Gaussian Processes Generated By Monotonically Modulated Stationary Kernels

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Abstract

This article examines Gaussian processes generated by monotonically modulating stationary kernels. An explicit isometry between the original and the modulated reproducing kernel Hilbert spaces is established, preserving eigenvalues and normalization. The expected number of zeros over the interval $[0, T]$ is shown to be exactly $\sqrt{-\ddot{K}(0)}\,(\theta(T) - \theta(0))$, where $\ddot{K}(0)$ is the second derivative of the kernel at zero and θ is the modulating function.

1 Introduction

This article explores the properties of Gaussian processes[2][1] generated by monotonically modulating the kernels of stationary Gaussian processes. The investigation centers on three key aspects: (1) the relationship between eigenfunctions of the covariance operators defined by the original and the modulated kernels, (2) the preservation of normalization and eigenvalues under modulation, and (3) the expected number of zeros of the resulting processes. Beginning with a precise definition of the class of modulating functions $\mathcal{F}$, the article proceeds to establish theorems on eigenfunction transformation, normalization preservation, and a formula for the expected value of the zero-counting function over $[0, T]$. These results provide a foundation for understanding how stationary Gaussian processes transform when modulated by monotonically increasing functions.

2 Main Results

Definition 1. *Let $\mathcal{F}$ denote the class of functions $\theta: \mathbb{R} \to \mathbb{R}$ which are:*

1. *piecewise continuous with piecewise continuous first derivative,*
2. *strictly monotonically increasing*

$$\theta(t) < \theta(s) \forall -\infty \leqslant t < s \leqslant \infty \tag{1}$$

Remark 2. The conditions in Definition 1 are somewhat redundant since a strictly monotonically increasing function must necessarily have a positive derivative.

Theorem 3. (Eigenfunctions) *For any stationary kernel $K(t,s)=K(|t-s|)$, the eigenfunctions of the integral covariance operator*

$$T_{K_\theta}[f](t)=\int_0^\infty K_\theta(|t-s|)\, f(s)\,\mathrm{d}\,s \tag{2}$$

defined by the θ-modulated kernel

$$K_\theta(t,s)=K(|\theta(t)-\theta(s)|) \tag{3}$$

are given $\forall\theta\in\mathcal{F}$ by

$$\phi_n(t)=\psi_n(\theta(t))\sqrt{\dot{\theta}(t)} \tag{4}$$

which satisfies the eigenfunction equation

$$\begin{aligned} T_{K_\theta}[\phi_n](t) &= \lambda_n\int_0^\infty K_\theta(|t-s|)\,\phi_n(s)\,\mathrm{d}\,s \\ &= \lambda_n\int_0^\infty K_\theta(|t-s|)\,\psi_n(\theta(s))\sqrt{\dot{\theta}(s)}\,\mathrm{d}\,s \\ &= \lambda_n\int_0^\infty K(|\theta(t)-\theta(s)|)\,\psi_n(\theta(s))\sqrt{\dot{\theta}(s)}\,\mathrm{d}\,s \\ &= \lambda_n\,\phi_n(t) \end{aligned} \tag{5}$$

where ψ_n are the normalized eigenfunctions of the covariance operator defined by the original unmodulated kernel $K(|t-s|)$ which satisfy

$$\begin{aligned} T_K[\psi_n](t) &= \lambda_n\int_0^\infty K(|t-s|)\,\psi_n(s)\,\mathrm{d}\,s \\ &= \lambda_n\,\psi_n(t) \end{aligned} \tag{6}$$

Proof. The eigenfunction equation for the modulated kernel's covariance operator is:

$$\int_{-\infty}^\infty K(|\theta(t)-\theta(s)|)\,\phi_n(s)\,d\,s=\lambda_n\,\phi_n(t) \tag{7}$$

The variables can be changed by substituting $u=\theta(s)$, $v=\theta(t)$:

$$\int_{-\infty}^\infty K(|v-u|)\frac{\phi_n(\theta^{-1}(u))}{\dot{\theta}(\theta^{-1}(u))}\,d\,u=\lambda_n\,\phi_n(\theta^{-1}(v)) \tag{8}$$

which is valid due to the strict monotonicity of θ which assures its invertability. Let

$$\psi_n(u) = \frac{\phi_n(\theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \tag{9}$$

Then:

$$\int_{-\infty}^{\infty} K(|v-u|)\, \psi_n(u)\, d\, u = \lambda_n\, \psi_n(v) \tag{10}$$

This is precisely the eigenfunction equation for the original kernel K's covariance operator. Therefore,

$$\phi_n(t) = \psi_n(\theta(t)) \sqrt{\dot{\theta}(t)} \tag{11}$$

are the eigenfunctions of the modulated kernel's covariance operator

$$T_{K_\theta}[\phi_n](t) \quad = \lambda_n \int_0^{\infty} K_\theta(|t-s|)\, \phi_n(s)\, \mathrm{d}\, s \tag{12}$$

and ψ_n are the eigenfunctions of the original kernel's covariance operator which satisfy

$$T_K[\psi_n](t) \quad = \lambda_n \int_0^{\infty} K(|t-s|)\, \psi_n(s)\, \mathrm{d}\, s \tag{13}$$ □

Corollary 4. (Eigenvalue Invariance) *The eigenvalues $\{\lambda_n\}$ of the modulated kernel K_θ's covariance operator are identical to those of the original kernel K's covariance operator.*

Proof. For normalized ψ_n:

$$\int_{-\infty}^{\infty} |\phi_n(t)|^2 d\, t = \int_{-\infty}^{\infty} |\, \psi_n(\theta(t))|^2 \dot{\theta}(t)\, d\, t \tag{14}$$

Under the change of variables $u = \theta(t)$:

$$\int_{-\infty}^{\infty} |\psi_n(u)|^2\, d\, u = 1 \tag{15}$$

Therefore the ϕ_n are already normalized without additional constants. □

Theorem 5. (Operator Conjugation) *The transformation operator*

$$M_\theta[\phi](t) = \sqrt{\dot{\theta}(t)}\;\phi(\theta(t)) \tag{16}$$

conjugates the integral covariance operator

$$T_K[\phi](t) = \int_0^\infty K(|t-s|)\;\phi(s)\,\mathrm{d}\,s \tag{17}$$

where the resulting conjugated operator is

$$\begin{aligned} T_{K_\theta}[\phi](t) &= M_\theta[T_K[M_\theta^{-1}[\phi]]](t) \\ &= M\left[\int_0^\infty K(|t-s|)\frac{\phi(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}}\,\mathrm{d}\,s\right](t) \\ &= \sqrt{\dot{\theta}(t)}\;\int_0^\infty K(|\theta(t)-\theta(s)|)\frac{\phi(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}}\,\mathrm{d}\,s \\ &= \int_0^\infty K(|\theta(t)-\theta(s)|)\;\phi(s)\,\mathrm{d}\,s \\ &= \int_0^\infty K_\theta(|t-s|)\;\phi(s)\,\mathrm{d}\,s \end{aligned} \tag{18}$$

providing an explicit isometry between the original and modulated kernel Hilbert spaces.

Proof. Observe that M has inverse operator

$$M^{-1}[\phi](t) = \frac{\phi(\theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \tag{19}$$

which follows from the invertibility of θ due to strict monotonicity and note that the last equality in Equation (18) follows from the change of variables $s \mapsto \theta(s)$ with Jacobian $\dot{\theta}(s)$, demonstrating that the conjugated operator is precisely the integral operator with modulated kernel $K(|\theta(t)-\theta(s)|)$. □

Theorem 6. (Expected Zero-Counting Function) *Let $\theta \in \mathcal{F}$ and let $K(\cdot)$ be any positive-definite, stationary covariance function, twice differentiable at 0. Consider the centered Gaussian process with covariance*

$$K_\theta(t,s) = K(|\theta(t) - \theta(s)|) \tag{20}$$

Then the expected number of zeros in $[0,T]$ is

$$\mathbb{E}[N([0,T])] = \sqrt{-\ddot{K}(0)}\ (\theta(T) - \theta(0)) \tag{21}$$

Proof. By the Kac-Rice formula[1, 10.3.1]:

$$\mathbb{E}[N([0,T])] = \int_0^T \sqrt{-\lim_{s \to t} \frac{\partial^2}{\partial t\, \partial s}\ K_\theta(s,t)}\ dt \tag{22}$$

Computing the mixed partial derivative and taking the limit as $s \to t$:

$$\lim_{s \to t} \frac{\partial^2}{\partial t\, \partial s}\ K_\theta(s,t) = -\ddot{K}(0)\, \dot{\theta}(t)^2 \tag{23}$$

Therefore

$$\mathbb{E}[N([0,T])] = \sqrt{-\ddot{K}(0)} \int_0^T \dot{\theta}(t)\ dt = \sqrt{-\ddot{K}(0)}\ (\theta(T) - \theta(0)) \tag{24}$$

so that

$$\begin{aligned} \sqrt{-\ddot{K}(0)}\ (\theta(T) - \theta(0)) &= \sqrt{-\ddot{K}(0)} \int_0^T \dot{\theta}(t)\ dt \\ &= \int_0^T \sqrt{-\ddot{K}(0)\dot{\theta}(t)^2}\ dt \\ &= \int_0^T \sqrt{-\lim_{s \to t} \frac{\partial^2}{\partial t\, \partial s}\ K(|\theta(t) - \theta(s)|)}\ dt \end{aligned} \tag{25}$$

which is precisely the Kac-Rice formula for the expected zero-counting function. □

3 Conclusion

The analysis presented in this article establishes several fundamental properties of Gaussian processes generated by monotonically modulated stationary kernels. Key results include: (1) a theorem demonstrating that the eigenfunctions of the covariance operator defined by the modulated kernel are compositions of the stationary kernel's covariance operator eigenfunctions with the modulating function, times the square root of the modulating function's derivative, (2) proof of normalization and eigenvalue preservation under this transformation, establishing an isometry between the original and the modulated reproducing kernel Hilbert spaces, and (3) a concise formula for the expected value of the zero-counting function of the monotonically transformed process, expressed in terms of the original kernel's second derivative at zero times the modulating function's values at the boundaries of the interval to which the expectation corresponds.

Bibliography

[1] Harald Cramér and M.R. Leadbetter. *Stationary and Related Processes: Sample Function Properties and Their Applications*. Wiley Series in Probability and Mathematical Statistics. 1967.

[2] A.M. Yaglom. *Correlation Theory of Stationary and Related Random Functions*, volume I: Basic Results of *Applied Probability*. Springer New York, 1987.