The Operational Matrix of the Random Wave Process

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Abstract

An expression for the convolution of a pair of spherical Bessel functions is determined.

1 Lemmas

Lemma 1. (Terminating Hypergeometric Series) *For any $p \in \mathbb{Z}_{\geq 0}$, the Gauss hypergeometric function terminates:*

$$ {}_2F_1(-p, b; c; z) = \sum_{k=0}^{p} \frac{(-p)_k (b)_k}{(c)_k \, k!} z^k \tag{1} $$

where $(a)_k = \prod_{i=0}^{k-1} (a + i)$

Proof. By definition, the Gauss hypergeometric series is:

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k \, k!} z^k \tag{2} $$

Setting $a = -p$ with $p \in \mathbb{Z}_{\geq 0}$, the Pochhammer symbol $(-p)_k$ becomes zero for all $k > p$. Explicitly:

$$ (-p)_k = \prod_{i=0}^{k-1} (-p + i) = \begin{cases} (-p)(-p+1)\cdots(-p+k-1), & k \leq p \\ 0 & k > p \end{cases} \tag{3} $$

Thus, the series terminates at $k = p$, yielding:

$$ {}_2F_1(-p, b; c; z) = \sum_{k=0}^{p} \frac{(-p)_k (b)_k}{(c)_k \, k!} z^k \tag{4} $$ □

Lemma 2. (Integral with Incomplete Gamma Function) *For $j \geq 0$,*

$$ \int_{-1}^{1} \left(\frac{1-x}{2} \right)^{j} e^{ixy} \, dx = \frac{e^{iy}}{2^j} \frac{\gamma(j+1, 2iy)}{(iy)^{j+1}} \tag{5} $$

where $\gamma(s, x)$ denotes the lower incomplete gamma function.

Proof. Substitute $t = \frac{1-x}{2} \Longrightarrow x = 1 - 2t$, $dx = -2\, dt$, adjusting limits:

$$ \int_{1}^{0} t^j e^{i(1-2t)y} (-2\, dt) = 2 e^{iy} \int_{0}^{1} t^j e^{-2iyt} \, dt \tag{6} $$

Let $u = 2iyt \Longrightarrow t = \frac{u}{2iy}$, $dt = \frac{du}{2iy}$:

$$ \frac{2 e^{iy}}{(2iy)^{j+1}} \int_{0}^{2iy} u^j e^{-u} \, du = \frac{e^{iy}}{2^j} \frac{\gamma(j+1, 2iy)}{(iy)^{j+1}} $$ □

Lemma 3. (Legendre Polynomial Representation) *The hypergeometric function quals the Legendre polynomial $P_m(x)$. The Legendre polynomials are hypergeometric functions*

$$ P_m(x) = {}_2F_1(-m, m+1; 1; \tfrac{1-x}{2}) \tag{7} $$

Proof. From the Rodrigues formula $P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m} (x^2 - 1)^m$ expand using the binomial theorem:

$$ (x^2 - 1)^m = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} x^{2k} \tag{8} $$

Differentiating m times yields terms proportional to x^k, matching the hypergeometric series:

$$ P_m(x) = {}_2F_1\left(-m, m+1; 1; \tfrac{1-x}{2} \right) $$ □

2 Main Theorem

Theorem 4. (Fourier Transform of Legendre Polynomial Products)

$$\begin{aligned} I_{m,n}(y) &= \int_{-1}^{1} P_m(x)\, P_n(x)\, e^{ixy}\, dx \\ &= e^{iy} \sum_{j=0}^{m+n} \frac{\Psi_j(m,n)}{2^j} \left[\frac{\gamma(j+1, 2\, i\, y)}{(i\, y)^{j+1}} \right] \end{aligned} \tag{9}$$

where

$$\Psi_j(m,n) = \frac{{}_4F_3\left(\begin{matrix} -m, m+1, -n, n+1 \\ 1, 1, j+1 \end{matrix} ; 1 \right)}{j!} \tag{10}$$

Proof.

Part 1: Integral Reduction
Expand $P_m(x)\, P_n(x)$ using Lemma 1:

$$P_m(x)\, P_n(x) = \sum_{k=0}^{m} \sum_{\ell=0}^{n} \frac{(-m)_k\, (m+1)_k\, (-n)_\ell\, (n+1)_\ell}{(1)_k (1)_\ell\, k! \ell!} \left(\frac{1-x}{2} \right)^{k+\ell} \tag{11}$$

Let $j = k + \ell$, valid for $0 \le k \le m$, $0 \le \ell \le n$. Then:

$$I_{m,n}(y) = \sum_{j=0}^{m+n} \underbrace{\sum_{k=\max(0,j-n)}^{\min(j,m)} \frac{(-m)_k\, (m+1)_k\, (-n)_{j-k}\, (n+1)_{j-k}}{(1)_k (1)_{j-k}\, k! (j-k)!}}_{\Psi_j(m,n)} \int_{-1}^{1} \left(\frac{1-x}{2} \right)^j e^{ixy}\, dx \tag{12}$$

Apply Lemma 2 to obtain the result.

Part 2: $\Psi_j(m,n)$ as a ${}_4F_3$ Function
Expand the ${}_4F_3$ series:

$${}_4F_3\left(\begin{matrix} -m, m+1, -n, n+1 \\ 1, 1, j+1 \end{matrix} ; 1 \right) = \sum_{k=0}^{j} \frac{(-m)_k\, (m+1)_k\, (-n)_k\, (n+1)_k}{(1)_k (1)_k\, (j+1)_k\, k!} \tag{13}$$

The main result follows. ☐