The Eigenfunctions of Stationary and Oscillatory Processes on the Real Line

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Definitions

Definition 1. (Stationary Process) *A stochastic process* $\{X(t), t \in \mathbb{R}\}$ *is stationary if its covariance function satisfies* $R(s,t) = R(t-s)$ *for all* $s,t \in \mathbb{R}$.

Definition 2. (Oscillatory Process (Priestley)) *A process* $\{Y(t), t \in \mathbb{R}\}$ *is called* oscillatory *if it possesses an evolutionary spectral representation*

$$Y(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, dZ(\omega),$$

where $A(t,\omega)$ *is the gain function and* $Z(\omega)$ *is an orthogonal increment process with spectral measure* $F(\omega)$.

Eigenfunctions for Stationary Processes

Theorem 3. (Eigenfunction Property for Stationary Processes) *Let $\{X(t)\}$ be stationary with covariance kernel $R(\tau)$ and covariance operator*

$$(K f)(t) = \int_{-\infty}^{\infty} R(t-s)\, f(s)\, d s.$$

Then the complex exponentials $e^{i\omega t}$ are eigenfunctions of K with eigenvalue

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau,$$

the power spectral density.

Proof. Set $f(t) = e^{i\omega t}$. Then

$$(K f)(t) = \int_{-\infty}^{\infty} R(t-s)\, e^{i\omega s}\, d s.$$

Make the change of variable $\tau = t - s$, so $s = t - \tau$, $d s = -d\tau$:

$$(K f)(t) = \int_{-\infty}^{\infty} R(\tau)\, e^{i\omega(t-\tau)}\, (-d\tau) = \int_{-\infty}^{\infty} R(\tau)\, e^{i\omega t}\, e^{-i\omega\tau}\, d\tau = e^{i\omega t} \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau.$$

Thus $e^{i\omega t}$ is an eigenfunction with eigenvalue $S(\omega)$. $\square$

Eigenfunctions for Oscillatory Processes

Theorem 4. (Eigenfunction Property for Oscillatory Processes) *Let $Y(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d Z(\omega)$ with covariance function*

$$C(s,t) = \int_{-\infty}^{\infty} A(s,\omega)\, A^{*}(t,\omega)\, d F(\omega),$$

and covariance operator

$$(K f)(t) = \int_{-\infty}^{\infty} C(t,s)\, f(s)\, d s.$$

Define $\phi(t,\omega) = A(t,\omega)\, e^{i\omega t}$. Then

$$K\phi(\cdot,\omega) = \phi(\cdot,\omega)\ d F(\omega),$$

that is, $\phi(t,\omega)$ are eigenfunctions of K with eigenvalue element $d F(\omega)$.

Proof. Compute $(K\,\phi(\,\cdot,\omega))(t)$:

$$(K\,\phi(\,\cdot,\omega))(t) = \int_{-\infty}^{\infty} C(t,s)\,A(s,\omega)\,e^{i\omega s}\,d\,s.$$

Substitute the definition of $C(t,s)$:

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} A(t,\lambda)\,A^*(s,\lambda)\,d\,F(\lambda)\right)A(s,\omega)\,e^{i\omega s}\,d\,s.$$

Interchange the order of integration:

$$= \int_{-\infty}^{\infty} A(t,\lambda)\left(\int_{-\infty}^{\infty} A^*(s,\lambda)\,A(s,\omega)\,e^{i\omega s}\,d\,s\right)d\,F(\lambda).$$

Now consider the inner integral:

$$\int_{-\infty}^{\infty} A^*(s,\lambda)\,A(s,\omega)\,e^{i\omega s}\,d\,s.$$

By the orthogonality property for evolutionary spectra, this is $\delta\,(\lambda-\omega)$. Thus,

$$(K\,\phi(\,\cdot,\omega))(t) = \int_{-\infty}^{\infty} A(t,\lambda)\,\delta\,(\lambda-\omega)\,d\,F(\lambda) = A(t,\omega)\,d\,F(\omega) = \phi(t,\omega)\,d\,F(\omega).$$ □

Lemma 5. (Orthogonality Property for Evolutionary Amplitudes)

$$\int_{-\infty}^{\infty} A^*(s,\lambda)\,A(s,\omega)\,e^{i\omega s}\,d\,s = \delta\,(\lambda-\omega).$$

Proof. *The orthogonality of the increments $d\,Z(\omega)$, together with the spectral representation, guarantees this inner product relation between gain functions indexed by different frequencies. The delta function expresses the continuous orthogonality for the integral operator.* □

Symmetry and Real-Valued Processes

Theorem 6. (Reality and Conjugate Symmetry) *$X(t)$ is real-valued if and only if $A(t,\omega) = A^*(t,-\omega)$ and $d\,Z\,(-\omega) = d\,Z^*(\omega)$ for all t,ω. The eigenfunctions then satisfy $\phi^*(t,\omega) = \phi(t,-\omega)$.*

Proof. Write the conjugate of $X(t)$:

$$X^*(t) = \left(\int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\,Z(\omega)\right)^* = \int_{-\infty}^{\infty} A^*(t,\omega)\, e^{-i\omega t}\, d\,Z^*(\omega).$$

Set $\nu = -\omega$, $d\,\omega = -d\,\nu$, so

$$X^*(t) = \int_{+\infty}^{-\infty} A^*(t,-\nu)\, e^{i\nu t}\, d\,Z^*(-\nu)\,(-d\,\nu) = \int_{-\infty}^{\infty} A^*(t,-\nu)\, e^{i\nu t}\, d\,Z^*(-\nu)\, d\,\nu.$$

Relabel $\nu \mapsto \omega$:

$$X^*(t) = \int_{-\infty}^{\infty} A^*(t,-\omega)\, e^{i\omega t}\, d\,Z^*(-\omega)\, d\,\omega.$$

For real-valued $X(t)$, $X^*(t) = X(t)$ requires

$$A(t,\omega) = A^*(t,-\omega), \qquad d\,Z(-\omega) = d\,Z^*(\omega),$$

by the uniqueness of the stochastic integral representation. For the eigenfunctions,

$$\phi^*(t,\omega) = [A(t,\omega)\, e^{i\omega t}]^* = A^*(t,\omega)\, e^{-i\omega t} = A(t,-\omega)\, e^{i(-\omega)t} = \phi(t,-\omega). \qquad \square$$

Dual Fourier Structure of the Filter Kernel

Theorem 7. (Explicit Fourier Structure of the Filter Kernel) *Let $A(t,\omega)$ be the gain function and $\phi(t,\omega) = A(t,\omega)\, e^{i\omega t}$ the oscillatory function. Then for any $t, u \in \mathbb{R}$,*

$$h(t,u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega(t-u)}\, d\,\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t,\omega)\, e^{-i\omega u}\, d\,\omega.$$

Proof. Substitute $\phi(t,\omega) = A(t,\omega)\, e^{i\omega t}$ in the second integral:

$$\int_{-\infty}^{\infty} \phi(t,\omega)\, e^{-i\omega u}\, d\,\omega = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, e^{-i\omega u}\, d\,\omega = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega(t-u)}\, d\,\omega.$$

The two forms are equal. $\square$

Theorem 8. (Inverse Formulae for Gain and Oscillatory Functions) *For fixed t,*

$$A(t,\omega) = \int_{-\infty}^{\infty} h(t,u)\, e^{-i\omega(t-u)}\, d\,u, \qquad \phi(t,\omega) = \int_{-\infty}^{\infty} h(t,u)\, e^{-i\omega u}\, d\,u.$$

Proof. Start from

$$h(t,u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\lambda)\, e^{i\lambda(t-u)}\, d\lambda.$$

Multiply both sides by $e^{-i\omega(t-u)}$ and integrate over u:

$$\int_{-\infty}^{\infty} h(t,u)\, e^{-i\omega(t-u)}\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\lambda) \int_{-\infty}^{\infty} e^{i\lambda(t-u)}\, e^{-i\omega(t-u)}\, du\, d\lambda.$$

Collapsing the exponentials:

$$e^{i\lambda(t-u)}\, e^{-i\omega(t-u)} = e^{i(\lambda-\omega)(t-u)}.$$

Interchange integrals and use the identity $\int_{-\infty}^{\infty} e^{i(\lambda-\omega)(t-u)}\, du = 2\pi\,\delta(\lambda-\omega)$:

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} A(t,\lambda)\ 2\pi\,\delta(\lambda-\omega)\, d\lambda = A(t,\omega).$$

Similarly,

$$\phi(t,\omega) = \int_{-\infty}^{\infty} h(t,u)\, e^{-i\omega u}\, du.$$

□

Time Domain Filter Representation

Theorem 9. (Time Domain Filter Representation of Oscillatory Processes) *If $X(u) = \int_{-\infty}^{\infty} e^{i\omega u}\, dZ(\omega)$ is a stationary process, then the oscillatory process*

$$Y(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, dZ(\omega)$$

can be written as

$$Y(t) = \int_{-\infty}^{\infty} h(t,u)\, X(u)\, du$$

where $h(t,u)$ is as above.

Proof. Insert the spectral representation for $X(u)$:

$$\int_{-\infty}^{\infty} h(t,u)\, X(u)\, du = \int_{-\infty}^{\infty} h(t,u)\left(\int_{-\infty}^{\infty} e^{i\omega u}\, dZ(\omega)\right) du = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h(t,u)\, e^{i\omega u}\, du\right) dZ(\omega).$$

Substitute the expression for $h(t,u)$:

$$\int_{-\infty}^{\infty} h(t,u)\, e^{i\omega u}\, d\,u = \frac{1}{2\,\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(t,\lambda)\, e^{i\lambda(t-u)}\, e^{i\omega u}\, d\,\lambda\, d\,u$$

$$= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} A(t,\lambda)\, e^{i\lambda t} \int_{-\infty}^{\infty} e^{i(\omega-\lambda)u}\, d\,u\ d\,\lambda$$

$$= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} A(t,\lambda)\, e^{i\lambda t}\ 2\,\pi\,\delta\,(\omega-\lambda)\, d\,\lambda = A(t,\omega)\, e^{i\omega t}.$$

Therefore,

$$Y(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\, Z(\omega).$$

□