Spectral Theory of Oscillatory Non-Stationary Processes and RKHS Framework

by Stephen Crowley

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1 Introduction and Fundamental Concepts

Definition 1. *[Stochastic Process] A stochastic process $\{X(t)\}_{t\in T}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of random variables indexed by $t \in T \subseteq \mathbb{R}$, where for each $\omega \in \Omega$, the function $t \mapsto X(t, \omega)$ is called a sample path or realization.*

Definition 2. *[Spectral Measure] For a second-order stationary process $\{X(t)\}$, the spectral measure μ is a finite, non-negative measure on $\mathbb{R}$ such that the covariance function admits the representation:*

$$R(s,t) = \mathbb{E}[X(s), X(t)] = \int_{-\infty}^{\infty} e^{i\omega(s-t)}\, d\,\mu(\omega) \tag{1}$$

The spectral measure is deterministic and characterizes the second-order properties of the entire process.

Definition 3. *[Random Measure] Let $d\,Z(\omega)$ be a complex-valued, orthogonal increment process on $\mathbb{R}$ satisfying:*

1. $\mathbb{E}[d\,Z(\omega)] = 0$ *for all* ω
2. $\mathbb{E}[d\,Z(\omega_1)\overline{d\,Z(\omega_2)}] = \delta\,(\omega_1 - \omega_2)\, d\,\mu(\omega_1)$
3. *For disjoint intervals* I_1, I_2*:* $\mathbb{E}[Z(I_1)\overline{Z(I_2)}] = 0$

This random measure $d\,Z(\omega)$ varies between sample paths and uniquely determines each realization.

Theorem 4. *[Spectral Representation Theorem] Every second-order stationary process $\{X(t)\}$ admits the representation:*

$$X(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, d\,Z(\omega) \tag{2}$$

where $d\,Z(\omega)$ is the random measure and the integral converges in mean-square.

Proof. Let $\{e_n\}$ be an orthonormal basis for $L^2(\mu)$. Define

$$Z_n = \langle X, e_n \rangle_{L^2(\mathbb{P})} \tag{3}$$

Then:

$$X(t) = \sum_{n=1}^{\infty} Z_n\, e_n(t) \tag{4}$$

$$= \sum_{n=1}^{\infty} Z_n \int_{\mathbb{R}} e_n(\omega)\, e^{i\omega t}\, d\,\mu(\omega) \tag{5}$$

$$= \int_{\mathbb{R}} e^{i\omega t} \left(\sum_{n=1}^{\infty} Z_n \, e_n(\omega) \right) d\,\mu(\omega) \tag{6}$$

$$= \int_{\mathbb{R}} e^{i\omega t} \, d\,Z(\omega) \tag{7}$$

where

$$d\,Z(\omega) = \left(\sum Z_n \, e_n(\omega) \right) d\,\mu(\omega) \tag{8}$$

defines the random measure. □

2 Priestley's Theory of Oscillatory Processes

Definition 5. *[Oscillatory Function] A function $\phi_t(\omega): \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is oscillatory if it can be written as:*

$$\phi_t(\omega) = A_t(\omega) \, e^{i\omega t} \tag{9}$$

where $A_t(\omega)$ is a quadratically integrable gain-function.

Definition 6. *[Oscillatory Process] A stochastic process $\{X(t)\}_{t \in \mathbb{R}}$ is oscillatory if there exists a family of oscillatory functions $\{\phi_t(\omega)\}$ and a random measure $d\,Z(\omega)$ such that:*

$$X(t) = \int_{\mathbb{R}} \phi_t(\omega) \, d\,Z(\omega) = \int_{\mathbb{R}} A_t(\omega) \, e^{i\omega t} \, d\,Z(\omega) \tag{10}$$

where the integral converges in mean-square for each t.

Theorem 7. *[Evolutionary Spectrum] For an oscillatory process with slowly varying amplitude $A_t(\omega)$, the evolutionary power spectrum is defined by:*

$$d\,H_t(\omega) = |A_t(\omega)|^2 \, d\,\mu(\omega) \tag{11}$$

where μ is the spectral measure of the random measure $d\,Z(\omega)$.

Proof. The second moment of the process at time t is:

$$\mathbb{E}[|X(t)|^2] = \mathbb{E}\left[\left| \int_{\mathbb{R}} A_t(\omega) \, e^{i\omega t} \, d\,Z(\omega) \right|^2 \right] \tag{12}$$

$$= \int_{\mathbb{R}} |A_t(\omega)|^2 \mathbb{E}[|d\,Z(\omega)|^2] \tag{13}$$

$$= \int_{\mathbb{R}} |A_t(\omega)|^2\, d\,\mu(\omega) \tag{14}$$

$$= \int_{\mathbb{R}} d\,H_t(\omega) \tag{15}$$

The evolutionary spectrum $d\,H_t(\omega)$ thus captures the time-varying spectral content. □

3 Reproducing Kernel Hilbert Spaces for Stochastic Processes

Definition 8. *[RKHS] A Hilbert space $\mathbb{H}$ of functions $f\colon \Omega \to \mathbb{C}$ is a reproducing kernel Hilbert space if there exists a function $K\colon \Omega \times \Omega \to \mathbb{C}$ (the reproducing kernel) such that:*

1. *For each $x \in \Omega$, $K(\cdot, x) \in \mathbb{H}$*
2. *For all $f \in \mathbb{H}$ and $x \in \Omega$: $f(x) = \langle f, K(\cdot, x)\rangle_{\mathbb{H}}$ (reproducing property)*

Theorem 9. *[Moore-Aronszajn Theorem] Every positive definite kernel $K\colon \Omega \times \Omega \to \mathbb{C}$ uniquely determines an RKHS $\mathbb{H}_K$ with reproducing kernel K.*

Proof. Define the pre-Hilbert space as the span of $\{K(\,\cdot, x)\colon x \in \Omega\}$ with inner product:

$$\left\langle \sum_{i=1}^{n} a_i\, K\!\left(\,\cdot, x_i), \sum_{j=1}^{m} b_j\, K(\cdot, y_j)\right. \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i \bar{b_j} K(x_i, y_j) \tag{16}$$

Complete this space to obtain $\mathbb{H}_K$. The reproducing property follows by construction. □

Definition 10. (Inverse Covariance Operator for Oscillatory Processes) *For an oscillatory process with representation*

$$X(t) = \int_{\mathbb{R}} A_t(\omega)\, e^{i\omega t}\, d\, Z(\omega) \tag{17}$$

and covariance function

$$R\,(s,t) = \int_{\mathbb{R}} A_s(\omega)\overline{A_t(\omega)} e^{i\omega(s-t)}\, d\, \mu(\omega) \tag{18}$$

the inverse covariance operator R^{-1} acts on functions $f \in L^2(\mathbb{R})$ by

$$\langle f, R^{-1}\, g\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \frac{\tilde{f}(\omega)\overline{\tilde{g}(\omega)}}{M(\omega)}\, d\, \mu(\omega) \tag{19}$$

where

$$\tilde{f}(\omega) = \int_{\mathbb{R}} f(t)\overline{A_t(\omega)} e^{-i\omega t}\, d\, t \tag{20}$$

and the time-averaged squared amplitude $M(\omega)$ is given by

$$M(\omega) = \lim_{T \to \infty} \frac{1}{2\,T} \int_{-T}^{T} |A_t(\omega)|^2\, d\, t \tag{21}$$

Definition 11. *[Covariance Kernel RKHS] For a stochastic process $\{X(t)\}$ with covariance function $R(s,t) = \mathrm{Cov}(X(s), X(t))$, the associated RKHS $\mathscr{H}_R$ consists of functions with finite norm:*

$$\|f\|_{\mathscr{H}_R}^2 = \iint_{\mathbb{R}^2} f(s)\overline{f(t)} d\, R^{-1}(s,t) \tag{22}$$

where R^{-1} is the inverse covariance operator.

Theorem 12. *[Sample Paths in RKHS] For a Gaussian process $\{X(t)\}$ with continuous covariance $R(s,t)$, the sample paths belong to $\mathscr{H}_R$ almost surely if and only if:*

$$\int_{\mathbb{R}} \frac{d\,\mu(\omega)}{1+\omega^2} < \infty \tag{23}$$

where μ is the spectral measure.

Proof. The condition ensures that $\mathbb{E}[\|X\|_{\mathscr{H}_R}^2] < \infty$. For Gaussian processes, this implies almost sure membership in $\mathscr{H}_R$ by the Fernique theorem and properties of

Gaussian measures on Banach spaces. □

4 Bandlimited Processes and the Sinc Kernel

Definition 13. *[Bandlimited Process] A stochastic process $\{X(t)\}$ is Ω-bandlimited if its spectral measure μ is supported on $[-\Omega, \Omega]$.*

Theorem 14. *[Sinc Kernel RKHS] The space of π-bandlimited functions forms an RKHS with reproducing kernel:*

$$K(s,t) = \mathrm{sinc}(s-t) = \frac{\sin\pi(s-t)}{\pi(s-t)} \tag{24}$$

and inner product:

$$\langle f, g, \_ \rangle \mathscr{H} = \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega)\overline{\hat{g}(\omega)}d\omega \tag{25}$$

Proof. For any π-bandlimited function f:

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(\omega)\, e^{i\omega t}\, d\omega && (26)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\infty}^{\infty} f(s)\, e^{-i\omega s}\, ds\right) e^{i\omega t}\, d\omega && (27)\\ &= \int_{-\infty}^{\infty} f(s)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega(t-s)}\, d\omega\right) ds && (28)\\ &= \int_{-\infty}^{\infty} f(s)\mathrm{sinc}(t-s)\, ds && (29)\\ &= \langle f, \mathrm{sinc}(\cdot - t), \_ \rangle \mathscr{H} && (30) \end{aligned}$$

Thus the sinc function serves as the reproducing kernel. □

Corollary 15. *[Sampling Theorem in RKHS] Every π-bandlimited function f can be perfectly reconstructed from its samples:*

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\mathrm{sinc}(t-n) \tag{31}$$

Theorem 16. *[Bandlimited Random Process Reconstruction] For a π-bandlimited stochastic process $\{X(t)\}$, each sample path can be reconstructed from its samples:*

$$X(t) = \sum_{n=-\infty}^{\infty} X(n)\mathrm{sinc}(t-n) \tag{32}$$

with probability 1, provided $\mathbb{E}[\sum_{n=-\infty}^{\infty} |X(n)|^2] < \infty$.

5 Frame Theory and Analysis-Synthesis Operators

Definition 17. *[Frame] A sequence $\{f_k\}_{k\in\mathcal{I}}$ in a Hilbert space $\mathscr{H}$ is a frame if there exist constants $A, B > 0$ such that:*

$$A\|f\|^2 \leq \sum_{k\in\mathcal{I}} |\langle f, f_k, | \rangle^2 \leq B\|f\|^2 \tag{33}$$

for all $f \in \mathscr{H}$. The constants A and B are called frame bounds.

Definition 18. (Analysis Operator) *For a frame $\{f_k\}$, the analysis operator $T: \mathscr{H} \to \ell^2(\mathcal{I})$ is defined by:*

$$Tf = \{c_k\}_{k\in\mathcal{I}} \quad \textit{where} \quad c_k = \langle f, f_k \rangle \tag{34}$$

Definition 19. (Synthesis Operator) *The synthesis operator $T^*: \ell^2(\mathcal{I}) \to \mathscr{H}$ is the adjoint of the analysis operator:*

$$T^*\{c_k\} = \sum_{k\in\mathcal{I}} c_k f_k \tag{35}$$

Theorem 20. *[Frame Operator Properties] The frame operator $S = T^*T: \mathscr{H} \to \mathscr{H}$ satisfies:*

1. *$Sf = \sum_{k\in\mathcal{I}} \langle f, f_k, f \rangle_k$*
2. *S is positive, self-adjoint, and invertible*
3. *$A\mathbf{I} \leq S \leq B\mathbf{I}$ where A, B are the frame bounds*
4. *The canonical dual frame is $\{\widetilde{f_k}\} = \{S^{-1} f_k\}$*

Proof. For any $f, g \in \mathscr{H}$:

For any $f, g \in \mathscr{H}$:

$$\begin{aligned}
\langle Sf, g\rangle &= \langle T^* Tf, g\rangle && (36)\\
&= \langle Tf, Tg\rangle && (37)\\
&= \sum_{k\in\mathcal{I}} \langle f, f_k\rangle \overline{\langle g, f_k\rangle} && (38)\\
&= \sum_{k\in\mathcal{I}} \langle f, f_k\rangle \langle f_k, g\rangle && (39)\\
&= \left\langle f, \sum_{k\in\mathcal{I}} \langle g, f_k\rangle f_k \right\rangle && (40)
\end{aligned}$$

Thus $Sf = \sum_{k\in\mathcal{I}} \langle f, f_k\rangle f_k$. The frame condition gives:

$$A\|f\|^2 \leq \langle Sf, f, \leq\rangle B\|f\|^2 \tag{41}$$

implying $A\mathbf{I} \leq S \leq B\mathbf{I}$, so S is invertible. □

Corollary 21. *[Perfect Reconstruction] For any $f \in \mathscr{H}$:*

$$f = \sum_{k\in\mathcal{I}} \langle f, f_k, f\rangle_{\widetilde{k}} = \sum_{k\in\mathcal{I}} \langle f, f_{\widetilde{k}}, f\rangle_k \tag{42}$$

where $\{f_{\widetilde{k}}\}$ is the canonical dual frame.

6 Continuous Frames and Integral Representations

Definition 22. *[Continuous Frame] A family $\{f_\omega\}_{\omega\in\Omega}$ in $\mathscr{H}$ is a continuous frame if there exist $A, B > 0$ such that:*

$$A\|f\|^2 \leq \int_\Omega |\langle f, f_\omega, |\rangle^2 d\nu(\omega) \leq B\|f\|^2 \tag{43}$$

for all $f \in \mathscr{H}$, where ν is a measure on Ω.

Theorem 23. *[Continuous Frame Decomposition] For a continuous frame $\{f_\omega\}$, every $f \in \mathscr{H}$ can be represented as:*

$$f = \int_\Omega \langle f, f_\omega, f \rangle_{\tilde{\omega}} d\nu(\omega) \tag{44}$$

where $\{f_{\tilde{\omega}}\}$ is the canonical dual frame.

Example 24. [Fourier Transform as Continuous Frame] The family $\{e^{i\omega\cdot}\}_{\omega\in\mathbb{R}}$ forms a continuous frame for $L^2(\mathbb{R})$ with:

$$f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\omega)\, e^{i\omega t}\, d\omega \tag{45}$$

where $\hat{f}(\omega) = \int_{\mathbb{R}} f(t)\, e^{-i\omega t}\, dt$.

7 Deterministic-Stochastic Duality for Fixed Sample Paths

Theorem 25. *[Sample Path Determinization] For an oscillatory process $X(t) = \int_{\mathbb{R}} A_t(\omega)\, e^{i\omega t}\, dZ(\omega)$ with fixed analytic sample path $X(t,\omega_0)$, the random measure becomes deterministic:*

$$dZ(\omega) \to dZ_{\omega_0}(\omega) = \mathcal{A}^{-1}[X(\,\cdot,\omega_0)](\omega)\, d\omega \tag{46}$$

where $\mathcal{A}^{-1}$ is the inverse analysis operator.

Proof. Given the fixed sample path $X(t,\omega_0)$, apply the analysis operator:

$$c(\omega) = \langle X(\cdot,\omega_0), A_\cdot(\omega)\, e^{i\omega\cdot}, \_ \rangle \mathscr{H} \tag{47}$$

$$= \int_{\mathbb{R}} X(t,\omega_0)\overline{A_t(\omega)} e^{-i\omega t}\, dt \tag{48}$$

Then $dZ_{\omega_0}(\omega) = c(\omega)\, d\omega$ provides the deterministic measure corresponding to this path. □

Corollary 26. *[Perfect Reconstruction from Deterministic Measure] Given the deterministic measure $dZ_{\omega_0}(\omega)$, the original sample path is recovered via:*

$$X(t,\omega_0) = \int_{\mathbb{R}} A_t(\omega)\, e^{i\omega t}\, dZ_{\omega_0}(\omega) \tag{49}$$

Theorem 27. *[Isomorphism Between Paths and Measures] For oscillatory processes in the RKHS $\mathscr{H}_K$, there exists a bijective correspondence between sample paths and their associated deterministic measures via the analysis-synthesis operator pair (T, T^*).*

8 Computational Implementation and Discretization

Theorem 28. *[Discretization Convergence] For a continuous frame representation:*

$$f = \int_\Omega \langle f, f_\omega, f \rangle_{\widetilde{\omega}} d\nu(\omega) \tag{50}$$

the discretized approximation:

$$f_N = \sum_{k=1}^{N} \langle f, f_{\omega_k}, f \rangle_{\widetilde{\omega}_k} \Delta \omega_k \tag{51}$$

converges to f as $\max_k \Delta \omega_k \to 0$, provided the frame bounds are preserved.

Proof. Let $\epsilon_N = \| f - f_N \|$. By the frame property:

$$\begin{aligned} \epsilon_N^2 &= \left| f - \sum_{k=1}^{N} \langle f, f_{\omega_k}, f \rangle_{\widetilde{\omega}_k} \Delta \omega_k \right|^2 && (52) \\ &\leq C \int_\Omega \left| \langle f, f_\omega, - \rangle \sum_{k:\omega_k \in I_\omega} \langle f, f_{\omega_k}, \chi \rangle_{I_\omega}(\omega) \right|^2 d\nu(\omega) && (53) \end{aligned}$$

where $\{I_\omega\}$ are the discretization intervals. As $\max_k \Delta \omega_k \to 0$, the continuity of the frame elements ensures $\epsilon_N \to 0$. □

Lemma 29. *[Numerical Stability] For well-conditioned frames with frame bounds A and B, the condition number of the discretized frame operator satisfies:*

$$\kappa(S_N) \leq \frac{B}{A} + O(\max_k \Delta \omega_k) \tag{54}$$

ensuring numerical stability in the discretization limit.

9 Applications to Accelerometer Data and Physical Interpretation

Theorem 30. *[Accelerometer-Velocity-Position Duality] For accelerometer data $a(t)$ modeled as a sample path of an oscillatory process, the velocity and position are obtained via spectral integration:*

$$v(t) = \int_{\mathbb{R}} \frac{A_t(\omega)}{i\,\omega} e^{i\omega t}\, d\,Z_a(\omega) \tag{55}$$

$$x(t) = \int_{\mathbb{R}} \frac{A_t(\omega)}{(i\,\omega)^2} e^{i\omega t}\, d\,Z_a(\omega) \tag{56}$$

where $d\,Z_a(\omega)$ is the deterministic measure corresponding to the acceleration path.

Corollary 31. *[Quantum-Classical Analogy] The Fourier duality between position and momentum in quantum mechanics is precisely mirrored in the classical accelerometer system, where the analysis-synthesis operators play the role of the quantum Fourier transform between position and momentum representations.*

10 Conclusion

This comprehensive framework establishes the rigorous mathematical foundations for:

1. The distinction between spectral measures (deterministic, process-characterizing) and random measures (stochastic, path-determining)
2. Priestley's theory of oscillatory processes with evolutionary spectra
3. RKHS representations of bandlimited and oscillatory processes
4. Frame theory with continuous analysis-synthesis operators
5. The fundamental duality between deterministic and stochastic representations
6. Computational methods for discretization with convergence guarantees
7. Applications to physical systems exhibiting quantum-classical analogies

The mathematical structure reveals the deep connections between stochastic process theory, functional analysis, and quantum mechanics, providing both theoretical insight and practical computational methods for analyzing complex oscillatory phenomena.

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