Eigenfunction Construction for Stationary Gaussian Processes

by Stephen Crowley

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1 Preliminaries

Definition 1

The Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are defined as:

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, d\, x \tag{1}$$

$$\mathcal{F}^{-1}[g](x) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} g(\omega)\, e^{i\omega x}\, d\,\omega \tag{2}$$

Definition 2

Let $K\,(x-y)$ be a stationary positive-definite kernel. By Bochner's theorem:

$$\begin{aligned} K\,(x-y) &= \mathcal{F}^{-1}[S](x-y) \\ &= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{i\omega\,(x-y)}\, S(\omega)\, d\,\omega \end{aligned} \tag{3}$$

where

$$\begin{aligned} S(\omega) &= \mathcal{F}[K](\omega) \\ &= \int_{-\infty}^{\infty} K(x)\, e^{-i\omega x}\, d\, x \end{aligned} \tag{4}$$

is the corresponding spectral density.

Definition 3

[Spectral Polynomials] *Let*

$$P_n(\omega) = \omega^n - \sum_{j=0}^{n-1} \frac{\int_{-\infty}^{\infty} P_j(\omega)\omega^j S(\omega) d\omega}{\langle P_j^2 \rangle} \tag{5}$$

be polynomials orthogonal with respect to the spectral density $S(\omega)$ and normalized so that $P_0(\omega) = 1$:

$$\int_{-\infty}^{\infty} P_m(\omega)\, P_n(\omega)\, S(\omega)\, d\,\omega = \delta_{mn} \tag{6}$$

Theorem 4

[Null Space Theorem] *The inverse Fourier transforms of the polynomials $P_n(\omega)$ orthogonal with respect to the spectral density $S(\omega)$*

$$f_n(x) = \mathcal{F}^{-1}[P_n(\omega)](x) \tag{7}$$

constitute the null-space of the kernel inner-product $\langle K, \cdot \rangle = \int_0^\infty K(x) f(x) \mathrm{d}x$ which is evinced by an application of Parseval's theorem

$$\begin{aligned} \langle K, \mathcal{F}^{-1}[P_n] \rangle &= \langle K, f_n \rangle \\ &= \int_0^\infty K(x) f_n(x) \mathrm{d}x \\ &= \langle \mathcal{F}^{-1}[S], \mathcal{F}^{-1}[P_n] \rangle \\ &= \langle S, P_n \rangle \\ &= 0 \end{aligned} \quad \forall n \geqslant 1 \tag{8}$$

Remark 5. The null-space of an operator is also called a kernel, but to avoid confusion with the kernel referring to the autocovariance kernel function of the Gaussian process integral covariance operator, the null-space terminology is preferred

Theorem 6

Let

$$f_k^{\perp}(x) = f_k(x) - \sum_{j=1}^{k-1} \frac{\langle f_k, f_j^{\perp} \rangle}{\| f_j^{\perp} \|^2} f_j^{\perp}(x) \tag{9}$$

be the orthogonal complement of the sequence of null space functions defined in Equation (7) then

$$K(x-y) = \sum_{n=0}^{\infty} \langle K, f_n^{\perp} \rangle f_n^{\perp}(x-y) \tag{10}$$

converges uniformly $\forall x - y \in \mathbb{R}$

Proof. Let

$$K_N(x-y) = \sum_{n=0}^{N} \langle K, f_n^{\perp} \rangle f_n^{\perp}(x-y) \tag{11}$$

then define the error

$$E_N(x) = K(x) - K_N(x)$$

whose L^2 norm has the upper bound

$$\|E_N\| \leqslant \langle K, f_N^{\perp} \rangle \tag{12}$$

since

$$\| f_k^{\perp} \| \leqslant 1 \tag{13}$$

by orthornormality. □

Remark 7. This is not a Mercer expansion. Notice that it is a sum over $\psi_n(x-y)$ not the product $\psi_n(x)\psi(y)$ which is the form it would have to have to be a Mercer expansion.

2 Uniform Basis of the Spectral Factor

Let $\{Q_n(\omega)\}_{n=0}^{\infty}$ be orthogonal polynomials with respect to $\sqrt{S(\omega)}$:

$$\langle Q_m, Q_n\rangle_{\sqrt{S}} = \int_{-\infty}^{\infty} Q_m(\omega)\, Q_n(\omega) \sqrt{S(\omega)}\, d\,\omega = \delta_{mn} \tag{14}$$

Define:

$$\xi_n(x) = \mathcal{F}^{-1}[Q_n(\omega)](x) \tag{15}$$

Apply Gram-Schmidt to $\{\xi_n\}$ to obtain orthonormal sequence $\{\phi_n\}$ via:

$$\phi_k(x) = \xi_k(x) - \sum_{j=1}^{k-1} \frac{\langle \xi_k, \phi_j\rangle}{\|\phi_j\|^2}\, \phi_j(x) \tag{16}$$

Then:

$$\begin{aligned} g(x) &= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{i\omega x} \sqrt{S(\omega)}\, d\,\omega \\ &= \sum_{n=0}^{\infty} \langle g, \phi_n\rangle\, \phi_n(x) \end{aligned} \tag{17}$$

where g is the spectral factor with $\mathcal{F}[g] = \sqrt{S}$ satisfying

$$\begin{aligned} g(x) * g(y) &= K(x-y) \\ &= \int_{-\infty}^{\infty} g(x+z)\overline{g(y-z)}\mathrm{d}z \end{aligned} \tag{18}$$

3 Eigenfunction Construction

By Fubini's theorem and uniform convergence:

$$\begin{aligned} K\,(x-y) &= (g * g)\,(x-y) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle g, \phi_m\rangle \langle g, \phi_n\rangle\, (\phi_m * \phi_n)\,(x-y) \end{aligned} \tag{19}$$

The eigenfunctions $\{f_n\}$ can be expressed in the uniform basis $\{\psi_n\}$ with finitely many terms:

$$f_n(x) = \sum_{k=0}^{n} c_{nk}\, \psi_k(x) \tag{20}$$

where

$$c_{nk} = \langle f_n, \psi_k \rangle \tag{21}$$

Substituting into Mercer expansion form:

$$\begin{aligned} K\,(x-y) &= \sum_{n=0}^{\infty} \lambda_n\, f_n(x)\, f_n(y) \\ &= \sum_{n=0}^{\infty} \lambda_n \sum_{k=0}^{n} \sum_{j=0}^{n} c_{nk}\, c_{nj}\, \psi_k(x)\, \psi_j(y) \end{aligned} \tag{22}$$

This double sum structure with coefficients is the spectral version of spatiotemporal inner product representation guaranteed by Moore-Aronszajn's theorem for reproducing kernel Hilbert spaces.

Theorem 8

(Triangularity of Eigenfunction Coefficients) *The coefficients $c_{n,k}$ in the eigenfunction expansion*

$$f_n(x) = \sum_{k=0}^{\infty} c_{n,k}\, \psi_k(x) \tag{23}$$

form a triangular matrix with $c_{n,k} = 0$ for $k > n$.

Proof. The spectral polynomials $P_n(\omega)$ are constructed recursively:

$$P_n(\omega) = \omega^n - \sum_{k=0}^{n-1} \alpha_{n,k}\, P_k(\omega) \tag{24}$$

where $\alpha_{n,k}$ are determined by orthogonality with respect to $S(\omega)$.

Taking the inverse Fourier transform:

$$\psi_n(x) = \mathcal{F}^{-1}[P_n(\omega)](x) = \mathcal{F}^{-1}[\omega^n](x) - \sum_{k=0}^{n-1} \alpha_{n,k}\,\psi_k(x) \tag{25}$$

The basis functions $\{\psi_n(x)\}$ thus satisfy:

$$\psi_n(x) = \phi_n(x) - \sum_{k=0}^{n-1} \alpha_{n,k}\,\psi_k(x) \tag{26}$$

where $\phi_n(x) = \mathcal{F}^{-1}[\omega^n](x)$. By construction, each $\psi_n(x)$ is expressed only in terms of $\{\psi_k(x)\}_{k=0}^{n-1}$. Therefore, when expressing eigenfunctions in this basis:

$$f_n(x) = \sum_{k=0}^{\infty} c_{n,k}\,\psi_k(x) \tag{27}$$

the coefficients $c_{n,k}$ must be zero for $k > n$, as these basis functions cannot appear in the expansion of ψ_n. $\square$