

The Density of The Product of A Pair Of Normally Distributed Standard Random Variables

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Let $Z = XY$ where X and Y are independent standard normally distributed random variables. To find the density $f_Z(z)$ it can be expressed in terms of the joint density of X and Y and the Dirac delta function:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, \frac{z}{x}) \delta(z - xy) dx \quad (1)$$

$$= \int_{-\infty}^{\infty} f_X(x) f_Y(\frac{z}{x}) \delta(z - xy) dx \quad (2)$$

Because X and Y are independent standard normals:

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (3)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \quad (4)$$

Substituting these in, we get:

$$f_Z(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-\frac{z^2}{2x^2}} \delta(z - xy) dx \quad (5)$$

Now, to deal with the delta function, we'll integrate over y from $-\infty$ to ∞ :

$$f_Z(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} \delta(z - xy) dy dx \quad (6)$$

Because of the delta function, only values where $y = \frac{z}{x}$ will contribute to the integral, so:

$$f_Z(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{x^2+z^2}{2x^2}} |x| dx \quad (7)$$

1 Substitution

Now, make a substitution: $u = x^2$ so $du = 2x dx$. Then:

$$f_Z(z) = \frac{1}{\pi} \int_0^{\infty} e^{-\frac{u+z^2}{2u}} \frac{du}{\sqrt{u}} \quad (8)$$

This is an integral representation of the Bessel function of the first kind of order 0 denoted $J_0(z)$ given by:

$$J_0(z) = \frac{1}{\pi} \int_0^{\infty} e^{-\frac{z^2}{2u}} \frac{du}{\sqrt{u}} \quad (9)$$

Comparing, we find:

$$f_Z(z) = |z| J_0(|z|) \quad (10)$$

So, the distribution of the product of two independent standard normal variables involves the Bessel function of the first kind of order 0.

2 Correlated But Independent Random Variables

When X and Y are independent standard normal random variables, their product $Z=XY$ does not follow a normal distribution. Instead, its probability density function $f_Z(z)$ is proportional to the $|z|J_0(|z|)$ where J_0 is the Bessel function of the first kind of order 0.

There is also a connection to the modified Bessel function of the second kind in the context of the product of correlated zero mean random variables. This is a related but different scenario. The distribution of the product of correlated zero-mean Gaussian variables does involve the modified Bessel function of the second kind.

The relationship between the Bessel functions and the product of Gaussian (random) variables is a deep and fascinating topic in probability and statistics. Both the Bessel function of the first kind and the modified Bessel function of the second kind can arise in different contexts when studying the products of or other nonlinear functions of Gaussian variables