Inverse Spectral Theory: Part I

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Abstract

Many self-adjoint operators appearing in mathematical physics and geometry have their spectral data: eigenvalues, informations of eigenvectors, scattering matrices. A natural attempt is the reconstruction of the original operator in terms of its spectral data. The precursor of this inverse spectral problem goes back at least to the Sturm–Liouville theory of differential operators. The systematic study of inverse spectral problems has become active from early 20th century, and the interest on this subject is unceasingly growing up since then.

The aim of this article is to give a brief survey of the inverse spectral problem for self-adjoint differential operators: boundary value problems and scattering problems for Schrödinger operators, Laplace–Beltrami operators on Riemannian manifolds. Both of the 1-dimensional and the multi-dimensional problems are discussed. There is so extensive literature on the inverse problem that our arguments must be restricted to limited aspects of the subject. The basic feature I would like to stress is:

One-dimensional spectral problems are smoothly deformable like C^∞-functions, while multi-dimensional problems are rigid like analytic functions (at least in Euclidean spaces).

Table of contents

1 The Inverse Eigenvalue Problem

1.1 Theorem of Borg–Levinson

Consider the simplest case of the Dirichlet (zero) boundary value problen $\forall x \in (0,1)$:

$$V(x)\, y(x) - \ddot{y}(x) = \lambda\, y(x) \tag{1}$$

where $V(x)$ is a real-valued potential, $y(x)$ is the wave function, and λ represents the eigenvalue.

The boundary conditions are given by:

$$y(0) = y(1) = 0 \tag{2}$$

If $V(x)$ is real-valued, the solution has a set of eigenvalues:

$$\lambda_1(V) < \lambda_2(V) < \cdots < \lambda_n(V) < \cdots \tag{3}$$

The first question of the inverse eigenvalue problem is:

Question: If $\lambda_n(V_1) = \lambda_n(V_2)$ for all $n \geq 1$, does V_1 coincide with V_2? The answer is easily seen to be **negative**. You have only to take $V_2(x) = V_1\,(1 - x) \neq V_1(x)$ as a counter example. A potential $V(x)$ is said to be *even* if $V(x) = V\,(1 - x)$. This parity is the only symmetry that the above Dirichlet problem has. The following theorem due to Borg and Levinson is the first uniqueness result up to symmetry.

Theorem 1

(Borg–Levinson) *If $V_1(x)$ and $V_2(x)$ are even then*

$$\lambda_n(V_1) = \lambda_n(V_2) \forall n \geqslant 1 \tag{4}$$

implies that $V_1(x) = V_2(x)$.

If the potential is not even, one needs some auxiliary condition to prove the uniqueness. The second theorem, also due to Borg and Levinson, shows that the values of the derivatives of the eigenfunctions at the boundary serve this purpose.

Let $y(x, \lambda)$ satisfy (1.1) and

$$y(0, \lambda) = 0 \tag{5}$$

$$\dot{y}(0, \lambda) = 1 \tag{6}$$

Put

$$k_n(V) = \dot{y}(1, \lambda_n(V)) \tag{7}$$

Theorem 2

(2nd Borg–Levinson) *If it is true that* $\forall n \geqslant 1$

$$\lambda_n(V_1) = \lambda_n(V_2) \tag{8}$$

and

$$k_n(V_1) = k_n(V_2) \tag{9}$$

then

$$V_1 = V_2 \tag{10}$$

1.2 Global structure of isospectral potentials

The set of potentials having the same eigenvalues is parametrized by $\{k_n(V)\}_{n=1}^{\infty}$, which makes it possible to deform continuously the potential keeping the eigenvalues fixed. Let us formulate it rigorously.

Let $L_R^2(0,1)$ be the set of all real-valued L^2-functions on $(0,1)$ then $\lambda_n(V)$ has an asymptotic expansion

$$\lambda_n(V) = n^2\pi^2 + \int_0^1 V(x)\,dx + h_n(V) \tag{11}$$

which satisifes

$$\sum_{n=1}^{\infty} h_n(V)^2 < \infty \forall V(x) \in L_{\mathbb{R}}^2(0,1) \tag{12}$$

With this in mind, we put

$$\mu_0(V) = \int_0^1 V(x)\,dx \tag{13}$$

and let $\mu(V) = (\mu_0(V), \mu_1(V), \mu_2(V), \ldots)$. Then μ is a map

$$\mu\colon L_R^2(0,1) \to \mathbb{R} \times l^2 := S \tag{14}$$

We also put $\kappa(V) = (\kappa_1(V), \kappa_2(V), \ldots)$, where

$$\kappa_n(V) = \log{(-1)^n k_n(V)} \tag{15}$$

and let l^2 be defined by

$$l^2 \ni \alpha = (\alpha_1, \alpha_2, \ldots) \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \alpha_n^2 < \infty \tag{16}$$

Theorem 3

$\kappa \times \mu$: $L^2_R(0, 1) \rightarrow l^2 \times S$ is a real analytic isomorphism.

Definition 4

*Here a map between real Hilbert spaces is said to be **real analytic** if it is continuously Fréchet differentiable on the complexification of the real Hilbert spaces.*

Theorem 1.3 characterizes the Dirichlet spectral data, the eigenvalues and the derivatives at the boundary of the eigenfunctions. Let

$$M(q) = \{V \in L_R^2(0, 1) \colon \lambda_n(q) = \lambda_n(V) \forall n \geqslant 1, q \in L_R^2(0, 1)\} \tag{17}$$

Theorem 5

$M(q)$ is a real analytic submanifold of $L^2_R(0, 1)$ and κ is a global coordinate system on $M(q)$.

Thus we have the following

Corollary 6

$V(x)$ can be continuously reshaped while keeping its eigenvalues fixed.

Proof. TODO: show how varying $\kappa_1(V)$ can accomplish this. □

1.2.1 Isospectral Deformation

It is well-known that for two bounded operators A and B

$$\sigma(A B) \setminus \{0\} = \sigma(B A) \setminus \{0\} \tag{18}$$

where $\sigma(A)$ denotes the spectrum of A. The proof of (2.1) is elementary for the finite dimensional case, and the infinite dimensional case uses the equation

$$\lambda(A B + \lambda)^{-1} + A(B A + \lambda)^{-1} B = 1 \tag{19}$$

For unbounded operators, (2.1) is also true if A is a densely defined closed operator and $B = A^*$. This is particularly useful in the application to one-dimensional problems. In fact, Crum[11] used this method, by a straightforward computation, to deform and remove eigenvalues of Sturm–Liouville operators.

The **commutation method** has a long history, precursors of which are seen in the works of Jacobi[41] and Darboux[12]. We elucidate it here formally.

Let $V \in L^2_R(0,1)$ and φ_n be a Dirichlet eigenfunction of $-d^2/dx^2 + V(x)$ with eigenvalue λ_n. Ignoring the question of domain of all relevant operators, we put formally

$$A = \varphi_n \frac{d}{dx}\left(\frac{1}{\varphi_n}\right) \tag{20}$$

$$A^* = -\frac{1}{\varphi_n}\frac{d}{dx}(\varphi_n \cdot) \tag{21}$$

Using

$$V\varphi_n - \varphi_n'' = \lambda_n \varphi_n \tag{22}$$

we have

$$A^* A = -\frac{d^2}{dx^2} + V(x) - \lambda_n \tag{23}$$

Therefore we will get $\sigma(A^* A) \setminus \{0\} = \sigma(A A^*) \setminus \{0\}$. However, the potential of $A A^*$ is not in L^2 in neighborhoods of the zeros of $\varphi_n(x)$. This causes troubles.

The remedy comes from the **double commutator**. Namely by taking

$$u = \frac{1}{\varphi_n}\left(a + b\int_0^x \varphi_n(t)^2\, dt\right) \tag{24}$$

and putting

$$B = u\frac{d}{dx}\left(\frac{1}{u}\right) \tag{25}$$

$$B^* = -\frac{1}{u}\frac{d}{dx}(u \cdot) \tag{26}$$

we have

$$A^* A = B^* B \tag{27}$$

Furthermore, we have

$$\frac{B^*}{u} = 0$$

This shows that $V(x)$ and $V(x) - \frac{d^2}{dx^2}\log(u\,\varphi_n)$ have the same Dirichlet eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$.

Let us note that this commutation method does not work in multidimension. We show in Section 4 that there is no analogue of isospectral deformation of $-\Delta + V$ like Theorem 1.4.

1.3 Inverse Scattering

1.3.1 Scattering problem

The concept of scattering experiment is as follows. One puts a target and projects a beam of particles. By observing the scattered particles, one tries to investigate the target. In the case of potential scattering in quantum mechanics, this process is described by the following Schrödinger equation in $\mathbb{R}^3$:

$$(-\Delta + V(x))\,\psi = E\,\psi, \tag{28}$$

where $E > 0$ denotes the energy of scattering particles and $V(x)$ is a real function, which is assumed to be rapidly decreasing.

The equation (3.1) has an infinite number of solutions. However, the solution corresponding to the above scattering process can be chosen uniquely under a suitable boundary condition at infinity, and has the following asymptotic expansion:

$$\psi \sim e^{i\sqrt{E}x\cdot\omega} + \frac{e^{i\sqrt{E}r}}{r}\,f(E;\theta,\omega) \tag{29}$$

as $r = |x| \to \infty$, $\theta = x/r$. Here the first term of the right-hand side of (3.2) represents the incident plane wave having direction $\omega \in S^2$, and the second term represents the scattered spherical wave. $f(E;\theta,\omega)$ is called the scattering amplitude. $|f(E;\theta,\omega)|^2$ is called the differential cross section and denotes the ratio of number of particles reflected to the θ direction to number of incident particles with direction ω. This is the physically observed quantity.

Time-dependent picture visualizes the scattering process more clearly. Let $H = -\Delta + V(x)$. Then the behavior of the particle is described by a solution of the time-dependent Schrödinger equation $i\,\hbar\,\partial_t\,u = H\,u$, namely $u(t) = e^{-iHt/\hbar}\,u$. In the remote past and the remote future, this particle runs very far from the scattering center. Since the potential $V(x)$ decays rapidly, the behavior of the particle is approximately governed by $H_0 = -\Delta$ as $t \to \pm\infty$. More precisely, there exist $u_\pm$ such that $|e^{-itH/\hbar}u - e^{-itH_0 u_\pm}| \to 0$ as $t \to \pm\infty$. This means that the incoming free particle $e^{-itH_0 u_-}$ is scattered into $e^{-itH_0 u_+}$ after the collision. The operator

$$S\colon u_- \mapsto u_+ \tag{30}$$

is called the *scattering operator*. The structure of S is seen more explicitly in...by passing to the Fourier transformation

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = (2\,\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-i\xi\cdot x}\,u(x)\,d\,x \tag{31}$$

In fact, we have

$$\hat{u}_{+}(\sqrt{E}\,\theta) = \hat{u}(\sqrt{E}\,\theta) - \frac{\sqrt{E}}{2\,\pi\,i}\int_{S^2} f(E;\theta,\omega)\,\hat{u}(\sqrt{E}\,\omega)\,d\,\omega \tag{32}$$

where $f(E;\theta,\omega)$ is the scattering amplitude which appeared in (3.2). This means that the operator

$$\hat{S} = \mathcal{F}\,S\,\mathcal{F}^{*} \tag{33}$$

acts only on the variable $\omega \in S^2$. This is physically natural, since $H_0\,\hat{S}\,\hat{S}^{*} = |\xi|^2$ and the energy is conserved during the scattering process.

Let $\hat{S}(E)$ be the integral operator on $L^2(S^2)$:

$$\hat{S}(E)\,\psi(\theta) = \psi(\theta) - \frac{\sqrt{E}}{2\,\pi\,i}\int_{S^2} f(E;\theta,\omega)\,\psi(\omega)\,d\,\omega \tag{34}$$

Then $\hat{S}$ is written as

$$(\hat{S}\,u)\,(\sqrt{E}\,\theta) = (\hat{S}(E)\,u)\,(\sqrt{E}\,\theta) \tag{35}$$

$\hat{S}(E)$ is a unitary operator on $L^2(S^2)$ and is called the S-matrix.

The inverse problem of scattering is now formulated as follows:

Given the scattering amplitude $f(E;\theta,\omega)$, reconstruct the potential $V(x)$.

Physically, the differential cross section is the only observable quantity. However, since $\hat{S}(E)$ is unitary, $f(E;\theta,\omega)$ satisfies an integral equation. Using this equation one can construct $f(E;\theta,\omega)$ from the square of its modulus when $|f(E;\theta,\omega)|$ is sufficiently small. [See Martin (Ref. 52).]

However, when the differential cross section is not small, this is no longer true. [See Newton (Ref. 60).]

1.3.2 Spherically symmetric potentials

When the potential $V(x)$ is spherically symmetric, $V(x) = V(|x|)$, the above problem is reduced to the one on the interval $(0,\infty)$ by the well-known procedure of partial wave expansion.Let $k = \sqrt{E}$ and $r = |x|$. Then the solution ψ of (3.1) depends only on r and the angle between ω and x. Letting θ denote this angle, we expand ψ as

$$\psi = \frac{1}{k\,r}\sum_{l=0}^{\infty}\,(2\,l+1)\,u_l(r,k)\,P_l(\cos\theta) \tag{36}$$

where P_l is the Legendre polynomial. Then by (3.1) and (3.2), $u_l(r,k)$ satisfies

$$-\frac{d^2}{dr^2}u_l(r,k)+\left[V(r)+\frac{l(l+1)}{r^2}\right]u_l(r,k)=k^2u_l(r,k) \tag{37}$$

$$u_l(0,k)=0 \tag{38}$$

$$u_l(r,k)\sim\sin(kr-l\pi/2+\delta_l) \quad \text{as} \quad r\to\infty \tag{39}$$

for some $\delta_l\in\mathbb{R}$. This is called the phase shift and the scattering amplitude is written as

$$f(k,\theta)=\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l\,P_l(\cos\theta) \tag{40}$$

In the following we pick up the case $l=0$.

1.3.3 Gel'fand–Levitan theory

Let us consider the scattering theory on the half line described by the following equation

$$-\frac{d^2}{dx^2}\psi(x,k)+V(x)\psi(x,k)=k^2\psi(x,k), \quad 0<x<\infty \tag{41}$$

$$\psi(0,k)=0, \quad k\in\mathbb{R} \tag{42}$$

If $V(x)$ decays sufficiently rapidly at infinity, the solution $\psi(x,k)$ has the following asymptotics

$$\psi(x,k)\approx C(k)\sin(kx-\delta(k))=\frac{C(k)}{2i}(e^{i\delta(k)}e^{ikx}-e^{-i\delta(k)}e^{-ikx}) \quad \text{as} \quad x\to\infty. \tag{43}$$

The term e^{ikx} represents the outgoing wave and the term e^{-ikx} represents the incoming wave. The correspondence $e^{i\delta(k)}\mapsto e^{-2i\delta(k)}$ then defines the S-matrix. Hence we put

$$S(k)=e^{-2i\delta(k)} \tag{44}$$

The inverse scattering problem is rephrased as

Given $S(k)$, *reconstruct* $V(x)$.

The first complete solution of the one-dimensional inverse problem was given by Gel'fand–Levitan. They reconstructed $V(x)$ from the spectral function, which is the density function in the generalized eigenfunction expansion theory for Sturm–Liouville operators. Krein and Marchenko completed the inverse scattering by showing the passage from $S(k)$ to the spectral function. The completed theory contains a characterization of the scattering matrix and the reconstruction procedure. It is summarized in the following theorem.

Theorem 7

(Gel'fand–Levitan–Krein–Marchenko) *In order that a $\mathbb{C}$-valued function $S(k)$ defined on $\mathbb{R}$ be the scattering matrix of a Schrödinger operator $H = -\frac{d^2}{d\,x^2} + V(x)$ on $(0, \infty)$ with Dirichlet boundary condition at 0 and a real-valued potential $V(x)$ satisfying*

$$\int_0^\infty |x\,V(x)|\,d\,x < \infty \tag{45}$$

it is necessary and sufficient that

- $S(k) \in C(\mathbb{R}), |S(k)| = 1, S(k) = S\,(-k)$
- $S(k) - 1 = \frac{e^{-i\,k\,t}}{2\,\pi\,i} \int_{-\infty}^{\infty} (F_1(t) + F_2(t))\,d\,t$

  where

  $F_1(t) \in L^1(\mathbb{R}), F_2(t) \in L^2(\mathbb{R})$ *and* $L^\infty(\mathbb{R})$
- $\arg S(0+) - \arg S(\infty) + \frac{\pi}{2}\,(S(0) - 1) = 2\,\pi\,m s$

where m is a non-negative integer.

If $m = 0$, the potential $V(x)$ is uniquely determined. If $m > 0$, there exists an m-parameter family of potentials having the same $S(k)$ as the S-matrix.

In the above theorem, m is the number of discrete eigenvalues of $-d^2/d\,x^2 + V(x)$. The deformation of the potential keeping the S-matrix fixed is carried out by the method of commutation in Section 2.

For the proof, see e.g., Faddeev (Ref. 20), or Marchenko (Ref. 51).

1.4 Generalized sine transformation

To study the spectral structure of H and the properties of S-matrix, the generalized sine transformation plays a key role. Let us recall that for a self-adjoint operator A, the absolutely continuous subspace for A, $\mathcal{A}_{ac}(A)$, is the set of all u such that $(E(\lambda)\,u, u)$ is absolutely continuous with respect to $d\,\lambda$, where

$$A = \int_{-\infty}^{\infty} \lambda\,d\,E(\lambda) \tag{46}$$

Now let $\psi(x, k)$ be the solution of the equation

$$-\psi''(x, k) + V(x)\,\psi(x, k) = k^2\,\psi(x, k) \forall x > 0 \tag{47}$$

with

$$\psi(0,k)=0, \quad \psi'(0,k)=1 \tag{48}$$

where $'=d/dx$. Then $\psi(x,k)$ behaves like

$$\psi(x,k)=\frac{A(k)}{k}\sin(kx-\delta(k))+o(1) \tag{49}$$

as $x\to\infty$. We define

$$\mathcal{F}u(k)=\int_0^\infty u(x)\,\psi^+(x,k)\,dx \tag{50}$$

$$\psi^+(x,k)=\frac{\psi(x,k)}{M(k)}, \quad M(k)=A(k)\,e^{i\delta(k)} \tag{51}$$

Then $\mathcal{F}$ is a unitary from $\mathcal{A}_{ac}(H)$ to $L^2((0,\infty);2\,k^2/\pi\,dk)$, and we have the inversion formula for $u\in\mathcal{A}_{ac}(H)$

$$u=\frac{2}{\pi}\int_0^\infty \psi^+(x,k)\,\mathcal{F}u(k)\,k^2\,dk \tag{52}$$

Moreover, $\mathcal{F}$ diagonalizes H:

$$(\mathcal{F}Hu)(k)=k^2\,(\mathcal{F}u)(k) \tag{53}$$

When $V(x)=0$, $\mathcal{F}$ reduces to the sine transformation

$$\mathcal{F}_0\,u(k)=\int_0^\infty u(x)\sin(kx)\frac{dx}{k} \tag{54}$$

1.5 The core of Gel'fand–Levitan theory

Let us explain the essence of Gel'fand–Levitan theory. Let $\psi(x,k)$ be as in (3.17) and (3.18). Then $\psi(x,k)$ is an even and entire function of $k\in\mathbb{C}$ satisfying

$$\psi(x,k)=\frac{\sin kx}{k}+o\left(\frac{e^{|\mathrm{Im}k|x}}{|k|}\right)\mathrm{as}|k|\to\infty \tag{55}$$

Here we recall the Paley–Wiener theorem. An entire function $F(z)$ is said to be of exponential type σ if for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$|F(z)|\le C_\varepsilon\,e^{(\sigma+\varepsilon)|z|}\forall z\in\mathbb{C} \tag{56}$$

Theorem 8

(Paley–Wiener) *$F(x)$ in $L^2(\mathbb{R})$ is extended to an entire function of exponential type σ if and only if there exists $h \in L^2(-\sigma, \sigma)$ such that*

$$F(z) = \int_{-\sigma}^{\sigma} h(\xi)\, e^{iz\xi}\, d\xi \tag{57}$$

By virtue of Paley–Wiener theorem and (3.25), $\psi(x,k)$ has the following representation

$$\psi(x,k) = \frac{\sin k x}{k} + \int_0^\infty K(x,y) \frac{\sin k y}{k}\, dy \tag{58}$$

We insert this expression to the equation (3.17). Then K is shown to satisfy the equation

$$(\partial_y^2 - \partial_x^2 + V(x))\, K(x,y) = 0 \tag{59}$$

The crucial fact is

$$\frac{d}{dx} K(x,x) = V(x) \tag{60}$$

One can further derive the following equation

$$K(x,y) + \Omega(x,y) + \int_0^\infty K(x,t)\, \Omega(t,y)\, dt = 0 \forall \quad x > y \tag{61}$$

where $\Omega(x,y)$ is a function constructed from the S-matrix and informations of bound states. This is called the Gel'fand–Levitan equation.

Thus the scenario of the reconstruction of $V(x)$ is as follows. From the scattering matrix and the bound states, one constructs $\Omega(x,y)$. Solving (3.29) one gets $K(x,y)$. The potential $V(x)$ is obtained by (3.28).

1.6 What is the hidden mechanism?

This is truly an ingenious trick and it is not easy to find the key fact behind their theory. It is Kay and Moses who studied an algebraic aspect of the Gel'fand–Levitan method.

Let H_0 and H be two self-adjoint operators. A unitary operator U from $\mathcal{A}_{ac}(H_0)$ to $\mathcal{A}_{ac}(H)$ is said to *intertwine* H_0 and H if it satisfies

$$H\, U = U\, H_0 \tag{62}$$

An important example of intertwining operator for $H_0 = -\Delta$ and $H = -\Delta + V(x)$ is the wave operator

$$W_\pm = s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0} \tag{63}$$

Another important example of intertwining operator is the so called spectral representation. Let A be self-adjoint and I be its absolutely continuous spectrum, $I = \sigma(\mathcal{A}_{ac}(A))$. A unitary operator T from $\mathcal{A}_{ac}(A)$ to $L^2(I; h)$, h being an auxiliary Hilbert space, is called a spectral representation of A if

$$(T\,A\,u)(\lambda) = \lambda\,(T u)(\lambda)\,\forall\lambda \in I, u \in \mathcal{A}_{ac}(A) \tag{64}$$

For $H_0 = -d^2/d\,x^2$ and $H = -d^2/d\,x^2 + V(x)$ on $(0, \infty)$, the above (generalized) sine transformations $\mathcal{F}_0$ and $\mathcal{F}$ are spectral representations. It is obvious that $U = \mathcal{F}^*\,\mathcal{F}_0$ intertwines H_0 and H.

It is also evident that if U intertwines $H_0 = -d^2/d\,x^2$ and $H = -d^2/d\,x^2 + V(x)$, then its kernel $U\,(x, y)$ satisfies

$$(-\partial_y^2 + V(x))\,U\,(x, y) = -\partial_y^2\,U\,(x, y) \tag{65}$$

The spectral representation for the one-dimensional Schrödinger operator H has the following distinguished property. Let $\psi^{(+)}(x, k)$ be as in (3.21). Let

$$U\,(x, y) = \int_0^\infty \psi^{(+)}(x, k)\sin k\,y\frac{2\,k^2}{\pi}\,d\,k, \qquad \rho(k) = \frac{2\,k^2}{\pi} \tag{66}$$

be the integral kernel of the intertwining operator $\mathcal{F}^*\,\mathcal{F}_0$. In view of the formula (3.26), we have

$$U\,(x, y) = \delta\,(x - y) + \eta\,(x - y)\,K(x, y) \tag{67}$$

where $\eta(t)$ is the Heaviside function. Namely, U is an integral operator of Volterra type. The key fact discovered by Kay–Moses is the following theorem.

Theorem 9

Let $H_0 = -\frac{d^2}{d\,x^2}$ be defined on $(0, \infty)$ with Dirichlet boundary condition, and let $H = H_0 + Q$ be a self-adjoint perturbation of H_0. Suppose $U = I + K$ intertwines H_0 and H and that U is Volterra, i.e. $K(x, y) = 0$ if $x < y$. Then Q is an operator of multiplication by

$$q(x) = 2\,\frac{d}{d\,x}\,K(x, x) \tag{68}$$

For $x > y$, the following equation holds

$$(\partial_y^2 - \partial_x^2)\,K(x, y) = q(x)\,K(x, y) \tag{69}$$

Proof. Let us content ourselves by the formal proof, since we are mainly interested in the algebraic or operator theoretical aspect of the Gel'fand–Levitan theory.

Since $U(x,y)=\delta(x-y)+\eta(x-y)\,K(x,y)$ satisfies the wave equation

$$(\partial_y^2-\partial_x^2)\,U(x,y)=-\int_0^\infty Q(x,z)\,U(z,y)\,dz \tag{70}$$

we have

$$\int_0^\infty Q(x,z)\,U(z,y)\,dz=2\,\delta(x-y)\,\frac{d}{dx}K(x,x)+\eta(x-y)\,q(x)\,C(x,y) \tag{71}$$

Since U is Volterra, so is U^{-1}. Hence by multiplying U^{-1} to the above equation we have

$$Q(x,y)=2\,\delta(x-y)\,\frac{d}{dx}K(x,x)+\eta(x-y)\,C(x,y) \tag{72}$$

However, since Q is self-adjoint, $C(x,y)=0$, which proves

$$Q(x,y)=2\,\delta(x-y)\,\frac{d}{dx}K(x,x)\quad\square \tag{73}$$

Let us summarize the above arguments. In the one-dimensional case, the generalized eigenfunction $\psi(x,k)$ of the Schrödinger operator $H=-d^2/dx^2+V(x)$ has the triangular expression (3.26). This makes the intertwining operator $\mathcal{F}^*\mathcal{F}_0$ into Volterra type. The potential $V(x)$ is reconstructed from the kernel of this Volterra operator.

An excellent exposition of the one-dimensional inverse scattering is the one given by Faddeev. A historical survey by Newton in the foreword of the monograph of Chadan–Sabatier contains an extensive literature up to 1977.