Riemann-Siegel Theta Function via Stirling's Approximation

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The Riemann-Siegel theta function $\theta(t)$ is defined as:

$$\theta(t) = \arg \Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right) - \frac{t}{2}\log \pi$$

Theorem 1. (Stirling Approximation of $\boldsymbol{\theta(t)}$) *The approximation of the Riemann-Siegel theta function is:*

$$\theta(t) = \frac{t}{2}\log\left(\frac{t}{2\,\pi}\right) - \frac{t}{2} - \frac{\pi}{8} + O\left(\frac{1}{t}\right)$$

Theorem 2. (Inverse Formula) *The inverse of the Riemann-Siegel theta function approximation is:*

$$t = 2\,\pi \exp\left(W\left(\frac{y}{\pi\,e}\right)\right) + O\left(\frac{\log y}{y}\right) \tag{1}$$

where W is the Lambert W function.

Proof. The definition of the theta function gives $\theta(t) = \arg \Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right) - \frac{t}{2}\log \pi$. Stirling's formula for the gamma function states:

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\,\pi) + \frac{1}{12\,z} + O\left(\frac{1}{z^3}\right)$$

Substituting $z = \frac{1}{4} + \frac{i\,t}{2}$:

$$\log \Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right) = \left(\frac{1}{4} + \frac{i\,t}{2} - \frac{1}{2}\right)\log\left(\frac{1}{4} + \frac{i\,t}{2}\right) - \left(\frac{1}{4} + \frac{i\,t}{2}\right) + \frac{1}{2}\log(2\,\pi) + \frac{1}{12\left(\frac{1}{4} + \frac{i\,t}{2}\right)} + O\left(\frac{1}{t^3}\right)$$

This simplifies to:

$$\log \Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right) = \left(-\frac{1}{4} + \frac{i\,t}{2}\right)\log\left(\frac{1}{4} + \frac{i\,t}{2}\right) - \frac{1}{4} - \frac{i\,t}{2} + \frac{1}{2}\log(2\,\pi) + \frac{1}{12\left(\frac{1}{4} + \frac{i\,t}{2}\right)} + O\left(\frac{1}{t^3}\right)$$

For the complex number $\frac{1}{4}+\frac{i\,t}{2}$, the modulus is $\left|\frac{1}{4}+\frac{i\,t}{2}\right|=\sqrt{\frac{1}{16}+\frac{t^2}{4}}=\frac{1}{2}\sqrt{\frac{1}{4}+t^2}$.

The argument is $\arg\left(\frac{1}{4}+\frac{i\,t}{2}\right)=\arctan\left(\frac{t/2}{1/4}\right)=\arctan(2\,t)$.

The logarithm of $\frac{1}{4}+\frac{i\,t}{2}$ in polar form equals:

$$\log\left(\frac{1}{4}+\frac{i\,t}{2}\right)=\log\left(\frac{1}{2}\sqrt{\frac{1}{4}+t^2}\right)+i\arctan(2\,t)$$

Taking the imaginary part of the Stirling expression and subtracting $\frac{t}{2}\log\pi$ gives:

$$\theta(t)=\frac{t}{2}\log\left(\frac{t}{2\,\pi}\right)-\frac{t}{2}-\frac{\pi}{8}+O\left(\frac{1}{t}\right)$$

For the inverse formula, set $y=\theta(t)$ and solve for t. The equation:

$$y=\frac{t}{2}\log\left(\frac{t}{2\,\pi}\right)-\frac{t}{2}-\frac{\pi}{8}+O\left(\frac{1}{t}\right)$$

With the substitution $u=\frac{t}{2\,\pi}$, this becomes:

$$y+\frac{\pi}{8}=\pi\,u\log u+O\left(\frac{1}{u}\right)$$

The solution utilizes the Lambert W function:

$$u=\exp\left(W\left(\frac{y+\frac{\pi}{8}}{\pi}\right)\right)$$

Converting back to $t=2\,\pi\,u$:

$$t=2\,\pi\exp\left(W\left(\frac{y}{\pi\,e}\right)\right)+O\left(\frac{\log y}{y}\right)$$

The error term follows from the asymptotic behavior of the Lambert W function. $\square$