Oscillatory Process Inversion

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Theorem 1. *[Inversion of Oscillatory Processes]Given*

$$Z(s) = \int_{-\infty}^{\infty} \phi_s(\lambda)\ d\,\Phi(\lambda) \tag{1}$$

where $\phi_s(\lambda)$ is measurable, invertible ($\phi_s(\lambda) \neq 0$ for all s, λ), and satisfies the ***quadratic integrability condition****:*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi_t(\lambda)|^2\ d\,t\ d\,\lambda < \infty \tag{2}$$

and $\Phi(\lambda)$ is a complex orthogonal random measure with

$$E(|d\,\Phi(\lambda)|^2) = d\,\mu(\lambda) \tag{3}$$

Define the stationary process:

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\ d\,\Phi(\lambda) \tag{4}$$

Define the inverse kernel:

$$b(t,s) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \frac{e^{i\lambda t}}{\phi_s(\lambda)}\ e^{-i\lambda s}\ d\,\lambda \tag{5}$$

Invertibility condition: *Require*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\frac{1}{\phi_t(\lambda)}\right|^2\ d\,t\ d\,\lambda < \infty \tag{6}$$

Then:

$$X(t) = \int_{-\infty}^{\infty} b(t,s)\ Z(s)\ d\,s \tag{7}$$

Proof.

- Expand the convolution:

$$\begin{aligned}\int_{-\infty}^{\infty} b(t,s)\ Z(s)\ d\,s\ &= \int_{-\infty}^{\infty}\left[\frac{1}{2\,\pi}\int_{-\infty}^{\infty}\frac{e^{i\lambda t}}{\phi_s(\lambda)}\,e^{-i\lambda s}\,d\,\lambda\right] \\ &\quad\times\left[\int_{-\infty}^{\infty}\phi_s(\omega)\,d\,\Phi(\omega)\right]d\,s\end{aligned} \tag{8}$$

- Exchange order of integration by Fubini's theorem (justified by the quadratic integrability conditions (6)):

$$= \frac{1}{2\,\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{i\lambda t}\,\frac{\phi_s(\omega)}{\phi_s(\lambda)}\,e^{-i\lambda s}\,d\,s\right]d\,\lambda\ d\,\Phi(\omega) \tag{9}$$

- Evaluate the inner integral over s:

$$\int_{-\infty}^{\infty} \frac{\phi_s(\omega)}{\phi_s(\lambda)}\, e^{-i\lambda s}\, d\, s \tag{10}$$

For $\phi_s(\lambda) = A_s(\lambda)\, e^{i\lambda s}$:

$$\frac{\phi_s(\omega)}{\phi_s(\lambda)}\, e^{-i\lambda s} = \frac{A_s(\omega)}{A_s(\lambda)}\, e^{i(\omega-\lambda)s} \tag{11}$$

The integral becomes:

$$\int_{-\infty}^{\infty} \frac{A_s(\omega)}{A_s(\lambda)}\, e^{i(\omega-\lambda)s}\;d\, s = 2\,\pi\;\delta\,(\omega-\lambda) \tag{12}$$

where this holds as a distributional identity under the stated integrability and invertibility conditions.

- Substitute (12) back:

$$\begin{aligned} \frac{1}{2\,\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\lambda t}\cdot 2\,\pi\,\delta\,(\omega-\lambda)\;d\,\lambda\;d\,\Phi(\omega) \;\; &= \int_{-\infty}^{\infty} e^{i\omega t}\;d\,\Phi(\omega) \\ &= X(t) \end{aligned} \tag{13}$$

□

Summary

Under the conditions:

1. $\int\int|\phi_t(\lambda)|^2\;d\,t\;d\,\lambda < \infty$
2. $\phi_s(\lambda) \neq 0$ everywhere
3. $\int\int|1/\phi_t(\lambda)|^2\;d\,t\;d\,\lambda < \infty$

The stationary process $X(t)$ is exactly inverted from the oscillatory process $Z(t)$ by equation (7) with kernel (5).