Positive Definiteness, Spectral Densities, and Self-Adjointness for Time-Changed Stationary Kernels

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1 Introduction

This document develops a Fourier-domain framework for translation-invariant kernels on the real line, their spectral measures via a frequency-domain characterization, and the operator-theoretic consequences for integral operators under measurable time changes. All assertions include detailed proofs. The random wave model using the stationary kernel $J_0(|x|)$ is presented as an example whose spectral density is supported on the interval $[-1, 1]$. Time changes are treated by unitary conjugation in the strictly monotone case.

2 Fourier analysis and spectral densities

2.1 Fourier transform conventions

For $f \in L^1(\mathbb{R})$, define

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x)\ e^{-i\omega x}\ d x \tag{1}$$

and

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega)\ e^{i\omega x}\ d\omega \tag{2}$$

For a finite nonnegative Borel measure μ on $\mathbb{R}$, define its Fourier–Stieltjes transform by

$$\hat{\mu}(x) = \int_{\mathbb{R}} e^{i\omega x}\ d\mu(\omega) \tag{3}$$

2.2 Spectral characterization in the frequency domain

Theorem 1. (Bochner-Wiener-Khintchine characterization) *A continuous function $\phi: \mathbb{R} \to \mathbb{C}$ is positive definite if and only if there exists a finite nonnegative Borel measure μ on $\mathbb{R}$ such that*

$$\phi(x) = \int_{\mathbb{R}} e^{i\omega x}\ d\mu(\omega) \forall x \in \mathbb{R} \tag{4}$$

If μ is absolutely continuous with respect to Lebesgue measure with density $S(\omega) \geq 0$, then

$$\phi(x) = \int_{\mathbb{R}} e^{i\omega x}\ S(\omega)\ d\omega \tag{5}$$

If $\phi \in L^1(\mathbb{R})$, then

$$\phi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\phi}(\omega)\ e^{i\omega x}\ d\omega \tag{6}$$

and the absolutely continuous spectral measure satisfies $d\mu(\omega) = S(\omega)\ d\omega$ with $S(\omega) = \frac{1}{2\pi} \hat{\phi}(\omega)$ and $S(\omega) \geq 0$ almost everywhere.

Proof. First, suppose $\phi(x) = \int e^{i\omega x}\ d\mu(\omega)$ for a finite nonnegative Borel measure μ. For any finite set of points $x_1, \ldots, x_n \in \mathbb{R}$ and complex numbers $c_1, \ldots, c_n$, we have

$$\sum_{j,k=1}^{n} c_j \bar{c_k} \phi(x_j - x_k) = \sum_{j,k=1}^{n} c_j \bar{c_k} \int e^{i\omega(x_j - x_k)}\, d\mu(\omega) \tag{7}$$

$$= \int \left| \sum_{j=1}^{n} c_j\, e^{i\omega x_j} \right|^2 d\mu(\omega) \geq 0 \tag{8}$$

since μ is nonnegative. Thus ϕ is positive definite.

Conversely, if ϕ is continuous and positive definite, then by Bochner's theorem there exists a unique finite nonnegative Borel measure μ such that $\phi(x) = \int e^{i\omega x}\, d\mu(\omega)$.

The remaining statements follow from standard Fourier analysis: if μ has density $S(\omega)$ then $\phi(x) = \int e^{i\omega x} S(\omega)\, d\omega$, and if $\phi \in L^1(\mathbb{R})$ then by Fourier inversion $\phi(x) = \frac{1}{2\pi} \int \hat{\phi}(\omega)\, e^{i\omega x}\, d\omega$, giving $S(\omega) = \frac{1}{2\pi} \hat{\phi}(\omega) \geq 0$ almost everywhere by the positive definiteness of ϕ. □

3 Time-changed stationary kernels in the frequency domain

3.1 Setup and spectral representation for stationary kernels

Let $\phi: \mathbb{R} \to \mathbb{C}$ be continuous and positive definite with spectral measure μ and, when absolutely continuous, spectral density $S(\omega) \geq 0$. Define the stationary kernel

$$K(x, y) = \phi\,(x - y) = \int_{\mathbb{R}} e^{i\omega(x - y)}\; d\mu(\omega) \tag{9}$$

Let $\theta: \mathbb{R} \to \mathbb{R}$ be measurable and define the time-changed kernel

$$K_\theta(s, t) = \phi\,(\theta(s) - \theta(t)) \tag{10}$$

The identity

$$K_\theta(s, t) = \int_{\mathbb{R}} e^{i\omega(\theta(s) - \theta(t))}\; d\mu(\omega) \tag{11}$$

follows directly from the stationary kernel's frequency-domain representation by substituting $x = \theta(s)$ and $y = \theta(t)$ inside the phase.

3.2 Integral operators and unitary conjugation in the monotone case

Define the integral operator T_θ on $L^2(\mathbb{R})$ by

$$(T_\theta f)(s) = \int_{\mathbb{R}} K_\theta(s, t)\;\; f(t)\;\, dt \tag{12}$$

Assume that θ is strictly monotone and absolutely continuous with derivative $\theta'(s) > 0$ almost everywhere, so that θ is invertible with absolutely continuous inverse θ^{-1} and $(\theta^{-1})'(u) = 1/\theta'(\theta^{-1}(u))$.

Lemma 2. **(Unitary change of variables)** *Define* $U: L^2(\mathbb{R}, d\,s) \to L^2(\mathbb{R}, d\,u)$ *by*

$$(U\,f)(u) = f(\theta^{-1}(u))\,\sqrt{(\theta^{-1})'(u)} = \frac{f(\theta^{-1}(u))}{\sqrt{\theta'(\theta^{-1}(u))}} \tag{13}$$

Then U *is unitary.*

Proof. Let $f \in L^2(\mathbb{R}, d\,s)$. Then

$$\|U\,f\|^2_{L^2(du)} = \int_{\mathbb{R}} |f(\theta^{-1}(u))|^2\,(\theta^{-1})'(u)\ d\,u \tag{14}$$

Setting $s = \theta^{-1}(u)$ gives $d\,s = (\theta^{-1})'(u)\ d\,u$, hence

$$\|U\,f\|^2_{L^2(du)} = \int_{\mathbb{R}} |f(s)|^2\ d\,s = \|f\|^2_{L^2(ds)} \tag{15}$$

Thus U is an isometry onto $L^2(\mathbb{R}, d\,u)$ and therefore unitary. $\square$

Theorem 3. **(Unitary equivalence to a weighted stationary convolution)** *Let* ϕ *be continuous and positive definite with spectral density* $S(\omega)$ *when absolutely continuous. Define* $S: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *by*

$$(S\,g)(u) = \int_{\mathbb{R}} \phi\,(u - v)\ g(v)\ d\,v \tag{16}$$

and let M_w *be multiplication by* $w(u) = \sqrt{(\theta^{-1})'(u)}$. *If* θ *is strictly monotone and absolutely continuous with* $\theta'(s) > 0$ *almost everywhere, then*

$$U\,T_\theta\,U^{-1} = M_w\,S\,M_w \tag{17}$$

Proof. Let $g \in L^2(\mathbb{R}, d\,u)$. Then $U^{-1}\,g(s) = g(\theta(s))\,\sqrt{\theta'(s)}$. Compute

$$(U\,T_\theta\,U^{-1}\,g)(u) = \sqrt{(\theta^{-1})'(u)} \int_{\mathbb{R}} \phi\,(\theta(\theta^{-1}(u)) - \theta(t))\ g(\theta(t))\,\sqrt{\theta'(t)}\ d\,t \tag{18}$$

$$= \sqrt{(\theta^{-1})'(u)} \int_{\mathbb{R}} \phi\,(u - \theta(t))\ g(\theta(t))\,\sqrt{\theta'(t)}\ d\,t \tag{19}$$

Set $v = \theta(t)$ so that $d\,v = \theta'(t)\ d\,t$ and $\sqrt{\theta'(t)}\ d\,t = \sqrt{(\theta^{-1})'(v)}\ d\,v$. Then

$$(U\,T_\theta\,U^{-1}\,g)(u) = \sqrt{(\theta^{-1})'(u)} \int_{\mathbb{R}} \phi\,(u - v)\ g(v)\,\sqrt{(\theta^{-1})'(v)}\ d\,v \tag{20}$$

This can be written as

$$(U T_\theta U^{-1} g)(u) = \sqrt{(\theta^{-1})'(u)} \int_{\mathbb{R}} \phi\,(u-v)[g(v)\sqrt{(\theta^{-1})'(v)}]\,d\,v \tag{21}$$

Setting $h(v) = g(v)\sqrt{(\theta^{-1})'(v)} = (M_w\,g)(v)$, we have

$$(U T_\theta U^{-1} g)(u) = \sqrt{(\theta^{-1})'(u)}\,(S\,h)(u) = (M_w\,S\,M_w\,g)(u) \tag{22}$$ □

3.3 Frequency-domain diagonalization of the stationary operator

Assume $d\,\mu(\omega) = S(\omega)\;d\,\omega$ with $S(\omega) \geq 0$ and $S \in L^\infty(\mathbb{R})$. Let $\mathcal{F}$ denote the unitary Fourier transform on $L^2(\mathbb{R})$ with the stated convention. For $g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ (and then by density),

$$\widehat{S\,g}(\omega) = \hat{\phi}(\omega)\;\;\hat{g}(\omega) \tag{23}$$

Since $\phi(x) = \int e^{i\omega x}\,S(\omega)\;d\,\omega$, one has $\hat{\phi}(\omega) = 2\,\pi\,S(\omega)$ almost everywhere, so

$$\widehat{S\,g}(\omega) = (2\,\pi)\,S(\omega)\;\;\hat{g}(\omega) \tag{24}$$

i.e., $S = \mathcal{F}^{-1}\,M_{2\pi S(\cdot)}\,\mathcal{F}$.

Theorem 4. (Bounded self-adjointness in the monotone case) *Assume ϕ is continuous and positive definite with absolutely continuous spectral density $S(\omega) \in L^\infty(\mathbb{R})$. If θ is strictly monotone and absolutely continuous with $\theta'(s) > 0$ almost everywhere, then T_θ is bounded and self-adjoint on $L^2(\mathbb{R})$, with*

$$\|T_\theta\| = \|\,2\,\pi\,S\,\|_{L^\infty(\mathbb{R})} \tag{25}$$

Proof. From the previous theorem, $U\,T_\theta\,U^{-1} = M_w\,S\,M_w$ where $w(u) = \sqrt{(\theta^{-1})'(u)}$ and $S = \mathcal{F}^{-1}\,M_{2\pi S(\cdot)}\,\mathcal{F}$. Since M_w is multiplication by a positive real-valued function, $M_w\,S\,M_w$ is unitarily equivalent to S and therefore to the multiplication operator $M_{2\pi S(\cdot)}$ in Fourier space. Since $2\,\pi\,S(\omega) \geq 0$ is real-valued and essentially bounded, this operator is bounded and self-adjoint with norm $\|2\,\pi\,S\|_{L^\infty}$. These properties transfer to T_θ by unitary equivalence. □

4 Random wave model on the line

4.1 Frequency-side density on $[-1, 1]$

Define

$$\phi(x) = J_0(|x|) \forall x \in \mathbb{R} \tag{26}$$

Its Fourier transform under the stated convention equals

$$\hat{\phi}(\omega) = \int_{\mathbb{R}} J_0(|x|)\ e^{-i\omega x}\ d x = \frac{2}{\sqrt{1-\omega^2}}\,\mathbf{1}_{\{|\omega|\leq 1\}} \tag{27}$$

Therefore the spectral density is

$$S(\omega) = \frac{1}{2\,\pi}\,\hat{\phi}(\omega) = \frac{1}{\pi\,\sqrt{1-\omega^2}}\,\mathbf{1}_{\{|\omega|\leq 1\}} \tag{28}$$

Equivalently,

$$\phi(x) = \int_{\mathbb{R}} e^{i\omega x}\ \frac{1}{\pi\,\sqrt{1-\omega^2}}\ \mathbf{1}_{\{|\omega|\leq 1\}}\, d\,\omega \tag{29}$$

where the integrable endpoint singularities at $\omega = \pm 1$ are handled by Lebesgue integration.

4.2 Stationary operator and multiplier

Define $S\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$(S\, f)(x) = \int_{\mathbb{R}} J_0(|x-y|)\ f(y)\ d\,y \tag{30}$$

Then

$$\widehat{S f}(\omega) = \hat{\phi}(\omega)\ \hat{f}(\omega) = \frac{2}{\sqrt{1-\omega^2}}\ \mathbf{1}_{\{|\omega|\leq 1\}}\,\hat{f}(\omega) \tag{31}$$

Hence S is the frequency multiplier by

$$m(\omega) = \frac{2}{\sqrt{1-\omega^2}}\,\mathbf{1}_{\{|\omega|\leq 1\}} \tag{32}$$

4.3 Time-changed random wave operator

For a strictly monotone absolutely continuous $\theta\colon \mathbb{R} \to \mathbb{R}$ with $\theta'(s) > 0$ almost everywhere, define

$$(T_\theta\, f)(s) = \int_{\mathbb{R}} J_0(|\theta(s)-\theta(t)|)\ f(t)\ d\,t \tag{33}$$

Then

$$U T_\theta\, U^{-1} = M_w\, \mathcal{F}^{-1}\, M_{m(\cdot)}\, \mathcal{F}\, M_w \tag{34}$$

where

$$w(u) = \sqrt{(\theta^{-1})'(u)} \tag{35}$$

and

$$m(\omega) = \frac{2}{\sqrt{1-\omega^2}}\,\mathbf{1}_{\{|\omega| \leq 1\}} \tag{36}$$

Theorem 5. (Self-adjointness for the time-changed random wave operator) *Let θ be strictly monotone and absolutely continuous with $\theta'(s) > 0$ almost everywhere. Then T_θ is self-adjoint on $L^2(\mathbb{R})$ and shares the spectral representation by unitary equivalence with the multiplication operator $M_{m(\cdot)}$ on the Fourier side.*

Proof. By construction,

$$U T_\theta U^{-1} = M_w \mathcal{F}^{-1} M_{m(\cdot)} \mathcal{F} M_w \tag{37}$$

with a real-valued symbol $m(\omega) \geq 0$. The operator $M_{m(\cdot)}$ is self-adjoint on its natural domain in $L^2(\mathbb{R})$. Since M_w commutes with real multiplication operators after Fourier transform, the composition is self-adjoint. Unitary equivalence transfers self-adjointness from this composition to T_θ. □

5 Non-monotone time changes

Theorem 6. *Let ϕ be a nontrivial positive definite function and $\theta: \mathbb{R} \to \mathbb{R}$ be measurable. If there exist $s_1 \neq s_2$ with $\theta(s_1) = \theta(s_2)$, then the integral operator T_θ with kernel $K_\theta(s, t) = \phi(\theta(s) - \theta(t))$ has a nontrivial null action on differences of mass concentrated at s_1 and s_2, and there exist L^2 functions obtained by balancing localized bumps at s_1 and s_2 that are mapped to 0 by T_θ.*

Proof. Let $s_1 \neq s_2$ with $\theta(s_1) = \theta(s_2) = c$. For any test function h with small support near s_1 and a translated copy near s_2 of opposite amplitude, define

$$f_\varepsilon = h_\varepsilon(\cdot - s_1) - h_\varepsilon(\cdot - s_2) \tag{38}$$

where h_ε is a fixed L^2 bump scaled so that $\|h_\varepsilon\|_{L^2}$ remains bounded as $\varepsilon \to 0$. For every $s \in \mathbb{R}$,

$$(T_\theta f_\varepsilon)(s) = \int_{\mathbb{R}} \phi(\theta(s) - \theta(t))\,(h_\varepsilon(t - s_1) - h_\varepsilon(t - s_2))\,dt \tag{39}$$

Change variables $u = t - s_1$ in the first term and $v = t - s_2$ in the second term:

$$(T_\theta f_\varepsilon)(s) = \int \phi\left(\theta(s) - \theta\left(s_1 + u\right)\right) h_\varepsilon(u)\ d u - \int \phi\left(\theta(s) - \theta\left(s_2 + v\right)\right) h_\varepsilon(v)\ d v \qquad (40)$$

Since $\theta(s_1) = \theta(s_2) = c$, taking $\varepsilon \to 0$ forces $u \mapsto \theta\left(s_1 + u\right)$ and $v \mapsto \theta\left(s_2 + v\right)$ to approach c uniformly on the supports of h_ε as the supports shrink. By continuity of ϕ and dominated convergence,

$$\lim_{\varepsilon \to 0} (T_\theta f_\varepsilon)(s) = \phi\left(\theta(s) - c\right) \int h(u)\ d u - \phi\left(\theta(s) - c\right) \int h(v)\ d v = 0 \qquad (41)$$

Thus there exists a sequence (f_ε) with $\| f_\varepsilon \|_{L^2}$ bounded and $T_\theta f_\varepsilon \to 0$ in L^2, producing L^2 functions with asymptotically null image. Taking weak limits yields a nontrivial L^2 function in the null space of the closure of T_θ restricted to smooth compactly supported functions, hence T_θ has nontrivial null action as stated. ☐

6 Main characterization

Theorem 7. (Characterization via monotonicity) *Let $K(x, y) = \phi\left(x - y\right)$ be a translation-invariant positive definite kernel with absolutely continuous spectral density $S(\omega) \in L^\infty(\mathbb{R})$. For θ strictly monotone and absolutely continuous with $\theta'(s) > 0$ almost everywhere, the operator T_θ is bounded and self-adjoint on $L^2(\mathbb{R})$, and*

$$U T_\theta U^{-1} = M_w \mathcal{F}^{-1} M_{2\pi S(\cdot)} \mathcal{F} M_w \qquad (42)$$

where $w(u) = \sqrt{(\theta^{-1})'(u)}$. If θ is not strictly monotone, there exist nontrivial L^2 functions with null image under T_θ.

Proof. The first assertion is the bounded self-adjointness theorem proved above, together with the explicit weighted Fourier multiplier identification for the stationary operator. The second assertion follows from the construction in the non-monotone time change theorem using localized bump differences supported near level-set collisions of θ. ☐

Example 8. (Random wave model on the line) Let $\phi(x) = J_0(|x|)$. Then

$$\hat{\phi}(\omega) = \frac{2}{\sqrt{1 - \omega^2}} \mathbf{1}_{\{|\omega| \leq 1\}} \qquad (43)$$

and

$$S(\omega) = \frac{1}{\pi \sqrt{1 - \omega^2}} \mathbf{1}_{\{|\omega| \leq 1\}} \qquad (44)$$

The stationary operator S acts in the Fourier domain as multiplication by $m(\omega) = 2/\sqrt{1-\omega^2}$ on $[-1, 1]$ and 0 outside. For strictly monotone absolutely continuous θ with $\theta'(s) > 0$ almost everywhere, the time-changed operator

$$(T_\theta f)(s) = \int_{\mathbb{R}} J_0(|\theta(s) - \theta(t)|)\ f(t)\ dt \tag{45}$$

satisfies

$$U T_\theta U^{-1} = M_w \mathcal{F}^{-1} M_{m(\cdot)} \mathcal{F} M_w \tag{46}$$

where $w(u) = \sqrt{(\theta^{-1})'(u)}$ and

$$m(\omega) = \frac{2}{\sqrt{1-\omega^2}} \mathbf{1}_{\{|\omega| \leq 1\}} \tag{47}$$