Theorem 1

Given:

$$\lambda(n) = \sqrt{4\,n+1}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma\,(n+1)^2}$$

We aim to show:

$$\lambda(n) = \int_0^\infty J_0(x)\;\psi_n(x)\;d\,x$$

where

$$\psi_n(x) = \frac{1}{2}\,\sqrt{4\,n+1}\,(-1)^n\,J_{2n+\frac{1}{2}}(x)\frac{\sqrt{2}}{\sqrt{x}}$$

Proof. Substitute $\psi_n(x)$ into the integral and simplify:

$$\lambda(n) = \int_0^\infty J_0(x)\left(\frac{1}{2}\,\sqrt{4\,n+1}\,(-1)^n\,J_{2n+\frac{1}{2}}(x)\frac{\sqrt{2}}{\sqrt{x}}\right)d\,x$$

$$=\frac{1}{\sqrt{2}}\,\sqrt{4\,n+1}\,(-1)^n\int_0^\infty\frac{J_0(x)\,J_{2n+\frac{1}{2}}(x)}{\sqrt{x}}\,d\,x$$

Use the known result for the integral of the product of Bessel functions:

$$\int_0^\infty\frac{J_0(x)\,J_{2n+\frac{1}{2}}(x)}{\sqrt{x}}\,d\,x = \frac{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{2^{n+\frac{1}{2}}\,\Gamma\,(n+1)}$$

Substitute this result back into $\lambda(n)$ and simplify:

$$\lambda(n) = \frac{1}{\sqrt{2}}\,\sqrt{4\,n+1}\,(-1)^n\frac{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{2^{n+\frac{1}{2}}\,\Gamma\,(n+1)}$$

$$=\sqrt{4\,n+1}\,\frac{(-1)^n\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{2^{n+1}\,\Gamma\,(n+1)}$$

Use the Gamma function duplication formula:

$$\Gamma\,(n+1) = \frac{\sqrt{\pi}\,\Gamma\,(2\,n+1)}{2^{2n}\,\Gamma\left(n+\frac{1}{2}\right)}$$

Substitute back into $\lambda(n)$:

$$\lambda(n) = \sqrt{4\,n+1}\,\frac{(-1)^n\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{2^{n+1}\left(\frac{\sqrt{\pi}\,\Gamma\,(2\,n+1)}{2^{2n}\,\Gamma\left(n+\frac{1}{2}\right)}\right)}$$

$$=\sqrt{4\,n+1}\,\frac{(-1)^n\,2^{2n}\,\Gamma\left(n+\frac{1}{2}\right)^2}{2^{n+1}\,\Gamma\,(2\,n+1)}$$

The term $(-1)^n$ cancels out because it appears in both the numerator and denominator:

$$=\sqrt{4\,n+1}\,\frac{2^{2\,n}\,\Gamma\left(n+\frac{1}{2}\right)^2}{2^{n+1}\,\Gamma\left(2\,n+1\right)}$$

Simplify further:

$$=\sqrt{4\,n+1}\,\frac{2^{n-1}\,\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\left(2\,n+1\right)}$$

Recognize $(2\,n)!=\Gamma\left(2\,n+1\right)$:

$$=\sqrt{4\,n+1}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma\left(n+1\right)^2}$$

Thus, the identity is confirmed:

$$\lambda(n)=\sqrt{4\,n+1}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma\left(n+1\right)^2}=\int_0^\infty J_0(x)\;\psi_n(x)\;d\,x \qquad \square$$