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Spectral Representation of the Correlation Function

Recall Khinchin's formula

$$B(\tau) = \int_{-\infty}^{\infty} e^{i\lambda\tau} d\,F(\omega) \tag{1}$$

for the correlation function $B(\tau)$ of a stationary random process $X(t)$ (see (2.52) above). The function $F(\omega)$ in this formula is a bounded monotone nondecreasing function of ω. It is clear that this function is determined by the formula for $B(\tau)$ only to within an arbitrary additional constant, which can be chosen so that $F(-\infty) = 0$.

In the real case, as is well known, $B(-\tau) = B(\tau)$ for all τ and, hence

$$B(\tau) = \frac{[B(\tau) + B(-\tau)]}{2} \tag{2}$$

Therefore in this case (2.52) can be rewritten in the form

$$B(\tau) = \int_0^{\infty} \cos(\omega\tau)\, d\,F(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} \cos(\omega\tau)\, d\,G(\omega) \tag{3}$$

where

$$d\,G(\omega) = d\,F(\omega) + d\,F(-\omega) \forall \omega > 0 \tag{4}$$

(see (2.53)). Moreover, it is easy to show that the relation $B(\tau) = B(-\tau)$ implies that the increments of $F(\omega)$ on intervals symmetric about the point $\omega = 0$ must coincide. Thus

$$d\,G(\omega) = 2\,d\,F(\omega) \tag{5}$$

for $\omega > 0$, and if the function $F(\omega)$ is continuous at the point $\omega = 0$, then

$$G(\omega) = 2\,F(\omega) + \text{const} \tag{6}$$

(However, if $F(\omega)$ is discontinuous at $\omega = 0$, then $G(\omega)$ will have the same, and not a two-fold jump at this point.) It is convenient to assume, when the function $G(\omega)$ is considered, that $G(-\omega) = 0$ (where $G(-\omega)$ is the value of $G(\omega)$ just before its possible jump at the point $\omega = 0$). Note that if $X(\tau)$ is μ of X, then

$$B(\tau) = b(\tau) + m^2 \tag{7}$$

, where $b(\tau)$ is a centered correlation function, which also permits representation in the form (2.52). Therefore, if $X(\tau) = m$, then the function $F(\omega)$ in (2.52) for $B(\tau)$ will necessarily satisfy the relation

$$F(\omega) - F(-\omega) > \frac{m^2}{2} \tag{8}$$

(i.e. $\omega = 0$ will be the discontinuity point of $F(\omega)$, and its jump at $\omega = 0$ will not be less than m^2). Further in this section, we shall again assume that $X(\tau) = 0$ (i.e. whenever $X(\tau) = m$, we shall assume that the value of m has already been subtracted from all the values of the process, so that $B(\tau)$ is actually a centered correlation function).

In practical applications of stationary random processes, the correlation function $B(\tau)$ usually tends to zero as $|\tau|\to\infty$ (cf. (1.36) in Sec. 4). Suppose that the absolute value of $B(\tau)$ falls off so rapidly as $|\tau|\to\infty$, that

$$\int_{-\infty}^{\infty}|B(t)|\,dt<\infty \quad (2.66) \tag{9}$$

(this condition is also usually fulfilled for practical situations). Then the function $B(t)$ can be represented as the Fourier integral

$$B(t)=\int_{-\infty}^{\infty}e^{i\omega t}f(\omega)\,d\omega \quad (2.67) \tag{10}$$

where $f(\omega)$ is a bounded and continuous function of ω. The representation (2.67) is also possible under some other conditions imposed on $B(t)$. For instance, (2.66) can be replaced by a less restrictive condition

$$\int_{-\infty}^{\infty}|B(t)|^2\,dt<\infty \quad (2.68) \tag{11}$$

but in this case the function $f(\omega)$ will no longer be necessarily continuous and bounded.[18] Formula (2.67) is a particular case of formula (2.52); it indicates that, subject to the condition (2.66) (or (2.68)),

$$F(\omega)=\int_{-\infty}^{\infty}f(\omega')\,d\omega' \quad (2.69) \tag{12}$$

where

$$f(\omega)=F'(\omega) \quad (2.70) \tag{13}$$

so that $F(\omega)$ is a differentiable function. In the case of a real process $X(t)$, where the function $B(t)$ is even, the function $f(\omega)$ will also be even, and the Fourier integral representation (2.67) can be rewritten as

$$B(t)=\int_{0}^{\infty}\cos(\omega t)\,g(\omega)\,d\omega \tag{14}$$

$$g(\omega)=2\,f(\omega)\;(2.71) \tag{15}$$

It is clear that in this case $g(\omega)=G'(\omega)$ and

$$G(\omega)=\int_{0}^{\omega'}g(\omega')\,d\omega' \tag{16}$$

where $G(\omega)$ is a function appearing in (2.53).

The Fourier representation (2.52) or (2.67) of the correlation function $B(t)$ is called the spectral representation of the correlation function. The function $F(\omega)$ appearing in this representation is called the spectral distribution function of the stationary random process $X(t)$. If, however, the relations (2.69) and (2.67) hold, then $f(\omega)$ is called the spectral density (*function*) of the process $X(t)$).

By virtue of (2.52)

$$\int_{-\infty}^{\infty}d\,F(\omega)=B(0)=\langle X(t)\rangle^2 \tag{17}$$

so that

$$\int_{-\infty}^{\infty} d\,F(\omega) = F(\omega) - F\,(-\infty) < \infty \tag{18}$$

We see that the spectral distribution function is necessarily bounded. If there exists a spectral density $f(\omega)$, then obviously

$$\int_{-\infty}^{\infty} f(\omega)\,d\,\omega = B(0) < \infty \tag{19}$$

thus the spectral density is always integrable. Moreover, since $F(\omega)$ is a monotone nondecreasing function, the spectral density $f(\omega) = F'(\omega)$ is everywhere non-negative:

$$f(\omega) \geq 0 \tag{20}$$

Conversely, if $f(\omega)$ is an integrable non-negative function, the function (2.69) will obviously be bounded and monotone nondecreasing, i.e. the function (2.67) will satisfy the conditions of Khinchin's theorem. Thus, Khinchin's theorem acquires a particularly simple form when applied to rapidly falling off functions $B(\tau)$ representable in the form of the Fourier integral (e.g. to functions satisfying the condition (2.66) or (2.68)): a rapidly falling off function $B(t)$ will be a correlation function if, and only if, its Fourier transform $f(\omega)$ is everywhere non-negative.

So far, we have not yet explained in this section how one proves Khinchin's theorem, which is so important for the entire content of the section; but this is discussed in Note 13, referring to Sec. 7. As indicated in that Note, Khinchin's theorem is a simple consequence of the following two statements, taken together:

(a) The class of functions $B(t)$, which are correlation functions of stationary random processes, coincides with the class of positive definite functions of the variable t (see above, Sec. 4 for a real case and Sec. 5 for a complex case).

(b) A continuous function $B(t)$ of the real variable t is positive definite if, and only if, it can be represented in the form (2.52), where $F(\omega)$ is bounded and nondecreasing (this statement was proved independently by Bochner and Khinchin, but was first published by Bochner and therefore is known as Bochner's theorem; see, e.g., Bochner (1959) and also Note 3 to Introduction).

In the preceding section it was emphasized that Khinchin's theorem lies at the basis of almost all the proofs of the spectral representation theorem for stationary random processes. It is, however, obvious that if we proved the spectral representation theorem without using Khinchin's theorem, this would also clearly imply the possibility of representing $B(t)$ in the form (2.52). Indeed, replacing $X\,(t+\tau)$ and $X(t)$ in the formula $B(t) = \langle X\,(t+\tau)\,X(t)\rangle$ by their spectral representation (2.61) and then using (2.1) by definition (2.62) of the corresponding Fourier–Stieltjes integral and the property (b') of the random function $Z(\omega)$, we obtain at once (2.52), where

$$F\,(\omega + \Delta\,\omega) - F(\omega) = |Z\,(\omega + \Delta\,\omega) - Z(\omega)|^2 \tag{21}$$

so that $F(\omega)$ is clearly a nondecreasing function. Formula (2.76) can also be written in the differential form:

$$\langle d\,Z(\omega)^2\rangle = d\,F(\omega) \tag{22}$$

Moreover, (2.77) can be combined with the property (b') of $Z(\omega)$ in the form of a single symbolic relation

$$\langle d\,Z(\omega)\,d\,Z(\omega')\rangle = \delta\,(\omega-\omega')\,d\,F(\omega)\,d\,\omega' \tag{23}$$

where $\delta(\omega)$ is the Dirac delta-function. It is easy to see that the substitution of (2.78) into the expression for the mean value of any double integral with respect to $d\,Z(\omega)$ and $d\,Z(\omega')$ gives the correct result. As the simplest example we consider the following derivation of Khinchin's formula (2.52):

$$\langle X\,(t+\tau)\,X(t)\rangle = \left\langle \int_{-\infty}^{\infty} e^{i\omega(t+\tau)}\,d\,Z(\omega)\int_{-\infty}^{\infty} e^{-i\omega' t}\,d\,Z(\omega')\right\rangle \tag{24}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega(t+\tau)-i\omega' t}\,\langle d\,Z(\omega)\,d\,Z(\omega')\rangle \tag{25}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega(t+\tau)-i\omega' t}\,\delta\,(\omega-\omega')\,d\,F(\omega)\,d\,\omega' \tag{26}$$

$$= \int_{-\infty}^{\infty} e^{i\omega\tau}\,d\,F(\omega) \tag{27}$$

Quite similarly, the following more general result can be derived:

$$\int_{-\infty}^{\infty} g(\omega)\,d\,Z(\omega)\int_{-\infty}^{\infty} h(\omega')\,d\,Z(\omega') = \int_{-\infty}^{\infty} g(\omega)\,h(\omega')\,\delta\,(\omega-\omega')\,d\,F(\omega)$$

where $g(\omega)$ and $h(\omega)$ are any two complex functions whose squared absolute values are integrable with respect to $d\,F(\omega)$. Note also that if the spectral density $f(\omega)$ exists, then the relations (2.77) and (2.78) obviously take the form

$$\langle d\,Z(\omega)^2\rangle = f(\omega)\,d\,\omega \tag{28}$$

$$\langle d\,Z(\omega)\,d\,Z(\omega')\rangle = \delta\,(\omega-\omega')\,f(\omega)\,d\,\omega\,d\,\omega' \tag{29}$$

Formulae (2.76)–(2.78) and (2.80)–(2.81) establish the relationship between the spectral representation of the correlation function (determined by the functions $F(\omega)$ and $f(\omega)$) and the spectral representation of the stationary random process $X(t)$ itself, which includes the random point function $Z(\omega)$ or the random interval function

$$Z\,(\Delta\,\omega) = Z(\omega_2) - Z(\omega_1) \tag{30}$$

where $\Delta\,\omega = [\omega_1, \omega_2]$. We shall see in Sec. 11 that this relationship gives physical meaning to Khinchin's mathematical theorem and permits one to verify it experimentally when the stationary process $X(t)$ is realized in the form of oscillations of some measurable physical quantity X. Now, however, we shall consider some more special questions bearing on the indicated relationship.

We begin with the particular case where the function $F(\omega)$ is a "step-function", which changes (namely, increases by a positive value) only at discrete discontinuity points but takes a constant value between any two discontinuities (cf. Figs. 1(a) and 1(b)). In this case (2.76) and (2.77) imply that the random interval function $Z(\Delta\omega)$ is concentrated entirely on a discrete set of discontinuity points ω_1, ω_2,... of the function $F(\omega)$, i.e. $Z(\Delta\omega) = 0$ for all intervals $\Delta\omega$ that do not contain any of these points. It is clear that then the spectral representation (2.58) reduces to the representation (2.31) or (2.38)) of the process $X(t)$ as a superposition of separate uncorrelated harmonic oscillations with random amplitudes and phases, i.e. $X(t)$ will be a process with a discrete spectrum in the sense explained in Sec. 7. Moreover, (2.52) shows that the corresponding correlation function $B(t)$ has the form (2.33) or, respectively, (2.39). We see that the representability of the correlation function $B(t)$ of the process $X(t)$ in the form of a sum of harmonics (or, in the real case, of cosine functions) implies that the corresponding random process $X(t)$ is a sum of uncorrelated random harmonic oscillations. This particular case of the general spectral representation theorem is due to Slutsky (see Note 7).

In the general case the nondecreasing function $F(\omega)$ can be represented as a sum of a continuous function $F_c(\omega)$ and a "step-function" $F_j(\omega)$ constructed from the "jumps" (discontinuities) of $F(\omega)$. By virtue of (2.76) and (2.77), the discontinuity point of $F_j(\omega)$ corresponds to the discontinuity point of the random function $Z(\omega)$, where changes jump-wise by a random variable X_k, such that

$$\langle X_k^2 \rangle = f_k \tag{31}$$

(here f_k is the increment of the function $F(\omega)$).

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*The terminology relating to spectral representations has not yet been finally established. For instance, the terms "spectral distribution function" and "spectral density (function)" are often used in relation to the function $G(\omega)$ and $g(\omega) = 2\,\pi\,\omega$ of the non-negative frequency ω. Moreover, the Fourier expansions in harmonics $e^{i\omega t}$ or $\cos(\omega\,t)$, where ω is the angular frequency (measured in "radians per second"), is often replaced by expansions in functions $e^{2\pi i\Omega t}$ or $\cos(2\,\pi\,\Omega\,t)$, where Ω is the ordinary frequency measured in Hertz (i.e. cycles per second). In this case the spectral density $f(\omega)$ must be replaced by the density $f_1(\omega) = 2\,\pi\,f\,(2\,\pi\,\omega)$, which is $2\,\pi$ times higher. (Note also that the term "power spectrum", or simply "spectrum", is widely used in all the applied literature instead of the term "spectral density" preferred by mathematicians. (The term "spectrum" has, in mathematics, another meaning, which will be explained below in this section.) As to the spectral distribution function, it is rarely used in the applied literature, but if used it is generally called the "integrated spectrum".)