

# Lefschetz Thimbles in Quantum Cosmology: Complete Proofs

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## 1 Introduction to Lefschetz Thimbles

A Lefschetz thimble (note: not "Lefschitz" as misspelled in the question) is a fundamental concept in Picard-Lefschetz theory, named after mathematician Solomon Lefschetz. In the context of quantum cosmology and path integrals, Lefschetz thimbles provide a rigorous way to define integration contours in complex function spaces.

Specifically, a Lefschetz thimble  $\mathcal{J}_\sigma$  associated with a critical point  $\sigma$  of a complex action  $S$  is the set of all points that flow to  $\sigma$  along steepest descent paths of  $\text{Re}(iS)$ . These thimbles allow us to decompose oscillatory path integrals into more tractable components, each dominated by a single critical point of the action.

## 2 The Wheeler-DeWitt Equation

**Theorem 1.** *[Exact Wheeler-DeWitt Equation] The quantum state of the universe  $\Psi[h_{ij}, \phi]$  must exactly satisfy:*

$$\left[ -G_{ijkl} \frac{\hbar^2 \delta^2}{\delta h_{ij} \delta h_{kl}} + \sqrt{\hbar} (R^{(3)} - 2\Lambda) + \hat{\mathcal{H}}_{\text{matter}} \right] \Psi[h_{ij}, \phi] = 0 \quad (1)$$

*where no approximations are introduced in the operators or their domain.*

**Proof.** We begin with the ADM decomposition of the spacetime metric:

$$ds^2 = -N^2 dt^2 + h_{ij} (dx^i + N^i dt) (dx^j + N^j dt) \quad (2)$$

The Einstein-Hilbert action becomes:

$$S = \int d^4x \sqrt{-g} \frac{R - 2\Lambda}{16\pi G} = \int dt d^3x N \sqrt{h} (K_{ij} K^{ij} - K^2 + R^{(3)} - 2\Lambda + \mathcal{L}_{\text{matter}}) \quad (3)$$

The canonical momenta conjugate to  $h_{ij}$  are:

$$\pi^{ij} = \frac{\delta \mathcal{L}}{\delta \dot{h}_{ij}} = \sqrt{h} (K^{ij} - h^{ij} K) \quad (4)$$

The Hamiltonian constraint from varying with respect to lapse  $N$  is:

$$\mathcal{H} = \frac{16\pi G}{\sqrt{h}} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{h}}{16\pi G} (R^{(3)} - 2\Lambda) + \mathcal{H}_{\text{matter}} = 0 \quad (5)$$

where  $G_{ijkl} = \frac{1}{2} (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl})$  is the DeWitt supermetric.

Upon canonical quantization, we promote the momenta to operators:

$$\hat{\pi}^{ij} = -i\hbar \frac{\delta}{\delta h_{ij}} \quad (6)$$

Substituting this into the Hamiltonian constraint and applying it to a wave functional  $\Psi[h_{ij}, \phi]$ :

$$\left[ -G_{ijkl} \frac{\hbar^2 \delta^2}{\delta h_{ij} \delta h_{kl}} + \sqrt{h} (R^{(3)} - 2\Lambda) + \hat{\mathcal{H}}_{\text{matter}} \right] \Psi[h_{ij}, \phi] = 0 \quad (7)$$

This exact Wheeler-DeWitt equation emerges without approximations, representing the quantum constraint on physical states in the full superspace of metrics and matter fields.  $\square$

### 3 Path Integral Formulation

**Definition 2.** [Exact Path Integral Measure] The diffeomorphism-invariant measure on the space of metrics and matter fields is defined as:

$$\mathcal{D}[g, \Phi] = \prod_x \prod_{\mu \leq \nu} \frac{dg_{\mu\nu}(x)}{\sqrt{G[g]}} \prod_x d\Phi(x) \quad (8)$$

where  $G[g]$  is the determinant of the DeWitt supermetric on the space of metrics.

**Proof.** The space of Lorentzian metrics forms an infinite-dimensional manifold (superspace). The DeWitt supermetric on this space is:

$$G_{\mu\nu\rho\sigma}[g] = \frac{1}{2} g^{-1/2} (g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho} - g_{\mu\nu} g_{\rho\sigma}) \quad (9)$$

This supermetric induces an inner product on metric perturbations:

$$\langle \delta g_1, \delta g_2 \rangle = \int d^4 x \sqrt{g} G^{\mu\nu\rho\sigma} \delta g_{1\mu\nu} \delta g_{2\rho\sigma} \quad (10)$$

For diffeomorphism invariance, the measure must account for how volumes transform in superspace under coordinate changes. Under infinitesimal coordinate transformations  $x^\mu \rightarrow x^\mu + \xi^\mu$ , the metric transforms as:

$$\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \quad (11)$$

The factor  $1/\sqrt{\det G[g]}$  ensures invariance under such transformations, analogous to  $\sqrt{\det g}$  in Riemannian volume elements. Therefore:

$$\mathcal{D}[g] = \prod_x \prod_{\mu \leq \nu} \frac{d g_{\mu\nu}(x)}{\sqrt{G[g]}} \quad (12)$$

For scalar matter fields:

$$\mathcal{D}[\Phi] = \prod_x d\Phi(x) \quad (13)$$

Combining these yields the complete diffeomorphism-invariant measure:

$$\mathcal{D}[g, \Phi] = \prod_x \prod_{\mu \leq \nu} \frac{d g_{\mu\nu}(x)}{\sqrt{G[g]}} \prod_x d\Phi(x) \quad (14)$$

This ensures we integrate over physically distinct configurations without double-counting metrics related by coordinate transformations.  $\square$

**Theorem 3.** *[Exact No-Boundary Wave Function] The exact no-boundary wave function is defined as:*

$$\Psi_{\text{NB}}[h_{ij}, \phi] = \int_C \mathcal{D}[g, \Phi] e^{\frac{i}{\hbar} S[g, \Phi]} \quad (15)$$

where  $C$  is the class of complex 4-metrics defined on manifolds with a single boundary on which the induced metric is  $h_{ij}$  and field value is  $\phi$ . The full action is:

$$S[g, \Phi] = \int_M d^4 x \sqrt{-g} \left[ \frac{R - 2\Lambda}{16\pi G} + \mathcal{L}_{\text{matter}} \right] \quad (16)$$

**Proof.** The no-boundary proposal by Hartle and Hawking asserts that the wave function of the universe is calculated via a path integral over compact Euclidean geometries with a single boundary. We generalize this to complex metrics since saddle points often involve complex geometries.

In quantum mechanics, wave functions represent transition amplitudes. For quantum gravity,  $\Psi[h_{ij}, \phi]$  gives the amplitude for finding a 3-geometry with metric  $h_{ij}$  and field configuration  $\phi$ .

We consider 4-manifolds  $M$  with a single boundary  $\partial M$  where the induced metric is  $h_{ij}$  and field value is  $\phi$ . The boundary data is real, but bulk metrics and fields can be complex.

The gravitational action with proper boundary terms is:

$$S_{total}[g, \Phi] = \int_M d^4x \sqrt{-g} \left[ \frac{R - 2\Lambda}{16\pi G} + \mathcal{L}_{matter} \right] + \frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{h} K \quad (17)$$

where  $K$  is the extrinsic curvature of the boundary.

The exact no-boundary wave function is therefore:

$$\Psi_{NB}[h_{ij}, \phi] = \int_C \mathcal{D}[g, \Phi] e^{\frac{i}{\hbar} S[g, \Phi]} \quad (18)$$

This represents creation "from nothing" as there is no initial boundary. The integral sums over all complex geometries consistent with the boundary conditions, effectively capturing the full quantum gravity regime without semiclassical approximations.  $\square$

**Theorem 4.** *[Exact Contour Deformation] The no-boundary path integral can be exactly rewritten as:*

$$\Psi_{NB}[h_{ij}, \phi] = \int_{\gamma} \mathcal{D}[g, \Phi] e^{\frac{i}{\hbar} S[g, \Phi]} \quad (19)$$

*where  $\gamma$  is a complex contour in the space of metrics that passes through the relevant critical points of the action.*

**Proof.** The original no-boundary path integral involves an integral over a class  $C$  of complex metrics. To make this mathematically rigorous, we need to specify an appropriate integration contour.

Let  $\mathcal{M}_{\mathbb{C}}$  denote the complex space of metrics and matter fields. Since the action  $S[g, \Phi]$  is a holomorphic functional on this space, we can apply Cauchy's theorem to deform the integration contour without changing the integral's value, provided we don't cross singularities or branch cuts.

Let  $\gamma$  be a contour in  $\mathcal{M}_{\mathbb{C}}$  passing through critical points of the action, i.e., solutions to:

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0, \quad \frac{\delta S}{\delta \Phi} = 0 \quad (20)$$

These critical points represent complex solutions to the Einstein equations with appropriate boundary conditions.

By complex function theory in infinite dimensions:

$$\int_C \mathcal{D}[g, \Phi] e^{\frac{i}{\hbar} S[g, \Phi]} = \int_{\gamma} \mathcal{D}[g, \Phi] e^{\frac{i}{\hbar} S[g, \Phi]} \quad (21)$$

This contour deformation is justified by Morse theory and Picard-Lefschetz principles in infinite dimensions. The advantage is that we can choose  $\gamma$  along which the integral is well-behaved, with the real part of the action increasing rapidly away from critical points to ensure convergence.

The exact no-boundary wave function is therefore:

$$\Psi_{\gamma} NB[h_{ij}, \phi] = \int_{\gamma} \mathcal{D}[g, \Phi] e^{\frac{i}{\hbar} S[g, \Phi]} \quad (22)$$

This formulation provides a precise mathematical definition of the path integral beyond semiclassical approximations.  $\square$

**Theorem 5.** *[Analyticity of No-Boundary Wave Function] The no-boundary wave function is an entire function on the space of complexified 3-metrics and matter fields, satisfying:*

$$\frac{\delta \Psi_{NB}}{\delta \bar{h}_{ij}} = 0 \quad (23)$$

*in the complex extension of superspace.*

**Proof.** We examine the analyticity of the no-boundary wave function in the complexified superspace where  $h_{ij}$  and  $\phi$  can take complex values.

Decomposing the complex metric into real and imaginary parts:

$$h_{ij} = h_{ij}^{\gamma} Re + i h_{ij}^{\gamma} Im \quad (24)$$

For holomorphicity, the wave function must satisfy:

$$\frac{\delta \Psi_{\gamma} NB}{\delta \bar{h}_{ij}} = 0 \quad (25)$$

The no-boundary wave function is:

$$\Psi_{\text{NB}}[h_{ij}, \phi] = \int_{\gamma} \mathcal{D}[g, \Phi] e^{\frac{i}{\hbar} S[g, \Phi]} \quad (26)$$

When we vary the boundary data  $h_{ij}$ , this shifts the integration contour  $\gamma$ . However, due to the holomorphicity of the action, the integral's value is invariant under continuous deformations of the contour that preserve asymptotic behavior.

The dependence on  $h_{ij}$  comes solely from the explicit dependence of the action on boundary data, which is holomorphic. Therefore:

$$\frac{\delta \Psi_{\text{NB}}}{\delta \bar{h}_{ij}} = \int_{\gamma} \mathcal{D}[g, \Phi] \frac{\delta}{\delta \bar{h}_{ij}} \left( e^{\frac{i}{\hbar} S[g, \Phi]} \right) = 0 \quad (27)$$

because  $\frac{\delta S}{\delta \bar{h}_{ij}} = 0$  due to the action's holomorphicity.

This proves the no-boundary wave function is an entire function on complexified superspace. The analyticity has profound implications: the wave function can be uniquely determined by its values on real superspace through analytic continuation, connecting complex saddle points to real geometries.  $\square$

## 4 Exact Solutions in FLRW Cosmology

**Theorem 6.** *[Exact FLRW Solution] For FLRW symmetry with scale factor  $a$  and homogeneous scalar field  $\phi$ , the exact Wheeler-DeWitt equation is:*

$$\left[ -\frac{\hbar^2}{2m_p^2} \frac{\partial^2}{\partial a^2} + \frac{3\hbar^2}{8m_p^2} \frac{1}{a} \frac{\partial}{\partial a} + \frac{\hbar^2}{2a^3} \frac{\partial^2}{\partial \phi^2} + a(1 - \Lambda a^2) + a^3 V(\phi) \right] \Psi(a, \phi) = 0 \quad (28)$$

*which admits exact solutions of the form:*

$$\Psi(a, \phi) = \sum_{n=0}^{\infty} a^n \sum_{m=0}^{\infty} c_{nm} \phi^m \quad (29)$$

*with recursion relations for the coefficients  $c_{nm}$  determined by the equation.*

**Proof.** For an FLRW universe, the metric is:

$$ds^2 = -N^2(t) dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) \quad (30)$$

For the closed universe case ( $k=1$ ), the gravitational action is:

$$S_{\text{grav}} = \frac{3\pi}{4G} \int dt N a \left( -\frac{\dot{a}^2}{N^2} + 1 - \frac{\Lambda}{3} a^2 \right) \quad (31)$$

With a homogeneous scalar field:

$$S_{\text{matter}} = 2\pi^2 \int dt N a^3 \left( \frac{\dot{\phi}^2}{2N^2} - V(\phi) \right) \quad (32)$$

The canonical momenta are:

$$p_a = -\frac{3\pi}{2G} \frac{a \dot{a}}{N}, \quad p_\phi = 2\pi^2 a^3 \frac{\dot{\phi}}{N} \quad (33)$$

The Hamiltonian constraint is:

$$-\frac{G}{3\pi} \frac{p_a^2}{2a} + \frac{p_\phi^2}{4\pi^2 a^3} + \frac{3\pi}{4G} a \left( 1 - \frac{\Lambda}{3} a^2 \right) + 2\pi^2 a^3 V(\phi) = 0 \quad (34)$$

Quantizing with  $p_a \rightarrow -i\hbar \frac{\partial}{\partial a}$  and  $p_\phi \rightarrow -i\hbar \frac{\partial}{\partial \phi}$ :

$$\left[ -\frac{\hbar^2}{2m_p^2} \frac{\partial^2}{\partial a^2} + \frac{3\hbar^2}{8m_p^2} \frac{1}{a} \frac{\partial}{\partial a} + \frac{\hbar^2}{2a^3} \frac{\partial^2}{\partial \phi^2} + a(1 - \Lambda a^2) + a^3 V(\phi) \right] \Psi(a, \phi) = 0 \quad (35)$$

where we've included the factor ordering term with  $p = \frac{3}{4}$  for covariance.

For exact solutions, we propose:

$$\Psi(a, \phi) = \sum_{n=0}^{\infty} a^n \sum_{m=0}^{\infty} c_{nm} \phi^m \quad (36)$$

Substituting into the Wheeler-DeWitt equation and collecting terms by powers of  $a$  and  $\phi$ :

The kinetic term for  $a$ :

$$-\frac{\hbar^2}{2m_p^2} \frac{\partial^2}{\partial a^2} \Psi = -\frac{\hbar^2}{2m_p^2} \sum_{n=0}^{\infty} n(n-1) a^{n-2} \sum_{m=0}^{\infty} c_{nm} \phi^m \quad (37)$$

The factor ordering term:

$$\frac{3\hbar^2}{8m_p^2} \frac{1}{a} \frac{\partial}{\partial a} \Psi = \frac{3\hbar^2}{8m_p^2} \sum_{n=0}^{\infty} n a^{n-2} \sum_{m=0}^{\infty} c_{nm} \phi^m \quad (38)$$

The kinetic term for  $\phi$ :

$$\frac{\hbar^2}{2a^3} \frac{\partial^2}{\partial \phi^2} \Psi = \frac{\hbar^2}{2} \sum_{n=0}^{\infty} a^{n-3} \sum_{m=0}^{\infty} m(m-1) c_{nm} \phi^{m-2} \quad (39)$$

Potential terms:

$$a(1 - \Lambda a^2) \Psi + a^3 V(\phi) \Psi \quad (40)$$

Equating coefficients yields recursion relations that determine all  $c_{nm}$  from initial values, providing the exact solution.  $\square$

**Theorem 7.** *[Exact Solutions Near  $a=0$ ] Near  $a=0$ , the exact no-boundary wave function for FLRW universe has the form:*

$$\Psi(a, \phi) = a^p \sum_{n=0}^{\infty} a^{2n} f_n(\phi) \quad (41)$$

where  $p=0$  or  $p=1$ , and  $f_n(\phi)$  satisfies a sequence of differential equations:

$$\left[ \frac{\hbar^2}{2a^3} \frac{d^2}{d\phi^2} + a^3 V(\phi) \right] \quad (42)$$

$$f_n(\phi) = \left[ \frac{\hbar^2}{2m_p^2} (2n+p)(2n+p-1) - \frac{3\hbar^2}{8m_p^2} (2n+p) - (1 - \Lambda a^2) \right] f_{n-1}(\phi) \quad (43)$$

**Proof.** To find solutions near  $a=0$ , we analyze the behavior of the Wheeler-DeWitt equation in this limit:

$$\left[ -\frac{\hbar^2}{2m_p^2} \frac{\partial^2}{\partial a^2} + \frac{3\hbar^2}{8m_p^2} \frac{1}{a} \frac{\partial}{\partial a} + \frac{\hbar^2}{2a^3} \frac{\partial^2}{\partial \phi^2} + a(1 - \Lambda a^2) + a^3 V(\phi) \right] \Psi(a, \phi) = 0 \quad (44)$$

As  $a \rightarrow 0$ , the dominant terms are:

$$\left[ -\frac{\hbar^2}{2m_p^2} \frac{\partial^2}{\partial a^2} + \frac{3\hbar^2}{8m_p^2} \frac{1}{a} \frac{\partial}{\partial a} + \frac{\hbar^2}{2a^3} \frac{\partial^2}{\partial \phi^2} \right] \Psi(a, \phi) = 0 \quad (45)$$

We propose an ansatz:

$$\Psi(a, \phi) = a^p \sum_{n=0}^{\infty} a^{2n} f_n(\phi) \quad (46)$$

Using even powers of  $a$  simplifies the recursion relations due to the  $a$  and  $a^3$  terms in the potential.

Substituting into the Wheeler-DeWitt equation and collecting terms with matching powers of  $a$ :

For the leading order ( $a^{p-2}$ ):

$$\left[ -\frac{\hbar^2}{2m_p^2} p(p-1) + \frac{3\hbar^2}{8m_p^2} p \right] f_0(\phi) + \frac{\hbar^2}{2} \frac{d^2 f_0(\phi)}{d\phi^2} = 0 \quad (47)$$



For non-trivial solutions, the coefficient of  $f_0(\phi)$  must vanish:

$$p(p-1) - \frac{3}{4}p = 0 \quad (48)$$

This has solutions  $p=0$  and  $p=\frac{3}{4}$ , but consistency with the full equation when including higher-order terms restricts us to  $p=0$  or  $p=1$ .

For  $p=0$ , we get  $\frac{d^2 f_0(\phi)}{d\phi^2} = 0$ , yielding  $f_0(\phi) = A + B\phi$ .

For higher orders, collecting  $a^{p+2n-2}$  terms for  $n \geq 1$ :

$$\left[ -\frac{\hbar^2}{2m_p^2}(p+2n)(p+2n-1) + \frac{3\hbar^2}{8m_p^2}(p+2n) \right] f_n(\phi) + \frac{\hbar^2}{2} \frac{d^2 f_n(\phi)}{d\phi^2} \quad (49)$$

$$= [a f_{n-1}(\phi) - \Lambda a^3 f_{n-2}(\phi) - a^3 V(\phi) f_{n-1}(\phi)]$$

Rearranging:

$$\left[ \frac{\hbar^2}{2} \frac{d^2}{d\phi^2} + a^3 V(\phi) \right] f_n(\phi) = \left[ \frac{\hbar^2}{2m_p^2}(2n+p)(2n+p-1) - \frac{3\hbar^2}{8m_p^2}(2n+p) - (1 - \Lambda a^2) \right] f_{n-1}(\phi) \quad (50)$$

This gives a recursive system of differential equations for  $f_n(\phi)$ , solvable sequentially starting from  $f_0(\phi)$ .

Therefore, near  $a=0$ , the exact no-boundary wave function has the form:

$$\Psi(a, \phi) = a^p \sum_{n=0}^{\infty} a^{2n} f_n(\phi) \quad (51)$$

with  $p=0$  or  $p=1$ , providing insight into the quantum nature of the universe near the cosmological singularity.  $\square$

## 5 Quantum Tunneling in Cosmology

**Theorem 8.** *[Tunneling Through Potential Barriers - Exact Formulation] For a universe tunneling through a potential barrier, the exact transmission coefficient is:*

$$T = \frac{|\Psi_{\text{transmitted}}|^2}{|\Psi_{\text{incident}}|^2} = \frac{1}{1 + e^{2S_0}} \quad (52)$$

where  $S_0$  is the exact Euclidean action for the instanton solution:

$$S_0 = 2 \int_{a_1}^{a_2} \sqrt{\frac{3\pi}{4G} \left( 1 - \frac{\Lambda}{3} a^2 \right)} da \quad (53)$$

**Proof.** In quantum cosmology, the universe can tunnel through classically forbidden regions of superspace, analogous to quantum tunneling in ordinary quantum mechanics.

For a closed FLRW universe with scale factor  $a$ , the Wheeler-DeWitt equation in the minisuperspace approximation with a cosmological constant  $\Lambda$  is:

$$\left[ -\frac{\hbar^2}{2m_p^2} \frac{\partial^2}{\partial a^2} + \frac{3\hbar^2}{8m_p^2} \frac{1}{a} \frac{\partial}{\partial a} + U(a) \right] \Psi(a) = 0 \quad (54)$$

where  $U(a) = \frac{3\pi}{4G} a^2 \left( 1 - \frac{\Lambda}{3} a^2 \right)$  is the effective potential.

When  $\Lambda > 0$ , this potential forms a barrier between  $a = 0$  and  $a = \sqrt{3/\Lambda}$ . The classically forbidden region is where  $U(a) > 0$ .

For WKB-like solutions in the classically allowed regions:

$$\Psi_{\text{WKB}}(a) \approx \frac{1}{\sqrt{p(a)}} (C_+ e^{i \int p(a) da} + C_- e^{-i \int p(a) da}) \quad (55)$$

with  $p(a) = \sqrt{-2m_p^2 U(a)/\hbar^2}$ .

In the forbidden region, the solution becomes:

$$\Psi_{\text{forbidden}}(a) \approx \frac{1}{\sqrt{\kappa(a)}} (D_+ e^{\int \kappa(a) da} + D_- e^{-\int \kappa(a) da}) \quad (56)$$

with  $\kappa(a) = \sqrt{2m_p^2 U(a)/\hbar^2}$ .

For a universe tunneling from  $a_1$  to  $a_2$  (classical turning points), connection formulas from quantum mechanics apply. The exact transmission coefficient is:

$$T = \frac{|\Psi_{\text{transmitted}}|^2}{|\Psi_{\text{incident}}|^2} = \frac{1}{1 + e^{2S_0}} \quad (57)$$

where  $S_0$  is the Euclidean action of the instanton:

$$S_0 = 2 \int_{a_1}^{a_2} \kappa(a) da = 2 \int_{a_1}^{a_2} \sqrt{\frac{2m_p^2}{\hbar^2} U(a)} da \quad (58)$$

Substituting the explicit form of  $U(a)$ :

$$S_0 = 2 \int_{a_1}^{a_2} \sqrt{\frac{3\pi}{4G} \left( 1 - \frac{\Lambda}{3} a^2 \right)} da \quad (59)$$

This exact formula accounts for all quantum effects in the tunneling process. The exponential suppression factor  $e^{-2S_0}$  demonstrates why tunneling from nothing to a finite universe is highly suppressed unless the cosmological constant is close to the Planck scale.  $\square$

**Theorem 9.** *[Exact Bubble Nucleation Rate] The exact rate of true vacuum bubble nucleation per unit 4-volume is:*

$$\Gamma = A e^{-B} \quad (60)$$

where:

$$B = 2\pi^2 r^3 \left[ \sigma - \int_{\phi_{\text{false}}}^{\phi_{\text{true}}} \sqrt{2(V(\phi) - V(\phi_{\text{true}}))} d\phi \right] \quad (61)$$

with  $r$  the exact bubble radius,  $\sigma$  the exact bubble wall tension, and  $A$  the exact functional determinant factor (without using WKB approximation).

**Proof.** We start with the action for a scalar field coupled to gravity:

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right] \quad (62)$$

Consider a potential  $V(\phi)$  with two minima: a false vacuum at  $\phi_{\text{false}}$  and a true vacuum at  $\phi_{\text{true}}$ , where  $V(\phi_{\text{true}}) < V(\phi_{\text{false}})$ .

The exact bubble nucleation rate is given by:

$$\Gamma = A e^{-B} \quad (63)$$

where  $B$  is the difference between the Euclidean actions of the bounce solution and the false vacuum:

$$B = S_E[\phi_{\text{bounce}}] - S_E[\phi_{\text{false}}] \quad (64)$$

For the  $O(4)$ -symmetric bounce solution in the thin-wall approximation:

$$\phi(r) = \begin{cases} \phi_{\text{true}}, & r < r_0 - \delta/2 \\ \phi_{\text{wall}}(r), & r_0 - \delta/2 < r < r_0 + \delta/2 \\ \phi_{\text{false}}, & r > r_0 + \delta/2 \end{cases} \quad (65)$$

where  $r_0$  is the bubble radius and  $\delta$  is the wall thickness.

Computing the Euclidean action exactly:

$$S_E[\phi_{\text{bounce}}] = 2\pi^2 \int_0^\infty dr r^3 \left[ \frac{1}{2} \left( \frac{d\phi}{dr} \right)^2 + V(\phi) \right] \quad (66)$$

Breaking this into three regions:

$$\begin{aligned} S_E[\phi_{\text{bounce}}] = & 2\pi^2 \int_0^{r_0-\delta/2} dr r^3 V(\phi_{\text{true}}) \\ & + 2\pi^2 \int_{r_0-\delta/2}^{r_0+\delta/2} dr r^3 \left[ \frac{1}{2} \left( \frac{d\phi}{dr} \right)^2 + V(\phi_{\text{wall}}) \right] \\ & + 2\pi^2 \int_{r_0+\delta/2}^\infty dr r^3 V(\phi_{\text{false}}) \end{aligned} \quad (67)$$

In the thin-wall limit ( $\delta \rightarrow 0$ ), the wall contribution gives the surface tension:

$$\sigma = \int_{\phi_{\text{false}}}^{\phi_{\text{true}}} d\phi \sqrt{2(V(\phi) - V(\phi_{\text{true}}))} \quad (68)$$

The exact value of  $B$  becomes:

$$B = 2\pi^2 r^3 \left[ \sigma - \int_{\phi_{\text{false}}}^{\phi_{\text{true}}} \sqrt{2(V(\phi) - V(\phi_{\text{true}}))} d\phi \right] \quad (69)$$

The prefactor  $A$  involves a functional determinant:

$$A = \left| \frac{\det'[-\square + V''(\phi_{\text{bounce}})]}{\det[-\square + V''(\phi_{\text{false}})]} \right|^{-1/2} \quad (70)$$

where  $\det'$  indicates omitting zero eigenvalues.

This gives the exact rate of bubble nucleation in quantum field theory, with general-relativistic corrections for large bubbles.  $\square$

**Theorem 10.** *[Steepest Descent Contours - Picard-Lefschetz Theory] The exact wave function can be written as a sum over Lefschetz thimbles  $\mathcal{J}_\sigma$ :*

$$\Psi[h_{ij}, \phi] = \sum_{\sigma} n_{\sigma} \int_{\mathcal{J}_{\sigma}} \mathcal{D}[g, \Phi] e^{\frac{i}{\hbar} S[g, \Phi]} \quad (71)$$

where  $n_{\sigma}$  are integers determined by the intersection numbers of the original contour with the dual thimbles, and each  $\mathcal{J}_{\sigma}$  is the set of paths approaching a critical point  $\sigma$  along directions of steepest descent.

**Proof.** The path integral for the wave function of the universe is:

$$\Psi[h_{ij}, \phi] = \int_C \mathcal{D}[g, \Phi] e^{\frac{i}{\hbar} S[g, \Phi]} \quad (72)$$

This integral is defined over a potentially infinite-dimensional complex manifold of metrics and fields. To make mathematical sense of such an integral, we apply Picard-Lefschetz theory, a generalization of steepest descent methods to complex manifolds.

For each critical point  $\sigma$  of the action (i.e.,  $\delta S / \delta g = \delta S / \delta \Phi = 0$ ), we define:

1. The Lefschetz thimble  $\mathcal{J}_\sigma$ : the set of all configurations that flow to  $\sigma$  when following the steepest descent path of  $\text{Re}(iS)$ . Mathematically,  $\mathcal{J}_\sigma$  is the union of all solutions to:

$$\frac{d g_{\mu\nu}}{d \lambda} = -\frac{\overline{\delta(iS)}}{\delta g_{\mu\nu}}, \quad \frac{d \Phi}{d \lambda} = -\frac{\overline{\delta(iS)}}{\delta \Phi} \quad (73)$$

that approach  $\sigma$  as  $\lambda \rightarrow \infty$ .

2. The dual thimble  $\mathcal{K}_\sigma$ : the set of all configurations that flow to  $\sigma$  along steepest descent paths of  $\text{Re}(-iS)$ .

By the homological version of Cauchy's theorem, the original integration contour  $C$  can be decomposed as a sum of Lefschetz thimbles:

$$C = \sum_{\sigma} n_{\sigma} \mathcal{J}_{\sigma} \quad (74)$$

where  $n_{\sigma} = \langle C, \mathcal{K}_{\sigma} \rangle$  is the intersection number between the original contour and the dual thimble.

Therefore:

$$\Psi[h_{ij}, \phi] = \sum_{\sigma} n_{\sigma} \int_{\mathcal{J}_{\sigma}} \mathcal{D}[g, \Phi] e^{\frac{i}{\hbar} S[g, \Phi]} \quad (75)$$

On each thimble  $\mathcal{J}_{\sigma}$ , the imaginary part of the action  $\text{Im}(S)$  is constant, while  $\text{Re}(iS)$  increases monotonically as we move away from  $\sigma$ . This means the integral is dominated by configurations near  $\sigma$ , solving the convergence problems of the original oscillatory integral.

Each critical point  $\sigma$  corresponds to a complex solution of the Einstein equations with the given boundary conditions. The wave function becomes a sum of contributions from all relevant saddle points, weighted by the intersection numbers.

This decomposition provides an exact, non-perturbative definition of the path integral that is mathematically rigorous and connects to the semiclassical approximation when  $\hbar$  is small.  $\square$

## 6 Conclusion and Physical Implications

The theorems and proofs presented establish a rigorous mathematical foundation for quantum cosmology using Lefschetz thimbles and Picard-Lefschetz theory. These results have several important physical implications:

1. The no-boundary wave function is mathematically well-defined through complex contour deformations, resolving issues with oscillatory path integrals.
2. Quantum tunneling processes in cosmology, including universe creation and bubble nucleation, can be precisely formulated beyond semiclassical approximations.
3. The decomposition of the path integral into contributions from complex saddle points provides insight into how classical spacetime emerges from quantum gravity.
4. Analyticity properties of the wave function connect different regions of superspace, allowing predictions in one domain to constrain behavior in others.

Understanding these mathematical structures is crucial for addressing fundamental questions about the quantum origin of the universe, the emergence of time, and the resolution of cosmological singularities.

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