phi(a) \tag{3}$$

for all $\phi \in \mathcal{S}(\mathbb{R})$. When $a = 0$, one writes $\delta = \delta_0$.

Definition 4 (Heaviside Step Function) *The Heaviside step function $H : \mathbb{R} \to \{0, 1\}$ is defined by*

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases} \tag{4}$$

Definition 5 (Distributional Derivative) *For a tempered distribution $T \in \mathcal{S}'(\mathbb{R})$, its distributional derivative $T' \in \mathcal{S}'(\mathbb{R})$ is defined by*

$$\langle T', \phi \rangle = -\langle T, \phi' \rangle$$

for all $\phi \in \mathcal{S}(\mathbb{R})$.

2 Basic Identities

Theorem 1 (Heaviside Derivative) *The Heaviside step function H satisfies*

$$H' = \delta \tag{5}$$

as distributions on $\mathcal{S}'(\mathbb{R})$.

Proof For all $\phi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}
\langle H', \phi \rangle &= -\langle H, \phi' \rangle &(6)\\
&= -\int_{-\infty}^{\infty} H(x)\phi'(x)\, dx &(7)\\
&= -\int_{0}^{\infty} \phi'(x)\, dx &(8)\\
&= -[\phi(x)]_0^\infty &(9)\\
&= -(\lim_{x\to\infty} \phi(x) - \phi(0)) &(10)\\
&= \phi(0) &(11)
\end{aligned}$$

where the limit vanishes since $\phi \in \mathcal{S}(\mathbb{R})$ decays rapidly at infinity. Thus

$$\langle H', \phi \rangle = \phi(0) = \langle \delta, \phi \rangle \tag{12}$$

□

Theorem 2 (Integral of Delta) *For any $a \in \mathbb{R}$ and $T \in \mathbb{R}$,*

$$\int_{-\infty}^{T} \delta(t-a)\, dt = H(T-a) \tag{13}$$

Proof Define

$$F(T) = \int_{-\infty}^{T} \delta(t-a)\, dt \tag{14}$$

Taking the distributional derivative with respect to T:

$$F'(T) = \frac{d}{dT}\int_{-\infty}^{T} \delta(t-a)\, dt = \delta(T-a) \tag{15}$$

Since $F(-\infty) = 0$ and

$$F'(T) = \delta(T-a) = H'(T-a) \tag{16}$$

from the previous theorem, one has

$$F(T) = H(T-a) + C \tag{17}$$

for some constant C. The boundary condition

$$F(-\infty) = 0 = H(-\infty) + C \tag{18}$$

implies $C = 0$, thus

$$F(T) = H(T-a) \tag{19}$$

□

3 Delta of a Smooth Function

Theorem 3 (Delta under Change of Variables) *Let $g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable with isolated, simple zeros $\{x_i\}$ such that $g(x_i) = 0$ and $g'(x_i) \neq 0$. Then the identity*

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|} \tag{20}$$

holds in $\mathcal{S}'(\mathbb{R})$.

Proof For $\phi \in \mathcal{S}(\mathbb{R})$,

$$\langle \delta(g(x)), \phi \rangle = \int_{-\infty}^{\infty} \phi(x)\delta(g(x))\, dx \tag{21}$$

Near each zero x_i, where g is locally monotone by the implicit function theorem, the change of variables $u = g(x)$ gives

$$\begin{aligned} \textstyle\int_{I_i} \phi(x)\delta(g(x))\, dx \quad &= \textstyle\int_{g(I_i)} \frac{\phi(g^{-1}(u))}{|g'(g^{-1}(u))|}\delta(u)\, du \\ &= \tfrac{\phi(x_i)}{|g'(x_i)|} \end{aligned} \tag{22}$$

by the sifting property of δ. Summing over all zeros yields

$$\langle \delta(g(x)), \phi \rangle = \sum_i \frac{\phi(x_i)}{|g'(x_i)|} = \left\langle \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}, \phi \right\rangle \tag{23}$$

Since this holds for all $\phi \in \mathcal{S}(\mathbb{R})$, the distributional equality follows. □

4 Counting Function for Level Crossings

Let $x : \mathbb{R} \to \mathbb{R}$ be continuously differentiable, and fix $u \in \mathbb{R}$. Assume the zeros of $g(t) := x(t) - u$ are isolated and simple; that is, for every zero t_i,

$$g'(t_i) = x'(t_i) \neq 0 \tag{24}$$

Definition 6 *[Level Crossing Counting Function] Define the counting function*

$$N(T) := \textit{the number of zeros } t_i \textit{of } x(t) - u \textit{with } t_i \leq T \tag{25}$$

Theorem 4 (Counting Function as Integral Over Delta) *For every $T \in \mathbb{R}$,*

$$N(T) = \int_{-\infty}^{T} |x'(t)| \delta(x(t) - u)\, dt \tag{26}$$

Proof Using the delta change of variables theorem with

$$g(t) = x(t) - u \tag{27}$$

one finds that

$$|x'(t)|\delta(x(t) - u) = |x'(t)| \sum_i \frac{\delta(t - t_i)}{|x'(t_i)|} \tag{28}$$

$$= \sum_i |x'(t)| \frac{\delta(t - t_i)}{|x'(t_i)|} \tag{29}$$

Since $x'(t_i) \neq 0$ by assumption, and $\delta(t - t_i)$ picks out the value at $t = t_i$,

$$\begin{aligned} |x'(t)|\delta(x(t) - u) \quad &= \textstyle\sum_i \frac{|x'(t_i)|}{|x'(t_i)|} \delta(t - t_i) \\ &= \textstyle\sum_i \delta(t - t_i) \end{aligned} \tag{30}$$

Therefore,

$$\begin{aligned} \textstyle\int_{-\infty}^{T} |x'(t)|\delta(x(t) - u)\, dt \quad &= \textstyle\sum_i \int_{-\infty}^{T} \delta(t - t_i)\, dt \\ &= \textstyle\sum_{t_i \leq T} 1 \\ &= N(T) \end{aligned} \tag{31}$$

□

Theorem 5 (Counting Function as Sum of Heaviside Steps) *The counting function (6) is given by*

$$N(T) = \sum_i H(T - t_i) \forall T \in \mathbb{R} \tag{32}$$

where the sum runs over all zero crossing times t_i.

Proof By definition of the Heaviside function,

$$H(T - t_i) = 1 \tag{33}$$

if and only if $T \geq t_i$, and

$$H(T - t_i) = 0 \tag{34}$$

otherwise. Therefore,

$$\begin{aligned} \textstyle\sum_i H(T - t_i) &= \textstyle\sum_{t_i \leq T} 1 \\ &= N(T) \end{aligned} \tag{35}$$

□

Theorem 6 (Equivalence of Representations) *The delta integral representation and the Heaviside step sum representation are equivalent:*

$$\int_{-\infty}^{T} |x'(t)| \delta(x(t) - u) \, dt = \sum_i H(T - t_i) \tag{36}$$

Proof This follows immediately from the two previous theorems, since both expressions equal $N(T)$. □