

# The Operational Matrix of the Random Wave Process

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## Abstract

An expression for the convolution of a pair of spherical Bessel functions is determined.

## 1 Lemmas

### Lemma 1

**(Terminating Hypergeometric Series)** For any  $p \in \mathbb{Z}_{\geq 0}$ , the Gauss hypergeometric function terminates:

$${}_2F_1(-p, b; c; z) = \sum_{k=0}^p \frac{(-p)_k (b)_k}{(c)_k k!} z^k \quad (1)$$

where  $(a)_k = \prod_{i=0}^{k-1} (a + i)$

**Proof.** By definition, the Gauss hypergeometric series is:

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k \quad (2)$$

Setting  $a = -p$  with  $p \in \mathbb{Z}_{\geq 0}$ , the Pochhammer symbol  $(-p)_k$  becomes zero for all  $k > p$ . Explicitly:

$$(-p)_k = \prod_{i=0}^{k-1} (-p + i) = \begin{cases} (-p)(-p+1) \cdots (-p+k-1), & k \leq p \\ 0 & k > p \end{cases} \quad (3)$$

Thus, the series terminates at  $k = p$ , yielding:

$${}_2F_1(-p, b; c; z) = \sum_{k=0}^p \frac{(-p)_k (b)_k}{(c)_k k!} z^k \quad (4) \quad \square$$

### Lemma 2

**(Integral with Incomplete Gamma Function)** For  $j \geq 0$ ,

$$\int_{-1}^1 \left( \frac{1-x}{2} \right)^j e^{ixy} dx = \frac{e^{iy}}{2^j} \frac{\gamma(j+1, 2iy)}{(iy)^{j+1}} \quad (5)$$

where  $\gamma(s, x)$  denotes the lower incomplete gamma function.

**Proof.** Substitute  $t = \frac{1-x}{2} \Rightarrow x = 1 - 2t$ ,  $dx = -2dt$ , adjusting limits:

$$\int_1^0 t^j e^{i(1-2t)y} (-2dt) = 2e^{iy} \int_0^1 t^j e^{-2iyt} dt \quad (6)$$

Let  $u = 2iyt \Rightarrow t = \frac{u}{2iy}$ ,  $dt = \frac{du}{2iy}$ :

$$\frac{2e^{iy}}{(2iy)^{j+1}} \int_0^{2iy} u^j e^{-u} du = \frac{e^{iy}}{2^j} \frac{\gamma(j+1, 2iy)}{(iy)^{j+1}} \quad \square$$

### Lemma 3

**(Legendre Polynomial Representation)** The hypergeometric function equals the Legendre polynomial  $P_m(x)$ . The Legendre polynomials are hypergeometric functions

$$P_m(x) = {}_2F_1(-m, m+1; 1; \frac{1-x}{2}) \quad (7)$$

**Proof.** From the Rodrigues formula  $P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m} (x^2 - 1)^m$  expand using the binomial theorem:

$$(x^2 - 1)^m = \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} x^{2k} \quad (8)$$

Differentiating  $m$  times yields terms proportional to  $x^k$ , matching the hypergeometric series:

$$P_m(x) = {}_2F_1\left(-m, m+1; 1; \frac{1-x}{2}\right) \quad \square$$

## 2 Main Theorem

### Theorem 4

#### (Fourier Transform of Legendre Polynomial Products)

$$\begin{aligned}
I_{m,n}(y) &= \int_{-1}^1 P_m(x) P_n(x) e^{ixy} dx \\
&= e^{iy} \sum_{j=0}^{m+n} \frac{\Psi_j(m,n)}{2^j} \left[ \frac{\gamma(j+1, 2iy)}{(iy)^{j+1}} \right] \\
&= e^{iy} \sum_{j=0}^{m+n} \frac{\frac{{}_4F_3\left(\begin{smallmatrix} -m, m+1, -n, n+1 \\ 1, 1, j+1 \end{smallmatrix}; 1\right)}{j!}}{2^j} \left[ \frac{\gamma(j+1, 2iy)}{(iy)^{j+1}} \right] \\
&= e^{iy} \sum_{j=0}^{m+n} \frac{{}_4F_3\left(\begin{smallmatrix} -m, m+1, -n, n+1 \\ 1, 1, j+1 \end{smallmatrix}; 1\right)}{j! 2^j} \frac{\gamma(j+1, 2iy)}{(iy)^{j+1}} \\
&= e^{iy} \sum_{j=0}^{m+n} \frac{\gamma(j+1, 2iy) {}_4F_3\left(\begin{smallmatrix} -m, m+1, -n, n+1 \\ 1, 1, j+1 \end{smallmatrix}; 1\right)}{j! 2^j (iy)^{j+1}}
\end{aligned} \tag{9}$$

where

$$\Psi_j(m,n) = \frac{{}_4F_3\left(\begin{smallmatrix} -m, m+1, -n, n+1 \\ 1, 1, j+1 \end{smallmatrix}; 1\right)}{j!} \tag{10}$$

**Proof.**

#### Part 1: Integral Reduction

Expand  $P_m(x) P_n(x)$  using Lemma 1:

$$P_m(x) P_n(x) = \sum_{k=0}^m \sum_{\ell=0}^n \frac{(-m)_k (m+1)_k (-n)_\ell (n+1)_\ell}{(1)_k (1)_\ell k! \ell!} \left(\frac{1-x}{2}\right)^{k+\ell} \tag{11}$$

Let  $j = k + \ell$ , valid for  $0 \leq k \leq m$ ,  $0 \leq \ell \leq n$ . Then:

$$I_{m,n}(y) = \sum_{j=0}^{m+n} \underbrace{\sum_{k=\max(0, j-n)}^{\min(j, m)} \frac{(-m)_k (m+1)_k (-n)_{j-k} (n+1)_{j-k}}{(1)_k (1)_{j-k} k! (j-k)!}}_{\Psi_j(m,n)} \int_{-1}^1 \left(\frac{1-x}{2}\right)^j e^{ixy} dx \tag{12}$$

Apply Lemma 2 to obtain the result.

**Part 2:  $\Psi_j(m, n)$  as a  ${}_4F_3$  Function**

Expand the  ${}_4F_3$  series:

$${}_4F_3\left(\begin{matrix} -m, m+1, -n, n+1 \\ 1, 1, j+1 \end{matrix}; 1\right) = \sum_{k=0}^j \frac{(-m)_k (m+1)_k (-n)_k (n+1)_k}{(1)_k (1)_k (j+1)_k k!} \quad (13)$$

The main result follows. □