Orthonormal Galerkin Method for Stationary Integral Covariance Operator Eigenfunction Expansions

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1 Given

1. $K(s,t) = K(t-s)$
2. $K(t-s) = \sum_{n=0}^{\infty} \psi_n(t-s)$ (uniformly convergent)
3. Eigenvalue equation: $\int_0^{\infty} K(t-s)\, \phi_k(t)\; d t = \lambda_k\, \phi_k(s)$
4. Eigenfunction expansion: $\phi_k(t) = \sum_{n=0}^{\infty} c_{n,k}\, \psi_n(t)$
5. The basis functions $\{\psi_n\}$ are orthonormal, i.e., $\int_0^{\infty} \psi_m(s)\, \psi_n(s)\; d s = \delta_{mn}$

2 Objective

Solve for the coefficient matrices $c_{n,k}$ for the eigenfunctions of the integral covariance operator

$$\int_0^{\infty} K(t-s)\, \phi_k(t)\; d t = \lambda_k\, \phi_k(s) \tag{1}$$

3 Proof

1. Substitute the eigenfunction expansion into the eigenvalue equation:

$$\int_0^{\infty} K(t-s) \sum_{n=0}^{\infty} c_{n,k}\, \psi_n(t)\; d t = \lambda_k \sum_{n=0}^{\infty} c_{n,k}\, \psi_n(s) \tag{2}$$

2. Use the uniform expansion of K:

$$\int_0^\infty \sum_{j=0}^{\infty} \psi_j\,(t-s) \sum_{n=0}^{\infty} c_{n,k}\,\psi_n(t)\ d\,t = \lambda_k \sum_{n=0}^{\infty} c_{n,k}\,\psi_n(s) \tag{3}$$

3. Apply Fubini's theorem (justified by uniform convergence):

$$\sum_{j=0}^{\infty} \sum_{n=0}^{\infty} c_{n,k} \int_0^\infty \psi_j\,(t-s)\,\psi_n(t)\ d\,t = \lambda_k \sum_{n=0}^{\infty} c_{n,k}\,\psi_n(s) \tag{4}$$

4. Define $G_{j,n}(s) = \int_0^\infty \psi_j\,(t-s)\,\psi_n(t)\ d\,t$:

$$\sum_{n=0}^{\infty} c_{n,k} \sum_{j=0}^{\infty} G_{j,n}(s) = \lambda_k \sum_{n=0}^{\infty} c_{n,k}\,\psi_n(s) \tag{5}$$

5. Project onto the basis $\{\psi_m(s)\}$. Multiply both sides by $\psi_m(s)$ and integrate over s:

$$\int_0^\infty \sum_{n=0}^{\infty} c_{n,k} \sum_{j=0}^{\infty} G_{j,n}(s)\,\psi_m(s)\ d\,s = \lambda_k \int_0^\infty \sum_{n=0}^{\infty} c_{n,k}\,\psi_n(s)\,\psi_m(s)\ d\,s \tag{6}$$

6. Assuming we can interchange summation and integration:

$$\sum_{n=0}^{\infty} c_{n,k} \sum_{j=0}^{\infty} \int_0^\infty G_{j,n}(s)\,\psi_m(s)\ d\,s = \lambda_k \sum_{n=0}^{\infty} c_{n,k} \int_0^\infty \psi_n(s)\,\psi_m(s)\ d\,s \tag{7}$$

7. Using the orthonormality of $\{\psi_n\}$, the right-hand side simplifies to $\lambda_k\,c_{m,k}$. Define:

$$b_{m,n} = \sum_{j=0}^{\infty} \int_0^\infty G_{j,n}(s)\,\psi_m(s)\ d\,s \tag{8}$$

8. Our equation becomes:

$$\sum_{n=0}^{\infty} b_{m,n}\,c_{n,k} = \lambda_k\,c_{m,k} \tag{9}$$

9. This is a standard eigenvalue problem:

$$B\,\vec{c}_k = \lambda_k\,\vec{c}_k \tag{10}$$

where $B = (b_{m,n})$ and $\vec{c}_k = (c_{0,k}, c_{1,k}, \ldots)^T$

4 Verification that Solutions are Eigenfunctions

We will now prove that the solutions obtained are indeed eigenfunctions of the original integral equation.

1. Let λ_k and $\vec{c}_k = (c_{0,k}, c_{1,k}, \ldots)^T$ be the eigenvalues and eigenvectors of the matrix equation:

$$B\,\vec{c}_k = \lambda_k\,\vec{c}_k \tag{11}$$

   where $B = (b_{m,n})$ as derived above.

2. We construct the functions $\phi_k(t)$:

$$\phi_k(t) = \sum_{n=0}^{\infty} c_{n,k}\,\psi_n(t) \tag{12}$$

3. Substitute this into the original integral equation:

$$\int_0^{\infty} K\,(t-s)\,\phi_k(t)\,d\,t = \int_0^{\infty} K\,(t-s)\left[\sum_{n=0}^{\infty} c_{n,k}\,\psi_n(t)\right] d\,t \tag{13}$$

4. Using the expansion of $K\,(t-s)$ and interchanging summations:

$$= \int_0^{\infty}\left[\sum_{j=0}^{\infty} \psi_j\,(t-s)\right]\left[\sum_{n=0}^{\infty} c_{n,k}\,\psi_n(t)\right] d\,t \tag{14}$$

$$= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} c_{n,k}\int_0^{\infty} \psi_j\,(t-s)\,\psi_n(t)\,d\,t \tag{15}$$

5. Recall our definitions:

$$G_{j,n}(s) = \int_0^{\infty} \psi_j\,(t-s)\,\psi_n(t)\,d\,t \tag{16}$$

$$b_{m,n} = \sum_{j=0}^{\infty}\int_0^{\infty} G_{j,n}(s)\,\psi_m(s)\,d\,s \tag{17}$$

6. Rewrite the left-hand side:

$$\sum_{n=0}^{\infty} c_{n,k}\left[\sum_{j=0}^{\infty} G_{j,n}(s)\right] \tag{18}$$

7. Project onto $\psi_m(s)$ by multiplying by $\psi_m(s)$ and integrating over s:

$$\begin{aligned}
&\int_0^\infty \psi_m(s)\left[\sum_{n=0}^\infty c_{n,k}\left[\sum_{j=0}^\infty G_{j,n}(s)\right]\right]d\,s && (19)\\
=&\sum_{n=0}^\infty c_{n,k}\left[\sum_{j=0}^\infty \int_0^\infty G_{j,n}(s)\,\psi_m(s)\,d\,s\right] && (20)\\
=&\sum_{n=0}^\infty c_{n,k}\,b_{m,n} && (21)\\
=&(B\,\vec{c}_k)_m && (22)\\
=&\lambda_k(\vec{c}_k)_m && (23)\\
=&\lambda_k\,c_{m,k} && (24)
\end{aligned}$$

8. On the other hand, projecting $\phi_k(s)$ onto $\psi_m(s)$:

$$\int_0^\infty \psi_m(s)\,\phi_k(s)\,d\,s=\int_0^\infty \psi_m(s)\left[\sum_{n=0}^\infty c_{n,k}\,\psi_n(s)\right]d\,s=c_{m,k} \tag{25}$$

9. Comparing the results from steps 7 and 8, we see that:

$$\int_0^\infty \psi_m(s)\left[\int_0^\infty K\,(t-s)\,\phi_k(t)\,d\,t\right]d\,s=\lambda_k\int_0^\infty \psi_m(s)\,\phi_k(s)\,d\,s \tag{26}$$

10. Since this holds for all m, and $\{\psi_m\}$ is a complete orthonormal basis, we conclude:

$$\int_0^\infty K\,(t-s)\,\phi_k(t)\,d\,t=\lambda_k\,\phi_k(s) \tag{27}$$

Therefore, the $\phi_k(s)$ constructed from the eigenvectors of B are indeed eigenfunctions of the original integral equation with eigenvalues λ_k.