Evaluation of an Integral Involving Hypergeometric Functions

by Stephen Crowley

Theorem 1.

$$
\begin{aligned}
I \; &= \int_{-1}^{1} {}_2F_1\!\left(-m, m+1; 1; \frac{1}{2} - \frac{x}{2}\right) {}_2F_1\!\left(-n, n+1; 1; \frac{1}{2} - \frac{x}{2}\right) e^{i x y}\, d\, x \\
&= e^{i y} \sum_{s=0}^{m+n} \frac{\Gamma(s+1)}{2^s (-i\, y)^{s+1}}\, {}_3F_2(-m, -n, -s; m+1, n+1; 1) \left[1 - e^{-2 i y} \sum_{j=0}^{s} \frac{(2\, i\, y)^j}{j!}\right]
\end{aligned} \tag{1}
$$

Proof.

The hypergeometric function ${}_2F_1(a, b; c; z)$ has the finite series representation:

$$
{}_2F_1(-p, b; c; z) = \sum_{k=0}^{p} \frac{(-p)_k (b)_k}{(c)_k\, k!}\, z^k \tag{2}
$$

when p is a non-negative integer. Here, $(a)_k = a\,(a+1)\,(a+2) \cdots (a+k-1)$ is the Pochhammer symbol. For the integral, expand both hypergeometric functions:

$$
{}_2F_1\!\left(-m, m+1; 1; \frac{1}{2} - \frac{x}{2}\right) = \sum_{k=0}^{m} \frac{(-m)_k\, (m+1)_k}{(1)_k\, k!} \left(\frac{1}{2} - \frac{x}{2}\right)^k \tag{3}
$$

$$
{}_2F_1\!\left(-n, n+1; 1; \frac{1}{2} - \frac{x}{2}\right) = \sum_{l=0}^{n} \frac{(-n)_l\, (n+1)_l}{(1)_l\, l!} \left(\frac{1}{2} - \frac{x}{2}\right)^l \tag{4}
$$

Substituting these into the integral:

$$
I = \int_{-1}^{1} \left[\sum_{k=0}^{m} \frac{(-m)_k\, (m+1)_k}{k!} \left(\frac{1}{2} - \frac{x}{2}\right)^k\right] \left[\sum_{l=0}^{n} \frac{(-n)_l\, (n+1)_l}{l!} \left(\frac{1}{2} - \frac{x}{2}\right)^l\right] e^{i x y}\, d\, x \tag{5}
$$

Expand the double sum:

$$
I = \sum_{k=0}^{m} \sum_{l=0}^{n} \frac{(-m)_k\, (m+1)_k}{k!} \frac{(-n)_l\, (n+1)_l}{l!} \int_{-1}^{1} \left(\frac{1}{2} - \frac{x}{2}\right)^{k+l} e^{i x y}\; d\, x \tag{6}
$$

Let $s = k + l$ and evaluate

$$I_s = \int_{-1}^{1} \left(\frac{1}{2} - \frac{x}{2}\right)^s e^{ixy}\, dx \tag{7}$$

Rewrite $\frac{1}{2} - \frac{x}{2}$:

$$\left(\frac{1}{2} - \frac{x}{2}\right)^s = \frac{1}{2^s} (1 - x)^s \tag{8}$$

Thus:

$$I_s = \frac{\int_{-1}^{1} (1 - x)^s e^{ixy}\, dx}{2^s} \tag{9}$$

Let $u = 1 - x$ so that $x = 1 - u$, $dx = -du$ and the limits of integration change:

$$x = -1 \Longrightarrow u = 2 \tag{10}$$

$$x = 1 \Longrightarrow u = 0 \tag{11}$$

thus the integral becomes:

$$I_s = \frac{\int_2^0 u^s e^{iy(1-u)} (-du)}{2^s} = \frac{\int_0^2 u^s e^{iy} e^{-iuy}\, du}{2^s} \tag{12}$$

and after factoring out e^{iy}:

$$I_s = \frac{e^{iy}}{2^s} \int_0^2 u^s e^{-iuy}\, du \tag{13}$$

Substituting the known integral into (13)

$$\int_0^2 u^s e^{-iuy}\, du = \frac{\Gamma(s+1)}{(-iy)^{s+1}} \left[1 - e^{-2iy} \sum_{j=0}^{s} \frac{(2iy)^j}{j!}\right] \tag{14}$$

gives

$$I_s = \frac{e^{iy} \Gamma(s+1)}{2^s (-iy)^{s+1}} \left[1 - e^{-2iy} \sum_{j=0}^{s} \frac{(2iy)^j}{j!}\right] \tag{15}$$

so that returning to the full integral:

$$I = \sum_{k=0}^{m} \sum_{l=0}^{n} \frac{(-m)_k (m+1)_k}{k!} \frac{(-n)_l (n+1)_l}{l!} \cdot \frac{e^{iy} \Gamma(k+l+1)}{2^{k+l} (-iy)^{k+l+1}} \left[1 - e^{-2iy} \sum_{j=0}^{k+l} \frac{(2iy)^j}{j!}\right] \tag{16}$$

Letting $s = k + l$ and rewriting the double sum:

$$I = e^{iy} \sum_{s=0}^{m+n} \frac{\Gamma(s+1)}{2^s (-i\,y)^{s+1}} \left[\sum_{k=\max(0,s-n)}^{\min(s,m)} \frac{(-m)_k\,(m+1)_k}{k!} \frac{(-n)_{s-k}\,(n+1)_{s-k}}{(s-k)!} \right] \left[1 - e^{-2iy} \sum_{j=0}^{s} \frac{(2\,i\,y)^j}{j!} \right] \tag{17}$$

and noting that the inner sum is a hypergeometric function:

$$\sum_{k=\max(0,s-n)}^{\min(s,m)} \frac{(-m)_k\,(m+1)_k}{k!} \frac{(-n)_{s-k}\,(n+1)_{s-k}}{(s-k)!} = {}_3F_2(-m,-n,-s;m+1,n+1;1) \tag{18}$$

it is seen that the result can be expressed as

$$\begin{aligned} I &= \int_{-1}^{1} {}_2F_1\!\left(-m,m+1;1;\frac{1}{2}-\frac{x}{2}\right) {}_2F_1\!\left(-n,n+1;1;\frac{1}{2}-\frac{x}{2}\right) e^{ixy}\,dx \\ &= e^{iy} \sum_{s=0}^{m+n} \frac{\Gamma(s+1)}{2^s (-i\,y)^{s+1}}\, {}_3F_2(-m,-n,-s;m+1,n+1;1) \left[1 - e^{-2iy} \sum_{j=0}^{s} \frac{(2\,i\,y)^j}{j!} \right] \end{aligned} \tag{19}$$

$\square$