Oscillatory Processes with Monotonic Phase Functions

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Definition 1. *[Oscillatory Process] Let $\{X_t\}_{t\in\mathbb{R}}$ be a complex second-order process. $\{X_t\}$ is called oscillatory if*

$$X_t = \int_{-\infty}^{\infty} \phi_t(\omega)\ d\,Z(\omega) \tag{1}$$

where $Z(\omega)$ has orthogonal increments where

$$E\,|d\,Z(\omega)|^2 = d\,\mu(\omega) \tag{2}$$

and

$$\phi_t(\omega) = A_t(\omega)\,e^{i\omega t} \tag{3}$$

Theorem 2. *Let $\theta\colon \mathbb{R}\to\mathbb{R}$ be smooth and strictly monotonically increasing. Define*

$$\phi_t(\omega) = e^{i\omega\theta(t)} \tag{4}$$

Set

$$A_t(\omega) = e^{i\omega(\theta(t)-t)} \tag{5}$$

Then the time-varying impulse response $h_t(u)$ defined by

$$A_t(\omega) = \int_{-\infty}^{\infty} e^{i\omega u}\,h_t(u)\ d\,u \tag{6}$$

is

$$h_t(u) = \delta\,(u - [\theta(t) - t]) \tag{7}$$

Proof. The inverse Fourier transform yields

$$\begin{aligned} h_t(u) &= \frac{1}{2\,\pi}\int_{-\infty}^{\infty} e^{-i\omega u}\, A_t(\omega)\ d\,\omega \\ &= \frac{1}{2\,\pi}\int_{-\infty}^{\infty} e^{-i\omega u}\, e^{i\omega\,(\theta(t)-t)}\, d\,\omega \\ &= \frac{1}{2\,\pi}\int_{-\infty}^{\infty} e^{i\omega\,(\theta(t)-t-u)}\, d\,\omega \\ &= \delta\,(\theta(t)-t-u) \\ &= \delta\,(u-[\theta(t)-t]) \end{aligned} \tag{8}$$

□

Corollary 3. *Let* $S_t = \int_{-\infty}^{\infty} e^{i\omega t}\, d\,Z(\omega)$. *Then*

$$X_t = \int_{-\infty}^{\infty} S_{t-u}\ h_t(u)\ d\,u = S_{2t-\theta(t)} \tag{9}$$

and

$$X_t = \int_{-\infty}^{\infty} e^{i\omega\theta(t)}\ d\,Z(\omega) \tag{10}$$

Proof. By the sifting property of the Dirac delta,

$$\begin{aligned} X_t &= \int_{-\infty}^{\infty} S_{t-u}\,\delta\,(u-[\theta(t)-t])\ d\,u \\ &= S_{t-[\theta(t)-t]} \\ &= S_{2t-\theta(t)} \end{aligned} \tag{11}$$

By direct substitution, the oscillatory representation holds by definition. □