Unitary Change-of-Variables Operators on $L^2(\mathbb{R})$

Definition 1. *A* change-of-variables operator *on $L^2(\mathbb{R},\mu)$ where μ is Lebesgue measure is a bounded linear operator $U\colon L^2(\mathbb{R})\to L^2(\mathbb{R})$ of the form*

$$(Uf)(x)=f(T(x))\cdot w(x) \tag{1}$$

for some measurable map $T\colon\mathbb{R}\to\mathbb{R}$ and measurable weight function $w\colon\mathbb{R}\to\mathbb{C}$ with $|w(x)|>0$ almost everywhere.

Theorem 2. *Let U be a change-of-variables operator as in Definition 1. Then U is unitary if and only if the following conditions hold:*

1. *$T\colon\mathbb{R}\to\mathbb{R}$ is a measurable bijection modulo null sets;*
2. *$\mu\circ T^{-1}\ll\mu$ and $\mu\ll\mu\circ T^{-1}$ (mutual absolute continuity);*
3. *$|w(x)|^2=\frac{d(\mu\circ T)}{d\mu}(x)$ almost everywhere;*
4. *$w(x)=\sqrt{\frac{d(\mu\circ T)}{d\mu}(x)}\cdot e^{i\theta(x)}$ for some measurable phase function $\theta\colon\mathbb{R}\to\mathbb{R}$.*

Furthermore, if T is differentiable almost everywhere with $T'(x)\neq 0$ a.e., then condition (3) becomes

$$|w(x)|^2=|T'(x)| \tag{2}$$

.

Proof. The proof proceeds by establishing necessity and sufficiency separately.

Necessity: Assume U is unitary. Since U is an isometry, for any $f\in L^2(\mathbb{R})$,

$$\|Uf\|_2^2=\|f\|_2^2 \tag{3}$$

Computing the left side:

$$\|Uf\|_2^2=\int_{\mathbb{R}}|f(T(x))|^2|w(x)|^2\,d\mu(x) \tag{4}$$

Define the measure ν by $d\nu = |w|^2 \, d\mu$. By the change-of-variables formula for the push-forward measure,

$$\int_{\mathbb{R}} |f(T(x))|^2 |\, w(x)\,|^2 \, d\mu(x) = \int_{\mathbb{R}} |\, f(y)|^2 \; d\,(T_* \nu)(y) \tag{5}$$

where $(T_* \nu)(A) = \nu(T^{-1}(A))$ for measurable sets A.

From equation (3), we require

$$\int_{\mathbb{R}} |f(y)|^2 \; d\,(T_* \nu)(y) = \int_{\mathbb{R}} |f(y)|^2 \; d\mu(y) \tag{6}$$

for all $f \in L^2(\mathbb{R})$.

This implies $T_* \nu = \mu$ as measures. Therefore, for any measurable set A,

$$\mu(A) = \nu(T^{-1}(A)) = \int_{T^{-1}(A)} |w(x)|^2 \; d\mu(x) \tag{7}$$

For U to be surjective (hence unitary rather than merely isometric), T must be invertible modulo null sets. This requires both directions of absolute continuity in condition (2).

By the Radon-Nikodym theorem, since $\mu \circ T^{-1} \ll \mu$, there exists $\rho \geq 0$ such that

$$\rho(y) = \frac{d\,(\mu \circ T^{-1})}{d\mu}(y) \tag{8}$$

The standard change-of-variables identity gives, for nonnegative measurable g,

$$\int_{\mathbb{R}} g(T(x)) \; d\mu(x) = \int_{\mathbb{R}} g(y)\; \rho(y) \; d\mu(y) \tag{9}$$

Comparing with the isometry requirement from equation (6), we obtain

$$\int_{\mathbb{R}} g(T(x)) |w(x)|^2 \; d\mu(x) = \int_{\mathbb{R}} g(y) \; d\mu(y) \tag{10}$$

This requires

$$|w(x)|^2 = \rho(T(x))^{-1} \tag{11}$$

almost everywhere. By the chain rule for Radon-Nikodym derivatives,

$$|w(x)|^2 = \frac{d\,(\mu \circ T)}{d\mu}(x) \tag{12}$$

The phase freedom in condition (4) follows from the fact that only $|w|^2$ is determined by the isometry condition.

Sufficiency: Conversely, assume conditions (1)-(4) hold. Define U as in Definition 1 with the specified T and w. The computation above shows that U is isometric. Since T is bijective modulo null sets with mutual absolute continuity, the operator U^* exists and is given by

$$(U^* g)(x) = g(T^{-1}(x)) \cdot \overline{w(T^{-1}(x))} \cdot \sqrt{\frac{d\,(\mu \circ T^{-1})}{d\,\mu}(x)} \cdot e^{-i\theta(T^{-1}(x))} \tag{13}$$

Direct computation verifies $U U^* = U^* U = I$, establishing unitarity.

The final statement regarding differentiable T follows from the fact that for such maps,

$$\frac{d\,(\mu \circ T)}{d\,\mu}(x) = |T'(x)| \tag{14}$$

by the classical change-of-variables theorem. □

Lemma 3. *If $T\colon\mathbb{R} \to \mathbb{R}$ is a measurable bijection that is differentiable almost everywhere, then T is either almost everywhere monotone increasing or almost everywhere monotone decreasing.*

Proof. Since T is a bijection of $\mathbb{R}$, the intermediate value theorem and injectivity require that T cannot change monotonicity on any interval where it is continuous. As T is differentiable almost everywhere, it is continuous almost everywhere, and the set where T' exists has full measure. On this set, T' cannot change sign without violating the bijection property, hence $T'(x) \geq 0$ almost everywhere or $T'(x) \leq 0$ almost everywhere. □