Unitary Time Changes of Stationary Processes Yield Oscillatory Processes

by Stephen Crowley

August 12, 2025

Definition 1. *[Unitary time change operator on $L^2(\mathbb{R})$] Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous with $\theta'(t) \neq 0$ almost everywhere. The unitary time change operator U_θ on $L^2(\mathbb{R})$ is defined by*

$$(U_\theta f)(t) := \sqrt{|\theta'(t)|}\; f(\theta(t)) \qquad \text{for } f \in L^2(\mathbb{R}) \tag{1}$$

Theorem 2. *[Unitarity of U_θ] The operator U_θ defined above is unitary on $L^2(\mathbb{R})$.*

Proof. Absolute continuity with $\theta'(t) \neq 0$ a.e. implies the change-of-variables formula

$$\int_{\mathbb{R}} |(U_\theta f)(t)|^2\, dt = \int_{\mathbb{R}} |\theta'(t)|\,|f(\theta(t))|^2\, dt = \int_{\mathbb{R}} |f(u)|^2\; du \tag{2}$$

so U_θ is isometric. Surjectivity follows from the same change-of-variables applied to $U_{\theta^{-1}}$, which exists almost everywhere under these hypotheses. Hence U_θ is unitary. $\square$

Definition 3. *[Oscillatory processes in the sense of Priestley] An oscillatory process Z is specified by a measurable gain function $A_t(\lambda)$ and has oscillatory function*

$$\varphi_t(\lambda) := A_t(\lambda)\; e^{i\lambda t} \tag{3}$$

The process Z has spectral representation

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda)\; \Phi(d\lambda) = \int_{\mathbb{R}} A_t(\lambda)\; e^{i\lambda t}\; \Phi(d\lambda) \tag{4}$$

where Φ is a complex orthogonal random measure on $\mathbb{R}$ with spectral measure F satisfying

$$E[\Phi(d\lambda)\, \overline{\Phi(d\mu)}] = 1_{\{\lambda = \mu\}}\; dF(\lambda) \tag{5}$$

The covariance kernel of Z is

$$R_Z(t, s) := E[Z(t)\overline{Z(s)}] = \int_{\mathbb{R}} A_t(\lambda)\; \overline{A_s(\lambda)}\; e^{i\lambda(t-s)}\; dF(\lambda) \tag{6}$$

Remark 4. [Real-valuedness condition] The oscillatory process Z is real-valued if and only if the gain satisfies conjugate symmetry:

$$A_t(-\lambda) = \overline{A_t(\lambda)} \quad \text{for } F\text{-almost every } \lambda, \text{for each fixed } t \tag{7}$$

Theorem 5. *[Unitary time change of stationary process yields oscillatory process] Let X be a zero-mean stationary Gaussian process with Cramér spectral representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t} \, \Phi(d\lambda) \tag{8}$$

where Φ is the same complex orthogonal random measure with spectral measure F as in the oscillatory definition. Let U_θ be a unitary time change operator as defined above. Then the transformed process

$$Z(t) := (U_\theta X)(t) = \sqrt{|\theta'(t)|} \, X(\theta(t)) \tag{9}$$

is an oscillatory process in the sense of Priestley with oscillatory function

$$\varphi_t(\lambda) = \sqrt{|\theta'(t)|} \, e^{i\lambda\theta(t)} \tag{10}$$

Proof. Starting from the stationary representation, we compute

$$Z(t) = \sqrt{|\theta'(t)|} \, X(\theta(t)) \tag{11}$$

$$= \sqrt{|\theta'(t)|} \int_{\mathbb{R}} e^{i\lambda\theta(t)} \, \Phi(d\lambda) \tag{12}$$

$$= \int_{\mathbb{R}} \sqrt{|\theta'(t)|} \, e^{i\lambda\theta(t)} \, \Phi(d\lambda) \tag{13}$$

Defining

$$\varphi_t(\lambda) := \sqrt{|\theta'(t)|} \, e^{i\lambda\theta(t)} \tag{14}$$

we have

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda) \, \Phi(d\lambda)$$

which is precisely the oscillatory form. The covariance kernel becomes

$$R_Z(t,s) = \int_{\mathbb{R}} \varphi_t(\lambda) \, \overline{\varphi_s(\lambda)} \, dF(\lambda) = \int_{\mathbb{R}} \sqrt{|\theta'(t)||\theta'(s)|} \, e^{i\lambda(\theta(t)-\theta(s))} \, dF(\lambda) \qquad \square$$

Theorem 6. *[Explicit gain function for unitary time change] In the setting of the previous theorem, the gain function for the oscillatory process*

$$Z(t) = (U_\theta X)(t) \tag{15}$$

is given by

$$A_t(\lambda) = \sqrt{|\theta'(t)|} \, e^{i\lambda(\theta(t)-t)} \tag{16}$$

The oscillatory function is

$$\varphi_t(\lambda) = A_t(\lambda) \, e^{i\lambda t} = \sqrt{|\theta'(t)|} \, e^{i\lambda\theta(t)} \tag{17}$$

and the covariance kernel takes the form

$$R_Z(t,s) = \int_{\mathbb{R}} A_t(\lambda) \, \overline{A_s(\lambda)} \, e^{i\lambda(t-s)} \, dF(\lambda) \tag{18}$$

Proof. From the previous theorem, we have

$$\varphi_t(\lambda) = \sqrt{|\theta'(t)|}\ e^{i\lambda\theta(t)} \tag{19}$$

Since the oscillatory function must satisfy

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{20}$$

one solves for the gain:

$$A_t(\lambda) = \frac{\varphi_t(\lambda)}{e^{i\lambda t}} = \frac{\sqrt{|\theta'(t)|}\ e^{i\lambda\theta(t)}}{e^{i\lambda t}} = \sqrt{|\theta'(t)|}\ e^{i\lambda(\theta(t)-t)}$$

and substitutes back into the covariance formula:

$$R_Z(t,s) = \int_{\mathbb{R}} \varphi_t(\lambda)\ \bar{\varphi}_s(\lambda)\}\, d\,F(\lambda) \tag{21}$$

$$= \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\ \overline{A_s(\lambda)\, e^{i\lambda s}}\, d\,F(\lambda) \tag{22}$$

$$= \int_{\mathbb{R}} A_t(\lambda)\, \overline{A_s(\lambda)}\, e^{i\lambda(t-s)}\, d\,F(\lambda) \tag{23}$$

$\square$