Dear professors Roger and Pol, let me introduce myself by saying that I read your book Stochastic Finite Elements: The Spectral Approach back in 2006 and thought it was fantastic and found myself revisiting it again during this last year when I was trying to understand a system where I calculated empirically the quantized variance structure function (the variogram) and was surprised to find the Bessel function of the first kind of order 0 there, clearly a scalar multiple of it. I set about trying to find the eigenfunctions of

$$\int J_0(x-y)f(x)\mathrm{d}x \tag{1}$$

and was shocked to find no such results!= available despite consulting every conceiveible reference on the topic of integral transforms.

well, I had an epiphany and came up with a technique that should solve any (positive definite of course) stationary covariance kernel (irrational, non-square-integrable is no problem). It goes like this

1. Identify the orthogonal polynomial sequence associated to the spectral density of the kernek $K = J_0$ which in this case is given by

$$S(\omega) = \int_0^\infty J_0(x)e^{ix\omega}\mathrm{d}x = \begin{cases} \frac{1}{\sqrt{1-\omega^2}} & \omega \in (-1,1) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

so that we identify Equation (1) with the Chebsyhev polynomials of the first kind whose orthogonality measure is given by the spectral density

$$\int_{-1}^1 T_n(\omega)T_m(\omega)S(\omega)\mathrm{d}w = \begin{cases} 0 & n \neq m \\ \pi & n = m = 0 \\ \frac{\pi}{2} & n = m \neq 0 \end{cases} \tag{3}$$

I had a hunch that if we take the (finite) Fourier transform of the Chebyshev type-I polynomials which is just the usual infinite Fourier transform with the integration restricted to the range $-1 \ldots 1$ since $T_n(x) = 0 \forall x \notin [-1,1]$

$$\hat{T}_n(y) = \int_{-1}^1 T_n(x)e^{ixy}\mathrm{d}x \tag{4}$$

and orthogonalize them so that our eigenfunctions are their orthogonal complement

$$\psi_n(y) = \hat{T}_n^{\perp}(y) \tag{5}$$

(via the Gram-Schmidt process)

$$\psi_n(y) = \hat{T}_n^{\perp}(y) = \hat{T}_n(y) - \sum_{m=1}^{n-1} \frac{\langle \hat{T}_m(y), \psi_m(y) \rangle}{\langle \hat{T}_m(y), \hat{T}_m(y) \rangle} \psi_m(y) \tag{6}$$

with respect to the unweighted standard Lebesgue inner product measure over 0 to ∞ that it would be related to the solution. I was pleasently surprised to notice that indeed the partial sums of the orthogonalized Fourier transforms of T_n are in fact proportional to the Fourier transforms of the Legendre polynomials over the same interval and are actually eigenfunctions of the given integral covariance operation in Equation (1) and that the eigenvalues are given by

$$\lambda_n = \int_{-\infty}^{\infty} J_0(x) \psi_n(x) \mathrm{d}x = \sqrt{\frac{4n+1}{\pi} \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2}} \tag{7}$$

(there is actually a more involved formula by letting m vary rather than fixed at 0 but my primary aim is to represent sample paths of the J_0 process presently) and eigenfunctions are given by

$$\begin{aligned} \psi_n(y) &= (-1)^n \sqrt{\frac{4n+1}{\pi}}\, j_{2n}(y) \\ &= (-1)^n \sqrt{\frac{4n+1}{\pi}} \sqrt{\frac{\pi}{2y}} J_{2n+\frac{1}{2}}(y) \\ &= (-1)^n \sqrt{\frac{4n+1}{2y}}\, J_{2n+\frac{1}{2}}(y) \end{aligned} \tag{8}$$

where $j_n(x)$ is the spherical Bessel function of the first kind

$$\begin{aligned} j_n(x) &= \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x) \\ &= \frac{1}{\sqrt{x}} \left(\sin(z)\, R_{n,\frac{1}{2}}(z) - \cos(z) R_{n,\frac{3}{2}}(z) \right) \end{aligned} \tag{9}$$

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and where $R_{n,v}(z)$ are the (misnomered) Lommel polynomials

$$R_{n,v}(z)=\frac{\Gamma(n+v)}{\Gamma(v)}\left(\frac{z}{2}\right)^{-n} {}_2F_3\left(\left[-\frac{n}{2},-\frac{n}{2}+\frac{1}{2}\right];[v,-n,-v+1-n];-z^2\right) \tag{10}$$

which are actually rational functions of z, not polynomial. It is the modified Lommel polynomials[1] which are actually polynomials defined by

$$h_{n,v}(z)=R_{n,v}\left(\frac{1}{z}\right) \tag{11}$$

All this leads to the uniformly convergent eigenfunction expansipon

$$\begin{aligned} J_0(x) &= \sum_{n=0}^{\infty} \lambda_n \psi_n(x) \\ &= \cdots. \end{aligned} \tag{12}$$

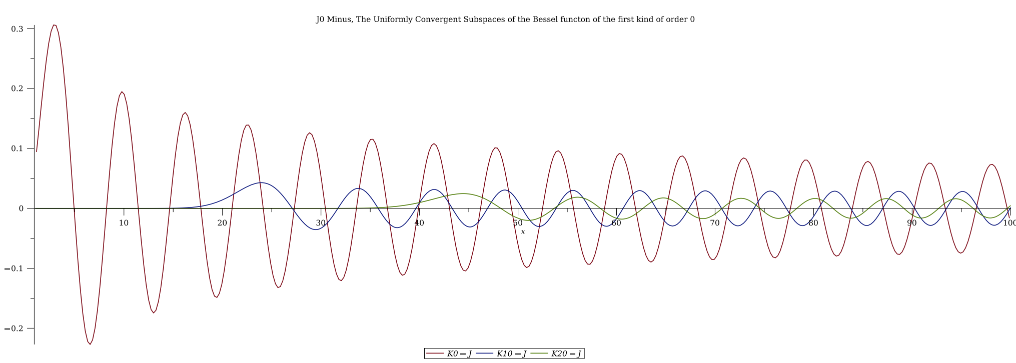

Figure 1. The Residual $J_0(x) - K_n(x)$ for $n = 0, 10, 20$

The question is, how to prove it? I am under the impression that this is noteworthy result and would like to get it published and hence I am reaching out to the most esteemed experts on the subject I can think of give that you wrote the book that I read to understand this method. It's an application of the Galerkin procedure described in the first part of Stochastic Finite Elements, except instead of just choosing a piecewise or other polynomial basis for the expansion, you apply the orthogonalization procedure to the Fourier transform of the orthogonal polynomials corresponding to the spectral density and thus the mass and stiffness matrices become the identity and $c_n = \lambda_n$ becomes an identity. I do not hold a degree, I'm just a guy who started programming computers when he was about 5 or 6 years old and ran into this interesting solution.

Bibliography

[1] R. Wong K.F. Lee. Asymptotic expansion of the modified lommel polynomials $h_{n,\nu}(x)$ and their zeros. *Proceedings of the American Mathematical Society*, 142(11):3953–3964, 2014.