Proof of the Triangular Type Covariance Factorization For Translation-Invariant Gaussian Process Kernels

BY STEPHEN CROWLEY

October 20, 2024

1 Proof: Kernel Spectral Representation

We will prove the following equality for a translation-invariant kernel $K(t-s)$ with even spectral density $S(\omega)$:

$$K(t-s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{i\omega(t-s)}\, d\omega = \int_{-\infty}^{\infty} h(t-u)\, h(s-u)\, du \tag{1}$$

where $h(t)$ is defined as:

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sqrt{S(\omega)}\, e^{i\omega t}\, d\omega \tag{2}$$

Proof. 1. Start with the spectral representation of the kernel:

$$K(t-s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{i\omega(t-s)}\, d\omega \tag{3}$$

2. Factor $\sqrt{S(\omega)}$:

$$K(t-s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\sqrt{S(\omega)}\, e^{i\omega t})\, (\sqrt{S(\omega)}\, e^{-i\omega s})\, d\omega \tag{4}$$

3. Define $h(t)$ as the inverse Fourier transform of $\sqrt{S(\omega)}$:

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sqrt{S(\omega)}\, e^{i\omega t}\, d\omega \tag{5}$$

4. By the Fourier transform pair property:

$$\sqrt{S(\omega)} = \int_{-\infty}^{\infty} h(\tau)\, e^{-i\omega\tau}\, d\tau \tag{6}$$

5. Substitute into the kernel expression:

$$K\,(t-s) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(\tau)\, e^{-i\omega(\tau-t)}\, d\,\tau \right) \times \left(\int_{-\infty}^{\infty} h(\sigma)\, e^{-i\omega(\sigma-s)}\, d\,\sigma \right) d\,\omega \tag{7}$$

6. Simplify and change order of integration:

$$K\,(t-s) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau)\, h(\sigma) \left(\int_{-\infty}^{\infty} e^{i\omega(t-\tau)}\, e^{-i\omega(s-\sigma)}\, d\,\omega \right) d\,\tau\, d\,\sigma \tag{8}$$

7. The ω integral is a Dirac delta function:

$$\int_{-\infty}^{\infty} e^{i\omega(t-\tau)}\, e^{-i\omega(s-\sigma)}\, d\,\omega = 2\,\pi\,\delta\,((t-\tau)-(s-\sigma)) \tag{9}$$

8. Simplify:

$$K\,(t-s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau)\, h(\sigma)\, \delta\,((t-\tau)-(s-\sigma))\, d\,\tau\, d\,\sigma \tag{10}$$

9. Evaluate the delta function:

$$K\,(t-s) = \int_{-\infty}^{\infty} h(\tau)\, h\,(\tau+(s-t))\, d\,\tau \tag{11}$$

10. Change of variables ($u = \tau + (s-t)$):

$$K\,(t-s) = \int_{-\infty}^{\infty} h\,(t-u)\, h\,(s-u)\, d\,u \tag{12}$$

This completes the proof $\square$