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USE OF THE BUBNOV-GALERKIN METHOD TO SKIN EFFECT PROBLEMS FOR THE CASE OF UNKNOWN BOUNDARY CONDITIONS

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Abstract - A method based on coupling the Bubnov-Galerkin method and the separation of variables method is described for the solution of alternating field problems in which the region of prime interest is embedded in an infinitely extending region where Laplace's equation holds. The proposed method can also be used when the stored energy associated with the exterior region and the boundary conditions at infinity are infinite. As an example illustrating the use of the method, the inner impedance of an elliptic conductor is calculated. By means of iterative numerical processes, the method enables the exact values of impedance to be obtained.

INTRODUCTION

In the case of conductors surrounded by a ferromagnetic substance (e.g. for conductors placed in slots of electric machines) we can determine boundary conditions and next calculate electromagnetic fields in these conductors by the variational or direct methods of functional analysis. In [1] there is a large list of articles concerning this subject matter.

In the case of fields in conductors embedded in an empty space the boundary conditions on their surfaces are mostly unknown, which complicates the use of variational and direct methods. Then such field problems with boundaries at infinity can be treated in finite element terms by constructing an element to model an extremely large annulus surrounding the region of interest [2]. Moreover, the finite element method for the bounded region of prime interest has been coupled with analytical solutions for the external infinitely extending region where Laplace's equation holds [3-6]. Minimized functionals appearing in these papers, however, have been written for the whole space, which causes the stored energy associated with the exterior region and the boundary conditions at infinity to be finite.

In order to avoid these restrictions we should write functionals and Bubnov-Galerkin equations systems for Poisson's and complex Helmholtz's equation, respectively, at an unknown Neumann's boundary condition only for the bounded region of interest and connect them skilfully with the separation of variables method for the external unbounded region where Laplace's equation holds. The aim of this paper is to give such a method. As an example illustrating the application of this method, the impedance of an elliptic conductor is determined.

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USE OF BUBNOV-GALERKIN METHOD FOR UNKNOWN BOUNDARY CONDITIONS

The paper [7] which describes the use of the Bubnov-Galerkin method for the calculation of conductors impedance in the case of known boundary conditions will be the starting point for the determination of a 2-dimensional quasistationary field at any infinitely long and straight non-magnetic conductor carrying an alternating electric current embedded in an empty space. The vector potential of this field has only the component A along the conductor which fulfils Laplace's equation outside the conductor and Helmholtz's equation in its interior

$$\nabla^2 A = jk^2 A, \quad k^2 = \omega\mu\gamma \tag{1}$$

where $j = \sqrt{-1}$ and ω, μ, γ denote the pulsation, permeability and conductivity, respectively.

We denote a cross-section of the conductor by D and the boundary of D by S. Next we surround the area D by a contour S_1 being a line x_1 = const or x_2 = const in the orthogonal system of coordinates (x_1, x_2), which allows the separation of variables for Laplace's equation. At last we introduce the region D_1 contained in the interior of S_1 and D_2 which is the whole exterior of S_1. The vector potential A_I in D_1 and A_{II} in D_2 will be determined by the Bubnov-Galerkin method and the separation of variables method, respectively.

An approximate value of A_I may be written in the form

$$A_I = \sum_{m=1}^{M} A_m \varphi_m , \tag{2}$$

where the sequence $\{\varphi_m\}$ consisting of linear independent elements is complete in the energetic space of the operator $-\nabla^2$. The constants A_m fulfil the following Bubnov-Galerkin equation system [7]

$$\int_{D_1} \operatorname{grad}\varphi_m^* \operatorname{grad}A_I \, dD + jk^2 \int_D \varphi_m^* A_I \, dD = \oint_{S_1} \varphi_m^* \psi(P) \, dS, \qquad \psi(P) = \left.\frac{\partial A_{II}}{\partial n}\right|_{S_1}, \quad P \in S_1 \tag{3}$$

where $\partial/\partial n$ is the derivative in the normal external direction to S_1 and z^* denotes conjugate of z.

Applying the separation of variables method to Laplace's equation in the coordinates (x_1, x_2), we obtain the unknown vector potential A_{II} appearing in (3) as follows

$$A_{II} = \sum_{n=0}^{N} C_n \psi_n , \tag{4}$$

where ψ_n are harmonic functions dependent on x_1 and x_2.

The vector potential A_{II} should satisfy the flux law

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$$-\oint_{S_1} \frac{\partial A_{II}}{\partial n} dS = \mu I \; , \qquad (5)$$

where I denotes the complex r.m.s. value of a current in the conductor.

Observe that (3) and (5) give M+1 equations. But the number of the unknown coefficients A_m and C_n is equal to M+N+1. Continuity of exterior and interior potentials make it possible to obtain the missing equations

$$A_I \big|_{S_1} = A_{II} \big|_{S_1} \; . \qquad (6)$$

In order to calculate C_n as functions of I and A_m, first we take advantage of (5), and next we multiply both sides of (6) by ψ_n and integrate over the boundary S_1. At last substituting C_n into (3) we obtain the system of equations which contain the coefficients A_m only. After solving this system the inner impedance of the conductor may be calculated.

If **E** and **H** denote the complex r.m.s. values of electric and magnetic fields, respectively, then the resistance R and the inner reactance X of the conductor have the forms

$$|I|^2 R = \gamma \int_D |\mathbf{E}|^2 dD, \quad |I|^2 X = \omega\mu \int_D |\mathbf{H}|^2 dD, \qquad (7)$$

where $|z|$ is the modulus of complex number z. Now on the basis of relationships between the vector potential, **E** and **H**, the inner impedance Z per unit length of the conductor may be written as follows

$$Z = R + jX = \frac{\omega}{\mu |I|^2} \int_D \left(k^2 |A_I|^2 + j\, |\mathrm{grad} A_I|^2\right) dD. \qquad (8)$$

In case when $S_1 = S$, multiplying (3) by A_m^* and summing up with regard to the subscript m from 1 to M and taking (8), it is possible to ascertain that

$$Z = \frac{j\omega}{\mu |I|^2} \oint_S A_I \frac{\partial A_{II}^*}{\partial n} dS. \qquad (9)$$

For the exact values of A_I and A_{II} the expression (9) directly follows from the Poynting vector and is also true when $S_1 \neq S$. However, the expressions (8) and (9) give different results for the approximate values of A_I and A_{II} provided that $S_1 \neq S$. Thus there is a problem which of the two formulas should be used. Since the approximate values of R and X given by (8) are proportional to the square of the norms of A_I in the space of quadratically integrable functions and in the energetic space of the operator $-\nabla^2$, respectively, they are convergent (for $M,N \to \infty$) to their exact values. Consequently, for $S_1 \neq S$ the expression (8) should be applied.

The above procedure can also be used for solving magnetostatic problems, i.e. when instead of the equation (1) we have Poisson's equation within the conductor

$$\nabla^2 A = -\mu J, \qquad (10)$$

where J denotes the component of current density along the conductor. The determination of the solution in the region D_1 for the Neumann's boundary condition (3) is based on seeking the function A_I in D_1 which will minimize the functional

$$F = \int_{D_1} \mathrm{grad}^2 A_I \, dD - 2\mu \int_D J A_I \, dD - 2 \oint_{S_1} A_I \Psi(P) \, dS. \quad (11)$$

In the same way as previously, the unknown coefficients C_n should be eliminated from (11). Then the functional F becomes a function of $A_1, A_2, \ldots A_M$ which is to be minimized.

The complete sequence $\{\varphi_m\}$ appearing in (2) can be determined by the Stone-Weierstrass Theorem. The sequence determined in this way permits an iterative process easily with respect to integers M and N and thereby the exact values of impedance can be calculated. If the boundary shapes of the conductor are complex, it is difficult to calculate the integrals occurring in (3). In this case it is more convenient to determine the approximating functions φ_m by the finite element method. However, we then obtain only an approximate value of impedance.

SKIN EFFECT IN AN ELLIPTIC CONDUCTOR

The cross-section of an elliptic conductor is shown in Fig.1.

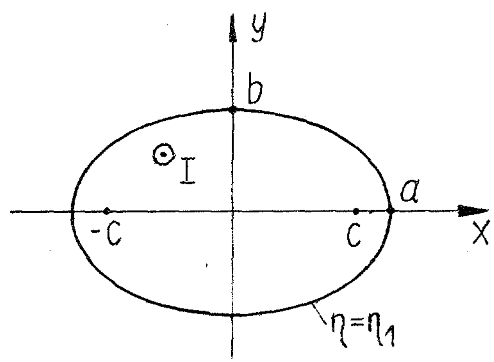

Fig.1. Cross-section of an elliptic conductor.

The considerations will be made in the elliptic system of coordinates (η,ψ), $0 \leq \eta < \infty$, $-\pi \leq \psi \leq \pi$, which are connected with the rectangular coordinates in the following way

$x = c \cosh\eta \cos\psi$, $y = c \sinh\eta \sin\psi$.

From the above expressions, the following useful relations may be derived.

$$c = a\left(1 - B^2\right)^{1/2} \; , \qquad (12)$$

$$e^{2\eta_1} = (1+B)/(1-B) \; , \qquad (13)$$

where $B = b/a$. (14)

The contours S and S_1 defined in Section 2 are assumed to be equal, i.e. they are an ellipse $\eta = \eta_1$. So A_I and A_{II} denote the vector potential within and outside the conductor, respectively.

The general solution of Laplace's equation due to the elliptic system of coordinates has the form [8]

$$A_{II} = C_0' + D_0'\eta + E_0'\psi + \sum_{n=1}^{\infty} \left(C_n' e^{-n\eta} + D_n' e^{n\eta}\right)\left(E_n' \cos n\psi + F_n' \sin n\psi\right).$$

Hence, observing that the vector potential is to fulfil the conditions of symmetry

$$A(\eta,\psi) = A(\eta,-\psi) = A(\eta,\pi-\psi) = A(\eta,-\pi+\psi), \qquad (15)$$

next noticing that A_{II} must be proportional to η at infinity and finally taking advantage of (5), we can write

$$A_{II}=\mu I\Big(-\frac{\eta}{2\pi}+C_o+\sum_{n=1}^{N}C_n e^{-2n\eta}\cos 2n\psi\Big). \qquad (16)$$

Thanks to the coefficient μI occuring in (16) the constants C_n are dimensionalless.

To determine the vector potential within the conductor, note that the functions $f_1=\exp 2(\eta-\eta_1)$ and $f_2=\cos 2\psi$ separate the points of the area $\Omega=\{(\eta,\psi):0\leqslant\eta\leqslant\eta_1,0\leqslant\psi\leqslant\pi/2\}$, i.e. for each pair of distinct points of Ω at least one of these functions has different values at these points. Consequently, by the Stone-Weierstrass Theorem, any complex continuous function fulfilling the condition (15) and definite on the area D of the conductor's cross-section can be uniformly approximated by linear combinations of terms of the double sequence $\{f_1^m f_2^n\}$. Since $\cos^n 2\psi$ is a linear combination of the functions $\cos 2i\psi$, $i=0,1,\ldots,n$, the above remark also applies to the sequence $\{f_1^m\cos 2n\psi\}$. Thus the vector potential within the conductor may be written in the following form

$$A_I=\mu I\sum_{m=0}^{M}\sum_{n=0}^{N}A_{m,n}e^{2m(\eta-\eta_1)}\cos 2n\psi\,. \qquad (17)$$

Of course, η_1 can be left out in (17). But if B is nearly equal to 1, then large numbers $\exp(2m\eta_1)$ occur in numerical calculations, which cause complications.

To determine the coefficients $A_{m,n}$, substitute (17) into (3), which becomes

$$\int_D\big(\mathrm{grad}\,\varphi^*_{i,l}\,\mathrm{grad}A_I+jk^2\varphi^*_{i,l}A_I\big)\,dD=\oint_S\varphi^*_{i,l}\frac{\partial A_{II}}{\partial n}\,dS, \qquad (18)$$

where $\varphi_{i,l}=\mu I\exp\big[2i(\eta-\eta_1)\big]\cos(2l\psi)$ and

$$\mathrm{grad}V=\frac{1}{e_1}\frac{\partial V}{\partial\eta}\,\mathbf{1}_\eta+\frac{1}{e_1}\frac{\partial V}{\partial\psi}\,\mathbf{1}_\psi\,, \qquad (19)$$

$$dD=e_1^2\,d\eta\,d\psi\,,\quad dS=e_1\,d\psi\,, \qquad (20)$$

$$e_1=c\big[(\cosh 2\eta-\cos 2\psi)/2\big]^{1/2}. \qquad (21)$$

Making use of (16),(17),(19),(20) and (21) we obtain the integrals appearing in (18)

$$\int_D\mathrm{grad}\,\varphi^*_{i,l}\,\mathrm{grad}A_I\,dD=4\pi(1+\delta_l)\mu^2|I|^2\sum_{m=0}^{M}A_{m,l}(im+l^2)h_{i+m}\,, \qquad (22)$$

$$\int_D\varphi^*_{i,l}A_I\,dD=\frac{1}{4}\pi c^2\mu^2|I|^2\Big[(1+\delta_l)\sum_{m=0}^{M}A_{m,l}\cdot(h_{i+m+1}/g+gh_{i+m-1})-\sum_{m=0}^{M}\sum_{n=0}^{N}A_{m,n}h_{i+m}\cdot(\delta_{l+n-1}+\delta_{l-n+1}+\delta_{l-n-1})\Big], \qquad (23)$$

$$\oint_S\varphi^*_{i,l}\frac{\partial A_{II}}{\partial n}\,dS=-\mu^2|I|^2(\delta_l+2\pi lC_l e^{-2l\eta_1})\,, \qquad (24)$$

where

$$\delta_n=\begin{cases}1 & \text{for } n=0\\ 0 & \text{for } n\neq 0\,,\end{cases} \qquad (25)$$

$$g=e^{-2n\eta_1}=(1-B)/(1+B), \qquad (26)$$

$$h_n=\int_0^{\eta_1}e^{2n(\eta-\eta_1)}\,d\eta=\begin{cases}\ln(1/g)/2 & \text{for } n=0\\ (1-g^n)/(2n) & \text{for } n\neq 0.\end{cases} \qquad (27)$$

The formula

$$\cos 2l\psi\cos 2n\psi\cos 2\psi=\frac{1}{4}\big[\cos 2(l+n+1)\psi+\cos 2(l+n-1)\psi+\cos 2(l-n+1)\psi+\cos 2(l-n-1)\psi\big]$$

has been used in the calculations of (23).

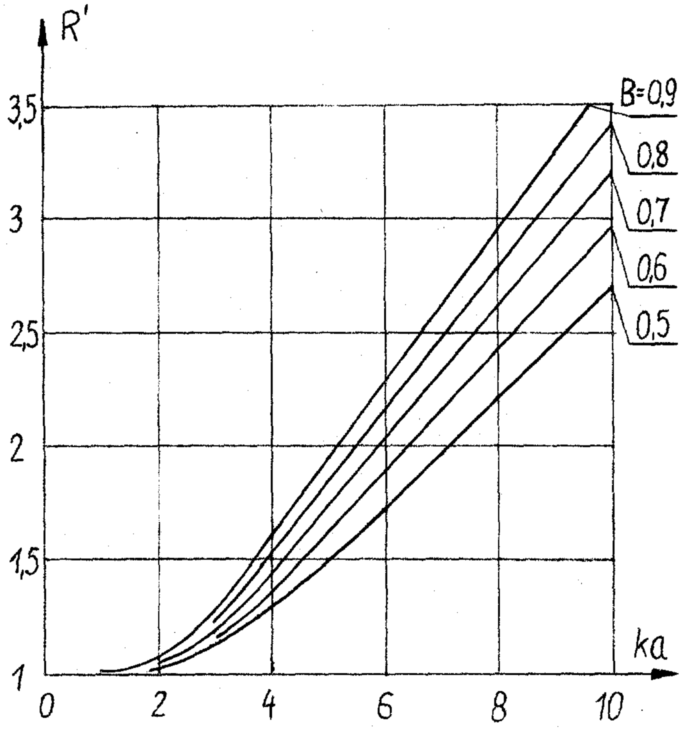

Fig.2. Graph of the relative resistance R'.

Continuity of exterior and interior potentials gives

$$C_n e^{-2n\eta_1}=\sum_{m=0}^{M}A_{m,n}\quad\text{for } n\neq 0\,. \qquad (28)$$

To obtain the final equation system describing $A_{m,n}$, substitute (12),(22),(26) and (28) into (18), which becomes

$$\sum_{m=0}^{M}\sum_{n=0}^{N}F_{i,l,m,n}A_{m,n}=-\frac{\delta_l}{4\pi}\,, \qquad (29)$$

where

$$F_{i,l,m,n}=\delta_{l-n}\big[(1+\delta_l)(im+l^2)h_{i+m}+l/2\big]+\frac{1}{16}j\big[ka(1+B)\big]^2\big[\delta_{l-n}(1+\delta_l)(h_{i+m+1}+g^2h_{i+m-1})-$$

$$-gh_{i+m}(\delta_{l+n-1}+\delta_{l-n+1}+\delta_{l-n-1})] \,. \qquad (30)$$

By virtue of (9) the relative inner impedance Z' can be calculated from the expression

$$Z'=R'+jX'=\frac{Z}{R_o}=j\frac{\pi k^2 ab}{\mu^2 |I|^2}\int_{-\pi}^{\pi} A_I \frac{\partial A^*_{II}}{\partial \eta}\Bigg|_{\eta=\eta_1} d\psi, \qquad (31)$$

where $R_o = 1/(\gamma \pi ab)$. (32)

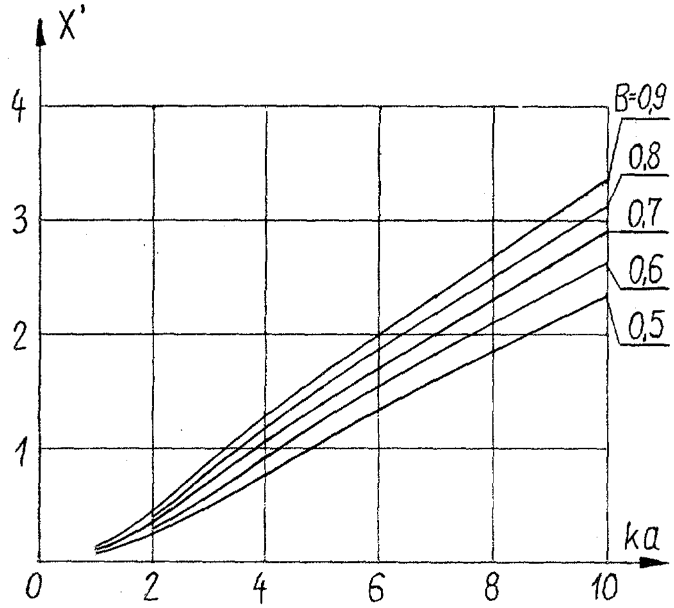

Fig.3. Graph of the relative reactance X'.

Hence taking advantage of (16),(17) and (28), we convince ourselves that Z' has the form

$$Z'=-j\pi k^2 a^2 B\left(\sum_{m=0}^{M} A_{m,0}+2\pi \sum_{n=1}^{N} n \left|\sum_{m=0}^{M} A_{m,n}\right|^2\right). \qquad (33)$$

The relative parameters R' and X' depend on integers M and N. So we can write

$R'=R'(M,N)$, $X'=X'(M,N)$.

An iterative process has been used with respect to integers M and N. Beginning with M=N=4, the calculations were continued as long as at least one of the following inequalities was satisfied

$$|R'(M,N) - R'(M-1,N-1)|\cdot 10^3 \geqslant R'(M,N) \,,$$

$$|X'(M,N) - X'(M-1,N-1)|\cdot 10^3 \geqslant X'(M,N) \,.$$

On the basis of these, the Figs.2 and 3 are drawn. It turns out that the required exactitude has already been reached at the second iteration, i.e. for M=N=5, for all values of ka and B.

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