# Infinite-Dimensional Stochastic Transforms and Reproducing Kernel Hilbert Spaces

## Overview

The paper by Jorgensen, Song, and Tian presents a comprehensive theoretical framework for constructing infinite-dimensional stochastic transforms at the intersection of Gaussian fields and reproducing kernel Hilbert spaces (RKHS). The authors develop a new infinite-dimensional Fourier transform in the general setting of Gaussian processes, serving to unify existing tools from infinite-dimensional analysis [1] [2].

## Main Theoretical Contributions

## General Transform for RKHS

The authors address a fundamental limitation of RKHS theory: while RKHS constitute versatile tools for applications in statistical inference, machine learning, PDEs, harmonic analysis, and stochastic processes, their realization requires an abstract completion step [1]. To overcome this, they present a general transform that applies universally to RKHS but provides more concrete presentations.

### Effective Sequences Framework

The cornerstone of their approach is the concept of "effective sequences" of orthogonal projections $(P_n)$ in a Hilbert space $\mathcal{H}$. For a sequence of projections, they define operators:

- $T_n = (1 - P_n)(1 - P_{n-1}) \cdots (1 - P_0)$
- $Q_n = P_n(1 - P_{n-1}) \cdots (1 - P_0)$

**Theorem 2.1** establishes the fundamental decomposition:
$\|x\|^2 = \|T_n x\|^2 + \sum_{k=0}^{n} \|Q_k x\|^2$

The system is called "effective" when $T_n \to 0$ strongly, which occurs if and only if:
$I = \sum_{j \in \mathbb{N}_0} Q_j^* Q_j$

This yields a generalized Parseval identity: $\|x\|^2 = \sum_{j \in \mathbb{N}_0} \|Q_j x\|^2$ [1].

## Stochastic Analysis and Gaussian Process Construction

The authors establish a connection between effective sequences and Gaussian processes. **Lemma 2.3** shows that for i.i.d. $N(0,1)$ random variables $\{Z_n\}$ and an effective system $\{Q_n\}$, the process:
$W(\cdot) = \sum_{n \in \mathbb{N}_0} Q_n Z_n(\cdot)$
defines an operator-valued Gaussian process with covariance $E(\langle W(\cdot)u, W(\cdot)v \rangle) = \langle u, v \rangle$ [1].

This construction bridges the gap between abstract RKHS theory and concrete probabilistic representations, providing a unified framework for analyzing positive definite kernels and their associated Gaussian processes[3] [4].

## The Isomorphism $T_K : L^2(K) \to \mathcal{H}(K)$

A central technical achievement is the construction of an isometric isomorphism between spaces of linear functionals and RKHS. The authors define:

**Definition 2.8**: $L^2(K)$ consists of continuous linear functionals on $\mathcal{H}(K)$
**Definition 2.9**: $M^2(K)$ consists of signed measures $\mu$ where $\iint \mu(ds)K(s,t)\mu(dt) < \infty$

The transform $T_K$ is defined as:

- For measures: $(T_K\mu)(t) = \int \mu(ds)K(s,t)$
- For functionals: $T_K(l) = F_l$ where $l(G) = \langle F_l, G \rangle_{\mathcal{H}(K)}$

**Theorem 2.25** establishes that $T_K$ maps $L^2(K)$ isometrically onto $\mathcal{H}(K)$, providing a complete characterization of RKHS in terms of more concrete functional spaces[1].

## Infinite-Dimensional Fourier Transform

## Construction and Properties

The authors develop an infinite-dimensional Fourier transform for Gaussian processes using the kernel $K(s,t) = e^{-\frac{1}{2}|s-t|}$, which serves as the covariance kernel for the complex process $e^{iX_t}$ where $X_t$ is standard Brownian motion[1].

**Definition 3.3** introduces the transform:
$T : L^2(\mathbb{P}) \to \mathcal{H}(K), \quad T(F)(t) = E[e^{-iX_t}F]$

This transform exhibits fundamental properties analogous to classical Fourier transforms:

- $T(e^{iX_t}) = K_t$ where $K_t(\cdot) = e^{-\frac{1}{2}|t-\cdot|}$
- $T^*(K_t) = e^{iX_t}$

**Corollary 3.5** establishes that $T$ is an isometric isomorphism from the complex Hilbert space $L^2(\mathbb{P})$ onto the real Hilbert space $\mathcal{H}(K)$[1].

## Extensions and Applications

The framework extends to Gaussian kernels $K_{Gauss}(x,y) = e^{-\frac{1}{2t}(x-y)^2}$, with the relationship:
$e^{-x^2/2t} = E[e^{ixX_{1/t}}]$
$e^{-(x-y)^2/2t} = E[e^{ixX_{1/t}}e^{-iyX_{1/t}}]$

For Hermite polynomial expansions, the authors derive:
$T(X_s^n)(t) = i^n e^{-t/2} s^{n/2} H_n(\sqrt{s})$
where $H_n$ are Hermite polynomials[1].

## Theoretical Significance

### Unification of Existing Tools

The paper unifies disparate approaches in infinite-dimensional analysis by establishing explicit connections between:

- Positive definite kernels and Gaussian processes via Kolmogorov's theorem [5]
- RKHS and concrete function spaces through isometric transforms [6] [7]
- Frame theory and stochastic analysis through effective sequences [8] [9]

### Applications to Machine Learning and Data Analysis

The framework provides theoretical foundations for:

- Kernel methods in machine learning [7] [10]
- Principal Component Analysis in infinite dimensions [11] [12]
- Sampling theory and approximation algorithms [1]
- Kaczmarz algorithms and optimization [13] [14]

### Connection to Broader Mathematical Frameworks

The work connects to several fundamental areas:

- Spectral theory of operators in Hilbert spaces [15]
- Stochastic calculus and infinite-dimensional analysis [16] [17]
- Harmonic analysis and boundary value problems [18] [19]
- Optimal transport and probability theory [20] [21]

### Mathematical Rigor and Innovation

The paper demonstrates mathematical rigor through:

- Precise characterization of effective sequences and their convergence properties
- Detailed construction of isometric isomorphisms between abstract and concrete spaces
- Extension of classical Fourier analysis to infinite-dimensional stochastic settings
- Integration of measure theory, functional analysis, and probability theory

The theoretical framework established provides a foundation for future research in kernel methods, stochastic analysis, and machine learning applications, offering both theoretical insights and practical computational tools for infinite-dimensional problems.

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1. https://link.springer.com/10.1007/s43670-023-00051-z
2. https://arxiv.org/pdf/2209.03801.pdf

3. https://www.opuscula.agh.edu.pl/vol39/4/art/opuscula_math_3930.pdf
4. https://www.semanticscholar.org/paper/e98fe2517254a7521df27479789b8afa8617fcf4
5. https://projecteuclid.org/euclid.aot/1512416211
6. https://www.semanticscholar.org/paper/24307a820fc0616f485f0aa65bc1171a5ac83583
7. https://www.semanticscholar.org/paper/71de5ab144545b697ccae97bc0a9dc67cb392cb2
8. https://www.worldscientific.com/doi/abs/10.1142/S0219691313500367
9. http://arxiv.org/pdf/1409.8310.pdf
10. https://arxiv.org/pdf/1807.02582.pdf
11. https://www.semanticscholar.org/paper/01b53923be220e621879086ef4d2f45d3d605f98
12. https://www.semanticscholar.org/paper/cd911dab4190bdc3adffe8bfc89c0fdfc3305ba3
13. https://link.springer.com/10.1007/s43036-020-00079-1
14. https://epubs.siam.org/doi/10.1137/21M1463306
15. http://arxiv.org/pdf/2203.17206.pdf
16. https://www.worldscientific.com/worldscibooks/10.1142/6680
17. https://arxiv.org/abs/2206.10166
18. http://www.tandfonline.com/doi/full/10.1080/01630563.2015.1062777
19. http://arxiv.org/pdf/1911.12344.pdf
20. https://arxiv.org/abs/2405.20630
21. https://www.semanticscholar.org/paper/e69de5214b0eb69ac45132b1b376552f35aad860