Invertibility of Oscillatory Processes

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1 Oscillatory Gaussian Processes

Definition 1

An oscillatory process $X(t)$ in Priestley's sense has the integral representation

$$X(t) = \int_{\infty}^{\infty} A(t, \lambda)\, e^{i\lambda t}\, d\, Z(\lambda) \tag{1}$$

where $A(t, \lambda)$ is the time-varying amplitude function and $d\, Z(\lambda)$ is an orthogonal increment process with

$$E\left[d\, Z(\lambda_1) \overline{d\, Z(\lambda_2)}\right] = \delta\left(\lambda_1 - \lambda_2\right) \mu\left(d\, \lambda_1\right) \tag{2}$$

for some measure μ.

1.1 Invertibility Conditions

Theorem 2

[Fundamental Invertibility Theorem] The oscillatory process $X(t)$ with amplitude $A(t, \lambda)$ allows the expression of the associated complex orthogonal random measure

$$d\, Z(\lambda) = \int_{-\infty}^{\infty} \overline{A(t, \lambda)} e^{-i\lambda t} X(t)\, d\, t \tag{3}$$

from a sample path realization $X(t)$ if and only if:

1. *$A(t, \lambda) \neq 0 \forall (t, \lambda)$ in the relevant domain*

   and

2. *The orthogonality condition holds:*

$$\int_{-\infty}^{\infty} \overline{A(t, \lambda_1)} A(t, \lambda_2)\, e^{i(\lambda_2 - \lambda_1)t}\, d\, t = \delta\left(\lambda_1 - \lambda_2\right) \tag{4}$$

Proof. From the representation

$$X(t) = \int_{-\infty}^{\infty} A(t, \lambda)\, e^{i\lambda t}\, d\, Z(\lambda) \tag{5}$$

, one seeks to obtain the expression for $d\,Z(\lambda)$. The orthogonality condition (4) ensures that the kernel functions form an orthonormal system. This allows the projection of $X(t)$ onto each frequency component.Multiply both sides by $\overline{A(t,\lambda_0)}e^{-i\lambda_0 t}$ and integrate over t

$$\begin{aligned}\int_{-\infty}^{\infty}\overline{A(t,\lambda_0)}e^{-i\lambda_0 t}\,X(t)\;d\,t\; &=\int_{-\infty}^{\infty}\overline{A(t,\lambda_0)}e^{-i\lambda_0 t}\int_{-\pi}^{\pi}A(t,\lambda)\,e^{i\lambda t}\;d\,Z(\lambda)\;d\,t\\ &=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\overline{A(t,\lambda_0)}A(t,\lambda)\,e^{i(\lambda-\lambda_0)t}\;d\,t\right]d\,Z(\lambda)\\ &=\int_{-\infty}^{\infty}\delta\,(\lambda-\lambda_0)d\,Z(\lambda)\\ &=\mathrm{d}\,Z(\lambda_0)\end{aligned}\tag{6}$$

where the second-to-last equality is due to

$$\int_{-\infty}^{\infty}\overline{A(t,\lambda_0)}A(t,\lambda)\,e^{i(\lambda-\lambda_0)t}\;d\,t=\delta\,(\lambda-\lambda_0)\tag{7}$$

which yields $d\,Z(\lambda_0)$ after application of the elementary Dirac delta function identity. □

Lemma 3

[Uniqueness of Inversion] The inversion formula (6) is unique under the given conditions.

Proof. Suppose two different inversion operators both recover $d\,Z(\lambda)$ from $X(t)$. Then their difference must annihilate all possible $X(t)$ while producing zero output, which implies they are identical by the non-degeneracy condition $A(t,\lambda)\neq 0$. □

2 References

Priestley, M.B. (1965). Evolutionary spectra and non-stationary processes. *Journal of the Royal Statistical Society: Series B*, 27(2), 204-237.

Priestley, M.B. (1981). *Spectral Analysis and Time Series*. Academic Press.