Uniform Convergence of an Eigenfunction Expansion for J_0

by Stephen Crowley <stephencrowley214@gmail.com>

I'm trying to prove that the eigenfunctions and eigenvalues of

$$\int_0^\infty J_0\,(x-y)\,\psi_n(x)\mathrm{d}x = \lambda_n \psi_n(x) \tag{1}$$

are given by

$$\psi_n(y) = (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\,j_{2n}(y) \tag{2}$$

and

$$\lambda_n = \int_{-\infty}^\infty J_0(x)\,\psi_n(x)\,\mathrm{d}x = \sqrt{\frac{4\,n+1}{\pi}}\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\,(n+1)^2} = \sqrt{\frac{4\,n+1}{\pi}}\,(n+1)^2_{-\frac{1}{2}} \tag{3}$$

and that the corresponding eigenfunction expansion converges uniformly on $(0,\infty)$

$$\begin{aligned}
J_0(x) &= \sum_{k=0}^\infty \lambda_k \psi_k(x) \\
&= \sum_{k=0}^\infty \sqrt{\frac{4\,n+1}{\pi}}\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\,(n+1)^2}(-1)^n \sqrt{\frac{4\,n+1}{\pi}}\,j_{2n}(y) \\
&= \sum_{k=0}^\infty \frac{4\,n+1}{\pi}\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\,(n+1)^2}(-1)^n\,j_{2n}(y)
\end{aligned} \tag{4}$$

If this was a bounded interval or the kernel was square integrable the Hilbert-Schmidt theorem or Mercer's theorem could be invoked that says a uniformly converging orthonormal sequence of translation invariant functions are necessarily the eigenfunctions of the corresponding integral covariance operator having the kernel to which the partial sums converge as the covariance kernel. The solution was derived by

1. Identifying the orthogonal polynomial sequence associated with the spectral density of the kernel K, which in the case where $K = J_0$ is given by

$$S(\omega) = \int_0^\infty J_0(x)\,e^{ix\omega}\,\mathrm{d}x = \begin{cases} \frac{1}{\sqrt{1-\omega^2}} & \omega \in (-1,1) \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

which is equal to the spectral density of the Gaussian process having the kernel $K(t, s) = J_0(t-s)$. Recalling the Chebyshev polynomials' orthogonality relation:

$$\int_{-1}^{1} T_n(\omega)\, T_m(\omega)\, S(\omega)\, \mathrm{d}\omega = \begin{cases} 0 & n \neq m \\ \pi & n = m = 0 \\ \frac{\pi}{2} & n = m \neq 0 \end{cases} \tag{6}$$

Calculate their (finite) Fourier transforms of the Chebyshev type-I polynomials(which is just the usual infinite Fourier transform with the integration restricted to the range $-1\ldots 1$ since $T_n(x) = 0 \forall x \notin [-1, 1]$) or equivalent the spectral density extended to take the value 0 outside $[-1, 1]$

$$\begin{aligned} \hat{T}_n(y) &= \int_{-\infty}^{\infty} e^{-ixy}\, T_n(x) d\, y = \int_{-1}^{1} e^{-ixy}\, T_n(x) d\, x \\ &= \int_{-\infty}^{\infty} e^{-ixy}\, {}_2F_1\left(\begin{matrix} n, & -n \\ & \frac{1}{2} \end{matrix} \middle| \frac{1}{2} - \frac{x}{2} \right) d\, x \\ &= \frac{i}{y}\, (e^{-iy} F_n^+(y) - e^{i(\pi n + y)} F_n^-(y)) \end{aligned} \tag{7}$$

where

$$F_n^{\pm}(y) = {}_3F_1\left(\begin{matrix} 1, & n, & -n \\ & & \frac{1}{2} \end{matrix} \middle| \frac{\pm i y}{2} \right) \tag{8}$$

Then use L^2 norm of $\hat{T}_n(y)$

$$|\hat{T}_n| = \sqrt{\int_{-\infty}^{\infty} \hat{T}_n(y)^2 \mathrm{d}y} = \sqrt{\frac{4(-1)^n \pi - (2n^2 - 1)}{4n^2 - 1}} \tag{9}$$

to define the normalized Fourier transforms[1] $Y_n(y)$ of $T_n(x)$ by

$$\begin{aligned} Y_n(y) &= \frac{\hat{T}_n(y)}{|\hat{T}_n|} \\ &= \frac{i}{y} \left(\frac{e^{-iy} F_n^+(y) - e^{i(\pi n + y)} F_n^-(y)}{\sqrt{\frac{4(-1)^n \pi - (2n^2 - 1)}{4n^2 - 1}}} \right) \end{aligned} \tag{10}$$

then orthogonalize them so that our eigenfunctions are recognized as the the orthogonal complement of the normalized Fourier transformed $Y_n(y)$ of the Type-1 Chebshev polynomials $T_n(x)$ (via the Gram-Schmidt process)

$$\psi_n(y) = Y_n^{\perp}(y) = Y_n(y) - \sum_{m=1}^{n-1} \frac{\langle Y_n(y), Y_m^{\perp}(y)\rangle}{\langle Y_m^{\perp}(y), Y_m^{\perp}(y)\rangle} Y_m^{\perp}(y) \tag{11}$$

with respect to the unweighted standard Lebesgue inner product measure over 0 to $\infty <$

$$\lambda_n = \int_{-\infty}^{\infty} J_0(x)\, \psi_n(x)\, \mathrm{d}x = \sqrt{\frac{4\,n+1}{\pi}}\, \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\,(n+1)^2} = \sqrt{\frac{4\,n+1}{\pi}}\, (n+1)^2_{-\frac{1}{2}} \tag{12}$$

where $(n+1)^2_{-\frac{1}{2}}$ is the Pochhammer symbol aka rising factorial. The eigenfunctions can be equivalently expressed as

$$\begin{aligned} \psi_n(y) &= (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, j_{2n}(y) \\ &= (-1)^n \sqrt{\frac{4\,n+1}{\pi}} \sqrt{\frac{\pi}{2\,y}}\, J_{2n+\frac{1}{2}}(y) \\ &= (-1)^n \sqrt{\frac{4\,n+1}{2\,y}}\, J_{2n+\frac{1}{2}}(y) \\ &= \sqrt{\frac{4\,n+1}{4\,\pi}} \int_{-1}^{1} P_{2n}(x)\, e^{ixy}\, \mathrm{d}x \end{aligned} \tag{13}$$

where $P_n(x)$ is the Legendre polynomials, $j_n(x)$ is the spherical Bessel function of the first kind,

$$j_n(x) = \sqrt{\frac{\pi}{2\,x}}\, J_{n+\frac{1}{2}}(x) = \frac{1}{\sqrt{x}} \left(\sin(z)\, R_{n,\frac{1}{2}}(z) - \cos(z)\, R_{n,\frac{3}{2}}(z)\right) \tag{14}$$

and where $R_{n,v}(z)$ are the (misnamed) Lommel polynomials[2]

$$R_{n,v}(z) = \frac{\Gamma\,(n+v)}{\Gamma(v)} \left(\frac{z}{2}\right)_2^{-n} F_3\!\left([-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}]; [v, -n, -v+1-n]; -z^2\right) \tag{15}$$

which are actually rational functions of z, not polynomial but rather "polynomial in $1/z''$. Uniform convergence would imply that

$$K_n(x,y) = \sum_{k=0}^{n} \lambda_k\, \psi_k\,(x-y) \tag{16}$$

such that

$$\lim_{n \to \infty} K_n(x, y) = J_0(x - y) \tag{17}$$

It is also the case that

$$\lambda_{m,n} = \int_{-\infty}^{\infty} K_m(x)\, \psi_n(x)\, \mathrm{d}x = \begin{cases} \lambda_n & n \leqslant m \\ 0 & n > m \end{cases} \tag{18}$$

Bibliography

[1] Atul Dixit, Lin Jiu, Victor H Moll, and Christophe Vignat. The finite Fourier transform of classical polynomials. *Journal of the Australian Mathematical Society*, 98:145–160, 2015.

[2] R. Wong K.F. Lee. Asymptotic expansion of the modified Lommel polynomials $h_{n,\nu}(x)$ and their zeros. *Proceedings of the American Mathematical Society*, 142(11):3953–3964, 2014.