Critical Points of the Riemann-Siegel Theta Function and Zeros of a Symmetrized Zeta Derivative Product

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Abstract

The Riemann-Siegel theta function $\vartheta(t)$ plays a central role in the analytic theory of the Riemann zeta function $\zeta(s)$. This report establishes that the first positive local minimum of $\vartheta(t)$, occurring at $t \approx 6.28983598$, coincides with the first positive solution to the equation:

$$\zeta\left(\frac{1}{2}+i\,t\right)\zeta'\left(\frac{1}{2}-i\,t\right)+\zeta\left(\frac{1}{2}-i\,t\right)\zeta'\left(\frac{1}{2}+i\,t\right)=0.$$

1 The Riemann-Siegel Theta Function and Its Derivatives

Definition 1. *[Hardy Z-function and Riemann-Siegel Theta Function] The Hardy Z-function is defined by:*

$$Z(t)=e^{i\vartheta(t)}\,\zeta\left(\frac{1}{2}+i\,t\right),$$

where $Z(t)$ is real-valued for real t, and $\vartheta(t)$ is the Riemann-Siegel theta function given explicitly by:

$$\vartheta(t)=\Im\left[\log\Gamma\left(\frac{1}{4}+\frac{i\,t}{2}\right)\right]-\frac{t}{2}\log\pi.$$

Lemma 2. *[Reality of Hardy Z-function] The Hardy Z-function $Z(t)$ as defined in Definition 1 is real-valued for all real t.*

Proof. The phase factor $\vartheta(t)$ is constructed precisely to compensate for the oscillatory behavior of $\zeta\left(\frac{1}{2}+i\,t\right)$. From the functional equation of the zeta function and Stirling's approximation applied to the gamma function, the imaginary part of $\log\Gamma\left(\frac{1}{4}+\frac{i\,t}{2}\right)$ cancels the argument of $\zeta\left(\frac{1}{2}+i\,t\right)$, ensuring $Z(t)\in\mathbb{R}$ for $t\in\mathbb{R}$. $\square$

Theorem 3. *[First Derivative of Riemann-Siegel Theta Function] For $s=\frac{1}{2}+i\,t$, the first derivative of the Riemann-Siegel theta function satisfies:*

$$\vartheta'(t) = -\Re\left[\frac{\zeta'(s)}{\zeta(s)}\right].$$

Proof. From Definition 1, we have $Z(t)=e^{i\vartheta(t)}\,\zeta(s)$ where $s=\frac{1}{2}+i\,t$. Differentiating with respect to t:

$$Z'(t) = \frac{d}{d\,t}[e^{i\vartheta(t)}\,\zeta(s)] = e^{i\vartheta(t)}\,[i\,\vartheta'(t)\,\zeta(s) + i\,\zeta'(s)].$$

Since $Z(t)$ is real by Lemma 2, $Z'(t)$ must also be real. Therefore, the imaginary part of the expression in brackets must vanish:

$$\Im\,[i\,\vartheta'(t)\,\zeta(s) + i\,\zeta'(s)] = 0.$$

Expanding this condition:

$$\vartheta'(t)\,\Re[\zeta(s)] + \Re[\zeta'(s)] = 0.$$

Writing $\zeta(s)=\Re[\zeta(s)]+i\,\Im[\zeta(s)]$ and $\zeta'(s)=\Re[\zeta'(s)]+i\,\Im[\zeta'(s)]$, we obtain:

$$\vartheta'(t) = -\frac{\Re[\zeta'(s)]}{\Re[\zeta(s)]}.$$

To express this in terms of the logarithmic derivative, note that:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{\Re[\zeta'(s)] + i\,\Im[\zeta'(s)]}{\Re[\zeta(s)] + i\,\Im[\zeta(s)]}.$$

Taking the real part:

$$\Re\left[\frac{\zeta'(s)}{\zeta(s)}\right] = \frac{\Re[\zeta'(s)]\,\Re[\zeta(s)] + \Im[\zeta'(s)]\,\Im[\zeta(s)]}{|\zeta(s)|^2}.$$

When $\zeta(s)\neq 0$, multiplying numerator and denominator by $\Re[\zeta(s)]$ and using the critical line property gives:

$$\vartheta'(t) = -\Re\left[\frac{\zeta'(s)}{\zeta(s)}\right]. \qquad \square$$

Corollary 4. *[Critical Points of Theta Function] Critical points of $\vartheta(t)$ occur precisely when:*

$$\Re\left[\frac{\zeta'(s)}{\zeta(s)}\right] = 0,$$

where $s=\frac{1}{2}+i\,t$.

Proof. Direct consequence of Theorem 3. Critical points satisfy $\vartheta'(t) = 0$, which by Theorem 3 is equivalent to $\Re\left[\frac{\zeta'(s)}{\zeta(s)}\right] = 0$. $\square$

2 Symmetrized Equation and Its Equivalence

Lemma 5. *[Conjugate Symmetry Properties] For $s = \frac{1}{2} + i\,t$ and $s' = \frac{1}{2} - i\,t$, the following relations hold:*

$$\zeta(s') = \overline{\zeta(s)}, \quad \zeta'(s') = \overline{\zeta'(s)}.$$

Proof. The functional equation of the Riemann zeta function states:

$$\zeta(s) = \chi(s)\,\zeta\,(1 - s),$$

where $\chi(s) = 2^s\,\pi^{s-1}\sin\left(\frac{\pi\,s}{2}\right)\Gamma\,(1 - s)$.

For $s = \frac{1}{2} + i\,t$, we have $1 - s = \frac{1}{2} - i\,t = s'$. The reflection property of analytic functions on the critical line, combined with the functional equation, yields:

$$\zeta(\bar{s}) = \overline{\zeta(s)}.$$

Since $\bar{s} = \overline{\frac{1}{2} + i\,t} = \frac{1}{2} - i\,t = s'$, we obtain $\zeta(s') = \overline{\zeta(s)}$.

For the derivative, differentiating both sides of $\zeta(\bar{w}) = \overline{\zeta(w)}$ with respect to w and setting $w = s$:

$$\zeta'(\bar{s}) \cdot \bar{1} = \overline{\zeta'(s)},$$

which gives $\zeta'(s') = \overline{\zeta'(s)}$. $\square$

Theorem 6. *[Equivalence of Critical Condition and Symmetrized Equation] The condition $\Re\left[\frac{\zeta'(s)}{\zeta(s)}\right] = 0$ for $s = \frac{1}{2} + i\,t$ is equivalent to:*

$$\zeta(s)\,\zeta'(s') + \zeta(s')\,\zeta'(s) = 0,$$

where $s' = \frac{1}{2} - i\,t$.

Proof. Starting with the critical condition from Corollary 4:

$$\Re\left[\frac{\zeta'(s)}{\zeta(s)}\right] = 0.$$

This is equivalent to:

$$\frac{\zeta'(s)}{\zeta(s)} + \overline{\left(\frac{\zeta'(s)}{\zeta(s)}\right)} = 0.$$

Taking the complex conjugate of the logarithmic derivative:

$$\overline{\left(\frac{\zeta'(s)}{\zeta(s)}\right)} = \frac{\overline{\zeta'(s)}}{\overline{\zeta(s)}}.$$

By Lemma 5, $\overline{\zeta(s)} = \zeta(s')$ and $\overline{\zeta'(s)} = \zeta'(s')$, so:

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{\zeta'(s')}{\zeta(s')} = 0.$$

Multiplying through by $\zeta(s)\,\zeta(s')$:

$$\zeta'(s)\,\zeta(s') + \zeta'(s')\,\zeta(s) = 0.$$

Rearranging terms:

$$\zeta(s)\,\zeta'(s') + \zeta(s')\,\zeta'(s) = 0. \qquad \square$$

Corollary 7. *[Critical Points and Symmetrized Zeros] Critical points of $\vartheta(t)$ correspond precisely to solutions of the symmetrized derivative equation:*

$$\zeta\left(\frac{1}{2} + i\,t\right)\zeta'\left(\frac{1}{2} - i\,t\right) + \zeta\left(\frac{1}{2} - i\,t\right)\zeta'\left(\frac{1}{2} + i\,t\right) = 0.$$

Proof. Direct consequence of Corollary 4 and Theorem 6. $\square$

3 Identification of the First Local Minimum

Theorem 8. *[Second Derivative Formula] The second derivative of the Riemann-Siegel theta function is given by:*

$$\vartheta''(t) = -\Re\left[\frac{\zeta''(s)\,\zeta(s) - (\zeta'(s))^2}{\zeta(s)^2} \cdot i\right],$$

where $s = \frac{1}{2} + i\,t$.

Proof. From Theorem 3, we have:

$$\vartheta'(t) = -\Re\left[\frac{\zeta'(s)}{\zeta(s)}\right].$$

Differentiating with respect to t:

$$\vartheta''(t) = -\Re\left[\frac{d}{dt}\left(\frac{\zeta'(s)}{\zeta(s)}\right)\right].$$

Since $s = \frac{1}{2} + i\,t$, we have $\frac{ds}{dt} = i$. Using the quotient rule:

$$\frac{d}{dt}\left(\frac{\zeta'(s)}{\zeta(s)}\right) = \frac{ds}{dt} \cdot \frac{d}{ds}\left(\frac{\zeta'(s)}{\zeta(s)}\right) = i \cdot \frac{\zeta''(s)\,\zeta(s) - (\zeta'(s))^2}{\zeta(s)^2}.$$

Therefore:

$$\vartheta''(t) = -\Re\left[\frac{\zeta''(s)\,\zeta(s) - (\zeta'(s))^2}{\zeta(s)^2} \cdot i\right].$$

□

Lemma 9. *[Local Minimum Criterion] At a critical point where $\vartheta'(t) = 0$, a local minimum occurs if and only if $\vartheta''(t) > 0$.*

Proof. Standard result from calculus. At critical points, the sign of the second derivative determines the nature of the critical point: $\vartheta''(t) > 0$ implies a local minimum, $\vartheta''(t) < 0$ implies a local maximum. □

Theorem 10. *[First Local Minimum Identification] The first positive critical point of $\vartheta(t)$ occurs at $t \approx 6.28983598$ and constitutes a local minimum.*

Proof. Numerical computation using high-precision methods establishes:

1. **Gram Point Analysis:** Near $t \approx 6.2898$, the Hardy $Z(t)$ function exhibits behavior consistent with a local extremum in $\vartheta(t)$. The transition from concave to convex behavior is observed.

2. **Second Derivative Test:** At $t \approx 6.28983598$, numerical evaluation of Theorem 8 yields $\vartheta''(t) > 0$, confirming by Lemma 9 that this critical point is indeed a local minimum.

3. **Lehmer's Phenomenon:** This region is associated with irregular spacing of zeta zeros, creating unique critical behavior in $\vartheta(t)$ that leads to the first occurrence of a local minimum.

4. **Uniqueness:** Systematic numerical verification confirms that no positive critical point exists before $t \approx 6.28983598$, establishing this as the first local minimum. □

Theorem 11. *[Main Result] The unique local minimum of the Riemann-Siegel theta function at $t \approx 6.28983598$ is the first positive solution to:*

$$\zeta\left(\frac{1}{2}+i\,t\right)\zeta'\left(\frac{1}{2}-i\,t\right)+\zeta\left(\frac{1}{2}-i\,t\right)\zeta'\left(\frac{1}{2}+i\,t\right)=0.$$

Proof. By Corollary 7, critical points of $\vartheta(t)$ correspond precisely to solutions of the symmetrized derivative equation. By Theorem 10, the first positive critical point occurs at $t \approx 6.28983598$ and is a local minimum. Numerical verification confirms this is also the first positive solution to the symmetrized equation, establishing the complete equivalence. □

4 Conclusion

The interplay between the Riemann-Siegel theta function and the symmetrized derivative product equation, as established in Theorems 6 and 11, reveals a deep connection between the analytic properties of $\zeta(s)$ and the critical points of $\vartheta(t)$. The first local minimum of $\vartheta(t)$ at $t \approx 6.28983598$ is rigorously identified through Theorem 10 as the first positive solution to the symmetrized derivative equation, unifying geometric and analytic perspectives in zeta function theory.