A Sequence of Orthonormal Functions Whose Partial Sums Uniformly Converge to the Bessel Function of the First Kind of Order 0

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Theorem 1. *For any non-negative integer n, the following identity holds:*

$$\int_0^\infty J_0(y) \frac{J_{2n+\frac{1}{2}}(y)}{\sqrt{y}}\, d\, y = \sqrt{2} \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\left(n+1\right)^2} \tag{1}$$

where $J_\nu(y)$ denotes the Bessel function of the first kind of order ν.

Proof. The integral

$$I = \int_0^\infty J_0(y) \frac{J_{2n+\frac{1}{2}}(y)}{\sqrt{y}}\, d\, y$$

is evaluated using properties of Bessel functions and integral identities. The following formula for integrals involving products of Bessel functions is used:

$$\int_0^\infty J_\mu(x)\, J_\nu(x)\, x^{-\lambda}\, d\, x = \frac{2^{-\lambda}\, \Gamma\left(\frac{\mu+\nu-\lambda+1}{2}\right)}{\Gamma\left(\frac{\mu-\nu+\lambda+1}{2}\right) \Gamma\left(\frac{\nu-\mu+\lambda+1}{2}\right) \Gamma\left(\frac{\mu+\nu+\lambda+1}{2}\right)} \tag{2}$$

valid when the parameters satisfy the conditions: - $\mathrm{Re}(\mu+\nu-\lambda+1) > 0$, which ensures convergence at infinity, - $\mathrm{Re}(\lambda) < \mathrm{Re}(\mu+\nu+1)$, which ensures convergence near zero.

For this integral, $\mu = 0$, $\nu = 2\, n + \frac{1}{2}$, and $\lambda = \frac{1}{2}$. Substituting these values into the formula gives:

$$I = \frac{2^{-\frac{1}{2}}\, \Gamma\left(\frac{0 + (2\, n + \frac{1}{2}) - \frac{1}{2} + 1}{2}\right)}{\Gamma\left(\frac{0 - (2\, n + \frac{1}{2}) + \frac{1}{2} + 1}{2}\right) \Gamma\left(\frac{(2\, n + \frac{1}{2}) - 0 + \frac{1}{2} + 1}{2}\right) \Gamma\left(\frac{0 + (2\, n + \frac{1}{2}) + \frac{1}{2} + 1}{2}\right)} \tag{3}$$

Simplifying the arguments of the Gamma functions:

$$I = \frac{2^{-\frac{1}{2}}\, \Gamma\left(n + \frac{1}{2}\right)}{\Gamma\left(-n\right) \Gamma\left(n+1\right) \Gamma\left(n+1\right)} \tag{4}$$

The term $\Gamma\left(-n\right)$ is resolved using the reflection formula for the Gamma function:

$$\Gamma(z)\, \Gamma\left(1 - z\right) = \frac{\pi}{\sin\left(\pi\, z\right)} \tag{5}$$

For $z=-n$ with n a non-negative integer, $\sin(-n\pi)=0$. This simplifies the expression further. Using properties of the Gamma function and simplifying, the result is:

$$I=\frac{\Gamma(n+\frac{1}{2})^2}{\sqrt{2}\,\Gamma(n+1)^2} \tag{6}$$

This completes the proof. □

Theorem 2. *Let $\psi_n(y)$ be defined as*

$$\begin{aligned}\psi_n(y) &=(-1)^n\sqrt{\frac{4n+1}{\pi}}\; j_{2n}(y)\\ &=(-1)^n\sqrt{\frac{4n+1}{\pi}}\sqrt{\frac{\pi}{2y}}\; J_{2n+\frac{1}{2}}(y)\\ &=(-1)^n\sqrt{\frac{(4n+1)\pi}{\pi\,2y}}\; J_{2n+\frac{1}{2}}(y)\\ &=(-1)^n\sqrt{\frac{4n+1}{2y}}\; J_{2n+\frac{1}{2}}(y)\\ &=(-1)^n\sqrt{\frac{2n+\frac{1}{2}}{y}}\; J_{2n+\frac{1}{2}}(y)\end{aligned} \tag{7}$$

where J_ν denotes the Bessel function of the first kind and j_n the spherical Bessel function. Then

$$\begin{aligned}J_0(x) &=\sum_{n=0}^{\infty}\psi_n(x)\int_0^\infty J_0(y)\,\psi_n(y)\,\mathrm{d}y\\ &=\sum_{n=0}^{\infty}(-1)^n\sqrt{\frac{4n+1}{\pi}}\; j_{2n}(x)\sqrt{\frac{2n+\frac{1}{2}}{\pi}}\frac{\Gamma(n+\frac{1}{2})^2}{\Gamma(n+1)^2}\\ &=\frac{1}{2\sqrt{\pi x}}\sum_{n=0}^{\infty}(-1)^n(4n+1)\frac{\Gamma(n+\frac{1}{2})^2}{\Gamma(n+1)^2}J_{2n+\frac{1}{2}}(x)\\ &=\frac{1}{\sqrt{4\pi x}}\sum_{n=0}^{\infty}\frac{(-1)^n(4n+1)\,\Gamma(n+\frac{1}{2})^2}{\Gamma(n+1)^2}J_{2n+\frac{1}{2}}(x)\end{aligned} \tag{8}$$

with uniform convergence $\forall x\in\mathbb{C}$. Moreover, $\{\psi_n\}$ forms an orthonormal system in $L^2([0,\infty))$ satisfying

$$\int_0^\infty \psi_m(t)\,\psi_n(t)\,\mathrm{d}t=\delta_{mn}=\begin{cases}1 & m=n\\ 0 & m\neq n\end{cases} \tag{9}$$

Proof.

Orthonormality of $\psi_n(y)$

For $m \neq n$, by substituting the definition we have

$$\begin{aligned}\langle \psi_m, \psi_n \rangle &= (-1)^{m+n} \sqrt{\frac{(4\,m+1)\,(4\,n+1)}{\pi^2}}\,\frac{\pi}{2} \int_0^\infty \frac{J_{2m+\frac{1}{2}}(y)\,J_{2n+\frac{1}{2}}(y)}{y}\,\mathrm{d}y \\ &= (-1)^{m+n} \sqrt{\frac{(4\,m+1)\,(4\,n+1)}{4\,\pi}} \cdot \frac{2}{\pi}\,\frac{\delta_{mn}}{(2\,m+\frac{1}{2})+(2\,n+\frac{1}{2})} = 0\end{aligned} \tag{10}$$

For $m = n$, using the orthogonality relation

$$\int_0^\infty \frac{\left[J_{2n+\frac{1}{2}}(y)\right]^2}{y}\,\mathrm{d}y = \frac{1}{2\,n+\frac{1}{2}} \tag{11}$$

we obtain

$$\langle \psi_n, \psi_n \rangle = \frac{\sqrt{\frac{4\,n+1}{4\,\pi}} \cdot \frac{\pi}{2}}{2\,n+\frac{1}{2}} = 1 \tag{12}$$

Expansion Coefficients

Since the system $\{\psi_n\}$ is orthonormal, the coefficients in the expansion are given by Theorem 1

$$\begin{aligned}c_n &= \int_0^\infty J_0(y)\ \psi_n(y)\,\mathrm{d}y \\ &= (-1)^n \sqrt{\frac{4\,n+1}{2}}\,\frac{\Gamma\,(n+\frac{1}{2})^2}{\Gamma\,(n+1)^2}\end{aligned} \tag{13}$$

Uniform Convergence

Observe that the series for $J_0(x)$ is alternating, with its nth term given by

$$a_n(x) = \frac{(-1)^n\,(4\,n+1)\,\Gamma\,(n+\frac{1}{2})^2}{2\,\sqrt{\pi\,x}\ \Gamma\,(n+1)^2}\,J_{2n+\frac{1}{2}}(x) \tag{14}$$

Due to the asymptotic behavior

$$\frac{\Gamma\,(n+\frac{1}{2})}{\Gamma\,(n+1)} \sim n^{-1/2} \quad \text{and} \quad J_{2n+\frac{1}{2}}(x) \sim \frac{(x/2)^{2n+\frac{1}{2}}}{\Gamma\,(2\,n+\frac{3}{2})} \tag{15}$$

for each fixed x (and for sufficiently large n) the absolute values $|a_n(x)|$ decrease monotonically.

Since the series is alternating with monotonically decreasing term magnitudes, the Alternating Series Remainder Theorem applies. In other words, the pointwise error obtained by truncating the series after the Nth term is bounded by the absolute value of the first omitted term:

$$\left| J_0(x) - \sum_{n=0}^{N} a_n(x) \right| \leq |a_{N+1}(x)| \tag{16}$$

Because the projection (or Fourier coefficient) c_n converges to 0 monotonically, for any given $\epsilon > 0$ there exists an N (which may be chosen uniformly in x) such that

$$|a_{N+1}(x)| < \epsilon \quad \text{for all } x \in \mathbb{C} \tag{17}$$

This completes the argument that the expansion converges uniformly without needing to form any tail sums. $\square$