Introduction

Oscillatory processes generalize stationary stochastic processes by allowing their spectral properties to evolve over time. Central to this representation is the gain function $A(t,\omega)$, a complex-valued function that works in conjunction with an underlying spectral density $S(\omega)$ to produce time-varying spectral characteristics. The magnitude $|A(t,\omega)|$ scales the spectral power at each frequency and time, while the argument $\arg A(t,\omega)$ introduces frequency-dependent phase shifts. The effective spectral density at time t becomes $|A(t,\omega)|^2 S(\omega)$, showing how the gain function and underlying spectral density work together multiplicatively.

Definition 1. (Stationary Process) *A stochastic process* $\{X(t), t\in\mathbb{R}\}$ *is stationary when its covariance* $R(s,t)$ *depends only on the lag:* $R(s,t)=R(t-s)$ *for all* $s,t\in\mathbb{R}$.

Definition 2. (Complex orthogonal random measure) *Let* $(E,\mathcal{E})$ *be a measurable space. A complex orthogonal random measure is a map* $\Phi\colon\mathcal{E}\to L^2(\Omega;\mathbb{C})$ *such that:*

1. *(Null and* σ*-additivity in* L^2*)* $\Phi(\varnothing)=0$, $\Phi(A\cup B)=\Phi(A)+\Phi(B)$ *for disjoint* A, $B\in\mathcal{E}$, *and for any disjoint sequence* $(A_n)_{n\geq 1}\subset\mathcal{E}$,

$$\Phi\left(\bigcup_{n\geq 1} A_n\right)=\sum_{n\geq 1}\Phi(A_n)\quad \text{in } L^2.$$

2. *(Zero mean and covariance) There exists a finite measure* μ *on* $(E,\mathcal{E})$ *such that, for all* $A,B\in\mathcal{E}$,

$$\mathbb{E}[\Phi(A)]=0,\qquad \mathbb{E}[\Phi(A)\,\overline{\Phi(B)}]=\mu(A\cap B).$$

In particular, for all $A\in\mathcal{E}$, $\mathbb{E}[|\Phi(A)|^2]=\mu(A)$, *and for disjoint* A,B *the increments are orthogonal in* L^2.

Theorem 3. (Spectral Representation of Oscillatory Processes) *A realization of an oscillatory process $Z(t)$ is one that satisfies*

$$Z(t) \;=\; \int_{-\infty}^{\infty} A_t(\omega)\; e^{i\omega t}\; d\,\Phi(\omega) \;=\; \int_{-\infty}^{\infty} h(t,u)\; X(u)\; d\,u, \tag{1}$$

where $A_t(\omega)$ is a gain function and Φ is a complex orthogonal random measure. The stationary reference process is

$$X(u) \;=\; \int_{-\infty}^{\infty} e^{i\omega u}\; d\,\Phi(\omega). \tag{2}$$

In the sense of Priestley's canonical definition, the oscillatory kernel h and the gain A_t form a Fourier pair (in the sense of distributions) with the convention

$$h(t,u) \;=\; \frac{1}{2\,\pi}\int_{-\infty}^{\infty} A_t(\lambda)\; e^{i\lambda(t-u)}\; d\,\lambda, \qquad A_t(\omega) \;=\; \int_{-\infty}^{\infty} h(t,u)\; e^{-i\omega(t-u)}\; d\,u. \tag{3}$$

If Z is real-valued, the conjugate symmetry conditions hold:

$$A_t(\omega) \;=\; A_t^*(-\omega), \qquad d\,\Phi\,(-\omega) \;=\; d\,\Phi^*(\omega). \tag{4}$$

Proof. Using (2) and Fubini/Tonelli in L^2,

$$\begin{aligned} Z(t) &= \int_{-\infty}^{\infty} h(t,u)\; X(u)\; d\,u = \int_{-\infty}^{\infty} h(t,u)\left(\int_{-\infty}^{\infty} e^{i\omega u}\; d\,\Phi(\omega)\right) d\,u \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t,u)\; e^{i\omega u}\; d\,u\; d\,\Phi(\omega). \end{aligned}$$

By the canonical Fourier relation (3),

$$\int_{-\infty}^{\infty} h(t,u)\; e^{i\omega u}\; d\,u = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} A_t(\lambda)\; e^{i\lambda t}\left(\int_{-\infty}^{\infty} e^{i(\omega-\lambda)u}\; d\,u\right) d\,\lambda = A_t(\omega)\; e^{i\omega t}.$$

Therefore $Z(t) = \int A_t(\omega)\; e^{i\omega t}\; d\,\Phi(\omega)$, proving (1). Real-valuedness follows from (4) by a standard change of variables. □

Theorem 4. (Eigenfunction Property for Stationary Processes) *Let $R(\tau)$ be a stationary covariance function and define the integral operator*

$$(K f)(t) \;=\; \int_{-\infty}^{\infty} R\,(t-s)\; f(s)\; d\,s. \tag{5}$$

Then

$$K\; e^{i\omega t} \;=\; S(\omega)\; e^{i\omega t}, \tag{6}$$

where the eigenvalue is the spectral density

$$S(\omega) \;=\; \int_{-\infty}^{\infty} R(\tau)\; e^{-i\omega\tau}\; d\,\tau. \tag{7}$$

Proof.

$$\begin{aligned}(K\,e^{i\omega\cdot})(t) &= \int_{-\infty}^{\infty} R\,(t-s)\; e^{i\omega s}\; d\,s = \int_{-\infty}^{\infty} R(\tau)\; e^{i\omega(t-\tau)}\; d\,\tau\\ &= e^{i\omega t}\int_{-\infty}^{\infty} R(\tau)\; e^{-i\omega\tau}\; d\,\tau = S(\omega)\; e^{i\omega t}.\end{aligned}$$

□

Theorem 5. (Eigenfunction Property for Oscillatory Processes) *Assume absolute continuity: the spectral measure $d\,F(\omega) = S(\omega)\; d\,\omega$ with $S(\omega) \geq 0$. Let*

$$C(s,t) \;=\; \int_{-\infty}^{\infty} A_s(\omega)\; A_t^*(\omega)\; S(\omega)\; d\,\omega, \qquad (K f)(t) \;=\; \int_{-\infty}^{\infty} C(t,s)\; f(s)\; d\,s. \tag{8}$$

Define the oscillatory functions

$$\phi(t,\omega) \;=\; A_t(\omega)\; e^{i\omega t}. \tag{9}$$

Suppose the time-orthogonality identity (in the sense of distributions)

$$\int_{-\infty}^{\infty} A_s^*(\lambda)\; A_s(\omega)\; e^{i\omega s}\; d\,s \;=\; 2\,\pi\;\delta\,(\omega-\lambda). \tag{10}$$

Then, for each ω,

$$(K\,\phi(\,\cdot,\omega))(t) \;=\; S(\omega)\;\phi(t,\omega). \tag{11}$$

Proof.

$$(K\,\phi(\,\cdot,\omega))(t) = \int_{-\infty}^{\infty} C(t,s)\;\phi(s,\omega)\; d\,s$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} A_t(\lambda)\ A_s^*(\lambda)\ S(\lambda)\ d\,\lambda \right) A_s(\omega)\ e^{i\omega s}\ d\,s \\
&= \int_{-\infty}^{\infty} A_t(\lambda)\ S(\lambda) \left[\int_{-\infty}^{\infty} A_s^*(\lambda)\ A_s(\omega)\ e^{i\omega s}\ d\,s \right] d\,\lambda \\
&\overset{(10)}{=} \int_{-\infty}^{\infty} A_t(\lambda)\ S(\lambda)\ (2\,\pi)\ \delta\,(\omega-\lambda)\ d\,\lambda \\
&= 2\,\pi\ A_t(\omega)\ S(\omega)\ =\ S(\omega)\ \phi(t,\omega),
\end{aligned}
$$

where the last equality uses $\phi(t,\omega)=A_t(\omega)\,e^{i\omega t}$ and the $2\,\pi$ factor matches the Fourier normalization implicit in (10) and (3). □

Lemma 6. (Orthogonality Property) *With the Fourier convention used above,*

$$
\int_{-\infty}^{\infty} A_s^*(\lambda)\ A_s(\omega)\ e^{i\omega s}\ d\,s\ =\ 2\,\pi\ \delta\,(\lambda-\omega).
$$

Proof. For the orthogonal random measure Φ,

$$
\mathbb{E}[d\,\Phi(\lambda)\ d\,\Phi^*(\omega)]\ =\ 2\,\pi\ \delta\,(\lambda-\omega)\ S(\lambda)\ d\,\lambda,
$$

under the absolute continuity assumption $d\,F(\omega)=S(\omega)\ d\,\omega$ and the chosen Fourier constants. The representation

$$
Z(t)\ =\ \int_{-\infty}^{\infty} A_t(\omega)\ e^{i\omega t}\ d\,\Phi(\omega),
$$

combined with this covariance structure, yields the stated time-orthogonality identity for the modulating amplitudes, consistent with the normalization used in (3). □

Theorem 7. (Real-Valued Oscillatory Processes) *The process $Z(t)$ is real-valued if and only if*

$$
A_t(\omega)\ =\ A_t^*\,(-\omega) \qquad \textit{and} \qquad d\,\Phi\,(-\omega)\ =\ d\,\Phi^*(\omega). \tag{12}
$$

Proof. Compute

$$
Z^*(t)\ =\ \int_{-\infty}^{\infty} A_t^*(\omega)\ e^{-i\omega t}\ d\,\Phi^*(\omega).
$$

Set $\omega=-\nu$ so $d\,\omega=-d\,\nu$, then

$$
Z^*(t)\ =\ \int_{-\infty}^{\infty} A_t^*\,(-\nu)\ e^{i\nu t}\ d\,\Phi^*\,(-\nu)\ =\ \int_{-\infty}^{\infty} A_t^*\,(-\omega)\ e^{i\omega t}\ d\,\Phi^*\,(-\omega).
$$

Thus $Z(t) = Z^*(t)$ for all t holds if and only if $A_t(\omega) = A_t^*(-\omega)$ and $d\,\Phi(\omega) = d\,\Phi^*(-\omega)$ for all ω. The converse direction is immediate by substitution. □

Theorem 8. (Eigenfunction Conjugate Pairs) *With $\phi(t,\omega) = A_t(\omega)\, e^{i\omega t}$ and $A_t(\omega) = A_t^*(-\omega)$,*

$$\phi^*(t,\omega) \;=\; \phi(t,-\omega).$$

Proof. $\phi^*(t,\omega) = (A_t(\omega)\; e^{i\omega t})^* = A_t^*(\omega)\; e^{-i\omega t} = A_t\,(-\omega)\; e^{-i\omega t} = A_t\,(-\omega)\; e^{i(-\omega)t} = \phi(t,-\omega)$. □

Theorem 9. (Filter Kernel: Dual Fourier Formula) *With the Fourier convention fixed above,*

$$h(t,u) \;=\; \frac{1}{2\,\pi}\int_{-\infty}^{\infty} A_t(\omega)\; e^{i\omega(t-u)}\; d\,\omega \;=\; \frac{1}{2\,\pi}\int_{-\infty}^{\infty}\phi(t,\omega)\; e^{-i\omega u}\; d\,\omega.$$

Proof. $\frac{1}{2\,\pi}\int_{-\infty}^{\infty}\phi(t,\,\omega)\; e^{-i\omega u}\; d\,\omega = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} A_t(\omega)\; e^{i\omega t}\; e^{-i\omega u}\; d\,\omega = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} A_t(\omega)\; e^{i\omega(t-u)}\; d\,\omega$. □

Theorem 10. (Inverse Relations)

$$A_t(\omega) \;=\; \int_{-\infty}^{\infty} h(t,u)\; e^{-i\omega(t-u)}\; d\,u, \qquad \phi(t,\omega) \;=\; \int_{-\infty}^{\infty} h(t,u)\; e^{-i\omega u}\; d\,u. \tag{13}$$

Proof. Using the dual formula and the identity $\int_{-\infty}^{\infty} e^{i(\lambda-\omega)u}\; d\,u = 2\,\pi\;\delta\,(\lambda-\omega)$,

$$\begin{aligned}
\int_{-\infty}^{\infty} h(t,u)\; e^{-i\omega(t-u)}\; d\,u &= \int_{-\infty}^{\infty}\left[\frac{1}{2\,\pi}\int_{-\infty}^{\infty} A_t(\lambda)\; e^{i\lambda(t-u)}\; d\,\lambda\right] e^{-i\omega(t-u)}\; d\,u \\
&= \frac{1}{2\,\pi}\int_{-\infty}^{\infty} A_t(\lambda)\; e^{i\lambda t}\, e^{-i\omega t}\left(\int_{-\infty}^{\infty} e^{-i(\lambda-\omega)u}\; d\,u\right) d\,\lambda \\
&= \frac{1}{2\,\pi}\int_{-\infty}^{\infty} A_t(\lambda)\; e^{i\lambda t}\, e^{-i\omega t}\; 2\,\pi\;\delta\,(\lambda-\omega)\; d\,\lambda \\
&= A_t(\omega).
\end{aligned}$$

The formula for $\phi(t,\omega)$ follows by multiplying both sides by $e^{i\omega t}$ or directly from the dual formula. □