L^2 Norm Preservation Under Smooth Bijective Unbounded Substitutions

by Stephen Crowley

July 30, 2025

Abstract

This paper establishes the fundamental connection between unitary bijections in L^2 spaces and measure-preserving transformations through weighted composition operators on unbounded domains. The investigation demonstrates that for C^1 bijective transformations $g: I \rightarrow J$ between unbounded intervals with positive derivative almost everywhere, L^2 norm preservation is achieved through the unitary change of variables operator $T_g f = f(g(y))\sqrt{g'(y)}$. The analysis proves that strictly increasing functions on unbounded domains yield bijective mappings onto unbounded ranges, providing the foundation for well-defined changes of variables in Lebesgue integration. The central result shows that the operator $T_g: L^2(J, d\,x) \rightarrow L^2(I, d\,y)$ constitutes an isometric isomorphism, with the square root of the Jacobian $\sqrt{g'(y)}$ serving as the unique scaling factor necessary for norm preservation. The necessity of this specific scaling is rigorously established through variational arguments, demonstrating that any alternative weighting function achieving the same isometric property must equal $\sqrt{g'(y)}$ almost everywhere. These findings bridge the change-of-variables formula in real analysis with the unitary structure of L^2 spaces, providing theoretical foundations for applications in ergodic theory and functional analysis on unbounded domains.

Table of contents

1 Introduction 2

2 Smooth Bijective Transformations and L^2 Norm Preservation 2

3 Norm-Preserving Substitution Operators: Measure-Preservation and Unitarity 3

4 Necessity and Canonicality of the Jacobian Weight 3

5 Unitary Operators, Invariant Measures, and Measure-Preservation . . . 4

6 Bibliography 5

Bibliography 5

1 Introduction

This document concerns the structure of L^2-norm-preserving operators induced on L^2 spaces by smooth, bijective, orientation-preserving substitutions $g: I \to J$ on (possibly unbounded) intervals $I, J \subseteq \mathbb{R}$. The topic is fundamental in ergodic theory and operator theory, as it precisely characterizes when a substitution operator corresponds to a unitary operator, and relates directly to the behavior of measures under measure-preserving bijections. The classical result is also crucial for understanding the behavior of the L^2 norm under change of variables. Canonicity and necessity of the Jacobian factor is established, and the role of unboundedness is treated from the start.

2 Smooth Bijective Transformations and L^2 Norm Preservation

Definition 1. *Let $I, J \subseteq \mathbb{R}$ be (possibly unbounded) open intervals. A map $g: I \to J$ is called a* smooth bijection *if g is:*

1. *Bijection between I and J,*

2. *Differentiable on I with $g'(y) > 0$ for almost every $y \in I$ (i.e., g is strictly increasing except possibly on a set of Lebesgue measure zero).*

Lemma 2. *[Bijectivity of Strictly Increasing Unbounded C^1 Maps] Let $I, J \subseteq \mathbb{R}$ be (possibly unbounded) open intervals. Suppose $g: I \to J$ is a C^1 function with $g'(y) > 0$ for all $y \in I$ except possibly a Lebesgue null set, and g is unbounded above and below on I. Then g is bijective onto $J = g(I)$, g^{-1} exists and is also strictly increasing and differentiable a.e.*

Proof. The function g is strictly increasing on every subset of I where $g'(y) > 0$; on the (at most measure-zero) set where $g'(y) = 0$, g remains monotonic and continuous by C^1 regularity. Since I is an interval and g is continuous and strictly increasing almost everywhere, g is injective by the intermediate value property of continuous strictly increasing functions.

Unboundedness of g on I implies that $g(I)$ is also an open interval in $\mathbb{R}$ (possibly the whole real line), so $g: I \to J$ is surjective. Thus, g is bijective from I onto $J = g(I)$. Its inverse $g^{-1}: J \to I$ is again continuous, strictly increasing (except possibly on a null set), and differentiable almost everywhere by the inverse function theorem. □

3 Norm-Preserving Substitution Operators: Measure-Preservation and Unitarity

Theorem 3. *[L^2 Norm Preservation via Jacobian Factor] Let $g: I \to J$ be a smooth bijection in the sense of Definition 1. For any $f \in L^2(J, dx)$, define*

$$\tilde{f}(y) := f(g(y))\sqrt{g'(y)}. \tag{1}$$

Then $\tilde{f} \in L^2(I, dy)$ and

$$\|\tilde{f}\|_{L^2(I,dy)} = \|f\|_{L^2(J,dx)}. \tag{2}$$

Proof. Since $g: I \to J$ is bijective, strictly increasing and differentiable almost everywhere with $g'(y) > 0$ a.e., the change of variables theorem applies (see e.g., [RoydenFitzpatrick], [Folland]).

For any $f \in L^2(J, dx)$,

$$\|\tilde{f}\|^2_{L^2(I,dy)} = \int_I |f(g(y))\sqrt{g'(y)}|^2 \, dy \tag{3}$$

$$= \int_I |f(g(y))|^2 \, g'(y) \, dy \tag{4}$$

By the change of variables formula for Lebesgue integrals, for any measurable function φ and bijective, strictly increasing g as in Lemma 2:

$$\int_I \varphi(g(y)) \, g'(y) \, dy = \int_J \varphi(x) \, dx. \tag{5}$$

Setting $\varphi(x) = |f(x)|^2$, one obtains

$$\int_I |f(g(y))|^2 g'(y) \, dy = \int_J |f(x)|^2 \, dx = \|f\|^2_{L^2(J,dx)} \tag{6}$$

Thus, $\|\tilde{f}\|_{L^2(I,dy)} = \|f\|_{L^2(J,dx)}$ as claimed. □

4 Necessity and Canonicality of the Jacobian Weight

Lemma 4. *[Density of Substitution Images] Let $g: I \to J$ be as in Theorem 3. Then the collection $\{f \circ g : f \in L^2(J, dx)\}$ is dense in $L^2(I, g'(y) \, dy)$.*

Proof. The transformation $T: L^2(J, d\,x) \to L^2(I, g'(y)\ d\,y)$ defined by $T(f) = f \circ g$ is an isometric isomorphism by the change of variables (5). The image of an isomorphism from a complete space is itself complete and thus dense. $\square$

Theorem 5. *[Necessity of the Square Root Jacobian Factor] Let $g: I \to J$ be as above. Suppose $\psi: I \to \mathbb{R}^+$ is measurable and for every $f \in L^2(J, d\,x)$,*

$$|f(g(\cdot)) \cdot \psi(\cdot)|_{L^2(I,dy)} = \|f\|_{L^2(J,dx)}. \tag{7}$$

Then $\psi(y) = \sqrt{g'(y)}$ for almost every $y \in I$.

Proof. Suppose (7) holds for all $f \in L^2(J, d\,x)$. Compute:

$$\int_I |f(g(y))|^2 |\ \psi(y)|^2\ d\,y = \|f\|^2_{L^2(J,dx)} \tag{8}$$

$$= \int_I |f(g(y))|^2\ g'(y)\ d\,y \tag{9}$$

Subtracting, for every f,

$$\int_I |f(g(y))|^2\,(|\psi(y)|^2 - g'(y))\ d\,y = 0 \tag{10}$$

By Lemma 4, the set $\{f(g(y))\}$ is dense in $L^2(I, g'(y)\,d\,y)$. Thus, for every $u \in L^2(I, g'(y)\,d\,y)$,

$$\int_I |u(y)|^2\,(|\psi(y)|^2 - g'(y))\ d\,y = 0 \tag{11}$$

By standard measure-theoretic arguments (cf. [Folland], p. 70), the only way for this to be true for all u is for $|\psi(y)|^2 = g'(y)$ almost everywhere. Since ψ is taken as non-negative, $\psi(y) = \sqrt{g'(y)}$ a.e. $\square$

5 Unitary Operators, Invariant Measures, and Measure-Preservation

Definition 6. *[Koopman Operator] Let $(X, \mathcal{B}, \mu)$ be a probability measure space, $T: X \to X$ a measurable bijection, and μ a T-invariant measure: for all $A \in \mathcal{B}$, $\mu(T^{-1}A) = \mu(A)$. The* Koopman operator *U_T is defined for measurable $f: X \to \mathbb{C}$ by*

$$(U_T f)(x) = f(Tx). \tag{12}$$

Theorem 7. *[Unitarity Corresponds to Measure-Preservation] The Koopman operator U_T on $L^2(X, \mu)$ is unitary if and only if T is invertible and both T and T^{-1} preserve the measure μ.*

Proof. If T is invertible and μ is T-invariant,

$$\|U_T f\|_{L^2(X,\mu)}^2 = \int_X |f(Tx)|^2 \, d\mu(x) = \int_X |f(x)|^2 \, d\mu(x)$$

where the last equality is by change of variables $x = T^{-1}(y)$ and measure-preservation, so U_T is an isometry. Surjectivity follows from invertibility of T and surjectivity of L^2 composition. Conversely, if U_T is unitary, then the above identity must hold for all f. Choosing indicator functions of sets A, it follows that $\mu(T^{-1}(A)) = \mu(A)$, so T preserves the measure. □

6 Bibliography

Bibliography

[RoydenFitzpatrick] H. L. Royden and P. M. Fitzpatrick, *Real Analysis*, Fourth Edition, Pearson, 2010.

[Folland] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, Second Edition, Wiley, 1999.

[Walters] P. Walters, *An Introduction to Ergodic Theory*, Springer, 1982.

[Halmos] P. R. Halmos, *Measure Theory*, Springer, 1974.

[EinsiedlerWard] M. Einsiedler and T. Ward, *Ergodic Theory with a View Towards Number Theory*, Springer, 2011.