On Various Definitions of the Envelope of a Random Process

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(Received 9 January 1985, and in revised form 7 May 1985)

Abstract

Statistical properties of the envelope definitions of Rice [rice1954], Crandall and Mark [crandall1963] and Dugundji [dugundji1958] are derived and compared. It is shown that the definitions of Rice [rice1954] and Dugundji [dugundji1958] are equivalent, which implies that the envelope of Rice [rice1954] is independent of the choice of a central frequency. This contradicts results which have appeared in the literature [lin1967, lin1976] and the reason for this contradiction is explained. The envelopes of Crandall and Mark [crandall1963] and Dugundji [dugundji1958] are found to have the same first order probability density function but different crossing rates and mean frequencies.

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1 Introduction

An extremely useful concept in the theory of random vibrations is that of the envelope process $a(t)$ associated with a random process $x(t)$. Physically, if $x(t)$ is reasonably narrow banded, the envelope process is a smooth curve joining the peaks of $x(t)$ as shown in Figure 1. Associated with the envelope process is a phase process such that $x(t)$ can be represented as a cosine curve having time varying amplitude (governed by the envelope process) and time varying frequency (governed by the phase process).

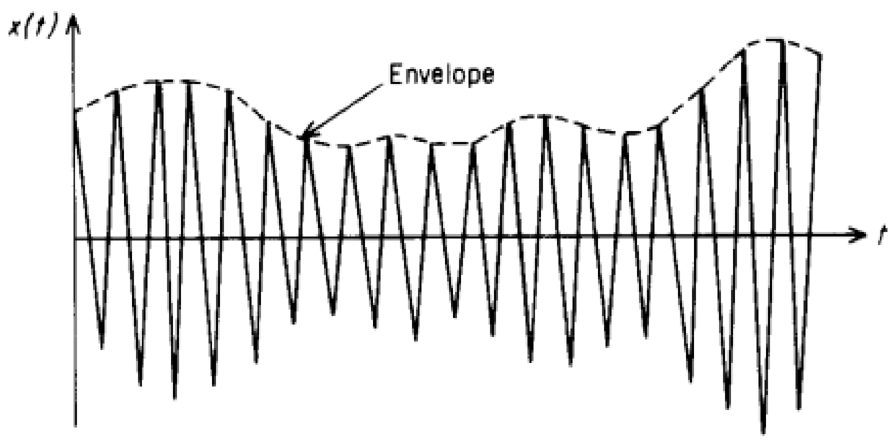

Figure 1. The envelope of a random proces

There exist a number of definitions for the envelope process, the three most notable of which are those due to Rice [rice1954], Crandall and Mark [crandall1963] and Dugundji [dugundji1958] (often attributed to Cramer and Leadbetter [cramer1967]). The Rice envelope [rice1954] is based upon the expansion of the process $x(t)$ about some central frequency ω_r, and is often considered to be the classical definition of the envelope. The envelope of Crandall and Mark [crandall1963] is an "energy envelope" and is defined in terms of $x(t)$ and its time derivative $\dot{x}(t)$. The envelope of Dugundji [dugundji1958] is derived from $x(t)$ and its Hilbert transform $\hat{x}(t)$. In what follows, the similarities and differences between these envelope definitions are discussed and various statistical properties are derived for each. It is shown that the envelope definitions of Rice [rice1954] and Dugundji [dugundji1958] are equivalent, contrary to some results in the literature [lin1967, lin1976].

2 Envelope Definitions

2.1 Envelopes Formed from a Complex Process

A random process $x(t)$ can be written as the real part of a complex process $z(t)$:

$$z(t) = x(t) + i\,y(t) \tag{1}$$

where $y(t)$ is some arbitrary random process. Using (1), $x(t)$ can be expressed as a cosine curve with time-varying amplitude and phase:

$$x(t) = a(t)\cos\phi(t) \tag{2}$$

$$a(t) = |z(t)| = \sqrt{x^2 + y^2} \tag{3}$$

$$\phi(t) = \tan^{-1}\left(\frac{y}{x}\right) \tag{4}$$

$a(t)$ and $\phi(t)$ are known as the random envelope and phase processes associated with $x(t)$. The process $y(t)$ must be chosen so that $a(t)$ is a smooth curve joining the peaks of $x(t)$. For a harmonic $x(t) = A\cos\omega t$, the required envelope is A, which implies $y(t) = \pm A\sin\omega t$. This can be related to $x(t)$ either as $y = \dot{x}/\omega$ or as the Hilbert transform $y = \hat{x}(t)$, where

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}\,d\tau \tag{5}$$

Based on this, two definitions for the envelope process are possible:

$$a_1(t) = \sqrt{x^2 + \left(\frac{\dot{x}}{\omega_c}\right)^2} \tag{6}$$

$$a_2(t) = \sqrt{x^2 + \hat{x}^2} \tag{7}$$

Equation (6) is the envelope definition of Crandall and Mark [crandall1963], while (7) is the envelope suggested by Dugundji [dugundji1958]. The choice of ω_c is discussed below.

For a stationary Gaussian process, the joint density function $p(\hat{x}, \dot{x})$ and the conditional density $p(\hat{x}|\dot{x})$ can be shown to be Gaussian with mean and variance:

$$\mathrm{E}[\hat{x}|\dot{x}] = -\left(\frac{m_1}{m_2}\right)\dot{x} \tag{8}$$

$$\mathrm{Var}[\hat{x}|\dot{x}] = m_0\,q^2 \tag{9}$$

where m_n is the nth spectral moment of the single-sided spectrum $S_{xx}(\omega)$:

$$m_n = \int_0^{\infty}\omega^n\,S_{xx}(\omega)\,d\omega \tag{10}$$

and

$$q^2 = 1 - \frac{m_1^2}{m_0\,m_2} \tag{11}$$

For the frequency ω_c in (6), two candidates are the mean frequency $\omega_1 = m_1/m_0$ and the mean zero-crossing frequency $\omega_0 = \sqrt{m_2/m_0}$. Using $\omega_c = \omega_0$ yields $\mathrm{E}[a_1^2(t)] = 2\,m_0$, matching the harmonic case. Thus, $\omega_c = \omega_0$ is used here.

2.2 The Envelope of Rice

Definition 1. *[Rice Envelope] Rice [rice1954] defined the envelope of a stationary random process $x(t)$ via its Fourier series expansion:*

$$x(t) = \sum_n \{a_n \cos \omega_n t + b_n \sin \omega_n t\} \tag{12}$$

where $\omega_n = 2\pi n/T$ and a_n, b_n are (ensemble) random variables. This can be rewritten as

$$x(t) = I_c(t) \cos \omega_r t - I_s(t) \sin \omega_r t \tag{13}$$

where

$$I_c(t) = \sum_n \{a_n \cos(\omega_n - \omega_r)\, t + b_n \sin(\omega_n - \omega_r)\, t\} \tag{14}$$

$$I_s(t) = \sum_n \{a_n \sin(\omega_n - \omega_r)\, t - b_n \cos(\omega_n - \omega_r)\, t\} \tag{15}$$

Thus,

$$x(t) = a(t) \cos[\omega_r t + \theta(t)] \tag{16}$$

$$a^2(t) = I_c^2(t) + I_s^2(t) \tag{17}$$

$$\theta(t) = \tan^{-1}\left(\frac{I_s(t)}{I_c(t)}\right) \tag{18}$$

Alternatively, $x(t)$ can be written as

$$x(t) = \mathrm{Re}\{z(t)\} \tag{19}$$

where

$$z(t) = \sum_n (a_n - i\, b_n)\, e^{i\omega_n t} \tag{20}$$

and it follows that

$$z(t) = x(t) + i\,\hat{x}(t) \tag{21}$$

so that

$$a^2(t) = x^2(t) + \hat{x}^2(t) \tag{22}$$

Thus, the Rice and Dugundji envelopes are equivalent.

3 The Statistics of the Envelope of Rice

For $x(t)$ Gaussian with zero mean, the joint probability density function (jpdf) $p(a, \dot{a}, \theta, \dot{\theta})$ can be derived by transforming the jpdf of $(I_c, I_s, \dot{I}_c, \dot{I}_s)$, which is a zero-mean Gaussian vector with covariance matrix S:

$$S = \begin{pmatrix} m_0 & 0 & 0 & M_2 \\ 0 & m_0 & -M_2 & 0 \\ 0 & -M_2 & M_1 & 0 \\ M_2 & 0 & 0 & M_1 \end{pmatrix} \tag{23}$$

where $M_2 = m_1 - \omega_r m_0$, $M_1 = m_2 - 2\,\omega_r m_1 + \omega_r^2 m_0$.

The transformation yields

$$p(a, \dot{a}, \theta, \dot{\theta}) = \frac{a^2}{4\,\pi^2\, m_0\, m_2\, q^2} \exp\left\{ -\frac{1}{2} \left[\frac{a^2}{m_0} + \frac{1}{q^2\, m_2} \left(\dot{a}^2 + a^2\, [\dot{\theta} - \frac{m_1}{m_0} + \omega_r]^2 \right) \right] \right\} \tag{24}$$

Integrating over θ and $\dot{\theta}$ gives

$$p(a, \dot{a}) = \frac{a}{\sqrt{2\,\pi\, m_2}\, q\, m_0} \exp\left\{ -\frac{1}{2} \left(\frac{a^2}{m_0} + \frac{\dot{a}^2}{q^2\, m_2} \right) \right\} \tag{25}$$

This result is independent of ω_r.

4 The Statistics of the Envelope of Dugundji

If $x(t)$ is Gaussian with zero mean, the vector $(x, \hat{x}, \dot{x}, \dot{\hat{x}})$ is Gaussian with covariance matrix

$$R = \begin{pmatrix} m_0 & 0 & 0 & m_2 \\ 0 & m_0 & -m_2 & 0 \\ 0 & -m_2 & m_1 & 0 \\ m_2 & 0 & 0 & m_1 \end{pmatrix}. \tag{26}$$

Transforming to $(a, \dot{a}, \phi, \dot{\phi})$ with $a(t)$ and $\phi(t)$ as in (2) (with $y(t) = \hat{x}(t)$) gives

$$p(a, \dot{a}, \phi, \dot{\phi}) = \frac{a^2}{4\,\pi^2\, m_0\, m_2\, q^2} \exp\left\{-\frac{1}{2}\left[\frac{a^2}{m_0} + \frac{1}{q^2\, m_2}\left(\dot{a}^2 + a^2\,[\dot{\phi} - \frac{m_1}{m_0}]^2\right)\right]\right\} \tag{27}$$

Integrating over ϕ and $\dot{\phi}$ gives the same $p(a, \dot{a})$ as in (25).

The mean rate at which the envelope crosses a level a with positive slope is

$$\nu_a^+ = \int_0^\infty \dot{a}\;\, p(a, \dot{a})\;\, d\,\dot{a} = \frac{1}{\sqrt{2\,\pi}}\sqrt{\frac{m_2}{m_0}}\; q\left(\frac{a}{\sqrt{m_0}}\right) \exp\left(-\frac{1}{2}\frac{a^2}{m_0}\right) \tag{28}$$

The maximum occurs at $a = \sqrt{m_0}$:

$$(\nu_a^+)_{\max} = \frac{1}{\sqrt{2\,\pi}}\sqrt{\frac{m_2}{m_0}}\; q\, e^{-1/2} \tag{29}$$

The envelope $a(t)$ has a Rayleigh distribution with mean $\sqrt{\pi\, m_0/2}$.

5 The Statistics of the Envelope of Crandall and Mark

For the envelope definition of Crandall and Mark [crandall1963] ($y = \dot{x}/\omega_0$), the following relations hold:

$$x(t) = a(t) \cos \phi(t) \tag{30}$$

$$\frac{\dot{x}(t)}{\omega_0} = a(t) \sin \phi(t) \tag{31}$$

Differentiating gives

$$\dot{x} = \dot{a} \cos \phi - a\, \dot{\phi} \sin \phi \tag{32}$$

$$\ddot{x}/\omega_0 = \dot{a} \sin \phi + a\, \dot{\phi} \cos \phi \tag{33}$$

From these,

$$\dot{\phi} = -\omega_0 + \left(\frac{\dot{a}}{a}\right) \cot \phi \tag{34}$$

The joint pdf of $(a, \dot{a}, \phi)$ is

$$p(a, \dot{a}, \phi) = \frac{a |\csc \phi|}{(2\,\pi)^{3/2}\, m_0\, \sigma\, \varepsilon} \exp\left\{-\left[\frac{a^2}{2\, m_0} + \frac{\dot{a}^2 \csc^2 \phi}{2\, \sigma^2\, \varepsilon^2}\right]\right\} \tag{35}$$

where $\sigma^2 = (m_4\, m_0)/m_2$ and $\varepsilon^2 = 1 - m_2^2/(m_0\, m_4)$

Integrating over $\dot{a}$ gives

$$p(a,\phi) = \frac{a}{2\,\pi\, m_0} \exp\left(-\frac{a^2}{2\, m_0}\right) \tag{36}$$

so a has a Rayleigh distribution and ϕ is uniform over $[0, 2\,\pi)$.

The mean rate at which the envelope crosses a level a with positive slope is

$$\nu_a^+ = \frac{4}{(2\,\pi)^{3/2}} \sqrt{\frac{m_4}{m_2}}\, \varepsilon \left(\frac{a}{\sqrt{m_0}}\right) \exp\left(-\frac{1}{2}\frac{a^2}{m_0}\right). \tag{37}$$

The ratio of mean crossing rates between Crandall and Mark and Dugundji is

$$\frac{(\nu_a^+)_{C-M}}{(\nu_a^+)_D} = \frac{2}{\pi} \sqrt{\frac{m_4\, m_0}{m_2^2}}\, \frac{\varepsilon}{q} = \frac{2}{\pi}\, \frac{\varepsilon}{q\,\sqrt{1-\varepsilon^2}} \tag{38}$$

The statistics of $\dot{\phi}$ are given by

$$p(a,\dot{\phi},\phi) = \frac{a^2 |\sec\phi|}{(2\,\pi)^{3/2}\, m_0\, \sigma\, \varepsilon} \exp\left\{-\left[\frac{a^2}{2\, m_0} + \frac{a^2\, (\dot{\phi}+\omega_0)^2 \sec^2\phi}{2\,\sigma^2\,\varepsilon^2}\right]\right\} \tag{39}$$

Integrating over a and ϕ gives

$$p(\dot{\phi}) = \frac{1}{4\,\pi} \sqrt{\frac{m_2}{m_4}}\, \frac{1}{\varepsilon}\, \Phi\left[\frac{\dot{\phi}+\omega_0}{\varepsilon} \sqrt{\frac{m_2}{m_4}}\right], \tag{40}$$

where

$$\Phi[m] = \int_0^{2\pi} \frac{|\sec\phi|\, d\,\phi}{(1+m^2 \sec^2\phi)^{3/2}}. \tag{41}$$

The mean value of $\dot{\phi}$ is $-\omega_0$, so the carrier frequency is $\omega_0 = \sqrt{m_2/m_0}$.

The cumulative distribution function is

$$P(\dot{\phi}) = \int_{-\infty}^{\dot{\phi}} p(\dot{\phi})\, d\,\dot{\phi} = \frac{1}{4\,\pi} \left\{\frac{4\,x}{\sqrt{1+x^2}}\, k\left(\frac{1}{\sqrt{1+x^2}}\right) + 2\,\pi\right\} \tag{42}$$

$$x = \sqrt{\frac{m_2}{m_4}}\, \frac{\dot{\phi}+\omega_0}{\varepsilon} \tag{43}$$

$$k(m) = \int_0^{\pi/2} \frac{d\,\phi}{\sqrt{1-m^2 \sin^2\phi}} \tag{44}$$

where $k(m)$ is the complete elliptic integral of the first kind [abramowitz1965].

6 Conclusions

The main conclusions are as follows (assuming $x(t)$ is stationary and Gaussian unless otherwise stated):

1. The definitions of Rice [rice1954] and Dugundji [dugundji1958] are equivalent, so the Rice envelope is independent of the choice of central frequency. The proof of equivalence is algebraic and does not depend on $x(t)$ being Gaussian.

2. The extent to which the envelope of Dugundji [dugundji1958] follows the peaks of $x(t)$ can be assessed from equation (11), which gives the mean and variance of $\hat{x}(t)$ when $\dot{x}(t)$ is specified (especially $\dot{x} = 0$).

3. The envelope definitions of Dugundji [dugundji1958] and Crandall and Mark [crandall1963] lead to the same joint pdf $p(a, \phi)$.

4. The mean or carrier frequency of the envelope of Dugundji [dugundji1958] is $\omega_1 = m_1/m_0$, while for Crandall and Mark [crandall1963] it is $\omega_0 = \sqrt{m_2/m_0}$.

5. The mean crossing rates of the envelopes of Dugundji [dugundji1958] and Crandall and Mark [crandall1963] are given by equations (28) and (37), respectively. Generally, the mean crossing rate of the latter is greater. The "slowly varying envelope" concept can be assessed from equation (29).

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