A Positive Definite Modulated Negative Exponential Kernel

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1 Basis for the Modified Riemann-Siegel Theta Function

The focus is on establishing a basis for the modified Riemann-Siegel theta function $\theta^*(t)$, ensuring it retains the desired properties of positive definiteness and monotonicity.

1.1 Key Properties

1. **Basis Construction**: We establish an orthonormal basis for the modified theta function $\theta^*(t)$, ensuring that it maintains the desired properties.

2. **Positive Definiteness**: The modification of $\theta(t)$ to $\theta^*(t)$ through reflection maintains positive definiteness, enabling us to derive a suitable kernel.

3. **Kernel Properties**: The modified theta function can be expressed in terms of this basis, allowing for the study of its properties and behavior.

1.2 Counting Function and Expectation

The expectation of the counting function is given by:

$$\mathbb{E}[N(T)] = \frac{\theta(T)}{\pi} + 1 \tag{1}$$

The full counting function includes the argument term:

$$N(T) = \frac{\theta(T)}{\pi} + \frac{S(T)}{\pi} + 1 \tag{2}$$

where $S(T) = \arg\zeta\left(\frac{1}{2} + iT\right)$ can be expressed as:

$$S(T) = \frac{\ln\zeta\left(\frac{1}{2} + iT\right) - \ln\overline{\zeta\left(\frac{1}{2} + iT\right)}}{2i} \tag{3}$$

For the associated kernel function:

$$K(t, s) = e^{-\frac{(\theta^*(t) - \theta^*(s))^2}{2}} \tag{4}$$

1.3 Implementation Steps

1. Establish the complete orthonormal basis for $\theta^*(t)$
2. Verify positive definiteness of the constructed kernel
3. Prove monotonicity properties
4. Develop numerical methods for computation

2 Normalization and Fourier Analysis of Non-Stationary Processes

2.1 Zero-Crossing Rate Normalization

Consider a process with zero crossing rates that increase as $|t| \to \infty$. Through normalization:

- Original process has increasing crossing rates
- Normalize to maintain unit rate across whole domain
- This enables Fourier transform analysis

2.2 Process Transformation

For a process with increasing zero crossing rates as $|t| \to \infty$:

The normalization procedure:

1. Apply normalization for unit zero crossing rate
2. Results in stationary process (constant crossing rate)
3. Enables Fourier transform analysis

Key insights:

- Original: non-stationary (increasing crossings)
- Post-normalization: stationary (unit crossing rate)
- Makes Fourier transform well-defined
- Spectral process exists for normalized version

The Fourier transform of the normalized process:

$$\tilde{Y}(\omega) = \int_{-\infty}^{\infty} Y_{normalized}(t)\, e^{-i\omega t}\, d\, t \tag{5}$$

exists because normalization creates a well-behaved stationary process with constant crossing rate properties.

2.3 Wigner-Ville Connection

The Wigner-Ville distribution:

$$W(t,\omega) = \int_{-\infty}^{\infty} x\left(t + \frac{\tau}{2}\right) x^*\left(t - \frac{\tau}{2}\right) e^{-i\omega\tau}\, d\,\tau \tag{6}$$

captures:

- Process that's "almost" stationary
- Triangular/bilinear structure
- Non-stationarity enters in controlled way
- Maintains certain symmetries despite non-stationarity