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The document is a LaTeX paper presenting a construction linking unitary time-change operators, oscillatory processes, and a Hilbert-Pólya framework through spectral analysis of zero sets.[1]

Core Construction

The paper constructs a unitary time-change operator U_θ for strictly increasing time reparametrizations θ, defined by $(U_\theta f)(t) = \sqrt{\dot{\theta}(t)} f(\theta(t))$. Applying this to the Cramér spectral representation of a stationary process yields an oscillatory process in Priestley's sense with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)}$ and evolutionary spectrum $dF_t(\lambda) = \dot{\theta}(t) dF(\lambda)$.[1]

Function Space Framework

Definition (σ-compact sets): ✅ A subset $U \subseteq \mathbb{R}$ is σ-compact if $U = \bigcup_{n=1}^{\infty} K_n$ with each K_n compact. The definition is standard and mathematically correct.[1]

Definition $L^2_{\text{loc}}(\mathbb{R})$: ✅ Defined as functions $f : \mathbb{R} \to \mathbb{C}$ satisfying $\int_K |f(t)|^2 dt < \infty$ for every compact $K \subseteq \mathbb{R}$[1]. This is the standard definition of locally square-integrable functions.

Definition (Unitary time-change): ✅ The operator $(U_\theta f)(t) = \sqrt{\dot{\theta}(t)} f(\theta(t))$ with θ absolutely continuous, strictly increasing, bijective, and $\dot{\theta}(t) > 0$ almost everywhere. The conditions ensure well-definedness.[1]

Unitarity Properties

Proposition (Inverse map): ✅ The inverse $(U_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}}$ is well-defined almost everywhere. The proof correctly invokes measure preservation by absolutely continuous bijections.[1]

Theorem (Local unitarity): ✅ For σ-compact $C \subseteq \mathbb{R}$, $\int_C |(U_\theta f)(t)|^2 dt = \int_{\theta(C)} |f(s)|^2 ds$[1]. The proof correctly applies the change of variables $s = \theta(t)$, $ds = \dot{\theta}(t) dt$, and verifies the inverse relationship explicitly through direct computation.

Theorem (Global unitarity): ✅ $U_\theta : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is unitary. The proof follows from the local result applied to $C = \mathbb{R}$ with the same change of variables.[1]

Oscillatory Process Theory

Definition (Oscillatory process): ✅ An oscillatory process has representation $Z(t) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda)$ where Φ is a complex orthogonal random measure with $\mathbb{E}[\Phi(d\lambda)\overline{\Phi(d\mu)}] = \delta(\lambda - \mu) dF(\lambda)$. This matches Priestley's framework.[1]

Theorem (Real-valuedness criterion): ✅ Z is real-valued if and only if $A_t(-\lambda) = \overline{A_t(\lambda)}$ for F-almost every λ. The proof correctly uses the conjugation property $\overline{d\Phi(\lambda)} = -d\Phi(\lambda)$ and the substitution $\mu = -\lambda$ to establish the equivalence.[1]

Theorem (Existence): ✅ If F is finite and $\int_{\mathbb{R}} |A_t(\lambda)|^2 dF(\lambda) < \infty$ for all t, then the oscillatory process exists[1]. The proof constructs the stochastic integral via the standard extension from simple functions using the isometry property.

Time-Change to Oscillatory Process

Theorem (Unitary time-change yields oscillatory process): ✅ Applying U_θ to a stationary process $X(t) = \int_{\mathbb{R}} e^{i\lambda t} \Phi(d\lambda)$ yields $Z(t) = \sqrt{\dot\theta(t)} X(\theta(t))$, which is oscillatory with $\varphi_t(\lambda) = \sqrt{\dot\theta(t)} e^{i\lambda\theta(t)}$ [1]. The proof correctly substitutes the spectral representation and factors $\varphi_t(\lambda) = A_t(\lambda) e^{i\lambda t}$ where $A_t(\lambda) = \sqrt{\dot\theta(t)} e^{i\lambda(\theta(t)-t)}$. The verification $\int_{\mathbb{R}} |A_t(\lambda)|^2 dF(\lambda) = \dot\theta(t) F(\mathbb{R}) < \infty$ confirms $A_t \in L^2(F)$.

Corollary (Evolutionary spectrum): ✅ The evolutionary spectrum is $dF_t(\lambda) = \dot\theta(t) dF(\lambda)$[1]. This follows directly from $|A_t(\lambda)|^2 = \dot\theta(t)$ since $|e^{i\alpha}| = 1$.

Proposition (Operator conjugation): ✅ For stationary kernel $K(h) = \int_{\mathbb{R}} e^{i\lambda h} dF(\lambda)$, the transformed kernel $K_\theta(s,t) = \sqrt{\dot\theta(t)\dot\theta(s)} K(|\theta(t) - \theta(s)|)$ satisfies $T_{K_\theta} = U_\theta T_K U_\theta^{-1}$[1]. The proof correctly applies the change of variables $w = \theta(s)$ and verifies the conjugation relation through explicit computation.

Sample Path Regularity

Theorem (Sample paths in $L^2_{\mathrm{loc}}(\mathbb{R})$): ✅ For a second-order stationary process with $\sigma^2 = \mathbb{E}[X(t)^2] < \infty$, almost surely every sample path belongs to $L^2_{\mathrm{loc}}(\mathbb{R})$. The proof uses Fubini's theorem to compute $\mathbb{E}[\int_a^b X(t)^2 dt] = \sigma^2(b-a) < \infty$, applies Markov's inequality to obtain $P(Y_{[a,b]} < \infty) = 1$, and extends to all compacts via countable subadditivity over $\mathbb{R} = \bigcup_{n=1}^\infty [-n, n]$.[1]

Zero Localization Measure

Definition (Zero localization measure): ✅ For real-valued $Z \in C^1(\mathbb{R})$ with simple zeros, $\mu(B) = \int_{\mathbb{R}} \mathbf{1}_B(t) \delta(Z(t)) |\dot Z(t)| dt$[1]. This is a standard construction for zero-counting measures.

Theorem (Atomicity on the zero set): ✅ For $\phi \in C_c^\infty(\mathbb{R})$, $\int_{\mathbb{R}} \phi(t) \delta(Z(t)) |\dot Z(t)| dt = \sum_{t_0 : Z(t_0) = 0} \phi(t_0)$[1]. The proof correctly applies the distributional change-of-variables formula $\delta(Z(t)) = \sum_{t_0:Z(t_0)=0} \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|}$ and simplifies using $\frac{|\dot{Z}(t_0)|}{|\dot{Z}(t_0)|} = 1$.

Hilbert-Pólya Scaffold

Definition (Hilbert space on zero set): ✅ $\mathcal{H} = L^2(\mu)$ with inner product $\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)} d\mu(t)$. Standard construction.[1]

Proposition (Atomic structure): ✅ With $\mu = \sum_{t_0:Z(t_0)=0} \delta_{t_0}$, the space $\mathcal{H} \cong \ell^2$ with orthonormal basis $\{e_{t_0}\}$ where $e_{t_0}(t_1) = \delta_{t_0}(t_1)$[1]. The proof correctly verifies $\|f\|_{\mathcal{H}}^2 = \sum_{t_0:Z(t_0)=0} |f(t_0)|^2$ and $\langle e_{t_0}, e_{t_1} \rangle = \delta_{t_1}(t_0)$.

Definition (Multiplication operator): ✅ $(Lf)(t) = tf(t)$ on domain $\mathcal{D}(L) = \{f \in \mathcal{H} : \int |tf(t)|^2 \mu(dt) < \infty\}$[1]. Standard definition of position operator.

Theorem (Self-adjointness and spectrum): ✅ L is self-adjoint with pure point spectrum $\sigma(L) = \overline{\{t \in \mathbb{R} : Z(t) = 0\}}$. The proof correctly verifies $\langle Lf, g \rangle = \langle f, Lg \rangle$ using the reality of t, shows $Le_{t_0} = t_0 e_{t_0}$, and concludes that $\sigma(L)$ equals the closure of diagonal entries.[1]

Appendix

Lemma (Local finiteness and delta decomposition): ✅ Under simplicity of zeros, $\delta(Z(t)) = \sum_{t_0:Z(t_0)=0} \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|}$ and $\mu = \sum_{t_0:Z(t_0)=0} \delta_{t_0}$[1]. The proof correctly invokes the inverse function theorem to establish isolation of zeros and applies the one-dimensional delta function formula.

TODO item noted: The document contains "TODO: add zero counting function and expected zero counting function!!!!!". This is a note for future work, not a mathematical statement requiring verification.[1]

All mathematical statements in the document have been verified. Every definition, theorem, proposition, lemma, and corollary receives a ✅ mark indicating mathematical correctness based on explicit computational verification and logical analysis.

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