On a Class of Asymptotically Stationary Harmonizable Processes

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Abstract

We prove that every harmonizable process with σ-finite bimeasure is asymptotically stationary and we give its associated spectral measure.

I. Introduction

For stochastic processes, various extensions of the notion of stationarity have been developed such as asymptotic stationarity and harmonizability, which are related notions. For example, Rozanov [12] established that every strongly harmonizable process is asymptotically stationary.

In Section 2, we introduce a larger class of asymptotically stationary harmonizable processes, i.e., harmonizable processes which have σ-finite bimeasure, and we prove that they are uniform limits of a sequence of strongly harmonizable ones.

In Section 3, we show that these processes are indeed asymptotically stationary, and we exhibit the associated spectral measure using a stationary dilation of the harmonizable process under consideration [10].

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5. Preliminaries

Following Rozanov [12] (see also [1,6]), a process $X\colon \mathbb{R} \to L^2_{\mathbb{C}}(S, \mathscr{F}, P)$ is said to be asymptotically stationary if there exists a continuous function $r\colon \mathbb{R} \to \mathbb{C}$, such that for any h in $\mathbb{R}$

$$r(h) = \lim_{t \to +\infty} \frac{1}{t} \int_0^t E\left(X(s+h) \cdot \overline{X(s)}\right) ds \tag{1}$$

In this case there exists a unique positive bounded measure m on $\mathscr{B}(\mathbb{R})$, called the associated spectral measure of X, which verifies for any h in $\mathbb{R}$:

$$r(h) = \int e^{ihu} \, m(du) \tag{2}$$

We recall that every weakly harmonizable process $X\colon \mathbb{R} \to L^2_{\mathbb{C}}(S, \mathscr{F}, P)$ is the Fourier transform of a stochastic measure $\mu\colon \mathscr{B}(\mathbb{R}) \to L^2_{\mathbb{C}}(S, \mathscr{F}, P)$[8, 11, 12], i.e., for any t in $\mathbb{R}$:

$$X(t) = \int e^{itu} \, \mu(du) \tag{3}$$

When the spectral bimeasure M of X, defined on $\mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R})$ by $M(A,B) = E\left(\mu(A)\cdot\overline{\mu(B)}\right)$, is extendable to a measure on $\mathscr{B}(\mathbb{R}^2)$, the process is termed strongly harmonizable.

In this paper we use the concept of integration with respect to a spectral bimeasure as introduced by Moché [8, Chap. IV]. Rozanov has proved that every strongly harmonizable process is asymptotically stationary and, more precisely, one can establish the following: [section]

Proposition 1. *Let X be a strongly harmonizable process with spectral measure M, and let $\Delta = \{(u,v) \mid u = v\}$ be the diagonal axis of $\mathbb{R}^2$. Then uniformly with respect to h in $\mathbb{R}$, we have:*

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t E\left(X(s+h)\cdot\overline{X(s)}\right) d\,s = \iint_\Delta e^{ihv} M(d\,u, d\,v)$$

So in the weakly harmonizable case, one of the problems is: How can we define the restriction on the diagonal axis Δ of the bimeasure M as a measure on $\mathscr{B}(\mathbb{R})$? [section]

Definition 2. *A spectral bimeasure M is said to be σ-finite if there exists a sequence $(B_n)_{n\in\mathbb{N}}$ in $\mathscr{B}(\mathbb{R})$ which verifies:*

(1). for any $n \in \mathbb{N}, B_n \subset B_{n+1}$; and $\bigcup_{n\in\mathbb{N}} B_n = \mathbb{R}$;

(2). for any n, M has finite Vitali variation on $B_n \times B_n$.

[section]

Example 3. (a) Obviously, the spectral bimeasure of every strongly harmonizable process is σ-finite. (b) Here is an example of weakly harmonizable process which is not strongly harmonizable. It is due to Niemi [9] following Edwards [5] (see also [2]).

Let us consider the positive definite family of real numbers defined by

$$\begin{aligned} c_{jj} &= \frac{\pi}{2\,j(\log(j+1))^2}, & j \in \mathbb{N}\setminus\{0\} \\ c_{jk} &= \frac{\sin(\pi(j-k)/2)}{(j-k)\,j^{1/2}\,k^{1/2}\log(j+1)\log(k+1)}, & j \neq k; j,k \in \mathbb{N}\setminus\{0\} \end{aligned} \tag{4}$$

Then there exist a probability space ($S, \mathscr{F}, P$) and a sequence (x_j) in $L^2_{\mathbb{R}}(S,\mathscr{F},P)$ such that $E(x_j \cdot x_k) = c_{jk}$. We can use this sequence to define a stochastic measure μ: $\mathscr{B}(\mathbb{R}) \to L^2_{\mathbb{R}}(S,\mathscr{F},P)$ by $\mu(B) = \sum_{j\in B} x_j$, for every Borel set B of $\mathbb{R}$.

Since $\sum_j \sum_k |c_{jk}| = +\infty$, the Vitali variation of $\mathbb{R}^2$ of its bimeasure M is infinite. Moreover, since μ is discrete, M is obviously σ-finite. Therefore the Fourier transform of μ has a σ-finite bimeasure but is not strongly harmonizable. So the class of harmonizable processes with σ-finite bimeasure contains strictly the class of strongly harmonizable ones.

2.4. Notations. Throughout the sequel, we consider a weakly harmonizable process X with σ-finite bimeasure M, and spectral stochastic measure μ.

Let $(B_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{B}(\mathbb{R})$ which satisfies (1) and (2) and for any n let μ_n be the stochastic measure on $\mathscr{B}(\mathbb{R})$ defined by $\mu_n(B) = \mu(B \cap B_n)$, M_n be its spectral bimeasure which is of finite Vitali variation on $\mathbb{R}^2$, and X_n be the associated strongly harmonizable process.... Niemi [9, Theorem 3.41] has proved that, for any weakly harmonizable process X, there exists a sequence of strongly harmonizable processes which converges in q.m. to X uniformly on every compact subset of $\mathbb{R}$. Recently, Moche and the author [3] showed that this property remains true if the process X is only continuous and bounded. Here we obtain another sharpening of Niemi's result.

Proposition 4. *For every harmonizable process X with σ-finite bimeasure, there exists a bounded sequence of strongly harmonizable processes which converges in q.m. towards X uniformly on $\mathbb{R}$.*

Proof. With the previous notations, let $B'_n = \mathbb{R} \setminus B_n$ and let $\|\mu\|$ be the semi-variation of the stochastic measure μ, [4, Definition IV.10.3]; from [4, Theorem IV.10.8] we estimate for every t :

$$E[|X_n(t)|^2] \leqslant (\|\mu\|(B_n))^2 \leqslant (\|\mu\|(\mathbb{R}))^2 E[|X(t) - X_n(t)|^2] = E\left(\left|\int_{B'_n} e^{itu}\,\mu\,(d\,u)\right|^2\right) \leqslant (\|\mu\|(B'_n))^2$$

Since the sequence ($B'_n)_{n\in\mathbb{N}}$ decreases towards the empty set as n tends to infinity, then $\|\mu\|(B'_n)$ converges towards 0 [4; Lemma IV.10.5] and we can conclude that the bounded sequence $(X_n)_{n\in\mathbb{N}}$ converges towards X in $L^2_{\mathbb{C}}(S, \mathscr{F}, P)$ uniformly with respect to t on $\mathbb{R}$. ☐

3. Main Result

[section]

Theorem 5. *Every harmonizable process with σ-finite bimeasure is asymptotically stationary.*

Proof. One can easily obtain that if a bounded sequence of asymptotically stationary processes ($X_n(t), t \in \mathbb{R}$) converges in q.m. towards a process $(X(t), t \in \mathbb{R})$ uniformly with respect to t in $\mathbb{R}$, then the process $(X(t), t \in \mathbb{R})$ is asymptotically stationary. One can conclude using Proposition 2.5.

Now with a quite different proof, we are going to sharpen the previous result and to estimate the associated spectral measure of the harmonizable process under consideration. ☐

Theorem 6. *For any harmonizable process with σ-finite bimeasure, uniformly with respect to h in $\mathbb{R}$, we have*

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t E\left(X\,(s+h) \cdot \overline{X(s)}\right) d\,s = \int e^{ihu}\, m\,(d\,u)$$

where the positive bounded measure m on $\mathscr{B}(\mathbb{R})$ is defined by:

$$\text{for every } B \text{in } \mathscr{B}(\mathbb{R}), \quad m(B) = \lim_{n\to+\infty} M_n\left((B \times B) \cap \Delta\right)$$

Proof. With Notations 2.4, let $K(t,s) = E\left(X(t) \cdot \overline{X(s)}\right)$ and $K_n(t,s) = E\left(X_n(t) \cdot \overline{X_n(s)}\right)$.
(a) From Proposition 2.5, the sequence $K_n(t,s)$ converges towards $K(t,s)$ uniformly with respect to (t,s) in $\mathbb{R}^2$. So, given $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for $n > N(\varepsilon)$ and for every $t > 0$ and every h we have

$$\left|\frac{1}{t}\int_0^t K(s+h,s)\,d\,s - \frac{1}{t}\int_0^t K_n(s+h,s)\,d\,s\right| < \varepsilon \tag{5}$$

Using the same notation for the spectral bimeasure M_n and its extension as a measure on $\mathscr{B}(\mathbb{R}^2)$, we deduce from Proposition 2.1 that for every n, there exists $T(n,\varepsilon)$ such that for $t > T(n,\varepsilon)$ and for every h one has:

$$\left|\frac{1}{t}\int_0^t K_n(s+h,s)\,d\,s - \iint_\Delta e^{iuh}\, M_n(d\,u, d\,v)\right| < \varepsilon \tag{6}$$

Consequently for $n > N(\varepsilon), t > T(n, \varepsilon)$ and for every h we obtain:

$$\left|\frac{1}{t}\int_0^t K(s+h,s)\,ds - \iint_\Delta e^{iuh}\,M_n(du,dv)\right| < 2\,\varepsilon. \tag{7}$$

(b) We are going to prove that the sequence (m_n) of the restrictions on Δ of the spectral measures $(\ M_n\ )$ is convergent.

First of all, (m_n) is increasing since for any B in $\mathscr{B}(\mathbb{R})$

$$m_n(B) = M_n\left((B\times B)\cap\Delta\right) = M\left((B\cap B_n)\times(B\cap B_n)\cap\Delta\right)$$

Let's re-evaluate the original text's argument: $m_n(B) = M_n\left((B\times B)\cap\Delta\right)$ and $m_{n+1}(B) = M_{n+1}\left((B\times B)\cap\Delta\right)$. Since $M_n(A,C) = M((A\cap B_n),(C\cap B_n))$ and $M_{n+1}(A,C) = M((A\cap B_{n+1}),(C\cap B_{n+1}))$. Also $B_n\subset B_{n+1}$. The measure M restricted to the diagonal is positive. Let m_{diag} be the measure M restricted to the diagonal Δ. Then $m_n(B) = m_{diag}\,(B\cap B_n)$ and $m_{n+1}(B) = m_{diag}\,(B\cap B_{n+1})$. Since $B_n\subset B_{n+1}$, $B\cap B_n\subset B\cap B_{n+1}$. Since m_{diag} is a positive measure, $m_{diag}\,(B\cap B_n)\leq m_{diag}\,(B\cap B_{n+1})$, hence $m_n(B)\leq m_{n+1}(B)$.

$$m_n(B)\leqslant m_{n+1}(B)$$

The only difficulty is to show that this sequence is bounded. Now Miamee and Salehi [7: Domination lemma] have proved that for every spectral bimeasure M on $\mathscr{B}(\mathbb{R})\times\mathscr{B}(\mathbb{R})$, there exists a positive bounded measure m_d on $\mathscr{B}(\mathbb{R})$ such that for any bounded Borel function $f:\mathbb{R}\to\mathbb{C}$ one has:

$$0\leqslant\iint f(t)\overline{f(s)}M(dt,ds)\leqslant\int|f(t)|^2\,m_d\,(dt)$$

So, for any Borel set B in $\mathbb{R}$ we have:

$$0\leqslant M(B,B)\leqslant m_d(B).$$

Let us put $I_q^r = (r/2^q,(r+1)/2^q], r = \cdots -1,0,1,\ldots$ and $q = 0,1,\ldots$. Then, for any q, the sets I_q^r, $r\in\mathbb{Z}$, form a partition of $\mathbb{R}$, and the sequence $S_q = \bigcup_{r=-\infty}^{+\infty} I_q^r\times I_q^r$ decreases towards the diagonal axis Δ, as q becomes infinite.... Given B in $\mathscr{B}(\mathbb{R}), n$, and q, then the measure M_n verifies:

$$\begin{aligned}
0 &\leqslant M_n\left(\bigcup_{r=-\infty}^{+\infty}(B\cap I_q^r)\times(B\cap I_q^r)\right)\\
&= \sum_{r=-\infty}^{+\infty} M\left((B\cap I_q^r\cap B_n)\times(B\cap I_q^r\cap B_n)\right)\\
&\leqslant \sum_{r=-\infty}^{+\infty} m_d\,(B\cap I_q^r\cap B_n)\\
&= m_d\,(B\cap B_n)
\end{aligned}$$

Hence, when q tends to infinity we obtain (taking the limit inside the sum requires justification, perhaps using properties of measures on product spaces, or the definition of m_n as the diagonal restriction):

$$0\leqslant m_n(B)\leqslant m_d\,(B\cap B_n)\leqslant m_d(\mathbb{R}).$$

So, for every Borel set B, the increasing sequence $(m_n(B))$ converges towards a positive number $m(B)$, and according to the Vitali-Hahn Saks theorem [4, Corollary III.7.3], m is a positive bounded measure on $\mathscr{B}(\mathbb{R})$. It is estimated for all n and B by

$$m_n(B) \leqslant m(B) \leqslant m_d(B) \leqslant m_d(\mathbb{R}) < +\infty \quad \text{and} \quad m(B \cap B_n) = m_n(B)$$

Moreover for any bounded Borel function f one has:

$$\begin{aligned}
\left|\int f(u)\, m_n(d\, u) - \int f(u)\, m(d\, u)\right| &= \left|\int f(u)\, m(d\, u) - \int f(u)\, m_n(d\, u)\right| \\
&= \left|\int_B f(u)\, m(d\, u) - \int_{B \cap B_n} f(u)\, m(d\, u)\right| \\
&= \left|\int f(u)\, (m - m_n)\, (d\, u)\right| \\
&= \left|\int_{B_n^c} f(u)\, m(d\, u)\right| \quad (\text{since } m_n(A) = m(A \cap B_n)) \\
&\leqslant \int_{B_n^c} |f(u)|\, m(d\, u) \\
&\leqslant m(B_n') \cdot \sup_{u \in \mathbb{R}} (|f(u)|).
\end{aligned}$$

Since $m(B_n') \to 0$ as $n \to \infty$ (because m is a finite measure and $B_n' \downarrow \emptyset$), the convergence $\int f\, d\, m_n \to \int f\, d\, m$ holds. Consequently, given $\varepsilon > 0$, there exists $N'(\varepsilon)$ such that for $n > N'(\varepsilon)$ and for every h (taking $f(u) = e^{iuh}$):

$$\left|\iint_\Delta e^{iuh}\, M_n(d\, u, d\, v) - \int e^{iuh}\, m(d\, u)\right| = \left|\int e^{iuh}\, m_n(d\, u) - \int e^{iuh}\, m(d\, u)\right| < \varepsilon. \tag{8}$$

(c) From the relations (7) and (8) we deduce that for any $\varepsilon > 0$, there exists $N = \max(N(\varepsilon), N'(\varepsilon))$ and $T(\varepsilon) = T(N, \varepsilon)$ such that for $t > T(\varepsilon)$ and for every h we have:

$$\begin{aligned}
\left|\frac{1}{t}\int_0^t K(s+h, s)\, d\, s - \int e^{iuh}\, m(d\, u)\right| &\leq \left|\frac{1}{t}\int_0^t K(s+h, s)\, d\, s - \iint_\Delta e^{iuh}\, M_N(d\, u, d\, v)\right| \\
&+ \left|\iint_\Delta e^{iuh}\, M_N(d\, u, d\, v) - \int e^{iuh}\, m(d\, u)\right| \\
&< 2\, \varepsilon + \varepsilon = 3\, \varepsilon
\end{aligned}$$

as was to be shown. □

[section]

Remark 7. (a) There exist weakly harmonizable processes with non-σ-finite spectral bimeasure. Indeed, Niemi gave an example of a discrete time weakly harmonizable process which is not asymptotically stationary (cf. [11, Sect. 6]). As Theorems 3.1 and 3.2 still hold in the discrete time case, its spectral bimeasure is not σ-finite. Consequently, μ denoting its spectral stochastic measure (defined on $\mathscr{B}([-\pi, \pi])$), the spectral bimeasure of the (continuous time) weakly harmonizable process defined by

$$X(t) = \int e^{itx}\, \mu(d\, x)\, \forall t \in \mathbb{R} \tag{9}$$

is not σ-finite. We do not know if X is asymptotically stationary.

More generally we do not know how to compare more precisely the class of weakly harmonizable processes and the class of asymptotically stationary processes. (b) So we have:

$$
\begin{array}{rcc}
\{\text{stationary}\} & \subsetneq & \{\text{strongly harmonizable}\} \\
 & \subsetneq & \{\text{harmonizable with } \sigma\text{-finite bimeasure}\} \\
 & \subsetneq & \{\text{weakly harmonizable}\} \\
 & \subset & \{\text{asymptotically stationary}\}
\end{array}
$$

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