Band Limited Processes and Spectral Bimeasures

Abstract

The following is an excerpt of [1, 6.2] by the amazing author M.M. Rao, a true master of abstraction for which I am thankful for. May he live to see my proof of Riemann's conjecture. And me too for that matter.

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1.1 Harmonizability

If $\{X_t, t \in \mathbb{R}\}$ is a mean zero stationary process with a continuous covariance $r(s,t) = \tilde{r}\,(s-t)$, then by the classical Bochner theorem

$$\tilde{r}\,(s-t) = \int_{\mathbb{R}} e^{i(s-t)u}\, d\, F(u) \qquad \forall s,t \in \mathbb{R} \tag{1}$$

where F is a bounded positive nondecreasing function, determining a (bounded) Borel measure. Such an F is termed the *spectral function* of the process, and if moreover the support of F is contained in a bounded interval $(-a,a)$, the process is usually called *band-limited*. More generally, suppose r is the covariance function of a second order process admitting a (two-dimensional) Fourier transform:

$$r(s,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{isx-ity}\, d\, F(x,y) \qquad \forall s,t \in \mathbb{R} \tag{2}$$

relative to a (necessarily positive definite) bimeasure F which however may only have a finite Fréchet (not Vitali) variation, so that the symbol is a (strict) MT-integral as noted in the preceding section, and the process is weakly harmonizable. Again F is termed a *spectral bimeasure*, and if its support is contained in a bounded rectangle $(-a, a) \times (-b, b)$, then the process (by analogy) can and will be termed band-limited. Thus the concept extends to harmonizable processes. The following result characterizes continuous covariances r that are Fourier transforms of such spectral bimeasures, and motivates how other extensions (e.g., the Cramér type processes) can be similarly analyzed.

Theorem 1

A continuous covariance function $r: \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is the Fourier transform of a bimeasure $F: B(\mathbb{R}) \times B(\mathbb{R}) \to \mathbb{C}$ (so that it is weakly harmonizable) iff $\|r\| < \infty$ where $f \in L^1(\mathbb{R})$ and

$$\hat{f}(t) = \int_{\mathbb{R}} e^{itx} f(x)\, dx \tag{3}$$

$$\|r\| = \sup \left\{ \left| \int_{\mathbb{R}} \int_{\mathbb{R}} r(s,t)\, f(s)\, g(t)\, ds\, dt \right| : \|f\|_\infty \leq 1, \|g\|_\infty \leq 1, f, g \in L^1(\mathbb{R}) \right\} \tag{4}$$

Here and below $L^P(\mathbb{R})$ is the Lebesgue space on the Lebesgue line $\mathbb{R}$.

Remark. *The condition that $\|r\| < \infty$ of (4) is termed weakly V-bounded and is an extended version of a one-dimensional concept originally formulated by Bochner [2]. The point of this result is that the (two-dimensional) Fourier transform is characterized by a simple analytical condition, namely (4), without reference to the "abstract" definition of harmonizability.*

Proof. The "if" part is immediate. Indeed, let r admit a representation (2). Then for any $f, g \in L^1(\mathbb{R})$ one has

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} r(s,t)\, f(s)\, g(t)\, ds\, dt &= \int_{\mathbb{R}} f(s)\, g(t) \times \left(\int_{\mathbb{R}} \int_{\mathbb{R}} e^{isx - ity}\, dF(x,y) \right) ds\, dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)\, g(y)\, dF(x,y) \end{aligned} \tag{5}$$

by a form of Fubini's theorem for the Morse-Transue-integrals,

$$\leq \|f\|_\infty \|g\|_\infty \|F\|_{\mathbb{R},\mathbb{R}} \tag{6}$$

by a property of the MT-integrals.

Here $\|F\|_{\mathbb{R},\mathbb{R}}$ is the Fréchet variation of the bimeasure (which is always finite) given by:

$$\|F\|_{\mathbb{R},\mathbb{R}} = \sup\left\{\sum_{i,j=1}^{n}\left|\int_{A_i}\int_{B_j} d\,F(x,y)\,a_i\,a_j\right|; A_i, B_j \in \mathbb{R}\text{disjoint}, |a_i| \leq 1, a_i \in \mathbb{C}\right\} \tag{7}$$

Thus (5) implies (4), so that $r(s,t)$ is (weakly) V-bounded.

For the converse suppose $\|r\| < \infty$. If $H\colon f \mapsto \hat{f}$ is the Fourier transform, consider the functionals

$$T\colon C_0(\mathbb{R}) \times C_0(\mathbb{R}) \to \mathbb{C} \tag{8}$$

defined by

$$T(f,g) = \mathcal{E}(H^{-1}(f), H^{-1}(g)) \tag{9}$$

where for $f, g \in L^1(\mathbb{R}), C_0(\mathbb{R})$ being the space of continuous functions vanishing off compact sets, and $\mathcal{E}(\cdot,\cdot)$ given by:

$$\mathcal{E}(f,g) = \int_{\mathbb{R}}\int_{\mathbb{R}} r\,(s,t)\;f(s)\;g(t)\,d\,s\,d\,t \tag{10}$$

This T is a bounded bilinear functional on $L^1(\mathbb{R}) \times L^1(\mathbb{R})$ since by (4)

$$\sup\{|T(f,g)|; \|f\|_\infty \leq 1, \|g\|_\infty \leq 1\} = \|r\| < \infty \tag{11}$$

Now T admits a bound preserving extension to all of $C_0(\mathbb{R}) \times C_0(\mathbb{R})$, endowed with the uniform norm, by the standard (Hahn-Banach type) extension results. Then by another representation theorem due to F. Riesz for multilinear functionals (cf., Dunford-Schwartz [1], Chapter VI, and Dobrakov [1] for extensions), there is a unique bounded bimeasure F, necessarily positive definite (cf., Chang and Rao [1], p.21), on $B(\mathbb{R}) \times B(\mathbb{R})$ such that the following holds:

$$T(f,g) = \int_{\mathbb{R}}\int_{\mathbb{R}} f^*(x)\;g(y)\,d\,F(x,y) \tag{12}$$

Then (6) and (12) imply

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} r\,(s,t)\;f(s)\;g(t)\,d\,s\,d\,t \;&= \mathcal{E}(f,g)\\
&= T(f,g)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{isx - ity}\,f(s)\;g(t)\,d\,s\,d\,t\,d\,F(x,y)
\end{aligned} \tag{13}$$

Subtracting and using again a form of Fubini's theorem, one gets

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left[r(s,t)-\int_{\mathbb{R}}\int_{\mathbb{R}}e^{isx-ity}\,dF(x,y)\right]f(s)\,g(t)\,ds\,dt=0 \tag{14}$$

Since $f,g\in L^1(\mathbb{R})$ are arbitrary, this implies that

$$\left[r(s,t)-\int_{\mathbb{R}}\int_{\mathbb{R}}e^{isx-ity}\,dF(x,y)\right]=0 \tag{15}$$

a.e., and because of the continuity of r it vanishes identically. Thus r admits the representation (2) □

Note 2. The preceding analysis also applies to the case that F is of bounded (Vitali) variation, with the following modifications. Consider r as a function on $\mathbb{R}^2$ and (4) replaced by $|r|(\mathbb{R}^2)<\infty$, where

$$|r|(\mathbb{R}^2)=\sup\{|\int_{\mathbb{R}^2}r(x)\,f(x)\,d\mu(x)|\colon f\in L^1(\mathbb{R}^2),\|f\|_\infty\le 1\} \tag{16}$$

with μ as the planar Lebesgue measure, and $f\colon\mathbb{R}^2\to\mathbb{C}$ (not necessarily of the product form $f(x_1,x_2)=f_1(x_1)\,f_2(x_2)$ for $x=(x_1,x_2)\in\mathbb{R}^2$). Then by another (Gleason) representation theorem applied to r, an $F\colon\mathbb{R}^2\to\mathbb{C}$ of bounded (Vitali) variation such that ($\tau\cdot x=$ is an $F'\colon\mathbb{R}^2\to\mathbb{C}$ of bounded (Vitali) variation such that ($\tau\cdot x=$ is an $F'\colon\mathbb{R}^2\to\mathbb{C}$ of bounded (Vitali) variation such that $(\tau\cdot x=(x_1-x_2)$ for $\tau=(s,-t)$

$$\tau(x)=\int_{\mathbb{R}^2}e^{i\tau\cdot x}\,dF(s,t) \tag{17}$$

and since r is positive definite, F is also.

1.1.1 Stochastic Fourier Representation of the X_t Process Itself

The preceding results implies the following:

Corollary 3

A continuous covariance function $r\colon\mathbb{R}^2\to\mathbb{C}$ is strongly harmonizable (or the Fourier transform of a positive definite function $F\colon\mathbb{R}^2\to\mathbb{C}$ of bounded (Vitali) variation satisfying (9)) if and only if $|r|(\mathbb{R}^2)<\infty$, or more explicitly,

$$\left|\int_{\mathbb{R}^2}r(s,t)\,f(s,t)\,d\mu(s,t)\right|\le K\|f\|_\infty\qquad\forall f\in L^1(\mathbb{R}^2) \tag{18}$$

for some constant $0<K<\infty$.

Note 4. Thus the class of processes for which a general band-limited concept can be defined is precisely the (weakly or strongly) harmonizable family. The importance of condition (4) or (18) is enhanced by an obtaining the corresponding *stochastic Fourier representation of the X_t-process itself*.

This is seen as follows. When r is ***V*-bounded**, one has for any bounded Borel f, g with compact supports,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} r(s,t)\, f(s)\, g(t)\, d\,s\, d\,t = \mathbb{E}\left[\int_{\mathbb{R}} X_s\, f(s)\, d\,s \int_{\mathbb{R}} X_t\, g(t)\, d\,t\right] \tag{19}$$

so that on taking $f = g$, this becomes

$$\mathbb{E}\left[\left(\int_{\mathbb{R}} X_s\, f(s)\, d\,s\right)^2\right] = \int_{\mathbb{R}}\int_{\mathbb{R}} r(s,t)\, f(s)\, f(t)\, d\,s\, d\,t \leq K\, \|f\|_\infty^2 \quad \text{by (4) [or (18)]} \tag{20}$$

for some $0 < K < \infty$. Hence

$$\left|\int_{\mathbb{R}} X_s\, f(s)\, d\,s\right|^2 \leq K\, \|f\|_\infty^2 \quad (11) \tag{21}$$

This implies that the set $\{\int_{\mathbb{R}} X_s\, f(s)\, d\,s,\, f \in L^1(\mathbb{R}), \|f\|_\infty \leq 1\}$ is *relatively weakly compact*. Consequently by another representation due to F. Riesz on $C_0(\mathbb{R}^2)$ to $L_0^2(P)$ (in both cases of (4) and (18)) there is a unique vector measure $Z\colon B(\mathbb{R}) \to L_0^2(P)$ such that

$$\int_{\mathbb{R}} X_s\, f(s)\, d\,s = T(f) = \int_{\mathbb{R}} f(t)\, d\,Z(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{itu}\, f(u)\, d\,u\, d\,Z(t) \tag{22}$$

where the left side is the standard vector Lebesgue [or the Bochner] integral and the right side is the Dunford-Schwartz's. Hence as before one has

$$\left[\int_{\mathbb{R}} X_t - \int_{\mathbb{R}} e^{itu}\, d\,Z(u)\right] f(t)\, d\,t = 0 \forall \quad f \in L^1(\mathbb{R}) \tag{23}$$

It follows that

$$X_t = \int_{\mathbb{R}} e^{itu}\, d\,Z(u) \forall \quad t \in \mathbb{R} \tag{24}$$

which is the integral representation of a weakly (or strongly) harmonizable process, and

$$\mathbb{E}\,(Z(A)\, Z(B)) = \int_A\int_B d\,F^*(s,t)$$

with F^* defining a bimeasure (a signed measure) on $B(\mathbb{R}^2)$.

Note 5. The importance of this result is that one can approximate e^{itu} by a (finite or infinite) series as in the proof of Theorem 1.3 (cf., eq(15)), leading to several such representations using different classical approximations and/or "sampling theorems". They give rise to *stochastic sampling versions* of various kinds. For instance, the covariance function r can be replaced by

$$r^*(s,t) = r(s,t)\, f(s)\, f(t) \tag{25}$$

for a measurable f such that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} r^*(s,t)\, d\,\mu(s)\, d\,\mu(t) < \infty \tag{26}$$

Then under similar conditions of V-boundedness r is a Fourier transform of a bimeasure, and more particularly, if $\hat{r}$ and $\hat{f}$ are such transforms themselves, then r is the Fourier transform of a convolution. So the band-limitedness of a process can be defined, and it leads to some new developments in (stochastic) sampling theory.

1.2 Analyticity of second order processes

For second order processes, especially the harmonizable classes, we shall consider the band-limitedness property and its relation to the analyticity of sample paths, which explains our preoccupation with the former in the last section. This point was already noted immediately after Corollary 1.2. The following result for stationary processes was observed by Belyaev [1], and for the (strongly) harmonizable case it was established by Swift [1].

Theorem 6

A strongly harmonizable process $\{X_t, t \in \mathbb{R}\}$ with spectral (signed) measure F is analytic in an open neighborhood of the origin iff F has a moment generating function in such a neighborhood of the complete plane. In particular, if the strongly harmonizable process is band-limited, then it is analytic in the complete plane.

Proof. The sufficiency is a consequence of Corollary 1.2. In fact, if F has a moment generating function, then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} e^{s_1 x + s_2 y}\, d\,F(x,y) < \infty \forall \quad |s_j| < a_j,\, j = 1, 2 \tag{27}$$

and hence F has all moments finite. Thus r is infinitely differentiable (by the dominated convergence theorem) so that it has an infinite Taylor series expansion converging (uniformly and absolutely) in the rectangle $(-a_1, a_1) \times (-a_2, a_2)$. So r is analytic, and then X_t has mean square derivatives of all orders. It is analytic as in Theorem 1.1.

For the converse, we assert that the analyticity of r in a rectangular region, as above, implies that its spectral measure function F satisfies (27). Since F determines a signed (or complex) measure on $\mathbb{R}^2$, by hypothesis, it is bounded and (by the Jordan decomposition) can be expressed as a linear combination of nonegative measure functions of bounded (Vitali) variation, $F_1,\ldots,F_4$. Hence, it suffices to establish (27) for one of them, say F_1. So let

$$r_1(s,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{isx-ity}\, d\,F_1(x,y) \tag{28}$$

and by hypothesis r_1 is analytic. Thus it admits a (uniformly and absolutely) convergent power series expansion

$$r_1(s_1,s_2) = \sum_{j,k=0}^{\infty} \frac{\partial^{j+k} r_1}{\partial s_1^j \partial s_2^k}(0,0)\frac{s_1^j}{j!}\frac{s_2^k}{k!} \tag{29}$$

for $|s_m| < p_m, m = 1, 2$, a rectangular neighborhood of $(0,0)$ in $\mathbb{R}^2$, following some standard results on characteristic functions of probability theory. Now r_1 of (28) is infinitely differentiable, and it is the Fourier transform of a bounded (positive) measure. This implies that F_1 has all (absolute) moments finite since the integral is in Lebesgue's sense. If

$$\alpha_{j,k} = \int_{\mathbb{R}}\int_{\mathbb{R}} x^j\, y^k\, d\,F_1(x,y) \tag{30}$$

then

$$\frac{\partial^{j+k} r_1}{\partial s_1^j \partial s_2^k}(0,0) = i^{j+k}\alpha_{j,k} \tag{31}$$

Because $\alpha_{2j,2k} \geq 0$, the absolute moments $\beta_{j,k}$ of F_1 are dominated by the even moments as follows. Using the elementary inequality

$$|a\,b| \leq (\frac{a^2+b^2}{2}) \tag{32}$$

one has

$$|x|^k|x_{k-1}|^j|y|^{j-1} \leq \frac{1}{4}(x^{2k}+x^{2k-2})\,(y^{2j}+y^{2j-2}) \tag{33}$$

and hence

$$\begin{aligned} \beta_{2k-1,2j-1} &\leq \frac{1}{(2\,k-1)!(2\,j-1)!} \\ &\leq \frac{1}{4}\,\alpha_{2k,2j}((2\,k)!(2\,j)!) + \frac{\alpha_{2k-2,2j-2}}{(2\,k-2)!(2\,j-2)!} \\ &\leq \frac{\alpha_{2k,2j}}{2\,k\cdot 2\,j-1} + \frac{\alpha_{2k-2,2j-2}}{2\,k+2\,j-2}\cdot\frac{1}{(2\,k-2)!(2\,j-2)!-1} \end{aligned} \tag{34}$$

Substituting this estimate and using the even coefficient terms of the absolutely convergent series (29), one finds that

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \beta_{j,k} \frac{s_1^{2j}}{(2j)!} \frac{s_2^{2k}}{(2k)!} \tag{35}$$

converges absolutely in the same open neighborhood of $(0,0)$ in $\mathbb{R}^2$. This implies immediately that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} e^{|\alpha s_1 + \beta s_2|} \, dF_1(x,y) < \infty \tag{36}$$

It follows from this that (27) holds for F_1 and by the initial reduction, it holds for F itself. Thus the condition is also necessary.

Finally, in the band-limited case, F has all moments finite (i.e., its moment generating function exists), and the sufficiency of the result implies that the process must be analytic □

Bibliography

[1] Malempati M. Rao. *Stochastic Processes: Inference Theory*. Springer Monographs in Mathematics. Springer, 2nd edition, 2014.