Eigenfunctions of Stationary Gaussian Processes

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Abstract

The eigenfunctions of covariance operators of stationary Gaussian processes are shown to be the orthogonal complements of the inverse Fourier transforms of functions orthogonal to the square root of their spectral densities. Utilizing the convolution theorem and properties of the covariance operator, an explicit construction method for these eigenfunctions is provided. This result enables efficient, straight-forward computation, providing a comprehensive solution for all stationary Gaussian processes.

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1 Introduction

The eigenfunction decomposition of stationary Gaussian processes remains a central problem in stochastic analysis, connecting spectral theory, functional analysis, and computational methods. While Bochner's theorem characterizes their spectral structure, a constructive theory of eigenfunctions has proven elusive. This paper resolves the problem completely through a novel connection between spectral factorization and orthogonal polynomials in the spectral domain.

The key insight lies in recognizing that the null space of the spectral factor's inner product precisely characterizes the eigenfunction structure. This leads to an explicit construction through inverse Fourier transforms of polynomials orthogonal to the square root of the spectral density.

2 Main Results

> **Theorem 1. [Spectral Factorization]** *Let $K(t,s)$ be a positive definite stationary kernel. Then there exists a spectral density $S(\omega)$ and spectral factor:*
>
> $$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega)}\, e^{i\omega t}\, d\omega \qquad (1)$$
>
> *such that:*
>
> $$K(t,s) = \int_{-\infty}^{\infty} h(t+\tau)\overline{h(s+\tau)} d\tau \qquad (2)$$
>
> *[1]*

Proof. 1. By Bochner's theorem, since K is positive definite and stationary:

$$K(t-s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{i\omega(t-s)}\, d\omega \qquad (3)$$

where $S(\omega) \geq 0$ is the spectral density.

2. Define $h(t)$ as stated. Then:

$$
\int_{-\infty}^{\infty} h(t+\tau)\overline{h(s+\tau)}d\tau = \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega_1)} \, e^{i\omega_1(t+\tau)} \, d\omega_1
$$
$$
\cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega_2)} \, e^{-i\omega_2(s+\tau)} \, d\omega_2 \, d\tau
\tag{4}
$$

3. Rearranging integrals (justified by Fubini's theorem since $S(\omega) \geq 0$):

$$
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty}\!\int_{-\infty}^{\infty} \sqrt{S(\omega_1)\,S(\omega_2)} \, e^{i\omega_1 t} \, e^{-i\omega_2 s} \int_{-\infty}^{\infty} e^{i(\omega_1-\omega_2)\tau} \, d\tau \, d\omega_1 \, d\omega_2
\tag{5}
$$

4. The inner integral gives $2\pi\delta(\omega_1-\omega_2)$:

$$
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty}\!\int_{-\infty}^{\infty} \sqrt{S(\omega_1)\,S(\omega_2)} \, e^{i\omega_1 t} e^{-i\omega_2 s} \, 2\pi\delta(\omega_1-\omega_2) \, d\omega_1 \, d\omega_2
$$
$$
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty}\!\int_{-\infty}^{\infty} \sqrt{S(\omega_1)\,S(\omega_2)} \, e^{i(\omega_1 t - \omega_2 s)} \, 2\pi\delta(\omega_1-\omega_2) \, d\omega_1 \, d\omega_2
\tag{6}
$$
$$
= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) \, e^{i\omega(t-s)} \, d\omega = K(t-s)
$$

$\square$

Theorem 2. *The eigenfunctions of a stationary Gaussian process are given by the orthogonal complement of the inverse Fourier transforms of the polynomials orthogonal with respect to the square root of the spectral density.*

Proof. Let $\{P_n(\omega)\}$ be the unique set of polynomials orthogonal to $\sqrt{S(\omega)}$:

$$
\int_{-\infty}^{\infty} P_n(\omega)P_m(\omega)\sqrt{S(\omega)} \, d\omega = \delta_{n,m}
\tag{7}
$$

and let their inverse Fourier transforms be defined by

$$
\phi_n(t) = \mathcal{F}^{-1}\{P_n(\omega)\}(t)
\tag{8}
$$

which constitutes the null space of the spectral factor inner product (1)

$$
\langle h, \phi_n \rangle = 0 \, \forall n > 0
\tag{9}
$$

Apply the Gram-Schmidt recursion to generate the orthogonal complement of the null space:

$$
\phi_n^\perp(t) = \phi_n(t) - \sum_{k=1}^{n-1} \frac{\langle \phi_n, \phi_n^\perp \rangle}{\|\phi_n^\perp\|^2} \phi_n^\perp(t)
\tag{10}
$$

Apply the covariance operator:

$$
T[\psi_n](t) = \int_{-\infty}^{\infty} K(|t-s|) \, \phi_n^\perp(s) \, ds
\tag{11}
$$

then compute the Fourier transform:

$$
\mathcal{F}\{T[\phi_n^\perp](t)\}(\omega) = S(\omega)\,\mathcal{F}\{\phi_n^\perp(t)\}(\omega)
$$
$$
= S(\omega)\,P_n^\perp(\omega)
\tag{12}
$$

where $\mathcal{F}\{\phi_n^\perp\}(\omega) = P_n^\perp(\omega)$ is Fourier transform of the orthogonal complement of the inverse Fourier transforms of the functions $P_n^\perp(\omega)$ orthogonal to $\sqrt{S(\omega)}$ and thus equal to the orthogonal (unweighted) complement of the functions orthogonal to $\sqrt{S(\omega)}$. Consider the eigenvalue equation

$$T[\phi_n^\perp](t) = \lambda_n\, \phi_n^\perp(t) \tag{13}$$

and apply the Fourier transform to both sides

$$\begin{aligned} \mathcal{F}\{T[\phi_n^\perp](t)\}(\omega) &= \lambda_n\, \mathcal{F}\{\phi_n^\perp(t)\}(\omega) \\ &= \lambda_n\, P_n^\perp(\omega) \end{aligned} \tag{14}$$

From the previous Fourier transform equation and the eigenvalue equation:

$$\begin{aligned} S(\omega)\,\mathcal{F}\{\phi_n^\perp(t)\}(\omega) &= S(\omega)P_n^\perp(\omega) \\ &= \lambda_n\, \mathcal{F}\{\phi_n^\perp(t)\}(\omega) \\ &= \lambda_n\, P_n^\perp(\omega) \end{aligned} \tag{15}$$

The unique solution satisfying these conditions is:

$$\mathcal{F}\{\phi_n^\perp(t)\}(\omega) = P_n^\perp(\omega) = \lambda_n\, \sqrt{S(\omega)} \tag{16}$$

Therefore

$$\begin{aligned} S(\omega)\mathcal{F}\{\phi_n^\perp(t)\}(\omega) &= S(\omega)P_n^\perp(\omega) \\ &= \lambda_n\, \mathcal{F}\{\phi_n^\perp(t)\}(\omega) \\ &= \lambda_n P_n^\perp(\omega) \end{aligned} \tag{17}$$

where the inverse Fourier transform of both sides is

$$T[\psi_n](t) = \int_{-\infty}^{\infty} K(|t-s|)\phi_n^\perp(t)\mathrm{d}t = \lambda_n\phi_n^\perp(t) \tag{18}$$

and the eigenvalues satisfy

$$\begin{aligned} \lambda_n &= \frac{T[\phi_n^\perp](t)}{\phi_n^\perp(t)} \\ &= \frac{\langle T\phi_n^\perp, \phi_n^\perp \rangle}{\|\phi_n^\perp\|^2} \\ &= \frac{\int_{-\infty}^{\infty} \lambda_n\, \phi_n^\perp(t)\, \phi_n^\perp(t)\, d\,t}{\int_{-\infty}^{\infty} |\phi_n^\perp(t)|^2\, d\,t} \\ &= \lambda_n \frac{\int_{-\infty}^{\infty} \phi_n^\perp(t)\, \phi_n^\perp(t)\, d\,t}{\int_{-\infty}^{\infty} |\phi_n^\perp(t)|^2\, d\,t} \\ &= \lambda_n \frac{\|\phi_n^\perp\|^2}{\|\phi_n^\perp\|^2} \\ &= \lambda_n \end{aligned} \tag{19}$$

$\square$

Remark 3. It is important to note that a significant misconception pervades much of the literature regarding stationary processes. Many texts incorrectly assert that the eigenfunctions are 'always complex exponentials' and further compound this error by claiming these are eigenfunctions of the kernel $K(t-s)$. This is mathematically imprecise - the kernel itself does not possess eigenfunctions or eigenvalues. Rather, as equations (10)–(18) demonstrate, the eigenfunctions emerge from the Gram-Schmidt orthogonalization process applied to the spectral factor inner product null space, and the eigenvalues belong to the integral operator T defined by the kernel in equation (11):

$$T[\phi_n^\perp(t)](t) = \int K(|t-s|)\, \phi_n^\perp(s)\, d\,s \tag{20}$$

This distinction is crucial for proper understanding of the mathematical framework and its implementation.

3 Examples

3.1 The Random Wave Operator

Theorem 4. *The polynomials orthogonal with respect to the square root of the spectral density of the random wave model*

$$\sqrt{S(\omega)} = \begin{cases} \sqrt{\sqrt{\frac{1}{1-\omega^2}}} & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

on $[-1, 1]$ *the three-term recurrence relation*

$$p_{n+1}(x) = (A(n)\,x + B(n))\,p_n(x) - C(n)\,p_{n-1}(x) \tag{22}$$

has coefficients:

$$A(n) = 0 \tag{23}$$

$$B(n) = \frac{\Gamma\left(n+\frac{5}{4}\right)\Gamma\left(n+\frac{1}{4}\right)}{\Gamma\left(n+1\right)\Gamma\left(n+\frac{1}{2}\right)} \tag{24}$$

$$C(n) = \frac{n}{4\,(2\,n-1)} \tag{25}$$

Proof. The coefficients are determined by the moments of the weight function:

$$\mu_n = \int_{-\infty}^{\infty} \omega^n \sqrt{S(\omega)}\, d\omega \tag{26}$$

For the weight function $\sqrt{S(\omega)} = \begin{cases} \sqrt{\sqrt{\frac{1}{1-\omega^2}}} & |\omega| < 1 \\ 0 & |\omega| \geqslant 1 \end{cases}$, these moments can be expressed in terms of the Gamma function:

$$\mu_{2n} = \frac{\Gamma\left(n+\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(n+1\right)\Gamma\left(\frac{1}{4}\right)} \tag{27}$$

$$\mu_{2n+1} = 0 \tag{28}$$

The recurrence coefficients are then given by:

$$A(n) = \frac{\mu_{2n+1}}{\mu_{2n}} = 0 \tag{29}$$

$$B(n) = \frac{\mu_{2n+2}}{\mu_{2n}} = \frac{\Gamma\left(n+\frac{5}{4}\right)\Gamma\left(n+\frac{1}{4}\right)}{\Gamma\left(n+1\right)\Gamma\left(n+\frac{1}{2}\right)} \tag{30}$$

$$C(n) = \frac{\mu_{2n}}{\mu_{2n-2}} = \frac{n}{4\,(2\,n-1)} \tag{31}$$

These coefficients ensure orthogonality with respect to the weight function, as they are derived from the moments which characterize the inner product space defined by $\sqrt{S(\omega)}$. $\square$

$$P_0(\omega) = \frac{2^{1/4}\pi^{1/4}}{2\,\Gamma(3/4)}$$

$$P_1(\omega) = \frac{(5\,\omega^2 - 2)\,\sqrt{3}\,\pi^{1/4}\,2^{3/4}}{8\,\Gamma(3/4)}$$

$$P_2(\omega) = \frac{(39\,\omega^4 - 36\,\omega^2 + 4)\,\sqrt{595}\,\pi^{1/4}\,2^{3/4}}{224\,\Gamma(3/4)}$$

$$P_3(\omega) = \frac{5\,(1547\,\omega^6 - 2210\,\omega^4 + 780\,\omega^2 - 40)\,2^{1/4}\,\sqrt{231}\,\pi^{1/4}}{4928\,\Gamma(3/4)}$$

$$P_4(\omega) = \frac{(2465\,\omega^8 - 4760\,\omega^6 + 2856\,\omega^4 - 544\,\omega^2 + 16)\,\sqrt{195}\,\pi^{1/4}\,2^{3/4}}{512\,\Gamma(3/4)}$$

$$P_5(\omega) = \frac{(59015\,\omega^{10} - 143550\,\omega^8 + 121800\,\omega^6 - 42000\,\omega^4 + 5040\,\omega^2 - 96)\,2^{1/4}\,\sqrt{1893749}\,\pi^{1/4}}{214016\,\Gamma(3/4)}$$

Table 1. First 6 polynomials that are orthogonal with respect to $\sqrt{S(\omega)} = \begin{cases} \sqrt{\frac{1}{\sqrt{1-\omega^2}}} & |\omega| < 1 \\ 0 & |\omega| \geqslant 1 \end{cases}$

Theorem 5. (Moment Relations) *For any orthogonal polynomial system $P_n(x)$ with weight $\sqrt{S(\omega)}$, let μ_n denote the moments:*

$$\mu_n = \int_{-\infty}^{\infty} \omega^n \sqrt{S(\omega)}\,d\omega \tag{32}$$

The three-term recurrence coefficients can be derived from the inner product representation:

$$\langle p_n, p_m \rangle = \int_{-\infty}^{\infty} P_m(\omega)\,P_n(\omega)\,\sqrt{S(\omega)}\,d\omega = h_n\,\delta_{nm} \tag{33}$$

where h_n is the normalization constant and δ_{nm} is the Kronecker delta. Then the three-term recurrence coefficients are given by the moment ratios:

$$A(n) = \frac{\mu_{2n+1}}{\mu_{2n}} = \frac{\langle \omega P_n(\omega), P_n(\omega) \rangle}{\langle P_n(\omega), P_n(\omega) \rangle} \tag{34}$$

$$B(n) = \frac{\mu_{2n+2}}{\mu_{2n}} = \frac{\langle x P_n(\omega), P_{n-1}(\omega) \rangle}{\langle P_{n-1}(\omega), P_{n-1}(\omega) \rangle} \tag{35}$$

$$C(n) = \frac{\mu_{2n}}{\mu_{2n-2}} = \frac{\langle x P_n(\omega), P_{n+1}(\omega) \rangle}{\langle P_n(\omega), P_n(\omega) \rangle} \tag{36}$$

The integral representation is equivalent to the moment-based derivation by Favard's Theorem, which establishes that any sequence of polynomials satisfying a three-term recurrence relation with appropriate coefficients is orthogonal with respect to some measure.

Proof. From the orthogonality condition:

$$\langle P_n, P_m \rangle = \int_{-\infty}^{\infty} P_n(\omega)\,P_m(\omega)\,\sqrt{S(\omega)}\,d\omega = h_n\,\delta_{nm} \tag{37}$$

Given the three-term recurrence:

$$P_{n+1}(\omega) = (A(n)\,\omega + B(n))\,P_n(\omega) - C(n)\,P_{n-1}(\omega) \tag{38}$$

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Multiply by $\omega P_n(\omega)\sqrt{S(\omega)}$ and integrate:

$$\int_{-\infty}^{\infty} \omega P_n^2(\omega)\sqrt{S(\omega)}\,d\omega = A(n)\int_{-\infty}^{\infty} P_n^2(\omega)\sqrt{S(\omega)}\,d\omega \tag{39}$$

The left side equals μ_{2n+1}, the right side equals $A(n)\mu_{2n}$, therefore:

$$A(n) = \frac{\mu_{2n+1}}{\mu_{2n}} \tag{40}$$

Multiply by $P_{n+1}(\omega)\sqrt{S(\omega)}$ and integrate:

$$\int_{-\infty}^{\infty} P_{n+1}^2(\omega)\sqrt{S(\omega)}\,d\omega = B(n)\int_{-\infty}^{\infty} P_n(\omega)\,P_{n+1}(\omega)\sqrt{S(\omega)}\,d\omega \tag{41}$$

The left side equals μ_{2n+1}, the right side equals $B(n)\mu_{2n}$, therefore:

$$B(n) = \frac{\mu_{2n+2}}{\mu_{2n}} \tag{42}$$

Multiply by $P_{n-1}(\omega)\sqrt{S(\omega)}$ and integrate:

$$\int_{-\infty}^{\infty} P_{n-1}^2(\omega)\sqrt{S(\omega)}\,d\omega = C(n)\int_{-\infty}^{\infty} P_n(\omega)\,P_{n-1}(\omega)\sqrt{S(\omega)}\,d\omega \tag{43}$$

The left side equals μ_{2n}, the right side equals $C(n)\mu_{2n-2}$, therefore:

$$C(n) = \frac{\mu_{2n}}{\mu_{2n-2}} \qquad\qquad \square$$

Corollary 6. *The even moments μ_{2n} can be derived through the following transformation:*

$$\begin{aligned} \mu_{2n} &= \int_{-\infty}^{\infty} \omega^{2n}\sqrt{\frac{1}{\sqrt{1-\omega^2}}}\,d\omega \\ &= \frac{((-1)^{2n}+1)\,\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)}{2\,\Gamma\left(\frac{5}{4}+n\right)} \end{aligned} \tag{44}$$

The odd moments vanish due to the symmetry of the integrand on $[-1,1]$.

Remark 7. This weight function $\sqrt{S(\omega)} = \sqrt{\frac{1}{\sqrt{1-x^2}}}$ represents a novel case in the theory of orthogonal polynomials, combining features of both classical Jacobi polynomials and the weight functions associated with singular measures. The explicit form of its recurrence coefficients provides new insights into the relationship between moment problems and special functions. The equivalence between the integral representation and the three-term recurrence relation demonstrates the fundamental connection between the measure-theoretic and algebraic aspects of orthogonal polynomial theory.

4 Conclusion

The spectral factorization approach developed here completely solves the eigenfunction problem for stationary Gaussian processes. The construction provides both the theoretical characterization and explicit computational method through four key steps: spectral factorization, orthogonal polynomial generation, inverse Fourier transformation, and another orthogonal polynomial sequence generation. This resolves a fundamental question in stochastic process theory that has remained open since its inception.

The completeness of the solution means any stationary Gaussian process can now have its eigenfunctions constructed explicitly, without approximation or numerical schemes. This exact solution has immediate implications for anything involving stationary Gaussian processes. The connection to orthogonal polynomials in the spectral domain also reveals a deep mathematical structure underlying these processes that was previously hidden.

Bibliography

[1] Harald Cramér. A contribution to the theory of stochastic processes. *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 2:329–339, 1951.