Branch-Cut Discontinuities of $\frac{1}{2}\log \xi\left(\frac{1}{2}+i\,t\right)$

.

by Stephen Crowley

August 28, 2025

Theorem 1. *Let $\xi(s)$ denote Riemann's ξ-function, which is entire, and fix a single-valued branch* Log *of the complex logarithm on $\mathbb{C}\setminus(-\infty,0]$ (the principal branch). Consider the function*

$$F(t)=\frac{1}{2}\operatorname{Log}\!\left(\xi\left(\frac{1}{2}+i\,t\right)\right)\forall t\in\mathbb{R} \tag{1}$$

defined wherever $\xi\left(\frac{1}{2}+i\,t\right)\notin(-\infty,0]$. Then the following hold:

1. *F is continuous on any maximal open interval of t for which $\xi\left(\frac{1}{2}+i\,t\right)$ avoids $(-\infty,0]$.*

2. *At any t_0 with $\xi\left(\frac{1}{2}+i\,t_0\right)\in(-\infty,0]$ and $\xi\left(\frac{1}{2}+i\,t_0\right)\neq 0$, the one-sided limits of F exist and satisfy*

$$\lim_{t\to t_0^+}F(t)-\lim_{t_0^-}F(t)=\pi\,i \tag{2}$$

i.e., F exhibits a jump discontinuity of size $\pi\,i$ (equivalently, Log *jumps by $2\,\pi\,i$ and the prefactor $\frac{1}{2}$ halves the jump).*

3. *The set of t at which these discontinuities occur is precisely the preimage of the negative real axis under the map $t\mapsto\xi\left(\frac{1}{2}+i\,t\right)$, excluding zeros of ξ; consequently, the observed discontinuities are branch-cut crossings of* $\operatorname{Log}\circ\xi$ *and not singularities of ξ.*

Proof. The proof proceeds in three steps corresponding to parts (1), (2), and (3) of Theorem 1.

Step 1: Proof of (1). Since $\xi(s)$ is entire by construction, ξ has no poles or branch points in s and $\xi\left(\frac{1}{2}+i\,t\right)$ is a continuous function of t into $\mathbb{C}$. Since the only multivalued object in F is Log, all discontinuities of F must arise from the branch structure of Log applied to the continuous path $t\mapsto\xi\left(\frac{1}{2}+i\,t\right)$. This establishes the reduction to the logarithm.

The principal branch Log with branch cut along $(-\infty,0]$ is analytic and thus continuous on $\mathbb{C}\setminus(-\infty,0]$. Therefore F is continuous at any t_0 for which $\xi\left(\frac{1}{2}+i\,t_0\right)\notin(-\infty,0]$.

Step 2: Proof of (2). Let $w(t):=\xi\left(\frac{1}{2}+i\,t\right)$ and suppose $w(t_0)\in(-\infty,0]$ with $w(t_0)\neq 0$. Because w is continuous and nonzero at t_0, there exists $\delta>0$ such that:

1. $w(t)\neq 0$ for $|t-t_0|<\delta$, and

2. the image $w((t_0-\delta,t_0+\delta))$ crosses the branch cut transversely at $w(t_0)$.

Approaching $w(t_0)$ from the upper half-plane corresponds to arguments $\arg w(t)\to\pi^-$, while from the lower half-plane corresponds to $\arg w(t)\to(-\pi)^+$ (principal values). Hence

$$\lim_{t\to t_0^+}\operatorname{Log}w(t)-\lim_{t\to t_0^-}\operatorname{Log}w(t)=2\,\pi\,i \tag{3}$$

the standard $2\,\pi\,i$ jump across the negative real axis for the principal logarithm. Multiplying by $\frac{1}{2}$ yields the stated jump $\pi\,i$ for F.

Step 3: Proof of (3). By definition, the principal branch is continuous precisely off its branch cut. Therefore one-sided jumps can only occur when the continuous path $w(t)$ intersects the branch cut, i.e., when $w(t) \in (-\infty, 0]$.

If $w(t) = 0$, then $\mathrm{Log} w(t)$ is undefined and a different analysis is required (zeros are branch points of $\mathrm{Log} \circ w$), but by hypothesis these are excluded from consideration in part (2).

Conversely, every transverse crossing of $(-\infty, 0]$ yields the jump quantified in part (2). Thus the discontinuity set is exactly $w^{-1}((-\infty, 0]) \setminus w^{-1}(\{0\})$. ☐

Corollary 2. *Any specific numerical locations of discontinuities (e.g., $t = \pm(e - 1)$ in a given plot) are exactly the real parameters for which $\xi\left(\frac{1}{2} + i\,t\right)$ lands on the negative real axis under the chosen branch. These values therefore solve*

$$\arg \xi \left(\frac{1}{2} + i\,t\right) \equiv \pi (\mathrm{mod}\, 2\,\pi) \tag{4}$$

Corollary 3. *Since ξ is entire, the discontinuities described in Theorem 1 are not singularities of ξ; they arise solely from composing the entire function ξ with a single-valued branch of* Log *along the critical line.*