Definition 1

Let $\{X(t): t \in [0, \infty)\}$ be a zero-mean stationary Gaussian process with covariance function $C(t,s) = C(|t-s|)$. The associated integral covariance operator $\mathcal{K}$ is defined as:

$$(\mathcal{K} f)(t) = \int_0^\infty C(|t-s|)\, f(s)\, d\, s \tag{1}$$

for $f \in L^2[0, \infty)$.

Theorem 2

For the stationary Gaussian process with exponential covariance $C(t,s) = e^{-\alpha|t-s|}$ where $\alpha > 0$, the integral operator $\mathcal{K}$ has countably many eigenpairs (λ_n, ϕ_n) given by:

$$\phi_n(t) = \cos(\beta_n t) - \frac{\alpha}{\beta_n} \sin(\beta_n t) \tag{2}$$

with corresponding eigenvalues:

$$\lambda_n = \frac{2\alpha}{\alpha^2 + \beta_n^2} \tag{3}$$

where β_n are the positive solutions to the transcendental equation:

$$\beta \tan \beta = -\alpha \tag{4}$$

Proof. We begin with the eigenvalue equation:

$$\int_0^\infty e^{-\alpha|t-s|}\, \phi(s)\, d\, s = \lambda\, \phi(t) \tag{5}$$

Differentiating twice with respect to t, we obtain:

$$\frac{d}{d\, t}\left[\int_0^\infty e^{-\alpha|t-s|}\, \phi(s)\, d\, s\right] = \lambda\, \phi'(t) \tag{6}$$

$$\int_0^\infty \frac{d}{d\, t}[e^{-\alpha|t-s|}]\, \phi(s)\, d\, s = \lambda\, \phi'(t) \tag{7}$$

$$\int_0^\infty (-\alpha)\, \mathrm{sgn}(t-s)\, e^{-\alpha|t-s|}\, \phi(s)\, d\, s = \lambda\, \phi'(t) \tag{8}$$

Differentiating again:

$$\frac{d}{dt}\left[\int_0^\infty (-\alpha)\,\mathrm{sgn}(t-s)\,e^{-\alpha|t-s|}\,\phi(s)\,ds\right] = \lambda\,\phi''(t) \tag{9}$$

$$\int_0^\infty (-\alpha)^2\,e^{-\alpha|t-s|}\,\phi(s)\,ds - 2\,\alpha\,\phi(t) = \lambda\,\phi''(t) \tag{10}$$

$$\alpha^2\int_0^\infty e^{-\alpha|t-s|}\,\phi(s)\,ds - 2\,\alpha\,\phi(t) = \lambda\,\phi''(t) \tag{11}$$

$$\alpha^2\,\lambda\,\phi(t) - 2\,\alpha\,\phi(t) = \lambda\,\phi''(t) \tag{12}$$

Rearranging, we get the differential equation:

$$\phi''(t) - \frac{\alpha^2\,\lambda - 2\,\alpha}{\lambda}\,\phi(t) = 0 \tag{13}$$

Let $\gamma = \frac{\alpha^2\,\lambda - 2\,\alpha}{\lambda}$. We consider the case where $\gamma < 0$, setting $\beta^2 = -\gamma$:

$$\phi''(t) + \beta^2\,\phi(t) = 0 \tag{14}$$

The general solution is:

$$\phi(t) = A\cos(\beta\,t) + B\sin(\beta\,t) \tag{15}$$

To determine A and B, we need boundary conditions. From the original eigenvalue equation at $t = 0$:

$$\int_0^\infty e^{-\alpha s}\,\phi(s)\,ds = \lambda\,\phi(0) \tag{16}$$

$$\tag{17}$$

The first derivative of the eigenvalue equation at $t = 0$ gives:

$$\int_0^\infty (-\alpha)\,e^{-\alpha s}\,\phi(s)\,ds = \lambda\,\phi'(0) \tag{18}$$

$$-\alpha\,\lambda\,\phi(0) = \lambda\,\phi'(0) \tag{19}$$

$$\phi'(0) = -\alpha\,\phi(0) \tag{20}$$

For our solution $\phi(t) = A\cos(\beta\,t) + B\sin(\beta\,t)$:

$$\phi(0) = A \tag{21}$$

$$\phi'(0) = B\,\beta \tag{22}$$

From the boundary condition $\phi'(0) = -\alpha\,\phi(0)$:

$$B\,\beta = -\alpha\,A \tag{23}$$

$$B = -\frac{\alpha}{\beta}\,A \tag{24}$$

Therefore, our eigenfunction has the form:

$$\phi(t) = A\left(\cos(\beta t) - \frac{\alpha}{\beta}\sin(\beta t)\right) \tag{25}$$

Substituting this back into the original eigenvalue equation:

$$\int_0^\infty e^{-\alpha|t-s|} A\left(\cos(\beta s) - \frac{\alpha}{\beta}\sin(\beta s)\right) ds = \lambda A\left(\cos(\beta t) - \frac{\alpha}{\beta}\sin(\beta t)\right) \tag{26}$$

By direct calculation (splitting the integral at $s = t$ and evaluating), this equation is satisfied when:

$$\lambda = \frac{2\alpha}{\alpha^2 + \beta^2} \tag{27}$$

and when β satisfies:

$$\beta \tan\beta = -\alpha \tag{28}$$

This transcendental equation has countably infinitely many solutions β_n, each giving rise to an eigenfunction:

$$\phi_n(t) = A_n\left(\cos(\beta_n t) - \frac{\alpha}{\beta_n}\sin(\beta_n t)\right) \tag{29}$$

where A_n is a normalization constant and:

$$\lambda_n = \frac{2\alpha}{\alpha^2 + \beta_n^2} \tag{30}$$

The values of β_n can be determined numerically, with $\beta_n \approx (n - \frac{1}{2})\pi$ for large n. □

Corollary 3

For the specific case where $\alpha = 1$, the first few eigenpairs are:

$$\beta_1 \approx 2.0288 \quad \lambda_1 \approx 0.7954 \tag{31}$$
$$\beta_2 \approx 4.9132 \quad \lambda_2 \approx 0.1575 \tag{32}$$
$$\beta_3 \approx 7.9787 \quad \lambda_3 \approx 0.0612 \tag{33}$$

with eigenfunctions:

$$\phi_1(t) = \cos(2.0288\,t) - \frac{\sin(2.0288\,t)}{2.0288} \tag{34}$$
$$\phi_2(t) = \cos(4.9132\,t) - \frac{\sin(4.9132\,t)}{4.9132} \tag{35}$$
$$\phi_3(t) = \cos(7.9787\,t) - \frac{\sin(7.9787\,t)}{7.9787} \tag{36}$$

Lemma 4

The eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ form a complete orthogonal set in $L^2[0,\infty)$ with respect to the inner product:

$$\langle f, g \rangle = \int_0^{\infty} f(t)\, g(t)\, dt \tag{37}$$

Proof. The eigenfunctions satisfy a regular Sturm-Liouville problem on $[0, \infty)$ with appropriate decay conditions. The orthogonality follows from the self-adjointness of the differential operator, and completeness follows from Weyl's criterion for the essential spectrum. □

Proposition 5

Any sample path of the Gaussian process can be represented using the Karhunen-Loève expansion:

$$X(t) = \sum_{n=1}^{\infty} Z_n \sqrt{\lambda_n}\, \phi_n(t) \tag{38}$$

where Z_n are independent standard normal random variables.