Eigenfunction Expansion for Translation-Invariant Kernels via Galerkin Method

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Definition 1. *For a translation-invariant kernel $K(x-y)$ on $\mathbb{R}^d$, its Gram matrix A with respect to a uniformly convergent orthonormal basis $\{\psi_j\}_{j=1}^{\infty}$ is:*

$$A_{ij} = \int_{\mathbb{R}^d} \psi_i(x-y)\, \psi_j(y)\, d y$$

Theorem 2. *The Gram matrix A can be expressed in terms of Fourier transforms:*

$$A_{ij} = \mathcal{F}^{-1}[\mathcal{F}[\psi_i]^* \cdot \mathcal{F}[\psi_j]]$$

where $\mathcal{F}$ denotes the Fourier transform, $\mathcal{F}^{-1}$ the inverse Fourier transform, and * the complex conjugate.

Proof. By the convolution theorem and Parseval's identity:

$$\begin{aligned}
A_{ij} &= \int_{\mathbb{R}^d} \psi_i(x-y)\, \psi_j(y)\, d y \\
&= (\psi_i * \psi_j)(x) \\
&= \mathcal{F}^{-1}[\mathcal{F}[\psi_i] \cdot \mathcal{F}[\psi_j]] \\
&= \mathcal{F}^{-1}[\mathcal{F}[\psi_i]^* \cdot \mathcal{F}[\psi_j]]
\end{aligned}$$

The last step follows from the fact that ψ_i is real-valued, so $\mathcal{F}[\psi_i] = \mathcal{F}[\psi_i]^*$. □

Theorem 3. *For a kernel $K(x-y) = \sum_{j=1}^{\infty} a_j\, \psi_j(x-y)$, the eigenfunctions ϕ_k and their corresponding eigenvalues λ_k are given by:*

$$\phi_k(x) = \sum_{j=1}^{\infty} b_{kj}\, \psi_j(x)$$

where the coefficients b_{kj} *satisfy:*

$$\sum_{j=1}^{\infty} a_i A_{ij} b_{kj} = \lambda_k b_{ki} \quad \textit{for all } i$$

Proof. Let $\phi_k(x) = \sum_{j=1}^{\infty} b_{kj} \psi_j(x)$ be an eigenfunction of K. Then:

$$\begin{aligned}
\lambda_k \phi_k(x) &= \int K(x-y)\,\phi_k(y)\,dy \\
&= \int \left(\sum_{i=1}^{\infty} a_i \psi_i(x-y)\right)\left(\sum_{j=1}^{\infty} b_{kj}\psi_j(y)\right) dy \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i b_{kj} \int \psi_i(x-y)\,\psi_j(y)\,dy \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i b_{kj} A_{ij} \\
&= \sum_{i=1}^{\infty} a_i \left(\sum_{j=1}^{\infty} A_{ij} b_{kj}\right)
\end{aligned}$$

Equating coefficients of $\psi_i(x)$ on both sides:

$$\lambda_k b_{ki} = a_i \sum_{j=1}^{\infty} A_{ij} b_{kj}$$

This is equivalent to the equation:

$$\sum_{j=1}^{\infty} a_i A_{ij} b_{kj} = \lambda_k b_{ki} \quad \text{for all } i$$

Thus, the eigenfunctions are given by the solutions of this equation system. □

Theorem 4. *The nth eigenfunction* $\phi_n(x)$ *of the kernel* $K(x-y) = \sum_{j=1}^{\infty} a_j \psi_j(x-y)$ *is given by:*

$$\phi_n(x) = \sum_{j=1}^{\infty} b_{nj} \psi_j(x)$$

where the coefficients b_{nj} *satisfy:*

$$\sum_{j=1}^{\infty} a_i A_{ij} b_{nj} = \lambda_n b_{ni} \quad \textit{for all } i$$