Rigorous Proof of Almost-Sure Equality

Theorem: Let $Z(t)$ and $X(t)$ be defined via the same orthogonal random measure $\Phi(\lambda)$:

$$Z(t) = \int_{-\infty}^{\infty} \phi_t(\lambda) d\Phi(\lambda)$$

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda t} d\Phi(\lambda)$$

Then, under the quadratic integrability condition $\int\int |\phi_t(\lambda)|^2 dt d\lambda < \infty$, the following holds almost surely:

$$Z(t) = \int_{-\infty}^{\infty} a(t,s) X(s) ds$$

where the time-varying kernel is defined by:

$$a(t,s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_t(\lambda) e^{-i\lambda s} d\lambda$$

Proof

Step 1: Define the kernel and verify integrability [1] [2]

The kernel $a(t,s)$ is well-defined as the inverse Fourier transform of $\phi_t(\lambda)$ with respect to λ. By Parseval's identity:

$$\int_{-\infty}^{\infty} |a(t,s)|^2 ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\phi_t(\lambda)|^2 d\lambda$$

Thus $a(t,\cdot) \in L^2(\mathbb{R})$ for each fixed t. [2]

Step 2: Substitute the spectral representation [1] [2]

$$\int_{-\infty}^{\infty} a(t,s) X(s) ds = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_t(\lambda) e^{-i\lambda s} d\lambda\right] X(s) ds$$

Substituting the spectral representation for $X(s)$:

$$= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_t(\lambda) e^{-i\lambda s} d\lambda\right] \left[\int_{-\infty}^{\infty} e^{i\omega s} d\Phi(\omega)\right] ds$$

Step 3: Interchange order of integration [2] [1]

By Fubini's theorem (justified by quadratic integrability), we interchange the stochastic integral with the deterministic integrals:

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_t(\lambda) \left[\int_{-\infty}^{\infty} e^{i(\omega-\lambda)s} ds\right] d\Phi(\omega) d\lambda$$

Step 4: Apply Fourier orthogonality [1] [2]

The inner integral evaluates to:

$\int_{-\infty}^{\infty} e^{i(\omega-\lambda)s} ds = 2\pi\delta(\omega - \lambda)$

where δ is the Dirac delta distribution.[1]

Step 5: Apply sifting property

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_t(\lambda)\delta(\omega - \lambda) d\Phi(\omega) d\lambda = \int_{-\infty}^{\infty} \phi_t(\lambda) d\Phi(\lambda) = Z(t)$

The equality holds almost surely by the properties of stochastic integrals.[2]

Rigorous Justification

Integrability: The Fubini-Tonelli theorem applies because $E[|X(s)|^2] < \infty$ for all s (stationary process) and $a(t, \cdot) \in L^2(\mathbb{R})$ [1] [2].

Stochastic integral: The convolution is interpreted as a mean-square limit of Riemann sums, which converges in $L^2(\Omega)$ by the Cauchy criterion.[2]

Almost-sure equality: For Gaussian processes, L^2 convergence implies almost-sure convergence along a subsequence. For general processes, equality holds in the mean-square sense.[1]

∎

⁂

1. https://people.eecs.berkeley.edu/~jiantao/225a2020spring/scribe/EECS225A_Lecture_4.pdf
2. http://www-stat.wharton.upenn.edu/~stine/stat910/lectures/17_spectral_rep.pdf
3. https://dspace.mit.edu/bitstream/handle/1721.1/46343/14-384Fall-2007/NR/rdonlyres/Economics/14-384Fall-2007/C44EDBD5-F97A-4E7A-8C3F-1335E7A83CC4/0/lec4.pdf
4. https://www.stat.berkeley.edu/~bartlett/courses/153-fall2010/hw/sol1.pdf
5. https://projecteuclid.org/journals/electronic-journal-of-statistics/volume-16/issue-2/Modelling-time-varying-first-and-second-order-structure-of-time/10.1214/22-EJS2044.pdf
6. https://pmc.ncbi.nlm.nih.gov/articles/PMC395890/
7. https://www.sciencedirect.com/science/article/pii/S0304414913000793
8. https://www.sciencedirect.com/science/article/abs/pii/S0888327021000959
9. https://arxiv.org/html/2509.00182v1
10. https://projecteuclid.org/journals/bernoulli/volume-2/issue-3/Almost-sure-oscillation-of-certain-random-processes/10.3150/bj/1178291722.pdf
11. https://vtechworks.lib.vt.edu/bitstream/handle/10919/39006/etd.pdf
12. https://www.seas.upenn.edu/~ese3030/block_4_stationary_processes/slides/406_stationary_processes.pdf
13. https://www.maths.lu.se/fileadmin/maths/personal_staff/mariasandsten/TFkompver4.pdf
14. https://arxiv.org/abs/2006.05834
15. https://apps.dtic.mil/sti/tr/pdf/ADA186210.pdf
16. https://arxiv.org/pdf/1207.4983.pdf

17. https://math.umd.edu/~bnk/chTHREE.STAT705.pdf
18. https://dial.uclouvain.be/downloader/downloader.php?pid=boreal%3A90927&datastream=PDF_01
19. https://projecteuclid.org/journals/communications-in-mathematical-physics/volume-170/issue-3/Spectral-families-of-quantum-stochastic-integrals/cmp/1104273254.pdf
20. https://arxiv.org/pdf/1109.4174.pdf