Proof of the Vitali–Hahn–Saks Theorem

Theorem 1. *[Vitali–Hahn–Saks] Let (X, Σ) be a measurable space, and let $\{\mu_n\}$ be a sequence of finite measures on (X, Σ). Suppose that for every set E in Σ, the limit $\lim_{n\to\infty} \mu_n(E)$ exists (finite or infinite). Then:*

1. *There exists a measure μ on (X, Σ) such that for every E in Σ: $\mu(E) = \lim_{n\to\infty} \mu_n(E)$*
2. *The sequence of measures $\{\mu_n\}$ is uniformly absolutely continuous with respect to μ.*
3. *The convergence of μ_n to μ is uniform on Σ.*

Proof. Step 1: Define the limit measure μ

For each $E \in \Sigma$, define $\mu(E) = \lim_{n\to\infty} \mu_n(E)$. We need to show that μ is indeed a measure.

a) Clearly, $\mu(\emptyset) = \lim_{n\to\infty} \mu_n(\emptyset) = 0$.

b) Countable additivity: Let $\{E_k\}$ be a sequence of disjoint sets in Σ. We need to show that $\mu(\bigcup_k E_k) = \sum_k \mu(E_k)$.

$$\begin{aligned}
\mu(\bigcup_k E_k) &= \lim_{n\to\infty} \mu_n(\bigcup_k E_k) \\
&= \lim_{n\to\infty} \sum_k \mu_n(E_k) \quad \text{(by countable additivity of } \mu_n) \\
&= \sum_k \lim_{n\to\infty} \mu_n(E_k) \quad \text{(by the monotone convergence theorem)} \\
&= \sum_k \mu(E_k)
\end{aligned}$$

Thus, μ is a measure on (X, Σ).

Step 2: Prove uniform absolute continuity

We'll use the following lemma:

Lemma 2. *For any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all n and all $E \in \Sigma$: If $\mu(E) < \delta$, then $\mu_n(E) < \varepsilon$.*

Suppose the lemma is false. Then there exists an $\varepsilon > 0$ such that for every $k \in \mathbb{N}$, we can find n_k and $E_k \in \Sigma$ with $\mu(E_k) < 1/k$ and $\mu_{n_k}(E_k) \geq \varepsilon$.

Define $F_k = \bigcup_{j\geq k} E_j$. Then $F_k \supseteq F_{k+1}$ and $\mu(F_k) \leq \sum_{j\geq k} \mu(E_j) < \sum_{j\geq k} 1/j \to 0$ as $k \to \infty$.

But for any k, $\mu_{n_k}(F_k) \geq \mu_{n_k}(E_k) \geq \varepsilon$.

This contradicts the fact that $\lim_{n\to\infty} \mu_n(F_k) = \mu(F_k)$ for all k. $\square$

Step 3: Prove uniform convergence

We'll use Egoroff's theorem, which states that if a sequence of measurable functions converges almost everywhere on a finite measure space, then it converges uniformly except on a set of arbitrarily small measure.

For each $E \in \Sigma$, define $f_E(n) = \mu_n(E)$. The sequence $\{f_E(n)\}$ converges for each E.

Let $\varepsilon > 0$. By the uniform absolute continuity proved in Step 2, there exists a $\delta > 0$ such that $\mu(A) < \delta$ implies $\mu_n(A) < \varepsilon / 3$ for all n.

Let $M = \mu(X)$. Choose a finite partition $\{P_1, ..., P_k\}$ of X with $\mu(P_i) < \delta$ for all i.

For each P_i, the sequence $f_{P_i}(n)$ converges. By Egoroff's theorem, there exists $A_i \subset P_i$ with $\mu(A_i) < \delta / k$ such that $f_{P_i}(n)$ converges uniformly on $P_i \setminus A_i$.

Let $A = \bigcup_i A_i$. Then $\mu(A) < \delta$, so $\mu_n(A) < \varepsilon / 3$ for all n.

For each i, choose N_i such that for $n, m \geq N_i$, $|f_{P_i \setminus A_i}(n) - f_{P_i \setminus A_i}(m)| < \varepsilon / 3\,k$.

Let $N = \max\{N_i\}$. Then for $n, m \geq N$ and any $E \in \Sigma$:

$$\begin{aligned}|\mu_n(E) - \mu_m(E)| &\leq |\mu_n(E \cap A) - \mu_m(E \cap A)| + \sum_i |\mu_n(E \cap (P_i \setminus A_i)) - \mu_m(E \cap (P_i \setminus A_i))| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon\end{aligned}$$

This proves uniform convergence.

Therefore, we have proved all three parts of the Vitali–Hahn–Saks theorem.