Orthonormality Of The Spectral Factor Convolution

Lemma 1. *[Orthonormality Preservation] Let $K(x,y)=J_0(|x-y|)$ be the Bessel kernel with spectral factor $\rho(y)=\mathcal{F}^{-1}\big(\sqrt{\mathcal{F}[K](\omega)}\big)$. Given an orthonormal basis $\{\psi_n\}_{n=1}^{\infty}$ for $L^2([0,\infty))$, the functions $\phi_n(x)=(\psi_n*\rho)(x)$ satisfy:*

$$\langle\phi_m,\phi_n\rangle_{\mathscr{H}_K}=\delta_{mn},$$

where $\mathscr{H}_K$ is the RKHS associated with K.

Proof. **Step 1: RKHS Inner Product Characterization**

For any $f,g\in\mathscr{H}_K$, the RKHS inner product for translation-invariant kernels is:

$$\langle f,g\rangle_{\mathscr{H}_K}=\int_{\mathbb{R}}\frac{\mathcal{F}[f](\omega)\overline{\mathcal{F}[g](\omega)}}{\mathcal{F}[K](\omega)}\,d\omega \tag{1}$$

Step 2: Spectral Representation of ϕ_n

Using the Fourier convolution theorem:

$$\begin{aligned}\mathcal{F}[\phi_n](\omega)=&\mathcal{F}\,[\psi_n*\rho](\omega) &(2)\\ =&\mathcal{F}[\psi_n](\omega)\cdot\mathcal{F}[\rho](\omega) &(3)\\ =&\mathcal{F}[\psi_n](\omega)\cdot\sqrt{\mathcal{F}[K](\omega)} &(4)\end{aligned}$$

Step 3: Inner Product Computation

Substitute into (1):

$$\begin{aligned}\langle\phi_m,\phi_n\rangle_{\mathscr{H}_K}=&\int_{\mathbb{R}}\frac{\big(\mathcal{F}[\psi_m](\omega)\sqrt{\mathcal{F}[K](\omega)}\big)\big(\overline{\mathcal{F}[\psi_n](\omega)}\sqrt{\mathcal{F}[K](\omega)}\big)}{\mathcal{F}[K](\omega)}\,d\omega &(5)\\ =&\int_{\mathbb{R}}\mathcal{F}[\psi_m](\omega)\overline{\mathcal{F}[\psi_n](\omega)}\,d\omega &(6)\end{aligned}$$

Step 4: Parseval's Theorem Application

By Plancherel's theorem (Parseval for L^2):

$$\int_{\mathbb{R}} \mathcal{F}[\psi_m](\omega)\overline{\mathcal{F}[\psi_n](\omega)}d\,\omega = \langle \psi_m, \psi_n \rangle_{L^2} = \delta_{mn} \tag{7}$$

Conclusion

Therefore:

$$\langle \phi_m, \phi_n \rangle_{\mathscr{H}_K} = \delta_{mn}$$

The convolution with ρ preserves orthonormality from L^2 to $\mathscr{H}_K$. $\square$