Equivalence Between Itô Processes and Their Characteristic Functions

The **characteristic function** provides a complete probabilistic description of an Itô process, encoding all moments and serving as a functional equivalent. For a general Itô process defined by:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$

its characteristic function $\phi(\theta, t) = \mathbb{E}[e^{i\theta X_t}]$ satisfies the partial differential equation derived from the **infinitesimal generator** $\mathcal{A}$:

$$\frac{\partial \phi}{\partial t} = \mathcal{A}\phi, \quad \text{where } \mathcal{A} = \mu(t, x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t, x)\frac{\partial^2}{\partial x^2}.$$

Explicit Form for Linear Drift and Diffusion

For processes with **linear coefficients** ($\mu(t, X_t) = \mu X_t + c$, $\sigma(t, X_t) = \sigma X_t$), the characteristic function becomes:

$$\phi(\theta, t) = \exp\left(i\theta X_0 e^{\mu t} - \frac{\theta^2\sigma^2}{4\mu}(e^{2\mu t} - 1) + i\theta\frac{c}{\mu}(e^{\mu t} - 1)\right).$$

Generalized Case via Feynman-Kac Formula

For nonlinear coefficients, the characteristic function solves:

$$\frac{\partial \phi}{\partial t} = \mu(t, x)\frac{\partial \phi}{\partial x} + \frac{1}{2}\sigma^2(t, x)\frac{\partial^2 \phi}{\partial x^2}, \quad \phi(\theta, 0) = e^{i\theta X_0}.$$

This PDE admits solutions of the form:

$$\phi(\theta, t) = \exp\left(i\theta\mathbb{E}[X_t] - \frac{\theta^2}{2}\mathrm{Var}(X_t) + \cdots\right),$$

where higher-order terms capture non-Gaussian features[1] [2].

Lamperti Transformation: Functional Equivalence for State-Dependent Volatility

The **Lamperti transform** converts Itô processes with state-dependent volatility into processes with unit diffusion:

$$Z_t = \int_{X_0}^{X_t} \frac{1}{\sigma(x)} dx.$$

The transformed process satisfies:

$$dZ_t = \left(\frac{\mu(X_t)}{\sigma(X_t)} - \frac{1}{2} \frac{\partial \sigma}{\partial x}\Big|_{X_t} \right) dt + dW_t.$$

This establishes equivalence between the original process X_t and the normalized process Z_t [3] [2].

Convolutional Representation

As shown in the search results[3] [4], the infinitesimal generator $\mathcal{A}$ of an Itô diffusion admits a **convolution-type form**:

$$\mathcal{A}f(x) = \int_{\mathbb{R}} \big(f(x+y) - f(x) - y f'(x) \big) \nu(dy),$$

where ν is the Lévy measure. This generalizes to:

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}} y \tilde{N}(ds, dy),$$

with $\tilde{N}$ being the compensated Poisson random measure[3] [5].

Key Equivalence Results

Itô Process Property	Equivalent Functional Representation	Source
Characteristic Function	Solves $\partial_t \phi = \mathcal{A}\phi$	[1] [6]
Lamperti Transform	Maps to unit volatility SDE	[3] [2]
Convolution Form	Lévy-Ito decomposition	[3] [5]

For specific cases like the **Ornstein-Uhlenbeck process**:

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t \implies \phi(\theta, t) = e^{i\theta\mu(1-e^{-\theta t}) - \frac{\sigma^2 \theta^2}{4\theta}(1-e^{-2\theta t})}.$$

This functional equivalence enables spectral analysis and path simulation via inverse Fourier transforms[7] [8].

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1. https://math.stackexchange.com/questions/2973003/characteristic-function-of-ito-process
2. https://www.csie.ntu.edu.tw/~lyuu/finance1/2012/20120418.pdf
3. https://arxiv.org/abs/1212.3603
4. https://eng.libretexts.org/Bookshelves/Mechanical_Engineering/System_Design_for_Uncertainty_(Hover_and_Triantafyllou)/02:_Linear_Systems/2.04:_The_Impulse_Response_and_Convolution
5. https://arxiv.org/abs/1306.1492
6. https://math.stackexchange.com/questions/2172025/estimate-transition-density-function-of-ito-diffusion-sde
7. https://en.wikipedia.org/wiki/Ornstein–Uhlenbeck_process
8. https://math.nyu.edu/~goodman/teaching/StochCalc2013/notes/Week8.pdf