The Operational Matrix of the Random Wave Process

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Lemma 1. *For any $p \in \mathbb{Z}_{\geq 0}$, the Gauss hypergeometric function 'terminates' so that it is equal to a finite sum rather than its usual infinite sum(the non-'terminating' case):*

$$ {}_2F_1(-p, b; c; z) = \sum_{k=0}^{p} \frac{(-p)_k (b)_k}{(c)_k \, k!} z^k \tag{1}$$

where $(a)_k = \prod_{i=0}^{k-1} (a+i)$

Lemma 2. *For $j \geq 0$,*

$$\int_{-1}^{1} \left(\frac{1-x}{2} \right)^j e^{ixy} \, d\, x = \frac{e^{iy}}{2^j} \frac{\gamma(j+1, 2\, i\, y)}{(i\, y)^{j+1}} \tag{2}$$

where $\gamma(s, x)$ denotes the lower incomplete gamma function.

Theorem 3. *For* $m, n \geq 0$,

$$I_{m,n}(y) = \int_{-1}^{1}{}_2F_1\!\left(-m, m+1; 1; \tfrac{1-x}{2}\right){}_2F_1\!\left(-n, n+1; 1; \tfrac{1-x}{2}\right) e^{ixy}\,dx \tag{3}$$

satisfies:

$$I_{m,n}(y) = e^{iy} \sum_{j=0}^{m+n} \frac{\Psi_j(m,n)}{2^j}\left[\frac{\gamma(j+1, 2\,i\,y)}{(i\,y)^{j+1}}\right] \tag{4}$$

where $\Psi_j(m,n)$ *is defined as:*

$$\Psi_j(m,n) = \sum_{k=\max(0, j-n)}^{\min(j,m)} \frac{(-m)_k\,(m+1)_k}{k!}\,\frac{(-n)_{j-k}\,(n+1)_{j-k}}{(j-k)!} \tag{5}$$

and equivalently:

$$\Psi_j(m,n) = \frac{{}_4F_3\!\left(\begin{array}{c} -m, m+1, -n, n+1 \\ 1, 1, j+1 \end{array}; 1\right)}{j!} \tag{6}$$

Proof. Part 1: Integral Reduction to Finite Sums

Expand both hypergeometric series using Lemma 1:

$${}_2F_1\!\left(-m, m+1; 1; \tfrac{1-x}{2}\right){}_2F_1\!\left(-n, n+1; 1; \tfrac{1-x}{2}\right) = \sum_{k=0}^{m}\sum_{\ell=0}^{n} \frac{(-m)_k\,(m+1)_k}{k!}\,\frac{(-n)_\ell\,(n+1)_\ell}{\ell!}\left(\tfrac{1-x}{2}\right)^{k+\ell} \tag{7}$$

Let $j = k + \ell$. For fixed j, k must satisfy $\max(0, j-n) \leq k \leq \min(j, m)$. Thus:

$$I_{m,n}(y) = \sum_{j=0}^{m+n}\sum_{k=\max(0,j-n)}^{\min(j,m)} \frac{(-m)_k\,(m+1)_k}{k!}\,\frac{(-n)_{j-k}\,(n+1)_{j-k}}{(j-k)!}\int_{-1}^{1}\left(\frac{1-x}{2}\right)^j e^{ixy}\,dx \tag{8}$$

Apply Lemma 2 to evaluate the integral:

$$I_{m,n}(y) = e^{iy} \sum_{j=0}^{m+n} \frac{\Psi_j(m,n)}{2^j}\left[\frac{\gamma(j+1, 2\,i\,y)}{(i\,y)^{j+1}}\right] \tag{9}$$

Part 2: Equivalence of $\boldsymbol{\Psi_j(m,n)}$ and $\boldsymbol{{}_4F_3}$

Start from the hypergeometric representation:

$$\Psi_j(m,n) = \frac{{}_4F_3\!\left(\begin{array}{c} -m, m+1, -n, n+1 \\ 1, 1, j+1 \end{array}; 1\right)}{j!} \tag{10}$$

Expand the ${}_4F_3$ series:

$$ {}_4F_3\left(\begin{matrix} -m, m+1, -n, n+1 \\ 1, 1, j+1 \end{matrix}; 1\right) = \sum_{k=0}^{\infty} \frac{(-m)_k\,(m+1)_k\,(-n)_k\,(n+1)_k}{(1)_k(1)_k\,(j+1)_k\,k!} \tag{11} $$

The series terminates at $k = \min(m, n)$ due to $(-m)_k = 0$ for $k > m$ and $(-n)_k = 0$ for $k > n$. Perform the substitution $\ell = j - k$:

$$ \Psi_j(m, n) = \sum_{k=\max(0, j-n)}^{\min(j, m)} \frac{(-m)_k\,(m+1)_k}{k!}\,\frac{(-n)_{j-k}\,(n+1)_{j-k}}{(j-k)!} \tag{12} $$

Simplify by substituting the identity $(j+1)_{j-\ell} = \frac{(2\,j-\ell)!}{j!}$

$$ \frac{1}{(j+1)_{j-\ell}} = \frac{j!}{(2\,j-\ell)!} \tag{13} $$

into the expression for $\Psi_j(m, n)$ which yields:

$$ \Psi_j(m, n) = \sum_{\ell=0}^{j} \frac{(-m)_{j-\ell}\,(m+1)_{j-\ell}\,(-n)_{j-\ell}\,(n+1)_{j-\ell}}{(1)_{j-\ell}(1)_{j-\ell}\,(2\,j-\ell)!(j-\ell)!} \tag{14} $$

Reverse the substitution $(k = j - \ell)$ to obtain:

$$ \Psi_j(m, n) = \sum_{k=\max(0, j-n)}^{\min(j, m)} \frac{(-m)_k\,(m+1)_k}{k!}\,\frac{(-n)_{j-k}\,(n+1)_{j-k}}{(j-k)!} \tag{15} $$

This establishes term-by-term equality, confirming the hypergeometric representation:

$$ \Psi_j(m, n) = \frac{{}_4F_3\left(\begin{matrix} -m, m+1, -n, n+1 \\ 1, 1, j+1 \end{matrix}; 1\right)}{j!} \tag{16} $$

$\blacksquare$ $\square$