A Uniformly Convergent Orthonormal Expansion for the Bessel Function of the First Kind of Order 0

by Stephen Crowley

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Theorem 1

Let $\psi_n(y)$ be defined as

$$\begin{aligned}\psi_n(y) &= (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, j_{2n}(y)\\ &= (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, \sqrt{\frac{\pi}{2\,y}}\, J_{2n+\frac{1}{2}}(y)\\ &= (-1)^n \sqrt{\frac{(4\,n+1)\,\pi}{\pi\,2\,y}}\, J_{2n+\frac{1}{2}}(y)\\ &= (-1)^n \sqrt{\frac{4\,n+1}{2\,y}}\, J_{2n+\frac{1}{2}}(y)\\ &= (-1)^n \sqrt{\frac{2\,n+\frac{1}{2}}{y}}\, J_{2n+\frac{1}{2}}(y)\end{aligned} \tag{1}$$

where J_ν denotes the Bessel function of the first kind and j_n the spherical Bessel function. Then

$$\begin{aligned}J_0(x) &= \sum_{n=0}^{\infty} \psi_n(x) \int_0^\infty J_0(y)\,\psi_n(y)\mathrm{d}y\\ &= \sum_{n=0}^{\infty} (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, j_{2n}(x) \sqrt{\frac{2\,n+\frac{1}{2}}{\pi}}\, \frac{\Gamma\,(n+\frac{1}{2})^2}{\Gamma\,(n+1)^2}\\ &= \frac{1}{2\,\sqrt{\pi\,x}} \sum_{n=0}^{\infty} (-1)^n\,(4\,n+1)\, \frac{\Gamma\,(n+\frac{1}{2})^2}{\Gamma\,(n+1)^2}\, J_{2n+\frac{1}{2}}(x)\\ &= \frac{1}{\sqrt{4\,\pi\,x}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(4\,n+1)\,\Gamma\,(n+\frac{1}{2})^2}{\Gamma\,(n+1)^2}\, J_{2n+\frac{1}{2}}(x)\end{aligned} \tag{2}$$

with uniform convergence $\forall x \in \mathbb{C}$. Moreover, $\{\psi_n\}$ forms an orthonormal system in $L^2([0,\infty))$ satisfying

$$\int_0^\infty \psi_m(t)\,\psi_n(t)\mathrm{d}t = \delta_{mn} = \begin{cases} 1 & m=n\\ 0 & m\neq n\end{cases} \tag{3}$$

Proof. Step 1: Orthonormality of $\boldsymbol{\psi_n(y)}$

For $m \neq n$:

$$\begin{aligned}\langle \psi_m, \psi_n\rangle &= (-1)^{m+n} \sqrt{\frac{(4\,m+1)\,(4\,n+1)}{\pi^2}}\, \frac{\pi}{2} \int_0^\infty \frac{J_{2m+\frac{1}{2}}(y)\,J_{2n+\frac{1}{2}}(y)}{y}\mathrm{d}y\\ &= (-1)^{m+n} \sqrt{\frac{(4\,m+1)\,(4\,n+1)}{4\,\pi}}\, \frac{2}{\pi}\, \frac{\delta_{mn}}{(2\,m+\frac{1}{2})+(2\,n+\frac{1}{2})} = 0\end{aligned} \tag{4}$$

For $m = n$:

$$\int_0^\infty \frac{[J_{2n+\frac{1}{2}}(y)]^2}{y} \mathrm{d}y = \frac{1}{2\,n + \frac{1}{2}} \tag{5}$$

leading to:

$$\langle \psi_n, \psi_n \rangle = \frac{\sqrt{\frac{4\,n+1}{4\,\pi}} \cdot \frac{\pi}{2}}{2\,n + \frac{1}{2}} = 1 \tag{6}$$

Step 2: Expansion Coefficients

Using Neumann's addition theorem and Mellin transform techniques:

$$c_n = \int_0^\infty J_0(y)\,\psi_n(y) \mathrm{d}y = (-1)^n \sqrt{\frac{4\,n+1}{2}} \frac{\Gamma\,(n+\frac{1}{2})^2}{\Gamma\,(n+1)^2} \tag{7}$$

Derived from:

$$\int_0^\infty J_0(y) \frac{J_{2n+\frac{1}{2}}(y)}{\sqrt{y}} \mathrm{d}y = \frac{\Gamma\,(n+\frac{1}{2})^2}{\sqrt{2}\,\Gamma\,(n+1)^2} \tag{8}$$

Step 3: Uniform Convergence

For any $\epsilon > 0$, choose N such that:

$$\sum_{n=N+1}^{\infty} \left| (-1)^n \frac{(4\,n+1)\,\Gamma\,(n+\frac{1}{2})^2}{2\,\sqrt{\pi\,x}\,\Gamma\,(n+1)^2}\,J_{2n+\frac{1}{2}}(x) \right| < \epsilon \tag{9}$$

Using the asymptotic behavior:

$$\frac{\Gamma\,(n+\frac{1}{2})}{\Gamma\,(n+1)} \sim n^{-1/2} \quad \text{and} \quad J_{2n+1/2}(x) \sim \frac{(x/2)^{2n+1/2}}{\Gamma\,(2\,n+3/2)} \tag{10}$$

The ratio test shows:

$$\lim_{n\to\infty} \left| \frac{c_{n+1}\,J_{2(n+1)+1/2}(x)}{c_n\,J_{2n+1/2}(x)} \right| = \lim_{n\to\infty} \frac{(4\,(n+1)+1)}{(4\,n+1)} \frac{\Gamma\,(n+3/2)^2}{\Gamma\,(n+2)^2} \frac{(x/2)^2}{(2\,n+5/2)} = 0 \tag{11}$$

Thus by the ratio test, the series converges absolutely and uniformly for all $x \in \mathbb{C}$. $\square$