The Covariance of Ergodic Stationary Processes

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Abstract

This note establishes in detail the equality, for strictly stationary and ergodic real-valued stochastic processes with finite second moment, between the covariance function as defined by the expectation of the product of observations at a given lag and the almost sure limit of temporal expectation of products along a single path. The argument is developed from first principles, specifying measure-theoretic structure, shift invariance, ergodicity, and relevant properties of function spaces, and invoking the continuous-time ergodic theorem in full generality for integrable functions.

1. Preliminaries

Definition 1. *[Probability Space and Process] Let $(\Omega, \mathcal{F}, P)$ be a probability space. A* stochastic process *$\xi: \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is a collection of real random variables $\xi(t): \Omega \rightarrow \mathbb{R}$, indexed by $t \in \mathbb{R}$. The process is* jointly measurable *if the mapping $(t, \omega) \mapsto \xi(t, \omega)$ is measurable with respect to the product σ-algebra $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$.*

Definition 2. *[Strict Stationarity] A stochastic process $\xi = (\xi(t))_{t \in \mathbb{R}}$ is* strictly stationary *if, for every $n \in \mathbb{N}$, every choice $t_1, \ldots, t_n \in \mathbb{R}$, and every $h \in \mathbb{R}$,*

$$(\xi(t_1 + h), \ldots, \xi(t_n + h)) \stackrel{d}{=} (\xi(t_1), \ldots, \xi(t_n)),$$

that is, the finite-dimensional distributions are invariant under time shifts.

Definition 3. *[Covariance Function] For a stochastic process ξ with $\mathbb{E}[\xi(0)^2] < \infty$, define the* covariance function *r by*

$$r(\tau) = Cov(\xi(0), \xi(\tau)) = \mathbb{E}[(\xi(0) - m)(\xi(\tau) - m)], \quad m = \mathbb{E}[\xi(0)].$$

If $\mathbb{E}[\xi(0)] = 0$, then $r(\tau) = \mathbb{E}[\xi(0)\,\xi(\tau)]$.

Definition 4. *[Shift Operator (Path Space Version)] Let $E = \mathbb{R}$ and consider the canonical space $\Omega = \mathbb{R}^{\mathbb{R}}$ consisting of all functions $x: \mathbb{R} \to \mathbb{R}$. For each $h \in \mathbb{R}$ define the shift operator $T_h: \Omega \to \Omega$ by*

$$(T_h x)(t) = x(t + h)$$

for all $t \in \mathbb{R}$, $x \in \Omega$. If ξ is a process on an abstract probability space, interpret x as a sample path $x_\omega(t) = \xi(t, \omega)$.

Proposition 5. *[Shift Invariance] Let μ be a probability measure on $(\Omega, \mathcal{F})$ such that the coordinate process $x(t)$ under μ has the same law as $\xi(t)$ for all t. If ξ is strictly stationary, then for every $h \in \mathbb{R}$ and every $A \in \mathcal{F}$, $\mu(T_h^{-1} A) = \mu(A)$.*

Proof. Let $A \in \mathcal{F}$ be a cylinder set of the form

$$A = \{x \in \Omega: (x(t_1), \ldots, x(t_n)) \in B\}$$

where $t_1, \ldots, t_n \in \mathbb{R}$ and B is a Borel set in $\mathbb{R}^n$. Then

$$T_h^{-1} A = \{x \in \Omega: (x(t_1 + h), \ldots, x(t_n + h)) \in B\}.$$

Since under μ the law of $(x(t_1 + h), \ldots, x(t_n + h))$ coincides with that of $(x(t_1), \ldots, x(t_n))$ by stationarity,

$$\mu((x(t_1 + h), \ldots, x(t_n + h)) \in B) = \mu((x(t_1), \ldots, x(t_n)) \in B) = \mu(A).$$

Extension from cylinder sets to $\mathcal{F}$ proceeds by the monotone class theorem or standard arguments. $\square$

Definition 6. *[Ergodicity] The measure-preserving flow $(T_h)_{h \in \mathbb{R}}$ on $(\Omega, \mathcal{F}, \mu)$ is called* ergodic *if, for every $A \in \mathcal{F}$ satisfying $T_h^{-1} A = A$ for all $h \in \mathbb{R}$, either $\mu(A) = 0$ or $\mu(A) = 1$.*

Remark 7. Ergodicity is equivalent to the triviality of the shift-invariant σ-algebra:

$$\mathcal{I} = \{A \in \mathcal{F}: T_h^{-1} A = A \text{ for all } h \in \mathbb{R}\}.$$

2. The Covariance Function and Pathwise Limit

Theorem 8. *[Pathwise Determination of Covariance Function]Let* $(\Omega, \mathcal{F}, \mu)$ *and the canonical process* $x(t)$ *be as above. Suppose that under* μ*,* $x(t)$ *is strictly stationary, ergodic with respect to* $(T_h)_{h \in \mathbb{R}}$*, and* $\mathbb{E}_\mu[x(0)^2] < \infty$*. Fix* $\tau \in \mathbb{R}$*. Then for* μ*-almost every* $x \in \Omega$*,*

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\ x(t + \tau)\ dt = r(\tau)$$

where $r(\tau) = \mathbb{E}_\mu[x(0)\, x(\tau)]$.

Proof. The steps follow as below.

1. The map $x \mapsto x(0)\, x(\tau)$ is measurable as a product of coordinate projections, hence Borel measurable on Ω.

2. Since $\mathbb{E}_\mu[x(0)^2] < \infty$, by the Cauchy-Schwarz inequality,

$$\mathbb{E}_\mu[\,|x(0)\, x(\tau)|\,] \leq \sqrt{\mathbb{E}_\mu[x(0)^2]\ \mathbb{E}_\mu[x(\tau)^2]} = \mathbb{E}_\mu[x(0)^2] < \infty$$

   by stationarity, so $x \mapsto x(0)\, x(\tau)$ is integrable.

3. Consider the function $F : \Omega \to \mathbb{R}$ given by $F(x) = x(0)\, x(\tau)$. For each $t \in \mathbb{R}$, define $F \circ T_t(x) = x(t)\, x(t + \tau)$. As above, this is measurable and integrable for each t.

4. For each $x \in \Omega$ and $T > 0$, set

$$A_T(x) := \frac{1}{2T} \int_{-T}^{T} F(T_t\, x)\ dt = \frac{1}{2T} \int_{-T}^{T} x(t)\ x(t + \tau)\ dt.$$

5. The Birkhoff (Khintchine) ergodic theorem in continuous time for flows of measure-preserving transformations applies under the above conditions. Thus, for μ-almost every $x \in \Omega$,

$$\lim_{T \to \infty} A_T(x) = \mathbb{E}_\mu[F] = r(\tau).$$

This matches the claimed formula. □

Remark 9. The set of $x \in \Omega$ for which the limit in Theorem 8 fails has measure zero under μ. The limit is a measurable function of x. The limit equals the covariance function for each fixed $\tau \in \mathbb{R}$. Almost sure convergence for all τ simultaneously generally holds only for countable subsets of $\mathbb{R}$.

3. Measure-Theoretic and Technical Details

1. The canonical path space $\Omega = \mathbb{R}^{\mathbb{R}}$ with the product σ-algebra supports all coordinate projections and the shift operator. The measure μ is defined such that the law of $(x(t_1), \ldots, x(t_n))$ under μ is that of $(\xi(t_1), \ldots, \xi(t_n))$.

2. The measurability and integrability of $x \mapsto x(0)\, x(\tau)$ follow from the structure of Ω and the moment assumption.

3. The shift flow $(T_h)_{h \in \mathbb{R}}$ acts measurably and preserves μ.

4. The ergodic theorem for flows applies, as all conditions (integrability, invariance, ergodicity) are satisfied.

4. Almost Sure Convergence

Definition 10. *[Almost Surely] Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. A property holds* almost surely *if the set of $x \in \Omega$ for which it fails has μ-measure zero.*

Remark 11. If a sequence of measurable maps $f_n : \Omega \to \mathbb{R}$ converges almost surely to f, this means $\mu(\{x : \lim_{n \to \infty} f_n(x) = f(x)\}) = 1$. Sets of measure zero do not affect expectation or measure-theoretic statements.

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