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Primes and the Lambert W function

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ABSTRACT:

The Lambert W function, implicitly defined by $W(x)e^{W(x)} = x$, is a “new” special function that has recently been the subject of an extended upsurge in interest and applications. In this note, I point out that the Lambert W function can also be used to gain a new perspective on the distribution of primes.

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KEYWORDS: Lambert W function; prime counting function; n 'th prime.

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1 Introduction

The Lambert W function, implicitly defined by $W(x)e^{W(x)} = x$, has a long and quite convoluted 250-year history, but only recently has it become common to view this function as one of the standard special functions [1]. Applications range widely [1, 2], from combinatorics (for instance in the enumeration of rooted trees) [1], to delay differential equations [1], to falling objects subject to linear drag [3], to the evaluation of the numerical constant in Wien's displacement law [4, 5], to quantum statistics [6], to constructing the “tortoise” coordinate for Schwarzschild black holes [7], *etcetera*. In this brief note I will indicate some apparently new applications of the Lambert W function to the distribution of primes, specifically to the prime counting function $\pi(x)$ and estimating the n 'th prime p_n .

2 The prime counting function $\pi(x)$

2.1 Upper bound

Theorem 1 *The prime counting function $\pi(x)$ satisfies*

$$\pi(x) < \frac{x}{W(x)} = e^{W(x)}; \quad (\forall x \geq 0). \quad (2.1)$$

Proof: First note that $x \geq p_{\pi(x)}$. Second recall the standard result that $p_n > n \ln n$ for $n \geq 1$. (See Rosser [8], or any standard reference book on prime numbers, for example [9, 10].) Then $x \geq p_{\pi(x)} > \pi(x) \ln \pi(x)$, so we have $x > \pi(x) \ln \pi(x)$. Invert, noting that the RHS is monotone increasing, to see that $\pi(x) < x/W(x)$, certainly for $\pi(x) \geq 1$ (corresponding to $x \geq 2$). Then explicitly check validity of the inequality on the domain $x \in [0, 2)$. Finally, use the definition of the Lambert W function to note $x/W(x) = e^{W(x)}$.

2.2 Asymptotics

Theorem 2 *The prime number theorem, $\pi(x) \sim x/\ln x$, is equivalent to*

$$\pi(x) \sim \frac{x}{W(x)} = e^{W(x)}; \quad (x \rightarrow \infty). \quad (2.2)$$

Proof: Trivial. Note that asymptotically $W(x) \sim \ln x$. (The only potential subtlety is that we are using the principal branch of the W function, denoted $W_0(x)$ whenever there is any risk of confusion [1].)

So we have *both* a strict upper bound on $\pi(x)$ *and* an asymptotic equality. See figure 1.

2.3 Lower bound

When it comes to developing an analogous lower bound on $\pi(x)$ the situation is more subtle. Consider the well-known upper bound on p_n [9–11]:

$$p_n < n(\ln n + \ln \ln n) = n \ln(n \ln n); \quad (n \geq 6). \quad (2.3)$$

Using logic similar to that of Theorem 1, we first note that $x < p_{\pi(x)+1}$, so it is easy to convert the above into the inequality

$$x < p_{\pi(x)+1} < [\pi(x) + 1] \ln\{[\pi(x) + 1] \ln[\pi(x) + 1]\}; \quad (\pi(x) \geq 5). \quad (2.4)$$

The function $z \rightarrow z \ln(z \ln z)$ is monotonic increasing on $z \in (1, \infty)$, and maps onto the range $(-\infty, \infty)$. So the inverse function $U(x)$, implicitly defined by

$$U(x) \ln\{U(x) \ln U(x)\} = x \quad (2.5)$$

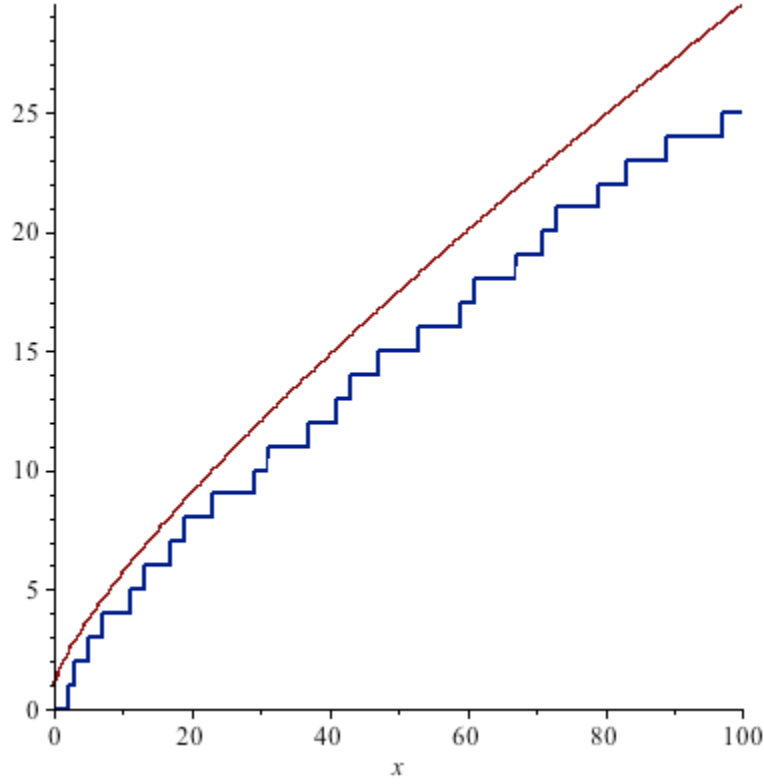


Figure 1. Prime counting function $\pi(x)$ and its upper bound $x/W(x)$ for $x \in (0, 100)$.

certainly exists over the entire domain $(-\infty, \infty)$, and satisfies

$$\pi(x) > U(x) - 1; \quad (\pi(x) \geq 5; \ x \geq 11). \quad (2.6)$$

Unfortunately while the inverse function $U(x)$ certainly exists, it has no simple closed form, either in terms of elementary functions or in terms of special functions (not even in terms of Lambert W). So to obtain a useful upper bound in terms of Lambert W our strategy must be more indirect.

Let us start with the elementary inequality

$$\ln x \leq \frac{x}{e}, \quad (2.7)$$

with equality only at $x = e$, and note that this implies

$$\ln x = \frac{\ln(x^\epsilon)}{\epsilon} \leq \frac{x^\epsilon}{\epsilon e}; \quad (\forall \epsilon > 0), \quad (2.8)$$

now with equality only at $x = e^{1/\epsilon}$. This inequality explicitly captures the well-known fact that the logarithm grows less rapidly than any positive power. Applied to the n 'th

prime this now implies

$$p_n < n \ln(n \ln n) \leq n \ln \left(\frac{n^{1+\epsilon}}{\epsilon e} \right) = n \{ (1+\epsilon) \ln n - 1 - \ln \epsilon \}; \quad (n \geq 6; \quad \forall \epsilon > 0). \quad (2.9)$$

This inequality is weaker than previously, equation (2.3), but much more tractable. Again using logic similar to that of Theorem 1, it is easy to convert this into the inequality

$$x < [\pi(x) + 1] \ln \left\{ \frac{[\pi(x) + 1]^{1+\epsilon}}{\epsilon e} \right\}; \quad (\pi(x) \geq 5; \quad \forall \epsilon > 0). \quad (2.10)$$

This is now easily inverted to obtain:

Theorem 3 *The prime counting function $\pi(x)$ satisfies*

$$\pi(x) > \frac{\frac{x}{1+\epsilon}}{W \left(\frac{x}{1+\epsilon} (\epsilon e)^{-1/(1+\epsilon)} \right)} - 1. \quad (2.11)$$

This inequality holds $\forall \epsilon > 0$, and at least for $\pi(x) \geq 5$, corresponding to $x \geq 11$. But, depending on ϵ , the domain of validity may actually be larger. That is, the inequality holds for $\pi(x) \geq n_0(\epsilon)$ with $n_0(\epsilon) \leq 5$, corresponding to $x \geq x_0(\epsilon)$ with $x_0(\epsilon) < 11$.

If for example we choose the specific case $\epsilon = e^{-1}$ we have (see figure 2):

$$\pi(x) > \frac{\frac{x}{1+e^{-1}}}{W \left(\frac{x}{1+e^{-1}} \right)} - 1; \quad (x \geq 5). \quad (2.12)$$

If instead we set $\epsilon = e^{-3}$ the domain of validity is now the entire positive half line (see figure 2):

$$\pi(x) > \frac{\frac{x}{1+e^{-3}}}{W \left(\frac{x}{1+e^{-3}} e^{2/(1+e^{-3})} \right)} - 1; \quad (x \geq 0). \quad (2.13)$$

This second special case, ($\epsilon = e^{-3}$), exhibits somewhat poorer bounding performance at intermediate values of x , but eventually overtakes the first special case, ($\epsilon = e^{-1}$), once $x \approx e^{2e+3} \approx 4600$, and then asymptotically provides a better bound. Numerous variations on this theme can also be constructed, amounting to different ways of approximating the logarithms in equation (2.3).

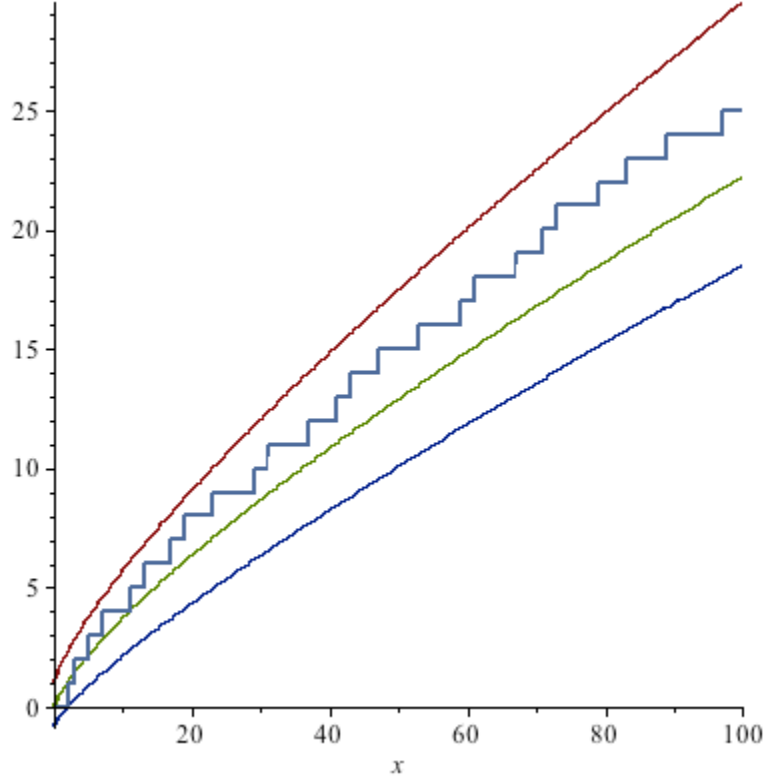


Figure 2. Prime counting function $\pi(x)$, its upper bound $x/W(x)$, and two specific lower bounds corresponding to $\epsilon = e^{-1}$ and $\epsilon = e^{-3}$, for $x \in (0, 100)$.

2.4 Implicit bounds

It is an old result (see for example Rosser [11]) that $\forall \epsilon > 0, \exists N(\epsilon) : \forall n \geq N(\epsilon)$

$$n\{\ln n + \ln \ln n - 1 + \epsilon\} < p_n < n\{\ln n + \ln \ln n - 1 + \epsilon\}. \quad (2.14)$$

Without an explicit calculation/specification of $N(\epsilon)$ these bounds are qualitative, rather than quantitative. Nevertheless it may be of interest to proceed as follows: Note that $\ln x$ is convex, and therefore

$$\ln \ln x < \ln \ln x_0 + \frac{\ln x - \ln x_0}{\ln x_0} \quad (x > 1; \ x_0 > 1). \quad (2.15)$$

Consequently

$$p_n < n \left\{ \ln n + \ln \ln x_0 + \frac{\ln n - \ln x_0}{\ln x_0} - 1 + \epsilon \right\} \quad (n \geq N(\epsilon); \ x_0 > 1). \quad (2.16)$$

That is

$$p_n < n \left\{ \left[1 + \frac{1}{\ln x_0} \right] \ln n + [\ln \ln x_0 - 2 + \epsilon] \right\} \quad (n \geq N(\epsilon); \ x_0 > 1). \quad (2.17)$$

Now *choose* x_0 to make the constant term vanish, that is set

$$\ln \ln x_0 = 2 - \epsilon; \quad \ln x_0 = e^{2-\epsilon}. \quad (2.18)$$

Then we have

$$p_n < \{1 + e^{-2+\epsilon}\} n \ln n; \quad (n \geq N(\epsilon); \ \epsilon > 0). \quad (2.19)$$

Inverting this, a minor variant of the arguments above immediately yields:

Theorem 4 $\forall \epsilon > 0, \exists M(\epsilon) : \forall x \geq M(\epsilon)$

$$\pi(x) > \frac{\frac{x}{1 + e^{-2+\epsilon}}}{W\left(\frac{x}{1 + e^{-2+\epsilon}}\right)} - 1. \quad (2.20)$$

It is now “merely” a case of estimating $M(\epsilon)$ to turn this into an explicit bound. For instance, it is known that for $\epsilon = 3$ we have $N(\epsilon = 3) = 55$, see Rosser [11]. Explicitly checking the domain of validity, this then corresponds to

Corollary 1

$$\pi(x) > \frac{\frac{x}{1 + e}}{W\left(\frac{x}{1 + e}\right)} - 1; \quad (x \geq 3). \quad (2.21)$$

Unfortunately, the bound is not particularly tight once x is large. A more recent result is that for $\epsilon = 1/2$ we have $N(\epsilon = 1/2) = 20$, see Rosser–Schoenfeld [12], equation (3.11) on page 69. This then corresponds to

Corollary 2

$$\pi(x) > \frac{\frac{x}{1 + e^{-3/2}}}{W\left(\frac{x}{1 + e^{-3/2}}\right)} - 1; \quad (x \geq 60). \quad (2.22)$$

In summary, we have used the Lambert W function to obtain a number of bounds, and some general classes of bounds, on the prime counting function $\pi(x)$ in terms of the Lambert W function $W(x)$. We shall now turn attention to the n ’th prime p_n .

3 The n 'th prime

3.1 Upper bound

Theorem 5 *The n 'th prime p_n satisfies*

$$p_n < -n W_{-1} \left(-\frac{1}{n} \right) \quad (n \geq 4). \quad (3.1)$$

Here $W_{-1}(x)$ is the second real branch of the Lambert W function, defined on the domain $x \in [-1/e, 0)$.

Proof: We start from the fact that for $x \in \mathbb{R}$ we have $\pi(x) > x/\ln x$ for $x \geq 17$. See Rosser–Schoenfeld [12], equation (3.5) on page 69.¹ Then, since $x \geq p_{\pi(x)}$ and $x/\ln x$ is monotone increasing for $x > e$, we have $\pi(x) > x/\ln x \geq p_{\pi(x)}/\ln p_{\pi(x)}$. This can be written as $n > p_n/\ln p_n$, this inequality certainly being valid for $p_n \geq 17$, corresponding to $n \geq 7$. Inverting, (and appealing to the monotonicity of $x/\ln x$), we have $p_n < -n W_{-1}(-1/n)$, certainly for $n \geq 7$. Explicitly inspecting $n \in \{1, 2, 3, 4, 5, 6\}$ shows that the actual range of validity is $n \geq 4$.

Corollary 3

$$p_n < -n W_{-1} \left(-\frac{1}{n+e} \right) \quad (n \geq 1). \quad (3.2)$$

Proof: Note that $-W_{-1}(x)$ is monotone increasing on $[-e^{-1}, 0)$. So we see that $-W_{-1}(-1/[n+e]) > -W_{-1}(-1/n)$, and the claimed inequality certainly holds for $n \geq 4$. For $n \in \{1, 2, 3\}$ verify the claimed inequality by explicit computation.

The virtue of this corollary is that it now holds for all positive integers. See figure 3. There are many other variations on this theme that one could construct.

3.2 Asymptotics

Theorem 6 *The prime number theorem, which can be written in the form $p_n \sim n \ln n$, is equivalent to*

$$p_n \sim -n W_{-1} \left(-\frac{1}{n} \right); \quad (n \rightarrow \infty). \quad (3.3)$$

Proof: Trivial. Consider the asymptotic result

$$W_{-1}(x) = \ln(-x) - \ln(-\ln(-x)) + o(1) \quad (x \rightarrow 0^-). \quad (3.4)$$

Then

$$-n W_{-1} \left(-\frac{1}{n} \right) = n \{ \ln n + \ln \ln n + o(1) \}. \quad (3.5)$$

¹Observe that since $\pi(x)$ changes in a stepwise fashion, and $x/\ln x$ is monotone, this is equivalent for $n \in \mathbb{Z}$ to $\pi(n) > (n+1)/\ln(n+1)$ for $n \geq 17$.

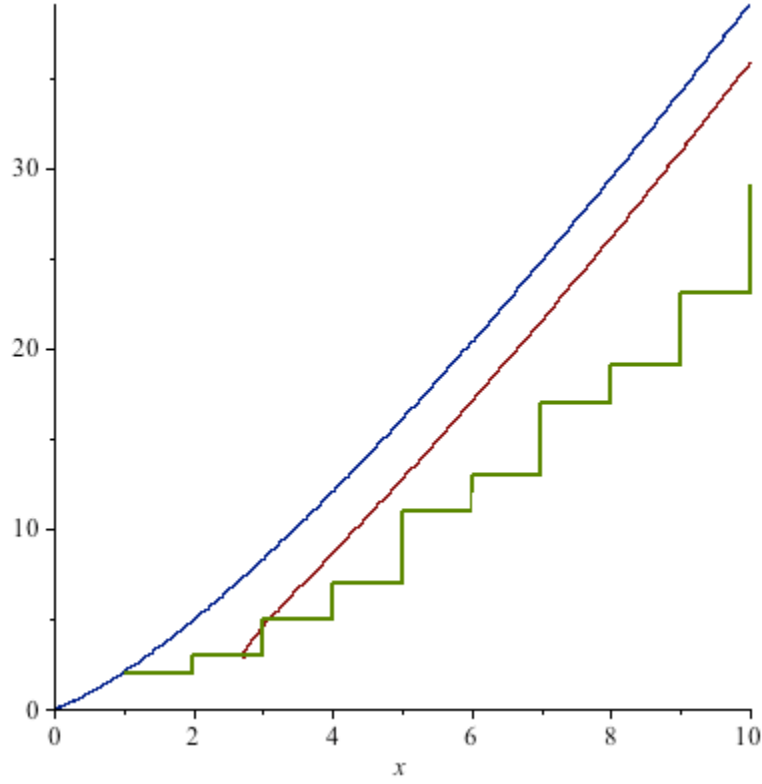


Figure 3. Two upper bounds on the n 'th prime p_n .

Comments: Note that use of the Lambert W function automatically yields the first two terms of the Cesaro–Cippola asymptotic expansion [13, 14]:

$$p_n = n\{\ln n + \ln \ln n - 1 + o(1)\}. \quad (3.6)$$

We can even obtain the first *three* terms of the Cesaro–Cippola asymptotic expansion by refining the prime number theorem slightly as follows:

Theorem 7

$$p_n \sim -n W_{-1} \left(-\frac{e}{n} \right); \quad (n \rightarrow \infty). \quad (3.7)$$

Finally, we note that use of the Lambert W function yields *both* a strict upper bound *and* an asymptotic result.

3.3 Lower bound

To obtain an lower bound on p_n we start with an upper bound on $\pi(x)$. Consider for instance the standard result [12]:

$$\pi(x) < \frac{x}{\ln x - \frac{3}{2}}; \quad (x > e^{3/2}) \quad (3.8)$$

Note that the RHS of this inequality is monotone increasing for $x > e^{5/2}$. Now we always have $p_{\pi(x)} \leq x < p_{\pi(x)+1}$, so

$$n < \frac{p_{n+1}}{\ln p_{n+1} - \frac{3}{2}}. \quad (3.9)$$

This holds at the very least for $p_n > e^{5/2}$, corresponding to $n \geq 6$, but an explicit check shows that it in fact holds for $n \geq 2$. This is perhaps more clearly expressed as

$$n - 1 < \frac{p_n}{\ln p_n - \frac{3}{2}}; \quad (n \geq 3). \quad (3.10)$$

Inverting, and noting the constraint arising from the domain of definition of W_{-1} , we now obtain:

Theorem 8

$$p_n > -(n-1) W_{-1} \left(-\frac{e^{3/2}}{n-1} \right); \quad (n \geq 14). \quad (3.11)$$

See figure (4). There are many other variations on this theme that one could construct.

3.4 Implicit bounds

It is an old result (see for example Rosser [11]) that $\forall \epsilon > 0, \exists N(\epsilon) : \forall n \geq N(\epsilon)$

$$\frac{x}{\ln x - 1 + \epsilon} < \pi(x) < \frac{x}{\ln x - 1 - \epsilon}. \quad (3.12)$$

Again note that without an explicit calculation of $N(\epsilon)$ these bounds are qualitative, rather than quantitative. Nevertheless it may be of interest to point out that a minor variant of the arguments above immediately yields:

Theorem 9 $\forall \epsilon > 0, \exists M(\epsilon) : \forall n \geq M(\epsilon)$

$$-n W_{-1} \left(-\frac{e^{1-\epsilon}}{n} \right) > p_n > -(n-1) W_{-1} \left(-\frac{e^{1+\epsilon}}{n-1} \right). \quad (3.13)$$

It is now “merely” a case of estimating $M(\epsilon)$ to turn these into explicit bounds. We have already seen that $\epsilon = 1$ provides a widely applicable upper bound, and $\epsilon = 1/2$ a widely applicable lower bound. Taking $\epsilon \rightarrow 0$ now makes it clear why

$$p_n \sim -n W_{-1} \left(-\frac{e}{n} \right); \quad (n \rightarrow \infty), \quad (3.14)$$

is such a good asymptotic estimate for p_n .

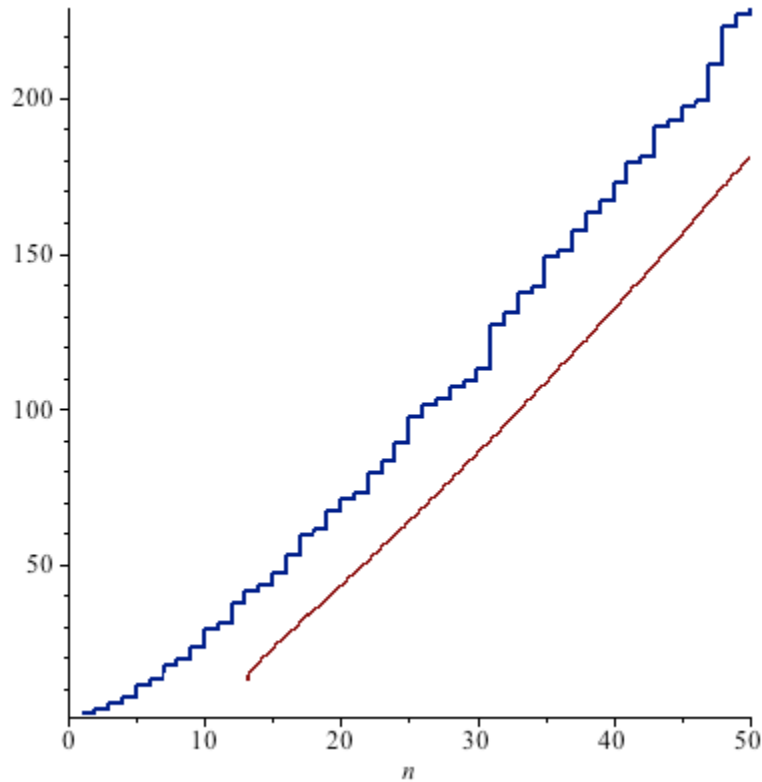


Figure 4. Lower bound on the n 'th prime p_n .

4 Discussion

While the calculations carried out above are very straightforward, almost trivial, it is perhaps the shift of viewpoint that is more interesting. The Lambert W function provides (in this context) a “new” special function to work with, one which may serve to perhaps simplify and unify many otherwise disparate results. It is perhaps worth noting that the infamous “ $\ln \ln x$ ” terms that infest the analytic theory of prime numbers will automatically appear as the sub-leading terms in asymptotic expansions of the Lambert W function.

Acknowledgments

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A Appendix

The Lambert W function is a multi-valued complex function defined implicitly by [1]

$$W(x) e^{W(x)} = x, \quad (\text{A.1})$$

There are two real branches: $W_0(x)$ defined for $x \in [-e^{-1}, \infty)$, and $W_{-1}(x)$ defined for $x \in [e^{-1}, 0)$. These two branches meet at the common point $W_0(-e^{-1}) = W_{-1}(-e^{-1}) = -1$. See figure 5. It is common to use $W(x)$ in place of $W_0(x)$ when there is no risk of confusion.

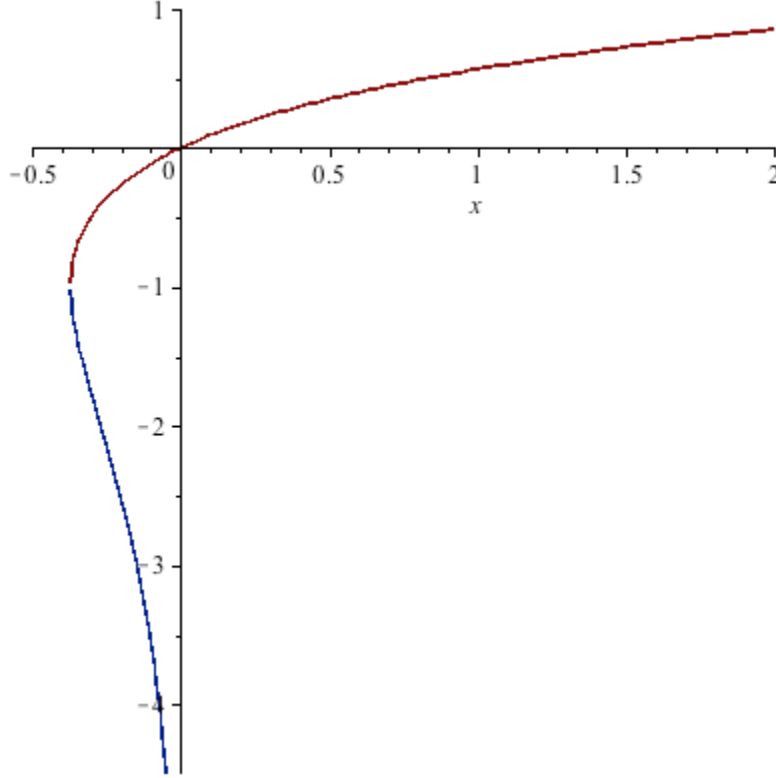


Figure 5. The two real branches of the Lambert W function.

Asymptotic expansions are:

$$W_0(x) = \ln x - \ln \ln x + o(1); \quad (x \rightarrow \infty); \quad (\text{A.2})$$

$$W_{-1}(x) = \ln(-x) - \ln(-\ln(-x)) + o(1); \quad (x \rightarrow 0^-). \quad (\text{A.3})$$

More details, and a Taylor expansion for $|x| < e^{-1}$, can be found in Corless *et al*, [1].

A key identity is:

$$\ln(a + bx) + cx = \ln d \quad \implies \quad x = \frac{1}{c} W \left(\frac{cd}{b} \exp \left[\frac{ac}{b} \right] \right) - \frac{a}{b}. \quad (\text{A.4})$$

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