VIII. Resolvent and Spectrum

Let T be a linear operator whose domain $D(T)$ and range $R(T)$ both lie in the same complex linear topological space X. We consider the linear operator

$$T_\lambda = \lambda I - T \tag{1}$$

where λ is a complex number and I the identity operator. The distribution of the values of λ for which T_λ has an inverse and the properties of the inverse when it exists, are called the *spectral theory* for the operator T. We shall thus discuss the general theory of the inverse of T_λ.

1. The Resolvent and Spectrum

Definition 1

If λ_0 is such that the range $R(T_{\lambda_0})$ is dense in X and T_{λ_0} has a continuous inverse $(\lambda_0 I - T)^{-1}$, we say that λ_0 is in the resolvent set $\rho(T)$ of T and denote this inverse $(\lambda_0 I - T)^{-1}$ by $R(\lambda_0; T)$ and call it the resolvent (at λ_0) of T. All complex numbers λ not in $\rho(T)$ form a set $\sigma(T)$ called the spectrum of T. The spectrum $\sigma(T)$ is composed of disjoint sets $P(T)$, $C(T)$ and $R(T)$ with the following properties:

- *The the point spectrum $P(T)$ of T is the totality of complex numbers λ for which T_λ does not have an inverse;*
- *The continuous spectrum $C(T)$ of T is the totality of complex numbers λ for which T_λ has a continuous inverse with domain dense in X*
- *The residual spectrum $R(T)$ of T is the totality of complex numbers λ for which T_λ has an inverse whose domain is not dense in of T is the totality of complex numbers λ*for which TÎ» does not have an inverse;*

From these definitions and the linearity of T we have the following

Proposition 2

A necessary and sufficient condition for λ_0 to be in $P(T)$ is that the equation $Tx = \lambda_0 x$ has a solution $x \neq 0$. In this case λ_0 is called an eigenvalue of T, and the corresponding eigenvector. The null space $N(\lambda_0 I - T)$ of T_λ is called the eigenspace of T corresponding to the eigenvalue λ_0. It consists of the vector or the totality of eigenvectors corresponding to λ_0. The dimension of the eigenspace corresponding to λ_0 is called the multiplicity of the eigenvalue λ_0.

Theorem 3

Let X be a complex B-space, and T a closed linear operator with its domain $D(T)$ and range $R(T)$ both in X. Then, for any λ_0 in $\rho(T)$,

$$\frac{1}{\lambda_0 I - T} \tag{2}$$

is an everywhere defined continuous linear operator.

Proof. Since λ_0 is in the resolvent set $\rho(T)$, $R(\lambda_0;T) = D((\lambda_0 I - T)^{-1})$ is dense in X in such a way that there exists a positive constant c for which

$\|(\lambda_0 \mathrm{I} - \mathrm{T})\, \mathrm{x}\| \geq \mathrm{c}\, \|\mathrm{x}\|$ whenever $\mathrm{x} \in \mathrm{D(T)}$. We have to show that $R\,(\lambda_0 I - T) = X$. But, if $s\text{-}\lim_{n\to\infty} x_n = y$ exists, then by the above inequality, $s\text{-}\lim_{n\to\infty} T x_n = T y$ exists, and so, by the closure property of T, we must have $(\lambda_0 I - T)\, x = y$. Hence, by the assumption that $R\,(\lambda_0 I - T)^{-1} = X$, we must have $R\,(\lambda_0 I - T) = X$. □

Example 4. If the space X is of finite dimension, then any bounded linear operator T is represented by a matrix (t_{ij}). It is known that the eigenvalues of T are obtained as the roots of the algebraic equation, the so-called *secular* or *characteristic equation* of the matrix (t_{ij}):

$$\det\,(\lambda_0 I_{ij} - t_{ij}) = 0 \tag{3}$$

where $\det\,(A)$ denotes the determinant of the matrix A.

Example 5. Let $X = L^2(-\infty,\infty)$ and let T be defined by

$$T\colon (x(t) \mapsto t\, x(t)) \tag{4}$$

that is, $D(T) = \{x(t)\colon x(t) \in L^2(-\infty,\infty)\}$ and $T x(t) = t\, x(t)$ for $x(t) \in D(T)$. Then every real number λ_0 is in $C_c(T)$.

Proof. The condition $(\lambda_0 I - T)\, x = 0$ implies $(\lambda_0 - t)\, x(t) = 0$ a.e., and so $x(t) = 0$ a.e. Thus $\rho\,(\lambda_0 - T) = \mathbb{C}$. The domain $D((\lambda_0 I - T)^{-1})$ comprises those $y(t) \in L^2(-\infty,\infty)$ which vanish identically in the neighbourhood of $t = \lambda_0$; the neighbourhood may vary with $y(t)$. Hence $D((\lambda_0 I - T)^{-1})$ is dense in $L^2(-\infty,\infty)$. It is easy to see that the norm of $(\lambda_0 I - T)^{-1}$ is not bounded on the totality of such $y(t)'\,s$. □

Example 6. Let X be the Hilbert space (ℓ^2), and let T_0 be defined by

$$T_0(\{s_1, s_2, \ldots\}) = (0, s_1, s_2, \ldots) \tag{5}$$

Then 0 is in the residual spectrum of T, since $R(T_0)$ is not dense in X.

Example 7. Let H be a self-adjoint operator in a Hilbert space X. Then the resolvent set $\rho(H)$ of H comprises all the complex numbers λ with $\mathrm{Im}(\lambda)\neq 0$, and the resolvent $R(\lambda; H)$ is a bounded linear operator with the estimate

$$\|R(\lambda;H)\|\leq\frac{1}{|\mathrm{Im}(\lambda)|} \tag{6}$$

Moreover,

$$\mathrm{Im}((\lambda I-H)x,x)=\mathrm{Im}(\lambda)\|x\|^2\forall x\in D(H) \tag{7}$$

Proof. If $x\in D(H)$, then (Hx,x) is real since $(Hx,x)=(x,Hx)$. Therefore we have (3) and so by Schwarz's inequality

$$\|(\lambda I-H)x\|\geqslant|(\lambda I-H)x|\geqslant\mathrm{Im}(\lambda)\,\|x\|^2 \tag{8}$$

which implies that

$$\|(\lambda I-H)x\|\geq|\mathrm{Im}(\lambda)|\|x\| \tag{9}$$

Hence the inverse $\frac{1}{\lambda I-H}$ exists if $\mathrm{Im}(\lambda)\neq 0$. Moreover, the range $R(\lambda I-H)$ is dense in X if $\mathrm{Im}(\lambda)\neq 0$. If otherwise, there would exist a $y\neq 0$ orthogonal to $R(\lambda I-H)$, i.e., $((\lambda I-H)x,y)=0\forall x\in D(H)$, and so $(x,(\bar{\lambda}I-H)y)=0\forall x\in D(H)$. Since the domain $D(H)$ of a self-adjoint operator H is dense in X, we must have $(\bar{\lambda}I-H)y=0$, that is, $Hy=\bar{\lambda}y$, contrary to the reality of the value (Hy,y).

Therefore, by the above theorem, we see that for any complex number λ with $\mathrm{Im}(\lambda)\neq 0$, the resolvent $R(\lambda;H)$ is a bounded linear operator with the estimate (2).

2. The Resolvent Equation and Spectral Radius

Theorem 8

Let T be a closed linear operator with domain and range both in a complex B-space X. Then the resolvent set $\rho(T)$ is an open set of the complex plane. In each component (the maximal connected sets) of $\rho(T)$, $R(\lambda;T)$ is a holomorphic function of λ.

Proof. By the Theorem of the preceding section, $R(\lambda;T)$ for $\lambda\in\rho(T)$ is an everywhere defined continuous operator. Let $\lambda_0\in\rho(T)$ and consider

$$S(\lambda)=R(\lambda_0;T)\left[I+\sum_{n=1}^{\infty}(\lambda_0-\lambda)^n R(\lambda_0;T)^n\right] \tag{10}$$

The series is convergent in the operator norm whenever $|\lambda_0 - \lambda|.\ \|R(\lambda_0; T)\| < 1$, and within this circle of the complex plane, the series defines a holomorphic function of λ. Multiplication by $(\lambda - T)$ on the left or right gives I so that the series $S(\lambda)$ actually represents the resolvent $R(\lambda; T)$. Thus we have proved that a circular neighbourhood of λ_0 belongs to $\rho(T)$ and $R(\lambda; T)$ are everywhere defined continuous operators, then the resolvent equation holds:

$$R(\lambda; T) - R(\mu; T) = (\mu - \lambda)\, R(\lambda; T)\, R(\mu; T) \tag{11}$$ □

Proof. We have

$$\begin{aligned} R(\lambda; T) &= R(\mu; T)\, [(\mu - T)\, R(\mu; T)] \\ &= R(\mu; T)\, [(\mu - \lambda)\, I + (\lambda - T)]\, R(\mu; T) \\ &= (\mu - \lambda)\, R(\lambda; T)\, R(\mu; T) \end{aligned} \tag{12}$$ □

Theorem 9

If T is a bounded linear operator on a complex B-space X, then the following limit exists:

$$\lim_{n \to \infty} \|T^n\|^{\frac{1}{n}} = r_0(T) \tag{13}$$

It is called the spectral radius of T, and we have

$$r_0(T) \leq \|T\| \tag{14}$$

If $|\lambda| > r_0(T)$, then the resolvent $R(\lambda; T)$ exists and is given by the series

$$R(\lambda; T) = \sum_{n=1}^{\infty} \lambda^{-n}\, T^{n-1} \tag{15}$$

which converges in the norm of operators.

Proof. Set $r = \inf_n \|T^n\|^{1/n}$. We have to show that $\lim_{n\to\infty} \|T^n\|^{1/n} \leq r$. For any $\epsilon > 0$, choose m such that $\|T^m\|^{1/m} \leq r + \epsilon$, for arbitrary n, write $n = p\, m + q$ where $0 \leq q < m$. Then, by $\|A\, B\| \leq \|A\| \|B\|$, we obtain

$$\|T^n\|^{1/n} \leq \|T^m\|^{p/n} \|T^q\|^{1/n} \leq (r + \epsilon)^{pm/n} \|T^q\|^{1/n} \tag{16}$$

Since $p\, m / n \to 1$ and $q / n \to 0$ as $n \to \infty$, we must have $\|T^n\|^{1/n} \leq r + \epsilon$. Since ϵ was arbitrary, we have proved $\lim_{n\to\infty} \|T^n\|^{1/n} \leq r$.

Since $\|T^n\| \le \|T\|^n$, we have $\lim_{n\to\infty} \|T^n\|^{1/n} \le \|T\|$. The series (5) is convergent in the norm of operators when $|\lambda| > r_0(T)$. For if $|\lambda| \ge r_0(T) + \epsilon > 0$, then, by (3), $\|R(\lambda; T)\|^{1/n} \le (r_0(T) + 2\,\epsilon)^{-n}$ for large n. Multiplication by $(\lambda\, I - T)$ on the left or right of this series gives I so that the series actually represents the resolvent $R(\lambda; T)$. □

Corollary 10

The resolvent set $\rho(T)$ is not empty when T is a bounded linear operator.

Theorem 11

For a bounded linear operator T in $L(X, X)$, we have

$$r_0(T) = \sup_{\lambda \in \sigma(T)} |\lambda| \tag{17}$$

Proof. By Theorem 3, we know that $R(\lambda; T)$ is holomorphic in λ when $|\lambda| > \sup_{\mu \in \sigma(T)} |\mu|$. Hence we have only to show that $r_0(T)$ is equal to $\sup_{\lambda \in \sigma(T)} |\lambda|$.

By Theorem 1, $R(\lambda; T)$ is holomorphic in λ where $|\lambda| > \sup_{\mu \in \sigma(T)} |\mu|$. Thus it admits a uniquely determined Laurent expansion in positive and non-positive powers of λ converging in the operator norm for $|\lambda| > \sup_{\mu \in \sigma(T)} |\mu|$. By Theorem 3, this Laurent series must coincide with

$$\sum_{n=1}^{\infty} \lambda^{-n} T^{n-1} \tag{18}$$

Hence $\lim_{n\to\infty} \frac{T^{n-1}}{\lambda^n} = 0$ if $|\lambda| > \sup_{\mu \in \sigma(T)} |\mu|$, and so, for any $\epsilon > 0$, we must have $\|T\| \le (\epsilon + \sup_{\mu \in \sigma(T)} |\mu|)^n$ for large n. This proves that

$$r_0(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \sup_{\lambda \in \sigma(T)} |\lambda| \tag{19}$$ □

Corollary 12

The series $\sum_{n=1}^{\infty} \lambda^{-n} T^{n-1}$ diverges if $|\lambda| < r_0(T)$.

Proof. Let γ be the smallest number ≥ 0 such that the series

$$\sum_{n=1}^{\infty} \lambda^{-n} T^{n-1} \tag{20}$$

converges in the operator norm for $|\lambda| > \gamma$. The existence of such an n is proved as for ordinary power series in λ^{-1}. Then, for $|\lambda| > \gamma$, $\lim_{n\to\infty} \lambda^{-n} T^{n-1} = 0$ and so, as in the proof of $r_0(T) \le \sup_{\mu \in \sigma(T)} |\mu|$, we must have $\lim_{n\to\infty} \|T^n\|^{1/n} \le \gamma$. This proves that $r_0(T) \le \gamma$.

□