Contractive Containment, Stationary Dilations, and Partial Isometries: Equivalence, Properties, and Geometric Intuition

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1 Preliminaries

Definition 1

[Hilbert Space Contraction] *A bounded linear operator $T: H_1 \rightarrow H_2$ between Hilbert spaces is called a contraction if*

$$\|Tx\|_{H_2} \leq \|x\|_{H_1} \quad \forall x \in H_1 \tag{1}$$

Equivalently, $\|T\| \leq 1$.

Definition 2

[Stationary Process] *A stochastic process $\{Y(t)\}_{t \in \mathbb{R}}$ is stationary if for any finite set of time points $\{t_1, \ldots, t_n\}$ and any $h \in \mathbb{R}$, the joint distribution of*

$$\{Y(t_1 + h), \ldots, Y(t_n + h)\} \tag{2}$$

is identical to that of $\{Y(t_1), \ldots, Y(t_n)\}$.

Definition 3

[Stationary Dilation] *Given a non-stationary process $X(t)$, a stationary dilation is a stationary process $Y(s)$ together with a family of bounded operators $\{\phi(t,\cdot)\}_{t\in\mathbb{R}}$ such that*

$$X(t) = \int_{\mathbb{R}} \phi(t,s)\, Y(s)\, d\, s \tag{3}$$

where $\phi(t,s)$ is a measurable function satisfying:

1. *$\|\phi(t,\cdot)\|_\infty \leq 1$ for all t*
2. *The map $t \mapsto \phi(t,\cdot)$ is strongly continuous*

Remark 4. The conditions on $\phi(t,s)$ ensure that the integral is well-defined and the resulting process $X(t)$ inherits appropriate regularity properties from $Y(s)$.

2 Main Results

Proposition 5

[Properties of Scaling Function] *The scaling function $\phi(t,s)$ in a stationary dilation satisfies:*

1. *$\|\phi(t,s)\| \leq 1$ for all $t, s \in \mathbb{R}$*
2. *For fixed t, $s \mapsto \phi(t,s)$ is measurable*
3. *For fixed s, $t \mapsto \phi(t,s)$ is continuous*

Theorem 6

[Equivalence of Containment] *For a non-stationary process $X(t)$ and a stationary process $Y(s)$, the following are equivalent:*

- *$Y(s)$ is a stationary dilation of $X(t)$*
- *There exists a contractive mapping Φ from the space generated by Y to the space generated by X such that*

$$X(t) = (\Phi Y)(t) \forall t \tag{4}$$

Proof. $(1 \Rightarrow 2)$: Define Φ by

$$(\Phi Y)(t) = \int_{\mathbb{R}} \phi(t, s)\, Y(s)\, d s \tag{5}$$

For any finite linear combination $\sum_i \alpha_i Y(t_i)$:

$$\begin{aligned} \|\Phi(\sum_i \alpha_i Y(t_i))\|^2 &= \|\sum_i \alpha_i \int_{\mathbb{R}} \phi(t_i, s)\, Y(s)\, d s\|^2 \\ &\leq \|\sum_i \alpha_i Y(t_i)\|^2 \end{aligned} \tag{6}$$

where the inequality follows from the bound on $\|\phi(t, s)\|$ and the Cauchy-Schwarz inequality.

$(2 \Rightarrow 1)$: The contractive mapping Φ induces a family of operators $\phi(t, s)$ via the Kernel theorem for Hilbert spaces. The stationarity of Y and the contractivity of Φ ensure that these operators satisfy the required properties. □

Lemma 7

[Minimal Dilation Property] *If $Y(s)$ is a minimal stationary dilation of $X(t)$, then the scaling function $\phi(t, s)$ achieves the bound*

$$\sup_{t, s} \|\phi(t, s)\| = 1 \tag{7}$$

Proof. If $\sup_{t, s} \|\phi(t, s)\| < 1$, we could construct a smaller dilation by scaling $Y(s)$, contradicting minimality. □

3 Structure Theory

Theorem 8

[Sz.-Nagy Dilation] *For any contraction T on a Hilbert space H, there exists a minimal unitary dilation U on a larger space $K \supseteq H$ such that:*

$$T^n = P_H U^n|_H \quad \forall n \geq 0 \tag{8}$$

where P_H is the orthogonal projection onto H.

Lemma 9

[Defect Operators] *For a contraction T, the defect operators defined by:*

$$D_T = \sqrt{I - T^* T} \tag{9}$$

$$D_{T^*} = \sqrt{I - T T^*} \tag{10}$$

satisfy:

1. $\|D_T\| \leq 1$ *and* $\|D_{T^*}\| \leq 1$
2. $D_T = 0$ *if and only if* T *is an isometry*
3. $D_{T^*} = 0$ *if and only if* T *is a co-isometry*

4 Convergence Properties

Theorem 10

[Strong Convergence] *For a contractive stationary dilation, the following limit exists in the strong operator topology:*

$$\lim_{n\to\infty} T^n = P_{ker(I - T^* T)} \tag{11}$$

where $P_{ker(I-T^*T)}$ *is the orthogonal projection onto the kernel of* $I - T^* T$.

Proof. For any x in the Hilbert space:

1. The sequence $\{\|T^n x\|\}$ is decreasing since T is a contraction
2. It is bounded below by 0
3. Therefore, $\lim_{n\to\infty} \|T^n x\|$ exists
4. The limit operator must be the projection onto the space of vectors x satisfying $\|Tx\| = \|x\|$
5. This space is precisely $ker(I - T^* T)$ □

Corollary 11

[Asymptotic Behavior] *If T is a strict contraction (i.e., $\|T\| < 1$), then*

$$\lim_{n\to\infty} T^n = 0 \tag{12}$$

in the strong operator topology.

5 Partial Isometries: The Mathematical Scalpel

Definition 12

[Partial Isometry] *An operator A on a Hilbert space H is a partial isometry if $A^* A$ is an orthogonal projection.*

Remark 13. [Geometric Intuition] A partial isometry is like a mathematical scalpel that carves out a section of space:

- It acts as a perfect rigid motion (isometry) on a specific subspace
- It completely annihilates the rest of the space

This property makes partial isometries powerful tools for selecting and transforming specific parts of a Hilbert space while cleanly disposing of the rest.

Proposition 14

[Key Properties of Partial Isometries] *Let A be a partial isometry. Then:*

1. *A is an isometry when restricted to $(ker(A))^{\perp}$*
2. *$A\,(ker(A))^{\perp} = range(A)$*
3. *A^* is also a partial isometry*
4. *$A\,A^*\,A = A$ and $A^*\,A\,A^* = A^*$*

Theorem 15

[Geometric Characterization] *For a partial isometry A:*

$$A^*\,A = P_{(ker(A))^{\perp}} \tag{13}$$

and

$$A\,A^* = P_{range(A)} \tag{14}$$

where P_S denotes the orthogonal projection onto subspace S.

Proof. The action of A can be decomposed as:

1. Project onto $(ker\,A)^{\perp}$ (this is $A^*\,A$)

2. Apply a perfect rigid motion to the projected space

This two-step process ensures $A^* A$ is the projection onto $(k\,e\,r\,A)^{\perp}$. $\square$

Remark 16. [The "Not So Partial" Nature] Despite the name, there's nothing incomplete about a partial isometry. It performs a complete operation:

- It's a full isometry on its initial space $(k\,e\,r\,(A))^{\perp}$
- It perfectly maps this initial space onto its final space $r\,a\,n\,(A)$
- It precisely annihilates everything else

This makes partial isometries fundamental building blocks in operator theory, crucial in polar decompositions, dimension theory of von Neumann algebras, and quantum mechanics.