The Eigenfunctions of $\int_0^\infty J_0(|x-y|)\, f(x) \mathrm{d}x$ and a Technique For Deriving The Eigenfunctions of Stationary Gaussian Process Integral Covariance Operators

by Stephen Crowley

stephencrowley214@gmail.com

September 16, 2024

Abstract

The null spaces of Gaussian process kernel inner product operators are shown to be the Fourier transforms of the polynomials orthogonal with respect to the spectral densities of the processes and it is furthermore shown that the orthogonal complements of the null spaces as given by the Gram-Schmidt recursions enumerate the products $g_k(t) = \sqrt{\frac{c_p}{c_q}} \frac{\prod_{i=1}^{n_k} (t - \alpha_{k,i})}{\prod_{j=1}^{m_k} (t - \beta_{k,j})} = f_k(t)\, f_k(s)$ of the eigenfunctions f_k of the corresponding integral covariance operators.

Let $C(x)$ be the covariance function of a stationary Gaussian process on $[0, \infty)$. Define the integral covariance operator T by:

$$(T f)(x) = \int_0^\infty C\,(x - y)\, f(y) \,\mathrm{d}y \tag{1}$$

Let $S(\omega)$ be the spectral density related to $C(x)$ by the Wiener-Khinchin theorem:

$$C(x) = \frac{1}{\pi} \int_{-\infty}^\infty e^{i\omega x}\, S(\omega) \,\mathrm{d}\omega \tag{2}$$

$$S(\omega) = \int_0^\infty C(x)\, e^{-ix\omega} \,\mathrm{d}x \tag{3}$$

Consider polynomials $\{p_n(\omega)\}$ orthogonal with respect to $S(\omega)$:

$$\int_{-\infty}^\infty p_n(\omega)\, p_m(\omega)\, S(\omega) \,\mathrm{d}\omega = \delta_{nm} \tag{4}$$

Define $r_n(x)$ as the inverse Fourier transforms of $p_n(\omega)$:

$$r_n(x) = \int_{-\infty}^\infty p_n(\omega)\, e^{i\omega x} \,\mathrm{d}\omega \tag{5}$$

Lemma 1. *The functions $r_n(x)$ form the null space of the kernel inner product:*

$$\int_0^\infty C(x)\, r_n(x)\, \mathrm{d}x = 0 \tag{6}$$

Proof. Let $C(x)$ and $r_n(x)$ be defined as:

$$C(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{iax}\, S(a)\, \mathrm{d}a \tag{7}$$

$$r_n(x) = \int_{-\infty}^{\infty} p_n(b)\, e^{ibx}\, \mathrm{d}b \tag{8}$$

where $S(a)$ is the spectral density and $p_n(b)$ are orthogonal polynomials with respect to $S(a)$. Note that $C(x)$ and $r_n(x)$ are even functions, as they depend on the difference between two variables.

Substitute the definitions of $C(x)$ and $r_n(x)$, and apply Fubini's theorem:

$$\begin{aligned}\int_0^\infty C(x)\, r_n(x)\, \mathrm{d}x &= \int_0^\infty \frac{1}{\pi} \int_{-\infty}^{\infty} e^{iax}\, S(a)\, \mathrm{d}a \int_{-\infty}^{\infty} p_n(b)\, e^{ibx}\, \mathrm{d}b\, \mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_n(b)\, S(a) \int_0^\infty e^{i(a+b)x}\, \mathrm{d}x\, \mathrm{d}b\, \mathrm{d}a\end{aligned}$$

Since $C(x)$ and $r_n(x)$ are even functions, we can write:

$$\int_0^\infty C(x)\, r_n(x)\, \mathrm{d}x = \frac{1}{2} \int_{-\infty}^{\infty} C(x)\, r_n(x)\, \mathrm{d}x \tag{9}$$

Now we have:

$$\begin{aligned}\frac{1}{2} \int_{-\infty}^{\infty} C(x)\, r_n(x)\, \mathrm{d}x &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_n(b)\, S(a) \int_{-\infty}^{\infty} e^{i(a+b)x}\, \mathrm{d}x\, \mathrm{d}b\, \mathrm{d}a \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_n(b)\, S(a)\, \delta(a+b)\, \mathrm{d}b\, \mathrm{d}a \\ &= \frac{1}{2} \int_{-\infty}^{\infty} p_n(-a)\, S(a)\, \mathrm{d}a\end{aligned}$$

By the orthogonality of $p_n(a)$ with respect to $S(a)$, we conclude:

$$\frac{1}{2} \int_{-\infty}^{\infty} p_n(-a)\, S(a)\, \mathrm{d}a = 0 \tag{10}$$

Thus, $\int_0^\infty C(x)\, r_n(x)\, \mathrm{d}x = 0$, which completes the proof. $\square$

1 Eigenfunctions from Orthogonalized Null Space

By orthogonalizing the null space $\{r_n(x)\}$, we obtain the eigenfunctions $\{\psi_n(x)\}$ of the covariance operator T. The orthogonalization process gives:

$$r_n^{\perp}(y) = \psi_n(x) = \sum_{k=0}^{n} a_{nk}\, r_k(x) = r_n(y) - \sum_{m=0}^{n-1} \frac{\langle r_n(y), r_m^{\perp}(y)\rangle}{\langle r_m^{\perp}(y), r_m^{\perp}(y)\rangle}\, r_m^{\perp}(y)$$

where the coefficients a_{nk} are given by:

$$a_{nk} = \begin{cases} 1 & \text{if } k = n \\ -\sum_{j=k}^{n-1} a_{nj}\, \langle r_n, \psi_j\rangle & \text{if } k\, n \\ 0 & \text{if } k > n \end{cases} \tag{11}$$

Theorem 2. *Let $\{\psi_n(x)\}$ be the orthogonal complement of $\{r_n(x)\}$. Then $\psi_n(x)$ are eigenfunctions of T, with eigenvalues:*

$$\lambda_n = \int_0^\infty C(z)\, \psi_n(z)\, \mathrm{d}z \tag{12}$$

Proof. This is not quite right, they have to be factorized as in Theorem 9. I think the infinite-dimensional version of this is the Hadamard product factorization? $\square$

Definition 3. *The spectral density of a stationary process is the Fourier transform of the covariance kernel due to Wiener-Khinchin theorem.*

Definition 4. *Let $S_n(x)$ be the orthogonal polynomials whose orthogonality measure is equal to the spectral density of the process. These polynomials shall be called the spectral polynomials corresponding to the process.*

Remark 5. If the spectral density does not equal the orthogonality measure of a known set of orthogonal polynomials then such a set can always be generated by applying the Gram-Schmidt process to the monomials so that they are transformed into a set that is orthogonal with respect any given spectral density (of a stationary process).

1.1 The Karhunen-Loeve Expansion

The Karhunen-Loeve expansion is a spectral representation theorem which expands the random process $w(x,\theta)$ in terms of a denumerable set of orthogonal random variables in the form

$$w(x,\theta) = \sum_{i=1}^{\infty} \mu_i(\theta)\, g_i(x) \tag{13}$$

where $\{\mu_i(\theta)\}$ is a set of random variable projections and $\{g_i(x)\}$ are the functions of the integral covariance operator associated to the Gaussian process having the covariance kernel of $w(x,\theta)$. Since equation (13) is a quantization of the random process it is a representation of the process $w(x,\theta)$ as a curve in the Hilbert space spanned by the set $\{g_i(x)\}$ expressed as a direct sum of orthogonal projections in this Hilbert space whose projections on successive basis vectors have magnitudes which are proportional to the corresponding eigenvalues of the covariance function associated with the eigenfunctions of the process $w(x,\theta)$.

1.1.1 Derivation

Theorem 6. *The random process $w(x,\theta)$ can be represented by the Karhunen-Loeve expansion defined by the Fourier-like series as*

$$w(x,\theta) = \sum_{n=0}^{\infty} \sqrt{\lambda_n}\, \xi_n(\theta)\, f_n(x) \tag{14}$$

where $\{\xi_n(\theta)\}$ is a said to be a set of 'random' variables to be determined by projecting the process $w(x,\theta)$ onto the $n-\mathrm{th}$ eigenfunction and λ_n is the n-th eigenvalue corresponding to the n-th eigenfunction of the corresponding integral covariance operator $\{f_n(x)\}$

Proof. Let $w(x,\theta)$ be a random process, function of the position vector x defined over the domain D, with θ belonging to the space of random events Ω. Let $\bar{w}(x)$ denote the expected value of $w(x,\theta)$ over all possible realizations of the process, and $C(x_1,x_2)$ denote its covariance function. By definition of the covariance function, it is bounded, symmetric and positive definite. Thus, it has the eigenfunction expansion

$$C(x_1,x_2) = \sum_{n=0}^{\infty} \lambda_n\, f_n(x_1)\, f_n(x_2) \tag{15}$$

where λ_n and $f_n(x)$ are the eigenvalue and the eigenfunction of the covariance kernel. And, specifically, that they are the solution to the integral equation

$$\int_D C(x_1,x_2)\, f_n(x_1)\, \mathrm{d}x_1 = \lambda_n\, f_n(x_2) \tag{16}$$

Due to the symmetry and the positive definiteness of the covariance kernel, its eigenfunctions are orthogonal and form a complete set. They can be normalized according to the following criterion

$$\int_D f_n(x)\, f_m(x)\, \mathrm{d}x = \delta_{nm} \tag{17}$$

where δ_{nm} is the Kronecker delta. Clearly, $w(x,\theta)$ can be written as

$$w(x,\theta) = \bar{w}(x) + \alpha(x,\theta) \tag{18}$$

where $\alpha(x,\theta)$ is a process with zero mean and covariance function $C(x_1,x_2)$. The process $\alpha(x,\theta)$ can be expanded in terms of the eigenfunctions $f_n(x)$ as

$$\alpha(x,\theta) = \sum_{n=0}^{\infty} \xi_n(\theta)\sqrt{\lambda_n}\, f_n(x) \tag{19}$$

Second order properties of the random variables ξ_n can be determined by multiplying both sides of equation (19) by $\alpha(x_2,\theta)$ and taking the expectation on both sides. Specifically, it is found that

$$\begin{aligned} C(x_1,x_2) &= \langle \alpha(x_1,\theta)\, \alpha(x_2,\theta)\rangle \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \langle \xi_n(\theta)\, \xi_m(\theta)\rangle \sqrt{\lambda_n \lambda_m}\, f_n(x_1)\, f_m(x_2) \end{aligned} \tag{20}$$

Then, multiplying both sides of equation (20) by $f_k(x_2)$, integrating over the domain$D$, and making use of the orthogonality of the eigenfunctions, yields

$$\begin{aligned} \int_D C(x_1,x_2)\, f_k(x_2)\, \mathrm{d}x_2 &= \lambda_k\, f_k(x_1) \\ &= \sum_{n=0}^{\infty} \langle \xi_n(\theta)\, \xi_k(\theta)\rangle \sqrt{\lambda_n \lambda_k}\, f_n(x_1) \end{aligned} \tag{21}$$

Multiplying once more by $f_l(x_1)$ and integrating over D gives

$$\int_D\int_D f_l(x_1)\, f_k(x_1)\, \mathrm{d}x_1 = \sum_{n=0}^{\infty} \langle \xi_n(\theta)\, \xi_k(\theta)\rangle \sqrt{\lambda_n \lambda_k}\, \delta_{nl} \tag{22}$$

Then, using equation (17) leads to

$$\lambda_k\, \delta_{kl} = \sqrt{\lambda_k \lambda_l}\, \langle \xi_k(\theta)\, \xi_l(\theta)\rangle \tag{23}$$

Equation (23) can be rearranged to give

$$\langle \xi_k(\theta)\, \xi_l(\theta)\rangle = \delta_{kl} \tag{24}$$

Thus, the random process $w(x,\theta)$ can be written as

$$w(x,\theta) = \bar{w}(x) + \sum_{n=0}^{\infty} \xi_n(\theta)\sqrt{\lambda_n}\, f_n(x) \tag{25}$$

where

$$\langle \xi_n(\theta) \rangle = 0 \tag{26}$$

$$\langle \xi_n(\theta)\, \xi_m(\theta) \rangle = \delta_{nm} \tag{27}$$

and λ_n, $f_n(x)$ are solution to equation (16). Truncating the series in equation (25) at the M^{th} term, gives

$$w(x,\theta) = \bar{w}(x) + \sum_{n=0}^{M} \xi_n(\theta)\sqrt{\lambda_n}\, f_n(x) \tag{28}$$

An explicit expression for $\xi_n(\theta)$ can be obtained by multiplying equation (19) by $f_n(x)$ and integrating over the domain D. That is,

$$\xi_n(\theta) = \frac{\int_D \alpha(x,\theta)\, f_n(x)\, \mathrm{d}x}{\sqrt{\lambda_n}} \tag{29} \quad \square$$

1.1.2 Uniqueness of the Expansion

Lemma 7. ***Uniqueness****: The random variables appearing in an expansion of the kind given by equation (18) are orthonormal if and only if the orthonormal functions $\{f_n(x)\}$ and the constants $\{\lambda_n\}$ are respectively the eigenfunctions and the eigenvalues of the covariance kernel as given by equation (16).*

Proof. The "if" part is an immediate consequence of equation (19). To show the "only if" part, equation (20) can be used with

$$\langle \xi_n(\theta)\, \xi_m(\theta) \rangle = \delta_{nm} \tag{30}$$

to obtain

$$C(x_1,x_2) = \sum_{n=0}^{\infty} \lambda_n\, f_n(x_1)\, f_n(x_2) \tag{31}$$

Multiplying both sides by $f_m(x_2)$ and integrating over D gives

$$\int_D C(x_1,x_2)\, f_m(x_2)\, \mathrm{d}x_2 = \sum_{n=0}^{\infty} \lambda_n\, f_n(x_1)\, \delta_{nm} = \lambda_m\, f_m(x_1) \tag{32} \quad \square$$

Theorem 8. *Let $Y_n(y)$ be the normalized Fourier transforms of the spectral polynomials $Y_n(y) = \frac{\hat{S}_n(y)}{|\hat{S}_n|}$ where the sequence $\hat{S}_n(y)$ of inverse Fourier transforms of the spectral polynomials $S_n(x)$ is given by*

$$\hat{S}_n(y) = \int_{-1}^{1} S_n(x)\, e^{ixy}\, \mathrm{d}x \tag{33}$$

The eigenfunctions of the integral covariance operator (52) are given by the products

$$\psi_n(x)\, \psi_n(y) = Y_n^{\perp}(x-y) \tag{34}$$

of the elements of orthogonal complement of the normalized Fourier transforms $Y_n(y)$ of the spectral polynomials (via the Gram-Schmidt process)

$$\begin{aligned} \psi_n(x)\, \psi_n(y) &= Y_n^{\perp}(x-y) \\ &= Y_n(x-y) - \sum_{m=0}^{n-1} \frac{\langle Y_m(x-y), Y_m^{\perp}(x-y)\rangle}{\langle Y_m^{\perp}(x-y), Y_m^{\perp}(x-y)\rangle} Y_m^{\perp}(x-y) \end{aligned} \tag{35}$$

Proof. ...there's some elegant way to do this that I can probably write down after my back surgery.. it involves Lemma 9 □

Lemma 9. *For any rational function*

$$f(t-s) = \frac{P(t-s)}{Q(t-s)} \tag{36}$$

, where P and Q are polynomials, there exist rational functions $g(t)$ and $g(s)$ such that

$$f(t-s) = g(t)\, g(s) \tag{37}$$

Proof. Let

$$P(t-s) = c_p \prod_{i=1}^{n} (t-s-\alpha_i) \tag{38}$$

and

$$Q(t-s) = c_q \prod_{j=1}^{m} (t-s-\beta_j) \tag{39}$$

then define

$$g(t) = \sqrt{\frac{c_p}{c_q}}\, \frac{\prod_{i=1}^{n}(t-\alpha_i)}{\prod_{j=1}^{m}(t-\beta_j)} \tag{40}$$

such that

$$
\begin{aligned}
g(t)\, g(s) &= \frac{c_p}{c_q} \frac{\prod_{i=1}^{n} (t-\alpha_i)}{\prod_{j=1}^{m} (t-\beta_j)} \frac{\prod_{i=1}^{n} (s-\alpha_i)}{\prod_{j=1}^{m} (s-\beta_j)} \\
&= \frac{c_p}{c_q} \frac{\prod_{i=1}^{n} (t-\alpha_i)\, (s-\alpha_i)}{\prod_{j=1}^{m} (t-\beta_j)\, (s-\beta_j)} \\
&= \frac{c_p \prod_{i=1}^{n} ((t-s)-\alpha_i)}{c_q \prod_{j=1}^{m} ((t-s)-\beta_j)} \\
&= f\,(t-s)
\end{aligned} \tag{41}
$$

For complex roots, we pair each α_i or β_j with its complex conjugate in the factorization of $g(t)$. This ensures that the product $(t-\alpha_i)\,(t-\bar{\alpha_i})$ results in a quadratic polynomial with real coefficients, making $g(t)$ a real-valued function. □

Example 10. Let the kernel function be given by $K(t,s) = J_0\,(t-s)$ then identify the orthogonal polynomial sequence associated with the spectral density

$$
S(\omega) = \int_0^{\infty} J_0(x)\, e^{ix\omega}\, \mathrm{d}x = \begin{cases} \dfrac{1}{\sqrt{1-\omega^2}} & \omega \in (-1,1) \\ 0 & \text{otherwise} \end{cases} \tag{42}
$$

as being the orthogonality measure of the Type-I Chebyshev polynomials $T_n(x)$ so that the orthogonal polynomial sequence is identified as

$$
S_n(x) = T_n(x) \tag{43}
$$

so that

$$
\int_{-1}^{1} S_n(\omega)\, S_m(\omega)\, S(\omega)\, \mathrm{d}\omega = \begin{cases} 0 & n \neq m \\ 2\,\pi & n = m = 0 \\ \pi & n = m \neq 0 \end{cases} \tag{44}
$$

The finite Fourier transforms of the Chebyshev polynomials are just the usual infinite Fourier transforms with the integration restricted to the range $-1 \ldots 1$ since $T_n(x) = 0 \forall x \notin [-1,1]$). Equivalently, the spectral density function can be extended to take the value 0 outside the interval $[-1,1]$.

$$
\begin{aligned}
\hat{T}_n(y) &= \int_{-\infty}^{\infty} e^{-ixy}\, T_n(x)\, \mathrm{d}y = \int_{-1}^{1} e^{-ixy}\, T_n(x)\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} e_2^{-ixy}\, F_1\left(\begin{matrix} n, & -n \\ & \frac{1}{2} \end{matrix} \,|\frac{1}{2} - \frac{x}{2} \right) \mathrm{d}x \\
&= \frac{i}{y}\, (e^{-iy}\, F_n^{+}(y) - e^{i(\pi n + y)}\, F_n^{-}(y))
\end{aligned} \tag{45}
$$

where

$$F_n^{\pm}(y) = {}_3F_1\left(\begin{array}{ccc} 1, & n, & -n \\ & & \frac{1}{2} \end{array} \middle| \frac{\pm i\, y}{2}\right) \tag{46}$$

the spectral polynomials S_n are given by the Type-I Chebyshev polynomials

$$S_n(x) = T_n(x) \tag{47}$$

and their normalization is

$$\begin{aligned} Y_n(y) &= \frac{\hat{T}_n(y)}{|\hat{T}_n|} \\ &= \frac{i}{y}\left(\frac{e^{-iy}\, F_n^{+}(y) - e^{i(\pi n + y)}\, F_n^{-}(y)}{\sqrt{\frac{4\,(-1)^n\,\pi - (2\,n^2 - 1)}{4\,n^2 - 1}}}\right) \end{aligned} \tag{48}$$

where the L^2 norm of $\hat{T}_n(y)$ is given by

$$\begin{aligned} |\hat{T}_n| &= \sqrt{\int_{-\infty}^{\infty} \hat{T}_n(y)^2\, \mathrm{d}y} \\ &= \sqrt{\frac{4\,(-1)^n\,\pi - (2\,n^2 - 1)}{4\,n^2 - 1}} \end{aligned} \tag{49}$$

Definition 11. *Let $j_n(x)$ is the spherical Bessel function of the first kind,*

$$\begin{aligned} j_n(z) &= \sqrt{\frac{\pi}{2\,z}}\, J_{n+\frac{1}{2}}(x) \\ &= \frac{1}{\sqrt{z}}\left(\sin(z)\, R_{n,\frac{1}{2}}(z) - \cos(z)\, R_{n-1,\frac{3}{2}}(z)\right) \end{aligned} \tag{50}$$

where $R_{n,v}(z)$ are the (misnamed) Lommel polynomials

$$R_{n,v}(z) = \frac{\Gamma(n+v)}{\Gamma(v)}\left(\frac{2}{z}\right)^n {}_2F_3\left(\left[-\frac{n}{2}, \frac{1}{2} - \frac{n}{2}\right]; [v, -n, 1 - v - n]; -z^2\right) \tag{51}$$

where ${}_2F_3$ is a generalized hypergeometric function. The "Lommel polynomials" are actually rational functions of z, not polynomial; but rather "polynomial in $\frac{1}{z}$".

Theorem 12. *The eigenfunctions of the stationary integral covariance operator*

$$[T\psi_n](x) = \int_0^\infty J_0\,(x-y)\,\psi_n(x)\,\mathrm{d}x = \lambda_n\,\psi_n(x) \tag{52}$$

are given by

$$Y_n^\perp\,(x-y) = \psi_n(x)\,\psi_n(y) = (-1)^n \sqrt{\frac{8\,n+2}{\pi}}\,j_{2n}\,(x-y) \tag{53}$$

and the eigenvalues are given by

$$\begin{aligned}\lambda_n &= \int_0^\infty J_0(x)\,\psi_n(x)\,\mathrm{d}x\\ &= \sqrt{\frac{2\,n+\frac{1}{2}}{\pi}}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\,(n+1)^2}\\ &= \sqrt{\frac{2\,n+\frac{1}{2}}{\pi}}\,(n+1)^2_{-\frac{1}{2}}\end{aligned} \tag{54}$$

where $(n+1)_{-\frac{1}{2}}$ *is the Pochhammer symbol (ascending/rising factorial).*

Proof. TODO: show that $Y_n^\perp(|x-y|) = \psi_n(x)\,\psi_n(y)$ converges uniformly by demonstrating that it is compact relative to the canonical metric induced by the process then apply Hilbert's proof of a theorem that was initiated by Schmidt as written in Whittaker and Watson's "A Course of Modern Analysis". Note that just because Hilbert and Schmidt are involved this does not mean this is the Hilbert-Schmidt class of operators; specifically, nothing in this proof depends on the boundedness of the domain or the square integrability of the kernel; both of which are significant limitations of the classical Mercer theorem. To see that unLet $\phi_n(\alpha)$ be a complete set of orthogonal functions satisfying the homogeneous integral equation with symmetric nucleus

$$\phi(\alpha) = \lambda \int_a^b K(\alpha,\xi)\,\phi(\xi)\,\mathrm{d}\xi \tag{55}$$

the corresponding characteristic numbers being $\lambda_1, \lambda_2, \lambda_3, \ldots$ Now suppose that the series $\sum_{n=1}^\infty \frac{\phi_n(\alpha)\,\phi_n(\gamma)}{\lambda_n}$ is uniformly convergent when $0 \leqslant a \leq \alpha \leq b \leqslant \infty$, $0 \leqslant a \leq \gamma \leq b \leqslant \infty$. Then it will be shown that

$$K(\alpha,\gamma) = \sum_{n=1}^\infty \frac{\phi_n(\alpha)\,\phi_n(\gamma)}{\lambda_n} \tag{56}$$

For consider the symmetric nucleus

$$H(\alpha,\gamma) = K(\alpha,\gamma) - \sum_{n=1}^{\infty} \frac{\phi_n(\alpha)\,\phi_n(\gamma)}{\lambda_n} \tag{57}$$

If this nucleus is not identically zero, it will possess at least one characteristic number μ. Let $\psi(\gamma)$ be any solution of the equation

$$\psi(\alpha) = \mu \int_a^b H(\alpha,\xi)\,\psi(\xi)\,\mathrm{d}\xi \tag{58}$$

which does not vanish identically. Multiply by $\phi_m(\alpha)$ and integrate term-by-term (which we may do since the series converges uniformly by hypothesis proved by other means), and get

$$\int_a^b \psi(\alpha)\,\phi_m(\alpha)\,\mathrm{d}\alpha = \mu \int_a^b \int_a^b \left[K(\alpha,\xi) - \sum_{n=1}^{\infty} \frac{\phi_n(\alpha)\,\phi_n(\xi)}{\lambda_n}\right] \psi(\xi)\,\phi_m(\alpha)\,\mathrm{d}\xi\,\mathrm{d}\alpha = 0 \tag{59}$$

Therefore $\psi(\alpha)$ is orthogonal to $\phi_1(\alpha), \phi_2(\alpha), \ldots$; and so taking the equation

$$\psi(\alpha) = \mu \int_a^b \left[K(\alpha,\xi) - \sum_{n=1}^{\infty} \frac{\phi_n(\alpha)\,\phi_n(\xi)}{\lambda_n}\right] \psi(\xi)\,\mathrm{d}\xi \tag{60}$$

we have

$$\psi(\alpha) = \mu \int_a^b K(\alpha,\xi)\,\psi(\xi)\,\mathrm{d}\xi \tag{61}$$

Therefore μ is a characteristic number of $K(\alpha,\gamma)$, and so $\psi(\alpha)$ must be a linear combination of the functions $\phi_n(\alpha)$ corresponding to this number; let

$$\psi(\alpha) = \sum_m a_m\,\phi_m(\alpha) \tag{62}$$

Multiply by $\phi_m(\gamma)$ and integrate; then since $\psi(\alpha)$ is orthogonal to all the functions $\phi_m(\alpha)$, we see that $a_m = 0$, so, contrary to hypothesis, $\psi(\alpha) = 0$. The contradiction implies that the nucleus $H(\alpha,\gamma)$ must be identically zero; that is to say, $K(\alpha,\gamma)$ can be expanded in the given series, if it is uniformly convergent.

2 Explanation

Given:

- An orthogonal set of functions $\phi_n(\alpha)$ defined over an interval $[a, b]$.

- A symmetric kernel $K(\alpha, \gamma)$ defined over $[a, b] \times [a, b]$.
- The series $\sum_{n=1}^{\infty} \frac{\phi_n(\alpha)\, \phi_n(\gamma)}{\lambda_n}$ uniformly converges to $K(\alpha, \gamma)$.

To Prove:

- The functions $\phi_n(\alpha)$ are the unique eigenfunctions of the integral operator with kernel $K(\alpha, \gamma)$.

Proof:

1. **Uniform Convergence of Series Representation:**

   - By hypothesis, the series

     $$K(\alpha, \gamma) = \sum_{n=0}^{\infty} \frac{\phi_n(\alpha)\, \phi_n(\gamma)}{\lambda_n} \tag{63}$$

     converges uniformly to $K(\alpha, \gamma)$.

   - This uniform convergence ensures that the series represents K accurately over the entire domain $[a, b] \times [a, b]$.

2. **Orthogonality and Completeness:**

   - The functions ϕ_n are orthogonal, meaning:

     $$\int_a^b \phi_m(\alpha)\, \phi_n(\alpha)\, \mathrm{d}\alpha = 0 \quad \text{for all } m \neq n \tag{64}$$

   - Orthogonality implies that no ϕ_n can be represented by a linear combination of other ϕ_ms in the set.

3. **Eigenfunction Equation:**

   - Each function ϕ_n satisfies the integral equation:

     $$\phi_n(\alpha) = \lambda_n \int_a^b K(\alpha, \xi)\, \phi_n(\xi)\, \mathrm{d}\xi \tag{65}$$

     defining them as eigenfunctions of K with corresponding eigenvalues λ_n.

4. **Uniqueness:**

   - Assume there exists another function $\psi(\alpha)$ that is not a linear combination of ϕ_n and also satisfies the integral equation for some λ:

$$\psi(\alpha) = \lambda \int_a^b K(\alpha, \xi)\, \psi(\xi)\, \mathrm{d}\xi \tag{66}$$

   - Multiply both sides by $\phi_m(\alpha)$ and integrate:

$$\int_a^b \psi(\alpha)\, \phi_m(\alpha)\, \mathrm{d}\alpha = \lambda \int_a^b \int_a^b K(\alpha, \xi)\, \psi(\xi)\, \phi_m(\alpha)\, \mathrm{d}\xi\, \mathrm{d}\alpha \tag{67}$$

   - Since ψ is orthogonal to all ϕ_n, the left-hand side is zero, implying $\psi(\alpha)$ must be zero by the completeness of ϕ_n.

5. **Conclusion:**

   - The set ϕ_n uniquely represents the kernel K via their series expansion. No other function set orthogonal to ϕ_n can satisfy the kernel's integral equation unless it is zero.

   - Therefore, $\phi_n(\alpha)$ are the unique eigenfunctions of the integral operator defined by $K(\alpha, \gamma)$. □

Theorem 13. *The series*

$$\begin{aligned} J_0(t) &= \sum_{k=0}^{\infty} \lambda_k\, \psi_k(t) \\ &= \sum_{k=0}^{\infty} \sqrt{\frac{2\,n + \frac{1}{2}}{\pi}} \sqrt{\frac{8\,n+2}{\pi}}\, (n+1)^2_{-\frac{1}{2}}\, (-1)^n\, j_{2n}(t) \\ &= \sum_{k=0}^{\infty} \frac{4\,n+1}{\pi}\, (n+1)^2_{-\frac{1}{2}}\, (-1)^n\, j_{2n}(t) \end{aligned} \tag{68}$$

converges uniformly for all complex t except the origin where it has a regular singular point where $\lim_{t \to 0} J_0(t) = 1$.

Proof.
$$\sqrt{\frac{2\,n + \frac{1}{2}}{\pi}} \sqrt{\frac{8\,n+2}{\pi}} = \frac{\sqrt{16\,n^2 + 8\,n + 1}}{\sqrt{\pi^2}} = \frac{\sqrt{(4\,n+1)\,(4\,n+1)}}{\sqrt{\pi\,\pi}} = \frac{4\,n+1}{\pi} \tag{69}$$

TODO..show that operator defined by Equation (52) is compact relative to the canonical metric induced by the covariance kernel $J_0(|x-y|)$ which can be demonstrated by calculating the metric entropy integral and showing it is finite for all positive epsilon □