# Orthonormality Of The Spectral Factor Convolution

**Lemma 1.** [Orthonormality Preservation] Let  $K(x, y) = J_0(|x - y|)$  be the Bessel kernel with spectral factor  $\rho(y) = \mathcal{F}^{-1}(\sqrt{\mathcal{F}[K](\omega)})$ . Given an orthonormal basis  $\{\psi_n\}_{n=1}^{\infty}$  for  $L^2([0,\infty))$ , the functions  $\phi_n(x) = (\psi_n * \rho)(x)$  satisfy:

$$\langle \phi_m, \phi_n \rangle_{\mathscr{H}_K} = \delta_{mn},$$

where  $\mathscr{H}_K$  is the RKHS associated with K.

#### Proof. Step 1: RKHS Inner Product Characterization

For any  $f, g \in \mathcal{H}_K$ , the RKHS inner product for translation-invariant kernels is:

$$\langle f, g \rangle_{\mathscr{H}_{K}} = \int_{\mathbb{R}} \frac{\mathcal{F}[f](\omega)\overline{\mathcal{F}[g](\omega)}}{\mathcal{F}[K](\omega)} d\omega \tag{1}$$

## Step 2: Spectral Representation of $\phi_n$

Using the Fourier convolution theorem:

$$\mathcal{F}[\phi_n](\omega) = \mathcal{F}[\psi_n * \rho](\omega) \tag{2}$$

$$= \mathcal{F}[\psi_n](\omega) \cdot \mathcal{F}[\rho](\omega) \tag{3}$$

$$= \mathcal{F}[\psi_n](\omega) \cdot \sqrt{\mathcal{F}[K](\omega)} \tag{4}$$

### **Step 3: Inner Product Computation**

Substitute into (1):

$$\langle \phi_m, \phi_n \rangle_{\mathcal{H}_K} = \int_{\mathbb{R}} \frac{\left( \mathcal{F}[\psi_m](\omega) \sqrt{\mathcal{F}[K](\omega)} \right) \left( \overline{\mathcal{F}[\psi_n](\omega)} \sqrt{\mathcal{F}[K](\omega)} \right)}{\mathcal{F}[K](\omega)} d\omega \tag{5}$$

$$= \int_{\mathbb{R}} \mathcal{F}[\psi_m](\omega) \overline{\mathcal{F}[\psi_n](\omega)} d\omega \tag{6}$$

## Step 4: Parseval's Theorem Application

By Plancherel's theorem (Parseval for  $L^2$ ):

$$\int_{\mathbb{R}} \mathcal{F}[\psi_m](\omega) \overline{\mathcal{F}[\psi_n](\omega)} d\omega = \langle \psi_m, \psi_n \rangle_{L^2} = \delta_{mn}$$
 (7)

### Conclusion

Therefore:

$$\langle \phi_m, \phi_n \rangle_{\mathscr{H}_K} = \delta_{mn}$$

The convolution with  $\rho$  preserves orthonormality from  $L^2$  to  $\mathscr{H}_K$ .