Theorem 1. **(Riemann Integral, Midpoint Definition)** *Let f be continuous on $[a, b]$. Then*

$$\int_a^b f(x)\ dx = \lim_{h \to 0} h \sum_{i=0}^{N-1} f\left(a + \left(i + \tfrac{1}{2}\right) \cdot h\right)$$

where

$$N = \frac{b-a}{h} \tag{1}$$

Proof. Since f is continuous on the compact set $[a, b]$, f is uniformly continuous on $[a, b]$.

By a standard theorem in real analysis, every uniformly continuous function is Riemann integrable. Therefore, f is Riemann integrable on $[a, b]$.

Let P_h be the uniform partition of $[a, b]$ into $N = \frac{b-a}{h}$ subintervals of width h, with mesh $|P_h| = h$. Consider the Riemann sum using midpoint tags:

$$S_h = h \sum_{i=0}^{N-1} f\left(a + \left(i + \tfrac{1}{2}\right) \cdot h\right) \tag{2}$$

Since f is Riemann integrable, for every $\epsilon > 0$, there exists $\delta > 0$ such that for any partition P with mesh $|P| < \delta$ and any choice of tags, the corresponding Riemann sum differs from $\int_a^b f(x)\ dx$ by less than ϵ.

Since $h \to 0$ implies $|P_h| = h \to 0$, we have

$$\lim_{h \to 0} S_h = \int_a^b f(x)\ dx \qquad \square$$