Construction of a Strictly Monotonically Increasing Version of the Riemann-Siegel $\boldsymbol{\theta}$ Function

by Stephen Crowley

Abstract

By defining $\theta_*'(t) := |\theta'(t)|$, we create a strictly increasing modified Riemann-Siegel theta function. This construction ensures positivity of the derivative everywhere while maintaining key analytic properties of the original function.

Core Construction

Let $t^* \approx 2\pi$ be the critical point where $\theta'(t^*) = 0$. Define:

$$\theta_*'(t) := |\theta'(t)| = \begin{cases} -\theta'(t) & \text{for } t < t^*, \\ \theta'(t) & \text{for } t \geq t^*. \end{cases}$$

Then, the modified function $\theta_*(t)$ is given by:

$$\theta_*(t) = \int_0^t |\theta'(\tau)| \, d\tau \tag{1}$$

Proof of Strict Monotonicity

1. Derivative Analysis

Case 1: $t < t^*$

The original derivative $\theta'(t) < 0$ by the Riemann-Siegel properties. Therefore:

$$\theta_*'(t) = -\theta'(t) > 0 \tag{2}$$

Case 2: $t \geq t^*$

In this regime, the original derivative $\theta'(t) > 0$. Thus:

$$\theta_*'(t) = \theta'(t) > 0 \tag{3}$$

Since $\theta_*'(t) > 0$ for all t, we conclude that $\theta_*(t)$ is strictly increasing.

2. Integral Argument

For any $t_2 > t_1$, we have:

$$\theta_*(t_2) - \theta_*(t_1) = \int_{t_1}^{t_2} \theta_*'(\tau)\, d\tau > 0 \tag{4}$$

because $\theta_*'(\tau) > 0$ for all τ.

Critical Point Handling

At $t = t^*$, the derivative satisfies:

$$\theta_*'(t^*) = |\theta'(t^*)| = 0. \tag{5}$$

This single zero does not affect the monotonicity of $\theta_*(t)$, as the integral remains strictly positive:

$$\int_{t^*-\epsilon}^{t^*+\epsilon} \theta_*'(\tau)\, d\tau = \int_{t^*-\epsilon}^{t^*} (-\theta'(\tau))\, d\tau + \int_{t^*}^{t^*+\epsilon} \theta'(\tau)\, d\tau > 0 \quad \forall \epsilon > 0 \tag{6}$$

Conclusion

This construction achieves the following:

- It guarantees strict monotonicity ($\theta_*'(t) > 0$ for all $t > 0$).
- It preserves the analytic properties of the Riemann-Siegel θ function.
- It creates a function $\theta_*(t)$ that is equivalent to $|\theta(t)|$.

Thus, $\theta_*(t) = |\theta(t)|$, as desired.