Spectral Support and Bandlimited Gaussian Processes

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1 Fundamental Definitions

Definition 1. *[Heaviside Step Function] The Heaviside step function* $H:\mathbb{R}\rightarrow\{0,1\}$ *is defined as*

$$H(x)=\begin{cases}1 & \text{if } x\geq 0\\ 0 & \text{if } x<0\end{cases} \tag{1}$$

Definition 2. *[Rectangular Function] The rectangular function* $\mathrm{rect}_{[a,b]}:\mathbb{R}\rightarrow\{0,1\}$ *for* $a<b$ *is defined as*

$$\mathrm{rect}_{[a,b]}(\omega)=H\left(\omega-a\right)-H\left(\omega-b\right) \tag{2}$$

which equals 1 for $\omega\in[a,b]$ *and 0 otherwise.*

Definition 3. *[Spectral Density] Let* $\{X_t\}_{t\in\mathbb{R}}$ *be a zero-mean, stationary Gaussian process with covariance function* $K(\tau)=\mathbb{E}\left[X_t\,X_{t+\tau}\right]$*. The spectral density* $S:\mathbb{R}\rightarrow[0,\infty)$ *is the Fourier transform of the covariance function:*

$$S(\omega)=\int_{-\infty}^{\infty}K(\tau)\,e^{-i\omega\tau}\,d\,\tau \tag{3}$$

provided this integral exists.

Definition 4. *[Spectral Support] The spectral support of a Gaussian process with spectral density* $S(\omega)$ *is the set*

$$\mathrm{supp}(S)=\overline{\{\omega\in\mathbb{R}:S(\omega)>0\}} \tag{4}$$

where $\bar{A}$ *denotes the closure of set* A*.*

Definition 5. *[Bandlimited Process] A stationary Gaussian process is called bandlimited if its spectral support is a compact subset of* $\mathbb{R}$*, i.e., if there exist constants* $a, b \in \mathbb{R}$ *with* $a < b$ *such that*

$$\operatorname{supp}(S) \subseteq [a, b] \tag{5}$$

and $\operatorname{supp}(S)$ *is closed and bounded.*

2 Main Results

Theorem 6. *[Sinc Kernel Spectral Density] Consider the covariance function*

$$K(\tau) = \frac{\sin(2\pi\tau)}{2\pi\tau} \tag{6}$$

with the convention that $K(0) = 1$*. The corresponding spectral density is*

$$S(\omega) = \frac{1}{2}\operatorname{rect}_{[-1,1]}(\omega) = \frac{1}{2}[H(\omega+1) - H(\omega-1)] \tag{7}$$

Proof. The Fourier transform of $K(\tau) = \frac{\sin(2\pi\tau)}{2\pi\tau}$ is computed as follows. Using the identity $\sin(2\pi\tau) = \frac{e^{i2\pi\tau} - e^{-i2\pi\tau}}{2i}$, one has

$$S(\omega) = \int_{-\infty}^{\infty} \frac{\sin(2\pi\tau)}{2\pi\tau} e^{-i\omega\tau} d\tau \tag{8}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i2\pi\tau} - e^{-i2\pi\tau}}{2i\tau} e^{-i\omega\tau} d\tau \tag{9}$$

$$= \frac{1}{4\pi i}\left[\int_{-\infty}^{\infty} \frac{e^{-i(\omega-2\pi)\tau}}{\tau} d\tau - \int_{-\infty}^{\infty} \frac{e^{-i(\omega+2\pi)\tau}}{\tau} d\tau\right] \tag{10}$$

Computing the Fourier transform of $\frac{\sin(a\tau)}{\pi\tau}$ directly: for $a > 0$,

$$\int_{-\infty}^{\infty} \frac{\sin(a\tau)}{\pi\tau} e^{-i\omega\tau} d\tau = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{ia\tau} - e^{-ia\tau}}{2i\tau} e^{-i\omega\tau} d\tau \tag{11}$$

$$= \frac{1}{2\pi i}\left[\int_{-\infty}^{\infty} \frac{e^{-i(\omega-a)\tau}}{\tau} d\tau - \int_{-\infty}^{\infty} \frac{e^{-i(\omega+a)\tau}}{\tau} d\tau\right] \tag{12}$$

Using the fact that $\int_{-\infty}^{\infty} \frac{e^{-i\alpha\tau}}{\tau} d\tau = -i\pi\operatorname{sgn}(\alpha)$ where sgn is the sign function, this evaluates to $\operatorname{rect}_{[-a,a]}(\omega)$. Therefore,

$$S(\omega) = \frac{1}{2}\operatorname{rect}_{[-1,1]}(\omega) = \frac{1}{2}[H(\omega+1) - H(\omega-1)] \tag{13}$$ □

Proposition 7. *[General Bandlimited Spectral Density] A Gaussian process is bandlimited with spectral support* $[a, b]$ *if and only if its spectral density can be written as*

$$S(\omega) = f(\omega) \cdot \mathrm{rect}_{[a,b]}(\omega) = f(\omega) \cdot [H(\omega - a) - H(\omega - b)] \tag{14}$$

for some non-negative function $f: [a, b] \to [0, \infty)$.

Proof. $(\Rightarrow)$ If the process is bandlimited with spectral support $[a, b]$, then $S(\omega) = 0$ for $\omega \notin [a, b]$. Define $f(\omega) = S(\omega)$ for $\omega \in [a, b]$ and extend arbitrarily to $\mathbb{R}$. Then $S(\omega) = f(\omega) \cdot \mathrm{rect}_{[a,b]}(\omega)$.

$(\Leftarrow)$ If $S(\omega) = f(\omega) \cdot \mathrm{rect}_{[a,b]}(\omega)$, then $S(\omega) = 0$ for $\omega \notin [a, b]$, implying $\mathrm{supp}(S) \subseteq [a, b]$. □

Example 8. [Band-pass Process] Consider a bandlimited process with spectral support $[-\Omega, -\omega_0] \cup [\omega_0, \Omega]$ where $0 < \omega_0 < \Omega$. The spectral density can be expressed as

$$\begin{aligned} S(\omega) &= f(\omega) \cdot [\mathrm{rect}_{[-\Omega, -\omega_0]}(\omega) + \mathrm{rect}_{[\omega_0, \Omega]}(\omega)] && (15) \\ &= f(\omega) \cdot [H(\omega + \Omega) - H(\omega + \omega_0) + H(\omega - \omega_0) - H(\omega - \Omega)] && (16) \end{aligned}$$

for some appropriate function f.

3 Conclusion

The spectral support serves as the fundamental concept for characterizing bandlimited Gaussian processes. The Heaviside step function provides a natural mathematical framework for expressing the boundaries of spectral support, enabling precise characterization of the frequency domain properties of such processes.