Generalized Solutions to the Zeta Function Integral Equation

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1 Introduction

Following the work of Rao [rao2025], this note establishes the complete characterization of generalized solutions to the integral equation arising from the equivalence between zeros of the Riemann zeta function and solutions to a convolution equation. Rao showed that the existence of nontrivial zeros $\zeta(\sigma + i\,t) = 0$ for $\sigma \in (0,1)$ is equivalent to the existence of nontrivial solutions to the integral equation

$$\int_{-\infty}^{\infty} K_\sigma(x-y)\,\phi(y)\;d\,y = 0 \tag{1}$$

where the kernel K_σ arises from the Fourier representation of the zeta function.

The kernel K_σ is defined explicitly through the integral representation

$$\zeta(s)\,(1-2^{1-s}) = \int_{\mathbb{R}} K_\sigma(u)\,e^{i\,t\,u}\;d\,u \tag{2}$$

for $s = \sigma + i\,t$, where

$$K_\sigma(u) = \frac{e^{\sigma u}}{e^{e^u}+1} \forall u \in \mathbb{R} \tag{3}$$

This kernel is obtained by the change of variables $u = \log x$ applied to the integral $\int_0^\infty \frac{x^{\sigma-1}}{e^x+1}\,e^{i t \log x}\;d\,x$ appearing in Rao's derivation.

Theorem 1. *[Complete Space of Generalized Solutions]*

Let

$$K_\sigma(u) = \frac{e^{\sigma u}}{e^{e^u} + 1} \forall \sigma \in (0,1) \tag{4}$$

Since $K_\sigma \in L^1(\mathbb{R})$, *its Fourier transform*

$$\widehat{K}_\sigma(t) = \int_{\mathbb{R}} K_\sigma(u)\, e^{-itu}\, d\,u \tag{5}$$

exists and is continuous. Define the zero set

$$Z_\sigma := \{t \in \mathbb{R}: \widehat{K}_\sigma(t) = 0\}. \tag{6}$$

The complete space of generalized solutions to the convolution equation

$$\int_{-\infty}^{\infty} K_\sigma\,(x-y)\,\phi(y)\; d\,y = 0, \quad \forall x \in \mathbb{R} \tag{7}$$

in the space of tempered distributions $\mathcal{S}'(\mathbb{R})$ *is*

$$\mathcal{N}_\sigma = \{\phi \in \mathcal{S}'(\mathbb{R}): supp(\hat{\phi}) \subseteq Z_\sigma\}. \tag{8}$$

Moreover, every solution $\phi \in \mathcal{N}_\sigma$ *admits the integral representation*

$$\phi(x) = \int_{Z_\sigma} e^{itx}\; d\,\mu(t) \tag{9}$$

where μ *is a complex tempered measure on* Z_σ.

Proof. The convolution operator $T: \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ defined by $T\phi = K_\sigma * \phi$ satisfies

$$\widehat{T\phi} = \widehat{K}_\sigma \cdot \hat{\phi} \tag{10}$$

in the sense of tempered distributions.

The equation $T\phi = 0$ is equivalent to $\widehat{K}_\sigma(t)\;\hat{\phi}(t) = 0$ as an identity of distributions. Since $\widehat{K}_\sigma$ is a continuous function, this occurs if and only if $\operatorname{supp}(\hat{\phi}) \subseteq Z_\sigma$.

For the integral representation, any tempered distribution ϕ with $\operatorname{supp}(\hat{\phi}) \subseteq Z_\sigma$ can be written as

$$\phi(x) = \int_{Z_\sigma} e^{itx}\; d\,\mu(t) \tag{11}$$

by the Bochner-Schwartz theorem, where μ is a tempered measure on Z_σ. The integral converges in $\mathcal{S}'(\mathbb{R})$ since for any test function $\psi \in \mathcal{S}(\mathbb{R})$,

$$\langle \phi, \psi \rangle = \int_{Z_\sigma} \hat{\psi}(t)\ d\,\mu(t) \tag{12}$$

is well-defined due to the rapid decay of $\hat{\psi}$.

Conversely, any such integral representation yields a solution since $\hat{\phi} = \mu$ as measures, and $\operatorname{supp}(\mu) \subseteq Z_\sigma$ implies $\widehat{K_\sigma} \cdot \hat{\phi} = 0$. □

Corollary 2. *[Application to Zeta Function Zeros] For the kernel K_σ defined above, the Fourier transform satisfies*

$$\widehat{K_\sigma}(t) = \frac{\zeta\,(\sigma + i\,t)\,(1 - 2^{1-(\sigma+it)})}{C_\sigma} \tag{13}$$

for some nonzero constant C_σ. Since $1 - 2^{1-s} \neq 0$ for s with $Re(s) \in (0,1)$, the zero set is

$$Z_\sigma = \{t \in \mathbb{R} \colon \zeta\,(\sigma + i\,t) = 0\} \tag{14}$$

Therefore, the complete space of generalized solutions is

$$\mathcal{N}_\sigma = \left\{ \phi(x) = \int_{\{t:\zeta(\sigma+it)=0\}} e^{itx}\ d\,\mu(t) \colon \mu \textit{is a tempered measure} \right\}. \tag{15}$$

Bibliography

[rao2025] Rao, M.M. Harmonic and Probabilistic Approaches to Zeros of Riemann's Zeta Function. Department of Mathematics, University of California, Riverside.