Measure-Preserving Bijective Time Changes of Stationary Gaussian Processes Constitute a Subclass of Oscillatory Gaussian Processes

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Abstract

This article establishes that Gaussian processes obtained through measure-preserving bijective unitary time transformations of stationary processes constitute a subclass of oscillatory processes in the sense of Priestley. The transformation $Z(t)=\sqrt{\dot{\theta}(t)}\,X(\theta(t))$, where $X(t)$ represents a realization of a stationary Gaussian process and θ denotes a strictly increasing C^1 differentiable monotonic function, yields an oscillatory process with evolutionary power spectrum $dF_t(\omega)=\dot{\theta}(t)\,d\mu(\omega)$. An explicit unitary transformation between the input stationary process and the transformed oscillatory process is established, preserving the L^2-norm and providing a complete spectral characterization.

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1 Introduction

The theory of non-stationary stochastic processes has found extensive applications in signal processing, time series analysis, and mathematical physics. Among the various classes of non-stationary processes, oscillatory processes as defined by Priestley provide a particularly elegant framework for understanding time-varying spectral characteristics. This work demonstrates that a specific subclass of oscillatory processes can be constructed through measure-preserving bijective time transformations of stationary Gaussian processes.

The main contribution establishes a fundamental connection between stationary and oscillatory processes through unitary operators that preserve essential geometric properties while introducing controlled non-stationarity. This approach provides both theoretical insights and practical methods for generating oscillatory processes with prescribed spectral evolution.

2 Mathematical Background

2.1 Stationary Gaussian Processes

Definition 1. *[Stationary Gaussian Process] A real-valued process* $\{X(t)\}_{t\in\mathbb{R}}$ *constitutes a stationary Gaussian process if it admits the continuous spectral representation*

$$X(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, d\,\Phi(\omega) \tag{1}$$

where $\Phi(\omega)$ *represents an orthogonal-increment process with spectral density*

$$E\,|d\,\Phi(\omega)|^2 = d\,\mu(\omega) = S(\omega) = \frac{\int_{-\infty}^{\infty} K(u)\, e^{-i\omega u}\, d\,u}{2\,\pi} = \dot{\mu}(\omega) \tag{2}$$

and μ *denotes an absolutely continuous measure on* $\mathbb{R}$*.*

2.2 Oscillatory Processes

Definition 2. *[Oscillatory Process] A complex-valued, second-order process* $\{Z(t)\}_{t\in\mathbb{R}}$ *is termed* oscillatory *if there exist*

1. *a family of oscillatory basis functions* $\{\phi_t(\omega)\}_{t\in\mathbb{R}}$ *with*

$$\phi_t(\omega) = A_t(\omega)\, e^{i\omega t} = \int_{-\infty}^{\infty} h(t,u)\, e^{i\omega u}\, d\,u \tag{3}$$

   and a corresponding family of gain functions

$$A_t(\omega) = \frac{\phi_t(\omega)}{e^{i\omega t}} \in L^2(\mu) \tag{4}$$

   with time-dependent filter given by

$$h(t,u) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} \phi_t(\lambda)\, e^{-i\lambda u}\, d\,\lambda \tag{5}$$

2. *a complex orthogonal random measure* $\Phi(\omega)$ *with*

$$E\,|d\,\Phi(\omega)|^2 = d\,\mu(\omega) = S(\omega) \tag{6}$$

such that

$$\begin{aligned} Z(t) &= \int_{-\infty}^{\infty} \phi_t(\omega)\, d\,\Phi(\omega) \\ &= \int_{-\infty}^{\infty} A_t(\omega)\, e^{i\omega t}\, d\,\Phi(\omega) \\ &= \int_{-\infty}^{\infty} h(t,u)\, X(u)\, d\,u \end{aligned} \tag{7}$$

where

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\, d\,\Phi(\lambda) \tag{8}$$

2.3 Time Scaling Functions

Definition 3. *[Scaling Functions] Let $\mathcal{F}$ denote the set of functions $\theta: \mathbb{R} \rightarrow \mathbb{R}$ satisfying*

1. *θ is absolutely continuous with*

$$\dot{\theta}(t) = \frac{d}{dt}\theta(t) \geq 0 \tag{9}$$

almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero

2. *θ is strictly increasing and bijective.*

Remark 4. The conditions in Definition 3 ensure that $\theta^{-1}(s)$ exists and is absolutely continuous. By the inverse function theorem for absolutely continuous functions,

$$\frac{d}{ds}(\theta^{-1})(s) = \frac{1}{\dot{\theta}(\theta^{-1}(s))} = \dot{\theta}(\theta^{-1}(s))^{-1} \tag{10}$$

for almost all s in the range of θ. The condition that $\dot{\theta}(t) = 0$ only on sets of measure zero ensures that $\frac{1}{\dot{\theta}(\theta^{-1}(s))}$ is well-defined almost everywhere.

3 The Unitary Time-Change Transformation

3.1 Definition of the Transformation Operator

Definition 5. *[Unitary Time-Change Operator] For $\theta \in \mathcal{F}$, the operator $M_\theta: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is defined by*

$$(M_\theta f)(t) = \sqrt{\dot{\theta}(t)}\; f(\theta(t)) \tag{11}$$

Definition 6. *[Unitarily Time-Changed Stationary Process] For $\theta \in \mathcal{F}$, applying the unitary time change operator M_θ from Definition 5 to a realization of a stationary process $X(t)$ from the ensemble $\{X(t)\}$ defines a realization of the unitarily time-changed process*

$$Z(t) = \sqrt{\dot{\theta}(t)}\; X(\theta(t)) \quad \forall t \in \mathbb{R} \tag{12}$$

Definition 7. *[Inverse Unitary Time-Change Operator] The inverse operator M_θ^{-1}: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ corresponding to the unitary time-change operator $(M_\theta f)(t)$ defined in Equation (11) is given by*

$$(M_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{13}$$

3.2 Well-Definedness and Unitarity Properties

Lemma 8. *[Well-Definedness of Inverse Operator] The operator M_θ^{-1} in Definition 7 is well-defined for all $\theta \in \mathcal{F}$.*

Proof. Since $\dot{\theta}(t) = 0$ only on sets of measure zero by Definition 3, and θ^{-1} maps sets of measure zero to sets of measure zero (as it preserves absolute continuity), the denominator $\sqrt{\dot{\theta}(\theta^{-1}(s))}$ is positive almost everywhere. The expression in equation (13) is therefore well-defined almost everywhere, which suffices for defining an element of $L^2(\mathbb{R})$. □

Theorem 9. *[Unitarity of Transformation Operator] The operator M_θ defined in equation (11) is unitary, i.e.,*

$$\int_{\mathbb{R}} |(M_\theta f)(t)|^2 dt = \int_{\mathbb{R}} |f(s)|^2 ds \quad \forall f \in L^2(\mathbb{R}) \tag{14}$$

Proof. Let $f \in L^2(\mathbb{R})$. The L^2-norm of $M_\theta f$ is computed as follows:

$$\int_{\mathbb{R}} |(M_\theta f)(t)|^2 dt = \int_{\mathbb{R}} \left| \sqrt{\dot{\theta}(t)}\, f(\theta(t)) \right|^2 dt \tag{15}$$

$$= \int_{\mathbb{R}} \dot{\theta}(t) |f(\theta(t))|^2 dt \tag{16}$$

Applying the change of variables $s = \theta(t)$, since θ is absolutely continuous and strictly increasing, the Jacobian is given by

$$ds = \dot{\theta}(t)\, dt \tag{17}$$

almost everywhere. As t ranges over $\mathbb{R}$, $s = \theta(t)$ ranges over $\mathbb{R}$ due to the bijectivity of θ. Therefore:

$$\int_{\mathbb{R}} \dot{\theta}(t) |f(\theta(t))|^2 dt = \int_{\mathbb{R}} |f(s)|^2 ds \tag{18}$$

This establishes equation (14). To complete the proof of unitarity, it remains to show that M_θ^{-1} is indeed the inverse of M_θ. For any $f \in L^2(\mathbb{R})$:

$$(M_\theta^{-1} M_\theta f)(s) = (M_\theta^{-1})\left[\sqrt{\dot{\theta}(\cdot)}\, f(\theta(\,\cdot\,))\right](s) \tag{19}$$

$$= \frac{\sqrt{\dot{\theta}(\theta^{-1}(s))}\, f(\theta(\theta^{-1}(s)))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{20}$$

$$= f(s) \tag{21}$$

where the last equality uses $\theta(\theta^{-1}(s)) = s$. Similarly, for any $g \in L^2(\mathbb{R})$:

$$(M_\theta M_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)}\, (M_\theta^{-1} g)(\theta(t)) \tag{22}$$

$$= \sqrt{\dot{\theta}(t)}\, \frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} \tag{23}$$

$$= \sqrt{\dot{\theta}(t)}\, \frac{g(t)}{\sqrt{\dot{\theta}(t)}} \tag{24}$$

$$= g(t) \tag{25}$$

Therefore

$$M_\theta M_\theta^{-1} = M_\theta^{-1} M_\theta = I \tag{26}$$

proving that M_θ is unitary. □

3.3 Measure Preservation Properties

Corollary 10. *[Measure Preservation] The transformation M_θ preserves the L^2-measure in the sense that for any measurable set $A \subseteq \mathbb{R}$*

$$\int_A |(M_\theta f)(t)|^2 dt = \int_{\theta(A)} |f(s)|^2 ds \tag{27}$$

Proof. The proof follows the same change of variables argument as in Theorem 9, applied to the characteristic function of the set A. □

Theorem 11. *[Variance Preservation] The transformation defined in equation (12) preserves the L^2-norm in the sense that*

$$\int_I \operatorname{var}(Z(t))\,dt = \int_{\theta(I)} \operatorname{var}(X(s))\,ds \tag{28}$$

for any measurable set $I \subseteq \mathbb{R}$.

Proof. Using the change of variables $s = \theta(t)$ with $ds = \dot{\theta}(t)\,dt$:

$$\begin{aligned}
\int_I \operatorname{var}(Z(t))\,dt &= \int_I \operatorname{var}\Big(\sqrt{\dot{\theta}(t)}\,X(\theta(t))\Big)\,dt && (29)\\
&= \int_I \dot{\theta}(t)\operatorname{var}(X(\theta(t)))\,dt && (30)\\
&= \int_{\theta(I)} \operatorname{var}(X(s))\,ds && (31)
\end{aligned}$$

□

4 Main Results

4.1 Oscillatory Representation

Theorem 12. *[Oscillatory Form] The process $\{Z(t)\}$ defined in equation (12) is oscillatory with oscillatory functions*

$$\phi_t(\omega) = A_t(\omega)\,e^{i\omega t} = \sqrt{\dot{\theta}(t)}\,e^{i\omega\theta(t)} \tag{32}$$

and gain functions

$$A_t(\omega) = \sqrt{\dot{\theta}(t)}\,e^{i\omega(\theta(t)-t)} \tag{33}$$

Proof. Applying the unitary time change operator $(M_\theta f)(t)$ from Definition 5 and substituting the spectral representation (1) of the stationary process $X(t)$:

$$\begin{aligned}
Z(t) &= (M_\theta X)(t) && (34)\\
&= \sqrt{\dot{\theta}(t)}\,X(\theta(t)) && (35)\\
&= \sqrt{\dot{\theta}(t)}\int_{-\infty}^{\infty} e^{i\omega\theta(t)}\,d\Phi(\omega) && (36)\\
&= \int_{-\infty}^{\infty} \sqrt{\dot{\theta}(t)}\,e^{i\omega\theta(t)}\,d\Phi(\omega) && (37)\\
&= \int_{-\infty}^{\infty} \phi_t(\omega)\,d\Phi(\omega) && (38)
\end{aligned}$$

where

$$\phi_t(\omega) = \sqrt{\dot{\theta}(t)}\, e^{i\omega\theta(t)} \tag{39}$$

To verify this constitutes an oscillatory representation according to Definition 2, the expression $\phi_t(\omega)$ must be written in the form of a function of the time-dependent gain $A_t(\lambda)$ as required:

$$\begin{aligned} \phi_t(\omega) =& A_t(\omega)\, e^{i\omega t} && (40)\\ =& \sqrt{\dot{\theta}(t)}\, e^{i\omega(\theta(t)-t)}\, e^{i\omega t} && (41)\\ =& \sqrt{\dot{\theta}(t)}\, e^{i\omega(\theta(t)-t+t)} && (42)\\ =& \sqrt{\dot{\theta}(t)}\, e^{i\omega\theta(t)} && (43) \end{aligned}$$

where

$$A_t(\omega) = \sqrt{\dot{\theta}(t)}\, e^{i\omega(\theta(t)-t)} \tag{44}$$

Since $\dot{\theta}(t) \geq 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of measure zero, the function $A_t(\omega)$ is well-defined almost everywhere. Moreover, $A_t(\cdot) \in L^2(\mu)$ for each t since:

$$\begin{aligned} \int_{-\infty}^{\infty} |A_t(\omega)|^2\, d\,\mu(\omega) =& \int_{-\infty}^{\infty} \dot{\theta}(t)\, d\,\mu(\omega) && (45)\\ =& \dot{\theta}(t) \int_{-\infty}^{\infty} d\,\mu(\omega) && (46)\\ =& \dot{\theta}(t)\, \mu(\mathbb{R}) < \infty && (47) \end{aligned}$$

where the finiteness follows from μ being a finite measure and $\dot{\theta}(t)$ being finite almost everywhere. □

4.2 Evolutionary Power Spectrum

Corollary 13. *[Evolutionary Spectrum] The evolutionary power spectrum is*

$$d\,F_t(\omega) = |A_t(\omega)|^2\, d\,\mu(\omega) = \dot{\theta}(t)\, d\,\mu(\omega) \tag{48}$$

Proof. By Definition 2 and the envelope from Equation (4), the evolutionary power spectrum is:

$$\begin{aligned} d\,F_t(\omega) =& |A_t(\omega)|^2\, d\,\mu(\omega) && (49)\\ =& \left|\sqrt{\dot{\theta}(t)}\, e^{i\omega(\theta(t)-t)}\right|^2 d\,\mu(\omega) && (50)\\ =& \dot{\theta}(t) |e^{i\omega(\theta(t)-t)}|^2\, d\,\mu(\omega) && (51)\\ =& \dot{\theta}(t)\, d\,\mu(\omega) && (52) \end{aligned}$$

since

$$|e^{i\alpha}|=1 \quad \forall \alpha \in \mathbb{R} \tag{53}$$ □

5 Advanced Properties

5.1 Operator Conjugation

Theorem 14. *[Operator Conjugation] Let T_K be the integral covariance operator defined by*

$$(T_K f)(t)=\int_{-\infty}^{\infty} K(|t-s|)\, f(s)\, d\, s \tag{54}$$

where $K(h)$ represents the stationary kernel

$$K(h)=\int_{-\infty}^{\infty} S(\lambda)\, e^{i\lambda h}\, d\, \lambda \tag{55}$$

and let T_{K_θ} be the integral covariance operator defined by

$$\begin{aligned}(T_{K_\theta} f)(t) &= \int_{-\infty}^{\infty} K_\theta(s,t)\, f(s)\, d\, s \\ &= \int_{-\infty}^{\infty} K(|\theta(t)-\theta(s)|)\sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\, f(s)\, d\, s\end{aligned} \tag{56}$$

for the unitarily time-changed kernel

$$K_\theta(s,t)=K(|\theta(t)-\theta(s)|)\sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \tag{57}$$

Then

$$T_{K_\theta}=M_\theta\, T_K\, M_\theta^{-1} \tag{58}$$

Proof. For any $g\in L^2(\mathbb{R})$, the computation of $(M_\theta\, T_K\, M_\theta^{-1}\, g)(t)$ proceeds as follows:

$$(M_\theta^{-1}\, g)(s)=\frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{59}$$

$$(T_K\, M_\theta^{-1}\, g)(t)=\int_{-\infty}^{\infty} K(|t-s|)\frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}}\, d\, s \tag{60}$$

Applying the change of variables $u = \theta^{-1}(s)$, so $s = \theta(u)$ and $d\,s = \dot{\theta}(u)\,d\,u$:

$$(T_K\,M_\theta^{-1}\,g)(t) = \int_{-\infty}^{\infty} K(|t - \theta(u)|)\frac{g(u)}{\sqrt{\dot{\theta}(u)}}\,\dot{\theta}(u)\,d\,u \tag{61}$$

$$= \int_{-\infty}^{\infty} K(|t - \theta(u)|)\,g(u)\sqrt{\dot{\theta}(u)}\,d\,u \tag{62}$$

Now applying M_θ:

$$(M_\theta\,T_K\,M_\theta^{-1}\,g)(t) = \sqrt{\dot{\theta}(t)}\,(T_K\,M_\theta^{-1}\,g)(\theta(t)) \tag{63}$$

$$= \sqrt{\dot{\theta}(t)}\int_{-\infty}^{\infty} K(|\theta(t) - \theta(u)|)\,g(u)\sqrt{\dot{\theta}(u)}\,d\,u \tag{64}$$

Applying the change of variables $s = \theta(u)$ in the reverse direction:

$$(M_\theta\,T_K\,M_\theta^{-1}\,g)(t) = \int_{-\infty}^{\infty} K(|\theta(t) - \theta(s)|)\sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\,g(s)\,d\,s \tag{65}$$

$$= (T_{K_\theta}\,g)(t) \tag{66}$$

This establishes the conjugation relation (58). □

5.2 Zero-Crossing Analysis

Theorem 15. *[Expected Zero-Counting Function] Let $\theta \in \mathcal{F}$ and let*

$$K(\tau) = \operatorname{cov}(X(t), X\,(t + \tau)) \tag{67}$$

be twice differentiable at $\tau = 0$. The expected number of zeros of the process Z_t in $[a, b]$ is

$$\mathbb{E}[N_{[a,b]}] = \sqrt{-\ddot{K}(0)}\,(\theta(b) - \theta(a)) \tag{68}$$

Proof. The covariance function of the time-changed process is

$$K_\theta(s, t) = \operatorname{cov}(Z_s, Z_t) = \sqrt{\dot{\theta}(s)\,\dot{\theta}(t)}\,K(|\theta(t) - \theta(s)|) \tag{69}$$

For the zero-crossing analysis, consideration of the normalized process and application of the Kac-Rice formula yields:

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{-\lim_{s \to t}\frac{\partial^2}{\partial\,s\,\partial\,t}\,K_\theta(s, t)}\,d\,t \tag{70}$$

Computing the mixed partial derivative:

$$\frac{\partial}{\partial t} K_\theta(s,t) = \frac{1}{2} \frac{\ddot{\theta}(t)}{\sqrt{\dot{\theta}(t)}} \sqrt{\dot{\theta}(s)}\, K(|\theta(t)-\theta(s)|) \tag{71}$$

$$+ \sqrt{\dot{\theta}(s)\,\dot{\theta}(t)}\, \dot{K}(|\theta(t)-\theta(s)|)\mathrm{sgn}(\theta(t)-\theta(s))\,\dot{\theta}(t) \tag{72}$$

Taking the limit as $s \to t$ and using the fact that $\dot{K}(0) = 0$ for stationary processes:

$$\lim_{s\to t} \frac{\partial^2}{\partial s\, \partial t} K_\theta(s,t) = \dot{\theta}(s)\,\dot{\theta}(t)\,\ddot{K}(0) \tag{73}$$

$$= \dot{\theta}(t)^2\,\ddot{K}(0) \tag{74}$$

Substituting into the Kac-Rice formula:

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{-\dot{\theta}(t)^2\,\ddot{K}(0)}\, d\,t \tag{75}$$

$$= \sqrt{-\ddot{K}(0)} \int_a^b \dot{\theta}(t)\, d\,t \tag{76}$$

$$= \sqrt{-\ddot{K}(0)}\, (\theta(b) - \theta(a)) \tag{77}$$

Here the second equality uses $\dot{\theta}(t) \geq 0$ almost everywhere. □

6 Conclusion

This analysis establishes that Gaussian processes generated by measure-preserving bijective time changes of stationary processes form a well-defined subclass of oscillatory processes. The key contributions include:

1. The construction of the unitary operator M_θ and its inverse, with proper treatment of the case where

$$\dot{\theta}(t) = 0 \tag{78}$$

on sets of measure zero

2. The explicit oscillatory representation with envelope function

$$A_t(\omega) = \sqrt{\dot{\theta}(t)}\, e^{i\omega(\theta(t)-t)} \tag{79}$$

3. The evolutionary power spectrum formula

$$d\,F_t(\omega) = \dot{\theta}(t)\,d\,\mu(\omega) \tag{80}$$

4. The operator conjugation relationship

$$T_{K_\theta} = M_\theta\,T_K\,M_\theta^{-1} \tag{81}$$

5. A closed-form expression for the expected zero count in terms of the range of the time scaling function

The theoretical framework developed provides both foundational insights into the relationship between stationary and oscillatory processes and practical methods for constructing oscillatory processes with prescribed spectral evolution characteristics.

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