Aronszajn's Theorem

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Aronszajn's theorem basically states that positive definite kernels are reproducing kernels and thereforefore a reproducing kernel Hilbert space for the kernel exists. [1, 5.1, p.252] [2, 9 Theorem 1, p.96]

Theorem 1

Aronszajn's Theorem:

Given a kernel function $K(x, y)$ defined $\forall x, y \in X$ where X is any set

1. ***Symmetry***: *$K(x, y) = \overline{K(y, x)}$ for all $x, y \in X$. (Here, $\bar{z}$ denotes the complex conjugate of z.) This definition is crucial as it extends the theorem to complex-valued functions, going beyond the common real-valued case.*
2. ***Positive Definiteness***: *For any finite set of points $\{x_1, x_2, \ldots, x_n\} \subset X$ and any set of complex numbers $\{c_1, c_2, \ldots, c_n\}$, the following inequality holds:*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \bar{c_j} K(x_i, x_j) \geq 0 \tag{1}$$

Then, a unique Hilbert space of functions $f: X \to \mathbb{C}$ (or $\mathbb{R}$ in the real case) exists, characterized by:

1. ***Reproducing Kernel Property***: *$K(x, \cdot)$ is in the Hilbert space for every x in X, and for every function f in the Hilbert space and for all $x \in X$, the reproducing property holds:*

$$f(x) = \langle f, K(x, \cdot) \rangle \tag{2}$$

This property enables the evaluation of functions in the Hilbert space at any point in X through inner products.

1. ***Spanning Property***: *The space is spanned by the functions $K(x, \cdot)$, meaning that every function in the Hilbert space can be approximated arbitrarily well by finite linear combinations of these basis functions.*

Proof.

1. **Construct the Hilbert Space H**:
   - Define $K_x = K(x, \cdot)$ for each $x \in X$. Let H_0 be the linear span of $\{K_x : x \in X\}$.

- Define an inner product on H_0 by:

$$\left\langle \sum_{j=1}^{n} b_j K_{y_j}, \sum_{i=1}^{m} a_i K_{x_i} \right\rangle_{H_0} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j K(y_j, x_i) \tag{3}$$

This inner product is symmetric due to the symmetry of K, and its non-degeneracy stems from K being positive definite. - The completion of H_0 with respect to this inner product forms H, consisting of functions like:

$$f(x) = \sum_{i=1}^{\infty} a_i K_{x_i}(x) \tag{4}$$

where the convergence is in the norm of H_0.

1. **Verification of Reproducing Property**:
   - $\forall f \in H$ and $\forall x \in X$:

$$\begin{aligned} \langle f, K_x \rangle_H &= \sum_{i=1}^{\infty} a_i \langle K_{x_i}, K_x \rangle_{H_0} \\ &= \sum_{i=1}^{\infty} a_i K(x_i, x) \\ &= f(x) \end{aligned} \tag{5}$$

This shows how the inner product in H can be used to evaluate functions at any point in X.

1. **Proof of Uniqueness**:
   - Assume G is another Hilbert space with K as its reproducing kernel. For every $x, y \in X$:

$$\begin{aligned} \langle K_x, K_y \rangle_H &= K(x, y) \\ &= \langle K_x, K_y \rangle_G \end{aligned} \tag{6}$$

This implies

$$\langle \cdot, \cdot \rangle_H = \langle \cdot, \cdot \rangle_G \tag{7}$$

on the span of $\{K_x : x \in X\}$, thus $H \subset G$. To show every element of G is in H, $\forall f \in G$, constitute f by combining $f_H \in H$ and $f_{H^\perp} \in H^\perp$:

$$\begin{aligned} f(x) &= \langle K_x, f \rangle_G \\ &= \langle K_x, f_H \rangle_G + \langle K_x, f_{H^\perp} \rangle_G \\ &= \langle K_x, f_H \rangle_G \\ &= \langle K_x, f_H \rangle_H \\ &= f_H(x) \end{aligned} \tag{8}$$

The inner product K_x with $f_{H^\perp} \in G$ is

$$\langle K_x, f_{H^\perp} \rangle_G = 0 \tag{9}$$

since $K_x \in H$. This demonstrates

$$f(x) = f_H(x) \forall x \in X \tag{10}$$

confirming the uniqueness of H. □

Bibliography

[1] Malempati M. Rao. *Stochastic Processes: Inference Theory*. Springer Monographs in Mathematics. Springer, 2nd edition, 2014.

[2] 吉田 耕作(Kōsaku Yosida). *Functional Analysis*. Classics in Mathematics. Springer Berlin Heidelberg, Reprint of the 1980 Edition edition, 1995.