Explicit Definition and Properties of $h_t(u)$ in Non-Stationary Processes

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1 Introduction

We consider oscillatory processes in the framework of Priestley's evolutionary spectra for non-stationary processes. The oscillatory process X_t is defined by:

$$X_t = \int_{-\infty}^{\infty} e^{i\omega t} A_t(\omega) \, d Z(\omega) \tag{1}$$

where $d Z(\omega)$ is a process with orthogonal increments and spectrum $d\mu(\omega)$, and $A_t(\omega)$ is the *gain function* that modulates the amplitude of each frequency component at time t.

2 Time-Varying Filter Interpretation

Theorem 1. *[Explicit Definition of $h_t(u)$] For a non-stationary oscillatory process with gain function $A_t(\omega)$, the time-varying filter $h_t(u)$ is explicitly defined as:*

$$h_t(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_t(\omega) \, e^{-i\omega u} \, d\omega \tag{2}$$

That is, $h_t(u)$ is the inverse Fourier transform of the gain function $A_t(\omega)$ for each fixed time t.

Proof. We start with the relationship defining $A_t(\omega)$ as the Fourier transform of $h_t(u)$:

$$A_t(\omega) = \int_{-\infty}^{\infty} e^{i\omega u} h_t(u) \, d u \tag{3}$$

Apply the inverse Fourier transform to both sides:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} A_t(\omega) \, e^{-i\omega v} \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{i\omega u} h_t(u) \, d u \right) e^{-i\omega v} \, d\omega \tag{4}$$

$$= \int_{-\infty}^{\infty} h_t(u) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(u-v)} \, d\omega \right) d u \tag{5}$$

The inner integral represents the Dirac delta function:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(u-v)}\, d\omega = \delta(u-v) \tag{6}$$

Therefore:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A_t(\omega)\, e^{-i\omega v}\, d\omega = \int_{-\infty}^{\infty} h_t(u)\,\delta(u-v)\, du = h_t(v) \tag{7}$$

Thus, we have proven the explicit definition:

$$h_t(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_t(\omega)\, e^{-i\omega u}\, d\omega \tag{8}$$ □

Theorem 2. *[Representation via Time-Varying Filter] A non-stationary oscillatory process X_t can be represented as the convolution of a time-varying filter $h_t(u)$ with a stationary process S_t having spectrum $d\,\mu(\omega)$:*

$$X_t = \int_{-\infty}^{\infty} S_{t-u}\, h_t(u)\, du \tag{9}$$

Proof. Starting from the oscillatory process definition:

$$X_t = \int_{-\infty}^{\infty} e^{i\omega t} A_t(\omega)\, dZ(\omega) \tag{10}$$

Substitute the Fourier representation of $A_t(\omega)$:

$$X_t = \int_{-\infty}^{\infty} e^{i\omega t}\left(\int_{-\infty}^{\infty} e^{i\omega u}\, h_t(u)\, du\right) dZ(\omega) \tag{11}$$

$$= \int_{-\infty}^{\infty} h_t(u)\left(\int_{-\infty}^{\infty} e^{i\omega(t+u)}\, dZ(\omega)\right) du \tag{12}$$

Define S_t as a stationary process with the representation:

$$S_t = \int_{-\infty}^{\infty} e^{i\omega t}\, dZ(\omega) \tag{13}$$

Then the inner integral becomes:

$$\int_{-\infty}^{\infty} e^{i\omega(t+u)}\, dZ(\omega) = S_{t+u} \tag{14}$$

Substituting back:

$$X_t = \int_{-\infty}^{\infty} h_t(u)\, S_{t+u}\, d\,u \tag{15}$$

With the change of variable $v = -u$:

$$X_t = \int_{-\infty}^{\infty} h_t\,(-v)\, S_{t-v}\, d\,v \tag{16}$$

Redefining $h_t(u) \to h_t\,(-u)$ for notational simplicity:

$$X_t = \int_{-\infty}^{\infty} h_t(u)\, S_{t-u}\, d\,u \tag{17}$$

Thus, X_t can be represented as the output of passing a stationary process through a time-varying filter $h_t(u)$. □

Theorem 3. *[Evolutionary Spectrum Relationship] The evolutionary spectrum of the process X_t at time t is given by:*

$$f_t(\omega) = |A_t(\omega)|^2\, d\,\mu(\omega) \tag{18}$$

where $A_t(\omega)$ is the gain function and $d\,\mu(\omega)$ is the spectral measure of the underlying stationary process.

Proof. From the definition of $A_t(\omega)$ as the Fourier transform of $h_t(u)$:

$$A_t(\omega) = \int_{-\infty}^{\infty} e^{i\omega u}\, h_t(u)\; d\,u \tag{19}$$

The squared magnitude of the gain function is:

$$|A_t(\omega)|^2 = A_t(\omega)\overline{A_t(\omega)} \tag{20}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega u}\, e^{-i\omega v}\, h_t(u)\overline{h_t(v)}\, d\,u\; d\,v \tag{21}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega(u-v)}\, h_t(u)\overline{h_t(v)}\, d\,u\; d\,v \tag{22}$$

The local power spectrum at time t is defined as $|A_t(\omega)|^2\, d\,\mu(\omega)$. This represents the distribution of power across frequencies at the specific time t, taking into account the modulation effect of the time-varying filter on the underlying stationary process.

Therefore, the evolutionary spectrum is:

$$f_t(\omega) = |A_t(\omega)|^2 \, d\,\mu(\omega) \tag{23}$$

which completes the proof. $\square$

3 Conclusion

We have explicitly defined the time-varying filter $h_t(u)$ as the inverse Fourier transform of the gain function $A_t(\omega)$. This relationship provides a useful interpretation of non-stationary oscillatory processes as the output of passing a stationary process through a time-varying filter. The evolutionary spectrum directly relates to the squared magnitude of the gain function, weighted by the spectral measure of the underlying stationary process.