Gaussian Processes, de Finetti's Theorem, and Path Integrals: Unifying Perspectives

by Stephen Crowley

Abstract

This document explores the deep connections between Gaussian processes, de Finetti's theorem, path integrals, and ergodic theory. We unify these areas through the concept of averaging over all possible realizations or paths. Beginning with de Finetti's representation $P(X \in A) = \int P(X \in A | f) \, d\mu(f)$ for Gaussian processes, we examine the Karhunen-Loève expansion $X(t) = \sum_i \sqrt{\lambda_i} \, Z_i \, \phi_i(t)$ and its relation to path integrals. We discuss measure equivalence across function spaces: $C([0, T])$ (continuous functions), $H_0^1([0, T])$ (Sobolev space of absolutely continuous functions vanishing at endpoints), and $\mathcal{S}'(\mathbb{R})$ (space of tempered distributions). We introduce advanced concepts like the Cameron-Martin and Girsanov theorems. The connection to ergodic theory is explored through Birkhoff's theorem, $\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \mathbb{E}[f | I](x)$, linking time and ensemble averages. We demonstrate how these relationships stem from fundamental symmetries in the underlying systems, providing a unifying framework for phenomena in mathematics and physics.

1 Introduction

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This document explores the deep connections between Gaussian processes, de Finetti's theorem, path integrals, and various areas of mathematical physics and stochastic analysis. We will show how these seemingly distinct areas are unified through the concept of averaging over all possible realizations or paths.

3 De Finetti's Theorem and Gaussian Processes

Theorem 1. *[De Finetti's Representation for Gaussian Processes] For a stationary Gaussian process $X(t)$, there exists a probability measure μ on the space of continuous functions such that:*

$$P(X \in A) = \int P(X \in A | f) \, d\mu(f)$$

where A is any event in the function space of continuous paths, and $P(X \in A | f)$ is the probability of the event A given a particular realization f.

Remark 2. This representation is fundamentally about averaging over all possible realizations of the process, weighted by their probability under the measure μ.

4 Karhunen-Loève Expansion and Averaging Over Realizations

The Karhunen-Loève expansion provides a concrete way to understand this averaging:

$$X(t) = \sum_i \sqrt{\lambda_i}\, Z_i\, \phi_i(t)$$

where λ_i are eigenvalues of the covariance operator, $\phi_i(t)$ are the corresponding eigenfunctions, and Z_i are independent standard normal random variables.

Theorem 3. *[Averaging Over Realizations] For any functional $F[X]$ of the Gaussian process:*

$$\mathbb{E}[F[X]] = \int F\left[\sum_i \sqrt{\lambda_i}\, z_i\, \phi_i(t)\right] \prod_i \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2}\, d z_i$$

This integral represents averaging $F[X]$ over all possible realizations of the process.

5 Connection to Path Integrals

The formulation of averaging over realizations has striking similarities with path integrals in physics and stochastic analysis.

5.1 Feynman Path Integral

In quantum mechanics, Feynman's path integral formulation expresses the probability amplitude of a particle moving from point a to point b as:

$$K(b, a) = \int \mathcal{D}[x(t)]\, e^{iS[x(t)]/\hbar}$$

where $S[x(t)]$ is the action functional and $\mathcal{D}[x(t)]$ represents integration over all possible paths.

5.2 Wiener Integral

For Brownian motion, the Wiener integral provides a similar formulation:

$$\mathbb{E}[F[W]] = \int F[w(t)]\, d\,\mu_W(w)$$

where μ_W is the Wiener measure on the space of continuous functions.

Remark 4. Both these formulations involve:

1. Integrating over all possible paths (realizations)

2. Weighting paths by appropriate probability measures

6 Unifying Perspective: Function Spaces and Measures

The connection between these formulations lies in the underlying function spaces and measures:

Definition 5. *[Relevant Function Spaces]*

1. $C([0,T])$*: Space of continuous functions on* $[0,T]$
2. $H_0^1([0,T])$*: Sobolev space of absolutely continuous functions vanishing at endpoints*
3. $\mathcal{S}'(\mathbb{R})$*: Space of tempered distributions (for quantum field theory)*

Theorem 6. *[Measure Equivalence] Under appropriate conditions, the following measures on* $C([0,T])$ *are equivalent:*

1. *Gaussian measure induced by a Gaussian process*
2. *Wiener measure*
3. *Path integral measure* $e^{-S[x]/\hbar}\mathcal{D}x$ *(after Wick rotation)*

7 Advanced Connections and Frameworks

7.1 Cameron-Martin Theorem

The Cameron-Martin theorem provides a way to understand how the measure of a Gaussian process changes under translations:

Theorem 7. *[Cameron-Martin] Let* μ *be the measure of a Gaussian process on* $C([0,T])$*, and* $h \in H$ *(the Cameron-Martin space). Then:*

$$\frac{d\mu_h}{d\mu}(f) = \exp\left(\int_0^T \dot{h}(t)\, dW_t - \frac{1}{2}\int_0^T \dot{h}(t)^2\, dt\right)$$

where μ_h *is the translated measure.*

7.2 Girsanov's Theorem

Girsanov's theorem extends this idea to changes of drift in stochastic differential equations:

Theorem 8. *[Girsanov] Let* W_t *be a Wiener process and* X_t *satisfy:*

$$dX_t = b(t, X_t)\, dt + dW_t$$

Then under a new measure $\mathbb{Q}$:

$$\frac{d\,\mathbb{Q}}{d\,\mathbb{P}} = \exp\left(\int_0^T b(t, X_t)\,d\,W_t - \frac{1}{2}\int_0^T b(t, X_t)^2\,d\,t\right)$$

X_t becomes a Wiener process under $\mathbb{Q}$.

7.3 Malliavin Calculus

Malliavin calculus provides a differential calculus on Wiener space, allowing us to define derivatives of functionals of Brownian motion:

Definition 9. *[Malliavin Derivative] For a smooth functional F of Brownian motion, its Malliavin derivative $D\,F$ is defined as:*

$$D\,F_t = \lim_{\epsilon \to 0} \frac{F\left(W + \epsilon\,1_{[0,t]}\right) - F(W)}{\epsilon}$$

8 Connection to Ergodic Theory

The concepts we've discussed - de Finetti's theorem, averaging over realizations of Gaussian processes, and path integrals - have a profound connection to ergodic theory. This connection provides another perspective on why averaging over realizations is so fundamental.

8.1 The Ergodic Theorem

Theorem 10. (Birkhoff's Ergodic Theorem) *Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving dynamical system with $\mu(X) = 1$. For any $f \in L^1(X, \mu)$,*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(T^k x\right) = \mathbb{E}[f|I](x)$$

for μ-almost every x, where I is the σ-algebra of T-invariant sets.

Remark 11. The ergodic theorem essentially states that, for ergodic systems, the time average of a function along a typical trajectory equals the space average over the entire phase space.

8.2 Connection to Gaussian Processes and de Finetti's Theorem

The connection between the ergodic theorem and our previous discussion lies in the equivalence between time averages and ensemble averages:

1. Time Average: In the context of stochastic processes, this corresponds to averaging a functional of the process over time.

2. Ensemble Average: This is equivalent to averaging over all possible realizations of the process, which is precisely what we've been discussing in the context of de Finetti's theorem and path integrals.

Theorem 12. (Ergodicity of Stationary Gaussian Processes) *A stationary Gaussian process is ergodic if and only if its spectral measure is continuous.*

For ergodic Gaussian processes, we have:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T F[X_t]\,dt=\mathbb{E}[F[X]]\quad\text{(almost surely)}$$

The right-hand side is precisely the ensemble average we've been discussing in the context of de Finetti's theorem and path integrals.

8.3 Implications for Path Integrals

In the context of path integrals, ergodicity provides a bridge between:

1. Averaging over all possible paths (ensemble average)
2. Long-time behavior of a single path (time average)

This connection is particularly important in statistical mechanics and quantum field theory, where it justifies the use of ensemble averages to compute observable quantities.

Example 13. (Feynman-Kac Formula) The Feynman-Kac formula provides a connection between path integrals and partial differential equations:

$$u(t,x)=\mathbb{E}_x\left[f(X_t)\exp\left(-\int_0^t V(X_s)\,ds\right)\right]$$

Here, the right-hand side is an average over all paths of a Brownian motion X_t starting at x. The ergodic properties of Brownian motion ensure that this ensemble average is meaningful and computable.

8.4 Ergodicity and de Finetti's Theorem

De Finetti's theorem can be viewed as a statement about the ergodic decomposition of exchangeable sequences:

Theorem 14. (Ergodic Decomposition of Exchangeable Sequences) *The measure μ in de Finetti's theorem represents the ergodic decomposition of the law of an exchangeable sequence. Each ergodic component corresponds to an i.i.d. sequence.*

This perspective provides a deeper understanding of why de Finetti's theorem involves averaging over all possible realizations: it's essentially decomposing the process into its ergodic components.

9 Unifying Perspective: Averaging, Ergodicity, and Symmetry

The connections we've explored - between de Finetti's theorem, Gaussian processes, path integrals, and ergodic theory - all revolve around a central theme: the equivalence between different types of averages due to underlying symmetries or invariances.

1. De Finetti's Theorem: Exchangeability (a type of symmetry) leads to representation as a mixture of i.i.d. sequences.

2. Gaussian Processes: Stationarity (time-translation invariance) often implies ergodicity, connecting time and ensemble averages.

3. Path Integrals: Integration over all paths reflects the symmetry of quantum mechanics under different trajectories.

4. Ergodic Theory: Time-invariance of the measure leads to equivalence of time and space averages.

This unifying perspective suggests that the deep connections we've explored are manifestations of fundamental symmetries in the underlying systems. It provides a powerful framework for understanding and analyzing a wide range of phenomena in mathematics, physics, and beyond.

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