Uniform Convergence of Orthonormal Basis Projections in RKHS

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Definition 1 (Reproducing Kernel Hilbert Space)**.** A Hilbert space H of functions on a set D is called a reproducing kernel Hilbert space (RKHS) if there exists a function $k : D \times D \to \mathbb{R}$ such that:

1. For every $x \in D$, the function $k_x(\cdot) = k(\cdot, x)$ belongs to H.
2. For every $x \in D$ and every $f \in H$, the reproducing property holds: $f(x) = \langle f, k_x \rangle_H$.

The function k is called the reproducing kernel of H.

Definition 2 (Orthonormal Basis in RKHS)**.** A sequence of functions $\{e_n\}_{n=1}^{\infty} \subset H$ is an orthonormal basis of the RKHS H if:

1. Orthonormality: For all indices n, m, $\langle e_n, e_m \rangle_H = \delta_{nm}$, where δ_{nm} is the Kronecker delta.
2. Completeness: The span of $\{e_n\}_{n=1}^{\infty}$ is dense in H, which means:
3. For any $f \in H$, if $\langle f, e_n \rangle_H = 0$ for all n, then $f = 0$.
4. Equivalently, every function $f \in H$ can be represented as

$$f = \sum_{n=1}^{\infty} \langle f, e_n \rangle_H e_n$$

with convergence in the H-norm:

$$\lim_{N \to \infty} \left\| f - \sum_{n=1}^{N} \langle f, e_n \rangle_H e_n \right\|_H = 0$$

5. Parseval's Identity: For any $f \in H$,

$$\|f\|_H^2 = \sum_{n=1}^{\infty} |\langle f, e_n \rangle_H|^2$$

In an RKHS, each basis function satisfies the reproducing property: $e_n(x) = \langle e_n, k(\cdot, x)\rangle_H$ for all $x \in D$.

Theorem 3. *Let H be a reproducing kernel Hilbert space (RKHS) on a set D with reproducing kernel k. Suppose that:*

1. *$\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of H as defined in Definition 2.*
2. *The kernel is uniformly bounded on D; that is, there exists a constant $M > 0$ such that*
$$\sup_{x\in D} \sqrt{k(x,x)} \le M.$$

Then for any function $f \in H$ with orthonormal expansion

$$f = \sum_{n=1}^{\infty} c_n e_n,$$

where $c_n = \langle f, e_n\rangle_H$, the partial sums

$$S_N f = \sum_{n=1}^{N} c_n e_n$$

converge uniformly to f on D; in other words,

$$\lim_{N\to\infty} \sup_{x\in D} \big|S_N f(x) - f(x)\big| = 0.$$

Proof. By the completeness property of the orthonormal basis (Definition 2), every function $f \in H$ can be represented by its orthonormal expansion that converges in the H-norm. Since H is an RKHS, the evaluation functional at any $x \in D$ is continuous. In particular, for each fixed x, there exists a constant (which can be taken as $\sqrt{k(x,x)}$) such that

$$\big|f(x) - S_N f(x)\big| = \Big|\langle f - S_N f, k(\cdot, x)\rangle_H\Big| \le \|f - S_N f\|_H \, \|k(\cdot, x)\|_H = \|f - S_N f\|_H \sqrt{k(x,x)}.$$

Taking the supremum over $x \in D$ yields

$$\sup_{x\in D} \big|f(x) - S_N f(x)\big| \le \|f - S_N f\|_H \, \sup_{x\in D} \sqrt{k(x,x)}.$$

By the boundedness assumption we have

$$\sup_{x\in D} \big|f(x) - S_N f(x)\big| \le M \, \|f - S_N f\|_H.$$

From the convergence property of orthonormal bases stated in Definition 2, we know that the partial sums converge in H-norm; that is,

$$\lim_{N\to\infty} \|f - S_N f\|_H = 0.$$

Now, for any given $\varepsilon > 0$, choose an index N_0 such that for all $N \geq N_0$,

$$\|f - S_N f\|_H < \frac{\varepsilon}{M}.$$

Then, for all $N \geq N_0$,

$$\sup_{x \in D} \big| f(x) - S_N f(x) \big| \leq M \, \|f - S_N f\|_H < M \left(\frac{\varepsilon}{M} \right) = \varepsilon.$$

This directly shows that

$$\lim_{N \to \infty} \sup_{x \in D} |S_N f(x) - f(x)| = 0,$$

which is the definition of uniform convergence. $\square$

Remark 4. The uniform boundedness condition on the kernel is essential. Without it, norm convergence in the RKHS would not necessarily imply uniform convergence of the function evaluations on the domain.

Remark 5. It is important to emphasize that the domain D in Theorem 3 is not required to be compact. The result holds for any domain, including unbounded domains such as $D = \mathbb{R}^n$ or $D = [0, \infty)$, provided that the kernel is uniformly bounded on that domain.

Remark 6. The uniform convergence described in Theorem 3 applies to any orthonormal basis when expanding functions in the RKHS H, whereas when expanding the reproducing kernel $k(x, y)$ itself, only the Mercer eigenbasis $\{e_n^*\}$, defined by the equation

$$\int_D k(x, y) e_n^*(y) \, dy = \lambda_n e_n^*(x),$$

converges uniformly, whereas non-Mercer orthonormal bases converge pointwisely.

References

Riesz, F. (1907). Sur les systèmes orthogonaux de fonctions. *Comptes rendus de l'Académie des sciences*, 144:615–619.

Fischer, E. (1907). Sur la convergence en moyenne. *Comptes rendus de l'Académie des sciences*, 144:1022–1024.

Aronszajn, N. (1950). Theory of reproducing kernels. *Transactions of the American Mathematical Society*, 68(3):337–404.

Berlinet, A. and Thomas-Agnan, C. (2004). *Reproducing Kernel Hilbert Spaces in Probability and Statistics*. Springer, Boston, MA.