Spectral Relations and Beat Frequency Analysis in Complex Fourier Transforms

by Analysis of Formulas 7.5.10, 7.5.11

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1 Fundamental Definitions

Definition 1. *[Beat Frequency] When two waves of slightly different frequencies f_1 and f_2 interfere, they produce a periodic variation in amplitude known as beats. The beat frequency is defined as:*

$$f_{\text{beat}} = |f_2 - f_1| \tag{1}$$

This represents the number of amplitude modulations (beats) per unit time observed in the interference pattern.

Definition 2. *[Carrier Frequency] In modulated wave systems, the carrier frequency is the base frequency of the unmodulated wave that serves as the medium for transmitting information. For two interfering waves with frequencies f_1 and f_2, the carrier frequency is defined as:*

$$f_{\text{carrier}} = \frac{f_1 + f_2}{2} \tag{2}$$

This represents the average frequency around which the beat modulation occurs.

2 Original Spectral Relations

From the theory of Fourier transforms for real-valued functions, we have the following fundamental relations:

$$F(\lambda_2) - F(\lambda_1) = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{-i\lambda_2 t} - e^{-i\lambda_1 t}}{-i\,t} \tau(t)\ d\,t \tag{3}$$

$$\xi(\lambda_2) - \xi(\lambda_1) = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{-i\lambda_2 t} - e^{-i\lambda_1 t}}{-i\,t} \xi(t)\ d\,t \tag{4}$$

These relations differ only by the functions being transformed: $\tau(t)$ in (7.5.10) and $\xi(t)$ in (7.5.11).

3 Trigonometric Expansions

Using Euler's formula $e^{-i\lambda t} = \cos(\lambda\,t) - i\sin(\lambda\,t)$, we can expand the complex exponentials.

Lemma 3. *[Trigonometric Form of Complex Exponential Difference] The difference of complex exponentials can be written as:*

$$e^{-i\lambda_2 t} - e^{-i\lambda_1 t} = [\cos(\lambda_2\,t) - \cos(\lambda_1\,t)] - i\,[\sin(\lambda_2\,t) - \sin(\lambda_1\,t)] \tag{5}$$

Proof. Direct application of Euler's formula:

$$e^{-i\lambda_2 t} - e^{-i\lambda_1 t} = [\cos(\lambda_2\,t) - i\sin(\lambda_2\,t)] - [\cos(\lambda_1\,t) - i\sin(\lambda_1\,t)] \tag{6}$$

$$= [\cos(\lambda_2\,t) - \cos(\lambda_1\,t)] - i\,[\sin(\lambda_2\,t) - \sin(\lambda_1\,t)] \tag{7}$$

□

4 Sum-to-Product Transformations

The trigonometric differences can be simplified using sum-to-product identities.

Lemma 4. *[Sum-to-Product Identities] For any real numbers A and B:*

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{8}$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{9}$$

Proof. Using the angle addition formulas:

$$\cos A = \cos\left(\frac{A+B}{2} + \frac{A-B}{2}\right) = \cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{10}$$

$$\cos B = \cos\left(\frac{A+B}{2} - \frac{A-B}{2}\right) = \cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{11}$$

Subtracting: $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$.

Similarly for sine:

$$\sin A = \sin\left(\frac{A+B}{2} + \frac{A-B}{2}\right) = \sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{12}$$

$$\sin B = \sin\left(\frac{A+B}{2} - \frac{A-B}{2}\right) = \sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{13}$$

Subtracting: $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$. □

5 Beat Frequency and Carrier Frequency Analysis

Theorem 5. *[Beat Frequency Decomposition] The complex exponential difference can be expressed in terms of beat and carrier frequencies:*

$$e^{-i\lambda_2 t} - e^{-i\lambda_1 t} = -2\,i\sin\left(\frac{(\lambda_2-\lambda_1)\,t}{2}\right)e^{-i\frac{(\lambda_2+\lambda_1)t}{2}} \tag{14}$$

where:

- *The beat angular frequency is* $\omega_{\text{beat}} = \frac{\lambda_2-\lambda_1}{2}$ *(corresponding to beat frequency* $f_{\text{beat}} = \frac{|\lambda_2-\lambda_1|}{4\,\pi}$*)*
- *The carrier angular frequency is* $\omega_{\text{carrier}} = \frac{\lambda_2+\lambda_1}{2}$ *(corresponding to carrier frequency* $f_{\text{carrier}} = \frac{\lambda_2+\lambda_1}{4\,\pi}$*)*

Proof. Applying the sum-to-product identities to the trigonometric form:

$$\cos(\lambda_2 t) - \cos(\lambda_1 t) = -2\sin\left(\frac{(\lambda_2 + \lambda_1)\, t}{2}\right)\sin\left(\frac{(\lambda_2 - \lambda_1)\, t}{2}\right) \tag{15}$$

$$\sin(\lambda_2 t) - \sin(\lambda_1 t) = 2\cos\left(\frac{(\lambda_2 + \lambda_1)\, t}{2}\right)\sin\left(\frac{(\lambda_2 - \lambda_1)\, t}{2}\right) \tag{16}$$

Therefore:

$$[\cos(\lambda_2 t) - \cos(\lambda_1 t)] - i\,[\sin(\lambda_2 t) - \sin(\lambda_1 t)] \tag{17}$$

$$= -2 \sin\left(\frac{(\lambda_2 + \lambda_1)\, t}{2}\right) \sin\left(\frac{(\lambda_2 - \lambda_1)\, t}{2}\right) - i \cdot 2\cos\left(\frac{(\lambda_2 + \lambda_1)\, t}{2}\right)\sin\left(\frac{(\lambda_2 - \lambda_1)\, t}{2}\right) \tag{18}$$

$$= -2\sin\left(\frac{(\lambda_2 - \lambda_1)\, t}{2}\right)\left[\sin\left(\frac{(\lambda_2 + \lambda_1)\, t}{2}\right) + i\cos\left(\frac{(\lambda_2 + \lambda_1)\, t}{2}\right)\right] \tag{19}$$

Using the identity $\sin\theta + i\cos\theta = i\,(\cos\theta - i\sin\theta) = i\, e^{-i\theta}$:

$$\sin\left(\frac{(\lambda_2 + \lambda_1)\, t}{2}\right) + i\cos\left(\frac{(\lambda_2 + \lambda_1)\, t}{2}\right) = i\, e^{-i\frac{(\lambda_2 + \lambda_1)\, t}{2}} \tag{20}$$

Therefore:

$$e^{-i\lambda_2 t} - e^{-i\lambda_1 t} = -2\, i\sin\left(\frac{(\lambda_2 - \lambda_1)\, t}{2}\right) e^{-i\frac{(\lambda_2 + \lambda_1)\, t}{2}} \tag{21}$$

The beat frequency arises from the $\sin\left(\frac{(\lambda_2 - \lambda_1)\, t}{2}\right)$ term, which modulates the amplitude with angular frequency $\frac{|\lambda_2 - \lambda_1|}{2}$, corresponding to beat frequency $f_{\text{beat}} = \frac{|\lambda_2 - \lambda_1|}{4\,\pi}$.

The carrier frequency comes from the $e^{-i\frac{(\lambda_2 + \lambda_1)\, t}{2}}$ term, which oscillates with angular frequency $\frac{\lambda_2 + \lambda_1}{2}$, corresponding to carrier frequency $f_{\text{carrier}} = \frac{\lambda_2 + \lambda_1}{4\,\pi}$. □

Corollary 6. *[Beat Frequency Interpretation of Spectral Relations] The spectral relations (7.5.10) and (7.5.11) can be rewritten as:*

$$F(\lambda_2) - F(\lambda_1) = \frac{-i}{\pi} \lim_{T \to \infty} \int_{-T}^{T} \frac{\sin\left(\frac{(\lambda_2 - \lambda_1)\, t}{2}\right)}{t} e^{-i\frac{(\lambda_2 + \lambda_1)\, t}{2}}\, \tau(t)\; d\, t \tag{22}$$

$$\xi(\lambda_2) - \xi(\lambda_1) = \frac{-i}{\pi} \lim_{T \to \infty} \int_{-T}^{T} \frac{\sin\left(\frac{(\lambda_2 - \lambda_1)\, t}{2}\right)}{t} e^{-i\frac{(\lambda_2 + \lambda_1)\, t}{2}}\, \xi(t)\; d\, t \tag{23}$$

This form explicitly shows how the spectral difference depends on:

- *The beat envelope function* $\frac{\sin\left(\frac{(\lambda_2 - \lambda_1)\, t}{2}\right)}{t}$
- *The carrier oscillation* $e^{-i\frac{(\lambda_2 + \lambda_1)\, t}{2}}$

Proof. Substituting the beat frequency decomposition into the original formulas:

$$F(\lambda_2) - F(\lambda_1) = \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} \frac{-2\, i \sin\left(\frac{(\lambda_2 - \lambda_1)\, t}{2}\right) e^{-i\frac{(\lambda_2+\lambda_1)\, t}{2}}}{-i\, t} \tau(t)\ d t \tag{24}$$

$$= \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} \frac{2 \sin\left(\frac{(\lambda_2 - \lambda_1)\, t}{2}\right) e^{-i\frac{(\lambda_2+\lambda_1)\, t}{2}}}{t} \tau(t)\ d t \cdot (-i) \tag{25}$$

$$= \frac{-i}{\pi} \lim_{T \to \infty} \int_{-T}^{T} \frac{\sin\left(\frac{(\lambda_2 - \lambda_1)\, t}{2}\right)}{t} e^{-i\frac{(\lambda_2+\lambda_1)\, t}{2}} \tau(t)\ d t \tag{26}$$

The same derivation applies to (7.5.11). □

6 Physical Interpretation

The beat frequency decomposition reveals crucial physical insights about the spectral relations:

- **Beat Envelope**: The factor $\sin\left(\frac{(\lambda_2 - \lambda_1)\, t}{2}\right)$ creates an amplitude modulation envelope with beat frequency $f_{\text{beat}} = \frac{|\lambda_2 - \lambda_1|}{4\pi}$. This envelope determines how rapidly the interference pattern oscillates between constructive and destructive interference.

- **Carrier Wave**: The factor $e^{-i\frac{(\lambda_2+\lambda_1)\, t}{2}}$ represents the carrier wave oscillating at the average frequency $f_{\text{carrier}} = \frac{\lambda_2 + \lambda_1}{4\pi}$. This carrier provides the fundamental oscillation that is modulated by the beat envelope.

- **Spectral Resolution**: The $\frac{\sin\left(\frac{(\lambda_2 - \lambda_1)\, t}{2}\right)}{t}$ term in the integral acts as a frequency resolution kernel. As $|\lambda_2 - \lambda_1| \to 0$, this kernel approaches a delta function, providing perfect frequency resolution.

- **Time-Frequency Uncertainty**: The beat structure demonstrates the fundamental time-frequency uncertainty principle in Fourier analysis - better frequency resolution (smaller $|\lambda_2 - \lambda_1|$) requires longer integration times T.

This decomposition is particularly valuable in signal processing, spectroscopy, and quantum mechanics where understanding the interference between close frequencies is essential for proper interpretation of measured spectra.