have

$$\int \phi(a) U(a,I)[g]\, da = \int \phi(a)[g \circ (a,I)^{-1}]\, da$$
$$= \phi \circ g = 0$$

since $\phi \circ g \in I_s \subseteq \mathcal{I}$. Since $g \in \mathcal{S}_1$ was arbitrary, it follows that $\int \phi(a) U(a,I)\, da = 0$. Therefore, $\operatorname{supp} \mathcal{F}^{-1} U(a,I) \subseteq V_+$, which is equivalent to the spectrum condition. For (5), if $\phi, \psi \in \mathcal{S}_1$ are spacelike separated, then

$$\|\Theta(\phi)\Theta(\psi)[g] - \Theta(\psi)\Theta(\phi)[g]\|^2$$
$$= w(g \otimes (\psi \otimes \phi - \phi \otimes \psi) \otimes (\psi \otimes \phi - \phi \otimes \psi) \otimes g) = 0$$

since $\psi \otimes \phi - \phi \otimes \psi \in I_c$. Condition (6) follows from (7.25). □

As a corollary we obtain the converse of Theorem 7.6.

Corollary 7.8. *If W_n, $n = 0, 1, \ldots$ is a sequence of tempered distributions on $\mathcal{S}(\mathbb{R}^{4n})$, $n = 0, 1, \ldots$, satisfying* W1–W6, *then there exists a unique (to within equivalence) Wightman quantum field (Θ, D, U, h) such that*

$$W_n(\phi_1 \otimes \cdots \otimes \phi_n) = \langle \Theta(\phi_1) \cdots \Theta(\phi_n) h, h \rangle.$$

PROOF. Define $w_h : \mathcal{S} \to C$ by (7.24). As in the proof of Theorem 7.7(a), w_h is an invariant state which annihilates the ideals I_s, I_c and satisfies (7.25). The result follows from Theorem 7.7(b). □

7.4. Fock Space

In this section we illustrate the concepts of the previous two sections for the important example of Fock space. Let $\mathcal{H}$ be a complex Hilbert space whose unit vectors represent the pure states for some quantum system. For illustrative purposes, suppose the quantum system consists of a single particle p. We would now like to describe the quantum system consisting of n particles all identical to p; for example, a system composed of n electrons. This system is represented by the tensor product $\mathcal{H}^n = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, where $\mathcal{H}_i = \mathcal{H}$, $i = 1, \ldots, n$. If $\phi_1, \phi_2 \in \mathcal{H}$ are unit vectors representing pure states of a particle, then one might think that $\phi_1 \otimes \phi_2 \in \mathcal{H}^2$ represents a system of two identical particles in which one particle is in state ϕ_1, and one is in state ϕ_2. But since the particles are indistinguishable, the state must not be altered upon an interchange of the two particles. Hence, we would obtain $\phi_2 \otimes \phi_1 = c\phi_1 \otimes \phi_2$ where $|c| = 1$. Since this is impossible if ϕ_1 and ϕ_2 are linearly independent, $\phi_1 \otimes \phi_2$ is not the correct form for a state consisting of two identical particles. What is needed is a unit vector $f(\phi_1, \phi_2) \in \mathcal{H}^2$ satisfying $f(\phi_2, \phi_1) = cf(\phi_1, \phi_2)$, where $|c| = 1$. Examples of such states are

$$a(\phi_1 \otimes \phi_2 + \phi_2 \otimes \phi_1) \tag{7.26}$$

and

$$b(\phi_1 \otimes \phi_2 - \phi_2 \otimes \phi_1), \tag{7.27}$$

where a and b are normalization constants. It has been found that the pure states of all known particles satisfy either (7.26) or (7.27). This is a law of physics called the *Pauli symmetrization principle*. Particles whose states satisfy (7.26) are called *bosons* and particles whose states satisfy (7.27) are called *fermions*. For concreteness we shall only consider bosons here. An analogous theory holds for fermions.

Let $\mathcal{H}$ be the Hilbert space for a single boson and $\mathcal{H}^n$ the Hilbert space for n identical bosons. Let A be the group of permutations of the set $\{1, \ldots, n\}$ and for $a \in A$ let $U(a)$: $\mathcal{H}^n \to \mathcal{H}^n$ be the linear operator defined by

$$U(a)\phi_1 \otimes \cdots \otimes \phi_n = \phi_{a(1)} \otimes \cdots \otimes \phi_{a(n)}. \tag{7.28}$$

It is not hard to show that $a \mapsto U(a)$ is a unitary representation of A on $\mathcal{H}^n$. Now define the linear operator

$$S_+^n = \frac{1}{n!} \sum_{a \in A} U(a) \tag{7.29}$$

on $\mathcal{H}^n$. It is easy to show that S_+^n is an orthogonal projection on $\mathcal{H}^n$ and that $S_+^n U(a) = U(a) S_+^n$ for all $a \in A$. The Pauli symmetrization principle says that a unit vector $\phi \in \mathcal{H}^n$ represents the state of n identical bosons if and only if $S_+^n \phi = \phi$. We write $\mathcal{H}_+^n = S_+^n \mathcal{H}^n$ and call $\mathcal{H}_+^n$ the *boson* (or *symmetrized*) subspace of $\mathcal{H}^n$. If $\phi_1, \ldots, \phi_n \in \mathcal{H}$ are unit vectors, then the unit vector

$$(\| S_+^n \phi_1 \otimes \cdots \otimes \phi_n \|)^{-1} S_+^n \phi_1 \otimes \cdots \otimes \phi_n$$

represents the state of a system of n identical bosons in which the ith boson is in state ϕ_i, $i = 1, \ldots, n$.

Now suppose we have a system of identical bosons, but the total number of bosons is unknown. Such situations are common is quantum field theory since interactions may occur in which particles are created or annihilated so the total number may change. If we define $\mathcal{H}_+^0 = C$, then this system is described by the direct sum Hilbert space

$$\mathcal{F}_+(\mathcal{H}) = \sum_{n=0}^{\infty} \mathcal{H}_+^n.$$

We call $\mathcal{F}_+(\mathcal{H})$ the *boson Fock space* (there is an analogous fermion Fock space which we shall not treat). The boson Fock space is a closed subspace of the *Fock space* $\mathcal{F}(\mathcal{H})=\Sigma\mathcal{H}^n$ and if we define the orthogonal projection $S_+=\Sigma S_+^n$, then $\mathcal{F}_+(\mathcal{H})=S_+\mathcal{F}(\mathcal{H})$. We identify $\mathcal{H}_+^n$ with the corresponding subspace of $\mathcal{F}_+(\mathcal{H})$ and write an element $\phi\in\mathcal{F}_+(\mathcal{H})$ as

$$\phi=\sum_{n=0}^{\infty}\phi^{(n)}, \tag{7.30}$$

where $\phi^{(n)}\in\mathcal{H}_+^n$ and $\Sigma\|\phi^{(n)}\|^2<\infty$. If ϕ in (7.30) is a unit vector corresponding to a pure state, then $\|\phi^{(n)}\|^2$ gives the probability that there are n bosons in the state corresponding to ϕ. The vector $\phi_0\in C$ given by $\phi_0=1$ is the *vacuum* vector representing the state with no bosons present.

If $\{u_k\}$ is an orthonormal basis for $\mathcal{H}$, then

$$\left\{u_{k_1}\otimes\cdots\otimes u_{k_n}:\{k_1,\ldots,k_n\}\subseteq\{1,2,\ldots\}\right\}$$

is an orthonormal basis for $\mathcal{H}^n$. Let n_i be the number of indices among the $k_1,\ldots,k_n$ in the vector $u_{k_1}\otimes\cdots\otimes u_{k_n}$ that equals i. Of course, $n_i=0$ except for finitely many i. By direct computation we have

$$\|S_+^n u_{k_1}\otimes\cdots\otimes u_{k_n}\|=\frac{n_1!n_2!\cdots n_i!\cdots}{n!}. \tag{7.31}$$

The unit vector

$$u(n_1,n_2,\ldots)=\left(\frac{n!}{n_1!n_2!\cdots}\right)^{1/2}S_+^n u_{k_1}\otimes\cdots\otimes u_{k_n} \tag{7.32}$$

represents the state in which there are n bosons, n_i of which are in the state u_i. Define the linear operator N on $\mathcal{F}_+(\mathcal{H})$ as follows. The domain $D(N)$ is

$$D(N)=\left\{\phi=\sum\phi^{(n)}\in\mathcal{F}_+(\mathcal{H}):\sum\|n\phi^{(n)}\|^2<\infty\right\}$$

and for $\phi\in D(N)$, $N\phi=\Sigma n\phi^{(n)}$. The operator N is self-adjoint and is called the *number of particles operator*.

For $\phi\in\mathcal{H}$, define the bounded linear operator $C(\phi)$: $\mathcal{H}^n\to\mathcal{H}^{n+1}$ by

$$C(\phi)\phi_1\otimes\cdots\otimes\phi_n=\phi\otimes\phi_1\otimes\cdots\otimes\phi_n.$$

Thus $\|C(\phi)\|=\|\phi\|$ and $C(\phi)$ corresponds to the creation of a particle in the state ϕ. The adjoint operator $C(\phi)^*:\mathcal{H}^{n+1}\to\mathcal{H}^n$ satisfies $C(\phi)^*=0$ if $\phi\in\mathcal{H}^0$ and

$$C(\phi)^*\phi_1\otimes\cdots\otimes\phi_n=\langle\phi,\phi_1\rangle\phi_2\otimes\cdots\otimes\phi_n$$

for $n\geq 1$. The operator $C(\phi)^*$ corresponds to the annihilation of a particle in the state ϕ. We define $C(\phi)$ and $C(\phi)^*$ on $\mathcal{F}(\mathcal{H})$ by

$$C(\phi)\left(\sum \phi^{(n)}\right)=\sum C(\phi)\phi^{(n)},$$

$$C(\phi)^*\left(\sum \phi^{(n)}\right)=\sum C(\phi)^*\phi^{(n)}.$$

Let D_0 be the dense subspace of $\mathcal{F}_+(\mathcal{H})$ defined by

$$D_0=\left\{\sum \phi^{(n)}\in\mathcal{F}_+(\mathcal{H}):\phi^{(n)}=0 \qquad \text{except for finitely many} \quad n\right\}.$$

For $\phi\in\mathcal{H}$ define the following operators on D_0:

$$\mathcal{A}(\phi)=S_+C(\phi)^*\sqrt{N}\;; \tag{7.33}$$

$$\mathcal{A}^*(\phi)=\sqrt{N}\;S_+C(\phi). \tag{7.34}$$

It can be shown that $\mathcal{A}(\phi)$ and $\mathcal{A}^*(\phi)$ are closed operators on D_0 and that $\mathcal{A}^*(\phi)$ is the adjoint of $\mathcal{A}(\phi)$. We call $\mathcal{A}(\phi)$ and $\mathcal{A}^*(\phi)$ *annihilation* and *creation* operators, respectively. It is not hard to show that $\mathcal{A}$ and $\mathcal{A}^*$ satisfy the following commutation relations on D_0:

$$[\mathcal{A}(\phi),\mathcal{A}(\psi)]=[\mathcal{A}^*(\phi),\mathcal{A}^*(\psi)]=0,$$

$$[\mathcal{A}(\phi),\mathcal{A}^*(\psi)]=\langle\psi,\phi\rangle I, \tag{7.35}$$

where $[A,B]=AB-BA$. Moreover, it is easy to show that

$$u(n_1,n_2,\ldots)=(n_1!n_2!\cdots)^{-1/2}[\mathcal{A}^*(u_{k_1})\mathcal{A}^*(u_{k_2})\cdots]\phi_0, \tag{7.36}$$

where $u(n_1,n_2,\ldots)$ is defined by (7.32). Since vectors of this form are dense in $\mathcal{F}_+(\mathcal{H})$ this shows that the entire boson Fock space is generated by creating bosons from the vacuum ϕ_0.

We now define the operators $p(\phi)$, $q(\phi)$ on D_0 by

$$p(\phi)=\frac{1}{\sqrt{2}\,i}[\mathcal{A}(\phi)-\mathcal{A}^*(\phi)],$$

$$q(\phi)=\frac{1}{\sqrt{2}}[\mathcal{A}(\phi)+\mathcal{A}^*(\phi)].$$

Then $p(\phi)$ and $q(\phi)$ are essentially self-adjoint and satisfy the commutation relations

$$[q(\phi),q(\psi)]=i\,\mathrm{Im}\langle\psi,\phi\rangle,$$

$$[p(\phi),p(\psi)]=i\,\mathrm{Im}\langle\psi,\phi\rangle,$$

$$[q(\phi),p(\psi)]=-i\,\mathrm{Im}\langle\psi,\phi\rangle. \tag{7.37}$$

If we define $W(\phi)=e^{iq(\phi)}$, then W satisfies (a), (b), (c) of Lemma 7.4 so W is a complex representation of the CCR. Moreover, it is not hard to show that W is cyclic. In a similar way, $e^{ip(\phi)}$ is a cyclic complex representation of the CCR. An explicit computation gives

$$\langle W(\phi)\phi_0, \phi_0\rangle = e^{-(1/4)\|\phi\|^2}. \tag{7.38}$$

Now let us suppose that $\mathcal{H}$ is a real Hilbert space. Then the $q(\phi), \phi \in \mathcal{H}$, are essentially self-adjoint and commute since $\mathrm{Im}\langle\psi,\phi\rangle = 0$. In fact, it can be shown that the spectral measures of the $q(\phi), \phi \in \mathcal{H}$, commute so the spectral measures generate a commutative C^*-algebra $\mathcal{A}$. It follows from the Gelfand–Naimark theorem [194] that there is an isometric isomorphism h from $\mathcal{A}$ onto the set of continuous functions $C(\Omega)$ on a compact Hausdorff space Ω. The state $A \mapsto \langle A\phi_0, \phi_0\rangle$ on $\mathcal{A}$ is mapped under h to a positive linear functional l on $C(\Omega)$ such that $l(1) = 1$. Applying the Riesz representation theorem, there exists a probability measure μ on the Borel sets $\mathcal{B}(\Omega)$ such that $\langle A\phi_0, \phi_0\rangle = \int h(A)\,d\mu$ for every $A \in \mathcal{A}$. Under the map h, $q(\phi)$ becomes a random variable $\Phi_{\mathcal{F}}(\phi)$ on the probability space $(\Omega, \mathcal{B}(\Omega), \mu)$ for every $\phi \in \mathcal{H}$. Since $q(\cdot)$ is linear, so is $\Phi_{\mathcal{F}}(\cdot)$ and it follows that $\Phi_{\mathcal{F}}$ is a random field. Moreover, the characteristic functional of $\Phi_{\mathcal{F}}$ satisfies

$$E\left[e^{i\Phi_{\mathcal{F}}(\phi)}\right] = \int e^{i\Phi_{\mathcal{F}}(\phi)}\,d\mu = \langle e^{iq(\phi)}\phi_0, \phi_0\rangle = e^{-(1/4)\|\phi\|^2}.$$

Thus $\Phi_{\mathcal{F}}$ is the unit Gaussian random field on $\mathcal{H}$ (except for the unimportant normalization constant $\frac{1}{4}$). We thus see that the boson Fock space over $\mathcal{H}$ is equivalent to the unit Gaussian random field on $\mathcal{H}$. More precisely, applying Theorem 6.13 we have the following result.

Theorem 7.9. *If $\mathcal{H}$ is a real Hilbert space then there is an isometry U from $\mathcal{F}_+(\mathcal{H})$ onto $\Gamma(\mathcal{H})$ such that* (a) $U\phi_0 = 1$, (b) $Uq(\phi)U^{-1} = \Phi_u(\phi)$, (c) $U\mathcal{H}_+^n = \Gamma_n$.

7.5. Euclidean Random Fields

In this section we give E. Nelson's method for constructing quantum fields from a certain class of random fields [199, 200]. Let $H^{-1}(\mathbb{R}^n), n \geq 2$, be the Hilbert space consisting of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ whose Fourier transforms $\hat{f}$ are functions and for which the norm

$$\|f\|_{-1}^2 = \int |\hat{f}(k)|^2 (k^2+1)^{-1}\,d^n k < \infty. \tag{7.39}$$

The Hilbert space $H^{-1}(\mathbb{R}^n)$ is an example of a Sobolev space. To see that $H^{-1}(\mathbb{R}^n)$ is a Hilbert space and to get an idea of where this space originates, we can write (7.39) in terms of the usual $L^2(\mathbb{R}^n)$ inner product

$$\langle f,g\rangle_{-1}=\langle \hat{f}(k), \frac{1}{k^2+1}\hat{g}(k)\rangle. \tag{7.40}$$

Since the Fourier transform of the Laplacian operator Δ goes over to multiplication by $-k^2$, taking the usual function inverse Fourier transform of (7.40) gives

$$\langle f,g\rangle_{-1}=\langle f,(-\Delta+1)^{-1}g\rangle.$$

Thus $H^{-1}(\mathbb{R}^n)$ is isomorphic to the Hilbert space of functions on $\mathbb{R}^n$ satisfying $\langle f,(-\Delta+1)^{-1}f\rangle<\infty$.

Equip $\mathbb{R}^n, n\geq 2$, with its usual inner product $x\cdot y=\sum_{i=1}^n x_i y_i$. The *Euclidean group* $\mathfrak{E}_n$ on $\mathbb{R}^n$ is the group of all nonsingular inhomogeneous linear transformations that preserve the inner product. By a *representation* of $\mathfrak{E}_n$ on a probability space (Ω,Σ,μ) we mean a group homomorphism $\beta\mapsto T_\beta$ of $\mathfrak{E}_n$ into the group of measure-preserving transformations on (Ω,Σ,μ) such that for all $u,v\in L^\infty(\Omega,\Sigma,\mu)$, $\beta\mapsto\int u(v\circ T_\beta)d\mu$ is measurable. A *covariant random field* on $H^{-1}(\mathbb{R}^n)$ is a random field $\Phi: H^{-1}(\mathbb{R}^n)\to R(\Omega,\Sigma,\mu)$ together with a representation T of $\mathfrak{E}_n$ on (Ω,Σ,μ) such that $\Phi(\phi)\circ T_\beta=\Phi(\phi\circ\beta)$ for every $\phi\in H^{-1}(\mathbb{R}^n)$ and $\beta\in\mathfrak{E}_n$. A covariant random field is *ergodic* if the only elements of $\Sigma/\mathcal{G}_\mu$ left invariant by T_β for every translation β are Ω and $\varnothing$.

Let $\Phi: H^{-1}(\mathbb{R}^n)\to R(\Omega,\Sigma,\mu)$ be a random field. For any open or closed set $A\subseteq\mathbb{R}^n$, let Σ_A be the σ-algebra generated by

$$\{\Phi(\phi): \operatorname{supp}\phi\subseteq A\}$$

and let E_A denote the conditional expectation $E(\cdot|\Sigma_A)$. We say that Φ is a *Markov* random field on $H^{-1}(\mathbb{R}^n)$ if for every closed set $A\subseteq\mathbb{R}^n$ and any Σ_{A^c}-measurable function f we have $E_A(f)=E_{\partial A}(f)$, where A^c denotes the complement of A and ∂A the boundary of A. Intuitively, the Markov condition says that if we wish to predict some aspect (namely, f) of the field's behavior inside A, then a knowledge Σ_{A^c} of the field outside A gives no more information than the knowledge $\Sigma_{\partial A}$ of the field on the boundary of A. This is a multi-dimensional generalization of the usual Markov condition for a stochastic process. For a stochastic process, $\mathbb{R}^n$ is replaced by $\mathbb{R}$ and is thought of as time. In this case, if $A=(-\infty,t]$, then the Markov condition states that the conditional expectation of future events given the past depends only on the present. More prosaically, the future is independent of the past given the present.

A random field Φ is *hermitian* if $\Phi(\bar{\phi})=\bar{\Phi}(\phi)$ for every ϕ. A *Euclidean random field* on $H^{-1}(\mathbb{R}^n)$ is a full, hermitian, covariant, ergodic, Markov random field on $H^{-1}(\mathbb{R}^n)$. Before we study the properties of Euclidean random fields, it is convenient to introduce some notation. Let (Φ, T) be a covariant random field. We denote by U_β the unitary representation of $\mathcal{E}_n$ on $L^2(\Omega, \Sigma, \mu)$ given by $(U_\beta f)(\omega) = f(T_\beta^{-1}\omega)$. By a standard theorem in the theory of representations of locally compact groups [194], $\beta \mapsto U_\beta$ is strongly continuous. We frequently denote an element of $\mathbb{R}^n$ by (x,t), where $x \in \mathbb{R}^{n-1}, t \in \mathbb{R}$, and we let $\mathbb{R}_0^{n-1}$ denote the hyperplane $\{(x,0): x \in \mathbb{R}^{n-1}\}$. We may think of t as the "time" coordinate. We let $\beta_t, t \in \mathbb{R}$, denote the translation $(x,s) \mapsto (x, s+t)$ and denote the corresponding unitary operator by $U_t = U_{\beta_t}$. Moreover, $\rho \in \mathcal{E}_n$ will be the time reflection $\rho(x,t) = (x, -t)$. We denote the projection operator $E_{\mathbb{R}_0^{n-1}}$ on $L^2(\Omega, \Sigma, \mu)$ by E_0 and the closed subspace $L^2(\Omega, \Sigma_{\mathbb{R}_0^{n-1}}, \mu) = E_0 L^2(\Omega, \Sigma, \mu)$ of $L^2(\Omega, \Sigma, \mu)$ by $\mathcal{K}$. Finally, we use the shorthand notation $E_t \equiv E_{\beta_t}(\mathbb{R}_0^{n-1}) = U_t E_0 U_t^{-1}$ and for $B \in \mathcal{B}(\mathbb{R}), E_B = E_{\mathbb{R}^{n-1} \times B}$.

In classical probability theory, Markov processes are naturally associated with contraction semigroups [65, 68]. We now show that there is an analogous result for covariant Markov random fields.

Theorem 7.10. *Let (Φ, T) be a covariant Markov random field on $H^{-1}(\mathbb{R}^n)$ and let $P_t = E_0 U_t | \mathcal{K}$. Then $\{P_t : t \geq 0\}$ is a strongly continuous self-adjoint contraction semigroup and $P_{-t} = P_t$ for every $t \in \mathbb{R}$.*

PROOF. Since $t \mapsto U_t$ is strongly continuous so is $t \mapsto P_t$ and since U_t is unitary, we have

$$P_t^* = (E_0 U_t E_0)^* = E_0 U_t^{-1} E_0 = P_{-t}.$$

Suppose that $f \in H^{-1}(\mathbb{R}^n)$ and $\operatorname{supp} f \subseteq \mathbb{R}_0^{n-1}$. We now show that $f \circ \rho = f$. Since $\operatorname{supp} f \subseteq \{(x,0): x \in \mathbb{R}^{n-1}\}$, it follows from Theorem 5.6 that there exist tempered distributions $g_j \in \mathcal{S}'(\mathbb{R}^{n-1}), 0 \leq j \leq m$, such that

$$f = \sum_{j=0}^{m} g_j \otimes \delta_0^{(j)}.$$

To find the Fourier transform of $\delta_0^{(j)}$ we have for any $\phi \in \mathcal{S}(\mathbb{R})$

$$\delta_0^{(j)\hat{}}(\phi) = \delta_0^{(j)}(\hat{\phi}) = \delta^{(j)}(2\pi)^{-1/2} \int e^{its}\phi(s)\, ds$$

$$= (2\pi)^{-1/2} \int (is)^j \phi(s)\, ds.$$

Hence $\delta_0^{(j)\hat{}} = (2\pi)^{-1/2}(is)^j$. We thus have

$$\| g_j \otimes \delta_0^{(j)} \|_{-1}^2 = (2\pi)^{-1} \int \int |\hat{g}_j(k)|^2 |s|^{2j} (k^2 + s^2 + 1) d^{n-1}k\, ds. \tag{7.41}$$

The s integration in (7.41) is finite if and only if $j = 0$. It follows that

$f=g_0\otimes\delta_0$ so clearly $f\circ\rho=f$. Hence, by covariance, we have

$$U_\rho\Phi(f)(\omega)=\Phi(f)\left(T_\rho^{-1}\omega\right)=\Phi(f)\circ T_\rho^{-1}(\omega)$$
$$=\Phi(f\circ\rho)(\omega)=\Phi(f)(\omega).$$

It follows that U_ρ leaves the range of E_0 pointwise invariant so that

$$U_\rho E_0=E_0U_\rho=E_0.$$

Hence,

$$P_{-t}=E_0U_{-t}E_0=E_0U_\rho U_tE_0=E_0U_tE_0=P_t.$$

We conclude that $P_t^*=P_t$ is self-adjoint for every $t\in\mathbb{R}$. Since $E_t=U_tE_0U_{-t}$ we have

$$U_tE_s=U_tU_sE_0U_{-s}=U_{t+s}E_0U_{-(t+s)}U_t=E_{t+s}U_t.$$

In particular,

$$U_tE_{-t}=E_0U_t,\qquad U_sE_0=E_sU_s. \tag{7.42}$$

Moreover, by the Markov condition, if $t,s\geq 0$, we have

$$E_{-t}E_0E_s=E_{-t}E_{(-\infty,0]}E_s=E_{-t}E_s. \tag{7.43}$$

Applying (7.42) and (7.43) for $t,s\geq 0$, we have

$$P_tP_s=E_0U_tE_0U_sE_0=E_0(E_0U_t)E_0(U_sE_0)E_0$$
$$=E_0U_tE_{-t}E_0E_sU_sE_0=E_0(U_tE_{-t})(E_sU_s)E_0$$
$$=E_0U_tU_sE_0=P_{t+s}.$$

Since $P_0=I$ and $\|P_t\|\leq 1,\{P_t:t\geq 0\}$ is a contraction semigroup. □

Corollary 7.11. *If Φ and P_t are defined as in the previous theorem, then there exists a unique positive self-adjoint operator H on $\mathcal{H}$ such that $P_t=e^{-|t|H},t\in\mathbb{R}$.*

PROOF. The proof follows from a standard result on self-adjoint contraction semigroups [131]. □

Following Nelson [199] we now show how a Wightman quantum field can be constructed from a Euclidean field Φ on $H^{-1}(\mathbb{R}^n)$. Construct the Hilbert space $\mathcal{H}$ and the positive self-adjoint operator as in Theorem 7.10 and Corollary 7.11. We first construct the "time zero" quantum field Θ_0. If $f\in\mathcal{S}(\mathbb{R}^{n-1})$ then $f\otimes\delta_0\in H^{-1}(\mathbb{R}^n)$. Indeed, since $\hat{\delta}_0=(2\pi)^{-1/2}$, we have

$$\|f\otimes\delta_0\|_{-1}^2=(2\pi)^{-1}\int\int|\hat{f}(k)|^2(k^2+s^2+1)^{-1}d^{n-1}k\,ds$$
$$\leq\int\int|\hat{f}(k)|^2(s^2+1)^{-1}d^{n-1}k\,ds\leq\|\hat{f}\|_\infty\int(s^2+1)^{-1}ds<\infty.$$

For $\phi \in \mathcal{S}(\mathbb{R}^{n-1})$, define $\Theta_0(\phi) = \Phi(\phi \otimes \delta_0)$. Since $\operatorname{supp} \phi \otimes \delta_0 \subseteq \mathbb{R}_0^{n-1}$, $\Theta_0(\phi)$ is a random variable which is measurable with respect to $\Sigma_{\mathbb{R}_0^{n-1}}$ and hence multiplication by the function $\Theta_0(\phi)$ is a linear operator (unbounded, in general) on some domain in $\mathcal{H} = L^2(\Omega, \Sigma_{\mathbb{R}_0^{n-1}}, \mu)$. In the sequel we consider $\Theta_0(\phi)$ to be such a linear operator. We define the "sharp time" quantum field $\Theta_t, t \in \mathbb{R}$, by $\Theta_t(\phi) = e^{itH}\Theta_0(\phi)e^{-itH}, \phi \in \mathcal{S}(\mathbb{R}^{n-1})$. Thus, for every $\phi \in \mathcal{S}(\mathbb{R}^{n-1}), t \in \mathbb{R}, \Theta_t(\phi)$ is a linear operator in $\mathcal{H}$ whose domain we shall define presently.

In order for $\Theta_t(\phi)$ to have a sufficiently large domain, Nelson introduces a mild regularity condition which we now discuss. For every integer k, let $\mathcal{H}^k$ be the completion of the domain $D(H^{k/2})$ in the norm $\|u\|_k = \|(I+H)^{k/2}u\|$ and let $\mathcal{H}^\infty = \bigcap \mathcal{H}^k, \mathcal{H}^{-\infty} = \bigcup \mathcal{H}^k$ [199]. All of these subspaces are dense in $\mathcal{H}$. By $\mathcal{L}(\mathcal{H}^k, \mathcal{H}^l)$ we mean the Banach space of bounded linear transformations from $\mathcal{H}^k$ to $\mathcal{H}^l$ equipped with the norms $\|\cdot\|_k, \|\cdot\|_l$, respectively. A Euclidean random field on $H^{-1}(\mathbb{R}^n)$ is *regular* if there exist integers k and l such that for every $\phi \in \mathcal{S}(\mathbb{R}^{n-1})$, $\Theta_0(\phi) \in \mathcal{L}(\mathcal{H}^k, \mathcal{H}^l)$ and $\phi \mapsto \Theta_0(\phi)$ is continuous. If Φ is regular, since e^{itH} leaves each $\mathcal{H}^k$ invariant and is unitary on it for every $t \in \mathbb{R}$, we have

$$\Theta_t(\phi) = e^{itH}\Theta_0(\phi)e^{-itH} \in \mathcal{L}(\mathcal{H}^k, \mathcal{H}^l)$$

for every $t \in \mathbb{R}$.

We are now ready to construct the quantum field Θ associated with Φ. For $\phi \in \mathcal{S}(\mathbb{R}^4)$, let $\phi_t(x) = \phi(x,t)$, $x \in \mathbb{R}^3$, $t \in \mathbb{R}$, and define

$$\Theta(\phi) = \int_{\mathbf{R}} \Theta_t(\phi_t)\, dt = \int_{\mathbf{R}} e^{itH}\Theta_0(\phi_t)e^{-itH}\, dt.$$

It is not hard to show that $\Theta(\phi) \in \mathcal{L}(\mathcal{H}^\infty)$. The next theorem shows that Θ is a Wightman quantum field.

Theorem 7.12 (Nelson). *If Φ is a regular Euclidean random field on $H^{-1}(\mathbb{R}^4)$, then Θ is a Wightman quantum field with vacuum vector* $\mathbf{1}$.

PROOF. We shall just give the idea of the proof; for details the reader is referred to [199]. First, the uniqueness of the vacuum follows almost immediately from ergodicity. For the remainder of the proof, form the expectation values

$$\mathcal{W}_n(\phi_1, \ldots, \phi_n) = \langle \Theta(\phi_1) \cdots \Theta(\phi_n)\mathbf{1}, \mathbf{1} \rangle$$

and the corresponding Wightman distributions given by

$$W_n(\phi_1 \otimes \cdots \otimes \phi_n) = \mathcal{W}_n(\phi_1, \ldots, \phi_n).$$

Positive definiteness (W1) and reality (W2) follow immediately. Relativistic covariance (W3) follows from the Euclidean covariance of Φ since in a certain sense M^4 is just $\mathbb{R}^4$ with t replaced by it. The spectrum condition (W4) is a consequence of the positivity of H and relativistic covariance. The most difficult part of the proof is locality (W5). Intuitively, the idea is the following. If ϕ_j and ϕ_{j+1} are spacelike separated, then by relativistic covariance, we may assume that they have supports in the hyperplane $\mathbb{R}_0^3$. But $\Theta = \Phi$ on this hyperplane and the values of Φ commute since they are random variables. □

B. Simon [250] has given a converse to Theorem 7.12. He has shown that a certain large class of Wightman quantum fields can be constructed from regular Euclidean random fields using Nelson's method.

Nelson's approach to the construction of quantum fields has many desirable features. Reducing the study of quantum fields to Euclidean random fields has the following advantages.

1. The values of a random field are random variables, so one does not have the cumbersome domain problems of unbounded operators. Moreover, the values of a random field always commute.
2. The Euclidean group is much easier to work with than the Poincaré group.
3. One can bring into play the powerful methods and results of probability theory. In particular, results analogous to those for Markov processes have already been widely exploited. Furthermore, using existence theorems of probability theory, the existence of random fields with specific properties are much easier to prove than the existence of quantum fields.

In Section 7.4 we showed that the unit Gaussian random field Φ over a real Hilbert space $\mathcal{H}$ is unitarily equivalent to the boson Fock space $\mathcal{F}_+(\mathcal{H})$ (in the sense of Theorem 7.9). Nelson has shown that the unit Gaussian random field over $H^{-1}(\mathbb{R}^4)$ is a regular Euclidean random field [200]. This not only shows that regular Euclidean random fields exist, but also that free quantum fields can be constructed from the unit Gaussian random field.

7.6. Notes and References

For a general discussion of the canonical commutation relations CCR, the canonical anticommutation relations and other aspects of algebraic quantum field theory we recommend [72]. Our approach to the CCR in Section 7.2 is similar to that in [6, 87]. For more details of the C^*-algebra framework applied to the CCR see [6–10, 13, 46, 61, 86, 127, 182, 231, 256, 257, 266]. The Wightman axioms are considered in detail in [23, 150, 161,