The Spectral Factorization Theorem for Stationary Gaussian Processes

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Theorem 1

(Spectral Factorization Theorem) *Let $K: \mathbb{R} \rightarrow \mathbb{R}$ be a positive definite stationary kernel function such that:*

1. $K \in L^1(\mathbb{R})$ *(integrability)*
2. K *is continuous*
3. *The spectral density $S(\omega)$ satisfies:*
   - $S(\omega) \geq 0$ *(non-negativity)*
   - $S(\omega) = S(-\omega)$ *(symmetry)*
   - $\int_{-\infty}^{\infty} |S(\omega)| \, d\omega < \infty$ *(integrability)*

Then there exists a real-valued function $h: \mathbb{R} \rightarrow \mathbb{R}$ such that:

$$K(t-s) = \int_{-\infty}^{\infty} h(t+\tau) \, h(s+\tau) \, d\tau \tag{1}$$

where

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega)} \, e^{i\omega t} \, d\omega \tag{2}$$

1 Proof

1.1 Initial Setup

By Bochner's theorem, since K is positive definite and stationary:

$$K(t-s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{i\omega(t-s)}\, d\omega$$

where $S(\omega) \geq 0$ and $S(\omega) = S(-\omega)$ for all $\omega \in \mathbb{R}$.

1.2 Construction of h

Define:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega)}\, e^{i\omega t}\, d\omega$$

Note that this is well-defined since $S(\omega) \geq 0$ and symmetric.

1.3 Convolution Analysis

Consider the convolution integral:

$$\int_{-\infty}^{\infty} h(t+\tau)\, h(s+\tau)\, d\tau$$

Substituting the definition of h:

$$\begin{aligned} K(t-s) &= \int_{-\infty}^{\infty} h(t+\tau)\, h(s+\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} \left(\frac{\int_{-\infty}^{\infty} \sqrt{S(\omega_1)}\, e^{i\omega_1(t+\tau)}\, d\omega_1}{2\pi} \right) \left(\frac{\int_{-\infty}^{\infty} \sqrt{S(\omega_2)}\, e^{i\omega_2(s+\tau)}\, d\omega_2}{2\pi} \right) d\tau \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{S(\omega_1)}\, \sqrt{S(\omega_2)}\, e^{i\omega_1 t}\, e^{i\omega_2 s}\, e^{i(\omega_1+\omega_2)\tau}\, d\tau\, d\omega_1\, d\omega_2 \end{aligned} \tag{3}$$

1.4 Application of Delta Function

The inner integral over τ yields:

$$\int_{-\infty}^{\infty} e^{i(\omega_1+\omega_2)\tau}\, d\tau = 2\pi\, \delta(\omega_1+\omega_2)$$

Therefore:

$$
\begin{aligned}
K\,(t-s) \; &= \frac{1}{2\,\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{S(\omega_1)}\,\sqrt{S(\omega_2)}\,e^{i\omega_1 t}\,e^{i\omega_2 s}\,\delta\,(\omega_1+\omega_2)\,d\,\omega_1\,d\,\omega_2 \\
&= \frac{1}{2\,\pi}\int_{-\infty}^{\infty} S(\omega)\,e^{i\omega\,(t-s)}\,d\,\omega
\end{aligned}
\tag{4}
$$

[1]

Bibliography

[1] Harald Cramér. A contribution to the theory of stochastic processes. *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 2:329–339, 1951.