The Hardy Z-Function as a Deterministic Oscillatory Weakly Harmonizable Process

Abstract

The Hardy Z-function admits an exact deterministic oscillatory weakly harmonizable representation. The spectral bimeasure has finite Fréchet variation and infinite Vitali variation, and the Morse-Transue integral reduces to a conditionally convergent sum.

1 Standard Definitions of Harmonizable Processes

1.1 Harmonizable Processes and Spectral Bimeasures

Definition 1 (Harmonizable Process [1, 2]). *A stochastic process $\{X(t) : t \in \mathbb{R}\}$ is harmonizable if there exists a complex-valued vector measure $\Phi : \mathcal{B}(\mathbb{R}) \to \mathcal{H}$ (taking values in a Hilbert space $\mathcal{H}$) such that*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, d\Phi(\lambda).$$

The spectral bimeasure is defined by $F(A, B) = \langle \Phi(A), \Phi(B)\rangle_{\mathcal{H}}$ for Borel sets $A, B \subset \mathbb{R}$.

1.2 Oscillatory Harmonizable Processes

Definition 2 (Oscillatory Harmonizable Process [4, 1]). *An oscillatory harmonizable process is a process of the form*

$$X(t) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t}\, d\Phi(\lambda),$$

where Φ is as in Definition 2.1 and $A_t(\lambda)$ is a deterministic time-varying gain function satisfying $A_t(\cdot) \in L^2(F)$ for each t.

1.3 Weak vs. Strong Harmonizability

Definition 3 (Variation Types [3, 2]). *For a bimeasure F:*

- *The Vitali variation is $|F|_{Vitali} = \sup \sum_{i,j} |F(A_i, A_j)|$ over finite partitions.*
- *The Fréchet variation is $|F|_{Fréchet} = \sup \left| \sum_{i,j} a_i \bar{a}_j F(A_i, A_j) \right|$ for $|a_i| \le 1$.*

A process is strongly harmonizable if $|F|_{Vitali} < \infty$ and weakly harmonizable if $|F|_{Fréchet} < \infty$.

1.4 Morse-Transue Integral

Definition 4 (Morse-Transue Integral [3, 2]). *Let F be a bimeasure. For functions $f, g : \mathbb{R} \to \mathbb{C}$, the MT integral exists if there exist simple functions $f_n \to f$, $g_n \to g$ such that the iterated integrals converge:*

$$\lim_{n\to\infty} \int \left| \int f_n(\lambda) F(d\lambda, \cdot) - \int f(\lambda) F(d\lambda, \cdot) \right| = 0,$$

with a symmetric condition for g. The integral is $\lim_{n\to\infty} \iint f_n(\lambda) g_n(\mu) F(d\lambda, d\mu)$.

For discrete bimeasures $F(A, B) = \sum_{(m,n)\in I(A)\times I(B)} w_{mn}$, the MT integral reduces to

$$\iint f(\lambda) g(\mu) F(d\lambda, d\mu) = \sum_{m,n=1}^{\infty} w_{mn} f(\lambda_m) g(\mu_n),$$

provided the series converges conditionally via Dirichlet's test.

2 Construction of the Hardy Z Representation

2.1 Spectral Gain and Spectral Measure

Define the time-varying gain function $A_t : \mathbb{R} \to \mathbb{C}$ as a spectral Shah (Dirac comb) function:

$$A_t(\lambda) = c_+(t) \sum_{n=1}^{\infty} \delta(\lambda + \log n) + c_-(t) \sum_{n=1}^{\infty} \delta(\lambda - \log n),$$

with scalar gains

$$c_+(t) = \frac{e^{i\theta(t)}}{2(1 - 2^{1/2-it})}, \qquad c_-(t) = \frac{e^{-i\theta(t)}}{2(1 - 2^{1/2+it})}.$$

Define the deterministic orthogonal spectral measure $\Phi : \mathcal{B}(\mathbb{R}) \to \ell^2(\mathbb{N})$ by

$$\Phi(E) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \left[\mathbf{1}_E(\log n) + \mathbf{1}_E(-\log n)\right] \mathbf{e}_n,$$

where $\{\mathbf{e}_n\}_{n\in\mathbb{N}}$ is the standard orthonormal basis of $\ell^2(\mathbb{N})$.

2.2 Spectral Bimeasure

The induced spectral bimeasure is

$$F(A, B) = \langle \Phi(A), \Phi(B)\rangle_{\ell^2} = \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{\sqrt{mn}} \left[\mathbf{1}_A(\log m)\mathbf{1}_B(\log n) + \mathbf{1}_A(-\log m)\mathbf{1}_B(-\log n)\right].$$

3 Main Theorem and Proof

Lemma 1 (Hardy-Z Identity)**.** *For all $t \in \mathbb{R}$,*

$$Z(t) = \mathrm{Re}\Big(e^{i\theta(t)}\zeta(1/2 + it)\Big) = \frac{e^{i\theta(t)}\eta(1/2 + it)}{2(1 - 2^{1/2-it})} + \frac{e^{-i\theta(t)}\eta(1/2 - it)}{2(1 - 2^{1/2+it})}.$$

Theorem 1 (Exact Oscillatory Weakly Harmonizable Representation)**.** *For all real t, define the $\ell^2(\mathbb{N})$-valued process*

$$Y(t) := \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t}\, d\Phi(\lambda),$$

and define the associated scalar process

$$X(t) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \Big[c_-(t) n^{it} + c_+(t) n^{-it}\Big].$$

Then $X(t) = Z(t)$ for all real t.

Proof. By the atomic structure of Φ and the sifting property of the Dirac delta, we evaluate the integral directly:

$$\begin{aligned}
\int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t}\, d\Phi(\lambda) &= \int_{\mathbb{R}} \Big[c_+(t) \sum_{m=1}^{\infty} \delta(\lambda + \log m) + c_-(t) \sum_{m=1}^{\infty} \delta(\lambda - \log m)\Big] e^{i\lambda t}\, d\Phi(\lambda) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \Big[c_+(t) e^{-it\log n} + c_-(t) e^{it\log n}\Big] \mathbf{e}_n \\
&= c_-(t) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} n^{it} \mathbf{e}_n + c_+(t) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} n^{-it} \mathbf{e}_n.
\end{aligned}$$

Therefore

$$Y(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \Big[c_-(t) n^{it} + c_+(t) n^{-it}\Big] \mathbf{e}_n.$$

Define the scalar process $X(t)$ as the conditionally convergent Dirichlet series:

$$X(t) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \Big[c_-(t) n^{it} + c_+(t) n^{-it}\Big] = c_-(t)\eta(1/2 - it) + c_+(t)\eta(1/2 + it),$$

where convergence follows from Dirichlet's test and the value agrees with the defining Dirichlet series $\eta(s) = \sum_{n\geq 1} (-1)^{n-1} n^{-s}$ at $s = 1/2 \mp it$.

By Lemma 1, $X(t) = Z(t)$ for all real t. □

4 Variation and Regularity Analysis

4.1 Variation Analysis

Lemma 2. *For the spectral bimeasure F, we have $|F|_{Vitali} = \infty$ and $|F|_{Fréchet} < \infty$.*

Proof. Vitali variation: Consider the partition $A_i = \{\log i\}$, $B_j = \{\log j\}$. Since $|F(A_i, B_j)| = \frac{1}{\sqrt{ij}}$, we have

$$|F|_{\text{Vitali}} \geq \sum_{i,j=1}^{\infty} \frac{1}{\sqrt{ij}} = \left(\sum_{n=1}^{\infty} n^{-1/2} \right)^2 = \infty.$$

Fréchet variation: For any sequence $|a_n| \leq 1$, let $S_N = \sum_{k=1}^{N} (-1)^{k-1} a_k$. By Abel summation,

$$\sum_{n=1}^{N} \frac{(-1)^{n-1} a_n}{\sqrt{n}} = \frac{S_N}{\sqrt{N}} + \sum_{n=1}^{N-1} S_n \left(n^{-1/2} - (n+1)^{-1/2} \right).$$

Since $|S_n| \leq 1$ and $\sum_{n=1}^{\infty} (n^{-1/2} - (n+1)^{-1/2}) = 1$, we have

$$\sup_{\{a_n\}} \left| \sum_{n=1}^{\infty} \frac{(-1)^{n-1} a_n}{\sqrt{n}} \right|^2 \leq 4 < \infty.$$

Thus $|F|_{\text{Fréchet}} < \infty$. $\square$

4.2 Gain Regularity

Lemma 3. *For each t, the gain A_t satisfies $A_t \in L^2(F)$ with*

$$\|A_t\|_{L^2(F)}^2 = (|c_+(t)|^2 + |c_-(t)|^2) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} < \infty.$$

Proof. The MT-integral defining $\|A_t\|_{L^2(F)}^2 = \int |A_t(\lambda)|^2 F(d\lambda, d\lambda)$ reduces to:

$$\int |A_t(\lambda)|^2 F(d\lambda, d\lambda) = (|c_+(t)|^2 + |c_-(t)|^2) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.$$

Since $c_\pm(t)$ are uniformly bounded and the alternating harmonic series converges, $\|A_t\|_{L^2(F)}^2 < \infty$. $\square$

5 Conclusion

The Hardy Z-function is exactly represented as a deterministic oscillatory weakly harmonizable process. The spectral bimeasure has bounded Fréchet variation and infinite Vitali variation, and the Morse-Transue integral reduces to a conditionally convergent Dirichlet series. This construction uses a Shah gain in the spectral variable and an atomic spectral measure, providing a mathematically rigorous representation.

References

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