

The Operational Matrix of the Random Wave Process

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Abstract

An expression for the convolution of a pair of spherical Bessel functions is determined.

1 Lemmas

Lemma 1

(Terminating Hypergeometric Series) For any $p \in \mathbb{Z}_{\geq 0}$, the Gauss hypergeometric function terminates:

$${}_2F_1(-p, b; c; z) = \sum_{k=0}^p \frac{(-p)_k (b)_k}{(c)_k k!} z^k \quad (1)$$

where $(a)_k = \prod_{i=0}^{k-1} (a + i)$

Proof. By definition, the Gauss hypergeometric series is:

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k \quad (2)$$

Setting $a = -p$ with $p \in \mathbb{Z}_{\geq 0}$, the Pochhammer symbol $(-p)_k$ becomes zero for all $k > p$. Explicitly:

$$(-p)_k = \prod_{i=0}^{k-1} (-p + i) = \begin{cases} (-p)(-p+1) \cdots (-p+k-1), & k \leq p \\ 0 & k > p \end{cases} \quad (3)$$

Thus, the series terminates at $k = p$, yielding:

$${}_2F_1(-p, b; c; z) = \sum_{k=0}^p \frac{(-p)_k (b)_k}{(c)_k k!} z^k \quad (4) \quad \square$$

Lemma 2

(Integral with Incomplete Gamma Function) For $j \geq 0$,

$$\int_{-1}^1 \left(\frac{1-x}{2} \right)^j e^{ixy} dx = \frac{e^{iy}}{2^j} \frac{\gamma(j+1, 2iy)}{(iy)^{j+1}} \quad (5)$$

where $\gamma(s, x)$ denotes the lower incomplete gamma function.

Proof. Substitute $t = \frac{1-x}{2} \implies x = 1 - 2t$, $dx = -2dt$, adjusting limits:

$$\int_1^0 t^j e^{i(1-2t)y} (-2dt) = 2e^{iy} \int_0^1 t^j e^{-2iyt} dt \quad (6)$$

Let $u = 2iyt \implies t = \frac{u}{2iy}$, $dt = \frac{du}{2iy}$:

$$\frac{2e^{iy}}{(2iy)^{j+1}} \int_0^{2iy} u^j e^{-u} du = \frac{e^{iy}}{2^j} \frac{\gamma(j+1, 2iy)}{(iy)^{j+1}} \quad \square$$

Lemma 3

(Legendre Polynomial Representation) The Legendre polynomials are hypergeometric functions

$$P_m(x) = {}_2F_1(-m, m+1; 1; \frac{1-x}{2}) \quad (7)$$

Proof. From the Rodrigues formula $P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m} (x^2 - 1)^m$ expand using the binomial theorem:

$$(x^2 - 1)^m = \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} x^{2k} \quad (8)$$

Differentiating m times yields terms proportional to x^k , matching the hypergeometric series:

$$P_m(x) = {}_2F_1\left(-m, m+1; 1; \frac{1-x}{2}\right) \quad \square$$

2 Main Theorem

Theorem 4

(Fourier Transform of Legendre Polynomial Products) *The Fourier transform of the product of a pair of Legendre polynomials is the convolution of a pair of spherical Bessel functions of the first kind which can be expressed as*

$$\begin{aligned} I_{m,n}(y) &= \int_{-1}^1 P_m(x) P_n(x) e^{ixy} dx \\ &= e^{iy} \sum_{j=0}^{m+n} \Psi_j(m, n) \frac{\gamma(j+1, 2iy)}{2^j (iy)^{j+1}} \end{aligned} \quad (9)$$

where

$$\Psi_j(m, n) = \frac{{}_4F_3\left(\begin{matrix} -m, m+1, -n, n+1 \\ 1, 1, j+1 \end{matrix}; 1\right)}{j!} \quad (10)$$

Expand both polynomials using their hypergeometric representations:

$$P_m(x) = \sum_{k=0}^m \frac{(-m)_k (m+1)_k}{(1)_k k!} \left(\frac{1-x}{2}\right)^k \quad (11)$$

$$P_n(x) = \sum_{\ell=0}^n \frac{(-n)_\ell (n+1)_\ell}{(1)_\ell \ell!} \left(\frac{1-x}{2}\right)^\ell \quad (12)$$

Their product becomes:

$$\begin{aligned} P_m(x) P_n(x) &= \sum_{k=0}^m \sum_{\ell=0}^n \frac{(-m)_k (m+1)_k (-n)_\ell (n+1)_\ell}{(1)_k (1)_\ell k! \ell!} \left(\frac{1-x}{2}\right)^{k+\ell} \\ &= \sum_{k=0}^m \sum_{\ell=0}^n \frac{(-m)_k (m+1)_k (-n)_\ell (n+1)_\ell}{k!^2 \ell!^2} \left(\frac{1-x}{2}\right)^{k+\ell} \end{aligned} \quad (13)$$

Reorganize using $j = k + \ell$, with explicit summation bounds:

$$P_m(x) P_n(x) = \sum_{j=0}^{m+n} \underbrace{\sum_{k=\max(0, j-n)}^{\min(j, m)} \frac{(-m)_k (m+1)_k (-n)_{j-k} (n+1)_{j-k}}{(1)_k (1)_{j-k} k! (j-k)!}}_{\Psi_j(m, n)} \left(\frac{1-x}{2}\right)^j \quad (14)$$

The inner sum coefficients are:

$$\Psi_j(m, n) = \sum_{k=\max(0, j-n)}^{\min(j, m)} \frac{(-m)_k (m+1)_k (-n)_{j-k} (n+1)_{j-k}}{(1)_k (1)_{j-k} k! (j-k)!} \quad (15)$$

Apply the identities:

$$(-n)_{j-k} = (-1)^{j-k} \frac{\Gamma(n+1)}{\Gamma(n-j+k+1)} \quad (16)$$

$$(n+1)_{j-k} = \frac{\Gamma(n+j-k+1)}{\Gamma(n+1)} \quad (17)$$

$$(-a)_n (a+1)_n = (-1)^n \frac{\Gamma(a+1)}{\Gamma(a-n+1)} \quad (18)$$

so that the substitutions

$$(-n)_{j-k} (n+1)_{j-k} = \frac{(-1)^{j-k} \Gamma(n+j-k+1)}{\Gamma(n-j+k+1)} \quad (19)$$

and $h = j - k$ can be made so that $\Psi_j(m, n)$ can be expressed as

$$\Psi_j(m, n) = (-1)^j \sum_{h=0}^{\min(n, j) - \max(0, j-m)} \frac{(-m)_{j-h} (m+1)_{j-h} \Gamma(n+h+1)}{(j-h)!^2 \Gamma(n-j+h+1) h!} \quad (20)$$

to reveal the hypergeometric form by reversing the index transformation:

$$\begin{aligned} \Psi_j(m, n) &= \frac{1}{j!} \sum_{k=0}^{\min(j, m, n)} \frac{(-m)_k (m+1)_k (-n)_k (n+1)_k}{(1)_k (1)_k (j+1)_k k!} \\ &= \frac{{}_4F_3\left(\begin{matrix} -m, m+1, -n, n+1 \\ 1, 1, j+1 \end{matrix}; 1\right)}{j!} \end{aligned} \quad (21)$$