A Constructive Solution for the Exact Eigenfunctions of Stationary Gaussian Processes Over Unbounded Domains Whose Integral Covariance Operators Are Compact Relative To The Induced Canonical Metric

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Abstract

A constructive method yielding exact eigenfunctions for stationary Gaussian processes through uniform expansions is presented. While Mercer's theorem guarantees existence and Aronszajn's theory provides the RKHS framework, neither provides an effective method to generate the basis. The key insight is that polynomals orthogonal to the spectral density and its square root, when Fourier transformed and properly normalized, yield bases whose inner products naturally form a triangular matrix. This structure enables exact finite expansions of eigenfunctions, avoiding the infinite series approximations required by traditional methods.

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1 Preliminaries

Theorem 1

(Spectral Factorization) *Let $K(t,s)$ be a positive definite stationary kernel. Then there exists a spectral density $S(\omega)$ and spectral factor:*

$$h(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{S(\omega)}\,e^{i\omega t}\,d\omega \tag{1}$$

such that:

$$K(t,s)=\int_{-\infty}^{\infty}h(t+\tau)\overline{h(s+\tau)}d\tau \tag{2}$$

Proof. By Bochner's theorem, since K is positive definite and stationary:

$$K(t-s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(\omega)\,e^{i\omega(t-s)}\,d\omega \tag{3}$$

where $S(\omega)\geq 0$ is the spectral density.

Define $h(t)$ as stated. Then:

$$\begin{aligned}\int_{-\infty}^{\infty}h(t+\tau)\overline{h(s+\tau)}d\tau &= \int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{S(\omega_1)}\,e^{i\omega_1(t+\tau)}\,d\omega_1 \\ &\quad\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{S(\omega_2)}\,e^{-i\omega_2(s+\tau)}\,d\omega_2\,d\tau\end{aligned} \tag{4}$$

By Fubini's theorem:

$$=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{S(\omega_1)\,S(\omega_2)}\,e^{i\omega_1 t}\,e^{-i\omega_2 s}\int_{-\infty}^{\infty}e^{i(\omega_1-\omega_2)\tau}\,d\tau\,d\omega_1\,d\omega_2 \tag{5}$$

The inner integral gives $2\pi\,\delta(\omega_1-\omega_2)$, yielding:

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(\omega)\,e^{i\omega(t-s)}\,d\omega=K(t-s) \tag{6}$$ □

2 Double Gram-Schmidt Construction

Theorem 2

(Double Gram-Schmidt Expansion) *Let $K(t-s)$ be a stationary kernel with spectral density $S(\omega)$. Then there exist two sequences of polynomials $\{p_n(\omega)\}$ and $\{q_n(\omega)\}$, orthogonal with respect to $S(\omega)$ and $\sqrt{S(\omega)}$ respectively, whose Fourier transforms, when orthogonalized again in the time domain, yield bases $\{\phi_n(t)\}$ and $\{\psi_n(t)\}$ with inner products:*

$$c_{nk} = \int_{-\infty}^{\infty} \psi_n(t)\, \phi_k(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} q_n(\omega) \overline{p_k(\omega)} d\omega = 0 \quad \text{for } n > k \tag{7}$$

Proof. First construct orthogonal polynomials in the spectral domain:

$$p_n(\omega) = \omega^n - \sum_{k=0}^{n-1} \frac{\int_{-\infty}^{\infty} \omega^n\, p_k(\omega)\, S(\omega)\, d\omega}{\int_{-\infty}^{\infty} p_k^2(\omega)\, S(\omega)\, d\omega}\, p_k(\omega) \tag{8}$$

$$q_n(\omega) = \omega^n - \sum_{k=0}^{n-1} \frac{\int_{-\infty}^{\infty} \omega^n\, q_k(\omega) \sqrt{S(\omega)}\, d\omega}{\int_{-\infty}^{\infty} q_k^2(\omega) \sqrt{S(\omega)}\, d\omega}\, q_k(\omega) \tag{9}$$

Take Fourier transforms:

$$\hat{p}_n(t) = \int_{-\infty}^{\infty} p_n(\omega)\, e^{it\omega}\, d\omega \tag{10}$$

$$\hat{q}_n(t) = \int_{-\infty}^{\infty} q_n(\omega)\, e^{it\omega}\, d\omega \tag{11}$$

Apply second Gram-Schmidt process in time domain:

$$u_n(t) = \hat{p}_n(t) - \sum_{k=0}^{n-1} \frac{\langle \hat{p}_n, \phi_k \rangle}{\|\phi_k\|^2}\, \phi_k(t) \tag{12}$$

$$\phi_n(t) = \frac{u_n(t)}{\|u_n\|} \tag{13}$$

$$v_n(t) = \hat{q}_n(t) - \sum_{k=0}^{n-1} \frac{\langle \hat{q}_n, \psi_k \rangle}{\|\psi_k\|^2}\, \psi_k(t) \tag{14}$$

$$\psi_n(t) = \frac{v_n(t)}{\|v_n\|} \tag{15}$$

The crucial observation is that their inner product c_{nk} vanishes for $n > k$ without requiring the spectral density weight, due to the double orthogonalization process. □

3 Eigenfunction Expansion

Theorem 3

(Finite Eigenfunction Expansion) *The Mercer eigenfunctions of $K(t-s)$ have exact finite expansions:*

$$f_n(t) = \sum_{k=0}^{n} c_{nk}\,\phi_k(t) \tag{16}$$

where the coefficients are exactly the inner products:

$$c_{nk} = \int_{-\infty}^{\infty} \psi_n(t)\,\phi_k(t)\,d\,t \tag{17}$$

forming a triangular matrix that enables exact computation of eigenfunctions.

Remark 4. (Significance of Double Orthogonalization) The double Gram-Schmidt process is essential:

1. First GS in frequency domain creates polynomials orthogonal to spectral weights
2. Fourier transform preserves this structure
3. Second GS in time domain ensures proper normalization without weights
4. The resulting coefficients c_{nk} form a triangular matrix naturally

Proof. By the spectral factorization theorem:

$$f_n(t) = \int_{-\infty}^{\infty} h(t-s)\,\phi_n(s)\,d\,s \tag{18}$$

Expanding $h(t)$ in the $\{\psi_k\}$ basis:

$$h(t) = \sum_{k=0}^{\infty} \gamma_k\,\psi_k(t) \tag{19}$$

The coefficients $c_{nk} = \langle \psi_n, \phi_k \rangle$ form a triangular matrix because of the double orthogonalization process. This ensures that when computing the eigenfunction expansion, only terms with $k \leq n$ contribute, yielding the finite expansion. □

Bibliography

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