The Spectrum of the Zero Crossing Number Operator

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Definition 1. *[Zero-Crossing Measure] Let $X(t, \omega)$ be a sample path of a Gaussian process with derivative $X'(t, \omega)$ existing almost surely where $\omega \in \Omega$ represents a specific sample path element from the ensemble of possible sample paths denoted by Ω. Define the zero-crossing measure μ_ω on $\mathbb{R}$ by:*

$$d\,\mu_\omega(s) = \delta(X(s, \omega))|X'(s, \omega)|\,d\,s \tag{1}$$

where δ is the Dirac delta function.

Definition 2. *[Zero-Crossing Spectral Operator] Define the zero-crossing spectral operator $T: L^2(\mathbb{R}, \mu_\omega) \to L^2(\mathbb{R}, \mu_\omega)$ by:*

$$(T f)(s) = s \cdot f(s) \tag{2}$$

Theorem 3. *[Zero-Crossing Spectrum] The operator T has spectrum given by:*

$$\sigma(T) = \overline{\{t \in \mathbb{R}: X(t, \omega) = 0\}} \tag{3}$$

where the overline denotes topological closure.

Proof. The operator T is multiplication by the function $m(s) = s$ on the measure space $(\mathbb{R}, \mu_\omega)$.

For multiplication operators on measure spaces, the spectrum is given by:

$$\sigma(T) = \text{essential range of } m \text{ with respect to } \mu_\omega \tag{4}$$

The essential range of $m(s) = s$ with respect to measure μ_ω is:

$$\text{ess ran}_{\mu_\omega}(s) = \{\lambda \in \mathbb{R}: \mu_\omega(\{s: |s - \lambda| < \epsilon\}) > 0 \text{ for all } \epsilon > 0\} \tag{5}$$

Since μ_ω is supported precisely on $Z_\omega = \{t \in \mathbb{R}: X(t, \omega) = 0\}$, we have:

$$\mu_\omega(\{s: |s - \lambda| < \epsilon\}) > 0 \text{ if and only if } (\lambda - \epsilon, \lambda + \epsilon) \cap Z_\omega \neq \emptyset \tag{6}$$

This occurs if and only if λ is in the closure of Z_ω.

Therefore:

$$\sigma(T) = \overline{Z_\omega} = \overline{\{t \in \mathbb{R} \colon X(t, \omega) = 0\}} \tag{7}$$

Equivalently, $\lambda \in \sigma(T)$ if and only if $(T - \lambda I)$ is not invertible, which occurs precisely when the multiplier $(s - \lambda)$ is not bounded away from zero on the support of μ_ω. $\square$