now prove it

Define H with the symmetric convention $H(0) = \frac{1}{2}$, so $H(x) = \frac{1}{2}(\operatorname{sgn} x + 1)$ and $1 - H(x) = \frac{1}{2}(1 - \operatorname{sgn} x)$ for all real x. The proposed form [1] [2]

$$\mathrm{incTheta}(t) = \theta(a) + [\theta(t) - \theta(a)]\ \operatorname{sgn}(t - a)$$

is equivalent to the Heaviside form

$$\mathrm{incTheta}(t) = 2\theta(a) - \theta(t) + 2[\theta(t) - \theta(a)]\, H(t - a),$$

since $\operatorname{sgn}(t - a) = 2H(t - a) - 1$ under $H(0) = \frac{1}{2}$. [2] [1]

Verification is pointwise by cases:

- Case $t > a$: $H(t - a) = 1$, $\operatorname{sgn}(t - a) = 1$. Then
$$2\theta(a) - \theta(t) + 2[\theta(t) - \theta(a)]\, H(t - a) = 2\theta(a) - \theta(t) + 2[\theta(t) - \theta(a)] = \theta(t),$$

and

$$\theta(a) + [\theta(t) - \theta(a)]\ \operatorname{sgn}(t - a) = \theta(a) + [\theta(t) - \theta(a)] = \theta(t),$$

matching the piecewise definition $\mathrm{incTheta}(t) = \theta(t)$ for $t > a$. [3] [1] [2]

- Case $t < a$: $H(t - a) = 0$, $\operatorname{sgn}(t - a) = -1$. Then
$$2\theta(a) - \theta(t) + 2[\theta(t) - \theta(a)]\, H(t - a) = 2\theta(a) - \theta(t),$$

and

$$\theta(a) + [\theta(t) - \theta(a)]\ \operatorname{sgn}(t - a) = \theta(a) - [\theta(t) - \theta(a)] = 2\theta(a) - \theta(t),$$

matching the piecewise definition $\mathrm{incTheta}(t) = 2\theta(a) - \theta(t)$ for $t < a$. [3] [1] [2]

- Case $t = a$: With the symmetric convention, $H(0) = \frac{1}{2}$ and $\operatorname{sgn}(0) = 0$. Both forms give
$$2\theta(a) - \theta(a) + 2[\theta(a) - \theta(a)]\tfrac{1}{2} = \theta(a), \quad \theta(a) + [\theta(a) - \theta(a)] \cdot 0 = \theta(a),$$

so the function is continuous with value $\theta(a)$ at $t = a$, consistent with the reflection definition at the critical point where $\theta'(a) = 0$. [1] [2] [3]

Hence the compact expression

$$\mathrm{incTheta}(t) = \theta(a) + [\theta(t) - \theta(a)]\ \operatorname{sgn}(t - a)$$

is exactly the reflected piecewise definition for all t, assuming the standard symmetric convention linking H and sgn. [2] [3] [1]

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1. https://en.wikipedia.org/wiki/Heaviside_function
2. https://mathworld.wolfram.com/HeavisideStepFunction.html
3. https://en.wikipedia.org/wiki/Riemann–Siegel_theta_function
4. https://www.youtube.com/watch?v=_tWOVabiHbk
5. https://math.mit.edu/classes/18.785/2021fa/LectureNotes17.pdf
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10. https://web.viu.ca/pughg/thesis.d/masters.thesis.pdf
11. https://tutorial.math.lamar.edu/classes/de/StepFunctions.aspx
12. https://www.youtube.com/watch?v=pgIMW2GsbQ8
13. https://reference.wolfram.com/language/ref/SiegelTheta.html
14. http://scipp.ucsc.edu/~haber/ph215/Stepfun18.pdf
15. https://en.wikipedia.org/wiki/Heaviside_step_function
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