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ON FLUCTUATIONS OF EIGENVALUES OF RANDOM HERMITIAN MATRICES

KURT JOHANSSON

1. Introduction. Let $\mathscr{H}_N$ be the set of all $N \times N$ hermitian matrices, $\Phi = (\Phi_{ij})_{1 \leqslant i,j \leqslant N}$, and let $\mathbb{P}_{2\ell}^{+}$ denote the set of all polynomials of degree 2ℓ with positive leading coefficient. On $\mathscr{H}_N$ we consider the probability measure

$$d\mu_{N,M}(\Phi) = \frac{1}{\mathscr{Z}_{N,M}^{2}} \exp(-M \operatorname{Tr} V(\Phi)) d\Phi, \tag{1.1}$$

where $V \in \mathbb{P}_{2\ell}^{+}$, $\ell \geqslant 1$, $M > 0$, and

$$d\Phi = \prod_{1 \leqslant i < j \leqslant N} d \operatorname{Re} \Phi_{ij} \, d \operatorname{Im} \Phi_{ij} \prod_{i=1}^{N} d\Phi_{ii}.$$

$\mathscr{Z}_{N,M}^{2}$ is a normalization constant (partition function). We will be interested in the spectral properties of matrices picked randomly with respect to (1.1). The measure (1.1) is invariant under conjugation by unitary matrices, but the matrix elements are not independent unless the polynomial V is quadratic. In the case of quadratic V, we get the Gaussian Unitary Ensemble (GUE), properly scaled. Our results will also apply to the corresponding invariant ensembles of symmetric matrices and so-called self-dual $2N \times 2N$ hermitian matrices. On the set of all $N \times N$ real symmetric matrices $\Phi = (\Phi_{ij})$, we put the probability measure

$$\frac{1}{\mathscr{Z}_{N,M}^{1}} \exp\left(-\frac{M}{2} \operatorname{Tr} V(\Phi)\right) \prod_{1 \leqslant i \leqslant j \leqslant N} d\Phi_{ij},$$

which is invariant under conjugation by orthogonal matrices. If V is quadratic, the matrix elements are independent, real, normal, random variables, and we get the Gaussian Orthogonal Ensemble (GOE). See [Me] for the definitions of self-dual hermitian matrices and the appropriate measure (symplectic ensembles). For historical background and physical motivation, see [Me] and [Po]. Random matrices are also relevant in connection with quantum chaos; see [GVZ] for reviews.

In the study of the spectral properties of random matrices, most of the interest has centered around the *local* fluctuation properties of the spectrum, for exam-

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ple, the distribution of the level spacing, that is, the distance between neighbouring eigenvalues. For recent interesting work on this problem, see [TW] and the review article of Tracy and Widom in [He]. A characteristic feature of the spectrum is that the eigenvalues are very regularly spaced. Pairs of eigenvalues lying close together, as well as large gaps, are rare; we see "level repulsion." The spectrum does not at all look like a Poisson process.

In this paper, we will be interested in the *global* fluctuation properties of the spectrum. If we take $V(t) = 2t^2$ and $M/N \to 1$ in (1.1), the asymptotic eigenvalue distribution is given by Wigner's semicircle law with density $u(t) = (2/\pi)\sqrt{1-t^2}$; that is, if $x_1, \dots, x_N$ are the eigenvalues, then $(1/N)\sum_\mu f(x_\mu) \to \int_{-1}^{1} f(t)u(t)\,dt$ as $N \to \infty$. What can we say about the fluctuations of the linear statistic $\sum_\mu f(x_\mu)$, the fluctuations around the semicircle law? The main result of the present paper is that for all sufficiently nice functions $h : \mathbb{R} \to \mathbb{R}$, satsfying $\int_{-1}^{1} f(t)u(t)\,dt = 0$, the sum $\sum_\mu f(x_\mu)$ converges in distribution to a normal random variable with mean zero and variance σ^2, which is a quadratic functional of h (see below). Note that there is no $1/\sqrt{N}$-normalization! Thus there must be very effective cancellation in the sum, which can be viewed as a global manifestation of the regularity of eigenvalue distribution.

One reason to expect that this central limit theorem should be true comes from the corresponding result for random unitary matrices. Consider the group $U(N)$ of all $N \times N$ unitary matrices, and let dU denote the normalized Haar measure on $U(N)$. Let $h : \mathbb{T} \to \mathbb{C}$ be continuous; $\mathbb{T} = [0, 2\pi]$ is the unit circle. From Weyl's integral formula for the integration of a class function on $U(N)$ (see [We1]; see also [We2, p. 197]), we have

$$\int_{U(N)} e^{\operatorname{Tr} h(U)}dU = \frac{1}{(2\pi)^N N!}\int_{\mathbb{T}^N} e^{\sum_{\mu=1}^N h(\theta_\mu)}\Delta(\theta)^2 d^N\theta, \tag{1.2}$$

where $\theta = (\theta_1, \dots, \theta_N)$, and

$$\Delta(\theta) = \prod_{1 \leqslant \mu < \nu \leqslant N} |e^{i\theta_\mu} - e^{i\theta_\nu}|. \tag{1.3}$$

Here $e^{i\theta_\mu}$ are the eigenvalues of U. (1.2) thus gives the probability density on $\mathbb{T}^N$ of the eigenvalues of a random unitary matrix from $U(N)$ with respect to Haar measure. This ensemble is also called the circular unitary ensemble (see [Dy] and [Me, Chapter 9]).

The integral on the right in (1.2) also appears in another context. Let $f \in L^1(\mathbb{T})$, and let $\hat{f}_k = (2\pi)^{-1}\int_0^{2\pi} f(\theta)e^{-ik\theta}d\theta$ be the Fourier coefficients of f. The Nth order Toeplitz determinant with generating function f is defined by

$$D_N(f) = \det(\hat{f}_{j-k})_{0 \leqslant j,k \leqslant N-1}.$$

Now (see [Sz1]),

$$D_N(f) = \frac{1}{(2\pi)^N N!} \int_{\mathbb{T}^N} \prod_{\mu=1}^{N} f(\theta_\mu) \Delta(\theta)^2 d^N\theta. \tag{1.4}$$

Szegö's asymptotic formula for Toeplitz determinants (see [Sz2]) says that if $\sum_{k=-\infty}^{\infty} |k||\hat{h}_k|^2 < \infty$, then

$$D_N(\exp h) = \exp\left(N\hat{h}_0 + \sum_{k=1}^{\infty} k\hat{h}_k \hat{h}_{-k} + o(1) \right) \tag{1.5}$$

as $N \to \infty$, where $\hat{h}_k$ are the Fourier coefficients of h.

Combining (1.2), (1.4), and (1.5), we see that the eigenvalues of a random unitary matrix will asymptotically be uniformly distributed on $\mathbb{T}$. Furthermore, we obtain a central limit theorem for the fluctuations around the uniform distribution. If $h : \mathbb{T} \to \mathbb{R}$ satisfies $\hat{h}_0 = 0$ and $A(h) = \sum_{k=1}^{\infty} |k||\hat{h}_k|^2 < \infty$, then

$$\sum_{\mu=1}^{N} h(\theta_\mu) \Rightarrow N(0, \sigma^2),$$

where $\sigma^2 = 2A(h)$, and $\Rightarrow$ means convergence in distribution. For a proof of Szegö's theorem based on (1.2) and the central limit theorem interpretation, see [Jo1]. See also [DS] and [Jo2] for more results on random matrices from the compact classical groups.

Let us return to random hermitian matrices. If the function $f : \mathscr{H}_N \to \mathbb{R}$ is invariant under conjugation by unitary matrices (i.e., $f(\Phi) = f(U\Phi U^{-1})$ for all $U \in U(N)$, $\Phi \in \mathscr{H}_N$), then (see [BIZ] and [HZ, p. 471])

$$\int_{\mathscr{H}_N} f(\Phi) d\mu_{N,M}(\Phi) = \int_{\mathbb{R}^N} f(D(x)) \rho_{N,M}(x) dx.$$

Here $D(x) = \mathrm{diag}(x_1, \ldots, x_N)$ is the diagonal matrix of eigenvalues of Φ, and

$$\rho_{N,M}(x) = \frac{1}{Z_{N,M}} \exp\left(\sum_{1 \leqslant i \neq j \leqslant N} \log|x_i - x_j| - M \sum_{j=1}^{N} V(x_j) \right) \tag{1.6}$$

is a probability density on $\mathbb{R}^N$, the density of the eigenvalues of the random hermtian matrix Φ. Let $E_{N,M}(\cdot)$ denote the expectation with respect to (1.6). The main result, Theorem 2.4, a strong Szegö theorem, says that, for sufficiently nice functions $h : \mathbb{R} \to \mathbb{R}$,

$$\lim_{N\to\infty}\left\{\log E_{N,N}\left[\exp\left(\sum_{j=1}^{N} h(x_j)\right)\right] - N\int_{\mathbb{R}} h(t)d\mu_V(t)\right\} = A(h), \qquad (1.7)$$

where $A(h)$ is an explicit quadratic functional of h (see (2.9)). $d\mu_V$ is a measure which gives the asymptotic eigenvalue distribution in the sense that if $u_{N,M}(t)$ is the 1-point function of (1.6), then

$$u_{N,M}(t)dt \to d\mu_V \qquad (1.8)$$

weakly as $N, M \to \infty$, $N/M \to 1$. We will be able to prove (1.7) for essentially all $V \in \mathbb{P}_{2\ell}^{+}$, $\ell \geqslant 1$, for which $\operatorname{supp}\mu_V$ is a single interval $[a, b]$. The leading asymptotics in (1.7) and (1.8) are true under much weaker conditions (see Theorem 2.1). In general, for polynomial V, $\operatorname{supp}\mu_V$ can consist of several intervals (see Example 3.2).

The measure (1.6) can be given a statistical mechanical interpretation. It is the canonical Gibbs measure, at inverse temperature $\beta = 2$, of a system of N unit charges interacting through a logarithmic Coulomb potential and confined by an external potential MV. $Z_{N,M}$ is the partition function of this Coulomb gas, also referred to as a *log-gas*. The eigenvalues of a random hermitian matrix thus behave as repelling charges confined by a potential. From the statistical mechanical point of view, it is natural to consider the Coulomb gas at arbitrary values of the inverse temperature β. That is, we consider the probability density

$$\rho_{N,M}^{\beta}(x) = \frac{1}{Z_{N,M}^{\beta}} e^{-\frac{\beta}{2}\left(\sum_{1\leqslant i\neq j\leqslant N}\log|x_i-x_j|^{-1}+M\sum_{j=1}^{N}V(x_j)\right)} \qquad (1.9)$$

on $\mathbb{R}^N$. The orthogonal ensembles of real symmetric matrices and the symplectic ensembles correspond to the inverse temperatures $\beta = 1$ and $\beta = 4$, respectively. If we take $\beta = 1$ and $V(t) = 2t^2$, we get the (scaled) eigenvalue density of the Gaussian Orthogonal Ensemble (GOE). We will prove that, for the same V's, an asymptotic formula similar to (1.7) holds for the density (1.9) also. The difference is that $A(h)$ in the right-hand side has to be multiplied by $2/\beta$, and $(2/\beta - 1)$ times a term that is linear in h has to be added (see Theorem 2.4).

The matrix measures with general polynomial V have received a lot of attention in theoretical physics in connection with so-called matrix models (see [FFS, Chapter 7] and [FGZ] for reviews). The large-N asymptotics of the partition function also have combinatorial applications (see [BIZ], [HZ], and [Ja]). Global fluctuations of the eigenvalues of large random matrices have been discussed recently in physics literature. Brézin and Zee [BZ] analyze the asymptotics of the 2-point correlation function for V even and $\beta = 2$. Beenakker gives a heuristic derivation for arbitrary V and β, and also gives the formula (2.13) for the variance of a linear statistic. In [Fo], the case of unitary matrices is also treated (cf. Remark 2.7). The Gaussian limit for a linear statistic in these models has been discussed heuristically in [P].

As a by-product of the asymptotic analysis used in the proof of (1.7), we can show some asymptotic results for the orthonormal polynomials $p_{n,M}(x)$, $n = 0, 1, \ldots$, with respect to the weight $\exp(-MV(x))$ on $\mathbb{R}$. These polynomials satisfy a recursion formula (see [Sz1])

$$xp_{n,M}(x) = R_{n+1,M}^{1/2} p_{n+1,M}(x) + S_{n,M} p_{n.M}(x) + R_{n,M}^{1/2} p_{n-1,M}(x). \tag{1.10}$$

For the same V's for which we can prove (1.7), we will show that

$$\lim_{\substack{N,M\to\infty \\ N/M\to 1}} R_{N,M} = R, \qquad \lim_{\substack{N,M\to\infty \\ N/M\to 1}} S_{N,M} = S,$$

where $R = (b-a)^2/16$ and $S = (a+b)/2$. Here $[a, b]$ is the support of μ_V. The existence of these limits is used in the theoretical physics literature on hermitian matrix models. This result is closely related to the now proved (see [Ma] and [LMS]) Freud conjecture, and for $V(t) = t^{2m}$, $m \geqslant 1$, equivalent to it.

2. Results. The leading-order asymptotics in (1.7), a weak-type Szegö theorem, can be proved in greater generality. A heuristic saddle point argument (see [BIPZ]) shows that the asymptotics of the 1-point functions $u_{N,M}(t)$ (i.e., the asymptotic distribution of the eigenvalues) should be given by a probability measure μ on $\mathbb{R}$ minimizing the energy functional

$$I_V[\mu] = \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\log|t-s|^{-1} + \frac{1}{2}V(t) + \frac{1}{2}V(s) \right) d\mu(t) d\mu(s). \tag{2.1}$$

See also Section 4.2 in [Me] on the classical fluid approximation to the Coulomb gas. Minimizing (2.1) is a weighted version of the classical logarithmic energy problem, and the existence of a unique minimizing measure is proved using methods analogous to those used by Frostman in classical potential theory. This problem has been investigated by Mhaskar and Saff in connection with some problems in weighted approximation (see [MS1] and [MS2]). The connection with (1.9) is that we can view

$$\delta_{N,V} = \left[\sup_{z\in\mathbb{R}^n} e^{-\left(\sum_{i\neq j} \log|x_i - x_j|^{-1} + N\sum_j V(x_j)\right)} \right]^{1/N(N-1)} \tag{2.2}$$

as a generalized *transfinite diameter*. The supremum in (2.2) is assumed at points (*generalized Fekete points*) with a distribution close to $d\mu_V$, and we have $-\log\delta_{N,V} \to I_V[\mu_V]$ as $N \to \infty$ (see Section 4).

The above ideas are the basis for the first theorem. Let $V : \mathbb{R} \to [0, \infty)$, and $h : \mathbb{R} \to \mathbb{R}$ be given. Consider the probability measure on $\mathbb{R}^N$ with density

$$\rho_{N,M}^{\beta,h}(x) = \frac{1}{Z_{N,M}^{\beta,h}(V)} e^{-\frac{\beta}{2}\left(\sum_{1\leqslant i\neq j\leqslant N} \log|x_i-x_j|^{-1}+M\sum_{j=1}^N V(x_j)\right)+\sum_{j=1}^N h(x_j)}, \tag{2.3}$$

and let $E_{N,M}^{\beta,h}(\cdot)$ denote expectation with respect to this probability measure on $\mathbb{R}^N$.

THEOREM 2.1. *Assume that $h : \mathbb{R} \to \mathbb{R}$ is continuous, $h(t) \leqslant C(V(t) + 1)$ for some constant C, and that $V : \mathbb{R} \to [0, \infty)$ satisfies that*

(a) *V is continuous;*

(b) *there is a constant $\delta > 0$ such that $V(t) \geqslant (1 + \delta)\log(t^2 + 1)$ for all sufficiently large t.*

Then there is a unique probability measure μ_V with compact support such that

$$\inf_{\mu\in\mathcal{M}^1(\mathbb{R})} I_V[\mu] = I_V[\mu_V], \tag{2.4}$$

where $\mathcal{M}^1(\mathbb{R})$ is the set of all Borel probability measures on $\mathbb{R}$. If g is bounded and continuous on E, then

$$\lim_{N\to\infty} \frac{1}{N} \log E_{N,M_N}^{\beta,h}\left(\exp\left(\sum_{j=1}^N g(x_j)\right)\right) = \int_{\mathbb{R}} g(t)d\mu_V(t), \tag{2.5}$$

provided $N/M_N \to 1$ as $N \to \infty$. Furthermore, if $u_{N,M}^{\beta,h}(t_1, \ldots, t_k)$ is the k-point function of (2.3), then

$$\lim_{N\to\infty} \int_{\mathbb{R}^k} \phi(t_1, \ldots, t_k)u_{N,M}^{\beta,h}(t_1, \ldots, t_k)dt_1 \cdots dt_k$$

$$= \int_{\mathbb{R}^k} \phi(t_1, \ldots, t_k)d\mu_V(t_1) \cdots d\mu_V(t_k) \tag{2.6}$$

for any continuous, bounded ϕ on $\mathbb{R}^k$.

Proof. See Section 4.

Remark 2.2. There is closely related and overlapping work concerning the problem addressed in Theorem 2.1 by Boutet de Monvel, Pastur, and Shcherbina [BPS]. They do not use the potential theoretic framework, but use ideas from mean-field theory. One of the main results of [BPS] will be important in the case of general β below.

Example 2.3. If $V(t) = 2t^2$, and $\beta = 2$, that is, we have (rescaled) GUE, then $d\mu_V(t) = (2/\pi)\sqrt{1-t^2}$, which is a classical result. This limit measure is called the Wigner semicircle law (see [Me]; see also Example 3.2.)

The derivative of the variational equation for the problem of minimizing (2.1) is

$$\int \frac{d\mu_V(t)}{t-s} = -\frac{1}{2}V'(s) \tag{2.7}$$

for all $s \in \operatorname{supp} \mu_V$. For a given polynomial $V \in \mathbb{P}^+_{2\ell}$, equation (2.7) may have several solutions. If $\operatorname{supp} \mu_V = [a,b]$ is a single interval, then

$$d\mu_V(t) = u(t)dt = \frac{1}{\pi} r(t)\sqrt{(t-a)(b-t)}1_{[a,b]}(t)dt, \tag{2.8}$$

where $1_X(\cdot)$ is the indicator function of the set X, and $r(t)$ is a polynomial of degree $2\ell - 2$ (see [Tr, p. 173] or [Mu, Chapter 11]).

To prove the second-order asymptotics in (1.7) and its generalizations to $\beta \neq 2$, we need some assumptions on h. The function $h : \mathbb{R} \to \mathbb{R}$ should not be too large for large values of t,

(i) $h(t) \leqslant C(V(t)+1)$ for some constant C, all $t \in \mathbb{R}$;

(ii) $|h'(t)| \leqslant q(t)$ for some polynomial $q(t)$ and all $t \in \mathbb{R}$.

Everything essential takes place in $\operatorname{supp} \mu_V$. So the regularity and size, apart from (i) and (ii), of $h(t)$ for large t should not be important. To formulate this precisely, for each $t_0 > 0$ let the function $\psi_{t_0} \in C^\infty$ be such that $\psi_{t_0}(t) = 1$ if $|t| \leqslant t_0$, and $\psi_{t_0}(t) = 0$ if $|t| \geqslant t_0 + 1$ and $0 \leqslant \psi_{t_0}(t) \leqslant 1$. Let $s > 0$ be given. We will assume that

(iii) for any $t_0 > 0$, there is an $\alpha > 0$ such that $h\psi_{t_0} \in H^{s+\alpha}$, where H^s, $s > 0$, is the standard L^2-Sobolev space.

Let $\mathscr{V}$ be the set of all polynomials of even degree with positive leading coefficient, such that μ_V is given by (2.8), and all zeros of $r(t)$ are nonreal. We can now state the following theorem.

THEOREM 2.4. *Assume that $V \in \mathscr{V}$ and h satisfies conditions (i)–(iii) above with $s = 2$ if $\beta = 2$, and $s = 17/2$ if $\beta \neq 2$. Then there is a signed measure ν_V on* $\operatorname{supp} \mu_V = [a,b]$, *which does not depend on h, such that*

$$\log E^\beta_{N,N}\left(e^{\sum_j h(x_j)}\right) - N\int_{\mathbb{R}} h(t)d\mu_V(t) \to \left(\frac{2}{\beta} - 1\right)\int_{\mathbb{R}} h(t)d\nu_V(t) + \frac{2}{\beta}A(h),$$

as $N \to \infty$, where

$$A(h) = \frac{1}{2}\int_a^b h(t)\delta^h(t)dt \tag{2.9}$$

with δ^h given by

$$\delta^h(t) := -\frac{1}{2\pi^2}\frac{1}{\sqrt{(t-a)(b-t)}}\int_a^b \frac{h'(s)\sqrt{(s-a)(b-s)}}{s-t}\,ds 1_{[a,b]}(t). \tag{2.10}$$

The quantity $A(h)$ can also be written

$$A(h) = \frac{1}{8}\sum_{k=1}^{\infty} k a_k^2 \tag{2.11}$$

with

$$a_k = \frac{2}{\pi}\int_0^{\pi} h\left(\frac{a+b}{2} + \frac{b-a}{2}\cos\theta\right)\cos k\theta d\theta. \tag{2.12}$$

Proof. See Section 3.

Remark 2.5. The signed measure ν_V is given by (3.54). If $V(t) = 2t^2$, then

$$d\nu_V(t) = \frac{1}{4}(\delta_0(t-1) + \delta_0(t+1)) - \frac{1}{2\pi}\cdot\frac{dt}{\sqrt{1-t^2}},$$

where δ_0 is the Dirac measure at the origin. If, in the case $\beta = 2$, we instead considered a Coulomb gas confined to the interval $[a,b]$, we would get the same second-order term, but the first-order term would be replaced by

$$\frac{N}{\pi}\int_a^b h(t)\frac{dt}{\sqrt{(t-a)(b-t)}}.$$

See [Jo1, p. 279], [Jo2], [Ge], and [Ok]. This shows that the global fluctuations show a kind of universality in the sense that the variance does not depend on the details of V. The fact that we assume different regularity of h for $\beta = 2$ and $\beta \neq 2$ is for purely technical reasons, and in fact the correct condition should be the finiteness of $A(h)$ in all cases.

Remark 2.6. In the case of a Coulomb gas on $\mathbb{T}$, as discussed in the Introduction, we also have an analogous result for any $\beta > 0$. If $E_N^{\beta}(\cdot)$ denotes expectation with respect to the probability density $Z_{N,\beta}^{-1}\Delta(\theta)^{\beta}$ on $[0,2\pi]^N$, with $\Delta(\theta)$ given by (1.3), then

$$\log E_N^{\beta}\left(\exp\left(\sum_{\mu} g(\theta_\mu)\right)\right) - \frac{N}{2\pi}\int_0^{2\pi} g(\theta)d\theta \to \frac{2}{\beta}\sum_{k=1}^{\infty} k\hat{g}_{-k}\hat{g}_k$$

as $n \to \infty$, where $\{\hat{g}_k\}$ are the complex Fourier coefficients of $g \in C^{1+\alpha}$, $\alpha > 0$. This is proved by slightly modifying the proof in [Jo1] for the case $\beta = 2$.

Remark 2.7. We can interpret Theorem 2.4 in the case $\beta = 2$ as a Szegö theorem in the following way. As in Section 1, we let $\{p_{n,N}(t)\}_{n=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to the weight $\exp(-NV(t))$ on $\mathbb{R}$. In $L^2(\mathbb{R})$, we let $\mathscr{P}_N$ denote the orthogonal projection onto the N-dimensional subspace spanned by $\phi_{n,N}(t) = p_{n,N}(t)\exp(-NV(t))$, $n = 0,\dots,N-1$. For $f \in L^{\infty}(\mathbb{R})$, we let M_f denote the multiplication operator $M_f\psi = f\psi$ on $L^2(\mathbb{R})$. The rank N *Toeplitz-type operator* is now defined by

$$T_N[f] = \mathscr{P}_N M_f \mathscr{P}_N.$$

Let $D_N[f] = \det T_N[f]$. Then (see [Sz1])

$$\frac{D_N[f]}{D_N[1]} = E_{N,N}^2\left(\prod_{j=1}^{N} f(x_j)\right),$$

and Theorem 2.4 is an analogue of Szegö's strong asymptotic formula for Toeplitz determinants. The limit (2.5) is the so-called weak Szegö theorem.

Let instead $\phi_n(t) = \exp(-x^2/2)H_n(x)$, $H_n(x)$ the nth-order Hermite polynomial, so that the basis functions do not depend on N. Furthermore, let $(M_N\psi)(x) = \exp(h(x/\sqrt{N}))\psi(x)$, so that the multiplication operator does depend on N. Then Theorem 2.4 gives

$$\log\det(\mathscr{P}_N M_N \mathscr{P}_N) - \frac{N}{\pi}\int_{-\sqrt{2}}^{\sqrt{2}} h(t)\sqrt{2-t^2}dt \to \frac{1}{8}\sum_{k=1}^{\infty} k a_k^2,$$

as $N \to \infty$ with $a_k = \frac{2}{\pi}\int_0^{\pi} h(\sqrt{2}\cos\theta)\cos k\theta d\theta$.

COROLLARY 2.8. *Consider the measure (1.1) on $\mathscr{H}_N$ with $N = M$. Put $X_k = \operatorname{Tr} T_k(\Phi) - N\alpha_k$, $k = 1,2,\dots,2m$, where T_k is the Chebyshev polynomial of degree k and $\alpha_k = \int_{\mathbb{R}} T_k(t)d\mu_V(t)$. Let $Y_1,\dots,Y_{2m}$ be independent standard normal random variables. Then*

$$(X_1, X_2, \dots, X_{2m}) \Rightarrow \left(\frac{1}{2}Y_1, \frac{\sqrt{2}}{2}Y_2, \dots, \frac{\sqrt{2m}}{2}Y_{2m}\right) \tag{2.13}$$

as $N \to \infty$, where $\Rightarrow$ denotes convergence in distribution of $\mathbb{R}^{2m}$-valued random variables.

Proof. See Section 3.

The analogous result also holds for the orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) ensembles with $\alpha_k = \int_{\mathbb{R}} = T_k(t)[d\mu_V(t) + (1/N)(2/\beta - 1)d\nu_V(t)]$ and the right-hand side of (2.13) multiplied by $\sqrt{2/\beta}$. See [DS] for an analogous result for random matrices from the compact classical groups.

The orthonormal polynomials $p_{n,M}$ satisfy the recursion relation (1.10). The 1-point function $u_{N,M}(t)$ can be expressed in terms of the orthonormal polynomials as follows (see [Me]):

$$u_{N,M}(t) = \frac{1}{N}\sum_{n=0}^{N-1} p_{n,M}(t)^2 e^{-MV(t)}. \tag{2.14}$$

Hence

$$p_{N,M}(t)^2 e^{-MV(t)} = (N+1)u_{N+1,M}(t) - Nu_{N,M}(t). \tag{2.15}$$

Combining (1.10) and (2.15), we can obtain asymptotic information about the recursion coefficients through asymptotic information on the moments of $u_{N,M}$. In this way, we can prove the following theorem.

THEOREM 2.9. *Assume that $V \in \mathscr{V}$, and let* $\operatorname{supp} \mu_V = [a, b]$. *Then*
(i) $R_{N,M_N} \to (b-a)^2/16$ *and*
(ii) $S_{N,M_N} \to (a+b)/2$
as $N \to \infty$, provided that $N/M_N \to 1$.

Proof. See Section 5.

As mentioned in the Introduction, this result is closely related to the Freud conjecture (see also Remark 5.1).

The proof of Theorem 2.4 is based on the following identity. Let $\lambda \in \mathbb{R}$ be a parameter and note that

$$\frac{d}{d\lambda}\log E_{N,M}^{\beta}\left(\exp\left(\lambda\sum_j h(x_j)\right)\right) = N\int_{\mathbb{R}} h(t)u_{N,M}^{\beta,\lambda h}(t)\,dt, \tag{2.16}$$

where $u_{N,M}^{\beta,\lambda h}(t)$ is the 1-point function of the probability density (2.3). Hence

$$E_{N,M}^{\beta}\left(\exp\left(\sum_j h(x_j)\right)\right) = \exp\left(N\int_0^1\left(\int_{\mathbb{R}} h(t)u_{N,M}^{\beta,\lambda h}(t)\,dt\right)d\lambda\right).$$

Thus, to get the next order asymptotics, we have to compute the second order asymptotics of $u_{N,M}^{\beta,\lambda h}(t)$. Let us first give a heuristic argument. Heuristically, we should have

$$\int_a^b \frac{u_{N,M}^h(t)}{t-s}dt \approx -\frac{1}{2}V'(s) + \frac{2}{\beta N}h'(s)$$

and hence, using (2.7),

$$\int_a^b \frac{N(u_{N,M}^h(t) - u(t))}{t-s}dt \approx \frac{2}{\beta}h'(s). \tag{2.17}$$

By [Tr, p. 173], this gives

$$N(u_{N,M}^h(t) - u(t)) \approx \frac{2}{\beta}\delta^h(t)$$

with $\delta^h(t)$ as in (2.10). This heuristic argument gives the correct answer when $\beta = 2$, but misses $d\nu_V(t)$ if $\beta \neq 2$.

The main problem is to make this rigorous. Our principal tool will be an equation which can be thought of as a kind of *variational equation for finite N*. This equation is obtained as follows. Let $\phi \in C^1(\mathbb{R})$ and assume that ϕ' is bounded from below. Make the change of variables $x_j = y_j + \lambda\phi(y_j)$ in the integral

$$Z_{N,M}^{\beta,h} = \int_{\mathbb{R}^N} e^{\frac{\beta}{2}\left(\sum_{i\neq j}\log|x_i - x_j| - M\sum_j V(x_j)\right) + \sum_j h(x_j)} d^N x.$$

We assume that $\lambda \geqslant 0$ is so small that $\lambda\phi'(y) > -1$ for all $y \in \mathbb{R}$. Since clearly $\frac{d}{d\lambda}\log Z_{N,M}^h|_{\lambda=0+} = 0$, differentiating the integral, after having made the change of variables, gives the identity

$$\frac{\beta}{2}N(N-1)\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\phi(t)-\phi(s)}{t-s}u_{N,M}^h(t,s)dt\,ds - \frac{\beta}{2}NM\int_{\mathbb{R}} V'(t)\phi(t)u_{N,M}^h(t)dt$$
$$+ N\int_{\mathbb{R}} h'(t)\phi(t)u_{N,M}^h(t)dt + N\int_{\mathbb{R}}\phi'(t)u_{N,M}^h(t)dt = 0, \tag{2.18}$$

where $u_{N,M}^h(t,s)$ is the 2-point function of (2.3). The idea of making the change of variables above comes from [Jo1], but the trick of differentiating with respect to a parameter in a change of variables, reparametrization invariance, is common in quantum field theory. Equation (2.18) is the basis for all the main results in this paper, and the next section is devoted to the analysis of this “variational formula.”

3. The variational formula and its consequences

3.1. The fundamental equation and first-order asymptotics. We will analyze the asymptotic properties of $u_{N,M}$ through its Stieltjes transform

$$U_{N,M}(z) = \int_{\mathbb{R}} \frac{u_{N,M}(t)}{z-t}\,dt,$$

defined for all $z \in \mathbb{C}\backslash\mathbb{R}$. From now on, we will usually omit the superscripts β and h. Put

$$k^h_{N,M}(t,s) = Nu^h_{N,M}(t)u^h_{N,M}(s) - (N-1)u^h_{N,M}(t,s).$$

For $z \in \mathbb{C}\backslash\mathbb{R}$, we define

$$\begin{aligned} K_{N,M}(z) &= N\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\frac{1}{(z-t)(z-s)} - \frac{1}{2}\frac{1}{(z-t)^2} - \frac{1}{2}\frac{1}{(z-s)^2} \right) k_{N,M}(t,s)\,dtds \\ &= -\frac{N}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(t-s)^2}{(z-t)^2(z-s)^2} k_{N,M}(t,s)\,dtds \qquad (3.1) \end{aligned}$$

and

$$H_{N,M}(z) = \int_{\mathbb{R}} \frac{h'(t)u_{N,M}(t)}{z-t}\,dt.$$

The real and imaginary parts of $t \to (z-t)^{-1}$, $z \in \mathbb{C}\backslash\mathbb{R}$, satisfy the conditions on ϕ in (2.18), and hence we can choose $\phi(t) = (z-t)^{-1}$. This gives the *fundamental equation*

$$\begin{aligned} NU_{N,M}(z)^2 &- MV'(z)U_{N,M}(z) + MT_{N,M}(z) \\ &= \frac{1}{N}K_{N,M}(z) - \frac{2}{\beta}H_{N,M}(z) + \left(\frac{2}{\beta} - 1\right)U'_N(z), \qquad (3.2) \end{aligned}$$

where

$$T_{N,M}(z) = \int_{\mathbb{R}} \frac{V'(z) - V'(t)}{z-t}\, u_{N,M}(t)\,dt \qquad (3.3)$$

is a polynomial of degree $2\ell - 2$ with leading coefficient $2\ell v_{2\ell}$.

Let

$$U(z) = \int_{\mathbb{R}} \frac{u(t)}{z-t}\,dt,$$

where $u(t)$ is the asymptotic eigenvalue distribution. From (2.6), it follows that $K_{N,M}(z)/N^2 \to 0$ for each $z \in \mathbb{C}\backslash\mathbb{R}$ as $N \to \infty$ and $N/M \to 1$. Taking the limit $N, M \to \infty$, $N/M \to 1$ in (3.2), we see that

$$U(z)^2 - V'(z)U(z) + T(z) = 0, \tag{3.4}$$

where

$$T(z) = \int_{\mathbb{R}} \frac{V'(z) - V'(t)}{z - t} u(t)dt. \tag{3.5}$$

Solving (3.4) gives

$$U(z) = \frac{1}{2}V'(z) - \sqrt{\frac{1}{4}V'(z)^2 - T(z)}, \tag{3.6}$$

where the square root should be defined so that $U(z) \sim 1/z$ as $|z| \to \infty$. Now,

$$u(t) = -\frac{1}{\pi} \lim_{y \to 0+} \operatorname{Im} U(x + iy),$$

and

$$\text{p.v.} \int_{\mathbb{R}} \frac{u(t)}{t - s} dt = -\frac{1}{2} V'(s) \tag{3.7}$$

for all $s \in \operatorname{supp} u$; that is, we obtain (2.7), so we see that (3.2) implies the variational equation (2.7). It is clear from (3.6) that supp u consists of finitely many intervals. The assumption $V \in \mathscr{V}$ means that

$$U(z) = \frac{1}{2}V'(z) - r(z)\sqrt{(z-a)(z-b)}, \tag{3.8}$$

and inversion of (3.7) (see [Tr, p. 178]) gives

$$r(z) = \frac{1}{2\pi} \int_a^b \frac{V'(z) - V'(t)}{z - t} \frac{dt}{\sqrt{(t-a)(b-t)}} \tag{3.9}$$

if supp $u = [a, b]$. Note that $r(z)$ is a polynomial of degree $2\ell - 2$ with leading coefficient $\ell v_{2\ell}$.

In general, it is not an easy problem to decide whether a given polynomial V belongs to $\mathscr{V}$. Let us give some examples.

PROPOSITION 3.1. *If $V \in \mathbb{P}_{2\ell}^{+}$ is convex, then $V \in \mathscr{V}$.*

Proof. To show that supp u is an interval, it suffices to show that it is connected since we know it is compact. Assume that supp $u \subseteq (-\infty, s_0] \cup [s_1, \infty)$, $s_0 < s_1$, where $s_0, s_1 \in$ supp u. Following [Ch, p. 265], we observe that equation (3.9) is valid at s_0 and s_1, and hence

$$\int \frac{u(t)}{t-s_0}\,dt = -\frac{1}{2}V'(s_0) \geqslant -\frac{1}{2}V'(s_1) = \int \frac{u(t)}{t-s_1}\,dt.$$

But

$$s \longrightarrow \int_{-\infty}^{s_0} \frac{u(t)}{t-s}\,dt + \int_{s_1}^{\infty} \frac{u(t)}{t-s}\,dt$$

is a strictly increasing function in (s_0, s_1), which gives a contradiction. Hence supp $u = [a, b]$ for some a, b. From (3.12) we see that $r(t) > 0$ for all $t \in \mathbb{R}$, since V is convex.

Example 3.2. We can use (3.8) and properties of $U(z)$ to get information about $u(t)$ and to compute a, b if supp $u = [a, b]$. Let $Q(z) = (1/4)V'(z)^2 - T(z)$. Then we have the following.

(i) All nonreal zeros of $Q(z)$ must have even multiplicity.

(ii) $\frac{1}{2}V'(z) - \sqrt{Q(z)} \sim 1/z$ as $|z| \to \infty$.

(iii) $\sqrt{Q(z)} = r(z)\sqrt{\prod_{j=1}^{s}(z-a_j)(z-b_j)}$, where $a_1 < b_1 < \cdots < a_s < b_s$ are the odd multiplicity real roots of $Q(z)$. The support of $u(t)$ is $\bigcup_{j=1}^{s}[a_j, b_j]$ and

$$u(t) = -\frac{1}{\pi}\lim_{y\to 0+} \operatorname{Im}\sqrt{Q(x+iy)}$$

is $\geqslant 0$ in the cuts $[a_j, b_j]$.

If, for example, V is even and of degree 4, we get the following result. After a rescaling of the problem, we can assume that

$$V(t) = \frac{1}{4}t^4 + \frac{c}{2}t^2 + d.$$

If $c > -2$, then

$$\text{supp } u = [-A, A], \qquad A = \sqrt{-\frac{2}{3}c + 2\sqrt{\frac{1}{9}c^2 + \frac{4}{3}}},$$

and

$$u(t) = \frac{1}{2\pi}\left(t^2 + \frac{1}{2}A^2 + c\right)\sqrt{A^2 - t^2}, \qquad t \in [-A, A].$$

If $c = -2$, then $u(t) = (1/2\pi)t^2\sqrt{4-t^2}\ 1_{[-2,2]}(t)$, and if $c < -2$, then $u(t) = (1/2\pi)|t|\sqrt{(t^2-a^2)(b^2-t^2)}$ for $t \in [-b,-a] \cup [a,b]$ and $u(t) = 0$ otherwise, where $a = \sqrt{-c-2}$ and $b = \sqrt{-c+2}$. Thus, $V \in \mathscr{V}$ if $c > -2$.

3.2. Second-order asymptotics. We turn now to the investigation of the next order asymptotics of $U_{N,M}(z)$. Let

$$D_{N,M}(z) = N(U_{N,M}(z) - U(z)).$$

It follows from (3.2) and (3.4) that $D_{N,M}(z)$ satisfies the equation

$$\begin{aligned}\frac{1}{N}D_{N,M}(z)^2 &+ (2U(z) - V'(z))D_{N,M}(z) + (N-M)(U_{N,M}(z) - U(z))\\ &+ (N-M)V'(z)U(z) + MT_{N,M}(z) - NT(z)\\ &= \frac{1}{N}K_{N,M}(z) - \frac{2}{\beta}H_{N,M}(z) + \left(\frac{2}{\beta} - 1\right)U_N'(z).\end{aligned} \tag{3.10}$$

Combining (3.4) and (3.8), we see that

$$V'(z)U(z) = -2r(z)\sqrt{(z-a)(z-b)}U(z) + 2T(z)$$

and

$$2U(z) - V'(z) = -r(z)\sqrt{(z-a)(z-b)}.$$

Write $\Delta N = N - M$ and

$$B_{N,M}(z) = 2\Delta NT(z) + MT_{N,M}(z) - NT(z),$$

so that $B_{N,M}(z)$ is a polynomial of degree $2\ell - 2$ with leading coefficient $2\ell\Delta Nv_{2\ell}$. Equation (3.10) can then be written

$$\begin{aligned}\frac{1}{N}D_{N,M}(z)^2 &- 2r(z)\sqrt{(z-a)(z-b)}(D_{N,M}(z) + \Delta NU(z)) + B_{N,M}(z)\\ &= \frac{1}{N}K_{N,M}(z) - \frac{2}{\beta}H_{N,M}(z) + \left(\frac{2}{\beta} - 1\right)U_N'(z).\end{aligned} \tag{3.11}$$

The last equation is the basis for the following central asymptotic result.

PROPOSITION 3.3. *Assume that V and h satisfy the conditions of Theorem 2.4.*

(i) *If $\beta = 2$ and $\Delta N = N - M$ is fixed, then for each $\delta > 0$*

$$D_{N,M}^{2,h}(z) \to -\Delta N U(z) + \frac{1}{\sqrt{(z-a)(z-b)}}\left(\Delta N + \frac{1}{2\pi}\int_a^b \frac{h'(t)\sqrt{(t-a)(b-t)}}{z-t}dt\right) \tag{3.12}$$

uniformly in $|\mathrm{Im}\, z| \geqslant \delta$ as $N \to \infty$.

(ii) *If $\beta \neq 2$, there is a signed measure $d\nu_V(t)$ on $[a,b]$ so that, for each $\delta > 0$,*

$$D_{N,N}^{\beta,h}(z) \to \left(\frac{2}{\beta} - 1\right)\int_{\mathbb{R}} \frac{d\nu_V(t)}{z-t} + \frac{1}{\beta\pi\sqrt{(z-a)(z-b)}}\int_a^b \frac{h'(t)\sqrt{(t-a)(b-t)}}{z-t}dt$$

uniformly in $|\mathrm{Im}\, z| \geqslant \delta$ as $N \to \infty$.

Proof. We postpone the proof of the general case to Section 3.4. Here we will give a proof in the case $V(t) = t^2$, $\beta = 2$, that is, rescaled GUE, which is considerably easier. For simplicity, we also assume $\Delta N = 0$. In this case, $B_{N,M}(z) \equiv 0$, and (3.11) gives

$$D_N(z) = \frac{\frac{1}{N}K_N(z) - H_N(z)}{-4\sqrt{(z-a)(z-b)} + U_N(z) - U(z)}.$$

(When $N = M$, we will often omit the subscript M.) It follows from the assumptions on h, Lemma 4.4, and (2.6), that $U_N(z) \to U(z)$ and

$$H_N(z) \to \int_{\mathbb{R}} \frac{h'(t)u(t)}{z-t}dt := H^h(t), \tag{3.13}$$

where $u(t) = (2/\pi)\sqrt{1-t^2}$, uniformly in $|\mathrm{Im}\, z| \geqslant \delta$. From this, we see that the result follows if we can show that $K_N(z)/N \to 0$ uniformly in $|\mathrm{Im}\, z| \geqslant \delta$.

If $p_{n,M}^h(t)$, $n = 0, 1, \ldots$, are the orthonormal polynomials with respect to the weight $\exp(-MV(t) + h(t))$ on $\mathbb{R}$, we have (see [Me])

$$k_{N,M}(t,s) = \frac{1}{N}\left(\sum_{n=0}^{N-1} p_{n,M}(t)p_{n,M}(s)\right)^2 \exp(-M(V(t) + V(s)) + h(s) + h(t)). \tag{3.14}$$

Now, if we use the Christoffel-Darboux formula (see [Sz1]), we find

$$k_{N,M}(t,s) = \frac{1}{N} R_{N,M} \frac{(p_{N-1,M}(t)p_{N,M}(s) - p_{N-1,M}(s)p_{N,M}(t))^2}{(s-t)^2} \times e^{-M(V(t)+V(s))+h(s)+h(t)}. \tag{3.15}$$

Inserting this in (3.1) and using the orthonormality of the polynomials and $1/|z-t| \leqslant 1/\delta$, we get

$$|K_{N,M}(z)| \leqslant R_{N,M}\delta^{-4}. \tag{3.16}$$

The result we want will now follow if we can show that the sequence $\{R_{N,N}\}_{N\geqslant 1}$ is bounded. Let

$$m_{N,M}(k) = \int_{\mathbb{R}} t^k u_{N,M}(t)dt, \quad k \geqslant 1, \tag{3.17}$$

be the moments of the density $u_{N,M}$; they are all finite by Lemma 4.4. From the recursion formula (1.10) and equation (2.15), it is not difficult to show that

$$(N+1)m_{N+1,M}(2) - Nm_{N,M}(2) = R_{N,M} + R_{N+1,M} + S_{N,M}^2. \tag{3.18}$$

If we take $\phi(t) = t$ in (2.18), we obtain

$$N(N-1) - 4NMm_{N,M}(2) + N\int_{\mathbb{R}} th'(t)u_{N,M}(t)dt + N = 0,$$

and hence

$$(N+1)m_{N+1,N}(2) - Nm_{N,N}(2) = \frac{2N+1}{4N} + \frac{N+1}{4N}\int_{\mathbb{R}} th'(t)u_{N+1,N}(t)dt - \frac{1}{4}\int_{\mathbb{R}} th'(t)u_{N,N}(t)dt.$$

If we use Lemma 4.4 and the assumptions on h, we see that this entails

$$|(N+1)m_{N+1,N}(2) - Nm_{N,N}(2)| \leqslant C$$

for all $N \geqslant 1$ and some constant C. Combining this with (3.18), we get $|R_{N,N}| \leqslant C$. This completes the proof of Proposition 3.3 in this special case.

The second term in the right-hand side of (3.12) is actually a Stieltjes transform. We have the following lemma.

LEMMA 3.4. *For all $z \in \mathbb{C}\backslash\mathbb{R}$,*

$$D^{2,h}(z) := \frac{1}{2\pi\sqrt{(z-a)(z-b)}}\int_a^b \frac{h'(t)\sqrt{(t-a)(b-t)}}{z-t}\,dt = \int_{\mathbb{R}} \frac{\delta^h(t)}{z-t}\,dt, \tag{3.19}$$

where

$$\delta^h(t) = -\frac{1}{2\pi^2\sqrt{(t-a)(b-t)}}\,\text{p.v.}\int_a^b \frac{h'(s)\sqrt{(s-a)(b-s)}}{s-t}\,ds 1_{[a,b]}(t).$$

Proof. For a function $\phi \in L^p$, $p > 1$, the Hilbert transform is defined by

$$H[\phi](t) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{\phi(s)}{s-t}\,ds.$$

Let $I = [a,b]$, $c = (a+b)/2$. Put

$$\phi_1(x) = -\frac{1}{2\pi\sqrt{(x-a)(b-x)}}1_I(x),$$

$$\phi_2(x) = h'(x)\sqrt{(x-a)(b-x)}1_I(x).$$

Then $\phi_1 \in L^{p_1}(\mathbb{R})$, $1 < p_1 < 2$, $\phi_2 \in L^{p_2}(\mathbb{R})$, $1 < p_2 < \infty$, and (see [Tr, p. 169])

$$H[\phi_1 H[\phi_2] + \phi_2 H[\phi_1]] = H[\phi_1]H[\phi_2] - \phi_1\phi_2.$$

Now, $H[\phi_1](x) = (2\pi)^{-1}\text{sgn}(x-c)[(x-a)(b-x)]^{-1/2}1_{I^c}(x)$, and hence

$$\begin{aligned} H[\delta^h](x) &= H[\phi_1 H[\phi_2]](x) = H[\phi_1 H[\phi_2]] + \phi_2 H[\phi_1]](x) \\ &= \frac{\text{sgn}(x-c)1_{I^c}(x)}{2\pi\sqrt{(x-a)(b-x)}}H[h'(t)\sqrt{(t-a)(b-t)}1_I(t)](x) + \frac{1}{2\pi}h'(x)1_I(x). \end{aligned}$$

Consequently,

$$-\pi H[\delta^h](x) - i\pi\delta^h(x) = \lim_{y\to 0+} D^{2,h}(x+iy),$$

and (3.19) follows.

3.3. Proof of Theorem 2.4. As discussed in Section 2, we will use equation (2.16) to prove Theorem 2.4. To do this, we need the following proposition.

PROPOSITION 3.5. *Assume that $\phi \in H^{s+\alpha}$, where $s = 2$ if $\beta = 2$, and $s = 17/2$ if $\beta \neq 2$, $\alpha > 0$. Suppose furthermore that V, h satisfy the conditions of Theorem 2.4. Then*

$$\lim_{N\to\infty} \int_{\mathbb{R}} \phi(t) N(u_{N,N}^{\beta,h}(t) - u(t))dt = \frac{2}{\beta}\int_{\mathbb{R}} \phi(t)\delta^h(t)dt + \left(\frac{2}{\beta} - 1\right) \int_{\mathbb{R}} \phi(t)d\nu_V(t). \tag{3.20}$$

Proof. Let $\varepsilon > 0$ be given. Define ϕ_A by $\hat{\phi}_A(\xi) = \hat{\phi}(\xi)1_{[-A,A]}(\xi)$, and choose A so large that

$$\int_{\mathbb{R}} (1 + 2|\xi|)^{2(s+\alpha)}|\hat{\phi}(\xi) - \hat{\phi}_A(\xi)|^2 d\xi \leqslant \varepsilon, \tag{3.21}$$

which is possible since $\phi \in H^{s+\alpha}$. Let $z = x + iy$, $y > 0$, fixed. Write $\delta_N(t) = \delta_N^{\beta,h}(t) = N(u_{N,N}^{\beta,h}(t) - u(t))$. Then, by Parseval's formula,

$$\begin{aligned}\int_{\mathbb{R}} \phi_A(t)\delta_N(t)dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}_A(-\xi)\hat{\delta}_N(\xi)d\xi = \frac{1}{\pi}\mathrm{Re}\int_0^{\infty} \hat{\phi}_A(-\xi)\hat{\delta}_N(\xi)d\xi \\ &= \frac{1}{\pi}\int_0^{\infty} (\hat{\phi}_A(-\xi)e^{y\xi})(\hat{\delta}_N(\xi)e^{-y\xi})d\xi. \end{aligned}\tag{3.22}$$

When $\mathrm{Im}\, z = y > 0$, we have

$$D_N(z) = \int_{\mathbb{R}} \frac{\delta_N(t)}{z - t}dt = -i\int_0^{\infty} \hat{\delta}_N(\xi)e^{i\xi z}d\xi,$$

and the inversion formula gives

$$-i\hat{\delta}_N(\xi)e^{-y\xi}1_{[0,\infty)}(\xi) = \widehat{D_N(\cdot + iy)}(\xi). \tag{3.23}$$

Since $\hat{\phi}_A$ has compact support, we can define

$$\Phi_A(z) = \int_{\mathbb{R}} \hat{\phi}_A(\xi)e^{i\xi z}d\xi,$$

and the inversion formula gives

$$\hat{\phi}_A(\xi)e^{\xi y} = \widehat{\Phi_A(\cdot - iy)}(\xi). \tag{3.24}$$

Equations (3.22)–(3.24) and Parseval's formula now give

$$\int_{\mathbb{R}} \phi_A(t)\delta_N(t)dt = -2\,\mathrm{Im}\int_{\mathbb{R}} \overline{\Phi_A(x-iy)}D_N(x+iy)dx.$$

Proposition 3.3 and Lemma 3.4 yield

$$\lim_{N\to\infty}\int_{\mathbb{R}} \phi_A(t)\delta_N(t)dt = -2\,\mathrm{Im}\int_{\mathbb{R}} \overline{\Phi_A(x-iy)}D^{\beta,h}(x+iy)dx,$$

where $D^{\beta,h}(z) = \int_{\mathbb{R}}(z-t)^{-1}d\lambda_V(t)$ with $d\lambda_V(t) = (\frac{2}{\beta}-1)d\nu_V(t) + \frac{2}{\beta}\delta^h(t)dt$. By suitably approximating $d\lambda_V(t)$ and reversing the argument above, we get (3.20) with $\phi = \phi_A$. The next step is to remove A. For this we need the following lemma.

LEMMA 3.6. *Let*

$$P_y(t) = \frac{y}{\pi(t^2+y^2)}, \qquad y > 0,$$

be the Poisson kernel for the upper half-plane. Suppose that V, h satisfy the assumptions of Theorem 2.4. Then there is a constant C such that

$$\|P_y * \delta_N^{\beta,h}\|_2^2 \leqslant C\max\left(\frac{\log(1/y)}{y^\kappa}, 1\right)$$

for all $y > 0$, where $\kappa = 4$ if $\beta = 2$ and $\kappa = 17$ if $\beta \neq 2$.

Proof. Since $\mathrm{Im}\, D_N(z) = \pi(P_y * \delta_N)(x)$, $z = x+iy$, we have

$$|(P_y * \delta_N)(x)| \leqslant \frac{1}{\pi}|D_N(x+iy)|.$$

The result for $\beta = 2$ now follows by using the estimate (3.40) below. For $\beta \neq 2$, we use instead the estimate (3.50).

Using this lemma, we can complete the proof of the proposition. If $s > 0$ and $f \in \mathscr{S}'$, then

$$\int_0^\infty e^{-y}y^{2s-1}\|P_y * f\|_2^2\, dy = \Gamma(2s)\int_{\mathbb{R}}(1+2|\xi|)^{-2s}|\hat{f}(\xi)|^2 d\xi. \qquad (3.25)$$

By Parseval's formula, the Cauchy-Schwarz inequality, (3.21), and (3.25), we have

$$\left|\int_{\mathbb{R}}(\phi_A(t)-\phi(t))\delta_N(t)dt\right| = \frac{1}{2\pi}\left|\int_{\mathbb{R}}\overline{(\hat{\phi}_A(\xi)-\hat{\phi}(\xi))}\hat{\delta}_N(\xi)d\xi\right|$$

$$= \frac{1}{2\pi}\left|\int_{\mathbb{R}}\overline{(\hat{\phi}_A(\xi)-\hat{\phi}(\xi))}(1+2|\xi|)^{s+\alpha}\hat{\delta}_N(\xi)(1+2|\xi|)^{-(s+\alpha)}d\xi\right|$$

$$\leqslant \frac{1}{2\pi}\left(\int_{\mathbb{R}}|\hat{\phi}_A(\xi)-\hat{\phi}(\xi)|(1+2|\xi|)^{2(s+\alpha)}\right)^{1/2}\left(\int_{\mathbb{R}}|\hat{\delta}_N(\xi)|^2(1+2|\xi|)^{-2(s+\alpha)}d\xi\right)^{1/2}$$

$$\leqslant \frac{\varepsilon}{2\pi\Gamma(2(s+\alpha))}\left(\int_0^\infty e^{-y}y^{2s-1+2\alpha}\|P_y*\delta_N\|_2^2dy\right)^{1/2}.$$

Now, by Lemma 3.6, this is bounded by a constant times ε, if $s \geqslant \kappa/2$, and together with (3.20) for $\phi = \phi_A$ and the assumptions on ϕ, we get the desired result.

Proof of Theorem 2.4. Given $t_0 > 0$, we can choose $\psi_{t_0} \in C^\infty$ such that $\psi_{t_0}(t) = 1$ if $t \in [-t_0, t_0]$ and $\psi_{t_0}(t) = 0$ if $|t| \geqslant t_0 + 1$. By Lemma 4.4, we can choose $t_0 > 0$ so that

$$\lim_{N\to\infty}\int_{\mathbb{R}} h(t)(1-\psi_{t_0}(t))\delta_N(t)dt = 0.$$

By assumption, $h\psi_{t_0} \in H^{s+\alpha}$, where $s = 2$ if $\beta = 2$ and $s = 17/2$ if $\beta \neq 2$. Thus, Proposition 3.5 gives

$$\lim_{N\to\infty}\int_{\mathbb{R}} h(t)\psi_{t_0}(t)\delta_N^{\beta,h}(t)dt = \frac{2}{\beta}\int_{\mathbb{R}} h(t)\delta^h(t)dt + \left(\frac{2}{\beta}-1\right)\int_{\mathbb{R}} h(t)d\nu_V(t). \tag{3.26}$$

If we define

$$F_N(\lambda) = \log E_N^\beta\left(\exp\left(\lambda\sum_j h(x_j)\right)\right) - \lambda\int_{\mathbb{R}} h(t)u(t)dt$$

for $\lambda \in \mathbb{R}$, then $F_N''(\lambda) \geqslant 0$ (see the proof of Lemma 3.11), and hence $F_N'(\lambda)$ is an increasing function. By (2.16) and (3.26), $F_N'(\lambda) \to \lambda B := F'(\lambda)$ for each $\lambda \in \mathbb{R}$, where B is the right-hand side of (3.26). Now $0 \leqslant F_N'(\lambda) - F_N'(0) \leqslant F_N'(1) - F_N'(0) \leqslant C$ for some constant C. Hence, by dominated convergence,

$$\lim_{N\to\infty} F_N(1) = \lim_{N\to\infty}\left[\int_0^1 F_N'(\lambda) - F_N'(0)d\lambda + F_N'(0)\right] = \int_0^1 \lambda B d\lambda = \frac{1}{2}B.$$

The identity

$$\frac{1}{2}\int_a^b h(t)\delta^h(t)dt = \frac{1}{8}\sum_{k=1}^{\infty} ka_k^2,$$

where a_k is given by (2.12), is obtained as follows. Make the change of variables $t = (a+b)/2 + (b-a)\tau/2$, and write $g(\tau) = h((a+b)/2 + (b-a)\tau/2)$. Then

$$\frac{1}{2}\int_a^b h(t)\delta^h(t)dt = -\frac{1}{4\pi^2}\int_{-1}^1 \frac{g(\tau)}{\sqrt{1-\tau^2}}\left(\int_{-1}^1 \frac{g'(\sigma)\sqrt{1-\sigma^2}}{\sigma-\tau}d\sigma\right)d\tau,$$

and $h(\tau) = \sum_{k=0}^{\infty} a_k T_k(\tau)$. Using $T_k'(\tau) = kU_{k-1}(\tau)$, where U_k, $k \geqslant 0$, are the Chebyshev polynomials of the second kind and

$$\frac{1}{\pi}\int_{-1}^1 \frac{U_{k-1}(\sigma)\sqrt{1-\sigma^2}}{\sigma-\tau}d\sigma = -T_k(\tau),$$

the identity follows by orthonormality. This completes the proof of Theorem 2.4.

3.4. Proof of Proposition 3.3. To prove this key proposition, we need a better understanding of equation (3.11). This is the purpose of the next lemma.

LEMMA 3.7. *Let $X_N(z)$, $A_N(z)$, $B_N(z)$, and $C_N(z)$, $N = 1, 2, \dots$, be sequences of functions analytic in $\mathbb{C}\backslash\mathbb{R}$. Let $\Omega_\delta = \{z \in \mathbb{C}; |\mathrm{Im}\, z| \geqslant \delta\}$ and let $D_r = \{z \in \mathbb{C}; |z| \leqslant r\}$. Assume that for each $\delta > 0$, the following five conditions hold.*

(1) $\frac{1}{N}X_N(z)^2 + A_N(z)X_N(z) + B_N(z) = C_N(z)$ *for all* $z \in \mathbb{C}\backslash\mathbb{R}$.

(2) $X_n(z)/N \to 0$ *uniformly in* Ω_δ.

(3) $A_N(z) \to r(z)a(z)$ *uniformly in* Ω_δ, *where $r(z)$ is a polynomial of degree L with no real roots and* $\inf_{z\in\Omega_\delta}|a(z)| > 0$.

(4) $B_N(z) = \sum_{j=0}^{L} \beta_{j,N} z^j$ *and* $\beta_{L,N} \to \beta_L$ *as* $N \to \infty$.

(5) *Let r_0 be such that $r(z) \neq 0$ if $|z| \geqslant r_0$. There is a function $K(\delta, N)$ satisfying $K(\delta, N)/N \to 0$ as $N \to \infty$ for each δ, so that $|C_N(z)| \leqslant K(\delta, N)$ for all $z \in \Omega_\delta \cap D_{2r_0}$ and $|C_N(z)/a(z)r(z)| \leqslant K(\delta, N)$, if $z \in \Omega_\delta \cap D_{r_0}^c$.*

Then there is a constant C and a $\delta_0 > 0$ so that

(i) $\max_{0\leqslant j\leqslant L}|\beta_{j,N}| \leqslant CK(\delta_0, N)$ *for all* $N \geqslant 1$.

If, furthermore, $C_N(z) \to C(z)$ uniformly in Ω_δ as $N \to \infty$ for any $\delta > 0$, then

(ii) $B_N(z) \to B(z)$ *as* $N \to \infty$, *and*

(iii) $X_N(z) \to (C(z) - B(z))(r(z)a(z))^{-1}$ *uniformly in* Ω_δ *as* $N \to \infty$ *for any* $\delta > 0$.

Proof. Choose $\delta_0 > 0$ so that all zeros of $r(z)$ lie in $\Lambda = \{z \in \mathbb{C}; |\mathrm{Im}\, z| > 2\delta_0, |z| < 3r_0/2\}$. Put

$$\eta_N = \sup_{z\in\Omega_{\delta_0}} |X_N(z)|,$$

which is finite by (2). By Cauchy's integral formula, there is a constant C such that

$$\max_{0 \leqslant j \leqslant L} \sup_{z \in \Lambda} \left| \frac{d^j}{dz^j} X_N(z) \right| \leqslant C\eta_N. \tag{3.27}$$

By Hurwitz' theorem, $A_N(z)$ has exactly L zeros, when $N \geqslant N_0$, in Λ (counting multiplicities), which converge to the zeros of $r(z)$. From (5) and Cauchy's integral formula, we get

$$\max_{0 \leqslant j \leqslant L} \sup_{z \in \Lambda} \left| \frac{d^j}{dz^j} C_N(z) \right| \leqslant CK(\delta_0, N) \tag{3.28}$$

for some constant C. We can write

$$B_N(z) = C_N(z) - A_N(z)X_N(z) - \frac{1}{N} X_N(z)^2. \tag{3.29}$$

An interpolation argument at the zeros of $A_N(z)$ using (4), (3.27), (3.28), and (3.29) and its derivatives, shows that

$$\max_{0 \leqslant j \leqslant L} |\beta_{j,L}| \leqslant C(K(\delta_0, N) + \eta_N^2/N) \tag{3.30}$$

for some constant C. We can also write

$$X_N(z) = \frac{C_N(z) - B_N(z) - X_N(z)^2/N}{A_N(z)}. \tag{3.31}$$

Let $\Lambda' = \Omega_{\delta_0} \backslash \Lambda$. By (3) and the definition of Λ, we can choose N_1 so that

$$|A_N(z)| \geqslant \frac{1}{2} |a(z)||r(z)| \geqslant C > 0$$

for all $N \geqslant N_1$ and $z \in \Lambda'$. Now, using (5), (3.30), (3.31), and $\deg B_N(z) \leqslant L = \deg r(z)$, we get

$$\sup_{z \in \Lambda'} |X_N(z)| \leqslant C(K(\delta_0, N) + \eta_N^2/N).$$

By the maximum principle, the same bound holds in Λ. Hence

$$\eta_N \leqslant C(K(\delta_0, N) + \eta_N^2/N)$$

or

$$\left(\frac{\eta_N}{N}\right)^2 - \frac{1}{C}\frac{\eta_N}{N} + \frac{K(\delta_0, N)}{N} \geqslant 0. \tag{3.32}$$

By (2) and (5), there is an N_2 such that $K(\delta_0, N)/N < 1/4C^2$ and $\eta_N < 1/2C$ if $N \geqslant N_2$. Inequality (3.32) then implies

$$\eta_N \leqslant CK(\delta_0, N) \tag{3.33}$$

for some constant C. Combining (3.30) and (3.33), we obtain (i) for all $N \geqslant \max(N_0, N_1, N_2)$ and for all $N \geqslant 1$ by adjusting the constant C.

The number δ_0 above was arbitrary, but sufficiently small, so for any fixed $\delta > 0$, sufficiently small, we see that (3.33) implies $X_N(z)^2/N \to 0$ uniformly in Ω_δ as $N \to \infty$. Assume now that $C_N(z)$ converges. An interpolation argument using (3.29) and the zeros of $A_N(z)$ shows that $B_N(z) \to B(z)$ as $N \to \infty$, where B is a polynomial of degree $\leqslant L$. Hence $B_N(z)/r(z)a(z)$ converges uniformly to $B(z)/r(z)a(z)$ in $\Lambda'' = \Omega_\delta \backslash \Lambda$. The limit of (3.29) shows that $B(z) - C(z)$ is zero at the zeros of $r(z)$, so $(C(z) - B(z))/r(z)a(z)$ is analytic in $\mathbb{C} \backslash \mathbb{R}$. Now, (3.31) shows that $X_N(z) \to (C(z) - B(z))/r(z)a(z)$ uniformly in Λ'' as $N \to \infty$, and hence, by the maximum principle, in Ω_δ. This completes the proof of the lemma.

By applying this lemma to the $D_N(z)$-equation (3.11), we can prove Proposition 3.3 in the case $\beta = 2$.

Proof of Proposition 3.3 (i). We can assume that $\deg V \geqslant 4$, since the quadratic case has already been dealt with. Take $X_N(z) = D_{N,M}(z)$, $A_N(z) = r(z)(-2\sqrt{(z-a)(z-b)})$ and $C_N(z) = 2r(z)\sqrt{(z-a)(z-b)}\Delta N U(z) + K_{N,M}(z)/N - H_{N,M}(z)$, where $\Delta N = N - M$ is fixed.

By (3.14) we have $k_N \geqslant 0$, and the definition of k_N gives $\int k_N = 1$. Consequently, (3.1) implies $|K_{N,M}(z)| \leqslant 2/\delta^2$ if $z \in \Omega_\delta$. The assumptions on h and Lemma 4.4 give $|H_{N,M}(z)| \leqslant C/\delta$ if $z \in \Omega_\delta$. We see that the conditions of Lemma 3.7 are satisfied with $K(\delta, N)$ independent of N. Now,

$$B_{N,M}(z) = \int_{\mathbb{R}} \frac{V'(z) - V'(t)}{z - t}\left(\Delta N u(t) + \frac{M}{N} N(u_{N,M}(t) - u(t)\right) dt.$$

This implies that $N(m_{N,M}(k) - m(k))$, $N \geqslant 1$, is a bounded sequence for $k = 0, 1, \ldots, 2\ell - 2$, by the next lemma. Here $m_{N,M}(k)$ are the moments of $u_{N,M}(t)$ (see (3.17)), and $m(k)$ are the moments of $u(t)$.

LEMMA 3.8. *Let $d\mu(s) \in \mathcal{M}^1(\mathbb{R})$ be such that all the moments*

$$\tau_k = \int_{\mathbb{R}} s^k d\mu(s)$$

$k = 0, \dots, 2\ell - 2$, exist, and let $V \in \mathbb{P}^+_{2\ell}$. Put

$$T(z) = \int \frac{V'(z) - V'(s)}{z - s}\, d\mu(s) = \sum_{k=0}^{2\ell-2} t_k z^k.$$

If $\tau = (\tau_0, \dots, \tau_{2\ell-2})$ and $t = (t_0, \dots, t_{2\ell-2})$, then there is an invertible matrix $\mathbb{A}$, depending only on V, such that

$$\tau = \mathbb{A} t.$$

Proof. This is a straightforward computation that gives $\mathbb{A} = \mathbb{V}^{-1}$, where

$$\mathbb{V} = \begin{pmatrix} 2v_2 & 3v_3 & 4v_4 & \cdots & (2\ell-1)v_{2\ell-1} & 2\ell v_{2\ell} \\ 3v_3 & 4v_4 & 5v_5 & \cdots & 2\ell v_{2\ell} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2\ell v_{2\ell} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

if $V(t) = \sum_{k=0}^{2\ell} v_k t^k$.

Continuing the proof of Proposition 3.3, we observe that $(N+1)m_{N+1,M}(2) - N m_{N,M}(2)$, $N \geqslant 1$, is a bounded sequence. Using (3.16) and (3.18), this implies

$$|K_{N,M}(z)| \leqslant C\delta^{-4}. \tag{3.34}$$

Consequently, $C_N(z) \to C(z) = 2r(z)\sqrt{(z-a)(z-b)}\Delta N U(z) - H^h(z)$ uniformly in Ω_δ as $N \to \infty$ with $H^h(z)$ given by (3.13). According to Lemma 3.7,

$$D^h_{N,M}(z) \to \frac{2r(z)\sqrt{(z-a)(z-b)}\Delta N U(z) - B(z) - H^h(z)}{-2r(z)\sqrt{(z-a)(z-b)}} \tag{3.35}$$

uniformly in Ω_δ as $N \to \infty$. Here $B(z)$ is a polynomial of degree $2\ell - 2$ and leading coefficient $2\ell \Delta N v_{2\ell}$. Now,

$$H^h(z) = -\frac{1}{\pi}\int_a^b \frac{r(z) - r(t)}{z - t} h'(t)\sqrt{(t-a)(b-t)}\,dt + r(z)\frac{1}{\pi}\int_a^b \frac{h'(t)\sqrt{(t-a)(b-t)}}{z-t}\,dt$$

$$= r_*(z) + r(z)\frac{1}{\pi}\int_a^b \frac{h'(t)\sqrt{(t-a)(b-t)}}{z-t}\,dt,$$

where $r_*(z)$ is a polynomial of degree $\leqslant 2\ell - 3$. Since the limit in the right-hand side of (3.35) is analytic in $\mathbb{C}\backslash\mathbb{R}$, we see that $(B(z) + r_*(z))/r(z)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$. This is only possible if $B(z) + r_*(z) = kr(z)$ for some constant k; comparing leading coefficients, we find $k = 2$. Hence the right-hand side of (3.35) equals

$$-\Delta N U(z) + \Delta N \frac{1}{\sqrt{(z-a)(z-b)}} + \frac{1}{2\pi\sqrt{(z-a)(z-b)}} \int_a^b \frac{h'(t)\sqrt{(t-a)(b-t)}}{z-t}\,dt,$$

and we have proved Proposition 3.3 in the case $\beta = 2$.

When $\beta \neq 2$, we cannot use orthogonal polynomials to control $K_N(z)$. Fortunately we are saved by a result of Boutet de Monvel, Pastur, and Shcherbina in [BPS]. Modifying their argument slightly, we get the following proposition.

PROPOSITION 3.9. *Assume that V and h satisfy the conditions of Theorem 2.1 and that V and h are locally Hölder continuous; that is, there is a $\gamma > 0$ such that for any $L > 0$,*

$$|V(t_1) - V(t_2)| \leqslant C(L)|t_1 - t_2|^\gamma$$

if $|t_1|, |t_2| \leqslant L$ and analogously for h. Then there is a constant C such that

$$|K_N^h(z)| \leqslant C\left(1 + \frac{1}{|\mathrm{Im}\, z|^\kappa}\right) N\omega_N \tag{3.36}$$

for all $z \in \mathbb{C}\backslash\mathbb{R}$, where $\omega_N = \log N$ and $\kappa = 3$.

We will give a proof of this proposition adapted to the present setting below. We write ω_N instead of $\log N$, since we will show later that (3.36) is true with a smaller ω_N but a larger κ.

Let $X_N(z) = D_N(z)$, $A_N(z) = r(z)\left(-2\sqrt{(z-a)(z-b)}\right)$, and

$$C_N(z) = \frac{1}{N} K_N(z) - \frac{2}{\beta} H_N(z) + \left(\frac{2}{\beta} - 1\right) U_N'(z). \tag{3.37}$$

Then we see, from (3.11) with $\Delta N = 0$ and $\deg B_N(z) \leqslant 2\ell - 3 < L = \deg r(z)$, that the conditions of Lemma 3.7 are satisfied. Using Proposition 3.9, we get the estimate

$$|C_N(z)| \leqslant C\left(1 + \frac{1}{|\mathrm{Im}\, z|^3}\right)(1 + \omega_N),$$

and since $|a(z)| = |-2\sqrt{(z-a)(z-b)}| \geqslant C\,|\mathrm{Im}\, z|^{1/2}$, we see that we can take

$$K(\delta, N) = C(1 + \delta^{-7/2})(1 + \omega_N).$$

Lemma 3.7 then implies that

$$|B_N(z)/r(z)| \leqslant C(1 + \omega_N) \tag{3.38}$$

for all $z \in S(y_0) = \{w;\ 0 < |\operatorname{Im} z| \leqslant y_0\}$ if y_0 is chosen sufficiently small; $r(z) \neq 0$ if $z \in S(y_0)$.

The estimates (3.36) and (3.38) are crucial in the proof of the next lemma, which gives an estimate of $D_N(z)$. We need this result also when $\beta = 2$, and in this case we are able to get a better result.

LEMMA 3.10. *Assume that V and h satisfy the conditions of Theorem 2.4. Then there is a constant C such that*

$$|D_N^{\beta,h}(z)| \leqslant C\frac{(1+|\operatorname{Im} z|^{-7/2})(1+\omega_N)}{|z-a|^{1/2}|z-b|^{1/2}} \tag{3.39}$$

for all $z \in \mathbb{C}\backslash\mathbb{R}$ and all $N \geqslant 1$. If $\beta = 2$, we have the better estimate

$$|D_N^{2,h}(z)| \leqslant \frac{C}{|z-a|^{1/2}|z-b|^{1/2}}\int_{\mathbb{R}} \frac{u_N(t)}{|z-t|^2}\,dt. \tag{3.40}$$

Proof. We will prove (3.39) and then, at the end, explain the changes necessary to obtain (3.40) in the case $\beta = 2$.

Equation (3.11) with $\Delta N = 0$ can be written

$$\frac{D_N(z)}{NR(z)}\left[\frac{D_N(z)}{NR(z)} - 2\right] = \frac{-B_N(z) + K_N(z)/N - H_N(z) + (1-2/\beta)U_N'(z)}{NR(z)^2},$$

where $R(z) = r(z)\sqrt{(z-a)(z-b)}$. Note that if $z \in S(y_0)$, then $|r(z)| \geqslant C > 0$. Using (3.36) and (3.38) to estimate the right-hand side, we find

$$\left|\frac{D_N(z)}{NR(z)}\right|\left|\frac{D_N(z)}{NR(z)} - 2\right| \leqslant \frac{C(1+|\operatorname{Im} z|^{-7/2})(1+\omega_N)}{N|R(z)||z-a|^{1/2}|z-b|^{1/2}} := \lambda_N(z). \tag{3.41}$$

We can choose N_0 so that, when $N \geqslant N_0$,

$$\left|\frac{D_N(w)}{NR(w)}\right| < \frac{1}{2} \tag{3.42}$$

for all w with $|\operatorname{Im} w| = y_0$. We now consider $z \in S(y_0)^+ = S(y_0) \cap \{\operatorname{Im} z > 0\}$; the other case is analogous.

Take $z = x + iy \in S(y_0)^+$ and let ℓ_z be the line segment from z to $w = x + iy_0$. From the definition of $\lambda_N(z)$ we see that there is a constant $c_0 > 0$ such that $|\lambda_N(\zeta)| \leqslant c_0|\lambda_N(z)|$ for all $\zeta \in \ell_z$. We now consider two cases.

(1) Assume $\lambda_N(z) < 1/c_0$. Then $\lambda_N(\zeta) < 1$ for all $\zeta \in \ell_z$. If $|D_N(z)/NR(z) - 2| < 1$, then $|D_N(z)/Nr(z)| > 1$, and by (3.42) and continuity, there is a $\zeta_0 \in \ell_z$ such that $|D_N(\zeta_0)/NR(\zeta_0)| = 1$. But $\lambda_N(\zeta_0) < 1$ and (3.41) cannot hold for $z = \zeta_0$.

Hence $|D_N(z)/NR(z) - 2| \geqslant 1$, and (3.41) implies

$$|D_N(z)| \leqslant N\lambda_N(z)|R(z)|.$$

(2) Assume now that $\lambda_N(z) \geqslant 1/c_0$. Let $c_1 = \max(4c_0, 1)$, and assume that

$$|D_N(z)/NR(z)| > c_1\lambda_N(z). \tag{3.43}$$

Then $|D_N(z)/NR(z) - 2| \geqslant c_1\lambda_N(z) - 2$, and (3.41) gives

$$\lambda_N(z) \geqslant c_1\lambda_N(z)(c_1\lambda_N(z) - 2),$$

but $c_1\lambda_N(z) \geqslant c_1/c_0 \geqslant 4$, which gives a contradiction. Hence (3.43) is false, and $|D_N(z)| \leqslant c_1 N\lambda_N(z)|R(z)|$.

This proves (3.39). If $\beta = 2$, we can use $k_N \geqslant 0$ and the inequality $(|z-t||z-s|)^{-1} \leqslant \frac{1}{2}|z-t|^{-2} + \frac{1}{2}|z-s|^{-2}$ in the definition (3.1) of $K_N(z)$ and the estimate of $B_N(z)$ to see that we can take

$$\lambda_N(z) = \frac{C}{N|R(z)||z-a|^{1/2}|z-b|^{1/2}} \int_{\mathbb{R}} \frac{u_N(t)}{|z-t|^2} dt$$

in this case. This completes the proof of the lemma.

To prove Proposition 3.3, that is, the convergence of $D_N(z)$, we need better estimates of $K_N(z)$. Next, we will show that we can use the estimates we already have to obtain a better estimate of $K_N(z)$.

LEMMA 3.11. *There is a constant C such that*

$$|K_N^h(z)| \leqslant C(1 + \omega_N^2)(1 + 1/|\operatorname{Im} z|^8) \tag{3.44}$$

for all $z \in \mathbb{C}\backslash\mathbb{R}$ and $N \geqslant 1$, where $\omega_N = \log N$.

Proof. Let $g \in C_b(\mathbb{R})$ and define

$$\begin{aligned} S_N(g) &= E_N^{\beta,h}\left[\left(\sum_{j=1}^N g(x_j) - E_N^{\beta,h}\left[\sum_{j=1}^N g(x_j)\right]\right)^2\right] \\ &= -N\int_{\mathbb{R}}\int_{\mathbb{R}} [g(t)g(s) - g(t)^2]k_N^{\beta,h}(t,s)\,dt\,ds. \end{aligned} \tag{3.45}$$

Write, for a fixed $z \in \mathbb{C}\backslash\mathbb{R}$,

$$\phi_z(t) = \operatorname{Re}\frac{\delta^2}{z-t}, \qquad \text{and} \qquad \psi_z(t) = \operatorname{Im}\frac{\delta^2}{z-t},$$

where $\delta = \operatorname{Im} z$. Then

$$\begin{aligned}\delta^4 K_N(z) &= -S_N(\phi_z + i\psi_z)\\ &= (-1+i)S_N(\phi_z) + (1+i)S_N(\psi_z) - iS_N(\phi_z + \psi_z)\end{aligned} \tag{3.46}$$

and $\|\phi_z'\|_\infty, \|\psi_z'\|_\infty \leqslant C$, where C is independent of z.

Let c_N, $N \geqslant 1$, be a given sequence of positive numbers and define, for real-valued $g \in C_b(\mathbb{R})$ and $w \in \mathbb{C}$,

$$F_N(w) = E_N^h\left[\exp\left(\frac{w}{c_N}\left(\sum_j g(x_j) - N\int g(t)u_N(t)dt\right)\right)\right].$$

Observe that for $w \in \mathbb{R}$,

$$\frac{d^2}{dw^2}(\log F_N)(0) = \frac{1}{c_N^2} S_N(g). \tag{3.47}$$

We would like to estimate the left-hand side. If $\lambda \in \mathbb{R}$, we have by Jensen's inequality

$$\begin{aligned}\frac{1}{E_N^h(e^{\lambda\sum_j g(x_j)})} &= E_N^{h+\lambda g}(e^{-\lambda\sum_j g(x_j)})\\ &\geqslant e^{-\lambda E_N^{h+\lambda g}\left(\sum_j g(x_j)\right)} = e^{-\lambda N\int_{\mathbb{R}} g(t)u_N^{h+\lambda g}(t)dt}.\end{aligned}$$

Consequently,

$$|F_N(w)| \leqslant \exp\left(\frac{\operatorname{Re} w}{c_N}\int_{\mathbb{R}} g(t)N(u_N^f(t) - u(t))dt\right),$$

where $f = h + (\operatorname{Re} w/c_N)g$. Note that

$$\int_{\mathbb{R}} \phi_z(t)N(u_N^f(t) - u(t))dt = \delta^2 \operatorname{Re} D_M^f(z).$$

Thus, if $g = \phi_z$, $g = \psi_z$, or $g = \phi_z + \psi_z$, then

$$|F_N(w)| \leqslant \exp\left(\frac{2\delta^2 \operatorname{Re} w}{c_N}|D_N^f(z)|\right). \tag{3.48}$$

By Lemma 3.10,

$$|D_N^f(z)| \leqslant C(1+\delta^{-4})(1+\omega_N),$$

and hence, by (3.48), with $c_N = 2\delta^2 C(1+\delta^{-4})(1+\omega_N)$, we have

$$|F_N(w)| \leqslant \exp(\operatorname{Re} w). \tag{3.49}$$

By (3.49) we have $|F_N(w)| \leqslant 3$ if $|w| = 1$, and thus by Cauchy's integral formula $|F_N'(z)| \leqslant 5$ if $|z| \leqslant 1/6$. Consequently, if $|w| \leqslant 1/6$, then

$$|F_N(w) - 1| = \left| \int_0^w F_N'(z)dz \right| \leqslant \frac{5}{6},$$

where we integrate along the line segment from the origin to w. Hence $|F_N(w)| \geqslant 1/6$ for $|w| \leqslant 1/6$, and if we put

$$G_N(z) = \int_0^z \frac{F_N'(w)}{F_N(w)} dw,$$

so that $G_N(z)$ is a logarithm of $F_N(z)$, then $|G_N(z)| \leqslant 5$ for $|z| \leqslant 1/6$ by the above estimates. Cauchy's integral formula now gives

$$|G_N''(0)| \leqslant 180.$$

Together with (3.46), (3.47), and the definition of c_N, this gives

$$|K_N(z)| \leqslant C\delta^{-4} c_N^2 \leqslant C(1+\delta^{-8})(1+\omega_N)^2,$$

and the lemma is proved.

Note that by the lemma, the estimate (3.36) holds with $\kappa = 8$ and $N\omega_N = 1 + (\log N)^2$, and hence instead of (3.39) we get the estimate

$$|D_N^h(z)| \leqslant C \frac{1 + |\operatorname{Im} z|^{-17/2}}{|z-a|^{1/2}|z-b|^{1/2}}. \tag{3.50}$$

We are now ready for the following proof.

Proof of Proposition 3.3(ii). We take as before $X_N(z) = D_N(z)$, $A_N(z) = r(z)a(z)$ with $a(z) = -2\sqrt{(z-a)(z-b)}$ and $C_N(z)$ given by (3.37). From (3.44) we see that

$$C_N(z) \to C(z) = -\frac{2}{\beta} H^h(z) + \left(\frac{2}{\beta} - 1\right) U'(z)$$

uniformly in $|\mathrm{Im}\, z| \geqslant \delta$ as $N \to \infty$ for each $\delta > 0$. Hence, by Lemma 3.7, $B_N(z) \to B(z)$ with $\deg B \leqslant 2\ell - 3$, and

$$D_N(z) \to D^{\beta,h}(z) = \frac{C(z) - B(z)}{r(z)a(z)}. \tag{3.51}$$

It remains to give a more explicit expression for the right-hand side of (3.51).

As in the proof of Proposition 3.3(i), we can write

$$H^h(z) = r_*(z) + r(z)\frac{1}{\pi}\int_a^b \frac{h'(t)\sqrt{(t-a)(b-t)}}{z-t}\,dt,$$

where $r_*(z)$ is a polynomial of degree $\leqslant 2\ell - 3$. From (3.8) we obtain

$$U'(z) = \frac{1}{2}V''(z) - r'(z)\sqrt{(z-a)(z-b)} - \frac{r(z)(2z-(a+b))}{2\sqrt{(z-a)(z-b)}}.$$

Consequently, by Lemma 3.4,

$$\begin{aligned} D^{\beta,h}(z) = D^{2,h}(z) + \left(\frac{2}{\beta} - 1\right)\Bigg[&\frac{2z-(a+b)}{4(z-a)(z-b)} \\ &- \frac{2\ell - 1}{2\sqrt{(z-a)(z-b)}} + \frac{q(z)}{2r(z)\sqrt{(z-a)(z-b)}} + \frac{r'(z)}{2r(z)}\Bigg], \end{aligned} \tag{3.52}$$

where $q(z)$ is a polynomial of degree $\leqslant 2\ell - 3$. Since $D^{\beta,h}(z)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$ and all the zeros of $r(z)$ are nonreal, we see from (3.52) that $q(z) + r'(z)\sqrt{(z-a)(z-b)} = r(z)g(z)$, where g is analytic in $\mathbb{C}\backslash\mathbb{R}$. Since $\deg q \leqslant \deg r - 1$, we see that q is uniquely determined by this condition. Hence if we give a polynomial q of degree $\leqslant 2\ell - 3$ such that

$$S(z) := \frac{q(z)}{2r(z)\sqrt{(z-a)(z-b)}} + \frac{r'(z)}{2r(z)}$$

is analytic in $\mathbb{C}\backslash\mathbb{R}$, this must be the q we want.

Let $z_1, \bar{z}_1, \ldots, z_p, \bar{z}_p$ be the different zeros of r, and let m_j be the multiplicity of z_j. Then

$$\frac{r'(z)}{2r(z)} = \sum_{j=1}^{p} m_j \frac{z - \mathrm{Re}\, z_j}{(z-z_j)(z-\bar{z}_j)}.$$

Let

$$A_j = \frac{1}{\pi}\int_a^b \frac{tdt}{|t-z_j|^2\sqrt{(t-a)(b-t)}}$$

and

$$B_j = \frac{1}{\pi}\int_a^b \frac{dt}{|t-z_j|^2\sqrt{(t-a)(b-t)}}.$$

The constants c_j and d_j are the unique solutions of the linear system

$$\begin{cases} A_j c_j + B_j d_j &= 1, \\ -|z_j|^2 B_j c_j + (A_j - (2\operatorname{Re} z_j)B_j)d_j &= -\operatorname{Re}\ z_j. \end{cases} \tag{3.53}$$

Put

$$q(z) = 2r(z)\sum_{j=1}^{p} m_j \frac{c_j + d_j}{(z-z_j)(z-\bar{z}_j)}.$$

Then

$$S(z) = \sum_{j=1}^{p} \frac{m_j}{(z-z_j)(z-\bar{z}_j)}\left[\frac{c_j + d_j}{\sqrt{(z-a)(z-b)}} + z - \operatorname{Re} z_j\right].$$

Now,

$$\begin{aligned} F_j(z) &= \frac{1}{\pi}\int_a^b \frac{c_j t + d_j}{|t-z_j|^2\sqrt{(t-a)(b-t)}}\frac{dt}{z-t} \\ &= \frac{1}{(z-z_j)(z-\bar{z}_j)}\left[\frac{c_j + d_j}{\sqrt{(z-a)(z-b)}} + z - \operatorname{Re} z_j\right] \end{aligned}$$

if c_j and d_j are given by (3.53). Thus,

$$S(z) = \sum_{j=1}^{p} m_j F_j(z).$$

Bringing everything together, we see that

$$D^{\beta,h}(z) = D^{2,h}(z) + \left(\frac{2}{\beta} - 1\right)\int_a^b \frac{d\nu_V(t)}{z-t},$$

where

$$d\nu_V(t) = \frac{1}{4}(\delta_a(t) + \delta_b(t)) - \left(\frac{2\ell - 1}{2\pi} - \sum_{j=1}^{p} m_j \frac{c_j t + d_j}{\pi |t - z_j|^2}\right) \frac{dt}{\sqrt{(t-a)(b-t)}}. \quad (3.54)$$

This completes the proof of Proposition 3.3(ii).

We still have to prove Proposition 3.9.

Proof of Proposition 3.9. The proof below is a modified version of the proof given in [BPS] adapted to the present setting. Let g be a complex-valued, bounded continuous function on $\mathbb{R}$ and put

$$\begin{aligned}\Sigma_N(g) &= E_N^{\beta,h}\left[\left|\sum_{j=1}^{N} g(x_j) - E_N^{\beta,h}\left[\sum_{j=1}^{N} g(x_j)\right]\right|^2\right] \\ &= N(N-1)\int_{\mathbb{R}}\int_{\mathbb{R}} g(t)\overline{g(s)}[u_N^{\beta,h}(t,s) - u_N^{\beta,h}(t)u_N^{\beta,h}(s)]dtds \\ &\quad + N\left[\int_{\mathbb{R}} |g(t)|^2 dt - \left|\int_{\mathbb{R}} g(t)u_N^{\beta,h}(t)dt\right|^2\right]. \end{aligned} \quad (3.55)$$

Comparing this with (3.45) and (3.47), we see that

$$|K_N(z)| = \left|S_N\left(\frac{1}{z-t}\right)\right| \leqslant \Sigma_N\left(\frac{1}{z-t}\right). \quad (3.56)$$

Let

$$\begin{aligned} H_1(x) &= \sum_j \log u_N^{\beta,h}(x_j), \\ H_2(x) &= \beta\left[\sum_{i<j} \log|x_i - x_j|^{-1} + \frac{N}{2}\sum_j V(x_j)\right] - \sum_j h(x_j), \end{aligned}$$

so that $\rho_N^{\beta,h}(x) = Z_N^{-1}\exp(-H_2(x))$ and $\lambda_N^{\beta,h}(x) := \exp(-H_1(x)) = \prod_j u_N^{\beta,h}(x_j)$. Let $E_N^*(\cdot)$ denote expectation with respect to the density λ_N. Jensen's inequality gives

$$\log\frac{1}{Z_N} = \log E_N(e^{H_2(x)-H_1(x)}) \geqslant E_N(H_2(x) - H_1(x))$$

and

$$\log Z_N = \log E_N^*(e^{H_1(x)-H_2(x)}) \geqslant E_N^*(H_1(x) - H_2(x));$$

that is,

$$E_N(H_2(x) - H_1(x)) \leqslant E_N^*(H_2(x) - H_1(x)). \tag{3.57}$$

Note that ρ_N and λ_N have the same 1-point functions, so (3.57) implies

$$N(N-1)\int_{\mathbb{R}}\int_{\mathbb{R}} \log|t-s|^{-1}[u_N(t,s) - u_N(t)u_N(s)]dtds \leqslant 0. \tag{3.58}$$

It follows from Lemma 4.4, (3.55), (3.56), and (3.58) that we can choose $L > 0$ so that if $|\operatorname{Im} z| \geqslant \delta > 0$, then

$$|K_N(z)| \leqslant N(N-1)\int_{-1/2}^{1/2}\left(\frac{1}{w-t}\right)\left(G_N\left(\frac{1}{\overline{w}-s}\right)\right)(t)dt + \frac{1}{\delta^2}e^{-N} \tag{3.59}$$

and

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \log|t-s|^{-1}[\rho_N(t,s) - \rho_N(t)\rho_N(s)]dtds \leqslant e^{-N}, \tag{3.60}$$

where $\rho_N(t,s) = u_N(Lt, Ls)$, $\rho_N(t) = u_N(Lt)$, $w = z/L$, and the operator G_N on $L^2(-1/2, 1/2)$ is defined by

$$\begin{aligned}(G_N f)(t) = &\int_{-1/2}^{1/2} f(s)[\rho_N(t,s) - \rho_N(t)\rho_N(s)]ds \\ &+ \frac{\rho_N(t)}{N-1}\left[\frac{1}{L}f(t) - \int_{-1/2}^{1/2}\rho_N(s)f(s)ds\right].\end{aligned}$$

The operator G_N is bounded from L^2 to itself, and from (3.55) we see that $G_N \geqslant 0$. Let us introduce some more operators on $L^2(-1/2, 1/2)$ that we will need. Let

$$p_w(t,s) = \frac{1}{(w-t)(\overline{w}-s)}$$

and

$$a_\varepsilon(t) = \begin{cases} \log(1/t) & \text{for } \varepsilon < t \leqslant 1, \\ \log(1/\varepsilon) - (t-\varepsilon)/\varepsilon & \text{for } 0 \leqslant t \leqslant \varepsilon, \end{cases}$$

for $0 < \varepsilon \leqslant 1/2$, and finally

$$b(t) = \begin{cases} \frac{3}{2} - 2t + \frac{1}{2}t^2 & \text{for } t \geqslant \frac{1}{2}, \\ \frac{5}{8} - \frac{3}{2}(t - \frac{1}{2}) & \text{for } 0 \leqslant t < \frac{1}{2}. \end{cases}$$

Let P_w, A_ε, and B be the bounded operators on $L^2(-1/2, 1/2)$ with kernels $p_w(t,s)$, $a_\varepsilon(|t-s|)$, and $b(|t-s|)$, respectively. Clearly, $P_w \geqslant 0$. Using the fact that if $\ell : [0,1] \to [0,\infty)$ is convex, then

$$\int_{-1}^{1} \ell(|x|) e^{ikx} dx \geqslant 0$$

for all integers k, we get

$$0 \leqslant B \leqslant A_{1/2} \leqslant A_\varepsilon \tag{3.61}$$

for $0 < \varepsilon \leqslant 1/2$ (see [BPS] for further details). Also, the operators P_w, B, and A_ε are trace class.

Since $\log|t-s|^{-1} \geqslant a_\varepsilon(|t-s|) \geqslant 0$ for $s,t \in [-1/2, 1/2]$, we see that (3.60) implies

$$\begin{aligned}
&\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} a_\varepsilon(|t-s|)[\rho_N(t,s) - \rho_N(t)\rho_N(s)]dtds \\
&\quad \leqslant e^{-N} + \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (\log|t-s|^{-1} - a_\varepsilon(|t-s|))\rho_N(t)\rho_N(s)dtds. \qquad (3.62)
\end{aligned}$$

We postpone the proof of the following lemma to Section 4.

LEMMA 3.12. *If $\varepsilon_N = 1/N^{3/\gamma}$, where γ is the Hölder exponent of V and h, then*

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (\log|t-s|^{-1} - a_{\varepsilon_N}(|t-s|))\rho_N(t)\rho_N(s)dtds \leqslant C\frac{\log N}{N}$$

for some constant C.

Since A_ε is trace class and G_N is bounded, $A_\varepsilon G_N$ is trace class, and (3.62), the definition of G_N, and Lemma 3.12 give

$$\operatorname{Tr} A_{\varepsilon_N} G_N \leqslant C\frac{\log N}{N} + \frac{a_{\varepsilon_N}(0)}{L(N-1)} \leqslant C\frac{\log N}{N}. \tag{3.63}$$

A straightforward computation gives

$$b_k = \frac{1}{2}\int_{-1}^{1} b(|t|)\cos \pi k t dt \geqslant \frac{1}{6\pi^2 k^2} \tag{3.64}$$

for all integers $k \neq 0$. Following [BPS], we now define

$$q(t) = \begin{cases} 2(t+1)(w+1/2)^{-1} & \text{for } 1 < t \leqslant -1/2, \\ (w-t)^{-1} & \text{for } -1/2 < t \leqslant 1/2, \\ 2(1-t)(w-1/2)^{-1} & \text{for } 1/2 < t < 1. \end{cases}$$

Using (3.64), we get

$$\begin{aligned} \sum_{-\infty}^{\infty} b_k^{-1}|q_k|^2 &\leqslant \frac{1}{b_0}|q_0| + 6\pi^2 \sum_{-\infty}^{\infty} k^2 |q_k|^2 \\ &\leqslant \frac{1}{b_0} + C\int_{-1}^{1} |q'(x)|^2 dx \leqslant CL^3(1+\delta^{-3}) \end{aligned} \tag{3.65}$$

if $|\operatorname{Im} z| \geqslant \delta$ $(w = z/l)$. Here, q_k are the Fourier coefficients of q as a 2-periodic function, and C is a numerical constant. If $f \in L^2(-1/2, 1/2)$ and f^* is the 2-periodic function defined by $f^*(t) = f(t)$ if $|t| \leqslant 1/2$ and $f^*(t) = 0$ if $1/2 \leqslant |t| \leqslant 1$, then

$$\begin{aligned} (P_w f, f) &= \left| \int_{-1}^{1} f^*(t)\overline{q(t)}dt \right|^2 = 4\left| \sum_{-\infty}^{\infty} b_k^{1/2} f_k b_k^{-1/2} \bar{q}_k \right|^2 \\ &\leqslant 4\left(\sum_{-\infty}^{\infty} b_k |f_k|^2 \right)\left(\sum_{-\infty}^{\infty} b_k^{-1} |q_k|^2 \right) \leqslant CL^3(1+\delta^{-3})(Bf, f), \end{aligned}$$

where f_k are the Fourier coefficients of f^*, $(\cdot,\cdot)$ is the scalar product on $L^2(-1/2, 1/2)$, and we have used the estimate (3.65) in the last inequality. Thus, by (3.61),

$$P_w \leqslant CL^3(1+\delta^{-3})B \leqslant CL^3(1+\delta^{-3})A_\varepsilon.$$

Since $G_N \geqslant 0$, the estimate (3.63) now gives

$$\operatorname{Tr} P_w G_N \leqslant CL^3\left(1 + \frac{1}{\delta^3}\right)\frac{\log N}{N}.$$

Combining this with (3.49), we get

$$|K_N(z)| \leqslant CL^3\left(1 + \frac{1}{\delta^3}\right) N \log N,$$

which completes the proof.

Proof of Corollary 2.8. Let $\xi = (\xi_1, \dots, \xi_{2m}) \in \mathbb{R}^{2m}$, and consider

$$E_N\left(\exp\left(-\sum_{k=1}^{2m} X_k - i\xi \cdot X\right)\right) := L_N(\xi).$$

We will prove that this converges to

$$\exp\left(\frac{1}{8}\sum_{k=1}^{2m} k(1+i\xi_k)^2\right)$$

as $N \to \infty$ for every $\xi \in \mathbb{R}^{2m}$. Corollary 2.8 then follows by standard arguments. Let $D = \{z \in \mathbb{C}; |z| < 2\} \times \{z \in \mathbb{C}; 1 < \operatorname{Re} z < 3\}$, an open subset of $\mathbb{C}^2$, and consider

$$F_N(z_1, z_2, \xi) = E_N\left[\exp\left(-\sum_{\mu=1}^{N}\left(\sum_{k=1}^{2m-1}(1 + z_1\xi_k)X_k(y_\mu) + z_2X_{2m}(y_\mu)\right)\right)\right]$$

so that $L_N(\xi) = F_N(i, 1 + \xi_{2m}i, \xi)$. F_N is holomorphic as a function of (z_1, z_2) in D. We have

$$|F_N(z_1, z_2, \xi)| \leqslant E_N\left[\exp\left(-\sum_{\mu=1}^{N}\sum_{k=1}^{2m-1}(1 + \operatorname{Re} z_1\xi_k)X_k(y_\mu) + \operatorname{Re} z_2X_{2m}(y_\mu)\right)\right].$$

The function in the exponent is a polynomial $h(y_\mu)$ of degree $2m$ which satisfies the conditions of Theorem 2.4. Since $\operatorname{Re} z_1 \in [-2, 2]$ and $\operatorname{Re} z_2 \in [1, 3]$, the proof of Theorem 2.4 shows that there is a constant such that

$$|F_N(z_1, z_2, \xi)| \leqslant C \tag{3.66}$$

for all $z \in D$ and all $N \geqslant 1$. (Note that $\int_{-1}^{1} h(y)u(y)dy = 0$.) If $(z_1, z_2) \in [-2, 2] \times [1, 3]$, then Theorem 2.4 implies that

$$F_N(z_1, z_2, \xi) \to \exp\left(\frac{1}{8}\sum_{k=1}^{2m-1} k(1 + z_1\xi_k)^2 + \frac{2m}{8}z_2^2\right) \tag{3.67}$$

as $N \to \infty$. Using (3.66), (3.67), and a standard compactness argument, we see that (3.67) holds uniformly in any compact subset of D. In particular,

$$L_N(\xi) \to e^{\frac{1}{8}\sum_{k=1}^{2m} k(1+i\xi_k)^2},$$

which is what we wanted to prove.

4. The analogue of the weak Szegö theorem. In this section, we will prove Theorem 2.1. We shall first outline and sketch the proofs of the potential theoretic results we will need. Let $E \subseteq \mathbb{R}$ be closed, and let $V : E \to [0, \infty]$ satisfy the following:

(i) V is lower semicontinuous (l.s.c);
(ii) $E_0 = \{x \in E; V(x) < \infty\}$ has positive logarithmic capacity; and
(iii) if E is unbounded and $\psi_V(t) = V(t) - \log(t^2 + 1)$, then

$$\lim_{|t| \to \infty, t \in E} \psi_V(t) = +\infty.$$

Without loss of generality, we can assume $V \geqslant 0$. Write

$$k_V(t,s) = \log|t-s|^{-1} + \frac{1}{2}V(t) + \frac{1}{2}V(s).$$

We have, for all $s, t \in E$,

$$\psi_V(t) \geqslant C \tag{4.1}$$

for some constant C, and

$$k_V(t,s) \geqslant \frac{1}{2}\psi_V(t) + \frac{1}{2}\psi_V(s), \tag{4.2}$$

since $\log|t-s|^{-1} \geqslant -\frac{1}{2}\log[(t^2+1)(s^2+1)]$ for all $s, t \in \mathbb{R}$. Consider the following weighted analogue of the classical logarithmic energy problem. Let $\mathscr{M}^1(E)$ be the set of all probability measures on E. Introduce the *energy functional*

$$I_V[\mu] = \int\int k_V(t,s)\,d\mu(t)\,d\mu(s).$$

It follows from (4.1) and (4.2) that $I_V[\mu]$ is well defined, possibly $= +\infty$. Put

$$F_V = \inf_{\mu \in \mathscr{M}^1(E)} I_V[\mu]. \tag{4.3}$$

Condition (ii) guarantees that $F_V < \infty$.

If $w = \exp(-1/2V)$, then w is an *admissible weight* in the terminology of [MS2], and according to [MS1] there is a unique measure μ_V for which the infimum in (4.3) is assumed. μ_V has compact support and finite logarithmic energy. The number $\exp(-F_V)$ is called the *w-capacity* of E. For completeness, we will outline a proof of these facts below. See [MS1] and [MS2] for further details and a more systematic development.

A sequence of measures $\{\mu_n\}_1^\infty$ in $\mathscr{M}^1(E)$ *converges weakly* to $\mu \in \mathscr{M}^1(E)$ if $\mu_n(f) \to \mu(f)$ for every $f \in C_b(E)$, the space of bounded continuous functions on E. The sequence $\{\mu_n\}_1^\infty$ is *tight* if, given $\varepsilon > 0$, there is an $A > 0$, such that $\mu_n(E \cap [-A, A]^c) < \varepsilon$, $n = 1, 2, \ldots$. A tight sequence in $\mathcal{M}^1(E)$ has a weakly converging subsequence (see [Bi]).

Proof of Theorem 2.1 (a). We first show that if $\mu_n \rightharpoonup \mu$, then

$$\liminf_{n\to\infty} I_V[\mu_n] \geqslant I_V[\mu]. \tag{4.4}$$

Since V is l.s.c. on E, there is an increasing sequence V_m of continuous functions on E such that $V_m \nearrow V$ on E. Now,

$$I_V[\mu_n] \geqslant I_{V_m}[\mu_n] \geqslant \int\int \min(L, k_{V_m}(t,s))d\mu_n(t)d\mu_n(s).$$

As $\min(L, k_{V_m}(t,s))$ is continuous on $E \times E$, we find

$$\liminf_{n\to\infty} I_V[\mu_N] \geqslant \int\int \min(L, k_{V_m}(t,s))d\mu(t)d\mu(s).$$

Letting $L \to \infty$ and $m \to \infty$ and using monotone convergence, we get (4.4).

Choose a sequence $\{\mu_n\}_1^\infty$ in $\mathcal{M}^1(E)$ such that

$$I_V[\mu_n] \leqslant F_V + 1/n. \tag{4.5}$$

It follows from (4.2) and (iii) that $\{\mu_n\}_1^\infty$ is tight. Hence there is a weakly converging subsequence with a weak limit $\mu \in \mathcal{M}^1(E)$, and $I_V[\mu] = F_V$ by (4.4) and (4.5); also $\int \psi_V d\mu < \infty$ by (4.2). Let D be any measurable subset of E with $\mu(D) > 0$. Put

$$\mu_\varepsilon = (1 + \varepsilon\mu(D))^{-1}(\mu + \varepsilon\mu|_D)$$

with $\varepsilon \in (-1, 1)$. Then $\mu_\varepsilon \in \mathcal{M}^1(E)$. If we let $f(\varepsilon) = I_V[\mu_\varepsilon]$, we must have $f'(0) = 0$, which gives

$$\int\int [k_V(s,t) - 2I_V[\mu]]d\mu(s)d\mu|_D(t) = 0.$$

Hence, by (4.2),

$$\int [\psi_V(t) - 2I_V[\mu] + \int \psi_V(s)d\mu(s)]d\mu|_D(t) \leqslant 0. \tag{4.6}$$

By (iii), we can choose T so that $\psi_V(t) - I_V[\mu] + \int \psi_V(s)d\mu(s) \geqslant 1$, when $|t| \geqslant T$. Hence, if $D \subseteq [-T, T]^c$, then (4.6) gives $0 \leqslant \mu(D) \leqslant 0$, a contradiction. Thus we must have $\operatorname{supp}\mu \subseteq [-T, T]$. Since $V \geqslant 0$ and $\operatorname{supp}\mu_V$ is compact, μ_V must have finite logarithmic energy and $\int V d\mu_V < \infty$.

Example 4.1. A condition like (iii) is necessary if we want μ_V to exist and have compact support. If $V(t) = (1-\varepsilon)\log(t^2+1)$, $\varepsilon > 0$, then it is not difficult to see that $M_V = -\infty$. Let $V(t) = \log(t^2+1)$. Then the change of variables $t = \tan(\theta/2)$, $s = \tan(\phi/2)$ transforms the minimum problem (4.3) to

$$\inf_{\nu \in \mathscr{M}^1(\mathbb{T})} \int_{\mathbb{T}} \int_{\mathbb{T}} \log|e^{i\theta} - e^{i\phi}|^{-1} d\nu(e^{i\theta}) d\nu(e^{i\phi}),$$

which has the unique solution: ν equals uniform measure on $\mathbb{T}$. Hence the unique minimizing measure for $V(t) = \log(t^2+1)$ is the Cauchy measure

$$d\mu_V(t) = \frac{1}{\pi}\frac{1}{1+t^2} dt,$$

which does not have compact support.

Let $x = (x_1, \ldots, x_N) \in E^N$. Define

$$K_{N,V}(x) = \sum_{1 \leqslant i \neq j \leqslant N} k_V(x_i, x_j),$$

and

$$d_{N,V} = \frac{1}{N(N-1)} \inf_{x \in E^N} K_{N,V}(x).$$

If $\delta_{N,V} = \exp(-d_{N,V})$, then $\inf_{N \geqslant 2} \delta_{N,V} = \delta_V$ is the analogue of the *transfinite diameter* in the present setting, and we have $\delta_V = \exp(-F_V)$ (see [MS2]). It is not difficult to show that $\delta_{N+1,V} \leqslant \delta_{N,V}$ (see [MS2, p. 120]), and hence $\{d_{N,V}\}_{N=2}^{\infty}$ is an increasing sequence. For arbitrary $x \in E^N$,

$$\frac{1}{N(N-1)} K_{N,V}(x) \geqslant d_{N,V}.$$

If we integrate this inequality with respect to $d\mu_V(x_1) \cdots d\mu_V(x_N)$, we get $F_V \geqslant d_{N,V}$ and hence

$$F_V \geqslant \lim_{N \to \infty} d_{N,V}. \tag{4.7}$$

Since $\psi_V(t) \to \infty$ as $|t| \to \infty$, (4.2) and V are l.s.c. on E, there is an $x^* \in E^N$ such that

$$d_{N,V} = \frac{1}{N(N-1)} K_{N,V}(x^*). \tag{4.8}$$

The points $x_1^*, \ldots, x_N^*$ are the analogues of the *Fekete points* of classical potential theory. Put $\lambda_N = N^{-1} \sum_k \delta_{x_k^*}$, where δ_a is a unit mass at a. Then

$$d_{N,V} \geqslant \frac{N^2}{N(N-1)} \iint \min(L, k_{V_m}(t,s)) d\lambda_N(t) d\lambda_N(s) - \frac{L}{N-1}. \tag{4.9}$$

From (4.7), (4.8), and (4.2), we find

$$F_V \geqslant d_{N,V} \geqslant 2 \int \psi_V(x) d\lambda_N(x),$$

and hence, by (iii), $\{\lambda\}$ is tight. Picking a weakly convergent subsequence of $\{\lambda_N\}$ with limit λ, we get from (4.9)

$$\liminf_{N \to \infty} d_{N,V} \geqslant I_V[\lambda] \geqslant F_V,$$

which together with (4.7) gives $d_{N,V} \to F_V$ and $\lambda_N \to \mu_V$ as $N \to \infty$.

The key to the proof of (2.5) and (2.6) is the following "large deviation" estimate. Note that

$$\sum_{i \neq j} \log |x_i - x_j| - N \sum_j V(x_j) = -K_{N,V}(x) - \sum_j V(x_j).$$

LEMMA 4.2. *Assume that the conditions of Theorem 2.1 are satisfied. Let $P_{N,M}^{\beta,h}$ be the probability measure with density (2.3), and let $\eta > 0$ be given. Put*

$$A_{N,\eta} = \left\{ x \in \mathbb{R}^N; \frac{1}{N^2} K_{N,V}(x) \leqslant F_V + \eta \right\}.$$

If $M_N/N \to 1$ as $N \to \infty$, then, for any $t \geqslant 0$,

$$P_{N,M_N}^{\beta,h}(\mathbb{R}^N \backslash A_{N,\eta+a}) \leqslant \exp\left(-\frac{\beta}{4} a N^2\right) \tag{4.10}$$

for all $N \geqslant N_0$, where N_0 depends on η but not on a.

Proof. The first step is to prove a lower bound on $Z_{N,M}^{\beta,h}$. Assume that ϕ is a probability density on $\mathbb{R}$ with compact support and finite entropy (i.e., $\phi \log \phi$ is integrable). Then we can write

$$\log Z_{N,M}^{\beta,h} = \log \int_{\mathbb{R}^N} \exp\left(-\frac{\beta}{2} K_{N,V}(x) + \frac{\beta}{2}(N - M - 1) \sum_j V(x_j) \right.$$
$$\left. + \sum_j (-\log \phi(x_j) + h(x_j)) \right) \prod_j \phi(x_j) d^N x$$

and use Jensen's inequality to obtain

$$\frac{1}{N^2}\log Z_{N,M}^{\beta,h} \geqslant -\frac{\beta N(N-1)}{2N^2} I_V[\phi] + \frac{\beta(N-M-1)N}{2N^2}\int_{\mathbb{R}} V(t)\phi(t)dt$$
$$+\frac{1}{N}\int_{\mathbb{R}}(h(t)-\log\phi(t))\phi(t)dt.$$

This gives

$$\liminf_{N\to\infty}\frac{1}{N^2}\log Z_{N,M}^{\beta,h} \geqslant -\frac{\beta}{2}I_V[\phi]. \tag{4.11}$$

Next, we want to show that, given $\varepsilon > 0$, we can choose ϕ_ε with finite entropy so that

$$I_V[\phi_\varepsilon] \leqslant F_V + \varepsilon/2. \tag{4.12}$$

Let us remark that if we know a priori that $d\mu_V$ is absolutely continuous with a Radon-Nikodym derivative $\phi = d\mu_V/dx$ having finite entropy, we can take this as ϕ_ε for any $\varepsilon > 0$.

We turn now to the construction of ϕ_ε. P. Deift, T. Kriecherbauer, and K. T.-R. McLaughlin suggested this construction to me. For any $\delta > 0$, set

$$\psi_\delta(x) = \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} d\mu_V(t).$$

Since $d\mu_V$ has compact support, so does ψ_δ, and since it is the convolution of the probability measure $d\mu_V$ with a probability density, ψ_δ is also a probability density. Standard arguments show that $\psi_\delta(x)dx$ tends weakly to $d\mu_V$ as $\delta \to 0$. We want to show that, in fact, $I_V[\psi_\delta] \to I_V[\mu_V] = F_V$ as $\delta \to 0$. Interchanging the order of integration gives

$$|I_V[\psi_\delta] - I_V[\mu_V]| \leqslant \left|\int V(t)\psi_\delta(t)dt - \int V(t)d\mu_V(t)\right| + \int\int H_\delta(t,s)d\mu_V(t)d\mu_V(s),$$

where

$$H_\delta(t,s) = \frac{1}{4\delta^2}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |\log|x-y+t-s| - \log|t-s||dxdy.$$

Since V is continuous and everything takes place in a compact set, the first term in the estimate goes to zero as $\delta \to 0$. If we put $\gamma = \delta/|t-s|$, then, by a change of variables,

$$H_\delta(t,s) = \frac{1}{4\gamma^2}\int_{-\gamma}^{\gamma}\int_{-\gamma}^{\gamma} |\log|u-v+1||dudv.$$

By making one more change of variables, it is not difficult to see that the last integral is

$$\leqslant \frac{1}{2\gamma}\int_{-2\gamma}^{2\gamma} |\log|1+x||dx,$$

which, in turn, is bounded by $c\log(1+\gamma)$ for a suitable numerical constant c. Thus

$$0 \leqslant \int\int H_\delta(t,s)d\mu_V(t)d\mu_V(s) \leqslant c\int\int \log\left(1+\frac{\delta}{|t-s|}\right)d\mu_V(t)d\mu_V(s).$$

The right-hand side goes to zero as $\delta \to 0$ by the dominated convergence theorem. To see this, note that, for $0 \leqslant \delta \leqslant 1$,

$$0 \leqslant \log\left(1+\frac{\delta}{|t-s|}\right) \leqslant \log 2 + \left|\log\frac{1}{|t-s|}\right|.$$

The right-hand side is integrable with respect to $d\mu_V(t)d\mu_V(s)$, since this measure has compact support and μ_V has finite logarithmic energy. Summing up, we have proved that $I_V[\psi_\delta] \to I[\mu_V] = F_V$ as $\delta \to 0$, and picking $\delta = \delta(\varepsilon)$ small enough, $\phi_\varepsilon = \psi_{\delta(\varepsilon)}$ will satisfy (4.12).

Combining (4.11) and (4.12), we get

$$\frac{1}{N^2}\log Z_{N,M}^{\beta,h} \geqslant -\frac{\beta}{2}(F_V+\varepsilon) \tag{4.13}$$

for all sufficiently large N.

The exponent in the density (2.3) can be written as follows:

$$\begin{aligned}
&\frac{\beta}{2}\sum_{j\neq k}\log|x_j-x_k| - \frac{M\beta}{2}\sum_j V(x_j)\\
&\quad = -\frac{\beta}{2}\sum_{j\neq k}k_V(x_j,x_k) + \frac{\beta}{2}(N-1-M)\sum_j V(x_j)\\
&\quad = -\frac{\beta}{2}(1-\gamma)K_{N,V}(x) - \frac{\beta}{2}(C_1+1)\sum_j V(x_j)\\
&\qquad -\frac{\beta}{2}\gamma\sum_{j\neq k}k_V(x_j,x_k) + \frac{\beta}{2}(N-M+C_1)\sum_j V(x_j),
\end{aligned} \tag{4.14}$$

where the constant $\gamma \in (0, 1/2)$ will be specified below, and C_1 is the constant in the estimate $h(t) \leqslant \beta C_1 V(t)/2$ assumed to hold for t large. It follows from (4.2) that the last two terms in (4.14) are

$$\begin{aligned}
&\leqslant \sum_j [-\gamma(N-1)\psi_V(x_j) + (N-M+C_1)V(x_j)]\\
&= \frac{\beta}{2}\sum_j [\gamma(N-1)\log(x_j^2+1) + (N-M+C_1-\gamma(N-1))V(x_j)].
\end{aligned} \tag{4.15}$$

By assumption, $V(x_j) \geqslant (1+\delta)\log(x_j^2+1)$ for all sufficiently large x_j, and thus there is a constant $C_2 \geqslant 0$ so that $V(x_j) \geqslant (1+\delta)\log(x_j^2+1) - C_2$ for all x_j. Since $M/N \to 1$ as $N \to \infty$, we have $N - M + C_1 - \gamma(N-1) \leqslant 0$ for all sufficiently large N. Hence the last expression in (4.15) is

$$\leqslant \frac{\beta}{2}[(N-M+C_1)(1+\delta) - \delta\gamma(N-1)]\sum_j \log(x_j^2+1)$$
$$+\frac{\beta}{2}[\gamma(N-1) - N + M - C_1]NC_2.$$

The coefficient in front of the sum is $\leqslant -4$ and if N is sufficiently large, then $\frac{\beta}{2}[\gamma(N-1) - N + M - C_1]NC_2 \leqslant \beta\gamma C_2 N^2$. Consequently, we see that

$$\frac{\beta}{2}\left[\sum_{j\neq k}\log|x_j - x_k| - M\sum_j V(x_j)\right] + \sum_j h(x_j)$$
$$\leqslant -\frac{\beta}{2}(1-\gamma)K_{N,V}(x) - 4\sum_j \log(x_j^2+1) + 2\gamma C_2 N^2 \tag{4.16}$$

for all sufficiently large N.

If $x \in E^N \backslash A_{N,\eta+a}$, then $K_{N,V} > (F_V + \eta + a)N^2$. Combining (4.13) and (4.16), we find

$$P_N(\mathbb{R}^N \backslash A_{N,\eta+a}) \leqslant \left(\int_{\mathbb{R}} \frac{dx}{(X^2+1)^4}\right)^N$$
$$\times \exp\left(\frac{\beta}{2}[-(1-\gamma)(F_V+\eta+a) + 2\gamma C_2 + F_V + \varepsilon]N^2\right)$$
$$\leqslant \exp\left(\frac{\beta}{2}\left[-\frac{a}{2} - \frac{\eta}{2} + \gamma(F_V + 2C_2)\right]N^2\right).$$

Here we have used $\gamma < 1/2$ and assumed that $\varepsilon \leqslant \eta/8$. If we choose γ so small that $-\eta/4 + \gamma(F_v + 2C_2) \leqslant 0$, then

$$P_N(E^N \backslash A_{N,\eta+a}) \leqslant \exp\left(-\frac{\beta}{4}aN^2\right)$$

for all $N \geqslant N_0$, where N_0 depends on η but not on a. The lemma is proved.

Proof of (2.5) and (2.6). Let $\phi : \mathbb{R}^k \to \mathbb{R}$ be bounded and continuous. Both (2.5) and (2.6) will follow if we can prove

$$\lim_{N\to\infty}\frac{1}{N}\log E^{\beta,h}_{N,M_N}\left(\exp\left(\frac{1}{N^{k-1}}\sum_{i_1,\dots,i_k}\phi(x_{i_1},\dots,x_{i_k})\right)\right)$$
$$= \int_{\mathbb{R}^k}\phi(t_1,\dots,t_k)d\mu_V(t_1)\cdots d\mu_V(t_k)$$

for any $k \geqslant 1$. Since $A_{N,2\eta}$ is compact and ϕ is continuous, there is a point $x^s \in A_{N,2\eta}$ such that

$$\sum_{i_1,\dots,i_k} \phi(x^s_{i_1},\dots,x^s_{i_k}) = \sup_{x\in A_{N,2\eta}} \sum_{i_1,\dots,i_k} \phi(x_{i_1},\dots,x_{i_k}).$$

Lemma 4.2 gives

$$\begin{aligned}
&\limsup_{N\to\infty}\frac{1}{N}\log E_N\left(\exp\left(\frac{1}{N^{k-1}}\sum_{i_1,\dots,i_k}\phi(x_{i_1},\dots,x_{i_k})\right)\right)\\
&= \limsup_{N\to\infty}\frac{1}{N}\log E_N\left(\exp\left(\frac{1}{N^{k-1}}\sum_{i_1,\dots,i_k}\phi(x_{i_1},\dots,x_{i_k})1_{A_{N,2\eta}}(x)\right)\right)\\
&\leqslant \limsup_{N\to\infty}\frac{1}{N^k}\sum_{i_1,\dots,i_k}\phi(x^s_{i_1},\dots,x^s_{i_k}) = \limsup_{N\to\infty}\int_{\mathbb{R}^k}\phi(t_1,\dots,t_k)d\nu^s_N(t_1)\cdots d\nu^s_N(t_k),
\end{aligned}$$

where $\nu^s_N = \frac{1}{N}\sum_j \delta_{x^s_j}$. Since $x^s \in A_{N,\eta}$, we have, by (4.2),

$$(N-1)\sum_j \psi_V(x^s_j) \leqslant K_{V,N}(x^s) \leqslant N^2(F_V+2\eta); \tag{4.17}$$

that is,

$$\int \psi_V d\nu^s_N \leqslant \frac{N}{N-1}(F_V+2\eta) \leqslant C.$$

Hence, using (iii), $\{\nu^s_N\}$ is tight and has a weakly convergent subsequence with limit ν^s_η, which could depend on η. Inequality (4.17) implies

$$\iint \min(L, k_V(\tau,\sigma))d\nu^s_N(\tau)d\nu^s_N(\sigma) \leqslant \frac{L}{N} + F_V + 2\eta.$$

Let $N \to \infty$ and then let $L \to \infty$; we get $I_V[\nu^s_\eta] \leqslant F_V + 2\eta$. We can now let $\eta \to 0$ along a sequence, and, picking a weakly converging subsequence, we obtain a limit measure ν^s satisfying $I_V[\nu^s] \leqslant F_V$, that is, $\nu^s = \mu_V$. An analogous argument shows that

$$\begin{aligned}
&\liminf_{N\to\infty}\frac{1}{N}\log E_N\left(\exp\left(\frac{1}{N^{k-1}}\sum_{i_1,\dots,i_k}\phi(x_{i_1},\dots,x_{i_k})\right)\right)\\
&\geqslant \int \phi(t_1,\dots,t_k)d\mu_V(t_1)\cdots d\mu_V(t_k),
\end{aligned}$$

and we are done.

We can also prove the existence of the so-called *free energy* [BIPZ].

COROLLARY 4.3. *Assume that V and h satisfy the conditions of Theorem 2.1 and $M_N/N \to 1$ as $N \to \infty$. Then, with $Z_{N,M}^{\beta,h}$ the partition function in (2.3), we have*

$$\lim_{N\to\infty}\left(-\frac{1}{N^2}\log Z_{N,M_N}^{\beta,h}\right) = \frac{\beta}{2}F_V = \frac{\beta}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}[\log|t-s|^{-1} + V(t)]d\mu_V(t)d\mu_V(s).$$

Proof. Letting $N \to \infty$ in (4.13) gives

$$\limsup_{N\to\infty}\left(-\frac{1}{N^2}\log Z_{N,N}^{\beta,h}\right) \leqslant \frac{\beta}{2}(F_V + \varepsilon)$$

for any $\varepsilon > 0$. On the other hand, it follows from (4.16) that for any fixed $\gamma \in (0,1/2)$,

$$Z_{N,M_N}^{\beta,h} \leqslant \exp\left(-\frac{\beta}{2}(1-\gamma)N(N-1)d_{N,V} + 2\gamma C_2 N^2\right)\left(\int_{\mathbb{R}}\frac{dx}{(1+x^2)^4}\right)^N$$

for all sufficiently large N. Since $d_{N,V} \to F_V$ as $N \to \infty$, we get

$$\liminf_{N\to\infty}\left(-\frac{1}{N^2}\log Z_{N,M_N}^{\beta,h}\right) \geqslant -\frac{\beta}{2}(1-\gamma)F_V + 2\gamma C_2.$$

Now, let $\gamma \to 0$ and the proof is finished.

Since $u_N(t)dt$ converges weakly to a measure with compact support, we would expect $u_N(t)$ to be small for large t and large N. An estimate is given by the next lemma.

LEMMA 4.4. *Assume that the conditions of Theorem 2.1 are satisfied and that $M_N/N \to 1$ as $N \to \infty$. Then there is a constant C so that*

(i) $u_{N,M_N}^{\beta,h}(t) \leqslant e^{CN}(1+t^2)^{\frac{\beta N}{2}}e^{-\frac{\beta M_N}{2}V(t)+h(t)}$ *and*

(ii) $u_{N,M_N}^{\beta,h}(t,s) \leqslant e^{CN}|t-s|^{\beta}(1+t^2)^{\frac{\beta N}{2}}(1+s^2)^{\frac{\beta N}{2}}e^{-\frac{\beta M_N}{2}(V(t)+V(s))+h(t)+h(s)}$ *for all t, $s \in \mathbb{R}$, and all $N \geqslant 1$.*

Proof. The 1-point function is given by

$$u_{N,M}(t) = \frac{Z_{N-1,M}}{Z_{N,M}}E_{N-1,M}\left(\prod_{j=1}^{N-1}|x_j - t|^{\beta}\right)e^{-\frac{\beta M}{2}V(t)+h(t)}. \tag{4.18}$$

We can write

$$\frac{Z_{N,M}}{Z_{N-1,M}} = E_{N-1,M}\left(\int_{\mathbb{R}}\exp\left(\beta\sum_{j=1}^{N-1}\log|t-x_j| - \frac{\beta M}{2}V(t) + h(t)\right)dt\right). \tag{4.19}$$

Let $Z = \int_{\mathbb{R}} \exp(-V(t))dt$. Then Jensen's inequality gives

$$\frac{1}{Z}\int_{\mathbb{R}} \exp\left(\beta \sum_{j=1}^{N-1} \log|t-x_j| - \frac{\beta M}{2} V(t) + h(t)\right) dt$$

$$\geqslant \exp\left[\frac{\beta}{Z}\int_{\mathbb{R}} \left(\sum_{j=1}^{N-1} \log|t-x_j| - \left(\frac{\beta M}{2} - 1\right) V(t) + h(t)\right) e^{-V(t)} dt\right]. \tag{4.20}$$

Now,

$$\int_{\mathbb{R}} \log|t-x_j| e^{-V(t)} dt \geqslant \int_{x_j-1}^{x_j+1} \log|t-x_j| e^{-V(t)} dt \geqslant \int_{x_j-1}^{x_j+1} \log|t-x_j| dt = -C, \tag{4.21}$$

since $V(t) \geqslant 0$. From (4.19)–(4.21), we get

$$\frac{Z_{N,M}}{Z_{N-1,M}} \geqslant e^{-CN} \tag{4.22}$$

for some constant C.

The inequality $(x-t)^2 \leqslant (1+x^2)(1+t^2)$ gives

$$E_{N-1,M}\left(\prod_{j=1}^{N-1} |x_j - t|^{\beta}\right) \leqslant (1+t^2)^{\frac{\beta N}{2}} E_{N-1,M}\left(\prod_{j=1}^{N-1} (1+x_j^2)^{\frac{\beta}{2}}\right). \tag{4.23}$$

Adding a constant to V does not change the right-hand side of (4.18), so we can assume that $V(t) \geqslant (1+\delta)\log(1+t^2)$ for all t. Using this and (4.2), we obtain

$$K_{N,V}(x) \geqslant \delta(N-1)\sum_j \log(1+x_j^2).$$

Thus, by Lemma 4.2, for any $A \geqslant 0$,

$$P_{N-1,M}\left(\sum_j \log(1+x_j^2) \geqslant AN\right) \leqslant P_{N-1,M}\big(K_{N,V}(x) \geqslant \delta A(N-1)N\big)$$
$$\leqslant \exp(-cAN^2) \tag{4.24}$$

for some $c \geqslant 0$ if A is sufficiently large. Hence

$$E_{N-1,M}\left(\prod_{j=1}^{N-1} (1+x_j^2)^{\frac{\beta}{2}}\right) \leqslant e^{CN}. \tag{4.25}$$

Combining (4.18) and (4.22)–(4.25), we get the desired estimate.

Estimate (ii) follows immediately. We have

$$u_{N,M}(t,s) = \frac{Z_{N-2,M}}{Z_{N,M}} E_{N-2,M}\left(\prod_{j=1}^{N-1} |x_j - t|^\beta |x_j - s|^\beta\right)$$
$$\times |t-s|^\beta e^{-\frac{\beta M}{2}(V(t)+V(s))+h(t)+h(s)}.$$

The Cauchy-Schwarz inequality gives

$$E_{N-2,M}\left(\prod_{j=1}^{N-1} |x_j - t|^\beta |x_j - s|^\beta\right)$$
$$\leqslant E_{N-2,M}\left(\prod_{j=1}^{N-1} |x_j - t|^{2\beta}\right)^{1/2} E_{N-2,M}\left(\prod_{j=1}^{N-1} |x_j - t|^{2\beta}\right)^{1/2},$$

so we can use the same estimates as above. The lemma is proved.

The last proof actually shows that we can assume that all $x_j \in [-L, L]$ for L sufficiently large. For a different proof of this, see [BPS, Lemma 1]. If $\Lambda_L = [-L, L]^{N-1}$, then

$$\frac{Z_{N-1,N}}{Z_{N,N}} E_{N-1,N}\left(\prod_{j=1}^{N-1} |x_j - t|^\beta 1_{\Lambda_L^c}(x)\right) e^{-\frac{\beta N}{2}V(t)+h(t)} \leqslant e^{-N} \tag{4.26}$$

when L is sufficiently large. This is seen as follows. The inequality $(x-t)^2 \leqslant (1+x^2)(1+t^2)$ and (4.24) show that we can choose A_0 so that the left-hand side of (4.26) is

$$\leqslant e^{(C+A_0)N} P_{N-1,N}(\Lambda_L^c)(1+t^2)^{\frac{\beta N}{2}} e^{-\frac{\beta N}{2}V(t)+h(t)} + \frac{1}{2}e^{-N}$$
$$\leqslant e^{C'N} P_{N-1,N}(\Lambda_L^c) + \frac{1}{2}e^{-N}. \tag{4.27}$$

In the last inequality, we used the assumptions on V and h. But, if $x \in \Lambda_L^c$, then $|x_k| \geqslant L$ for some k, and hence

$$P_{N-1,N}(\Lambda_L^c) \leqslant N \int_{|t|\geqslant L} u_N(t)dt \leqslant \frac{1}{2}e^{-C'N-N}$$

if we choose L sufficiently large, which together with (4.27) gives the estimate (4.26).

We now have the results we need to prove Lemma 3.12.

Proof of Lemma 3.12. The idea of the proof comes from [BPS], but since [BPS] does not give the details in the case we need, we present a proof. The key point is the following estimate of how $u_N(t)$ changes if we perturb t slightly.

Claim. If $t \in [-L, L]$ and $|\delta| \leqslant 1/N^{2/\gamma}$, where γ is the Hölder exponent of V and h in Proposition 3.9, then there is a constant C such that

$$u_N(t) \leqslant C u_N(t+s) + O(e^{-N}) \tag{4.28}$$

when $|s| \leqslant \delta$.

If we accept the claim, we can prove the lemma as follows.

$$\begin{aligned}
&\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (\log|t-s|^{-1} - a_\varepsilon(|t-s|))\rho_N(t)\rho_N(s)dtds \\
&\leqslant \frac{1}{L^2}\int_{-L/2}^{L/2}\left(\int_{y-\varepsilon L}^{y+\varepsilon L} |\log|x-y||u_N(x)dx\right)u_N(y)dy \\
&= \frac{1}{2\delta}\int_{-L/2}^{L/2}\left(\int_{y-\varepsilon L}^{y+\varepsilon L} |\log|x-y||u_N(x)\int_{x-\delta}^{x+\delta} dtdx\right)u_N(y)dy.
\end{aligned}$$

By the claim, the last expression is

$$\begin{aligned}
&\leqslant \frac{1}{2\delta L^2}\int_{-L/2}^{L/2}\left(\int_{y-\varepsilon L}^{y+\varepsilon L} |\log|x-y|| \int_{x-\delta}^{x+\delta} (Cu_N(t) + O(e^{-N}))\, dtdx\right)u_N(y)dy \\
&\leqslant \frac{C}{2\delta}\int_{-L/2}^{L/2}\left(\int_{y-\varepsilon L}^{y+\varepsilon L} |\log|x-y||dx\right)u_N(y)dy \leqslant \frac{C\varepsilon\log(1/\varepsilon L)}{\delta}.
\end{aligned}$$

Now, let $\varepsilon L = 1/N^{3/\gamma}$ and $\delta = 1/1/N^{2/\gamma}$, and the lemma is proved.

It remains to prove the claim. The Hölder continuity of V and h gives

$$\sup_{\substack{|t|\leqslant 2L \\ |s|\leqslant\delta}} |(NV(t) - h(t)) - (NV(s) - h(s))| \leqslant C_L \delta^\gamma N. \tag{4.29}$$

Now, if we choose L sufficiently large, we can use (4.26) and get

$$\begin{aligned}
u_N(t) = \frac{1}{Z_N} e^{-NV(t)+h(t)} &\int_{[-L,L]^{N-1}} \prod_{j=1}^{N-1} |x_j - t|^\beta \prod_{1\leqslant i<j\leqslant N-1} |x_i - x_j|^\beta \\
&\times e^{-\sum_{j=1}^{N-1}(NV(x_j)-h(x_j))} d^{N-1}x + O(e^{-N}).
\end{aligned} \tag{4.30}$$

If we make the change of variables $x_j = y_j - s$ in (4.30) and use (4.29), we obtain

$$u_N(t) \leqslant e^{C_L\delta^\gamma N^2} u_N(t+s) + O(e^{-N})$$

and we have proved (4.28) since $\delta^\gamma N^2 \leqslant 1$.

5. Asymptotics of the recursion coefficients. In this section, we will prove Theorem 2.9. We will assume that $\deg V \geqslant 4$. By iterating the recursion formula (1.10), it is possible to express $t^k p_{N,M}(t)$ as a linear combination of $p_{j,M}(t)$, $j = N-k, \dots, N+k$. A nice way to keep track of the coefficients was given in [BIZ]. Let $p : \{0, 1, \dots, k\} \to \mathbb{N}$ be a *path function* satisfying $|p(j+1) - p(j)| \leqslant 1$. The weight of p is defined by

$$W_M[p] = \prod_{j=0}^{k-1} w_M[p(j), p(j+1) - p(j)],$$

where the 1-step weight, $w_M[j,k]$, is $R_{j+1,M}^{1/2}$ if $k=1$, $S_{j,M}$ if $k=0$, and $R_{j,M}^{1/2}$ if $k=-1$. Then (see [BIZ])

$$t^k p_{N,M}(t) = \sum_{j=N-k}^{N+k} \sum_{\substack{\text{path } p \\ p(0)=N, p(k)=j}} W_M[p] p_{j,M}(t). \tag{5.1}$$

Using the orthonormality, we find

$$\int_{\mathbb{R}} t^k p_{N,M}(t)^2 e^{-MV(t)} dt = \sum_{\substack{\text{path } p \\ p(0)=N, p(k)=N}} W_M[p]. \tag{5.2}$$

Equations (2.5) and (5.2) imply

$$(N+1) m_{N+1,M}(k) - N m_{N,M}(k) = \sum_{\substack{\text{path } p \\ p(0)=N, p(k)=N}} W_M[p]. \tag{5.3}$$

Write $\alpha_{N,M}(k) = (N+1) m_{N+1,M}(k) - N m_{N,M}(k)$. Equation (5.3) now gives

$$\alpha_{N,M}(1) = S_{N,M}, \tag{5.4a}$$

$$\alpha_{N,M}(2) = R_{N,M} + R_{N+1,M} + S_{N,M}^2, \tag{5.4b}$$

$$\begin{aligned}\alpha_{N,M}(4) = {} & R_{N+1,M} R_{N+2,M} + R_{N+1,M}^2 + 2R_{N,M} R_{N+1,M} + R_{N,M}^2 \\ & + R_{N-1,M}^2 + 6\bar{S}_{N,M}^2 R_{N+1,M} + 6\tilde{S}_{N,M}^2 R_{N,M} + S_{N,M}^4,\end{aligned} \tag{5.4c}$$

where

$$\bar{S}_{N,M}^2 = \frac{1}{6}(3S_{N,M}^2 + 2S_{N,M} S_{N+1,M} + S_{N+1,M}^2),$$

and

$$\tilde{S}_{N,M}^2 = \frac{1}{6}(3S_{N,M}^2 + 2S_{N,M} S_{N-1,M} + S_{N-1,M}^2).$$

Let $[a,b]$ be the support of μ_V, $V \in \mathscr{V}$. Put $S = (a+b)/2$ and $R = (b-a)^2/16$. We want to investigate the asymptotics of $\alpha_{N,M}(k)$. Define

$$G_{N,M}(z) = (N+1)U_{N+1,M}(z) - NU_{N,M}(z).$$

Using the fundamental equation (3.2) with $\beta = 2$, we see that $G_{N,M}$ satisfies the equation

$$\frac{1}{N+1}G_{N,M}(z)^2 + \left(\frac{2N}{N+1}U_{N,M}(z) - \frac{M}{N+1}V'(z)\right)G_{N,M}(z)$$
$$+ M\left(T_{N+1,M}(z) - \frac{N}{N+1}T_{N,M}(z)\right) = \frac{1}{N+1}(K_{N+1,M}(z) - K_{N,M}(z)). \quad (5.5)$$

Consider a sequence (N,M), $M = M_N$ such that $N/M_N \to 1$ as $N \to \infty$. Write $A_N(z) = \frac{2N}{N+1}U_{N,M}(z) - \frac{M}{N+1}V'(z)$, $B_N(z) = M(T_{N+1,M}(z) - \frac{N}{N+1}T_{N,M}(z))$, and $C_N(z) = \frac{1}{N+1}(K_{N+1,M}(z) - K_{N,M}(z))$. Note that

$$B_{N,M}(z) = \frac{M}{N+1}\int_{\mathbb{R}} \frac{V'(z) - V'(t)}{z-t}((N+1)u_{N+1,M}(t) - Nu_{N,M}(t))dt, \quad (5.6)$$

so $B_N(z)$ is a polynomial of degree $2\ell - 2$ with leading coefficient $(M/(N+1))2\ell v_{2\ell}$. Put $X_N = G_{N,M}$. From (3.34) and (3.8), it follows that the conditions in Lemma 3.7 are satisfied with $a(z) = -2r(z)\sqrt{(z-a)(z-b)}$ and $C_N(z) \to 0$ uniformly in Ω_δ. Thus,

$$G_{N,M}(z) \to \frac{B(z)}{2r(z)\sqrt{(z-a)(z-b)}} := G(z),$$

where $B(z)$ is a polynomial of degree $2\ell - 2$ with leading coefficient $2\ell v_{2\ell}$. Since $G(z)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$, we must have $B(z) = 2r(z)$. Thus,

$$G(z) = \frac{1}{\sqrt{(z-a)(z-b)}} = \frac{1}{\pi}\int_a^b \frac{1}{z-t}\cdot\frac{dt}{\sqrt{(t-a)(b-t)}}.$$

Now, by (3.9),

$$B(z) = 2r(z) = \frac{1}{\pi}\int_a^b \frac{V'(z) - V'(t)}{z-t}\cdot\frac{dt}{\sqrt{(t-a)(b-t)}}.$$

Since $B_{N,M}(z) \to B(z)$, Lemma 3.8 gives, provided $\deg V \geqslant 6$,

$$\alpha_{N,M}(k) \to \frac{1}{\pi}\int_a^b \frac{t^k}{\sqrt{(t-a)(b-t)}}\,dt \tag{5.7}$$

for $k = 1, 2, 4$. If $\deg V = 4$, we get (5.7) for $k = 1, 2$ and we can take $\phi(t) = t$ and $h = 0$ in (2.18) to get the result for $k = 4$. From (5.7) we find

$$\alpha_{N,M}(1) \to S, \tag{5.8a}$$

$$\alpha_{N,M}(2) \to 2R + S^2, \tag{5.8b}$$

$$\alpha_{N,M}(4) \to S^4 + 12S^2R + 6R^2. \tag{5.8c}$$

It follows immediately from (5.4a) and (5.8a) that $S_{N,M} \to S$ as $N \to \infty$, and we have proved one half of Theorem 2.9. Write $\beta_{N,M} = R_{N,M} + R_{N+1,M}$ and $\gamma_{N,M} = R_{N+1,M}R_{N+2,M} + R_{N+1,M}^2 + 2R_{N,M}R_{N+1,M} + R_{N,M}^2 + R_{N-1,M}^2$. Now, (5.4 a–c) and (5.8 a–c) imply

$$\beta_{N,M} \to 2R, \qquad \gamma_{N,M} \to 6R^2 \tag{5.9}$$

as $N \to \infty$. Note that

$$\left(R_{N,M} + \frac{1}{4}(\beta_{N+1,M} - 2\beta_{N,M} - \beta_{N-1,M})\right)^2 = \frac{1}{16}(\beta_{N+1,M} - 2\beta_{N,M} - \beta_{N-1,M})^2 - \frac{1}{2}(\gamma_{N,M} - \beta_{N+1,M}\beta_{N,M}). \tag{5.10}$$

By (5.9), the right-hand side of (5.10) goes to zero as $N \to \infty$, and hence (5.9) and (5.10) yield $R_{N,M} \to R$. This completes the proof of Theorem 2.9.

Remark 5.1. The now proved Freud conjecture says that if $R_N = k_{N-1}^2/k_N^2$, where k_N is the leading coefficient in the Nth degree orthonormal polynomial with respect to the weight $\exp(-|x|^{2\alpha})$, $\alpha > 0$, on $\mathbb{R}$, then

$$\lim_{N\to\infty} N^{-2/\alpha}R_N = \frac{1}{4}\left(\frac{\Gamma(\alpha)}{2^{\alpha-2}\Gamma(\alpha/2)^2}\right)^2.$$

In the case $\alpha \in \mathbb{Z}^+$, this follows, by rescaling, from Theorem 2.9. This case was first proved by Magnus, who also settled the case of a weight $\exp(-Q(t))$, $Q \in \mathbb{P}_{2\ell}^+$ (see [Ma]). This last case can also be proved by the methods of this paper. A rescaling gives the weight $\exp(-NV_N(t))$, where

$$V_N(t) = q_{2\ell}t^{2\ell} + q_{2\ell-1}N^{-\frac{2\ell-1}{2\ell}}x^{2\ell-1} + \cdots.$$

If we use this N-dependent polynomial above, only the leading term $q_{2\ell}x^{2\ell}$ will be important for the R_N-asymptotics. The case of noninteger α, and other more general weight functions, was proved in [LMS].

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Department of Mathematics, Royal Institute of Technology, S-100 44 Stockholm, Sweden; kurtj@math.kth.se