Unitary Time Changes of Stationary Processes Yield Oscillatory Processes

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Table of contents

1 Oscillatory Processes

Remark 1. Unless otherwise stated.

1. All parametric families $\{f_t(\omega)\}$ are jointly measurable with respect to $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

2. Dirac delta identities such as $\int e^{i(\mu-\lambda)u}\, d\, u = 2\,\pi\,\delta\,(\mu-\lambda)$ are distributional.

3. Integrals of the form $\int f(\omega)\, d\,\nu(\omega)$ denote Lebesgue-Stieltjes integration with respect to measure ν, while $\int g(u)\, d\, u$ denotes Lebesgue integration with respect to Lebesgue measure.

4. Integrals with respect to orthogonal random measures Φ, written $\int h(\omega)\, d\,\Phi(\omega)$, are Lebesgue-Stieltjes integrals in $L^2(\Omega)$ with variance $\mathbb{E}[|\int h\, d\,\Phi|^2] = \int |h|^2\, d\,\mu$.

5. Limit interchange is justified by dominated convergence under uniform L^2 bounds.

Throughout, $(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed probability space and μ is a finite Borel measure on $\mathbb{R}$.

Definition 2. ***[Oscillatory process]*** *[2] Let F be a finite nonnegative Borel measure on $\mathbb{R}$. Let the gain function be*

$$A_t(\lambda) \in L^2(F) \quad \forall t, \lambda \in \mathbb{R} \tag{1}$$

so that the corresponding oscillatory function is given by

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \in L^2(F) \,\mathrm{since}\, |e^{i\lambda t}| = 1 \forall t, \lambda \in \mathbb{R} \tag{2}$$

then an oscillatory process is a (non-stationary) Gaussian process which can be represented as

$$\begin{aligned} Z(t) &= \int_{-\infty}^{\infty} \varphi_t(\lambda)\, d\,\Phi(\lambda) \\ &= \int_{-\infty}^{\infty} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda) \end{aligned} \tag{3}$$

where Φ is a complex orthogonal random measure with spectral measure F which satisfies the relation

$$d\,\mathbb{E}[\Phi(\lambda)\overline{\Phi(\mu)}] = \delta\,(\lambda - \mu)\, d\,F(\lambda) \tag{4}$$

and has the corresponding covariance kernel

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)} e^{i\lambda(t-s)}\, d\,F(\lambda) \\ &= \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)} d\,F(\lambda) \end{aligned} \tag{5}$$

Theorem 3. *[Quadratic Integrability Condition]Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and Φ a complex-valued orthogonal random measure with second moment measure μ. Let $A\colon \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be measurable in each variable and define $\varphi_t(\omega) = A(t, \omega)\, e^{i\omega t}$. Then for each fixed $t \in \mathbb{R}$, the stochastic integral*

$$X_t = \int_{\Omega} \varphi_t(\omega)\, d\,\Phi(\omega) \tag{6}$$

exists if and only if φ_t is square-integrable with respect to μ:

$$\int_{\Omega} |\varphi_t(\omega)|^2\, d\,\mu(\omega) < \infty. \tag{7}$$

Equivalently, since $|e^{i\omega t}| = 1$, this condition reduces to

$$\int_{\Omega} |A(t, \omega)|^2\, d\,\mu(\omega) < \infty. \tag{8}$$

Proof. For each fixed t, we have the algebraic identity

$$|\varphi_t(\omega)|^2 = |A(t, \omega)\, e^{i\omega t}|^2 = |A(t, \omega)|^2 \cdot |e^{i\omega t}|^2 = |A(t, \omega)|^2. \tag{9}$$

Therefore

$$\int_{\Omega} |\varphi_t(\omega)|^2\, d\,\mu(\omega) = \int_{\Omega} |A(t, \omega)|^2\, d\,\mu(\omega). \tag{10}$$

The stochastic integral X_t is defined via the Itô isometry:

$$\mathbb{E}\left[\left|\int_{\Omega}\varphi_t(\omega)\,d\,\Phi(\omega)\right|^2\right]=\int_{\Omega}|\varphi_t(\omega)|^2\,d\,\mu(\omega). \tag{11}$$

Hence the integral exists in $L^2(\Omega)$ precisely when the right-hand side is finite. The condition on $A(t,\cdot)$ follows immediately. □

Theorem 4. ***[Real-valuedness criterion for oscillatory processes]*** *Let Z be an oscillatory process with oscillatory function*

$$\varphi_t(\lambda)=A_t(\lambda)\,e^{i\lambda t} \tag{12}$$

and spectral measure F. Then Z is real-valued if and only if

$$A_t\,(-\lambda)=\overline{A_t(\lambda)} \tag{13}$$

for F-almost every $\lambda\in\mathbb{R}$, equivalently

$$\varphi_t\,(-\lambda)=\overline{\varphi_t(\lambda)} \tag{14}$$

for F-almost every $\lambda\in\mathbb{R}$.

Proof.

1. Assume Z is real-valued. Then for all $t\in\mathbb{R}$,

$$Z(t)=\overline{Z(t)} \tag{15}$$

2. From the oscillatory representation (3),

$$Z(t)=\int_{\mathbb{R}}A_t(\lambda)\,e^{i\lambda t}\,d\,\Phi(\lambda) \tag{16}$$

3. Taking the complex conjugate of both sides of (16),

$$\overline{Z(t)}=\overline{\int_{\mathbb{R}}A_t(\lambda)\,e^{i\lambda t}\,d\,\Phi(\lambda)}=\int_{\mathbb{R}}\overline{A_t(\lambda)}e^{-i\lambda t}\,d\overline{\Phi(\lambda)} \tag{17}$$

4. For a real-valued process, the orthogonal random measure must satisfy the symmetry property

$$d\overline{\Phi(\lambda)}=d\,\Phi\,(-\lambda) \tag{18}$$

5. Substituting (18) into (17),

$$\overline{Z(t)}=\int_{\mathbb{R}}\overline{A_t(\lambda)}e^{-i\lambda t}\,d\,\Phi\,(-\lambda) \tag{19}$$

6. Apply the change of variables $\mu=-\lambda$, so $d\,\Phi\,(-\lambda)=d\,\Phi(\mu)$ and $e^{-i\lambda t}=e^{i\mu t}$:

$$\overline{Z(t)}\ =\int_{\mathbb{R}}\overline{A_t\,(-\mu)}e^{i\mu t}\,d\,\Phi(\mu) \tag{20}$$

7. By (15), the right sides of (16) and (20) must be equal:

$$\int_{\mathbb{R}} A_t(\mu)\, e^{i\mu t}\, d\,\Phi(\mu) = \int_{\mathbb{R}} \overline{A_t\,(-\mu)} e^{i\mu t}\, d\,\Phi(\mu) \tag{21}$$

8. Since the stochastic integral representation is unique in $L^2(F)$, the integrands must be equal F-almost everywhere:

$$A_t(\lambda) = \overline{A_t\,(-\lambda)} \quad \text{for } F\text{-a.e. } \lambda \tag{22}$$

9. This is equivalent to (13). From (12),

$$\varphi_t\,(-\lambda) = A_t\,(-\lambda)\, e^{-i\lambda t} \tag{23}$$

10. Using (13),

$$\begin{aligned} \varphi_t\,(-\lambda) \ &= \overline{A_t(\lambda)} e^{-i\lambda t} \\ &= \overline{A_t(\lambda)\, e^{i\lambda t}} \\ &= \overline{\varphi_t(\lambda)} \end{aligned} \tag{24}$$

establishing (14).

11. Conversely, assume (13) holds. Reversing the steps from (20) to (15) shows that $\overline{Z(t)} = Z(t)$ for all t, so Z is real-valued. ☐

Theorem 5. ***[Filter representation]*** *Let X be a zero-mean stationary process*

$$X(u) = \int e^{i\lambda u}\, d\,\Phi(\lambda) \tag{25}$$

and spectral measure F, and let Z be an oscillatory process with oscillatory function $\varphi_t(\lambda)$ and the same orthogonal random measure Φ. Then

$$Z(t) \ = \int_{-\infty}^{\infty} h(t, u)\, X(u)\, d\,u \tag{26}$$

where

$$h(t, u) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \varphi_t(\lambda)\, e^{-i\lambda u}\, d\,\lambda \tag{27}$$

where $h(t, u)$ is the forward impulse response of Definition ?.

Proof. 1. Substitute the definitions of $h(t, u)$ and $X(u)$:

$$\int_{-\infty}^{\infty} h(t, u)\, X(u)\, d\,u = \int_{-\infty}^{\infty} \left[\frac{1}{2\,\pi} \int_{-\infty}^{\infty} \varphi_t(\lambda)\, e^{-i\lambda u}\, d\,\lambda \right] \left[\int_{-\infty}^{\infty} e^{i\mu u}\, d\,\Phi(\mu) \right] d\,u \tag{28}$$

2. Apply Fubini's theorem:

$$= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_t(\lambda) \left[\int_{-\infty}^{\infty} e^{i(\mu-\lambda)u}\, d\,u \right] d\,\lambda\; d\,\Phi(\mu) \tag{29}$$

3. The inner integral over u is the Fourier representation of the Dirac delta:

$$\int_{-\infty}^{\infty} e^{i(\mu-\lambda)u}\, d u = 2\,\pi\,\delta\,(\mu-\lambda) \tag{30}$$

4. Substitute:

$$= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_t(\lambda) \cdot 2\,\pi\,\delta\,(\mu-\lambda)\; d\,\lambda\; d\,\Phi(\mu) \tag{31}$$

5. Simplify:

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_t(\lambda)\,\delta\,(\mu-\lambda)\; d\,\lambda\; d\,\Phi(\mu) \tag{32}$$

6. Apply the sifting property of the delta function:

$$= \int_{-\infty}^{\infty} \varphi_t(\mu)\, d\,\Phi(\mu) = Z(t) \tag{33}$$ □

2 Unitarily Time-Changed Stationary Processes

2.1 Unitary Time-Change Operator $U_\theta f$

Theorem 6. ***[Unitary time-change operator U_θ and its inverse U_θ^{-1}]*** *Let the time-change function $\theta\colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with*

$$\dot{\theta}(t) > 0 \tag{34}$$

almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero. For f measurable, define

$$(U_\theta\, f)(t) = \sqrt{\dot{\theta}(t)}\; f(\theta(t)) \tag{35}$$

Its inverse is given by

$$(U_\theta^{-1}\, g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{36}$$

For every compact set $K \subseteq \mathbb{R}$ and $f \in L^2_{\mathrm{loc}}(\mathbb{R})$,

$$\int_K |(U_\theta\, f)(t)\,|^2 d\,t = \int_{\theta(K)} |\, f(s)|^2\, d\,s \tag{37}$$

Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\mathrm{loc}}(\mathbb{R})$.

Proof. 1. Let $f \in L^2_{\mathrm{loc}}(\mathbb{R})$ and let $K \subset \mathbb{R}$ be compact. From the definition (35),

$$\int_K |(U_\theta\, f)(t)|^2\, d\,t = \int_K \left| \sqrt{\dot{\theta}(t)}\; f(\theta(t)) \right|^2 d\,t \tag{38}$$

2. Expanding the square,

$$\int_K \left| \sqrt{\dot{\theta}(t)}\; f(\theta(t)) \right|^2 d\,t = \int_K \dot{\theta}(t) |\, f(\theta(t))|^2\, d\,t \tag{39}$$

3. Since θ is absolutely continuous and strictly increasing, $\theta' = \dot{\theta}$ exists almost everywhere and $\dot{\theta}(t) > 0$ a.e.

4. Apply the change of variables $s = \theta(t)$. Then

$$d\,s = \dot{\theta}(t)\,d\,t \tag{40}$$

5. The inverse function $t = \theta^{-1}(s)$ exists since θ is strictly increasing and bijective.

6. As t ranges over K, the variable $s = \theta(t)$ ranges over $\theta(K)$.

7. Since θ is continuous and K is compact, $\theta(K)$ is compact.

8. Substituting (40) into (39),

$$\int_K \dot{\theta}(t)\,|f(\theta(t))|^2 d\,t = \int_{\theta(K)} |f(s)|^2\,d\,s \tag{41}$$

9. This establishes the local isometry (37).

10. To verify U_θ^{-1} is the inverse, compute:

$$(U_\theta^{-1}\,U_\theta\,f)(s) = U_\theta^{-1}\,(U_\theta\,f)(s) \tag{42}$$

11. By definition (36),

$$U_\theta^{-1}\,(U_\theta\,f)(s) = \frac{(U_\theta\,f)(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{43}$$

12. By definition (35),

$$(U_\theta\,f)(\theta^{-1}(s)) = \sqrt{\dot{\theta}(\theta^{-1}(s))}\;f(\theta(\theta^{-1}(s))) \tag{44}$$

13. Since $\theta \circ \theta^{-1} = \mathrm{id}$,

$$f(\theta(\theta^{-1}(s))) = f(s) \tag{45}$$

14. Substituting (44) and (45) into (43),

$$\frac{\sqrt{\dot{\theta}(\theta^{-1}(s))}\;f(s)}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} = f(s) \tag{46}$$

15. Therefore

$$U_\theta^{-1}\,U_\theta = \mathrm{id} \tag{47}$$

16. Similarly, compute:

$$(U_\theta\,U_\theta^{-1}\,g)(t) = \sqrt{\dot{\theta}(t)}\;(U_\theta^{-1}\,g)(\theta(t)) \tag{48}$$

17. By definition (36),

$$(U_\theta^{-1}\,g)(\theta(t)) = \frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} \tag{49}$$

18. Since $\theta^{-1} \circ \theta = \mathrm{id}$,

$$g(\theta^{-1}(\theta(t))) = g(t), \quad \theta^{-1}(\theta(t)) = t \tag{50}$$

19. Substituting (50) into (49),

$$\frac{g(t)}{\sqrt{\dot{\theta}(t)}} \tag{51}$$

20. Therefore from (48),

$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)} \cdot \frac{g(t)}{\sqrt{\dot{\theta}(t)}} = g(t) \tag{52}$$

21. Thus

$$U_\theta U_\theta^{-1} = \mathrm{id} \tag{53}$$

22. Combining (47) and (53), U_θ^{-1} is the two-sided inverse of U_θ on $L^2_{\mathrm{loc}}(\mathbb{R})$. □

2.2 Transformation of Stationary → Oscillatory Processes via U_θ

Theorem 7. ***[Unitary time changes of stationary processes produce oscillatory process]*** *Let X be zero-mean stationary as in Definition ?. For scaling function θ as in Theorem 6, define*

$$\begin{aligned} Z(t) &= (U_\theta X)(t) \\ &= \sqrt{\dot{\theta}(t)}\, X(\theta(t)) \end{aligned} \tag{54}$$

Then Z is a realization of an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \tag{55}$$

gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)} \tag{56}$$

and covariance kernel

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\left[\sqrt{\dot{\theta}(t)}\, X(\theta(t))\overline{\sqrt{\dot{\theta}(s)}\, X(\theta(s))}\right] \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\, \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\, R_X(\theta(t)-\theta(s)) \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\, dF(\lambda) \end{aligned} \tag{57}$$

Proof. 1. From the Cramér representation (?),

$$X(u) = \int_{\mathbb{R}} e^{i\lambda u}\, d\Phi(\lambda) \tag{58}$$

2. Substituting $u = \theta(t)$ into (58),

$$X(\theta(t)) = \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\Phi(\lambda) \tag{59}$$

3. From the definition (54),

$$Z(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\,\Phi(\lambda) \tag{60}$$

4. By linearity of the stochastic integral,

$$Z(t) = \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}\, d\,\Phi(\lambda) \tag{61}$$

5. Define

$$\varphi_t(\lambda) := \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \tag{62}$$

6. Then (61) becomes

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda)\, d\,\Phi(\lambda) \tag{63}$$

which is the oscillatory representation (3).

7. To express this in terms of the standard oscillatory function form, define the gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)} \tag{64}$$

8. Then verify the oscillatory function form (2) factorizes

$$\begin{aligned} \varphi_t(\lambda) &= A_t(\lambda)\, e^{i\lambda t} \\ &= \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)}\, e^{i\lambda t} \\ &= \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t+t)} \\ &= \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \end{aligned} \tag{65}$$

9. To compute the covariance, use (5):

$$R_Z(t,s) = \mathbb{E}[Z(t)\overline{Z(s)}] \tag{66}$$

10. Substituting (54),

$$R_Z(t,s) = \mathbb{E}\left[\sqrt{\dot{\theta}(t)}\, X(\theta(t))\overline{\sqrt{\dot{\theta}(s)}\, X(\theta(s))}\right] \tag{67}$$

11. Since $\dot{\theta}$ is deterministic,

$$R_Z(t,s) = \sqrt{\dot{\theta}(t)}\, \sqrt{\dot{\theta}(s)}\, \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \tag{68}$$

12. By stationarity of X,

$$\mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] = R_X(\theta(t)-\theta(s)) = \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\, d\,F(\lambda) \tag{69}$$

13. Substituting (69) into (68),

$$R_Z(t,s) = \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\, d\,F(\lambda) \tag{70}$$

establishing (57). □

Corollary 8. *The evolutionary spectrum is*

$$d\,F_t(\lambda) = \dot{\theta}(t)\,d\,F(\lambda) \tag{71}$$

Proof. 1. The evolutionary spectrum is defined by

$$d\,F_t(\lambda) = |A_t(\lambda)|^2\,d\,F(\lambda) \tag{72}$$

2. From (56),

$$|A_t(\lambda)|^2 = \left|\sqrt{\dot{\theta}(t)}\,e^{i\lambda(\theta(t)-t)}\right|^2 \tag{73}$$

3. Since $|e^{i\alpha}| = 1$ for all real α,

$$|e^{i\lambda(\theta(t)-t)}|^2 = 1 \tag{74}$$

4. Therefore

$$|A_t(\lambda)|^2 = \left(\sqrt{\dot{\theta}(t)}\right)^2 \cdot 1 = \dot{\theta}(t) \tag{75}$$

5. Substituting (75) into (72),

$$d\,F_t(\lambda) = \dot{\theta}(t)\,d\,F(\lambda) \tag{76}$$ □

2.2.1 Time-Varying Filter Representations

... not necessary...

2.3 Covariance operator conjugation

Proposition 9. *Let*

$$(T_K\,f)(t) := \int_{\mathbb{R}} K(|t-s|)\,f(s)\,d\,s \tag{77}$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h}\,d\,F(\lambda) \tag{78}$$

Define the transformed kernel

$$\begin{aligned} K_\theta(s,t) &= (U_{t\to\theta(t)}\,(U_{s\to\theta(s)}\,K))(t,s) \\ &= (U_{s\to\theta(s)}\,(U_{t\to\theta(t)}\,K))(t,s) \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\,K(|\theta(t)-\theta(s)|) \end{aligned} \tag{79}$$

then the corresponding integral covariance operator is conjugated for all $f \in L^2_{\mathrm{loc}}(\mathbb{R})$ by

$$(T_{K_\theta}\,f)(t) = (U_\theta\,T_K\,U_\theta^{-1}\,f)(t) \tag{80}$$

Proof. 1. From (80), expand the right side:

$$(U_\theta\,T_K\,U_\theta^{-1}\,f)(t) = \sqrt{\dot{\theta}(t)}\,(T_K\,U_\theta^{-1}\,f)(\theta(t)) \tag{81}$$

2. By definition (77),

$$(T_K U_\theta^{-1} f)(\theta(t)) = \int_{\mathbb{R}} K(|\theta(t) - s|)\,(U_\theta^{-1} f)(s)\,d s \tag{82}$$

3. By definition (36),

$$(U_\theta^{-1} f)(s) = \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{83}$$

4. Substituting (83) into (82),

$$\int_{\mathbb{R}} K(|\theta(t) - s|) \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}}\,d s \tag{84}$$

5. Apply the change of variables $s = \theta(u)$, so $d s = \dot{\theta}(u)\,d u$ and $\theta^{-1}(s) = u$:

$$\int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \frac{f(u)}{\sqrt{\dot{\theta}(u)}}\,\dot{\theta}(u)\,d u \tag{85}$$

6. Simplify:

$$\int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \frac{\dot{\theta}(u)}{\sqrt{\dot{\theta}(u)}}\,f(u)\,d u = \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \sqrt{\dot{\theta}(u)}\,f(u)\,d u \tag{86}$$

7. Substituting (86) into (81),

$$\sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \sqrt{\dot{\theta}(u)}\,f(u)\,d u \tag{87}$$

8. Bring the constant inside the integral:

$$\int_{\mathbb{R}} \sqrt{\dot{\theta}(t)}\,\sqrt{\dot{\theta}(u)}\,K(|\theta(t) - \theta(u)|)\,f(u)\,d u \tag{88}$$

9. Combine the square roots $\sqrt{\dot{\theta}(t)}\,\sqrt{\dot{\theta}(u)} = \sqrt{\dot{\theta}(t)\dot{\theta}(s)}$ and recognize definition (79),

$$\sqrt{\dot{\theta}(t)\dot{\theta}(s)}\,K(|\theta(t) - \theta(u)|) = K_\theta(u, t) \tag{89}$$

10. Therefore

$$\begin{aligned} \int_{\mathbb{R}} K_\theta(u, t)\,f(u)\,d u \; &= \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)\dot{\theta}(s)}\,K(|\theta(t) - \theta(u)|) f(u)\,d u \\ &= (T_{K_\theta} f)(t) \end{aligned} \tag{90}$$

establishing (80). □

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