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Hans Rademacher

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Editors' Preface

At the time of Professor Rademacher's death early in 1969, there was available a complete manuscript of the present work. The editors had only to supply a few bibliographical references and to correct a few misprints and errors. No substantive changes were made in the manuscript except in one or two places where references to additional material appeared; since this material was not found in Rademacher's papers, these references were deleted.

The editors are grateful to Springer-Verlag for their helpfulness and courtesy.

Rademacher started work on the present volume no later than 1944; he was still working on it at the inception of his final illness. It represents the parts of analytic number theory that were of greatest interest to him. The editors, his students, offer this work as homage to the memory of a great man to whom they, in common with all number theorists, owe a deep and lasting debt.

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I. Analytic Tools

Chapter 1

Bernoulli Polynomials and Bernoulli Numbers

1. The binomial coefficients

The summing of the first n natural numbers, or squares, or cubes, is a rather elementary problem in number theory and leads to the well known formulae

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2},$$

$$\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6},$$

$$\sum_{n=1}^{N} n^3 = \frac{N^2(N+1)^2}{4}. \tag{1.1}$$

Of interest is the fact

$$\sum_{n=1}^{N} n^3 = \left(\sum_{n=1}^{N} n\right)^2. \tag{1.11}$$

The generalization of the above formulae for higher powers is connected with the name of Jacob Bernoulli (1654—1705).

It is advisable not to treat the power functions directly but to discuss functions which lend themselves better to a process of summation.

Such functions are the binomial coefficients

$$\binom{x}{j} = \frac{x(x-1)\cdots(x-j+1)}{1\cdot 2\cdots j}, \tag{1.2}$$

for which we derive immediately

$$\binom{x}{j} = \binom{x+1}{j+1} - \binom{x}{j+1}, \tag{1.3}$$

so that

$$\sum_{n=M}^{N-1} \binom{n}{j} = \binom{N}{j+1} - \binom{M}{j+1}.$$

We have therefore only to express the powers x^q as combinations of binomial coefficients.

The binomial coefficients are polynomials which take integer values for integer values of the variable x. Moreover they form a basis for all such polynomials. Indeed, let $f(x)$ be an "integer valued" polynomial of degree n, and let us put

$$f(x) = A_0 + A_1\binom{x}{1} + \cdots + A_n\binom{x}{n}. \tag{1.4}$$

For the determination of the A_j we have

$$\begin{aligned} f(0) &= A_0, \\ f(1) &= A_0 + A_1, \\ f(2) &= A_0 + A_1\binom{2}{1} + A_2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ f(n) &= A_0 + A_1\binom{n}{1} + A_2\binom{n}{2} + \cdots + A_{n-1}\binom{n}{n-1} + A_n. \end{aligned} \tag{1.5}$$

This system of linear equations shows that the A_j are uniquely determined and are integers, since the $f(j)$ are integers by assumption.

We can stop the right hand side in (1.4) after the binomial coefficient $\binom{x}{n}$, since higher binomial coefficients would produce a polynomial of degree higher than n, which cannot agree with a polynomial of degree n for infinitely many integer values of x. If we therefore would replace (1.4) formally by

$$f(x) = \sum_{j\geqq 0} A_j\binom{x}{j}, \quad \binom{x}{0} = 1, \tag{1.7}$$

an extension of (1.5) to higher values of the argument would necessarily lead to

$$A_{n+1} = A_{n+2} = \cdots = 0.$$

We apply now these reasonings to special polynomials, the powers themselves.

$$x^q = \sum_{j\geqq 0} A_{qj}\binom{x}{j}, \quad q = 1, 2, \ldots, \tag{1.8}$$

where we actually have a finite sum on the right-hand side, since we know

$$A_{qj} = 0 \quad \text{for } j > q. \tag{1.81}$$

We can include the case $q = 0$ in (1.8) if we choose

$$A_{00} = 1, \qquad A_{0j} = 0 \qquad \text{for } j > 0. \tag{1.82}$$

If in (1.8) we put $x = 0$ for $q \geqq 1$ it follows from (1.5) that[1]

$$A_{q0} = 0, \qquad q > 0. \tag{1.83}$$

The powers of x must agree on both sides of (1.8). In particular x^q appears on the right-hand side only in the term for $j = q$, and we have thus

$$A_{qq} = q!\,. \tag{1.84}$$

Those A_{qj} which are not mentioned as vanishing in (1.81), (1.82), (1.83) are positive integers. This can be seen from the expression

$$A_{qj} = \sum_{\substack{l_k > 0 \\ l_1 + l_2 + \cdots l_j = q}} \frac{q!}{l_1!\, l_2! \cdots l_j!}, \qquad q = 1, 2, \ldots, \tag{1.85}$$

a sum of multinomial coefficients. Expression (1.85) can be obtained from the multinomial theorem together with (1.8). We do not need (1.85) for our purpose and leave its proof to the reader as an exercise.

Another expression for the A_{qj} is obtained by writing (1.8) for $x = l = 0, 1, 2, \ldots, k \leqq q$ and summing after multiplication with $(-1)^l \binom{k}{l}$:

$$\begin{aligned}
\sum_{l=0}^{k} (-1)^l \binom{k}{l} l^q &= \sum_{l=0}^{k} (-1)^l \binom{k}{l} \sum_{j=0}^{q} A_{qj} \binom{l}{j} \\
&= \sum_{j=0}^{q} A_{qj} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \binom{l}{j} \\
&= \sum_{j=0}^{q} A_{qj} \sum_{j \leqq l \leqq k} (-1)^l \frac{k!}{l!\,(k-l)!} \, \frac{l!}{j!\,(l-j)!} \\
&= \sum_{j=0}^{k} A_{qj} \frac{k!}{j!\,(k-j)!} \sum_{j \leqq l \leqq k} (-1)^l \frac{(k-j)!}{(k-l)!\,(l-j)!} \\
&= \sum_{j=0}^{k} A_{qj} \binom{k}{j} \sum_{j \leqq l \leqq k} (-1)^l \binom{k-j}{l-j}.
\end{aligned}$$

It is apparent from the second member in this chain of equations that the inner sum for $j = k$ has to be read as $(-1)^k$. Otherwise we have

$$\sum_{j \leqq l \leqq k} (-1)^l \binom{k-j}{l-j} = (-1)^j \sum_{\lambda=0}^{k-j} (-1)^\lambda \binom{k-j}{\lambda} = 0\,.$$

[1] The numbers $\mathfrak{S}_q^j = A_{qj}/j\,!$ are called Stirling numbers of the second kind.

Thus we obtain finally

$$\sum_{l=0}^{k} (-1)^l \binom{k}{l} l^q = (-1)^k A_{qk} . \tag{1.91}$$

If we write this in the form

$$A_{qk} = \sum_{m=0}^{k} (-1)^m \binom{k}{m} (k-m)^q$$

we recognize that A_{qk} appears as the k-th difference of x^q:

$$A_{qk} = \Delta^k x^q \big|_{x=0} , \tag{1.92}$$

where Δ is the operator which applied on $f(x)$ produces $\Delta f(x) = f(x+1) - f(x)$.

2. The Bernoulli polynomials

With this knowledge about the A_{qj} we return to the problem of the summation of n^q. First we rewrite (1.8) as

$$x^q = \sum_{j \geqq 0} A_{qj} \left\{ \binom{x+1}{j+1} - \binom{x}{j+1} \right\} . \tag{2.1}$$

Taking integer values for x and summing, we have

$$\sum_{n=M}^{N-1} n^q = \sum_{j \geqq 0} A_{qj} \left\{ \binom{N}{j+1} - \binom{M}{j+1} \right\} .$$

We introduce now the Bernoulli polynomials

$$B_r(y) = r \sum_{j \geqq 0} A_{r-1,j} \binom{y}{j+1} + C_r, \qquad r = 1, 2, \dots . \tag{2.2}$$

The additive constant C_r we keep at our disposal in order to simplify later some formulae[1]. Then we can write the preceding equation as

$$\sum_{n=M}^{N-1} n^q = \frac{1}{q+1} \{ B_{q+1}(N) - B_{q+1}(M) \} , \tag{2.3}$$

where the later choice of C_r is irrelevant.

We have to study now the Bernoulli polynomials in order to make the

[1] The notation of the Bernoulli polynomials is not fully standardized. Some authors omit the C_r, some others the factor r. Each of these notations has its own advantages and disadvantages. N. Nielsen in his classical treatise [44] denotes by $n!\, B_n(x-1)$ the polynomials denoted here by $B_n(x)$.

summation formula (2.3) more significant.

From (2.2) we obtain immediately

$$B_r(0) = C_r, \quad r = 1, 2, \ldots, \tag{2.41}$$

$$B_r(1) = C_r, \quad r = 2, 3, \ldots, \tag{2.42}$$

$$B_1(1) = 1 + C_1, \tag{2.43}$$

the latter two equations by means of (1.82) and (1.83).

Through the definition (2.2) we can restate (2.1) in the form

$$x^q = \frac{1}{q+1}\left(B_{q+1}(x+1) - B_{q+1}(x)\right). \tag{2.5}$$

By differentiation we obtain

$$q\,x^{q-1} = \frac{1}{q+1}\left(B'_{q+1}(x+1) - B'_{q+1}(x)\right), \quad q = 1, 2, \ldots,$$

which compared with (2.5) leads to

$$B_q(x+1) - B_q(x) = \frac{1}{q+1}\left(B'_{q+1}(x+1) - B'_{q+1}(x)\right)$$

or

$$\frac{1}{q+1} B'_{q+1}(x+1) - B_q(x+1) = \frac{1}{q+1} B'_{q+1}(x) - B_q(x).$$

This equation shows that the polynomial

$$\frac{1}{q+1} B'_{q+1}(x) - B_q(x), \quad q = 1, 2, \ldots$$

has the period 1. But a periodic polynomial can only be a constant:

$$\frac{1}{q+1} B'_{q+1}(x) - B_q(x) = K_q, \quad q = 1, 2, \ldots.$$

We choose now the constant C_q which is available in $B_q(x)$ so that K_q becomes 0. We have then

$$\frac{1}{q+1} B'_{q+1}(x) = B_q(x). \tag{2.6}$$

Now (2.2) together with (1.82) gives

$$B_1(x) = x + C_1,$$

and from (2.6) and the boundary condition (2.41) we obtain by integration

$$\frac{1}{2!} B_2(x) = \frac{x^2}{2!} + \frac{C_1}{1!}x + \frac{C_2}{2!}$$

and similarly in consecutive steps

$$\frac{1}{q!} B_q(x) = \frac{x^q}{q!} + \frac{C_1}{1!} \frac{x^{q-1}}{(q-1)!} + \cdots + \frac{C_{q-1}}{(q-1)!} \frac{x}{1!} + \frac{C_q}{q!}. \quad (2.71)$$

The constants C_q are in turn completely determined by the condition (2.42) which reads here

$$0 = \frac{1}{q!} + \frac{C_1}{1!(q-1)!} + \cdots + \frac{C_{q-1}}{(q-1)!1!}, \qquad q \geqq 2.$$

This is a recursion formula for the C_q. They are called the *Bernoulli numbers,* and we designate them from now on as $B_q = C_q$. Through multiplication with $q!$ the previous formula becomes, in the new notation,

$$0 = 1 + \binom{q}{1} B_1 + \binom{q}{2} B_2 + \cdots + \binom{q}{q-1} B_{q-1}, \qquad q \geqq 2, \quad (2.72)$$

which yields in particular

$$B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \ldots .$$

Thus (2.71) with $C_q = B_q$ gives the complete definition of the Bernoulli polynomials.

Incidentally, (2.6) shows that the functions $f_q(x) = \frac{1}{q!} B_q(x)$ form an Appell sequence (cf. [1]).

Returning to (2.3) we have now for $q \geqq 1$

$$\sum_{n=1}^{N-1} n^q = \frac{1}{q+1} \{B_{q+1}(N) - B_{q+1}(1)\}$$

and also

$$\sum_{n=-N+1}^{-1} n^q = \frac{1}{q+1} \{B_{q+1}(0) - B_{q+1}(1-N)\}.$$

Since

$$\sum_{n=-N+1}^{-1} n^q = (-1)^q \sum_{n=1}^{N-1} n^q,$$

we find, in view of (2.41) and (2.42), with $C_q = B_q$

$$B_{q+1}(1-N) - B_{q+1} = (-1)^{q+1} (B_{q+1}(N) - B_{q+1}).$$

Here we have two polynomials of degree $q + 1$ which agree for the infinitely many values of all natural numbers N. They must therefore be identical in x:

$$B_{q+1}(1-x) - B_{q+1} = (-1)^{q+1}\left(B_{q+1}(x) - B_{q+1}\right).$$

Through differentiation and (2.6) we obtain

$$B_q(1-x) = (-1)^q B_q(x)\,, \tag{2.8}$$

which implies for $x = 0$

$$B_q(1) = (-1)^q B_q(0)\,, \tag{2.81}$$

and thus, because of (2.41) and (2.42) for $q \geqq 2$,

$$B_q\left(1-(-1)^q\right) = 0\,.$$

We infer therefore

$$B_{2k+1} = 0 \text{ for } k \geqq 1\,, \tag{2.9}$$

whereas we had already found

$$B_1 = -1/2\,.$$

3. Zeros of the Bernoulli polynomials

None of the B_{2k} vanish, and they are of alternating signs, as we shall show presently. From (2.8) we conclude

$$B_{2k+1}(1/2) = 0, \qquad k \geqq 0\,. \tag{3.1}$$

Leaving $B_1(x) = x - 1/2$ aside, we have also

$$B_{2k+1}(0) = B_{2k+1}(1) = 0, \qquad k \geqq 1\,, \tag{3.2}$$

because of (2.9) and (2.8). We show now that these statements exhibit all the zeros in $0 \leqq x \leqq 1$. Indeed, suppose that $B_{2k+1}(x)$ had the four zeros 0, α_1, α_2, 1 with

$$0 < \alpha_1 < \alpha_2 < 1\,,$$

where one of the numbers α_1, α_2 may be 1/2. Then, according to Rolle's theorem, there would exist three numbers β_1, β_2, β_3 such that

$$0 < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \beta_3 < 1\,,$$

for which

$$0 = B'_{2k+1}(\beta_j) = B_{2k}(\beta_j), \qquad j = 1, 2, 3.$$

Rolle's theorem would then again ensure the existence of two numbers γ_1, γ_2 such that

$$\beta_1 < \gamma_1 < \beta_2 < \gamma_2 < \beta_3$$

with

$$0 = B'_{2k}(\gamma_j) = B_{2k-1}(\gamma_j)\,, \quad j = 1, 2,$$

which would ascribe to $B_{2k-1}(x)$ the same property which we assumed for $B_{2k+1}(x)$. Hence all $B_{2k+1}(x)$ for $k \geqq 1$ would have 4 zeros at least in $0 \leqq x \leqq 1$, which is impossible for $B_3(x)$. The same argument shows also that 0, 1/2, 1 are *simple* zeros of $B_{2k+1}(x)$.

It follows now that $B_{2k}(x) - B_{2k}$ does not change sign between 0 and 1, since a zero ξ, $0 < \xi < 1$

$$B_{2k}(\xi) - B_{2k} = 0$$

together with

$$B_{2k}(0) - B_{2k} = B_{2k}(1) - B_{2k} = 0$$

(see (2.71) and (2.8)) would produce two zeros of $B'_{2k}(x) = B_{2k-1}(x)$ between 0 and 1, which we have just disproved.

Because of $B_{2k+1}(0) = B_{2k+1}(1/2) = 0$ there exists a β, $0 < \beta < 1/2$, such that $B'_{2k+1}(\beta) = B_{2k}(\beta) = 0$. Now, since $B_{2k}(x) - B_{2k}$ does not change sign in (0, 1) we have, for $0 < x < 1$,

$$(B_{2k}(x) - B_{2k})\,(B_{2k}(\beta) - B_{2k}) > 0\,,$$

or according to (2.71)

$$\left\{\binom{2k}{2} B_{2k-2}\,x^2 + \binom{2k}{4} B_{2k-4}\,x^4 + \cdots\right\} B_{2k} < 0\,, \qquad 0 < x < 1\,.$$

From this we derive

$$B_{2k-2}\,B_{2k} < 0$$

by first factoring out x^2 and then letting x approach zero.

We found $B_2 = 1/6 > 0$ and have therefore

$$(-1)^{k-1} B_{2k} > 0\,, \quad k \geqq 1\,. \tag{3.3}$$

One more remark about the $(B_{2k}(x) - B_{2k})$. They vanish at $x = 0, 1$. But we have also after (3.2)

$$B'_{2k}(0) = B_{2k-1}(0) = 0\,, \quad k \geqq 2\,,$$
$$B'_{2k}(1) = B_{2k-1}(1) = 0\,,$$

so that 0 and 1 are double roots of $B_{2k}(x) - B_{2k}$, $k \geqq 2$. Now $B_2(x) - B_2$, of degree 2, has only the simple zeros 0 and 1, and is therefore of the form $C\,x(x-1)$. Thus all the $B_{2k}(x) - B_{2k}$ for $k \geqq 2$ must have $(B_2(x) - B_2)^2$ as factor, and $B_4(x) - B_4$ of degree 4 must be

$C(B_2(x) - B_2)^2$. Since moreover (2.71) shows that the $B_q(x)$ are monic polynomials, we have

$$B_4(x) - B_4 = (B_2(x) - B_2)^2,$$

which explains the occurrence of (1.11) in view of (2.3).

4. The Bernoulli numbers

The recursion formula (2.72) for the Bernoulli numbers can be written in symbolic form as

$$B^q = (1 + B)^q, \quad q \geqq 2, \tag{4.1}$$

where the right-hand member is understood to be expanded by the binomial theorem with the convention that every power B^k be replaced by B_k. The Bernoulli polynomials from (2.71) are to be rewritten as

$$B_q(x) = \sum_{k=0}^{q} \binom{q}{k} B_k x^{q-k}, \tag{4.2}$$

where we have set $B_0 = 1$, and (2.41), (2.42) appear as

$$B_q(0) = B_q(1) = B_q, \quad q \geqq 2. \tag{4.3}$$

From (2.2) we have

$$B_q(x) = q \sum_{j=0}^{q-1} A_{q-1,j} \binom{x}{j+1} + B_q, \quad q = 1, 2, \ldots. \tag{4.4}$$

Let us compare now the coefficients of powers of x in (4.2) and (4.4). The coefficients of x^0 are identical, those of the highest power x^q we have already compared in (1.84). Besides these only the linear term in (4.2) can be easily determined since

$$\binom{x}{j+1} = \frac{x(x-1)\cdots(x-j)}{(j+1)!} = (-1)^j \frac{x}{j+1} + \cdots.$$

We obtain thus

$$B_{q-1} = \sum_{j=0}^{q-1} A_{q-1,j} \frac{(-1)^j}{j+1}, \tag{4.5}$$

which agrees for $q = 1$ with our convention $B_0 = 1$ because of (1.82). By means of (1.91) we arrive at the explicit formula for B_q:

$$B_q = \sum_{j=1}^{q} \frac{1}{j+1} \sum_{l=1}^{j} (-1)^l \binom{j}{l} l^q, \quad q \geqq 1. \tag{4.6}$$

This formula shows that the denominator of B_q can at most be the least common multiple of $2, 3, \ldots, q+1$.

The literature (see e.g. Saalschütz [64] and Nielsen [44]) contains quite a number of types of recursion formulae as well as of "independent" representations.

The first eight Bernoulli numbers of even index are

$$B_2 = \frac{1}{6}, \qquad B_4 = -\frac{1}{30}, \qquad B_6 = \frac{1}{42}, \qquad B_8 = -\frac{1}{30},$$

$$B_{10} = \frac{5}{66}, \qquad B_{12} = -\frac{691}{2730}, \qquad B_{14} = \frac{6}{7}, \qquad B_{16} = -\frac{3617}{510}.$$

5. The von Staudt-Clausen theorem

The just mentioned question of the denominator of B_q is fully answered by the following

Theorem. *The denominator of the Bernoulli number B_{2k} is the product of those different primes p for which $p - 1$ divides $2k$, and more precisely*

$$B_{2k} = G_{2k} - \sum_{(p-1)|2k} \frac{1}{p}, \tag{5.1}$$

where G_{2k} is a certain integer.

For the proof we go back to the numbers A_{qj}. We need two lemmas.

Lemma 1. *The "Stirling numbers of the second kind"*

$$\mathfrak{S}_q^j = A_{qj}/j!$$

are integers.

Proof. With the abbreviation

$$(x)_1 = x,\ (x)_2 = x(x-1), \ldots, (x)_n = x(x-1)\cdots(x-n+1)$$

we can write (1.8) in the form

$$x^q = \sum_{j=1}^{q} \frac{A_{qj}}{j!}(x)_j = \sum_{j=1}^{q} \mathfrak{S}_q^j (x)_j, \qquad q > 0.$$

Now, comparing the coefficients of x^q on both sides we find $\mathfrak{S}_q^q = 1$. Then we have

$$x^q - (x)_q = \sum_{j=1}^{q-1} \mathfrak{S}_q^j (x)_j.$$

On the left-hand side we have a polynomial with integer coefficients, of degree $q - 1$. Comparison of the coefficients of x^{q-1} yields $\mathfrak{S}_q^{q-1}$ as integer and so on. The cases $A_{q0}/0!$ not covered by this argument show also integers. This proves the lemma. $\square$

Lemma 2.

$$A_{2k,j} \equiv \begin{cases} -1 & (\operatorname{mod} j+1),\ \textit{if } j+1 \textit{ is prime and } j \mid 2k\,, \\ 0 & (\operatorname{mod} j+1),\ \textit{otherwise}\,. \end{cases}$$

Proof. Let first $j+1$ be a prime number and let j be a divisor of $2k$. Then, by Fermat's theorem,

$$l^j \equiv 1 \ (\operatorname{mod} j+1) \text{ for } l = 1, 2, \ldots, j\,,$$

and thus also

$$l^{2k} \equiv 1 \ (\operatorname{mod} j+1)\,.$$

Through (1.91) we find then

$$A_{2k,j} \equiv (-1)^j \sum_{l=1}^{j} (-1)^l \binom{j}{l} \ (\operatorname{mod} j+1)$$

$$\equiv (-1)^j \sum_{l=0}^{j} (-1)^l \binom{j}{l} - (-1)^j = -(-1)^j \ (\operatorname{mod} j+1)\,.$$

This proves the first statement of the lemma for odd primes $j+1$, but also for $j+1=2$, since $1 \equiv -1$ (mod 2).

Now, secondly, let $j+1$ again be a prime, but $j \nmid 2k$. This excludes $j=1, j=2$ so that $j+1$ is an odd prime >3. It excludes also $j=2k$, and therefore only $j<2k$ is of interest since $A_{2k,j}=0$ for $j>2k$ anyway. We have again, after Fermat,

$$l^{2k} \equiv l^{2k-j} \ (\operatorname{mod} j+1)\,.$$

Using this in (1.91) we obtain

$$A_{2k,j} \equiv A_{2k-j,j} \ (\operatorname{mod} j+1)\,,$$

where $2k-j$ is again even. This can be iterated. If we choose[1]

$$\varrho = \left[\frac{2k}{j}\right]$$

we have $0 < 2k - \varrho j < j$ and thus

$$A_{2k,j} \equiv A_{2k-\varrho j,j} \ (\operatorname{mod} j+1)$$

and therefore in view of (1.81)

$$A_{2k,j} \equiv 0 \ (\operatorname{mod} j+1)\,.$$

If, thirdly, $j+1$ is composite and $\geqq 6$ then obviously

$$(j+1) \mid j!\,.$$

---

[1] As customary $[x]$ means for real x that integer for which $[x] \leqq x < [x]+1$.

Now Lemma 1 says $A_{qj} \equiv 0 \pmod{j!}$ so that a fortiori $A_{2k,j} \equiv 0 \pmod{j+1}$, $j+1$ composite, $\geqq 6$. There remains the single case $j+1=4$, $j=3$ to be considered. Here we have directly after (1.91)

$$\begin{aligned} A_{2k,3} &= -\sum_{l=1}^{3} (-1)^l \binom{3}{l} l^{2k} = 3 - 3\cdot 2^{2k} + 3^{2k} \\ &= 3(3^{2k-1}+1) - 3\cdot 2^{2k} \equiv 0 \pmod 4 . \end{aligned}$$

This proves Lemma 2 completely. $\square$

Now we have from (4.5)

$$B_{2k} = \sum_{j=0}^{2k} A_{2k,j} \frac{(-1)^j}{j+1} .$$

For those j classified in Lemma 2 under "otherwise" the summand will be an integer, also for $j=0$. For the summands for which $j+1$ is a prime number p and $j=p-1$ a divisor of $2k$ Lemma 2 shows

$$\frac{A_{2k,j}}{j+1} = M_p - \frac{1}{p} ,$$

where M_p is a certain integer. We obtain thus

$$B_{2k} = \sum_{(p-1)|2k} \frac{(-1)^p}{p} + \text{integer} .$$

This is already the von Staudt-Clausen theorem since $(-1)^p = -1$ for all primes except 2 and since for $p=2$ the fractions $1/2$ and $-1/2$ differ by an integer.

Equation (5.1) shows by the way that the denominators of all Bernoulli numbers B_{2k} contain the factor 6.

Exercise 1. Prove that for any prime number q

$$q(q^{2k}-1)B_{2k}$$

is an integer.

Exercise 2. Using (1.8) to express x^q and x^{q+1} and comparing, prove that

$$A_{q+1,j} = j(A_{q,j-1} + A_{q,j}) .$$

This formula can be used for a different proof of Lemma 1.

Very little is known in general about the *numerators* of the Bernoulli numbers. (We shall mention one property of the numerators later (§ 12)). However, they are of great interest in themselves. In particular they appear in Kummer's theory of Fermat's last theorem. Extensive tables of Bernoulli numbers have been prepared because of their many applications.

6. A multiplication formula for the Bernoulli polynomials

From (2.5) we obtain for $x = n + j/k$

$$(kn + j)^q = \frac{k^q}{q+1}\left\{B_{q+1}\left(n + \frac{j}{k} + 1\right) - B_{q+1}\left(n + \frac{j}{k}\right)\right\},$$

which is valid for $q \geqq 0$. Summation over n and then over j yields

$$\sum_{n=M}^{N-1}\sum_{j=0}^{k-1}(kn + j)^q = \frac{k^q}{q+1}\sum_{j=0}^{k-1}\left\{B_{q+1}\left(N + \frac{j}{k}\right) - B_{q+1}\left(M + \frac{j}{k}\right)\right\}.$$

On the left side we have now

$$\sum_{m=Mk}^{Nk-1} m^q .$$

By means of (2.3) we obtain therefore

$$B_{q+1}(Nk) - k^q\sum_{j=0}^{k-1} B_{q+1}\left(N + \frac{j}{k}\right) = B_{q+1}(Mk) - k^q\sum_{j=0}^{k-1} B_{q+1}\left(M + \frac{j}{k}\right).$$

The polynomial

$$B_{q+1}(kx) - k^q\sum_{j=0}^{k-1} B_{q+1}\left(x + \frac{j}{k}\right),$$

which is at most of degree $q + 1$, has for all integers $x = N$ the same value and must therefore be a constant. Differentiation has therefore the result

$$kB'_{q+1}(kx) - k^q\sum_{j=0}^{k-1} B'_{q+1}\left(x + \frac{j}{k}\right) = 0 .$$

In view of (2.6) we have thus proved the

Theorem.

$$B_q(kx) = k^{q-1}\sum_{j=0}^{k-1} B_q\left(x + \frac{j}{k}\right) \tag{6.1}$$

for $q = 1, 2, \ldots$. It is also true for $q = 0$, if we put $B_0(x) = 1$, in agreement with (2.6) *and* (4.2). *Putting $k = 2$, $x = 0$, $q = 2m$ we obtain, by observing also* (4.3), *the result*

$$B_{2m}\left(\frac{1}{2}\right) = \left(\frac{1}{2^{2m-1}} - 1\right)B_{2m}(0) = \left(\frac{1}{2^{2m-1}} - 1\right)B_{2m}, \tag{6.2}$$

which supplements (3.1). *It remains also true for $m = 0$ with our stipulation $B_0(x) = B_0 = 1$.*

Chapter 2

The Euler-MacLaurin Sum Formula

7. Use of the Bernoulli polynomials

Let $f(x)$ be continuous with as many continuous derivatives as required. Noticing $B_1'(x) = 1$ we obtain through integration by parts

$$\int_0^1 f(x)\,dx = [B_1(x)\,f(x)]_0^1 - \int_0^1 B_1(x)\,f'(x)\,dx\,.$$

Using (2.6) we repeat the process with the result

$$\int_0^1 f(x)\,dx = \sum_{r=1}^{q}(-1)^{r-1}\left[\frac{B_r(x)}{r!}\,f^{(r-1)}(x)\right]_0^1 + (-1)^q\int_0^1 \frac{B_q(x)}{q!}\,f^{(q)}(x)\,dx\,.$$

The term with the exceptional $B_1(x) = x - 1/2$ we treat separately. Otherwise we apply (4.3). After some rearrangements we obtain

$$f(1) = \int_0^1 f(x)\,dx + \sum_{r=1}^{q}(-1)^r\,\frac{B_r}{r!}\,\{f^{(r-1)}(1) - f^{(r-1)}(0)\} + (-1)^{q-1}\int_0^1 \frac{B_q(x)}{q!}\,f^{(q)}(x)\,dx\,. \tag{7.1}$$

If we replace here $f(x)$ by the function $f(n-1+x)$ we get

$$f(n) = \int_0^1 f(n-1+x)\,dx + \sum_{r=1}^{q}(-1)^r\,\frac{B_r}{r!}\,\{f^{(r-1)}(n) - f^{(r-1)}(n-1\} + (-1)^{q-1}\int_0^1 \frac{B_q(x)}{q!}\,f^{(q)}(n-1+x)\,dx\,.$$

We now sum from $a+1$ to b, where a and b are integers, so that we have

$$\sum_{n=a+1}^{b} f(n) = \int_a^b f(x)\,dx + \sum_{r=1}^{q}(-1)^r\,\frac{B_r}{r!}\,\{f^{(r-1)}(b) - f^{(r-1)}(a)\} + R_q\,, \tag{7.2}$$

where the remainder term is

$$R_q = (-1)^{q-1} \sum_{n=a+1}^{b} \int_0^1 \frac{B_q(x)}{q!} f^{(q)}(n-1+x)\,dx$$

$$= \frac{(-1)^{q-1}}{q!} \int_a^b B_q(x-[x])\, f^{(q)}(x)\,dx\,. \tag{7.21}$$

The formula (7.2) with (7.21) is the Euler-MacLaurin sum formula.

8. Fourier expansions of the Bernoulli polynomials

The functions

$$\psi_q(t) = B_q(t-[t])\,, \quad q \geqq 1 \tag{8.1}$$

are periodic of period 1. With the exception of $\psi_1(t)$, which has jumps at all integers, they are continuous, since $B_q(0) = B_q(1)$ for $q \geqq 2$. They are each of bounded variation in a finite interval and thus permit a Fourier expansion

$$\psi_q(t) = \frac{a_0^{(q)}}{2} + \sum_{n=1}^{\infty} (a_n^{(q)} \cos 2\pi n t + b_n^{(q)} \sin 2\pi n t) \tag{8.2}$$

with

$$a_n^{(q)} = 2 \int_0^1 B_q(x) \cos 2\pi n x\,dx\,, \tag{8.21}$$

$$b_n^{(q)} = 2 \int_0^1 B_q(x) \sin 2\pi n x\,dx\,. \tag{8.22}$$

We treat first the case $n = 0$ separately:

$$a_0^{(q)} = 2 \int_0^1 B_q(x)\,dx = \frac{2}{q+1} \int_0^1 B'_{q+1}(x)\,dx$$

$$= \frac{2}{q+1} \left(B_{q+1}(1) - B_{q+1}(0)\right) = 0, \quad q = 1, 2, \ldots\,. \tag{8.3}$$

Let us now consider $n > 0$. Integration by parts furnishes

$$a_n^{(q)} = 2 \left[B_q(x) \frac{\sin 2\pi n x}{2\pi n}\right]_0^1 - \frac{2}{2\pi n} \int_0^1 B'_q(x) \sin 2\pi n x\,dx$$

$$= -\frac{2q}{2\pi n} \int_0^1 B_{q-1}(x) \sin 2\pi n x\,dx = -\frac{q}{2\pi n} b_n^{(q-1)}\,, \qquad q \geqq 2 \tag{8.41}$$

and in particular

$$a_n^{(1)} = 0, \quad n \geqq 1\,. \tag{8.42}$$

Similarly, for $q \geqq 2$:

$$b_n^{(q)} = -2\left[B_q(x)\frac{\cos 2\pi n x}{2\pi n}\right]_0^1 + \frac{2}{2\pi n}\int_0^1 B_q'(x)\cos 2\pi n x\,dx$$

$$= \frac{2q}{2\pi n}\int_0^1 B_{q-1}(x)\cos 2\pi n x\,dx = \frac{q}{2\pi n}\,a_n^{(q-1)}, \quad n \geqq 1\,. \tag{8.51}$$

For $q = 1$, with $B_1(x) = x - 1/2$:

$$b_n^{(1)} = -2\left[B_1(x)\frac{\cos 2\pi n x}{2\pi n}\right]_0^1$$

$$+\frac{2}{2\pi n}\int_0^1 \cos 2\pi n x\,dx = -\frac{2}{2\pi n}\,. \tag{8.52}$$

From these equations we obtain by an obvious mathematical induction

$$\left.\begin{aligned} a_n^{(2k-1)} &= 0\,, & b_n^{(2k-1)} &= (-1)^k\frac{2(2k-1)!}{(2\pi n)^{2k-1}}\,,\\ a_n^{(2k)} &= (-1)^{k-1}\frac{2(2k)!}{(2\pi n)^{2k}}, & b_n^{(2k)} &= 0\,, \end{aligned}\right\}\quad \begin{aligned} &k \geqq 1\,,\\ &n \geqq 0\,.\end{aligned}$$

Collecting these results we obtain

$$\psi_{2k-1}(t) = B_{2k-1}(t - [t])$$

$$= 2(-1)^k(2k-1)!\sum_{n=1}^{\infty}\frac{\sin 2\pi n t}{(2\pi n)^{2k-1}}, \quad k \geqq 1, \tag{8.61}$$

$$\psi_{2k}(t) = B_{2k}(t - [t]) = 2(-1)^{k-1}(2k)!\sum_{n=1}^{\infty}\frac{\cos 2\pi n t}{(2\pi n)^{2k}}, \tag{8.62}$$

where, however, for $\psi_1(t)$ integer values have to be exempted, because of the discontinuity of $\psi_1(t)$ at $t = 0$. The Fourier series yields here

$$\frac{1}{2}\left(\psi_1(-0) + \psi_1(+1)\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\right) = 0\,.$$

9. Sums of reciprocal powers

In view of (4.3) the formula (8.61) shows again $B_{2k-1} = 0$ for $k \geqq 2$; and (8.62) yields

$$\frac{B_{2k}}{(2k)!} = \frac{2(-1)^{k-1}}{(2\pi)^{2k}}\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\,, \tag{9.1}$$

which confirms (3.3). Moreover, the B_{2k} being rational numbers, we see that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \pi^{2k} \times \text{rational number} ,$$

a result first obtained by Euler. In particular we have, with $B_2 = 1/6$, $B_4 = -1/30$,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} , \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} . \tag{9.2}$$

For the reciprocal odd powers no analogous formulae are known. However, if we put $t = 1/4$ in (8.61) we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2k-1}} &= (-1)^k \frac{(2\pi)^{2k-1}}{2(2k-1)!} B_{2k-1}\left(\frac{1}{4}\right) \\ &= \pi^{2k-1} \times \text{rational number} . \end{aligned} \tag{9.3}$$

In particular, with $B_1(t) = t - 1/2$, $B_3(t) = t(t-1)(t-1/2)$ we find

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} ,$$

the well-known Gregory-Leibniz formula, and

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32} .$$

We use now the Fourier expansions for $\psi_q(t)$ for an appraisal of the magnitude of the ψ_q and the B_{2k}. From (8.61), (8.62) and (9.2) we infer, for $q \geqq 2$,

$$|B_q(t-[t])| \leqq \frac{2(q!)}{(2\pi)^q} \sum_{n=1}^{\infty} \frac{1}{n^q} \leqq \frac{2(q!)}{(2\pi)^q} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{12} \frac{q!}{(2\pi)^{q-2}} . \tag{9.4}$$

The formula (9.1) implies

$$|B_{2k}| = 2 \frac{(2k)!}{(2\pi)^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} , \tag{9.5}$$

and therefore

$$2 \frac{(2k)!}{(2\pi)^{2k}} < |B_{2k}| \leqq 2 \frac{(2k)!}{(2\pi)^{2k}} \sum_{1}^{\infty} \frac{1}{n^2} = \frac{(2k)!}{12(2\pi)^{2k-2}} , \tag{9.6}$$

which shows that the B_{2k} increase in absolute value after $B_6 = 1/42$. From (9.4) and (9.5) we obtain then

$$|B_{2k}(t-[t])| \leqq |B_{2k}| \leqq \frac{(2k)!}{12(2\pi)^{2k-2}}. \tag{9.7}$$

10. The generating function of the Bernoulli numbers

We apply the Euler-MacLaurin formula (7.2), (7.21) to the function

$$f(x) = e^{xz},$$

where z is taken as a parameter. We have with $a = 0$, $b = N$

$$\sum_{n=1}^{N} e^{nz} = \int_0^N e^{xz}\,dx + \sum_{r=1}^{q} (-1)^r \frac{B_r}{r!} z^{r-1}\{e^{Nz}-1\} + R_q,$$

$$R_q = \frac{(-1)^{q-1}}{q!} \int_0^1 B_q(x) z^q \sum_{n=1}^{N} e^{(n-1+x)z}\,dx. \tag{10.1}$$

Let us take

$$\operatorname{Re}(z) \neq 0 \tag{10.2}$$

so that $|e^z| \neq 1$. We have then

$$\sum_{n=1}^{N} e^{nz} = e^z \frac{e^{Nz}-1}{e^z-1},$$

$$\int_0^N e^{xz}\,dx = \frac{1}{z}(e^{Nz}-1),$$

$$\sum_{n=1}^{N} e^{(n-1+x)z} = e^{xz}\frac{e^{Nz}-1}{e^z-1},$$

all with non-vanishing denominators. All terms in (10.1) have thus the factor $e^{Nz}-1$ in common, which we cancel, in view of (10.2). This leads to

$$\frac{e^z}{e^z-1} = \frac{1}{z} + \sum_{r=1}^{q} (-1)^r \frac{B_r}{r!} z^{r-1} + \frac{(-1)^{q-1}}{q!} \frac{z^q}{e^z-1} \int_0^1 B_q(x) e^{xz}\,dx.$$

The integral here we estimate by means of (9.4)

$$\frac{1}{q!}\left|\int_0^1 B_q(x) e^{xz}\,dx\right| \leqq \frac{1}{q!}\int_0^1 |B_q(x)|\,e^{|z|}\,dx \leqq \frac{e^{|z|}}{12(2\pi)^{q-2}},$$

which shows that

$$\frac{z^q}{q!}\int_0^1 B_q(x) e^{xz}\,dx \to 0$$

with $q \to \infty$ for

$$|z| < 2\pi. \tag{10.3}$$

We obtain therefore the series

$$z \frac{e^z}{e^z - 1} = \sum_{r=0}^{\infty} (-1)^r \frac{B_r}{r!} z^r , \tag{10.4}$$

with $B_0 = 1$.

This series can be considered as the generating function for the Bernoulli numbers B_r, which we have defined in (3.4) and (4.11). Since

$$z \frac{e^z}{e^z - 1} - \frac{z}{2} = \frac{z}{2} \frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}}$$

is an even function, (10.4) shows again that

$$B_1 = -1/2, \quad B_{2k+1} = 0, \quad k \geqq 1 .$$

Thus (10.4) can be replaced by

$$z \frac{e^z}{e^z - 1} = \sum_{r=0}^{\infty} \frac{B_r}{r!} z^r + z$$

or

$$\frac{z}{e^z - 1} = \sum_{r=0}^{\infty} \frac{B_r}{r!} z^r . \tag{10.5}$$

It is now evident that the restriction (10.3) is unavoidable since the function in (10.4) has poles at $z = \pm 2\pi i$. The estimate (9.6) shows moreover that the series (10.4) is divergent on the whole circle of convergence.

However, the restriction (10.2) can now be lifted since on both sides in (10.4) we have functions which are analytic in $|z| < 2\pi$.

11. Tangent and cotangent coefficients

The power series for $\cot z$ and $\tan z$ can be immediately derived from (10.5). We have first

$$\frac{z}{2} \frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} = \frac{z}{2} \frac{e^z + 1}{e^z - 1} = z \frac{e^z}{e^z - 1} - \frac{z}{2} = -\frac{z}{2} + \sum_{r=0}^{\infty} (-1)^r \frac{B_r}{r!} z^r$$

and therefore

$$\begin{aligned} \cot u &= i \frac{e^{iu} + e^{-iu}}{e^{iu} - e^{-iu}} = \frac{1}{u} \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} (2iu)^{2k} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k)!} 2^{2k} u^{2k-1} , \end{aligned} \tag{11.1}$$

where we have taken into account $B_1 = -1/2$ and $B_{2k+1} = 0$ for $k \geqq 1$.

For the tangent expansion we begin with the identities

$$\frac{z}{2}\,\frac{e^z-1}{e^z+1} = z\,\frac{e^z}{e^z+1} - \frac{z}{2} = z\,\frac{e^{2z}-e^z}{e^{2z}-1} - \frac{z}{2}$$

$$= 2z\,\frac{e^{2z}}{e^{2z}-1} - z\,\frac{e^z}{e^z-1} - \frac{z}{2}$$

and obtain then from (10.4)

$$\frac{z}{2}\,\frac{e^{z/2}-e^{-z/2}}{e^{z/2}+e^{-z/2}} = -\frac{z}{2} + \sum_{r=0}^{\infty}(-1)^r\frac{B_r}{r!}(2z)^r - \sum_{r=0}^{\infty}(-1)^r\frac{B_r}{r!}z^r$$

$$= \sum_{k=0}^{\infty}\frac{B_{2k}}{(2k)!}(2^{2k}-1)z^{2k}\,.$$

This we apply to

$$\tan u = \frac{1}{i}\,\frac{e^{iu}-e^{-iu}}{e^{iu}+e^{-iu}}$$

with the result

$$\tan u = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_{2k}}{(2k)!}(2^{2k}-1)\,2^{2k}u^{2k-1}\,.$$

It has been customary to single out the "tangent coefficients" defined by

$$T_k = (-1)^{k-1}\frac{B_{2k}}{2k}(2^{2k}-1)\,2^{2k}\,, \tag{11.2}$$

which according to (3.3) are all positive, and to write thus

$$\tan u = \sum_{k=1}^{\infty}T_k\,\frac{u^{2k-1}}{(2k-1)!}\,. \tag{11.3}$$

12. A theorem by Frobenius about the numerators of the Bernoulli numbers

We shall see that the tangent coefficients are all integers. We need first a

Lemma. *If* $A(x) = \sum_{n=0}^{\infty}\frac{a_n}{n!}x^n$, $B(x) = \sum_{n=0}^{\infty}\frac{b_n}{n!}x^n$ *are two formal* *power series and $b_0 \neq 0$, then their quotient is*

$$C(x) = \frac{A(x)}{B(x)} = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n ,$$

where the c_n are given by the recursion formula

$$a_n = \sum_{m=0}^{n} \binom{n}{m} c_m b_{n-m} . \tag{12.1}$$

The proof, by means of Cauchy multiplication of power series, is straightforward and can be omitted.

Remark. If in particular the a_n and b_n are all integers and $b_0 = 1$ then (12.1) shows that the c_n are also integers.

Now this situation prevails for the tangent coefficients since

$$\tan u = \frac{A(x)}{B(x)}$$

with

$$A(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \quad B(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.$$

The tangent coefficients are therefore integers.

Exercise. Prove from the recursion formula specialized for the tangent coefficients that, with the exception of $T_1 = 1$, the T_k are *even.*

If we now write the Bernoulli numbers

$$B_{2k} = (-1)^{k-1} \frac{N_{2k}}{D_{2k}}$$

the denominators D_{2k} are fully described by the von Staudt-Clausen theorem. The formula (11.2) gives now some slight information about the numerators N_{2k}. Since T_k is an integer the greatest odd factor of $2k$ must divide $N_{2k}(2^{2k} - 1)$; if it is prime to $2^{2k} - 1$ it must therefore be a factor of N_{2k}. In particular if $k = p$ is an odd prime number we have

$$2^{p-1} \equiv 1 \pmod{p}$$

and hence

$$2^{2p} - 1 \equiv 3 \pmod{p} .$$

Therefore, if $k = p > 3$, it follows that p must be a divisor of N_{2p}. Indeed e.g. $B_{10} = 5/66$, $B_{14} = 7/6$. This result is a special case of the following theorem of Frobenius [10], esp. p. 27:

Theorem. *The denominator of B_r/r contains no other primes than the denominator of B_r itself.*

Proof. We copy the important features of the formula for $\tan u$, replacing only the number 2 by an arbitrary integer m, which means that we study

$$F(z) = \sum_{r=0}^{\infty} (-1)^r \frac{B_r}{r!} (m^r - 1) z^r = mz \frac{e^{mz}}{e^{mz} - 1} - z \frac{e^z}{e^z - 1}, \qquad (12.2)$$

after (10.4). We have then

$$\begin{aligned} F(z) &= z \frac{m e^{mz} - (e^{mz} + e^{(m-1)z} + \cdots + e^z)}{e^{mz} - 1} \\ &= z \frac{(e^{mz} - e^{(m-1)z}) + (e^{mz} - e^{(m-2)z}) + \cdots + (e^{mz} - e^z)}{e^{mz} - 1} \\ &= z \frac{(e^z - 1)\{e^{(m-1)z} + e^{(m-2)z}(e^z + 1) + \cdots + e^z(e^{(m-2)z} + \cdots + e^z + 1)\}}{e^{mz} - 1} \\ &= z \frac{e^z + 2e^{2z} + \cdots + (m-1)e^{(m-1)z}}{1 + e^z + e^{2z} + \cdots + e^{(m-1)z}}. \end{aligned}$$

Then

$$\frac{1}{z} F(mz) = \frac{e^{mz} + 2e^{2mz} + \cdots + (m-1)e^{(m-1)mz}}{\frac{1}{m}(1 + e^{mz} + e^{2mz} + \cdots + e^{(m-1)mz})}$$

is of the form considered in the lemma, with a_n and b_n integers and in particular $b_0 = 1$, as the expansions of numerator and denominator into power series show. If we put

$$\frac{1}{z} F(mz) = \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n$$

we have, as remarked after the lemma, the c_n are integers. On the other hand, from (12.2) we take

$$\frac{1}{z} F(mz) = \sum_{r=1}^{\infty} (-1)^r \frac{B_r}{r!} (m^r - 1) m^r z^{r-1},$$

so that we conclude that

$$c_{r-1} = (-1)^r \frac{B_r}{r} (m^r - 1) m^r$$

is an integer.

Suppose now that p is a prime factor of r, but not contained in the denominator of B_r. After von Staudt-Clausen this means that $(p-1) \nmid r$. We take now m as a primitive root modulo p. Then

$$m^k \equiv 1 \pmod{p}$$

happens only for k which are multiples of $p - 1$, and therefore

$$m^r \not\equiv 1 \pmod{p}.$$

Therefore $(m^r - 1)m^r$ is not divisible by p. If now p^α is the highest power of p contained in r, this p^α must thus divide N_r in order to make the above c_{r-1} an integer, so that p^α is not a factor of the denominator of B_r/r. This proves the theorem. ▯

13. The generating function of the Bernoulli polynomials

The summation in (10.1) can be carried out without the Euler-MacLaurin formula:

$$\sum_{n=0}^{N-1} e^{nz} = \sum_{n=0}^{N-1} \sum_{r=0}^{\infty} \frac{(nz)^r}{r!} = \sum_{r=0}^{\infty} \frac{z^r}{r!} \sum_{n=0}^{N-1} n^r$$

or after (2.3)

$$\frac{e^{Nz} - 1}{e^z - 1} = \sum_{r=0}^{\infty} \frac{z^r}{(r+1)!} \{B_{r+1}(N) - B_{r+1}(0)\}. \tag{13.1}$$

This identity holds for all complex z and all positive integers N. However, if we restrict z to

$$|z| < 2\pi$$

then (13.1) holds also for all complex values t replacing N. Indeed, a power series expansion

$$\frac{e^{tz} - 1}{e^z - 1} = \sum_{r=0}^{\infty} A_r(t) z^r \tag{13.2}$$

is valid for $|z| < 2\pi$ since the left-hand member denotes a regular function in the circle. The equation

$$\sum_{m=1}^{\infty} \frac{(tz)^m}{m!} = \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{r=0}^{\infty} A_r(t) z^r$$

leads, through a comparison of coefficients, to a recursion formula for the $A_r(t)$ which shows that they are polynomials in t. Now comparing (13.1) and (13.2) we see that $A_r(t)$ and $\frac{1}{(r+1)!}\left(B_{r+1}(t) - B_{r+1}(0)\right)$ agree for the infinitely many positive integral values for t. They must therefore be identical. We have thus for any complex t and $|z| < 2\pi$

$$z\frac{e^{tz} - 1}{e^z - 1} = \sum_{q=1}^{\infty} \frac{z^q}{q!} \{B_q(t) - B_q(0)\}.$$

Now, because of $B_0 = B_0(t) = 1$, $B_q(0) = B_q$, we have in view of (10.5)

$$\sum_{q=1}^{\infty} \frac{B_q(0) z^q}{q!} = \frac{z}{e^z - 1} - 1$$

and consequently

$$\frac{z e^{tz}}{e^z - 1} = \sum_{q=0}^{\infty} \frac{B_q(t) z^q}{q!} . \tag{13.3}$$

This equation gives the generating function for the Bernoulli polynomials $B_q(t)$.

14. The secant coefficients or Euler numbers

We can use (13.3) for the calculation of the secant coefficients or Euler numbers, which have some historical interest but will appear also later in some formulae for the representation of numbers as sums of squares. We start with

$$\sec u = \frac{2}{e^{iu} + e^{-iu}} = 2\frac{e^{iu} - e^{-iu}}{e^{2iu} - e^{-2iu}} = 2\frac{e^{3iu} - e^{iu}}{e^{4iu} - 1} .$$

Putting now $t = 3/4$ and $t = 1/4$ in (13.3) we obtain, using also (2.8),

$$\begin{aligned} \frac{e^{3z/4} - e^{z/4}}{e^z - 1} &= \sum_{q=1}^{\infty} \frac{1}{q!} \left(B_q\left(\frac{3}{4}\right) - B_q\left(\frac{1}{4}\right) \right) z^{q-1} \\ &= -2 \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} B_{2k+1}\left(\frac{1}{4}\right) z^{2k} , \end{aligned}$$

and thus with $z = 4iu$

$$\sec u = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{4^{2k+1}}{(2k+1)!} B_{2k+1}\left(\frac{1}{4}\right) u^{2k} . \tag{14.1}$$

We define now the secant coefficients or Euler numbers by

$$\sec u = \sum_{k=0}^{\infty} \frac{E_k}{(2k)!} u^{2k} \tag{14.2}$$

so that

$$E_k = (-1)^{k+1} \frac{4^{2k+1}}{2k+1} B_{2k+1}\left(\frac{1}{4}\right) . \tag{14.3}$$

That the E_k are all integers can be seen from $E_0 = 1$ and the recursion formula

$$\sum_{j=0}^{n} \binom{2n}{2j} (-1)^j E_{n-j} = 0, \quad n > 0 , \tag{14.4}$$

which follows directly from sec $u = (\cos u)^{-1}$ or

$$\sum_{k=0}^{\infty} \frac{E_k}{(2k)!} u^{2k} \cdot \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} u^{2j} = 1 .$$

Moreover, the E_k are all *positive*. We know that for $k \geqq 1$ the $B_{2k+1}(x)$ have zeros at $x = 0$ and $1/2$ and none in between. Therefore the sign of $B_{2k+1}(x)$ for $0 < x < 1/2$ is the same as that of $B'_{2k+1}(0) = B_{2k}(0)$ and in particular

$$B_{2k+1}\left(\frac{1}{4}\right) \cdot B_{2k}(0) = B_{2k+1}\left(\frac{1}{4}\right) \cdot B_{2k} > 0 .$$

Therefore, after (3.3)

$$B_{2k+1}\left(\frac{1}{4}\right)(-1)^{k+1} > 0, \quad k \geqq 1 .$$

This is true also for $B_1(1/4) = -1/4$. The positivity of E_k is then seen through (14.3).

A relation between the Euler numbers and Bernoulli numbers is given by (14.3) which in view of (4.2) explicitly reads

$$\begin{aligned} E_k &= \frac{(-1)^{k+1}}{2k+1} \sum_{j=0}^{2k+1} \binom{2k+1}{j} 4^j B_j \\ &= \frac{(-1)^{k+1}}{2k+1} \left\{1 + \binom{2k+1}{1} 4 B_1 + \sum_{h=1}^{k} \binom{2k+1}{2h} 4^{2h} B_{2h}\right\}. \end{aligned} \tag{14.5}$$

In order to establish also a connection with the tangent coefficients T_k we rewrite (11.2) as

$$4^{2h} B_{2h} = (-1)^{h+1} 2h T_h + 2^{2h} B_{2h} ,$$

through which (14.5) goes over into

$$\begin{aligned} (-1)^{k+1} E_k &= \frac{1}{2k+1} \left\{1 + \binom{2k+1}{1} 4 B_1 + \sum_{h=1}^{k} \binom{2k+1}{2h} 2^{2h} B_{2h}\right\} \\ &\quad + \frac{1}{2k+1} \sum_{h=1}^{k} (-1)^{h+1} \binom{2k+1}{2h} 2h T_h \\ &= \frac{1}{2k+1} 2^{2k+1} B_{2k+1}\left(\frac{1}{2}\right) + 2 B_1 + \sum_{h=1}^{k} (-1)^{h+1} \binom{2k}{2h-1} T_h . \end{aligned}$$

Finally, because of (3.1) and $B_1 = -1/2$, we have

$$(-1)^k E_k = 1 + \sum_{h=1}^{k} (-1)^h \binom{2k}{2h-1} T_h . \tag{14.6}$$

This formula shows that all E_k are odd (see exercise in § 12). The first few, derived from (14.4), are

$$E_1 = 1, \quad E_2 = 5, \quad E_3 = 61, \quad E_4 = 1385.$$

The Euler numbers (see (14.3)) permit us to rewrite (9.3) as

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2k+1}} = \frac{E_k}{2^{2k+2}\,(2k)!}\,\pi^{2k+1}, \qquad k \geqq 0\,. \tag{14.7}$$

This yields an estimate of the E_k:

$$\frac{2}{3}\,\frac{2^{2k+2}\,(2k)!}{\pi^{2k+1}} < E_k < \frac{2^{2k+2}\,(2k)!}{\pi^{2k+1}}\,. \tag{14.8}$$

15. Stirling's formula

This application of the Euler-MacLaurin sum formula is well known. We include it here in order to have later some of its special features available for our arguments.

For the function $f(x) = \log x$ formula (7.2) yields, with $q = 2m$ and the separate treatment of $B_1 = -1/2$,

$$\log N! = \sum_2^N \log n = \int_1^N \log x\,dx + \frac{1}{2}\log N$$
$$+ \sum_{j=1}^{m} \frac{B_{2j}}{(2j)!}(2j-2)!\left\{\frac{1}{N^{2j-1}} - 1\right\} + \int_1^N \frac{B_{2m}(x-[x])}{2m}\,x^{-2m}\,dx,$$

$$\log N! = \left(N + \frac{1}{2}\right)\log N - N + \sum_{j=1}^{m} \frac{B_{2j}}{(2j-1)\,2j}\,N^{-2j+1}$$
$$- \Omega_{2m}(N) + K_m, \qquad m \geqq 1 \tag{15.1}$$

with

$$\Omega_{2m}(N) = \frac{1}{2m}\int_N^{\infty} B_{2m}(x-[x])\,x^{-2m}\,dx\,, \tag{15.11}$$

$$K_m = \frac{1}{2m}\int_1^{\infty} B_{2m}(x-[x])\,x^{-2m} - \sum_{j=1}^{m} \frac{B_{2j}}{(2j-1)\,2j}\,. \tag{15.12}$$

Here K_m is free of N, but actually also independent of m, for (15.1) and (15.11) show that the following limit exists and that

$$\lim_{N\to\infty}\left(\log N! - \left(N + \frac{1}{2}\right)\log N + N\right) = K_m\,.$$

which proves that

$$K_m = K \tag{15.2}$$

does not depend on m at all.

We can consider $\Omega_{2m}(N)$ as a remainder term which we estimate by means of (9.7) as

$$|\Omega_{2m}(N)| \leqq \frac{|B_{2m}|}{2m} \int_N^\infty x^{-2m}\,dx = \frac{|B_{2m}|}{(2m-1)\,2m} N^{-2m+1}. \tag{15.3}$$

This is, as function of N, of the magnitude of the last term of the sum in (15.1) and cannot be essentially improved, as we shall presently see. It does not go to zero with increasing m, and (15.1) would for $m \to \infty$ furnish a divergent series. However it is for each fixed m an *asymptotic formula*, for increasing N. Let us put the last term of the sum in (15.1) and $-\Omega_{2m}(N)$ together as an error term

$$\begin{aligned} R_{2m} &= \frac{B_{2m}}{(2m-1)\,2m} N^{-2m+1} - \Omega_{2m}(N) \\ &= -\frac{1}{2m} \int_N^\infty \big(B_{2m}(x-[x]) - B_{2m}\big) x^{-2m}\,dx\,. \end{aligned}$$

Because of inequality (15.3) we see that R_{2m} has the sign of B_{2m}. We may therefore set

$$R_{2m} = \vartheta_m \frac{B_{2m}}{(2m-1)\,2m} N^{-2m+1}, \qquad 0 < \vartheta_m\,. \tag{15.4}$$

Because of this and (15.2), formula (15.1) goes over into

$$\begin{aligned} \log N! = K + \left(N + \frac{1}{2}\right) \log N - N \\ + \sum_{j=1}^{m-1} \frac{B_{2j}}{(2j-1)\,2j} N^{-2j+1} + R_{2m}\,. \end{aligned} \tag{15.5}$$

Let us replace here m by $(m+1)$. Then comparison shows that

$$R_{2m} = \frac{B_{2m}}{(2m-1)\,2m} N^{-2m+1} + R_{2m+2}\,,$$

where according to (15.4)

$$R_{2m+2} = \vartheta_{m+1} \frac{B_{2(m+1)}}{(2m+1)\,(2m+2)} N^{-2m-1}, \quad 0 < \vartheta_{m+1}\,.$$

Since B_{2m} and $B_{2(m+1)}$ have opposite signs, so have therefore R_{2m} and R_{2m+2}. Combining the last two formulae we can write

$$R_{2m} = \frac{B_{2m}}{(2m-1)\,2m} N^{-2m+1} \left\{1 - \vartheta_{m+1} \left| \frac{B_{2(m+2)}}{B_{2m}} \right| \frac{(2m-1)\,2m}{(2m+1)\,(2m+2)} N^{-2} \right\}.$$

Comparing this with (15.4) we see that ϑ_m is represented by the expression in { }, so that $\vartheta_m < 1$. We have thus

$$0 < \vartheta_m < 1 .$$

For the expansion (15.5) this means that the remainder term R_{2m} is a positive fraction of the first neglected term. Divergent series with this property have sometimes been called "semi-convergent series."

From (15.1) and (15.2) we have in particular for $m = 1$ with $B_2 = 1/6$

$$\log N! = K + \left(N + \frac{1}{2}\right) \log N - N + \frac{\vartheta}{12N}, \quad 0 < \vartheta < 1 . \tag{15.51}$$

The equations (15.5) and (15.51) give the well-known *Stirling's formula,* if we also show that

$$K = \log \sqrt{2\pi} . \tag{15.52}$$

This is done by the classical Wallis formula

$$\frac{\pi}{2} = \lim_{N\to\infty} \prod_{n=1}^{N} \frac{4n^2}{4n^2 - 1} = \lim_{N\to\infty} \frac{2^{4N}(N!)^4}{((2N)!)^2(2N+1)} .$$

Using here the exponential form of (15.51)

$$N! = e^K N^{N+1/2} e^{-N} e^{\vartheta/12N} \tag{15.6}$$

we obtain

$$\frac{\pi}{2} = \lim_{N\to\infty} \frac{e^{4K} 2^{4N} N^{4N+2} e^{-4N} e^{\vartheta_1/3N}}{e^{2K}(2N)^{4N+1} e^{-4N} e^{\vartheta_2/12N}(2N+1)}$$

$$= \lim_{N\to\infty} e^{2K} \frac{N^2}{2N(2N+1)} = \frac{1}{4} e^{2K} ,$$

which implies (15.52).

16. A further application

If the Euler-MacLaurin formula is applied to $f(x) = 1/x$ we obtain by a procedure completely analogous to that carried out in the previous section

$$\sum_{n=1}^{N} \frac{1}{n} = \log N + \gamma + \frac{1}{2N} - \sum_{j=1}^{m-1} \frac{B_{2j}}{2j} N^{-2j} - \vartheta \frac{B_{2m}}{2m} N^{-2m},$$

$$0 < \vartheta < 1 . \tag{16.1}$$

Here γ, the so-called Euler-Mascheroni constant, corresponds to the K in the case above. Nothing is known about the arithmetical character of γ or about any relations to other well-known constants. As (16.1) shows, it can be defined through

$$\gamma = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - \log N \,. \tag{16.2}$$

For $N = 1$, $m = 3$ (16.1) gives some information about the size of γ:

$$\gamma = \frac{1}{2} + \frac{B_2}{2} + \frac{B_4}{4} + \vartheta\,\frac{B_6}{6} = \frac{23}{40} + \frac{\vartheta}{252}\,, \qquad 0 < \vartheta < 1\,,$$

$$0.575 < \gamma < 0.579\,.$$

A more precise determination of γ, which can be obtained from (16.1) by use of higher values of N and m, is

$$\gamma = 0.577215665\ldots.$$

17. A historical remark

The Euler-MacLaurin formula (7.2) was first derived without the remainder term R_q, which from the modern point is of course the most essential part of it. The classical approach was something like this (cf. p. 314 (1)):

Take $f(x)$ as a polynomial, so that high enough derivatives vanish and therefore no questions about error terms in the Taylor expansion will arise. Let $f(x)$ be of degree m, and put for abbreviation

$$\Delta f(0) = f(1) - f(0)\,.$$

Then the Taylor expansion gives

$$\begin{aligned}
\int_0^1 f(x)\,dx &= \frac{f(0)}{1!} + \frac{f'(0)}{2!} + \frac{f''(0)}{3!} + \cdots + \frac{f^{(m)}(0)}{(m+1)!}\,,\\
\Delta f(0) &= \frac{f'(0)}{1!} + \frac{f''(0)}{2!} + \cdots + \frac{f^{(m)}(0)}{m!}\,,\\
\Delta f'(0) &= \frac{f''(0)}{1!} + \cdots + \frac{f^{(m)}(0)}{(m-1)!}\,,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\,,\\
\Delta f^{(m-1)}(0) &= \frac{f^{(m)}(0)}{1!}\,.
\end{aligned}$$

The goal is then to eliminate here $f'(0), f''(0), \ldots, f^{(m)}(0)$. This is done by means of the Bernoulli numbers for which (2.72) can be taken as a definition. We multiply the $(j+1)$st line by $B_j/j!$, taking $B_0 = 1$, and add:

$$\begin{aligned}\int_0^1 f(x)\,dx + \sum_{j=1}^{m} \frac{B_j}{j!} \Delta f^{(j-1)}(0) &= f(0) + \sum_{q=1}^{m} f^{(q)}(0) \sum_{j=0}^{q} B_j \frac{1}{j!(q+1-j)!} \\ &= f(0) + \sum_{q=1}^{m} \frac{f^{(q)}(0)}{(q+1)!} \sum_{j=0}^{q} \binom{q+1}{j} B_j \\ &= f(0)\end{aligned}$$

after (2.72), so that

$$f(0) = \int_0^1 f(x)\,dx + \sum_{j=1}^{m} \frac{B_j}{j!} \Delta f^{(j-1)}(0)\,. \tag{17.1}$$

Since $f(x)$ is a polynomial of degree m, one verifies that (17.1) agrees with (7.1) for $q = m+1$, provided that one also knows $B_1 = -1/2$, $B_{2k+1} = 0$ for $k \geqq 1$; this can also be obtained from (2.71). The transition to the summation formula is then done as before by replacing $f(x)$ by $f(n-1+x)$ and so on.

Chapter 3

The Γ-function and Mellin's Theorem

18. Definition of the Γ-function

The factorial function

$$m! = 1 \cdot 2 \cdots m \tag{18.1}$$

is only defined for natural numbers m. It fits into most arguments to put also $0! = 1$, as we have done throughout the previous chapters. The problem to extend the factorial function to other values has of course no unique answer, for if we would have found any definition for $x!$ for continuously varying x any other definition differing from it by a function vanishing at all non-negative integers (e.g. $\sin \pi x$) will solve the interpolation problem just as well. One has therefore to look for some plausible reason to choose a more or less natural extension of the function $m!$

In the form (18.1) the definition of $m!$ is quite inflexible. We try to push the difficulty farther out towards infinity by the following device.

$$\begin{aligned} m! &= \frac{(m+N)!}{(m+1)(m+2)\cdots(m+N)} \\ &= \frac{1}{m+1}\cdot\frac{2}{m+2}\cdots\frac{N}{m+N}\cdot(N+1)(N+2)\cdots(N+m) \\ &= \prod_{\nu=1}^{N}\frac{\nu}{m+\nu}\cdot(N+1)^m\cdot\frac{(N+1)(N+2)\cdots(N+m)}{(N+1)^m}. \end{aligned}$$

This holds for any choice of the positive integer N.

Since the last factor tends to 1 if $N\to\infty$ we have

$$m! = \lim_{N\to\infty}(N+1)^m\prod_{\nu=1}^{N}\frac{\nu}{m+\nu}.$$

Here we can distribute the factor $(N+1)^m$ over the factors under the product sign if we observe the identity

$$N+1 = \prod_{\nu=1}^{N}\frac{\nu+1}{\nu}.$$

Then we get

$$m! = \lim_{N\to\infty}\prod_{\nu=1}^{N}\frac{\nu}{\nu+m}\left(\frac{\nu+1}{\nu}\right)^m = \prod_{\nu=1}^{\infty}\frac{\nu}{\nu+m}\left(\frac{\nu+1}{\nu}\right)^m,$$

which our arguments have shown to be convergent for all positive integers m; it is trivially also convergent for $m=0$ and furnishes the desired $0!=1$.

We replace now m by an arbitrary complex number s and define

$$s! = \prod_{\nu=1}^{\infty}\frac{\nu}{\nu+s}\left(\frac{\nu+1}{\nu}\right)^s. \tag{18.2}$$

This will solve the interpolation problem for the factorial function if we can prove that the product in (18.2) is convergent. The convergence behavior of (18.2) is the same as that of

$$\sum_{\nu=1}^{\infty}\left(s\log\left(1+\frac{1}{\nu}\right)-\log\left(1+\frac{s}{\nu}\right)\right).$$

Let us consider only the domains $|s|\leqq R$ and, dropping a finite number of terms of the series, confine our attention to the series

$$\sigma = \sum_{\nu\geqq M}\left(s\log\left(1+\frac{1}{\nu}\right)-\log\left(1+\frac{s}{\nu}\right)\right),$$

with $M = 2\max(R, 1)$. Then

$$\sigma = \sum_{\nu \geqq M} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k\,\nu^k}(s - s^k) = \sum_{\nu \geqq M} \frac{1}{\nu^2} \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} \frac{s - s^k}{\nu^{k-2}}.$$

Now the inner sum can be estimated through

$$\left| \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} \frac{s - s^k}{\nu^{k-2}} \right| \leqq RM \sum_{k=2}^{\infty} \frac{1}{k} \frac{1}{2^{k-2}} < RM,$$

uniformly for $|s| \leqq R$, and the series σ converges therefore absolutely and uniformly for s in the circle $|s| \leqq R$, and forms therefore an analytic function for $|s| < R$.

The finitely many terms of (18.2) which we have suppressed through $\nu \geqq M$ form a meromorphic function with simple poles at the negative integers. Therefore (18.2) defines $s!$ as a meromorphic function in the whole complex s-plane with simple poles at the negative integers.

Our definition of $s!$ is that of Gauss. It is, however, now customary not to use the function $s!$ defined in (18.2) but the function

$$\Gamma(s) = \frac{1}{s} \prod_{\nu=1}^{\infty} \frac{\nu}{\nu + s} \left(\frac{\nu + 1}{\nu} \right)^s, \tag{18.3}$$

which is connected with $s!$ through

$$s! = s\Gamma(s). \tag{18.4}$$

We leave it to the reader to prove, by means of (16.2), that the definition (18.3) is equivalent with Weierstrass' definition

$$\Gamma(s) = \frac{1}{s} e^{-\gamma s} \prod_{\nu=1}^{\infty} \frac{\nu}{\nu + s} e^{s/\nu}. \tag{18.5}$$

19. Functional equations of $\Gamma(s)$

Since we have $m! = m \cdot (m - 1)!$ we expect now

$$\Gamma(s + 1) = s\Gamma(s) \tag{19.1}$$

for all complex s for which $\Gamma(s)$ is regular. In order to prove this we have to show

$$\frac{1}{s + 1} \prod_{\nu=1}^{\infty} \frac{\nu}{\nu + s + 1} \left(\frac{\nu + 1}{\nu} \right)^{s+1} = \prod_{\nu=1}^{\infty} \frac{\nu}{\nu + s} \left(\frac{\nu + 1}{\nu} \right)^s$$

or

$$\frac{1}{s+1} \prod_{\nu=1}^{\infty} \frac{\nu+s}{\nu+s+1} \cdot \frac{\nu+1}{\nu} = 1 .$$

Now indeed

$$\frac{1}{s+1} \prod_{\nu=1}^{N} \frac{\nu+s}{\nu+s+1} \cdot \frac{\nu+1}{\nu} = \frac{1}{N+s+1} \cdot \frac{N+1}{1},$$

which for $N \to \infty$ proves the previous equation and hence (19.1).

Exercise. Show by means of (19.1) that $\Gamma(s)$ has a simple pole at $s = -m$, $m = 0, 1, 2, \ldots$ with the residue

$$\frac{(-1)^m}{m!} .$$

Also of interest is the product

$$\Gamma(s) \cdot \Gamma(-s) = -\frac{1}{s^2} \prod_{\nu=1}^{\infty} \frac{\nu}{\nu+s} \cdot \frac{\nu}{\nu-s},$$

or, with the use of (19.1) with $-s$ instead of s,

$$\Gamma(s) \cdot \Gamma(1-s) = \frac{1}{s} \prod_{\nu=1}^{\infty} \frac{\nu^2}{\nu^2 - s^2},$$

which in view of

$$\sin z = z \prod_{\nu=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 \nu^2}\right)$$

leads to

$$\Gamma(s)\, \Gamma(1-s) = \frac{\pi}{\sin \pi s} . \tag{19.2}$$

This shows $\Gamma(1/2) = \sqrt{\pi}$, where the $+$ sign is ensured by (18.3).

Besides the basic properties (19.1) and (19.2) a multiplicative property deserves interest. We form, for an integer $k > 1$, directly from (18.3)

$$\Gamma(s)\, \Gamma\left(s + \frac{1}{k}\right) \cdots \Gamma\left(s + \frac{k-1}{k}\right) = \frac{k}{ks} \cdot \frac{k}{ks+1} \cdots \frac{k}{ks+k-1}$$

$$\times \prod_{\nu=1}^{\infty} \frac{(k\nu)^k}{(ks+k\nu)\,(ks+k\nu+1) \cdots (ks+k\nu+k-1)} \left(\frac{\nu+1}{\nu}\right)^{ks+(k-1)/2} .$$

This we compare with

$$\Gamma(ks) = \frac{1}{ks}\cdot\frac{1}{ks+1}\cdot\frac{2}{ks+2}\cdots\frac{k-1}{ks+k-1}\left(\frac{2}{1}\cdot\frac{3}{2}\cdots\frac{k}{k-1}\right)^{ks}$$
$$\times\prod_{\nu=1}^{\infty}\frac{k\nu\cdot(k\nu+1)\cdots(k\nu+k-1)}{(ks+k\nu)(ks+k\nu+1)\cdots(ks+k\nu+k-1)}$$
$$\times\left(\frac{(k\nu+1)}{k\nu}\cdot\frac{k\nu+2}{k\nu+1}\cdots\frac{k\nu+k}{k\nu+k-1}\right)^{ks}$$
$$= k^{ks}\frac{1\cdot 2\cdots(k-1)}{ks(ks+1)\cdots(ks+k-1)}$$
$$\times\prod_{\nu=1}^{\infty}\frac{k\nu\cdots(k\nu+k-1)}{(ks+k\nu)\cdots(ks+k\nu+k-1)}\left(\frac{\nu+1}{\nu}\right)^{ks},$$

which is nothing else than (18.3) for ks and each k terms grouped together. Division of the previous equations furnishes

$$\frac{\Gamma(s)\,\Gamma\left(s+\frac{1}{k}\right)\cdots\Gamma\left(s+\frac{k-1}{k}\right)}{\Gamma(ks)}$$
$$= \lim_{N\to\infty}\frac{k^{k-ks}}{1\cdot 2\cdots(k-1)}\prod_{\nu=1}^{N}\frac{(k\nu)^k}{k\nu(k\nu+1)\cdots(k\nu+k-1)}\left(\frac{\nu+1}{\nu}\right)^{(k-1)/2}$$
$$= \lim_{N\to\infty}\frac{k^{-ks}k^{(N+1)k}\,N!^k(N+1)^{(k-1)/2}}{(kN+k-1)!}$$
$$= \lim_{N\to\infty}k^{-ks}\frac{k^{(N+1)k}N!^k}{(kN)!}\cdot\frac{(N+1)^{(k-1)/2}}{(kN+1)\cdots(kN+k-1)}.$$

Applying here Stirling's formula (15.6) we get after some simplifications

$$\Gamma(s)\,\Gamma\left(s+\frac{1}{k}\right)\cdots\Gamma\left(s+\frac{k-1}{k}\right) = (2\pi)^{(k-1)/2}k^{(1/2)-ks}\,\Gamma(ks). \tag{19.3}$$

This is Gauss' multiplication formula. The special case $k = 2$ goes back to Legendre, who derived it from the definition of the Γ-function by Euler's integral, which we shall meet later.

20. Application of the Euler-MacLaurin sum formula

Starting from $\Gamma(s) = (s-1)\,\Gamma(s-1)$ we get from (18.3)

$$\log\Gamma(s) = \lim_{N\to\infty}\left\{(s-1)\log(N+1) - \sum_{\nu=1}^{N}\log\frac{\nu+s-1}{\nu}\right\}. \tag{20.1}$$

For the definition of the logarithm we take on both sides of the equation the principal branch in a plane which is cut along the negative real axis. This makes $\log \Gamma(s)$ as well as $\log (\nu + s - 1)/\nu$ unique since neither $\Gamma(s)$ nor $(\nu - 1 + s)/\nu$ have zeros or poles outside the negative real axis. This definition of $\log (\nu - 1 + s)/\nu$ was implicit in the investigation of convergence in § 18 through the use of the logarithmic series.

The sum in (20.1) calls for the application of the Euler-MacLaurin formula with

$$f(x) = \log (x + s - 1) - \log x\,,$$

$$f^{(k)}(x) = (-1)^{k-1}(k-1)!\left\{\frac{1}{(x+s-1)^k} - \frac{1}{x^k}\right\}.$$

We obtain from (7.2), (7.21) for $q = 2m$, $a = 1$, $b = N$

$$\sum_{\nu=1}^{N} \log \frac{\nu + s - 1}{\nu} = \log s + \sum_{\nu=2}^{N} \{\log (\nu + s - 1) - \log \nu\}$$

$$= \log s + \int_1^N (\log (x + s - 1) - \log x)\, dx$$

$$+ \frac{1}{2}\{\log (N + s - 1) - \log N - \log s\}$$

$$+ \sum_{j=1}^{m} \frac{B_{2j}}{(2j-1)\,2j}\left\{\frac{1}{(N+s-1)^{2j-1}} - \frac{1}{N^{2j-1}} - \frac{1}{s^{2j-1}} + 1\right\}$$

$$+ \frac{1}{(2m)!}\int_1^N B_{2m}(x - [x])\,(2m-1)!\left(\frac{1}{(x+s-1)^{2m}} - \frac{1}{x^{2m}}\right) dx\,.$$

This, inserted in (20.1), leads to

$$\log \Gamma(s) = -\frac{1}{2}\log s + \sum_{j=1}^{m} \frac{B_{2j}}{(2j-1)\,2j}\left(\frac{1}{s^{2j-1}} - 1\right)$$

$$- \frac{1}{2m}\int_1^\infty B_{2m}(x - [x])\left(\frac{1}{(x+s-1)^{2m}} - \frac{1}{x^{2m}}\right) dx$$

$$+ \lim_{N\to\infty}\left\{(s-1)\log (N+1)\right.$$

$$- \int_1^N (\log (x + s - 1) - \log x)\, dx$$

$$\left. - \frac{1}{2}\log \frac{N + s - 1}{N}\right\}.$$

Here we have

$$\lim_{N\to\infty}\{\cdots\} = \lim_{N\to\infty}\Big\{(s-1)\log N + (s-1)\log\frac{N+1}{N}$$
$$- (N+s-1)\log(N+s-1) + (N+s-1)$$
$$+ s\log s - s + N\log N - N + 1$$
$$- \frac{1}{2}\log\left(1+\frac{s-1}{N}\right)\Big\}$$
$$= s\log s - \lim_{N\to\infty}\left\{(N+s-1)\log\left(1+\frac{s-1}{N}\right)\right\}$$
$$= s\log s - s + 1,$$

so that we obtain

$$\log\Gamma(s) = \left(s-\frac{1}{2}\right)\log s - s + \sum_{j=1}^{m}\frac{B_{2j}}{(2j-1)\,2j}\frac{1}{s^{2j-1}}$$
$$- \frac{1}{2m}\int_1^\infty B_{2m}(x-[x])\frac{1}{(x+s-1)^{2m}}\,dx + K_m,$$

where K_m includes all terms which are free of s. Now we see that for any fixed $m \geqq 1$ the sum and the integral term go to 0 for $s \to +\infty$ so that

$$K_m = \lim_{s\to\infty}\left(\log\Gamma(s) - \left(s-\frac{1}{2}\right)\log s + s\right),$$

which shows that firstly K_m does not depend on m and secondly, because of $\Gamma(N+1) = N!$ and of (15.51), (15.52)

$$K_m = \log\sqrt{2\pi}.$$

The final formula is therefore

$$\log\Gamma(s) = \frac{1}{2}\log(2\pi) + \left(s-\frac{1}{2}\right)\log s - s$$
$$+ \sum_{j=1}^{m}\frac{B_{2j}}{(2j-1)\,2j}\frac{1}{s^{2j-1}} - \frac{1}{2m}\int_0^\infty\frac{B_{2m}(x-[x])}{(x+s)^{2m}}\,dx, \quad (20.2)$$

for all s which are not zero or negative real. This formula is an extension of the logarithmic Stirling formula for the factorial (15.5) with (15.52).

21. Asymptotic behavior of $\Gamma(s)$

In (20.2) we take $m = 1$ and have

$$\log \Gamma(s) = \frac{1}{2} \log (2\pi) + \left(s - \frac{1}{2}\right) \log s - s - \Omega_2(s) \tag{21.1}$$

with

$$\Omega_2(s) = \frac{1}{2} \int_0^\infty \frac{B_2(x - [x]) - B_2}{(x + s)^2} \, dx \, .$$

We know from § 3 that $B_2(x) - B_2$ does not change sign between 0 and 1, and indeed we have after (4.2)

$$B_2(x) - B_2 = x^2 - x$$

and thus

$$\frac{1}{2} \left| B_2(x - [x]) - B_2 \right| \leqq \frac{1}{8} ,$$

$$|\Omega_2(s)| \leqq \frac{1}{8} \int_0^\infty \frac{dx}{|x + s|^2} \, .$$

Let us put

$$s = \sigma + it \, .$$

Since we avoid $s = 0$ and the negative real axis, $t = 0$ can occur here only for $\sigma > 0$. Then

$$|\Omega_2(s)| \leqq \frac{1}{8} \int_0^\infty \frac{dx}{(\sigma + x)^2 + t^2} = \begin{cases} \dfrac{1}{8\sigma} & \text{for } t = 0, \quad \sigma > 0 \, , \\[2ex] \dfrac{1}{8|t|} \arctan \dfrac{|t|}{\sigma} \, , & t \neq 0 \end{cases} \tag{21.2}$$

with the understanding that

$$0 \leqq \arctan \frac{|t|}{\sigma} = |\arg s| < \pi \, .$$

For higher m a similar and more precise result may be obtained from (20.2), but (21.1) together with (21.2) will suffice for our purposes.

Most important is the behavior of $\Gamma(s)$ in a "vertical" strip

$$a \leqq \sigma = \operatorname{Re}(s) \leqq b \tag{21.31}$$

for large t, where we can assume from the beginning

$$|t| \geqq 1\,. \tag{21.32}$$

Let us take $t \geqq 1$. We have then from (21.1) and (21.2)

$$\log \Gamma(s) = \frac{1}{2}\log(2\pi) + \left(\sigma + it - \frac{1}{2}\right)\left(\log \sqrt{\sigma^2 + t^2} + i \arctan \frac{t}{\sigma}\right)$$
$$- (\sigma + it) + \vartheta \cdot \frac{1}{8t} \arctan \frac{t}{\sigma}\,,$$

where ϑ is a certain complex number with

$$|\vartheta| \leqq 1\,.$$

Now we have

$$\log \sqrt{\sigma^2 + t^2} = \frac{1}{2}\log\left(1 + \frac{\sigma^2}{t^2}\right) + \log t = \log t + O\left(\frac{1}{t^2}\right)$$

and

$$\arctan \frac{t}{\sigma} = \frac{\pi}{2} - \arctan \frac{\sigma}{t} = \frac{\pi}{2} - \frac{\sigma}{t} + O\left(\frac{1}{t^3}\right)$$

because of (21.3) for $t \to \infty$. Thus we obtain

$$\log \Gamma(s) = \frac{1}{2}\log(2\pi) - \frac{\pi}{2}t + \left(\sigma - \frac{1}{2}\right)\log t$$
$$+ it(\log t - 1) + i\left(\sigma - \frac{1}{2}\right)\frac{\pi}{2} + O\left(\frac{1}{t}\right), \tag{21.4}$$

or in exponential form:

$$\Gamma(s) = \sqrt{2\pi}\, e^{-\frac{\pi}{2}t}\, t^{\sigma - \frac{1}{2}}\, e^{it(\log t - 1)}\, e^{\frac{\pi i}{2}\left(\sigma - \frac{1}{2}\right)}\left(1 + O\left(\frac{1}{t}\right)\right), \tag{21.5}$$

for $a \leqq \operatorname{Re} s \leqq b$, $t \geqq 1$. Since $\Gamma(s)$ and $\log \Gamma(s)$ are real for positive s, they take conjugate complex values in conjugate complex points. For $t < -1$ the asymptotic formulae for $\Gamma(s)$ and $\log \Gamma(s)$ are obtained from (21.4) and (21.5) by going over to the conjugate complex values.

For many applications it is sufficient to use instead of (21.5) only

$$\Gamma(s) = O\left(e^{-(\pi/2)|t|}\, |t|^{\sigma - (1/2)}\right), \tag{21.51}$$

$$\frac{1}{\Gamma(s)} = O\left(e^{(\pi/2)|t|}\, |t|^{(1/2) - \sigma}\right) \tag{21.52}$$

under the conditions (21.31), (21.32).

22. A lemma

It will be useful to have ready the following

Lemma. *The equality*

$$\int_0^\infty \left(\int_0^\infty F(x, y)\, dx\right) dy = \int_0^\infty \left(\int_0^\infty F(x, y)\, dy\right) dx \tag{22.1}$$

is certainly valid under the following conditions:

(1) $F(x, y)$ *is two-dimensionally continuous in the quadrant* $x \geqq 0$, $y \geqq 0$.

(2) *The integrals*

$$\int_0^\infty F(x, y)\, dx \text{ and } \int_0^\infty F(x, y)\, dy$$

exist both and converge uniformly in every finite range of their parameters y *and* x *respectively.*

(3) *There exist two positive functions* $G(X)$, $H(Y)$ *with* $\lim\limits_{X\to\infty} G(X) = 0$, $\lim\limits_{Y\to\infty} H(Y) = 0$

so that

$$\left|\int_0^{X'}\int_Y^{Y'} F(x, y)\, dx\, dy\right| \leqq H(Y), \textit{for all } X', \textit{and all } Y' \geqq Y,$$

$$\left|\int_X^{X'}\int_0^{Y'} F(x, y)\, dx\, dy\right| \leqq G(X), \textit{for all } Y', \textit{and all } X' \geqq X.$$

Proof. We make use only of the simple facts that (i) the Riemann integral of a continuous function over a finite rectangle is equal to iterated single integrals, and in either order of the variables, and (ii) that a limit with respect to a parameter and the integration can be interchanged in their order if the limit is uniform with respect to the variable of integration.

We establish now the existence of the iterated integrals (22.1). We have, since the inner integral exists according to (2),

$$\begin{aligned}
\left|\int_Y^{Y'}\left(\int_0^\infty F(x, y)\, dx\right) dy\right| &= \left|\int_Y^{Y'}\left(\lim_{X'\to\infty}\int_0^{X'} F(x, y)\, dx\right) dy\right| \\
&= \left|\lim_{X'\to\infty}\int_Y^{Y'}\left(\int_0^{X'} F(x, y)\, dx\right) dy\right| && \text{after (ii)} \\
&= \lim_{X'\to\infty}\left|\int_Y^{Y'}\int_0^{X'} F(x, y)\, dx\, dy\right| && \text{after (i)} \\
&\leqq H(Y),
\end{aligned}$$

which because of $H(Y) \to 0$ ensures the existence of the integral on the left-hand side of (22.1). For the same reason the right-hand integral exists since our conditions are symmetric in x and y.

The equality (22.1) is proved if we show that

$$\lim_{\substack{X\to\infty \\ Y\to\infty}} \int_0^X \int_0^Y F(x, y)\, dx\, dy$$

exists independently of the manner in which X and Y go to infinity. In other words, it suffices to show that to any $\varepsilon > 0$ there exists an M, such that

$$D = \left| \int_0^X \int_0^Y F(x, y)\, dx\, dy - \int_0^{X'} \int_0^{Y'} F(x, y)\, dx\, dy \right| < \varepsilon$$

of X, Y, X', Y' are all $>M$. Now

$$\begin{aligned} D &= \left| \int_0^{X'} \int_{Y'}^{Y} F(x, y)\, dx\, dy + \int_{X'}^{X} \int_0^{Y} F(x, y)\, dx\, dy \right| \\ &\leqq G(\min(Y, Y')) + H(\min(X, X')), \end{aligned}$$

which proves $D < \varepsilon$ for M large enough, and thus proves the lemma. []

Remark. The lemma does not require absolute convergence of the integrals. The reader may show as an exercise that it permits to prove the equality

$$\int_1^\infty \left(\int_1^\infty \sin x^2 y^3\, dx \right) dy = \int_1^\infty \left(\int_1^\infty \sin x^2 y^3\, dy \right) dx ,$$

where the inner integrals are *not* absolutely convergent. (Hint: Use the second mean value theorem in order to establish (3)).

That the order of integration can be relevant is illustrated by the example

$$F(x, y) = \frac{x - y}{(x + y + 1)^3} .$$

Here conditions (1) and (2) are fulfilled. However,

$$\int_0^\infty \left(\int_0^\infty \frac{x - y}{(x + y + 1)^3} dx \right) dy = \frac{1}{2}, \quad \int_0^\infty \left(\int_0^\infty \frac{x - y}{(x + y + 1)^3} dy \right) dx = -\frac{1}{2}.$$

Here the inner integrals are absolutely convergent but the double integral is not. We have here also, e.g.,

$$\lim_{M\to\infty} \int_0^M \int_0^M \frac{x - y}{(x + y + 1)^3}\, dx\, dy = 0 ,$$

which is evident because of antisymmetry of the integrand with respect to the line $x = y$. This example shows that some condition like (3) is indispensable.

If we have absolute convergence of the double integral, then the equation (22.1) is guaranteed by Fubini's theorem, in which case (3) is eo ipso fulfilled. A particularly simple condition of this sort is expressed by the

Corollary. *The equation* (22.1) *is valid if the condition* (1) *of the Lemma is retained, and* (2) *and* (3) *are replaced by: there exist two positive functions* $g_1(x)$, $g_2(y)$ *such that for* $x \geqq 0$, $y \geqq 0$

$$|F(x, y)| \leqq g_1(x)\, g_2(y)$$

and that

$$\int_0^\infty g_1(x)\, dx, \qquad \int_0^\infty g_2(y)\, dy$$

exist.

23. The Mellin formula

We return to the Γ-function. Equation (18.3) shows that $\Gamma(z)$ is meromorphic with simple poles at $0, -1, -2, \ldots$.

If we draw now a simple closed rectifiable curve C in the half-plane $\mathrm{Re}(z) > 0$, in which $\Gamma(z)$ is regular, enclosing in its interior a point $s = \sigma + it$, we have

$$\Gamma(s) = \frac{1}{2\pi i} \int_C \frac{\Gamma(z)}{z - s}\, dz\,.$$

As curve C we choose the rectangle with the vertices

$$a - i\omega, \quad b - i\omega, \quad b + i\omega, \quad a + i\omega$$

with

$$0 < a < \mathrm{Re}(s) = \sigma < b, \qquad 0 < |\mathrm{Im}(s)| = |t| < \omega\,,$$

so that we obtain

$$\Gamma(s) = \frac{1}{2\pi i} \left\{ \int_{a-i\omega}^{b-i\omega} + \int_{b-i\omega}^{b+i\omega} + \int_{b+i\omega}^{a+i\omega} + \int_{a+i\omega}^{a-i\omega} \frac{\Gamma(z)}{z - s}\, dz \right\}$$

$$= \frac{1}{2\pi i} \{I_1 + I_2 + I_3 + I_4\}\,,$$

say. Now, according to (21.51)

$$|I_1| < C \frac{(b - a)\, e^{-(\pi/2)\omega}\, \omega^{b-1/2}}{\omega - |t|} \to 0$$

as $\omega \to \infty$. The same estimate holds for I_3. Moreover (21.51) shows also that I_2 and I_4 tend to limits as $\omega \to \infty$, and we have therefore

$$\Gamma(s) = \frac{1}{2\pi i}\left\{\int_{b-i\infty}^{b+i\infty} - \int_{a-i\infty}^{a+i\infty} \frac{\Gamma(z)}{z-s}\,dz\right\}$$

$$= \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{\Gamma(z)}{z-s}\,dz + \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\Gamma(z)}{s-z}\,dz\,,$$

where the integrals are taken along lines parallel to the imaginary axis. In the first of these integrals we have $\operatorname{Re}(z-s) = b-\sigma > 0$ and can therefore put

$$\frac{1}{z-s} = \int_1^\infty u^{s-z-1}\,du\,,$$

which is an absolutely convergent integral; in the second we have $\operatorname{Re}(s-z) = \sigma - a > 0$, which permits us to put

$$\frac{1}{s-z} = \int_0^1 u^{s-z-1}\,du\,,$$

so that we obtain

$$\Gamma(s) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \Gamma(z)\left(\int_1^\infty u^{s-z-1}\,du\right)dz$$

$$+ \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Gamma(z)\left(\int_0^1 u^{s-z-1}\,du\right)dz\,.$$

We can interchange the order of integration in both double integrals, in virtue of the corollary of § 22 and the estimates (21.51) for $\Gamma(z)$ and trivial ones for the inner integrals. Indeed, if we write $z = x + iy$ we see that the corollary is fulfilled for the first integral with

$$g_1(y) = C e^{-(\pi/2)|y|}\,(1+|y|)^{b-1/2}\,, \qquad g_2(u) = u^{\sigma-b-1}$$

and similarly for the second integral. We obtain therefore

$$\Gamma(s) = \frac{1}{2\pi i}\int_1^\infty u^{s-1}\left(\int_{b-i\infty}^{b+i\infty} \Gamma(z)\,u^{-z}\,dz\right)du$$

$$+ \frac{1}{2\pi i}\int_0^1 u^{s-1}\left(\int_{a-i\infty}^{a+i\infty} \Gamma(z)\,u^{-z}\,dz\right)du. \tag{23.1}$$

Now, for any fixed $u > 0$ the function

$$\Gamma(z)\,u^{-z}$$

is regular in the half-plane $\mathrm{Re}(z) > 0$ and therefore

$$\int_{a-i\infty}^{a+i\infty} \Gamma(z)\, u^{-z}\, dz = \int_{b-i\infty}^{b+i\infty} \Gamma(z)\, u^{-z}\, dz\,,$$

as can be seen by integrating $\Gamma(z)\, u^{-z}$ first around a rectangle of vertices $a - i\omega$, $b - i\omega$, $b + i\omega$, $a + i\omega$ and the letting ω tend to infinity, as we have done a moment ago. The inner integrals in (23.1) represent therefore the same function of u

$$I(u; a) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(z)\, u^{-z}\, dz\,, \tag{23.2}$$

and we have

$$\Gamma(s) = \frac{1}{2\pi i} \int_0^{\infty} u^{s-1}\, I(u; a)\, du\,. \tag{23.3}$$

We investigate now the function $I(u; a)$. For this purpose we need the residue $R_{-\nu}$ of $\Gamma(z)\, u^{-z}$ at $z = -\nu$, $\nu = 0, 1, 2, 3, \ldots$. We have

$$R_{-\nu} = \lim_{z\to-\nu} (z + \nu)\, \Gamma(z)\, u^{-z} = \lim_{z\to-\nu} \frac{(z+\nu)\, \Gamma(z+\nu) u^{-z}}{z(z+1)\cdots(z+\nu-1)}$$

$$= \lim_{z\to-\nu} \frac{\Gamma(z+\nu+1) u^{-z}}{z(z+1)\cdots(z+\nu-1)} = \frac{(-1)^\nu u^\nu}{\nu!}\,,$$

where we have repeatedly applied (19.1) and used $\Gamma(1) = 1$. We now integrate $\Gamma(z)\, u^{-z}$ around the rectangle of vertices $-m - 1/2 - i\omega$, $a - i\omega$, $a + i\omega$, $-m - 1/2 + i\omega$, which will give the sum of the residues of the enclosed poles multiplied by $2\pi i$. The estimate (21.51) shows again that we can push ω to ∞. This yields, in the notation (23.2),

$$I(u; a) = \sum_{\nu=0}^{m} \frac{(-u)^\nu}{\nu!} + I\left(u; -m - \frac{1}{2}\right). \tag{23.4}$$

Here we have

$$I\left(u; -m - \frac{1}{2}\right) = \frac{1}{2\pi i} \int_{-m-1/2-i\infty}^{-m-1/2+i\infty} \Gamma(z)\, u^{-z}\, dz$$

$$= \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \Gamma(w - m - 1)\, u^{-w+m+1}\, dw$$

$$= \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\Gamma(w)}{(w-1)(w-2)\cdots(w-m-1)} u^{-w+m+1}\, dw$$

and therefore

$$\left|I\left(u; -m-\frac{1}{2}\right)\right| \leqq \frac{u^{m+1/2}}{2\pi \cdot \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{2m+1}{2}} \int_{-\infty}^{\infty} \left|\Gamma\left(\frac{1}{2}+iy\right)\right| dy\,,$$

which shows that

$$\lim_{m\to\infty} I\left(u; -m-\frac{1}{2}\right) = 0$$

for every $u > 0$. If therefore in (23.4) we let m go to infinity, we obtain

$$I(u; a) = \sum_{\nu=0}^{\infty} \frac{(-u)^\nu}{\nu!}\,,$$

or

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(z)\, u^{-z}\, dz = e^{-u}, \quad a > 0, \quad u > 0\,. \tag{23.5}$$

This is *Mellin's formula.* Inserting this in (23.3) we arrive at

$$\Gamma(s) = \int_0^\infty e^{-u} u^{s-1}\, du\,, \tag{23.6}$$

which is *Euler's Γ-integral,* derived under the condition $\mathrm{Re}(s) > 0$.

The Γ-function is often defined by (23.6). We have, however, preferred Gauss's definition (18.3) and derived (23.6) as a consequence. This has the advantage that (18.3) defines the Γ-function in the whole s-plane, whereas (23.6) was proved only for $\mathrm{Re}(s) > 0$, and this is an unavoidable restriction since the Γ-integral is not convergent at its lower end 0 for $\mathrm{Re}(s) \leqq 0$.

We have derived Mellin's formula (23.5) only for positive u. It is, however, valid also in the half-plane

$$\mathrm{Re}(u) > 0\,. \tag{23.7}$$

In order to prove this assertion we have to show only that the left-hand member of (23.5) represents an analytic function in the domain (23.7). Let us put

$$u = r \cdot e^{i\varphi}$$

and consider the region

$$0 < \varrho \leqq r\,, \quad |\varphi| \leqq \frac{\pi}{2} - \delta < \frac{\pi}{2} \tag{23.8}$$

(which for suitable ϱ and δ includes any given point in (23.7)). We have on the path of integration in (23.5) with $z = a + iy$

$$\begin{aligned} |\Gamma(z)\, u^{-z}| &= |\Gamma(a + iy)\, e^{-(a+iy)(\log r + i\varphi)}| \\ &< C e^{-(\pi/2)|y|} (1 + |y|)^{a-1/2} e^{-a \log r + y\varphi} \\ &< C e^{-\delta |y|} (1 + |y|)^{a-1/2} \varrho^{-a}\,, \end{aligned}$$

which shows uniform convergence of the integral in the region (23.8) and thus analyticity in any point of the half-plane $\mathrm{Re}(u) > 0$. But the then equality in (23.5) is established in this half-plane since the two members of the equation agree on the positive real axis.

24. Hankel's formula

Euler's integral (23.6) converges only for $\mathrm{Re}(s) > 0$. It will be useful to obtain a related integral valid for all complex values of s. We consider the "loop integral" in the w-plane

$$G(s) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^{w} w^{-s}\, dw\,. \tag{24.1}$$

It is understood here that the w-plane has been cut along the negative real axis, that w^{-s} represents the principal branch

$$w^{-s} = e^{-s(\log|w| + i \arg w)} \quad \text{with} \quad |\arg w| \leqq \pi\,,$$

the special notation of the integral (introduced by G. N. Watson) indicating that the path of integration is a loop beginning at $-\infty$ on the lower border of the cut, going around the point 0 in the positive sense to the upper border and ending there at $-\infty$. The particular shape of the loop is irrelevant since the function $e^{w} w^{-s}$ is regular in the cut w-plane. The convergence of the integral is best seen if we take a loop of a special shape, consisting of three pieces:

(1) w from $-\infty$ to $-\varepsilon$ on the lower border of the cut, i.e. with $\arg w = -\pi$;

(2) a circle of radius ε about 0;

(3) w on the upper border of the cut, $\arg w = +\pi$, from $-\varepsilon$ to $-\infty$.

We have then

$$\begin{aligned} G(s) &= \frac{1}{2\pi i} \int_{-\infty}^{-\varepsilon} e^{w} e^{-s(\log|w| - i\pi)}\, dw + \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\varepsilon e^{i\varphi}} e^{-s(\log \varepsilon + i\varphi)}\, \varepsilon e^{i\varphi}\, d\varphi \\ &\quad + \frac{1}{2\pi i} \int_{-\varepsilon}^{-\infty} e^{w} e^{-s(\log|w| + i\pi)}\, dw \\ &= \frac{\sin \pi s}{\pi} \int_{\varepsilon}^{\infty} e^{-x} x^{-s}\, dx + \frac{\varepsilon^{1-s}}{2\pi} \int_{-\pi}^{\pi} e^{\varepsilon e^{i\varphi}} e^{(1-s)i\varphi}\, d\varphi\,. \end{aligned}$$

Here only entire functions of s appear before and under the integral signs. It is readily seen that the convergence of the improper integral is uniform in any bounded region of the s-plane. Therefore $G(s)$ is an entire function of s.

Now we have

$$\left|\frac{\varepsilon^{1-s}}{2\pi}\int_{-\pi}^{\pi} e^{\varepsilon e^{i\varphi}}\, e^{(1-s)i\varphi}\, d\varphi\right| \leqq \varepsilon^{1-\sigma} e^{\varepsilon} e^{\pi|t|}\,,$$

which for

$$\operatorname{Re}(s) = \sigma < 1 \tag{24.2}$$

converges to 0 as $\varepsilon \to 0$. This condition on σ ensures also that

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty} e^{-x} x^{-s}\, dx = \int_{0}^{\infty} e^{x} x^{-s}\, dx\,,$$

so that we get in view of (23.6) and (19.2)

$$G(s) = \frac{\sin \pi s}{\pi}\,\Gamma(1-s) = \frac{1}{\Gamma(s)}\,.$$

This has so far only been proved under condition (24.2). However, since both members of this equation are entire functions, the equation must hold for all s.

We have thus proved Hankel's formula

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{w} w^{-s}\, dw\,. \tag{24.3}$$

25. An application to Bessel functions

The Bessel functions are defined by

$$J_\nu(z) = \left(\frac{z}{2}\right)^{\nu} \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{z}{2}\right)^{2n}}{n!\,\Gamma(\nu+n+1)}\,. \tag{25.1}$$

It is evident that $J_\nu(z)$ is an entire function for ν a non-negative integer. This is also the case for a negative integer ν, if the zeros of $\Gamma(\nu+n+1)^{-1}$ for $n = 0, \ldots, |\nu| - 1$ are used to cancel the first terms of the series, which then begins with $n = |\nu|$. For non-integral $\nu, J_\nu(z)$ has a branch point at $z = 0$.

The application of Hankel's formula (24.3) gives

$$J_\nu(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^\nu \int_{-\infty}^{(0+)} e^w w^{-\nu-1} \sum_{n=0}^{\infty}(-1)^n \frac{w^{-n}(z/2)^{2n}}{n!} dw$$

$$= \frac{\left(\frac{z}{2}\right)^\nu}{2\pi i} \int_{-\infty}^{(0+)} e^{w-(z^2/4w)} w^{-\nu-1} dw . \tag{25.21}$$

For $\mathrm{Re}(\nu) > 0$ the loop path can be spread open into a vertical line lying in the right half-plane:

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{w-(z^2/4w)} w^{-\nu-1} dw , \quad \mathrm{Re}(\nu) > 0, \quad c > 0 . \tag{25.22}$$

The same treatment can be applied to the Bessel function of "purely imaginary argument"[1]

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n}}{n!\,\Gamma(\nu+n+1)} = \frac{\left(\frac{z}{2}\right)^\nu}{2\pi i} \int_{-\infty}^{(0+)} e^{w+(z^2/4w)} w^{-\nu-1} dw , \tag{25.3}$$

where for $\mathrm{Re}(\nu) > 0$ the loop integral can again be converted into an integral from $c - i\infty$ to $c + i\infty$.

26. The Fourier integral

The two functions e^{-u} and $\Gamma(s)$, connected with each other by the equations (23.5) and (23.6) form an instance of a pair of "reciprocal functions" in the sense of the theory given by H. Mellin [40]. Since however these two integrals are of a different sort, one on the positive real axis, the other on a parallel to the imaginary axis, it is preferable to introduce the Fourier integrals, which are of a more symmetric nature and to obtain later Mellin's results by a change of variables.

Let $F(z)$ be given as a regular analytic function in the strip

$$a < x = \mathrm{Re}(z) < b , \tag{26.11}$$

with $z = x + iy$. Also given are two real numbers $\alpha < \beta$. Let $F(z)$ have the property that for any $\delta > 0$, $\varepsilon > 0$ and all z satisfying

$$a + \varepsilon \leqq x \leqq b - \varepsilon \tag{26.12}$$

[1] A misleading name, since z in turn can take complex values.

we have

$$|F(z)| < K e^{-\gamma y} \tag{26.13}$$

for all γ fulfilling

$$\alpha + \delta \leqq \gamma \leqq \beta - \delta . \tag{26.14}$$

The constant K may depend on δ and ε.

We define now, with $w = u + iy$, the function

$$\frac{\Phi(w)}{\sqrt{2\pi}} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(z)\, e^{-iwz}\, dz , \tag{26.2}$$

where c is a real number satisfying

$$a + \varepsilon \leqq c \leqq b - \varepsilon . \tag{26.21}$$

The integral exists if we restrict w by

$$\alpha + 2\delta \leqq u \leqq \beta - 2\delta . \tag{26.22}$$

Indeed, we have

$$|F(z)\, e^{-iwz}| < K e^{-\gamma y} \left| e^{-i(u+iv)(c+iy)} \right| = K e^{-(\gamma-u)y} e^{cv}$$

and since (26.13) is valid for *all* γ in the interval (26.14)

$$|F(z)\, e^{-iwz}| \leqq K e^{cv} \inf e^{-(\gamma-u)y} \leqq K e^{cv} e^{-\delta|y|} \tag{26.23}$$

because of (26.22). This ensures the uniform convergence of the integral in the interval (26.22). Therefore $\Phi(w)$ is analytic and regular in the interval, and since $\delta > 0$ is arbitrary also in $\alpha < \operatorname{Re}(w) < \beta$. Moreover $\Phi(w)$ is independent of c in its range (26.21). This can be seen if the integration of $F(z)\, e^{-iwz}$ is first carried out around a rectangle of vertices $c - i\omega$, $c_1 - i\omega$, $c_1 + i\omega$, $c + i\omega$, where c_1 is also in the interval (26.21). The estimate (26.23) shows then that the parts of the integral on the horizontal sides go to zero as $\omega \to \infty$, so that

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(z)\, e^{-iwz}\, dz = \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} F(z)\, e^{-iwz}\, dz .$$

In order to have a brief way to refer to such a procedure we shall in the sequel simply say that the path of integration in (26.2) can be "shifted" from the abscissa c to another abscissa c_1.

The estimate (26.23) yields also

$$|\Phi(w)| < \frac{2}{\sqrt{2\pi}} K \frac{1}{\delta} e^{cv} = K_1 e^{cv} \tag{26.3}$$

for all c and u satisfying (26.21) and (26.22) respectively with K_1 depending on δ and ε.

We form now

$$J(\zeta)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Phi(w)\,e^{i\zeta w}\,dw\,, \tag{26.4}$$

where γ, being a special admissible value of u, must fulfill

$$\alpha+2\delta\leqq\gamma\leqq\beta-2\delta\,. \tag{26.5}$$

We intend to prove

$$J(\zeta)=\frac{F(\zeta)}{\sqrt{2\pi}} \tag{26.6}$$

in the strip

$$a+2\varepsilon\leqq\xi\leqq b-2\varepsilon \tag{26.61}$$

for the variable $\zeta=\xi+i\eta$.

Now the convergence of the integral $J(\zeta)$ is proved by means of (26.3), (26.21), (26.5) just as that of $\Phi(w)$ was derived from (26.13), (26.14), (26.22) and it is independent of γ in the range (26.5).

We insert now $\Phi(w)$ from (26.2) in (26.4), choosing thereby c in two different ways consistent with (26.21).

$$\frac{J(\zeta)}{\sqrt{2\pi}}=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma}e^{i\zeta w}\left\{\frac{1}{2\pi i}\int_{b-\varepsilon-i\infty}^{b-\varepsilon+i\infty}F(z)\,e^{-iwz}\,dz\right\}dw$$
$$+\frac{1}{2\pi i}\int_{\gamma}^{\gamma+i\infty}e^{i\zeta w}\left\{\frac{1}{2\pi i}\int_{a+\varepsilon-i\infty}^{a+\varepsilon+i\infty}F(z)\,e^{-iwz}\,dz\right\}dw\,.$$

The estimate (26.23) together with the restrictions (26.5), (26.61) imply now in the two integrals with

$$w=\gamma+iv,\qquad v\leqq 0,\qquad z\,b=-\,\varepsilon+iy$$

and

$$w=\gamma+iv,\qquad v\geqq 0,\qquad z=a+\varepsilon+iy$$

that

$$|F(z)\,e^{iw(\zeta-z)}|<Ke^{-\delta|y|}e^{-\varepsilon|v|}e^{-\gamma\eta}\,,$$

so that the corollary of § 22 permits the interchange of the integrations. We have then

$$\frac{J(\zeta)}{\sqrt{2\pi}}=\frac{1}{2\pi i}\int_{b-\varepsilon-i\infty}^{b-\varepsilon+i\infty}F(z)\left\{\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma}e^{(i\zeta-z)w}\,dw\right\}dz$$
$$+\frac{1}{2\pi i}\int_{a+\varepsilon-i\infty}^{a+\varepsilon+i\infty}F(z)\left\{\frac{1}{2\pi i}\int_{\gamma}^{\gamma+i\infty}e^{i(\zeta-z)w}\,dw\right\}dz=I_1+I_2\,,$$

say. The inner integrals can be calculated, and we obtain

$$\sqrt{2\pi}\,J(\zeta) = \frac{1}{2\pi i}\int_{b-\varepsilon-i\infty}^{b-\varepsilon+i\infty}\frac{F(z)\,e^{i(\zeta-z)\gamma}}{z-\zeta}\,dz$$

$$-\frac{1}{2\pi i}\int_{a+\varepsilon-i\infty}^{a+\varepsilon+i\infty}\frac{F(z)\,e^{i(\zeta-z)\gamma}}{z-\zeta}\,dz\,.$$

This shows that $\sqrt{2\pi}\,J(\zeta)$ is equal to the residue of

$$\frac{F(z)\,e^{i(\zeta-z)\gamma}}{z-\zeta}$$

at $z-\zeta$, which proves (26.6).

Since $\delta > 0$, $\varepsilon > 0$ are arbitrary, we realize that in (26.22), (26.51), (26.61) the numbers 2δ and 2ε can again be replaced by δ and ε if we only adjust the constant K_1 appropriately.

Summarizing, we can state therefore the

Theorem. *Let $a < b$, $\alpha < \beta$ be two pairs of real numbers and let $F(z)$ be regular analytic in the strip*

$$a < x < b \tag{26.71}$$

with $z = x + iy$, satisfying there for every $\varepsilon > 0$ the inequality

$$|F(z)| < K e^{-\gamma y} \tag{26.72}$$

for all z in the strip

$$a + \varepsilon \leqq x \leqq b - \varepsilon \tag{26.73}$$

and all γ in the interval

$$\alpha + \delta \leqq \gamma \leqq \beta - \delta\,, \tag{26.74}$$

where the constant K may depend on δ and ε. Then the integral

$$\frac{1}{\sqrt{2\pi}}\,\Phi(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)\,e^{-iwz}\,dz \tag{26.75}$$

exists for any c in

$$a < c < b\,, \tag{26.76}$$

is independent of c and is regular analytic in the strip

$$\alpha < u < \beta \tag{26.81}$$

of the variable $w = u + iv$. *For any* $\delta > 0$, $\varepsilon > 0$ *the estimate*

$$|\Phi(w)| < K^* e^{cv} \tag{26.82}$$

holds in the intervals

$$\alpha + \delta \leqq u \leqq \beta - \delta\,, \tag{26.83}$$

$$a + \varepsilon \leqq c \leqq b - \varepsilon\,, \tag{26.84}$$

K^* *depending on* δ *and* ε, *and the equation*

$$\frac{1}{\sqrt{2\pi}} F(z) = \frac{1}{2\pi i} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} \Phi(w)\, e^{izw}\, dw \tag{26.85}$$

holds for all γ *in*

$$\alpha < \gamma < \beta\,. \tag{26.86}$$

The sets of formulae (26.71) to (26.76) and (26.81) to (26.86) correspond to each other and differ only by a sign in the variable of the exponential functions. They go over into each other through the following substitutions

$$\begin{array}{lll}
z \to -w, & x + iy \to -u - iv\,, & \\
w \to z\,, & u + iv \to x + iy\,, & \\
F \to -\Phi\,, & \Phi \to F\,, & \\
\alpha \to -b, & \gamma \to -c, & \beta \to -a\,, \\
a \to -\beta, & c \to \gamma, & b \to -\alpha\,, \\
\varepsilon \to \delta, & \delta \to \varepsilon\,. &
\end{array}$$

Since we have proved that the first set implies the second set, this symmetry shows now that the second set conversely implies the first set of formulae.

Suppose first that $a < 0 < b$, $\alpha < 0 < \beta$. Then we can choose $c = \gamma = 0$ in (26.75) and (26.85) respectively, and we can then restrict ourselves to the imaginary axis $z = iy$, $w = iv$. If we introduce

$$\Phi(iv) = \Psi(v)\,, \quad F(iy) = G(y)\,,$$

we obtain the pair of reciprocal formulae

$$\Psi(v) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} G(y)\, e^{ivy}\, dy\,, \qquad G(y) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \Psi(v)\, e^{-iyv}\, dv\,. \tag{26.9}$$

The function $\Psi(v)$ is called the *Fourier transform* of $G(y)$; the second formula then shows $G(-y)$ as the Fourier transform of $\Psi(v)$.

We have proved the equations (26.9) only for functions $\Psi(v)$ and $G(y)$ which are analytic in strips surrounding the v-axis and the y-axis respectively and which fulfill there certain growth conditions implied in (26.82), (26.84) and in (26.72), (26.74). The formulae have, however, a much wider range of validity. They can be applied to functions $\Psi(v)$ and $G(y)$ defined only for the real variables v and y under certain conditions, for which we refer to the literature on Fourier integrals. Our theory for (26.9) developed here has the same relation to the special theory of Fourier integrals as the theory of the Laurent series has to the theory of Fourier series of a real variable.

27. Mellin's formulae

The purpose of the discussions of the previous section was to derive Mellin's theory of which (23.5) and (23.6) in the theory of the Γ-function are a special example.

Let us assume now

$$\alpha = -\beta < u < \beta \tag{27.1}$$

and let us make the substitutions

$$e^{iw} = \zeta, \qquad \varrho = e^{-v}, \qquad u = \varphi\,, \qquad \zeta = \varrho e^{i\varphi}\,. \tag{27.2}$$

We also write

$$s = \sigma + it$$

instead of $z = x + iy$. The substitution (27.2) maps the parallel strip (27.1) in the w-plane on the angle

$$|\varphi| = |\arg\zeta| < \beta \tag{27.3}$$

in the ζ-plane. Moreover we write

$$\frac{1}{\sqrt{2\pi}}\,\Phi(w) = \frac{1}{\sqrt{2\pi}}\,\Phi\left(\frac{\log\zeta}{i}\right) = \Psi(\zeta)\,.$$

The theorem of § 26 states that if $F(s)$ is regular in the strip

$$a < \delta = \mathrm{Re}(s) < b$$

satisfying there for every $\delta > 0$, $\varepsilon > 0$ the inequality

$$|F(s)| < Ke^{-(\beta-\delta)|t|}$$

for all s in the strip $a + \varepsilon \leqq \sigma \leqq b - \varepsilon$, where K may depend on δ and ε, then the integral

$$\Psi(\zeta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, \zeta^{-s}\, ds \tag{27.4}$$

exists for any c in the interval $a < c < b$, and $\Psi(\zeta)$ is independent of c and regular analytic in the angle $|\varphi| < \beta$. Moreover

$$|\Psi(\zeta)| < K^* |\zeta|^{-c}$$

for $|\varphi| \leqq \beta - \delta$, $a + \varepsilon \leqq c \leqq b - \varepsilon$, K^* depending on δ and ε, and

$$F(s) = \int_0^\infty \Psi(\zeta)\, \zeta^{s-1}\, d\zeta\,, \tag{27.5}$$

where the path of the integration may be any ray issuing from 0 under an angle φ satisfying (27.3).

In particular we see that Euler's and Mellin's integrals (23.6) and (23.5) are the special cases

$$F(s) = \Gamma(s)\,, \qquad \Psi(\zeta) = e^{-\zeta}$$

of (27.5) and (27.4), with $a = 0$, b positive arbitrary, $\beta = \pi/2$.

28. Some further examples of Mellin's formulae

(I) We take a parameter $p = m + in$, which we shall keep fixed, and define

$$F(s) = \Gamma(s)\, \Gamma(p - s)\,. \tag{28.1}$$

If $m = \mathrm{Re}(p) > 0$ then $F(s)$ is regular for

$$0 < \mathrm{Re}(s) < m\,,$$

and it follows from (21.51)

$$|F(s)| < K e^{-(\pi-\delta)|t|} \tag{28.11}$$

in $0 < \varepsilon \leqq \mathrm{Re}(s) \leqq m - \varepsilon$. Mellin's theory then shows that the integral

$$\Psi(\zeta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\, \Gamma(p - s)\, \zeta^{-s}\, ds \tag{28.2}$$

exists for any c in $0 < c < m$ and that $\Psi(\zeta)$ is regular in the angle $|\arg \zeta| < \pi$, i.e. in the whole ζ-plane with the exception of the negative real axis. Moreover we have

$$\Gamma(s)\, \Gamma(p - s) = \int_0^\infty \Psi(\zeta)\, \zeta^{s-1}\, d\zeta\,. \tag{28.3}$$

Let us for the moment confine ζ to the region

$$0 < \varrho \leqq |\zeta| < 1\,, \qquad |\arg \zeta| < \pi - 2\delta\,. \tag{28.4}$$

Since using (28.1) and (21.51) it is seen that (28.11) is valid in any vertical strip, provided $|t| > |n| + 1$, we have with a properly adjusted K,

$$|F(s)\,\zeta^{-s}| < K\,|\zeta|^{-\sigma} e^{-\delta|t|}\,, \tag{28.5}$$

and we can therefore "shift" the path of integration in (28.2) to the left from c to $c - k$ if we take into account the poles of the integrand which we pass in the process. We may assume here $0 < c < 1$. Observing the residues of $\Gamma(s)$ at the poles (see exercise § 19) we obtain in this way

$$\Psi(\zeta) = \sum_{\nu=0}^{k-1} \frac{\Gamma(p+\nu)}{\nu!}(-\zeta)^{\nu} + I(\zeta, c-k)$$

with

$$I(\zeta, c-k) = \frac{1}{2\pi i}\int_{c-k-i\infty}^{c-k+i\infty} \Gamma(s)\,\Gamma(p-s)\,\zeta^{-s}\,ds\,,$$

where the path of integration is obviously free of poles. Now we have

$$\begin{aligned} I(\zeta, c-k) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s-k)\,\Gamma(p-s+k)\,\zeta^{-s+k}\,ds \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\,\Gamma(p-s) \\ &\qquad \times \frac{(p-s)\,(p-s+1)\cdots(p-s+k-1)}{(s-1)\,(s-2)\cdots(s-k)}\,\zeta^{-s+k}\,ds\,. \end{aligned}$$

A simple estimation of the integrand, using (28.4) and (28.5), shows now that

$$\lim_{k\to\infty} I(\zeta; c-k) = 0$$

and uniformly in the region (28.4). We thus have there

$$\Psi(\zeta) = \sum_{\nu=0}^{\infty} \frac{\Gamma(p+\nu)}{\nu!}(-\zeta)^{\nu} = \Gamma(p)\sum_{\nu=0}^{\infty}\binom{-p}{\nu}\zeta^{\nu} = \Gamma(p)\,(1+\zeta)^{-p}\,.$$

But we have seen that $\Psi(\zeta)$ is regular in the ζ-plane cut along the negative real axis so that

$$\Psi(\zeta) = \Gamma(p)\,(1+\zeta)^{-p}$$

is true for $|\arg \zeta| < \pi$. The formulae (28.2) and (28.3) are now explicitly

$$\frac{1}{(1+\zeta)^p} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\,\Gamma(p-s)}{\Gamma(p)} \zeta^{-s}\,ds\,, \tag{28.61}$$

$$\frac{\Gamma(s)\,\Gamma(p-s)}{\Gamma(p)} = \int_0^\infty \frac{\zeta^{s-1}}{(1+\zeta)^p}\,d\zeta \tag{28.62}$$

valid for $0 < c < \mathrm{Re}(p)$, $0 < \mathrm{Re}(s) < \mathrm{Re}(p)$, $|\arg \zeta| < \pi$. The formula (28.62) contains Euler's B-integral, which is usually written

$$\frac{\Gamma(s)\,\Gamma(p-s)}{\Gamma(p)} = \int_0^1 x^{s-1}(1-x)^{p-s-1}\,dx$$

obtained from (28.62) through the substitution $\zeta/(1+\zeta) = x$.

(II) A similar procedure yields the formula

$$2\zeta^{p/2} K_p(2\sqrt{\zeta}) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \Gamma(s)\,\Gamma(p+s)\,\zeta^{-s}\,ds\,, \tag{28.71}$$

where

$$K_p(z) = \frac{\pi}{2} \frac{I_{-p}(z) - I_p(z)}{\sin p z}\,, \tag{28.72}$$

p not an integer, $\mathrm{Re}(p) > 0$, $|\arg \zeta| < \pi$, and where $I_p\ (z)$ is defined by the series in (25.3). The proof may be left as an exercise to the reader. Mellin's theory gives then the reciprocal formula to (28.71)

$$\Gamma(s)\,\Gamma(s+p) = 2\int_0^\infty K_p(2\sqrt{\zeta})\,\zeta^{(p/2)+s-1}\,d\zeta, \quad \mathrm{Re}(s) > 0\,. \tag{28.73}$$

Equation (28.71) shows immediately that $K_p(z)$ is continuous in the variable p also at p integer. We define therefore $K_p(z)$ for integral values of p as the limit in (28.72) if p tends to a positive integer (see [76], p. 78).

(III) *Hecke's formula.* Let α, β be complex numbers and u, v positive real numbers. Then Euler's formula (23.6) yields immediately (by obvious substitutions)

$$\frac{\Gamma(s+\alpha)}{u^{s+\alpha}} = \int_0^\infty e^{-ux}\,x^{s+\alpha-1}\,dx\,, \qquad \frac{\Gamma(s+\beta)}{v^{s+\beta}} = \int_0^\infty e^{-vy}\,y^{s+\beta-1}\,dy$$

for such s which fulfill

$$\mathrm{Re}(s) > -\min(A, B) = -\mu \text{ with } A = \mathrm{Re}(\alpha), B = \mathrm{Re}(\beta)\,. \tag{28.74}$$

By multiplication we obtain (because of absolute convergence of the single integrals) the double integral:

$$\frac{\Gamma(s+\alpha)\,\Gamma(s+\beta)}{u^{s+\alpha} v^{s+\beta}} = \int_0^\infty \int_0^\infty e^{-ux-vy}\,(xy)^{s-1} x^\alpha y^\beta \,dx\,dy\,.$$

Here we introduce new variables

$$\zeta = xy, \quad \vartheta = \left(\frac{y}{x}\right)^{1/2},$$

or

$$x = \zeta^{1/2}\vartheta^{-1}, \quad y = \zeta^{1/2}\vartheta\,.$$

In view of the Jacobian

$$\frac{\partial(x,y)}{\partial(\zeta,\vartheta)} = \begin{vmatrix} \frac{1}{2}\zeta^{-1/2}\vartheta^{-1} & \frac{1}{2}\zeta^{-1/2}\vartheta \\ -\zeta^{1/2}\vartheta^{-2} & \zeta^{1/2} \end{vmatrix} = \vartheta^{-1}$$

we obtain

$$\begin{aligned} \frac{\Gamma(s+\alpha)\,\Gamma(s+\beta)}{u^{s+\alpha} v^{s+\beta}} &= \int_0^\infty \int_0^\infty \exp\left(-u\,\zeta^{1/2}\vartheta^{-1} - v\,\zeta^{1/2}\vartheta\right) \\ &\quad \times \zeta^{s-1+\frac{\alpha+\beta}{2}}\,\vartheta^{\beta-\alpha-1}\,d\zeta\,d\vartheta \\ &= \int_0^\infty \Psi(\zeta)\,\zeta^{s-1}\,d\zeta \qquad (28.81) \end{aligned}$$

with

$$\Psi(\zeta) = \zeta^{(\alpha+\beta)/2} \int_0^\infty \exp\left(-u\,\zeta^{1/2}\vartheta^{-1} - v\,\zeta^{1/2}\vartheta\right)\vartheta^{\beta-\alpha-1}\,d\vartheta\,. \qquad (28.82)$$

The application of Mellin's theory will give

$$\Psi(\zeta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s+\alpha)\,\Gamma(s+\beta)}{u^{s+\alpha} v^{s+\beta}}\,\zeta^{-s}\,ds\,, \qquad (28.83)$$

if we can establish an interval (a, b) and a positive angle ψ so that for all c in

$$a+\varepsilon \leqq c \leqq b-\varepsilon \qquad (28.84)$$

and all ζ in the angle

$$|\arg\zeta| \leqq \psi - \delta$$

we have

$$|\Psi(\zeta)| < K\,|\zeta|^{-c},$$

where K may depend on δ, ε as well as on the parameters α, β, u, v. Such an estimate will have to be based on the definition (28.82) of $\Psi(\zeta)$. As ψ we can take π and have then

$$\operatorname{Re}(\zeta^{1/2}) \geqq |\zeta|^{1/2} \sin\frac{\delta}{2} = \lambda .$$

Then (28.82) leads to

$$\begin{aligned}
|\Psi(\zeta)| &\leqq |\zeta^{(\alpha+\beta)/2}| \int_0^\infty e^{-u\lambda\vartheta^{-1}-v\lambda\vartheta}\,\vartheta^{B-A-1}\,d\vartheta \\
&< K_1 |\zeta|^{(A+B)/2} \int_0^1 e^{-u\lambda\vartheta^{-1}}\,\vartheta^{B-A-1}\,d\vartheta \\
&\quad + K_1 |\zeta|^{(A+B)/2} \int_1^\infty e^{-v\lambda\vartheta}\vartheta^{B-A-1}\,d\vartheta \\
&= K_1 |\zeta|^{(A+B)/2} (u\lambda)^{B-A} \int_{u\lambda}^\infty e^{-r} r^{A-B-1}\,dr \\
&\quad + K_1 |\zeta|^{(A+B)/2} (v\lambda)^{A-B} \int_{v\lambda}^\infty e^{-r} r^{B-A-1}\,dr = U + V ,
\end{aligned}$$

say. Now for $|\zeta| \geqq 1$ we have

$$U < K_2 |\zeta|^B e^{-(u\lambda)/2} \int_{u\lambda}^\infty e^{-r/2} r^{A-B-1}\,dr < K_3 |\zeta|^B e^{-k|\zeta|^{1/2}} < K_4 |\zeta|^{-c}$$

for c in any finite interval (28.84), where all constants may depend on u, A, B, a, b, δ, ε but not on ζ. For $0 < |\zeta| < 1$ we have

$$\begin{aligned}
U &< K_2 |\zeta|^B \left\{ \int_{u\sin\delta}^\infty e^{-r} r^{A-B-1}\,dr + \int_{u\lambda}^{u\sin\delta} r^{A-B-1}\,dr \right\} \\
&< K_5 |\zeta|^B + K_6 |\zeta|^B \int_{|\zeta|^{1/2}}^1 r^{A-B-1}\,dt ,
\end{aligned}$$

with $t = r\,|\zeta|^{1/2}/u\lambda$.

Now, as can be seen by direct computation,

$$|\zeta|^B \int_{\zeta^{1/2}}^1 r^{A-B-1}\,dr \leqq K_7 |\zeta|^{\mu-\epsilon}, \qquad 0 < |\zeta| < 1$$

in all three cases $A > B$, $A = B$, $A < B$. Therefore, with μ defined by (28.74),

$$|U| < K_8 |\zeta|^{\mu-\epsilon} ,$$

and a fortiori

$$|U| < K\,|\zeta|^{-c} \tag{28.85}$$

for any c which lies in the interval

$$-\mu + \varepsilon \leqq c \leqq b - \varepsilon\,,$$

and this is, with suitable K, now also true for $|\zeta| \geqq 1$ as we have seen. A similar discussion goes through for V, and thus (28.83) is proved for $|\arg\zeta| \leqq \pi - \delta$.

Hecke's formula is obtained as the special case $\zeta = 1$:

$$\int_0^\infty e^{-u\vartheta^{-1}-v\vartheta}\,\vartheta^{\beta-\alpha-1}\,d\vartheta = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s+\alpha)\,\Gamma(s+\beta)}{u^{s+\alpha}v^{s+\beta}}\,ds\,. \tag{28.9}$$

Hecke has made important applications of this formula in the theory of real quadratic number fields [21], p. 114. It can be greatly generalized to serve a similar purpose for algebraic number fields of higher degree [54], p. 352.

The special case $\alpha = \beta = 0$, $v = 1$ of (28.9) is found in Mellin's paper [40], p. 326. Also, it may be observed that in the case $\alpha = p$, $\beta = 0$, $u = \zeta$, $v = 1$ the right-hand members of (28.9) and (28.71) differ only by the factor ζ^{-p}. We obtain thus a new integral representation for the function K_p:

$$K_p(z) = \frac{1}{2}\left(\frac{z}{2}\right)^p \int_0^\infty e^{-\vartheta-(z^2/4\vartheta)}\,\frac{d\vartheta}{\vartheta^{p+1}}\,,$$

which because of $z^2/4 = \zeta = u$ is at first only proved for positive real z and can then by analytic continuation be extended to the domain $|\arg z| < \pi/2$.

Chapter 4

The Phragmén-Lindelöf Theorem

29. The main theorem

The maximum principle for analytic functions of a complex variable, which states that the absolute value of an analytic function attains its maximum on the boundary, is derived from the fact that the function cannot have a maximum of its absolute value in any interior point. It is thus only proved for compact regions. And indeed it is not valid for non-compact regions, as the following example shows. We consider $f(s) = e^{\cos s}$ in the strip S: $-\pi/2 \leqq \sigma \leqq \pi/2$. It is obviously regular in the strip and on its boundaries. On the boundaries we have $f(\pm\pi/2 + it) = e^{\mp\sin it} = e^{\mp i\sinh t}$ and thus $|f(\pm\pi/2 + it)| = 1$. However $f(it) = e^{\cosh t}$

$\geqq e^{(1/2)\exp|t|}$ which tends to ∞ as $|t| \to \infty$. This example is also instructive in so far as it shows the least order of growth that a function bounded on the boundary must have if it does not remain bounded in the interior of the strip.

The situation is covered by the famous theorem of Phragmén and Lindelöf ([58]):

Theorem. *Let $f(s)$ be regular and analytic in the strip $S(a, b): a \leqq \sigma \leqq b$. Let us assume $|f(s)| \leqq 1$ on the boundaries $\sigma = a$ and $\sigma = b$, and moreover $f(s) = O(e^{e^{k|t|}})$ for $t \to \infty$ with $0 < k < \pi/(b-a)$. Then $|f(s)| \leqq 1$ also in the interior of $S(a, b)$.*

Proof. In view of a simple linear transformation there is no loss of generality in taking $a = -\pi/2$, $b = \pi/2$, $0 < k < 1$. We put

$$k_1 = \frac{1+k}{2}, \qquad \cos\frac{k_1\pi}{2} = \varkappa > 0\,. \tag{29.1}$$

We have now, for $-\pi/2 \leqq \sigma \leqq \pi/2$,

$$\left|e^{-\cos k_1 s}\right| = \left|e^{-\cos k_1(\sigma+it)}\right| = e^{-\cos k_1\sigma \cosh k_1 t} \leqq e^{-\varkappa\cosh k_1 t} < e^{-(\varkappa/2)e^{k_1|t|}}\,. \tag{29.2}$$

Now let $s_0 = \sigma_0 + it_0$ be any interior point of $S(-\pi/2, \pi/2)$. We set, with an arbitrary $\varepsilon > 0$,

$$F_\varepsilon(s) = f(s)\,e^{-\varepsilon\cos k_1 s}\,. \tag{29.3}$$

We choose $T > |t_0|$ and consider $F_\varepsilon(s)$ in the rectangle $R_T: |\sigma| \leqq \pi/2$, $|t| \leqq T$, in the interior of which s_0 is situated. On the boundary $\sigma = \pm\pi/2$ we have because of the hypothesis about $f(s)$ and from (29.2) that $|F_\varepsilon(\pm\pi/2 + it)| < 1$. Furthermore we have on the horizontal boundaries of R

$$|F_\varepsilon(\sigma \pm iT)| < C e^{e^{k|T|} - (\varepsilon\varkappa/2)e^{k_1|T|}}\,.$$

Since according to (29.1) $0 < k < k_1$ we see that $F_\varepsilon(\sigma \pm iT) \to 0$ for $T \to \infty$. We can therefore enlarge T so much that

$$|F_\varepsilon(\sigma \pm iT)| < 1\,, \qquad -\pi/2 \leqq \sigma \leqq \pi/2\,.$$

Since we have now $|F_\varepsilon(s)| < 1$ on the boundary of the rectangle R, we have, according to the principle of the maximum, also $|F_\varepsilon(s)| < 1$ in the interior of R and especially

$$|F_\varepsilon(s_0)| < 1\,.$$

Now, after (29.3)

$$|f(s_0)| = |F_\varepsilon(s_0)| \cdot \left|e^{\varepsilon\cos k_1 s_0}\right| < e^{\varepsilon\cosh t_0}\,.$$

This inequality, being valid for any $\varepsilon > 0$, yields now with $\varepsilon \to 0$

$$|f(s_0)| \leqq 1 ,$$

which proves the theorem. $\square$

30. A theorem of the Phragmén-Lindelöf type for subharmonic functions

We do not need here the concept of a subharmonic function in its full generality[1]. It will suffice to deal only with subharmonic functions of class C''. Throughout this chapter Δ stands for the Laplacian operator $\Delta = D_\sigma^2 + D_t^2$. We shall make use of the following two propositions:

(1) For a function $u(\sigma, t)$ defined in the region D to be subharmonic of class C'' it is sufficient (and necessary) that in D

$$\Delta u \geqq 0 .$$

Every harmonic function is therefore also subharmonic.

(2) Let D be a region whose closure $\bar{D} = D + B$ is compact. If $h(\sigma, t)$ is harmonic in D and continuous in $\bar{D}$ and $u(\sigma, t)$ subharmonic in D and continuous in D, and if on the boundary B of D we have $u \leqq h$, then $u \leqq h$ also in the interior of D.

In particular, a subharmonic function u is $\leqq 0$ in the interior if it is $\leqq 0$ on the boundary. The property (2) is not necessarily valid in non-compact regions, not even for harmonic functions. The counter-example is related to that of § 29. The real part of an analytic function is a harmonic function. We take the strip $S(-\pi/2, \pi/2)$ and in it the harmonic function

$$H(\sigma, t) = \operatorname{Re} \cos(\sigma + it) = \cos\sigma \cosh t .$$

We have $H(\pm\pi/2, t) = 0$, whereas $H(0, t) = \cosh t \to +\infty$ as $|t| \to \infty$.

It will turn out that a subharmonic function vanishing on the boundary of $S(-\pi/2, \pi/2)$ must either be $\leqq 0$ in the interior or go to positive infinity at least with the growth $Ce^{|t|}$ as $|t| \to \infty$. The situation is described by the following

Theorem. *Let $u(\sigma, t)$ be subharmonic in the strip $S(a, b): a \leqq \sigma \leqq b$ and let $u(a, t) \leqq 0$, $u(b, t) \leqq 0$. If then moreover in the interior $u(\sigma, t) < Ce^{k|t|}$ with a certain $C > 0$, $0 < k < \pi/(b-a)$, then $u(\sigma, t) \leqq 0$ in the interior of $S(a, b)$.*

Proof. Although the proof runs parallel to the preceding one it may be worthwhile to write out its details. Using a linear substitution we can

[1] For a detailed account of the theory see, e.g., [58].

assume without loss of generality $a = -\pi/2$, $b = \pi/2$, $0 < k < 1$. We use the notations of (29.1) and consider then, with $0 < k_1 < 1$ and $k_1 > k$,

$$H_1(\sigma, t) = \operatorname{Re} \cos k_1 s = \cos k_1 \sigma \cosh k_1 t \geqq \varkappa \cosh k_1 t \geqq \frac{\varkappa}{2} e^{k_1|t|} .$$

Let $s_0 = \sigma_0 + it_0$ be an interior point of $S(-\pi/2, \pi/2)$, kept fixed in the proof. Then we choose $T > |t_0|$, so that s_0 lies in the interior of the rectangle $R_T: |\sigma| \leqq \pi/2$, $|t| \leqq T$. We put, with an arbitrary $\varepsilon > 0$,

$$U_\varepsilon(\sigma, t) = u(\sigma, t) - \varepsilon H_1(\sigma, t),$$

which is again subharmonic. We have on the vertical boundaries of R_T

$$U_\varepsilon(\pm\pi/2, t) < 0$$

and on the horizontal boundaries

$$U_\varepsilon(\sigma, \pm T) = u(\sigma, \pm T) - \varepsilon H_1(\sigma, \pm T) < C e^{kT} - \frac{\varepsilon}{2} e^{k_1 T} < 0$$

if only T is chosen sufficiently large (depending, of course, on ε). Since $U_\varepsilon(\sigma, t)$ is negative on the boundaries of R_T we have also

$$U_\varepsilon(\sigma_0, t_0) < 0$$

for the interior point ε_0. Then

$$u(\sigma_0, t_0) = U_\varepsilon(\sigma_0, t_0) + \varepsilon H_1(\sigma_0, t_0) < \varepsilon H_1(\sigma_0, t_0) .$$

Since this inequality is valid for every $\varepsilon > 0$ we conclude

$$u(\sigma_0, t_0) \leqq 0 ,$$

which finishes the proof. []

31. The Poisson integral formula for a strip

Let $f(\vartheta)$ be real and continuous in $0 \leqq \vartheta \leqq 2\pi$ with the exception of at most finitely many points and integrable in the interval. Then in the circle $|z| < 1$ the Poisson formula

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\vartheta) \frac{1 - r^2}{1 - 2r\cos(\varphi - \vartheta) + r^2} d\vartheta$$

gives the harmonic function $h(z)$ with the prescribed boundary values $f(\varphi)$ as $r \to 1$ in $z = re^{i\varphi}$. The formula can be rewritten as

$$h(z) = \frac{1}{2\pi} \int_{|\zeta|=1} g(\zeta) \frac{1 - z\bar{z}}{1 - (z\bar{\zeta} + \bar{z}\zeta) + z\bar{z}} \frac{d\zeta}{i\zeta}$$

with $g(\zeta) = g(e^{i\vartheta}) = f(\vartheta)$, and then

$$h(z) = \frac{1}{2\pi} \int\limits_{|\zeta|=1} g(\zeta) \operatorname{Re} \frac{1+z\bar{\zeta}}{1-z\bar{\zeta}} \frac{d\zeta}{i\zeta}.$$

A conformal mapping carries a harmonic function into a harmonic function of the image region. We put

$$z = \frac{1}{i} \frac{e^{is}-1}{e^{is}+1}, \qquad \zeta = \frac{1}{i} \frac{e^{iw}-1}{e^{iw}+1}.$$

The circle $|z| \leqq 1$ goes over into the strip $S(-\pi/2, \pi/2): -\pi/2 \leqq \sigma \leqq \pi/2$, the points $z = \pm i$ corresponding to $s = \pm i\infty$; the variable of integration $w = u + iv$ runs on $u = \pm\pi/2$, v from $-\infty$ to $+\infty$. A straightforward computation gives

$$\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}} \frac{d\zeta}{i\zeta} = \frac{e^{i(s-\bar{w})}+1}{e^{is}+e^{-i\bar{w}}} \frac{dv}{\sin w} \tag{31.1}$$

with $dw = i dv$.

For $w = \pi/2 + iv$ we obtain

$$\operatorname{Re} \frac{1+z\bar{\zeta}}{1-z\bar{\zeta}} \frac{d\zeta}{i\zeta} = \operatorname{Re} \frac{-i e^{is-v}+1}{e^{is}-i e^{-v}} \frac{dv}{\cosh v} \tag{31.21}$$

and after some simplification

$$\operatorname{Re} \frac{1+z\bar{\zeta}}{1-z\bar{\zeta}} \frac{d\zeta}{i\zeta} = \frac{\cos\sigma}{\cosh(v-t)-\sin\sigma} dv,$$

and similarly for $w = -\pi/2 + iv$

$$\operatorname{Re} \frac{1+z\bar{\zeta}}{1-z\bar{\zeta}} \frac{d\zeta}{i\zeta} = \operatorname{Re} \frac{i e^{is-v}+1}{e^{is}+i e^{-v}} \frac{dv}{\cosh v} = \frac{-\cos\sigma}{\cosh(v-t)+\sin\sigma} dv. \tag{31.22}$$

If we now put

$$h\left(\frac{1}{i} \frac{e^{is}-1}{e^{is}+1}\right) = H(\sigma, t),$$

$$g\left(\frac{1}{i} \frac{ie^{-v}-1}{ie^{-v}+1}\right) = A(v), \quad g\left(\frac{1}{i} \frac{-ie^{-v}-1}{-ie^{-v}+1}\right) = B(v),$$

then

$$H(\sigma, t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} A(v) \frac{\cos\sigma}{\cosh(v-t)+\sin\sigma} dv$$

$$+ \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} B(v) \frac{\cos\sigma}{\cosh(v-t)-\sin\sigma} dv \tag{31.3}$$

for $-\pi/2 < \sigma < \pi/2$, where $H(\sigma, t)$ has the boundary values $A(t)$, $B(t)$ as $\sigma \to -\pi/2$ or $\pi/2$ respectively. For the validity of the formula it suffices that $A(v)$, $B(v)$ have at most finitely many discontinuities, are integrable and of order $O(e^{k|v|})$, $|v| \to \infty$, $0 < k < 1$.

In order to adjust (31.1) for a strip $S(a, b)$ we define

$$\omega(\sigma, t) = \frac{\sin \pi\sigma}{\cosh \pi t - \cos \pi\sigma}. \tag{31.4}$$

Then

$$\begin{aligned} H^*(\sigma, t) = &\frac{1}{2(b-a)} \int_{-\infty}^{\infty} A(v)\, \omega\left(\frac{\sigma - a}{b-a}, \frac{v-t}{b-a}\right) dv \\ &+ \frac{1}{2(b-a)} \int_{-\infty}^{\infty} B(v)\, \omega\left(\frac{b-\sigma}{b-a}, \frac{v-t}{b-a}\right) dv \end{aligned} \tag{31.5}$$

is harmonic in $S(a, b)$. This formula goes over into (31.3) for $a = -\pi/2$, $b = \pi/2$ [1]. We need here $A(v)$, $B(v) = O(e^{k|v|})$, $0 < k < \pi/(b-a)$.

32. A lemma

We make use of (31.5) to prove the following

Lemma. *Let a, b, Q, γ, δ be real numbers,*

$$-Q \leqq a < b, \quad \gamma \leqq \delta.$$

Then there exists in the strip $S(a, b): a \leqq \mathrm{Re}(s) = \sigma \leqq b$ an analytic function $\varphi(s)$ with the boundary properties

$$\begin{aligned} |\varphi(s)| &= |Q + s|^{\gamma} \quad \text{on} \quad \mathrm{Re}(s) = a, \\ |\varphi(s)| &= |Q + s|^{\delta} \quad \text{on} \quad \mathrm{Re}(s) = b \end{aligned} \tag{32.1}$$

and so that in $S(a, b)$

$$|\varphi(s)| \geqq |Q + s|^{l(\sigma)}, \tag{32.21}$$

with

$$l(\sigma) = \gamma \frac{b - \sigma}{b - a} + \delta \frac{\sigma - a}{b - a}, \tag{32.22}$$

and so that

$$\varphi(s) = O(|t|^c), \quad |t| \to \infty \tag{32.3}$$

[1] Formula (31.3) appears in [17], p. 550. See also [80].

with a certain $c > 0$. In the case $a + Q = 0$ there may occur a singularity of $\varphi(s)$ at $s = a$ on the boundary.

Proof. If $\gamma = \delta$ the function $\varphi(s) = (Q + s)^{\gamma} = (Q + s)^{\delta}$ furnishes the solution, where here and in the following $(Q + s)^{\gamma}$ is understood as the principal branch of the function.

We can thus assume $\gamma < \delta$. We construct now by means of (31.5) the harmonic function $H^*(\sigma, t)$ in the strip $S(a, b)$ with

$$A^*(t) = \gamma \log |Q + a + it| = \frac{1}{2}\gamma \log((Q + a)^2 + t^2),$$

$$B^*(t) = \delta \log |Q + b + it| = \frac{1}{2}\delta \log((Q + b)^2 + t^2). \tag{32.4}$$

The conditions for $A(t)$, $B(t)$ are amply fulfilled. The logarithmic singularity which arises for $Q + a = 0$ in $A^*(t)$ at $t = 0$ does not disturb the integrability.

Using the abbreviation

$$\omega_1(\sigma, t; v) = \omega\left(\frac{\sigma - a}{b - a}, \frac{v - t}{b - a}\right), \quad \omega_2(\sigma, t; v) = \omega\left(\frac{b - \sigma}{b - a}, \frac{v - t}{b - a}\right)$$

we have the estimate

$$|H^*(\sigma, t)| < C \int_{-\infty}^{\infty} |\log((Q + a)^2 + v^2)|\, \omega_1(\sigma, t; v)\, dv$$

$$+ C \int_{-\infty}^{\infty} |\log((Q + b)^2 + v^2)|\, \omega_2(\sigma, t; v)\, dv = I_1 + I_2 .$$

Now

$$I_1 = -C \int_{(Q+a)^2+v^2 \leq 1} \log((Q + a)^2 + v^2)\, \omega_1 dv + C \int_{(Q+a)^2+v^2>1} \log((Q + a)^2 + v^2)\, \omega_1 dv$$

$$= -C \int_{-\infty}^{\infty} \log((Q + a)^2 + v^2)\, \omega_1 dv + 2C \int_{(Q+a)^2+v^2>1} \log((Q + a)^2 + v^2)\, \omega_1 dv$$

$$< -C \int_{-\infty}^{\infty} \log((Q + a)^2 + v^2)\, \omega_1 dv$$

$$+ 2C \int_{-\infty}^{\infty} \log((Q + 1 + a)^2 + v^2)\, \omega_1\, dv .$$

Similarly

$$I_2 < -C \int_{-\infty}^{\infty} \log((Q + b)^2 + v^2)\, \omega_2\, dv$$

$$+ 2C \int_{-\infty}^{\infty} \log((Q + 1 + b)^2 + v^2)\, \omega_2\, dv .$$

But $\log((Q + a)^2 + t^2)$, $\log((Q + b)^2 + t^2)$ are the boundary values of the harmonic function $\log|Q + s|^2$, whereas $\log((Q + 1 + a)^2 + t^2)$, $\log((Q + 1 + b)^2 + t^2)$ are the boundary values of $\log|Q + 1 + s|^2$. The integrals therefore represent these harmonic functions, so that we obtain

$$\begin{aligned} |H^*(\sigma, t)| &< 2C(b - a)\{2\log|Q + 1 + s|^2 - \log|Q + s|^2\} \\ &= O(\log|t|), \qquad |t| \to \infty . \end{aligned} \tag{32.5}$$

We construct now an analytic function $G^*(s)$, regular in the interior of $S(a, b)$, of which $H^*(\sigma, t)$ is the real part. This we could do by finding the imaginary part of $G^*(s)$ as a conjugate harmonic function to $H^*(\sigma, t)$. But in our case it is simpler to drop in (31.21) and (31.22) the symbol Re in order to obtain an analytic function in s. Instead of (31.3) we have then

$$\begin{aligned} G(s) = {} & \frac{1}{2\pi} \int_{-\infty}^{\infty} A(v) \frac{i e^{is-v} + 1}{e^{is} + i e^{-v}} \frac{dv}{\cosh v} \\ & + \frac{1}{2\pi} \int_{-\infty}^{\infty} B(v) \frac{-i e^{is-v} + 1}{e^{is} - i e^{-v}} \frac{dv}{\cosh v}, \end{aligned}$$

which we adjust to the strip $a \leqq \sigma \leqq b$ by putting

$$G^*(s) = G\left(\frac{\pi}{b-a}\left(s - \frac{a+b}{2}\right)\right),$$

$$A(v) = A^*\left(\frac{b-a}{\pi} v\right), \qquad B(v) = B^*\left(\frac{b-a}{\pi} v\right).$$

Then we have

$$\operatorname{Re} G^*(s) = H^*(\sigma, t) .$$

We now put

$$\varphi(s) = \exp G^*(s) . \tag{32.6}$$

This $\varphi(s)$ satisfies (32.1), (32.3) because $|\varphi(s)| = \exp H^*(\sigma, t)$ in view of (32.4), (32.5).

We now consider the function

$$U(\sigma, t) = (\lambda\sigma + \mu)\log|Q + s| = \frac{1}{2}(\lambda\sigma + \mu)\log((Q + \sigma)^2 + t^2). \tag{32.7}$$

Since $U(\sigma, t)$ is of the form $f \cdot g$, and thus

$$\varDelta U = f\varDelta g + 2(f_\sigma g_\sigma + f_t g_t) + g\varDelta f,$$

where Δf, Δg, f_t vanish in our case, we have

$$\Delta U(\sigma, t) = 2\lambda \frac{Q+\sigma}{(Q+\sigma)^2 + t^2} .$$

We choose now λ, μ so that $U(\sigma, t)$ satisfies the boundary conditions (32.4) for $H^*(\sigma, t)$, which means

$$\lambda a + \mu = \gamma , \quad \lambda b + \mu = \delta$$

or

$$\lambda = \frac{\delta - \gamma}{b - a} , \quad \mu = \frac{\gamma b - \delta a}{b - a} .$$

The conditions for a, b, γ, δ show that $\lambda > 0$ (we consider only $\delta > \gamma$) and $Q + \sigma \geqq 0$, so that

$$\Delta U(\sigma, t) \geqq 0 .$$

This shows that $U(\sigma, t)$ is a *subharmonic function* in $S(a, b)$. Therefore $D(\sigma, t) = U(\sigma, t) - H^*(\sigma, t)$ is also subharmonic, with boundary values 0. Since $D(\sigma, t) = O(\log |t|)$, $|t| \to \infty$ we can apply the theorem of § 30 and conclude $D(\sigma, t) \leqq 0$ or

$$U(\sigma, t) \leqq H^*(\sigma, t) .$$

Looking back at the definitions (32.6) and (32.7) we find

$$|Q + s|^{\lambda\sigma + \mu} \leqq |\varphi(s)| ,$$

which is (32.21). This proves the lemma. $\square$

33. A generalization of the Phragmén-Lindelöf theorem

Theorem. *Let $f(s)$ be regular analytic in the strip $S(a, b)$, $-Q \leqq a < b$, and satisfy there with a certain $C > 0$, and $0 < b < \pi/(b-a)$*

$$|f(s)| < C e^{e^{k|t|}} . \tag{33.1}$$

Assume moreover

$$|f(s)| \leqq \begin{cases} A\, |Q + s|^{\alpha} & \text{for } \mathrm{Re}(s) = a , \\ B\, |Q + s|^{\beta} & \text{for } \mathrm{Re}(s) = b \end{cases} \tag{33.2}$$

with

$$\alpha \geqq \beta . \tag{33.3}$$

Then throughout the strip $S(a, b)$

$$|f(s)| \leqq A^{(b-\sigma)/(b-a)} B^{(\sigma-a)/(b-a)} |Q + s|^{\alpha(b-\sigma)/(b-a) + \beta(\sigma-a)/(b-a)}. \quad (33.4)$$

For $a + Q = 0$, $f(s)$ *may have a singularity at* $s = a$.

Proof. We consider the function $\varphi(s)$ of the lemma of § 32 with

$$\gamma = -\alpha, \quad \delta = -\beta \quad (33.5)$$

(which satisfy the condition $\gamma \leqq \delta$). Then we form the regular function

$$F(s) = f(s)\,\varphi(s) E^{-1} e^{-\nu s}, \quad (33.6)$$

where E and ν are constants determined by

$$A = E e^{\nu a}, \quad B = E e^{\nu b}. \quad (33.7)$$

Wee see that

$$|F(s)| < C_1 e^{e^{k_1|t|}}$$

with $0 < k < k_1 < \pi/(b - a)$. The conditions (32.1), (33.2), (33.5), (33.7) give then

$$|F(a + it)| \leqq 1, \quad |F(b + it)| \leqq 1,$$

from which we infer after the Phragmén-Lindelöf theorem of § 29

$$|F(s)| \leqq 1$$

throughout $S(a, b)$. This means

$$|f(s)| \leqq E e^{\nu\sigma} |\varphi(s)|^{-1},$$

which because of (32.21) proves (33.4) and thus the theorem. ▯

Our theorem is not only a generalization of the Phragmén-Lindelöf theorem, but yields also Hadamard's three-circle theorem as a special case.

Suppose indeed that $g(z)$ is regular in the annulus $0 < r \leqq |z| \leqq R$, and that

$$|g(r e^{i\vartheta})| \leqq A, \quad |g(R e^{i\vartheta})| \leqq B. \quad (33.8)$$

We put $z = e^s$ and set $r = e^a$, $R = e^b$. Then

$$g(z) = g(e^s) = h(s)$$

is regular in the strip $S(a, b)$ and has there the period $2\pi i$, thus fulfilling certainly (33.1). Our theorem yields

$$|h(s)| \leqq A^{(b-\sigma)/(b-a)} B^{(\sigma-a)/(b-a)}$$

and thus as corollary

Hadamard's three-circle theorem. *Let $g(z)$ be regular in the annulus $0 < r \leqq |z| \leqq R$ and fulfill the boundary conditions* (33.8). *Then we have in the annulus*

$$|g(z)| \leqq A^{\lambda_1(z)} B^{\lambda_2(z)}$$

with

$$\lambda_1(z) = \frac{\log \frac{R}{|z|}}{\log \frac{R}{r}}, \qquad \lambda_2(z) = \frac{\log \frac{|z|}{r}}{\log \frac{R}{r}}.$$

34. Applications to the Γ-function

The generalized Phragmén-Lindelöf theorem has wide applications in analytic number theory. We give here the following simple consequence.

Theorem A. *Let $0 \leqq c \leqq 1$. Then, for $R(s) \geqq (1 - c)/2$ we have*

$$\left| \frac{\Gamma(s+c)}{\Gamma(s)} \right| \leqq |s|^c. \tag{34.1}$$

Remark. It is easily seen from (21.1) that

$$\left| \frac{\Gamma(s+c)}{\Gamma(s)\, s^c} \right|$$

is bounded in the right half-plane. But it is not obvious that this bound is 1. This bound evidently cannot be improved for $c = 1$.

Proof. We choose $c > 0$, and s_0 with $\sigma_0 + c > 0$. Then

$$\left| \frac{\Gamma(s_0 + c)}{\Gamma(s_0)} \right| = \left| \frac{\Gamma(c + 2\sigma_0 - s_0)}{\Gamma(s_0)} \right|$$

since $\overline{\Gamma(s)} = \Gamma(\bar{s})$. We consider now

$$f(s) = \frac{\Gamma(c + 2\sigma_0 - s)}{\Gamma(s)}$$

and choose the strip $S(a, b)$ in the following way. Firstly because $|\Gamma(s)| = |\Gamma(\bar{s})|$,

$$|f(a + it)| = \left| \frac{\Gamma(c + 2\sigma_0 - a + it)}{\Gamma(a + it)} \right|.$$

Here we demand $c + 2\sigma_0 - a = a + 1$, which means

$$a = \frac{c-1}{2} + \sigma_0,$$

so that

$$|f(a+it)| = \left|\frac{\Gamma(a+1+it)}{\Gamma(a+it)}\right| = |a+it|.$$

Secondly we want

$$|f(b+it)| = \left|\frac{\Gamma(c+2\sigma_0-b+it)}{\Gamma(b+it)}\right| = 1,$$

which demands $c + 2\sigma_0 - b = b$ or

$$b = \frac{c}{2} + \sigma_0 .$$

Our theorem can now be applied with $Q = 0, \alpha = 1, \beta = 0, A = B = 1$. We need only $0 \leqq a + Q = a$, or

$$\frac{1-c}{2} \leqq \sigma_0 .$$

The theorem of § 33 yields now

$$|f(s)| \leqq |s|^{2(c/2+\sigma_0-\sigma)}$$

for $(c-1)/2 + \sigma_0 \leqq \sigma \leqq c/2 + \sigma_0$. The special value $s = s_0$ is therefore admissible, and we obtain

$$|f(s_0)| = \left|\frac{\Gamma(c+s_0)}{\Gamma(s_0)}\right| \leqq |s_0|^c,$$

which proves the theorem. ▯

The restriction $\sigma \geqq (1-c)/2$ may be stronger than is needed for (34.1). However, some restriction stronger than $\sigma \geqq 0$ is necessary as the examples

$$\left|\frac{\Gamma\left(\frac{1}{3}+\frac{i}{\sqrt{3}}\right)}{\Gamma\left(\frac{i}{\sqrt{3}}\right)\left(\frac{i}{\sqrt{3}}\right)^{1/3}}\right| = 1.0017\ldots, \quad \left|\frac{\Gamma\left(\frac{1}{4}+\frac{i}{\sqrt{3}}\right)}{\Gamma\left(\frac{i}{\sqrt{3}}\right)\left(\frac{i}{\sqrt{3}}\right)^{1/4}}\right| = 1.0142\ldots$$

show, which I obtain from a table of $\log \Gamma(s)$ (see [39]).

A second example for the theorem in § 33 is given by the following

Theorem B. *For* $-1/2 \leqq \sigma \leqq 1/2$ *we have*

$$\left|\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right| \leqq \left|\frac{1+s}{2}\right|^{1/2-\sigma} \tag{34.2}$$

(see [50], where a more general theorem is proved as Lemma 1 on p. 197).

Remark. Equality holds here on the boundaries $\sigma = \pm 1/2$.

Proof. We consider

$$f(s) = \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$$

in the strip $S(-1/2, 1/2)$ in which it is regular. We have here again because of $|\Gamma(s)| = |\Gamma(\bar{s})|$,

$$\left|f\left(-\frac{1}{2}+it\right)\right| = \left|\frac{\Gamma\left(\frac{3}{4}-\frac{it}{2}\right)}{\Gamma\left(-\frac{1}{4}+\frac{it}{2}\right)}\right| = \left|-\frac{1}{4}+\frac{it}{2}\right|$$

and

$$\left|f\left(\frac{1}{2}+it\right)\right| = \left|\frac{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}\right| = 1.$$

In other words, we have

$$|f(s)| = \begin{cases} \left|\frac{1+s}{2}\right| & \text{for } \operatorname{Re}(s) = -1/2, \\ 1 & \text{for } \operatorname{Re}(s) = 1/2. \end{cases}$$

Our theorem in § 33 with $a = -1/2$, $b = 1/2$, $\alpha = 1$, $\beta = 0$, $Q = 1$, $A = 1/2$, $B = 1$ therefore yields (34.2). ▯

Chapter 5

The Poisson Sum Formula and Applications

35. The theorem

Let $f(x)$ be continuous for all x. We shall presently subject this function to some further restrictions. The problem is to evaluate

$$S = \sum_{n=-\infty}^{\infty} f(n).$$

Poisson's basic idea is to introduce a variable and to consider

$$S(u) = \sum_{n=-\infty}^{\infty} f(n+u) \tag{35.1}$$

under suitable conditions of convergence. It is clear that

$$S(u+1) = S(u)\,, \tag{35.11}$$

since the effect of an increase of u by 1 on the right-hand member of (35.1) is the same as a change of n into $n+1$, which leaves the set of integers $n\,(-\infty < n < \infty)$ unchanged.

After this preliminary orientation we formulate Poisson's theorem in a form which will be more than general enough to cover all our applications in the sequel.

Theorem A. *Let $f(x)$ be twice continuously differentiable in $-\infty < x < +\infty$ and let*

$$\int_{-\infty}^{\infty} f(x)\,dx \quad \textit{and} \quad \int_{-\infty}^{\infty} |f''(x)|\,dx \tag{35.2}$$

exist. Then

$$S = \sum_{n=-\infty}^{\infty} f(n) \tag{35.3}$$

converges, and we have

$$S = \sum_{k=-\infty}^{\infty} A_k \tag{35.41}$$

with

$$A_k = \int_{-\infty}^{\infty} f(u)\,e^{-2\pi iku}\,du\,. \tag{35.42}$$

Proof. The first step will be to establish

$$\lim_{x\to\pm\infty} f'(x) = 0\,, \quad \lim_{x\to\pm\infty} f(x) = 0\,. \tag{35.5}$$

Indeed we have from (35.2)

$$\lim_{X\to\infty} \int_a^X f''(x)\,dx = \int_a^\infty f''(x)\,dx$$

or

$$\lim_{X\to\infty} f'(X) = \int_a^\infty f''(x)\,dx + f'(a) = L\,,$$

say. That here $L = 0$ follows now from

$$\lim_{X\to\infty} \int_X^{X+1} f(x)\, dx = 0\,, \tag{35.51}$$

since this implies also

$$0 = \lim_{X\to\infty} \int_X^{X+1} \bigl(f(x+1) - f(x)\bigr)\, dx = \lim_{X\to\infty} \int_X^{X+1} f'(\xi)\, dx$$
$$= \lim_{X\to\infty} f'(\xi)^* = L,$$

where ξ and ξ^* are certain mean values between X and $X + 2$. The same argument works, of course, for $X \to -\infty$.

Now again (35.51) shows that

$$f(\xi) \to 0$$

for certain mean values $\xi = \xi_X$, $X \leqq \xi_X \leqq X + 1$. But $0 \leqq \xi_X - X \leqq 1$ and

$$f(X) = f(\xi_X) + f'(\xi^*)\,(X - \xi_X)\,,$$

and from $f(\xi_X) \to 0$, $f'(\xi^*) \to 0$ follows $f(X) \to 0$ as $X \to \infty$.

We consider now first the sum

$$T = \sum_{n=-\infty}^{\infty} f'(n) = \int_{-\infty}^{\infty} f'(x)\, dx + \sum_{n=-\infty}^{\infty} \left\{ f'(n) - \int_{n-1}^{n} f'(x)\, dx \right\}. \tag{35.6}$$

Here

$$\int_{-\infty}^{\infty} f'(x)\, dx = [f(x)]_{-\infty}^{\infty} = 0\,,$$

and we obtain

$$T = \sum_{n=-\infty}^{\infty} \int_{n-1}^{n} \bigl(f'(n) - f'(x)\bigr)\, dx$$
$$= \sum_{n=-\infty}^{\infty} \int_{n-1}^{n} dx \left(\int_x^n f''(y)\, dy \right)$$
$$= \sum_{n=-\infty}^{\infty} \iint_{n-1 \leqq x \leqq y < n} f''(y)\, dx\, dy$$
$$= \sum_{n=-\infty}^{\infty} \int_{n-1}^{n} (y - n + 1)\, f''(y)\, dy\,,$$

which is majorized by

$$\sum_{n=-\infty}^{\infty} \int_{n-1}^{n} |f''(y)|\, dy = \int_{-\infty}^{\infty} |f''(y)|\, dy\,,$$

so that the convergence of T in (35.6) is proved.

We treat similarly

$$S = \sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} f(x)\,dx + S_1$$

with

$$S_1 = \sum_{n=-\infty}^{\infty} \left\{f(n) - \int_{n-1}^{n} f(x)\,dx\right\}.$$

Because of (35.2) the convergence of S is equivalent to that of S_1. We find

$$S_1 = \sum_{n=-\infty}^{\infty} \int_{n-1}^{n} (y - n + 1) f'(y)\,dy$$

$$= \sum_{n=-\infty}^{\infty} \left\{\frac{1}{2} f'(n) - \frac{1}{2} \int_{n-1}^{n} (y - n + 1)^2 f''(y)\,dy\right\}$$

$$= \frac{1}{2} T - \frac{1}{2} \sum_{n=-\infty}^{\infty} \int_{n-1}^{n} (y - n + 1)^2 f''(y)\,dy\,,$$

where the last sum is again majorized by

$$\int_{-\infty}^{\infty} |f''(y)|\,dy\,.$$

Thus the convergence of S is also established.

In all these arguments the sums S and T could have been replaced respectively by

$$\Sigma f(n + u)\,, \quad \Sigma f'(n + u)$$

and we would at the same time have shown that these sums are uniformly convergent in any finite interval $a \leqq u \leqq b$ since this is true for the integrals

$$\int_{-\infty}^{\infty} f(x + u)\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} |f''(x + u)|\,dx\,.$$

In particular therefore the function

$$S(u) = \sum_{n=-\infty}^{\infty} f(n + u)$$

is continuously differentiable, $S'(u)$ being obtained by termwise differentiation. Also

$$S'(u) = \sum_{-\infty}^{\infty} f'(n + u)$$

is continuous.

Under these circumstances and because of (35.11), $S(u)$ can be expanded in a Fourier series

$$S(u) = \sum_{k=-\infty}^{\infty} A_k e^{2\pi i k u}$$

with

$$A_k = \int_0^1 S(v)\, e^{-2\pi i k v}\, dv = \int_0^1 \sum_{n=-\infty}^{\infty} f(n+v)\, e^{-2\pi i k v}\, dv\,.$$

Since in view of uniform convergence integration and summation can be interchanged we obtain

$$A_k = \sum_{n=-\infty}^{\infty} \int_0^1 f(n+v)\, e^{-2\pi i k v}\, dv = \sum_{n=-\infty}^{\infty} \int_n^{n+1} f(w) e^{-2\pi i k w}\, dw$$

$$= \int_{-\infty}^{\infty} f(w)\, e^{-2\pi i k w}\, dw\,,$$

which shows (35.42). Putting now $u = 0$ we have completed the proof of the theorem. ▯

A slight generalization of Theorem A, in which $f(x)$ can have discontinuities, is the following

Theorem B [42][1]. *Let $f(x)$ be of bounded variation for $|x| \leqq M$ and let $f(x)$ be twice differentiable for $|x| \geqq M$. Let moreover*

$$\int_{-\infty}^{\infty} f(x)\, dx \quad \textit{and} \quad \int_M^{\infty} |f''(x)|\, dx\,, \quad \int_{-\infty}^{-M} |f''(x)|\, dx \tag{35.7}$$

exist. If we put $f^*(x) = \frac{1}{2}\left(f(x+0) + f(x-0)\right)$ *then*

$$S = \sum_{-\infty}^{\infty} f^*(n) \tag{35.8}$$

is convergent and

$$S = \sum_{k=-\infty}^{\infty} A_k \tag{35.9}$$

with A_k given in (35.42).

---

[1] The conditions $f(x) \to 0$, $f'(x) \to 0$ as $|x| \to \infty$ are not needed since they are implied by (35.2) as our proof shows.

Indeed, the previous proof still shows the uniform convergence of

$$\sum_{|n|>M+2} f'(n+u) \quad \text{and} \quad S(u) = \sum_{-\infty}^{\infty} f(n+u)$$

for $-1/2 \leqq u \leqq 3/2$. Then we can write

$$S(u) = \sum_{|n|\leqq M+2} f(n+u) + \sum_{|n|>M+2} f(n+u)\,,$$

which shows that $S(u)$ is of bounded variation, since the first sum of the right side is of bounded variation by hypothesis and the second one is also because it has a continuous and therefore bounded derivative in $[0, 1]$. Since $S(n+1) = S(n)$ as before, we have the Fourier expansion

$$S^*(u) = \sum_{k=-\infty}^{\infty} A_k e^{2\pi i k u}$$

with $S^*(u) = \frac{1}{2}\,(S(u+0) + S(u-0))$, and for $u = 0$ in particular (35.8).

36. Application: A transformation formula for a ϑ-function

We record here briefly the well-known application of Poisson's sum formula to the transformation of $\vartheta_3(v\,|\,\tau)$.

Let us consider the sum

$$\psi(t, \alpha) = \sum_{n=-\infty}^{\infty} e^{-\pi(n+\alpha)^2 t}$$

with real or complex α and with $t > 0$. The convergence of the sum is evident. Also the conditions (35.2) for $f(x) = e^{-\pi(x+\alpha)^2 t}$ are fulfilled. We have thus

$$\psi(t, \alpha) = \sum_{k=-\infty}^{\infty} A_k$$

with

$$A_k = \int_{-\infty}^{\infty} e^{-\pi(x+\alpha)^2 t} e^{-2\pi i k x}\, dx = e^{2\pi i k\alpha - (\pi k^2/t)} \int_{-\infty}^{\infty} e^{-\pi t(x+\alpha+(ik/t))^2}\, dx$$

$$= e^{2\pi i k\alpha - (\pi k^2/t)} \int e^{-\pi t z^2}\, dz\,,$$

where the integral is extended along a line parallel to the real axis with

$$\operatorname{Im}(z) = \operatorname{Im}(\alpha) + \frac{k}{t}\,.$$

However, the integral can be shifted back to the real axis as simple estimates of the integrand at $\pm\infty$ show. Then we have

$$A_k = e^{2\pi i k\alpha - (\pi k^2/t)} \int_{-\infty}^{\infty} e^{-\pi t x^2}\, dx = \frac{K}{\sqrt{\pi t}}\, e^{2\pi i k\alpha - (\pi k^2/t)}, \tag{36.1}$$

where

$$K = \int_{-\infty}^{\infty} e^{-u^2}\, du .$$

This integral is well known. Its value will however appear presently as a result of our reasoning. Putting things together we have obtained

$$\psi(t, \alpha) = \sum_{n=-\infty}^{\infty} e^{-\pi(n+\alpha)^2 t} = \frac{K}{\sqrt{\pi t}} \sum_{k=-\infty}^{\infty} e^{-(\pi k^2/t) + 2\pi i k\alpha} .$$

The special case $t = 1$, $\alpha = 0$ is

$$\psi(1, 0) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2} = \frac{K}{\sqrt{\pi}} \sum_{k=-\infty}^{\infty} e^{-\pi k^2},$$

which shows

$$K = \sqrt{\pi},$$

and thus we have in general

$$\psi(t, \alpha) = \sum_{n=-\infty}^{\infty} e^{-\pi(n+\alpha)^2 t} = \frac{1}{\sqrt{t}} \sum_{k=-\infty}^{\infty} e^{-(\pi k^2/t) + 2\pi i k\alpha}$$

$$= \frac{e^{-\pi\alpha^2 t}}{\sqrt{t}}\, \psi\left(\frac{1}{t}, -i\alpha t\right). \tag{36.2}$$

So far we have taken t as positive. The series in (36.2) however remain convergent for

$$\operatorname{Re}(t) > 0 \tag{36.3}$$

and represent analytic functions of t. The equation holds therefore under the wider condition (36.3) with the understanding that the principal branch of $\sqrt{t}$ has to be taken, since for $t > 0$ we had $\sqrt{t} > 0$, as the meaning of the substitution of the variable of integration in (36.1) shows.

It is customary to define

$$\vartheta_3(v \mid \tau) = \sum_{-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n v} \tag{36.4}$$

for v, τ complex and in particular

$$\operatorname{Im}(\tau) > 0 . \tag{36.5}$$

Through the substitutions

$$-t = i\tau, \qquad \alpha = \frac{v}{\tau}$$

the conditions (36.3) and (36.5) are in agreement and we obtain from (36.2)

$$e^{\pi i v^2/\tau}\,\vartheta_3(v\,|\,\tau) = \frac{1}{\sqrt{-i\tau}}\,\vartheta_3\left(\frac{v}{\tau}\,\middle|\,-\frac{1}{\tau}\right), \tag{36.6}$$

a transformation formula for $\vartheta_3(v\,|\,\tau)$.

We make further an important observation. Let us put $\operatorname{Re}(1/t) = W$, $\operatorname{Im}(\alpha) = A$; then

$$\left|\sum_{-\infty}^{\infty} e^{-\pi n^2/t + 2\pi i n\alpha} - 1\right| \leqq 2\sum_{n=1}^{\infty} e^{-\pi n^2 W + 2\pi n|A|} < 2\sum_{n=1}^{\infty} e^{-\pi n(W-2|A|)}$$

$$= 2\,\frac{e^{-\pi(W-2|A|)}}{1 - e^{-\pi(W-2|A|)}} \to 0$$

as $W \to \infty$. From (36.2) we derive therefore

$$\lim_{t\to 0} \sqrt{t}\,\psi(t, \alpha) = \lim_{t\to 0} \sqrt{t} \sum_{n=-\infty}^{\infty} e^{-\pi(n+\alpha)^2 t} = 1 \tag{36.7}$$

provided $\operatorname{Re}(1/t) \to \infty$ with $t \to 0$.

37. Lipschitz's formula

Another application of Poisson's formula leads to the

Theorem [36]. *Assume*

$$\operatorname{Re}(z) > 0, \qquad 0 < \alpha \leqq 1, \qquad \operatorname{Re}(s) > 1.$$

Then

$$\frac{(2\pi)^s}{\Gamma(s)} \sum_{m=0}^{\infty} (m+\alpha)^{s-1} e^{-2\pi z(m+\alpha)} = \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n\alpha}}{(z+ni)^s}. \tag{37.1}$$

Under the condition $0 < \alpha < 1$ *this formula holds in the larger half-plane* $\operatorname{Re}(s) > 0$.

Proof. We take first, for our proof, $\operatorname{Re}(s) > 1$. Then

$$S = \sum_{-\infty}^{\infty} \frac{e^{2\pi i n\alpha}}{(z+ni)^s} \tag{37.2}$$

is absolutely convergent. Here the expression $(z+ni)^s$ denotes the principal value, i.e.

$$(z+ni)^s = |z+ni|^s e^{is\arg(z+ni)}$$

with

$$|\arg(z + ni)| < \frac{\pi}{2}.$$

The conditions of Theorem A in § 35 are satisfied for

$$f(x) = \frac{e^{2\pi i x\alpha}}{(z + xi)^s}.$$

We have therefore

$$S = \sum_{m=-\infty}^{\infty} A_m$$

with

$$A_m = \int_{-\infty}^{\infty} \frac{e^{2\pi i v\alpha}}{(z + vi)^s} e^{-2\pi i m v}\, dv.$$

We put $z + vi = w$:

$$A_m = \frac{1}{i} \int_{z-i\infty}^{z+i\infty} \frac{e^{2\pi(w-z)(\alpha-m)}}{w^s} dw = \frac{1}{i} e^{2\pi z(m-\alpha)} \int_{z-i\infty}^{z+i\infty} \frac{e^{-2\pi w(m-\alpha)}}{w^s} dw$$

or, replacing m by $-m$:

$$A_{-m} = \frac{1}{i} e^{-2\pi z(m+\alpha)} \int_{z-i\infty}^{z+i\infty} \frac{e^{2\pi w(m+\alpha)}}{w^s} dw\,. \tag{37.3}$$

We take first $m + \alpha \leqq 0$. In the integral, for $\Omega > 0$

$$\int_{z-i\Omega}^{z+i\Omega} \frac{e^{2\pi w(m+\alpha)}}{w^s} dw$$

we replace the path of integration by a half-circle into the right half-plane. As $\Omega \to \infty$ the integral then goes to zero so that

$$A_{-m} = 0 \quad \text{for } m < 0\,.$$

In the case $m + \infty > 0$ we treat

$$I = \frac{1}{i} \int_{z-i\infty}^{z+i\infty} \frac{e^{2\pi w(m+\alpha)}}{w^s} dw = \frac{(2\pi)^s (m+\alpha)^{s-1}}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{\zeta}}{\zeta^s} d\zeta\,,$$

with $a = 2\pi(m + \alpha)\operatorname{Re}(z)$. For $\operatorname{Re}(s) > 1$ the path of the integral can be deformed into a loop

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{\zeta}}{\zeta^s} d\zeta = \frac{1}{2\pi i} \int_{-\infty}^{(0^+)} \frac{e^{\zeta}}{\zeta^s} d\zeta = \frac{1}{\Gamma(s)}\,,$$

the Hankel integral (24.3). We have thus

$$A_{-m} = e^{-2\pi z(m+\alpha)} \frac{(2\pi)^s (m+\alpha)^{s-1}}{\Gamma(s)}$$

for $m \geqq 0$ and thus have proved (37.1), firstly only for $0 \leqq \alpha \leqq 1$, $\operatorname{Re}(s) > 1$.

For $0 < \alpha < 1$, however, the formula remains true for $\operatorname{Re}(s) > 0$. The left member is then analytic in the whole s-plane. The right-hand sum converges for $\operatorname{Re}(s) > 0$ and represents there an analytic function. For this we have to prove only that

$$S_{N,P} = \sum_{n=N+1}^{N+P} \frac{e^{2\pi i n\alpha}}{(z+ni)^s} \to 0$$

with $N \to \infty$, $P > 0$. Indeed

$$S_{N,P} = \sum_{n=N+1}^{N+P} \frac{s_n - s_{n-1}}{(z+ni)^s}$$

with

$$s_n = \sum_{\nu=0}^{n} e^{2\pi i\nu\alpha} = \frac{e^{2\pi i(n+1)\alpha} - 1}{e^{2\pi i\alpha} - 1}, \qquad |s_n| \leqq \frac{2}{|e^{\pi i\alpha} - e^{-\pi i\alpha}|} = \frac{1}{\sin \pi\alpha}.$$

Thus

$$S_{N,P} = \sum_{n=N+1}^{N+P} s_n \left(\frac{1}{(z+ni)^s} - \frac{1}{(z+(n+1)i)^s} \right) - \frac{s_N}{(z+(N+1)i)^s} + \frac{s_{N+P}}{(z+(N+P+1)i)^s}.$$

Now

$$\left| \frac{1}{(z+ni)^s} - \frac{1}{(z+(n+1)i)^s} \right| = \left| \frac{1}{s} \int_n^{n+1} \frac{dv}{(z+vi)^{s+1}} \right|$$
$$< e^{|t|\pi} |s|^{-1} \frac{1}{(x^2 + (n - |y|)^2)^{(\sigma+1)/2}}$$

with $z = x + iy$, $n > |y|$. Therefore, for $N > |y|$,

$$|S_{N,P}| < C \frac{1}{|s|} \sum_{n=N+1}^{\infty} \frac{1}{(x^2 + (n-|y|)^2)^{(\sigma+1)/2}},$$

which indeed goes to 0 as $N \to \infty$, and uniformly for all s with $\sigma \geqq \delta > 0$. By analytic continuation (37.1) holds therefore also for $\operatorname{Re}(s) > 0$ in case $0 < \alpha < 1$. ▯

II. Special Functions

Chapter 6

The Riemann ζ-function

38. Definition of the ζ-function and its analytic continuation

In his famous memoir "Über die Anzahl der Primzahlen unterhalb einer gegebenen Grösse," which has become the basis of the whole analytic theory of the distribution of primes, Riemann defined the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} . \tag{38.1}$$

The series is absolutely convergent for

$$\sigma = \mathrm{Re}(s) > 1 . \tag{38.2}$$

The series for $\zeta(s)$ is majorized by $\sum n^{-\sigma}$, which shows that the convergence of (38.1) is uniform in every half-plane $\mathrm{Re}(s) \geqq 1 + \delta > 1$. Thus $\zeta(s)$ is regular in the half-plane (38.2).

If we write

$$\zeta(s) = 1 + \lim_{N\to\infty} \sum_{n=2}^{N} n^{-s}$$

we have a sum to which we can apply the Euler-MacLaurin sum formula (7.2), (7.21) with

$$f(x) = x^{-s}, \quad f^{(k)}(x) = (-1)^k s(s+1) \cdots (s+k-1)\, x^{-s-k} .$$

We obtain thus, treating the term with $B_1 = -1/2$ separately,

$$\begin{aligned}\zeta(s) - 1 = \lim_{N\to\infty} \Bigg\{ & \int_1^N x^{-s}\, dx + \frac{1}{2}(N^{-s} - 1) \\ & - \sum_{r=2}^{q} \frac{B_r}{r!} s(s+1) \cdots (s+r-2)\,(N^{-s-r+1} - 1) \\ & - \frac{1}{q!} s(s+1) \cdots (s+q-1) \int_1^N B_q(x - [x])\, x^{-s-q}\, dx \Bigg\} .\end{aligned}$$

Under condition (38.2) the limit can be performed termwise with the result

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{r=2}^{q} \frac{B_r}{r!} s(s+1)\cdots(s+r-2)$$

$$- \frac{1}{q!} s(s+1)\cdots(s+q-1) \int_1^\infty B_q(x-[x])\, x^{-s-q}\, dx\,. \quad (38.3)$$

The integral converges not only in the half plane $\mathrm{Re}(s) > 1$ but also for

$$\mathrm{Re}(s) > 1 - q\,, \quad (38.4)$$

and uniformly in each compact region in that half-plane. Thus (38.3) furnishes an analytic continuation of $\zeta(s)$ into the half-plane (38.4), showing as only singularity the pole at $s = 1$. But q is an arbitrary natural number. Therefore $\zeta(s)$ can be continued by means of (38.3) to any point in the s-plane, and $\zeta(s)$ emerges as a meromorphic function with the simple pole at $s = 1$.

Some special values of $\zeta(s)$ in the newly gained territory are of interest. For $s = 0$ we obtain

$$\zeta(0) = -1/2\,. \quad (38.5)$$

Let us further investigate $s = -k$ with a positive integer k. If we choose $q = k + 1$, then the integral will have a vanishing factor and we retain only

$$\zeta(-k) = \frac{-1}{k+1} + \frac{1}{2} - \sum_{r=2}^{k+1} \frac{B_r}{r!} k(k-1)\cdots(k-r+2)\,.$$

If we multiply both sides by $k + 1$ and remember $B_1 = -1/2$ we have

$$(k+1)\,\zeta(-k) = -\sum_{r=0}^{k+1} \binom{k+1}{r} B_r = -B_{k+1}(1) = -B_{k+1}\,, \quad (38.6)$$

in view of (4.2) and (4.3). Thus $\zeta(s)$ vanishes at the even negative numbers, and

$$\zeta(-2m+1) = -\frac{1}{2m} B_{2m}\,. \quad (38.6)$$

The zeros of $\zeta(s)$ at $s = -2m$ are often called the "trivial zeros" as they are so easily found.

39. Two special integrals

For a further discussion of (38.3) we need the

Lemma. *For* $0 < \operatorname{Re}(s) < 1$ *we have*

$$\Gamma(s) \cos \frac{\pi}{2} s = \int_0^\infty y^{s-1} \cos y \, dy \,, \tag{39.1}$$

$$\Gamma(s) \sin \frac{\pi}{2} s = \int_0^\infty y^{s-1} \sin y \, dy \,. \tag{39.2}$$

Equation (39.2) is valid also for $-1 < \operatorname{Re}(s) \leqq 0$.

Proof. We integrate $z^{s-1} e^{-z}$ around a square Q of vertices 0, M, $M + iM$, iM, of which the vertex 0, however, is cut off by a small quarter circle of radius ε. Thus we have

$$\begin{aligned} 0 &= \int_Q z^{s-1} e^{-z} \, dz = \int_\varepsilon^M x^{s-1} e^{-x} \, dx + \int_M^{M+iM} z^{s-1} e^{-z} \, dz + \int_{M+iM}^{iM} z^{s-1} e^{-z} \, dz \\ &\quad + \int_M^\varepsilon (iy)^{s-1} e^{-iy} \, i \, dy + \int_{\pi/2}^0 \varepsilon^{s-1} e^{i(s-1)\varphi} e^{-\varepsilon e^{i\varphi}} \, d(\varepsilon e^{i\varphi}) \\ &= I_1 + I_2 + I_3 + I_4 + I_5 \,, \end{aligned} \tag{39.3}$$

say. Everywhere the principal value of z^{s-1} is taken so that in particular in I_4

$$(iy)^{s-1} = y^{s-1} e^{(i\pi/2)(s-1)}.$$

Now we have, with $0 < \sigma < 1$,

$$\begin{aligned} |I_2 + I_3| &\leqq e^{-M} e^{|t| (\pi/4)} \int_0^M (M^2 + y^2)^{(\sigma-1)/2} \, dy \\ &\quad + e^{|t| (\pi/2)} \int_0^M (x^2 + M^2)^{(\sigma-1)/2} e^{-x} \, dx \\ &< e^{-M} e^{|t| (\pi/4)} M \cdot M^{\sigma-1} + e^{|t| (\pi/2)} M^{\sigma-1} \end{aligned}$$

so that $I_2 + I_3 \to 0$ as $M \to \infty$.

Furthermore

$$|I_5| < \varepsilon^\sigma \frac{\pi}{2} e^{(\pi/2)|t|}$$

and thus $I_5 \to 0$ as $\varepsilon \to 0$.

Thus (39.3) shows

$$\lim_{\substack{\varepsilon \to \infty \\ M \to \infty}} (I_1 + I_4) = 0$$

or, explicitly[1],

$$\int_0^\infty x^{s-1} e^{-x} \, dx = i e^{(\pi i/2)(s-1)} \int_0^\infty y^{s-1} e^{-iy} \, dy$$

or

$$e^{-\pi i s/2} \Gamma(s) = \int_0^\infty y^{s-1} e^{-iy} \, dy \, .$$

If we carry out the same reasoning but using the square of vertices 0, M, $M - iM$, $-iM$, the square again being indented at 0 by a quarter-circle of radius ε, we obtain

$$e^{\pi i s/2} \Gamma(s) = \int_0^\infty y^{s-1} e^{iy} \, dy \, .$$

Addition and subtraction of the last two equations then will furnish (39.1) and (39.2). In (39.2), moreover, both members of the equation are analytic also for $-1 < \mathrm{Re}(s) \leqq 0$, since $\Gamma(s) \sin(\pi/2) s$ has no pole at $s = 0$ and since the improper integral converges at the lower end. By virtue of analytic continuation we have thus (39.2) valid in the strip $-1 < \mathrm{Re}(s) < 1$. ▯

40. Riemann's functional equation for $\zeta(s)$

We return now to (38.3) and put there $q = 3$, obtaining

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \frac{B_2}{2} s$$
$$- \frac{1}{6} s(s+1)(s+2) \int_1^\infty B_3(x - [x]) \, x^{-s-3} \, dx \, , \tag{40.1}$$

valid for

$$\mathrm{Re}(s) > -2 \, . \tag{40.2}$$

Now

$$\int_0^1 B_3(x) \, x^{-s-3} \, dx$$

exists for $\sigma < -1$, since $B_3(x) = x(x-1)(x-1/2)$ has a factor x. Considering also (40.2) we assume now

$$-2 < \mathrm{Re}(s) < -1 \, . \tag{40.3}$$

Repeated integration by parts now yields (in view of (2.6))

$$\frac{1}{6} s(s+1)(s+2) \int_0^1 B_3(x) \, x^{-s-3} \, dx = -\frac{B_2}{2} s - \frac{1}{2} - \frac{1}{s-1}$$

[1] Mellin's theory does not quite cover this formula. It would permit only the rotation of the path of integration through an angle ψ, $|\psi| < \pi/2$.

so that (40.1) can be replaced by

$$\zeta(s) = -\frac{1}{6}s(s+1)(s+2)\int_0^\infty B_3(x-[x])\,x^{-s-3}\,dx$$

in the strip (40.3).

We take now (8.61) into account and have

$$\zeta(s) = -2s(s+1)(s+2)\int_0^\infty \sum_{n=1}^\infty \frac{\sin 2\pi n x}{(2\pi n)^3}\,x^{-s-3}\,dx$$

$$= -2s(s+1)(s+2)\sum_{n=1}^\infty \frac{1}{(2\pi n)^3}\int_0^\infty \frac{\sin 2\pi n x}{x^{s+3}}\,dx. \tag{40.4}$$

In order to justify interchange of summation and integration we put

$$S_\varepsilon = \sum_{n=1}^\infty \frac{1}{(2\pi n)^3}\int_0^\varepsilon \frac{\sin 2\pi n x}{x^{s+3}}\,dx, \qquad S_\omega = \sum_{n=1}^\infty \frac{1}{(2\pi n)^3}\int_\omega^\infty \frac{\sin 2\pi n x}{x^{s+3}}\,dx.$$

We have

$$\left|\int_0^\varepsilon \frac{\sin 2\pi n x}{x^{s+3}}\,dx\right| < \int_0^\varepsilon \frac{2\pi n x}{x^{\sigma+3}}\,dx = 2\pi n\,\frac{\varepsilon^{-\sigma-1}}{-\sigma-1}$$

and thus

$$|S_\varepsilon| < \frac{\varepsilon^{-\sigma-1}}{-\sigma-1}\sum_1^\infty \frac{1}{(2\pi n)^2} \to 0 \quad \text{as } \varepsilon \to 0,$$

since $-\sigma - 1 > 0$. Also

$$\left|\int_\omega^\infty \frac{\sin 2\pi n x}{x^{s+3}}\,dx\right| < \int_\omega^\infty \frac{dx}{x^{\sigma+3}} = \frac{\omega^{-\sigma-2}}{\sigma+2}$$

and thus

$$|S_\omega| < \frac{\omega^{-\sigma-2}}{\sigma+2}\sum_1^\infty \frac{1}{(2\pi n)^3} \to 0 \quad \text{as } \omega \to \infty,$$

since $-\sigma - 2 < 0$.

Then we can write

$$\int_0^\infty \sum_{n=1}^\infty \frac{\sin 2\pi n x}{(2\pi n)^3}\,x^{-s-3}\,dx = \lim_{\substack{\varepsilon\to 0\\ \omega\to\infty}} \int_\varepsilon^\omega \sum_{n=1}^\infty$$

$$= \lim_{\substack{\varepsilon\to 0\\ \omega\to\infty}} \sum_{n=1}^\infty \frac{1}{(2\pi n)^3}\int_\varepsilon^\omega \frac{\sin 2\pi n x}{x^{s+3}}\,dx$$

since the convergence of the sum is uniform for $\varepsilon \leqq x \leqq \omega$. Continuing:

$$= \lim_{\substack{\varepsilon \to 0 \\ \omega \to \infty}} \left\{ \sum_{n=1}^{\infty} \frac{1}{(2\pi n)^3} \int_0^{\infty} \frac{\sin 2\pi n x}{x^{s+3}} \, dx - S_\varepsilon - S_\omega \right\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(2\pi n)^3} \int_0^{\infty} \frac{\sin 2\pi n x}{x^{s+3}} \, dx ,$$

which settles the validity of (40.4). If we rewrite (40.4) as

$$\zeta(s) = -2s(s+1)(s+2) \sum_{n=1}^{\infty} \frac{1}{(2\pi n)^{1-s}} \int_0^{\infty} y^{-s-3} \sin y \, dy$$

we can apply (39.2) since $-1 < \operatorname{Re}(-s-2) < 0$ and obtain

$$\zeta(s) = -2s(s+1)(s+2)\, \Gamma(-s-2) \sin \frac{\pi}{2}(-s-2) \sum_{n=1}^{\infty} \frac{1}{(2\pi n)^{1-s}}$$

and finally

$$\zeta(s) = 2^s \pi^{s-1}\, \Gamma(1-s) \sin \frac{\pi s}{2}\, \zeta(1-s) . \tag{40.5}$$

This is Riemann's functional equation for $\zeta(s)$. It is proved here at the outset only under the restriction (40.3), but because both sides of the functional equation show meromorphic functions it is valid in the whole s-plane[1].

The functional equation can, after Riemann, be put into a more illuminating form if we make use of Legendre's duplication formula

$$\Gamma(s)\, \Gamma(s+1/2) = 2\pi^{1/2} 2^{-2s}\, \Gamma(2s) ,$$

which is the special case $k = 2$ of (19.3). If we replace here $2s$ by $1-s$ we get

$$\Gamma(1-s)\pi^{1/2} 2^s = \Gamma\left(\frac{1-s}{2}\right) \Gamma\left(1 - \frac{s}{2}\right) = \frac{\pi}{\sin \frac{\pi s}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} ,$$

the latter after (19.2). Then (40.5) goes over into

$$\pi^{-s/2}\, \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{(s-1)/2}\, \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) . \tag{40.6}$$

[1] A similar proof was given by Schnee [65]. In Schnee's proofs the first Bernoulli polynomial is used which causes more difficulties in the interchange of summation and integration.

In other words, the function

$$\Phi(s) = \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{40.71}$$

fulfills the functional relation

$$\Phi(s) = \Phi(1-s)\,. \tag{40.72}$$

The role of the "trivial zeros" of $\zeta(s)$ is evident in (40.6). For $\operatorname{Re}(s) < 0$ we have $\operatorname{Re}(1-s) > 1$, so that the right-hand member of the equation is regular. However, $\Gamma(s/2)$ has poles for $s = -2, -4, -6, \ldots$ which are just neutralized by the zeros of $\zeta(s)$ there. Only for $s = 0$ we have a pole of first order on both sides of (40.6) since $\zeta(0) \neq 0$.

Exercise. Prove formula (38.6) by means of (40.5).

41. Another proof for the functional equation of $\zeta(s)$

The Lipschitz formula (37.1) can be used for another method to continue $\zeta(s)$ over the whole s-plane and to derive at the same time the functional equation. Let us consider, for $\operatorname{Re}(z) > 0$, $\operatorname{Re}(s) > 1$

$$\varphi(s, z) = \sum_{-\infty}^{\infty} \frac{1}{(z+ni)^s} = \frac{(2\pi)^s}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-1} e^{-2\pi n z}\,,$$

after (37.1). The right-hand member shows that $\varphi(s, z)$ is an entire function in s for each z in $\operatorname{Re}(z) > 0$. Therefore also for $z > 0$

$$Z(s, z) = \varphi(s, z) - z^{-s}$$

is an entire function of s, where in $z^{-s} = \exp(-s \log z)$ we take $\log z$ real. For $\operatorname{Re}(s) > 1$ this function can be represented as

$$Z(s, z) = \sum_{n=1}^{\infty}\left(\frac{1}{(z+ni)^s} + \frac{1}{(z-ni)^s}\right) \tag{41.1}$$

and for all s as

$$Z(s, z) = -z^{-s} + \frac{(2\pi)^s}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-1} e^{-2\pi n z}\,. \tag{41.2}$$

Now (41.1) shows that $Z(s, z)$ is also analytic in z for $|z| < 1$, since none of the denominators shows a logarithmic branch-point in that circle. Therefore the Taylor expansion

$$Z(s, z) = Z(s, 0) + \frac{z}{1!}\left(\frac{\partial Z(s, z)}{\partial z}\right)_{z=0} + \frac{z^2}{2!}\left(\frac{\partial^2 Z(s, z)}{\partial z^2}\right)_{z=0} + \cdots \tag{41.3}$$

holds for $|z| < 1$. Now (41.1) and (41.2) both show that

$$\frac{\partial Z(s, z)}{\partial z} = -sZ(s+1, z),$$

so that (41.3) can be written as

$$Z(s, z) = Z(s, 0) - \frac{zs}{1!} Z(s+1, 0) + \frac{z^2 s(s+1)}{2!} Z(s+2, 0) \ldots .$$

Now (41.1) shows, since we always have taken the principal branch of the logarithm, for $\operatorname{Re}(s) > 1$

$$Z(s, 0) = \sum_1^\infty \frac{1}{n^s} (e^{-\pi i s/2} + e^{\pi i s/2}) = 2\zeta(s) \cos \frac{\pi s}{2}.$$

We obtain therefore

$$Z(s, z) = 2\zeta(s) \cos \frac{\pi s}{2} - 2\frac{zs}{1!} \zeta(s+1) \cos \frac{\pi(s+1)}{2} + 2\frac{z^2 s(s+1)}{2!} \zeta(s+2) \cos \frac{\pi(s+2)}{2} - + \cdots . \tag{41.4}$$

For $0 < z < 1$ both equations (41.2) and (41.4) are valid, the first one for all s, the second one for $\operatorname{Re}(s) > 1$. Since in (41.4), $Z(s, z)$ is holomorphic for all s and since

$$\zeta(s+1) \cos \frac{\pi(s+1)}{2}, \quad \zeta(s+2) \cos \frac{\pi(s+2)}{2}, \ldots \tag{41.5}$$

are defined by the ζ-series for $\operatorname{Re}(s) > 0$ we see that $\zeta(s) \cos \pi s/2$ in (41.4) is also regular for $\operatorname{Re}(s) > 0$. But then the functions (41.5) are regular for $\operatorname{Re}(s) > -1$, which makes $\zeta(s) \cos \pi s/2$ also regular for $\operatorname{Re}(s) > -1$. This in turn shows that the functions (41.5) are regular for $\operatorname{Re}(s) > -2$ and so on.

We have therefore proved that

$$\zeta(s) \cos \frac{\pi s}{2}$$

can be continued through the whole s-plane as an entire function. The convergence of (41.4) is maintained for all s in this process and is uniform for $|s| \leqq R$, since the functions $\zeta(s+k) \cos \pi(s+k)/2$, $k = 0, 1, 2, \ldots$ remain bounded in a compact region. It would, by the way, suffice to choose a fixed value of z, say $z = 1/2$ for this whole argument.

Now let $\operatorname{Re}(s) < 0$, $0 < z < 1$. From (41.4) we deduce

$$\lim_{z \to 0} Z(s, z) = Z(s, 0) = 2\zeta(s) \cos \frac{\pi s}{2}$$

and from (41.2) by means of Abel's limit theorem

$$\lim_{z\to 0} Z(s,z) = \frac{(2\pi)^s}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-1} .$$

The last two equations together yield

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\,\Gamma(s)\cos\frac{\pi s}{2}\,\zeta(s) ,$$

which is equivalent to (40.5). The functional equation is thus proved again first for $\mathrm{Re}(s) < 0$, and then by analytic continuation for all s.

42. Connection between the ζ-function and a ϑ-function

There are some more proofs of Riemann's functional equation known (see e.g. [74]). Riemann himself provided two in his memoir, and a third one was edited by Siegel from Riemann's manuscripts [69]. Among Riemann's proofs the second one has so far been the most fruitful one, and we shall give it here, in view of its importance. We change the proof slightly, using Mellin's formula, which will save us some discussions of convergence. We have from (23.5)

$$e^{-\pi n^2 u} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(z)\,(\pi n^2 u)^{-z}\,dz, \qquad a > 0, \quad u > 0$$

and thus

$$\sum_{n=1}^{\infty} e^{-\pi n^2 u} = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \int_{a-i\infty}^{a+i\infty} \Gamma(z)\,(\pi n^2 u)^{-z}\,dz .$$

For $a > 1/2$ we can interchange summation and integration, since after (21.51) the integral and sum converge absolutely. With

$$\psi(u) = \sum_{-\infty}^{\infty} e^{-\pi n^2 u}$$

we obtain thus

$$\psi(u) - 1 = \frac{2}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(z)\pi^{-z}\,\zeta(2z)\,u^{-z}\,dz , \qquad a > 1/2 . \tag{42.1}$$

Then Mellin's theory (§ 27) permits us to write

$$2\pi^{-z}\,\Gamma(z)\,\zeta(2z) = \int_0^{\infty} (\psi(u) - 1)\,u^{z-1}\,du , \tag{42.2}$$

and with the substitution $z = s/2$, when $\operatorname{Re}(s) > 1$,

$$2\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty (\psi(u) - 1)\,u^{s/2-1}\,du = \int_0^1 + \int_1^\infty$$

$$= -\frac{2}{s} + \int_0^1 \psi(u)\,u^{s/2-1}\,du$$

$$+ \int_1^\infty (\psi(u) - 1)\,u^{s/2-1}\,du\,,$$

where $\int_0^1$ exists, since after (36.7)

$$\psi(u) = O(u^{-1/2}), \qquad u \to 0\,.$$

We use now (36.2) with $\alpha = 0$ and obtain

$$\int_0^1 \psi(u)\,u^{s/2-1}\,du = \int_0^1 \psi\left(\frac{1}{u}\right) u^{s/2-3/2}\,du$$

$$= \int_1^\infty (\psi(v) - 1)\,v^{-s/2-1/2}\,dv + \int_1^\infty v^{-s/2-1/2}\,dv$$

and thus

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = -\frac{1}{s} - \frac{1}{1-s} + \frac{1}{2}\int_1^\infty (\psi(u) - 1)$$
$$\times\ (u^{s/2-1} + u^{(1-s)/2-1})\,du\,. \tag{42.3}$$

Now the right-hand member is a meromorphic function of s with the only poles at $s = 0$, $s = 1$, so that this equation provides an analytic continuation of the left-hand member as a meromorphic function over the whole s-plane, and $\zeta(s)$ appears as meromorphic, only with the simple pole at $s = 1$. Moreover, the right-hand member of (42.3) is symmetric in s and $1 - s$. It implies therefore the Riemann functional equation

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \Phi(s) = \Phi(1 - s)\,. \tag{42.4}$$

This proof is based on

$$\psi(t) = \frac{1}{\sqrt{t}}\,\psi\left(\frac{1}{t}\right), \tag{42.5}$$

or, which is the same, Jacobi's transformation equation

$$\vartheta_3(0\,|\,\tau) = \frac{1}{\sqrt{-i\tau}}\,\vartheta_3\left(0\,\middle|\,-\frac{1}{\tau}\right).$$

Conversely, Riemann's functional equation (42.4) has (42.5) as a consequence, if moreover

$$\zeta(s) \sim \frac{1}{s-1}, \quad s \to 1, \tag{42.6}$$

and for $-1 \leqq \operatorname{Re}(s) \leqq 2$

$$\zeta(s) = O(|t|^3), \quad |t| \to \infty \tag{42.7}$$

are known. Both (42.6) and (42.7) follow from (40.1), (40.2) (but can also be inferred from (42.4) as we shall see later). It is easily seen that (42.3) gives the residues of $\Phi(s)$ at $s = 0$, $s = 1$ as -1, $+1$ respectively.

Now, replacing z by $s/2$ we can write (42.1) as

$$\psi(u) - 1 = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Phi(s) u^{-s/2}\, ds, \tag{42.8}$$

where we have chosen $a = 1$. Substituting here $1 - s$ for s we have

$$\begin{aligned} \psi(u) - 1 &= \frac{1}{2\pi i} \int_{-1-i\infty}^{-1+i\infty} \Phi(1-s) u^{(s-1)/2}\, ds \\ &= u^{-1/2} \frac{1}{2\pi i} \int_{-1-i\infty}^{-1+i\infty} \Phi(s) u^{s/2}\, ds \end{aligned}$$

in view of (42.4). The integral here will be similar to that in (42.8) if we shift the path of integration from the abscissa -1 to $+2$. This is permissible because of (42.7) and the behavior of $\Gamma(s)$ with $|t| \to \infty$. However, we pass over the two poles at $s = 0$ and $s = 1$, the residues of which we just mentioned. Thus we have

$$\psi(u) - 1 = u^{-1/2} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Phi(s) u^{s/2}\, ds + u^{-1/2} - 1 .$$

We realize that this integral arises from that in (42.8) by the substitution of u^{-1} for u. Therefore we have

$$\psi(u) = u^{-1/2}\left(\psi\left(-\frac{1}{u}\right) - 1\right) + u^{-1/2} = u^{-1/2}\psi\left(-\frac{1}{u}\right),$$

which is (42.5).

Riemann's functional equation and Jacobi's ϑ-transformation formula are therefore equivalent under the Mellin transformation.

43. Estimation of $\zeta(s)$ in a vertical strip

For $\sigma > 1$ we have evidently from the definition (38.1)

$$|\zeta(s)| \leqq \zeta(\sigma). \tag{43.1}$$

The functional equation in the form (40.5) yields

$$|\zeta(s)| \leqq 2^\sigma \pi^{\sigma-1} |\Gamma(1-s)| e^{\pi|t|/2} \zeta(1-\sigma)$$

for $1-\sigma > 1$ or $\sigma < 0$, and because of (21.51)

$$|\zeta(s)| = O(|t|^{1/2-\sigma}), \quad a \leqq \sigma \leqq -\delta < 0, \quad |t| \to \infty. \tag{43.2}$$

The results (43.1) and (43.2) leave out the strip $0 \leqq \sigma \leqq 1$. Now the estimate (42.7) covering that strip will suffice for our purposes. But much more can be obtained from the Phragmén-Lindelöf theorem.

The functional equation (40.6) furnishes

$$|\zeta(s)| = \pi^{\sigma-1/2} \left| \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \right| |\zeta(1-s)|$$

and Theorem B of § 34, for $-1/2 \leqq \sigma \leqq 1/2$,

$$|\zeta(s)| \leqq (2\pi)^{\sigma-1/2} |1+s|^{1/2-\sigma} |\zeta(1-s)|$$

and in particular for $0 < \eta \leqq 1/2$

$$|\zeta(-\eta+it)| \leqq (2\pi)^{-\eta-1/2} |1-\eta+it|^{1/2+\eta} \zeta(1+\eta).$$

This together with (43.1) gives our appraisal of $\zeta(s)$ on the lines $\sigma = -\eta$, $\sigma = 1+\eta$. But the Phragmén-Lindelöf theorem can be applied only to regular functions. We consider therefore $\zeta(s)(s-1)$ and have

$$\begin{aligned} |\zeta(1+\eta+it)(\eta+it)| &\leqq \zeta(1+\eta)|\zeta+it| \\ &< \zeta(1+\eta)|1+(1+\eta+it)| \end{aligned}$$

and

$$\begin{aligned} &|\zeta(-\eta+it)(-\eta-1+it)| \\ &\leqq (2\pi)^{-\eta-1/2}|1-\eta+it|^{1/2+\eta}|1+\eta+it|\,\zeta(1+\eta) \\ &\leqq 3(2\pi)^{-\eta-1/2}|1-\eta+it|^{3/2+\eta}\zeta(1+\eta) \end{aligned}$$

since

$$\left|\frac{1+\eta+it}{1-\eta+it}\right| \leqq \frac{1+\eta}{1-\eta} \leqq 3 \quad \text{for} \quad 0<\eta \leqq 1/2 .$$

Thus

$$f(s) = \frac{\zeta(s)\,(s-1)}{\zeta(1+\eta)}$$

has the boundary properties

$$|f(s)| \leqq \begin{cases} 3\,(2\pi)^{-\eta-1/2}\,|1+s|^{3/2+\eta} & \text{for } \operatorname{Re}(s) = -\eta\,, \\ |1+s| & \text{for } \operatorname{Re}(s) = 1+\eta\,. \end{cases}$$

For $a = -\eta$, $b = 1+\eta$, $Q = 1$, $\alpha = 3/2+\eta$, $\beta = 1$, $A = 3\,(2\pi)^{-\eta-1/2}$, $B = 1$ we have thus in the strip $S(-\eta, 1+\eta)$

$$\begin{aligned} |\zeta(s)| &\leqq \left(3\,(2\pi)^{-\eta-\frac{1}{2}}\right)^{\frac{1+\eta-\sigma}{1+2\eta}} \left|1+s\right|^{\left(\frac{3}{2}+\eta\right)\frac{1+\eta-\sigma}{1+2\eta}+\frac{\sigma+\eta}{1+2\eta}} \\ &\qquad \times |1-s|^{-1}\,\zeta(1+\eta) \\ &< 3\left|\frac{1+s}{1-s}\right| \left|\frac{1+s}{2\pi}\right|^{\frac{1+\eta-\sigma}{2}} \zeta(1+\eta)\,. \end{aligned} \tag{43.3}$$

If we define $\mu(\sigma)$ as the greatest lower bound of all γ for which

$$\zeta(\sigma+it) = O(|t|^{\gamma})$$

is correct, then (43.1), (43.3), (43.2) show that

(I) $\mu(\sigma) = 0$ for $\sigma \geqq 1$,

(II) $\mu(\sigma) \leqq 1/2 - \sigma/2$ for $0 \leqq \sigma \leqq 1$,

(III) $\mu(\sigma) = 1/2 - \sigma$ for $\sigma < 0$.

Theorem § 33 shows that $\mu(\sigma)$ must be a *convex function* of σ. Therefore (I) cannot be improved since $\zeta(\sigma+it) \to 1$ as $\sigma \to \infty$. Also (III) cannot be sharpened as (21.52) shows. However, (II) is not the best result known. An account of $\mu(\sigma)$, called the Lindelöf function, is given in [74]. We have $\mu(1/2) \leqq 1/4$, and it is known that $\mu(1/2) < 1/6$. Lindelöf has conjectured $\mu(1/2) = 0$. If this were so, the graph of $\mu(\sigma)$ would consist of 2 straight lines: $\mu(\sigma) = 0$ for $\sigma \geqq 1/2$, $\mu(\sigma) = 1/2 - \sigma$ for $\sigma \leqq 1/2$.

Chapter 7

About the Prime-number Theorem and the Zeros of the ζ-function

44. The Euler product

The fundamental theorem of number theory, proved essentially by Euclid, states that every natural number can be decomposed in only one manner into a product of powers of different primes. We can therefore write, for $\mathrm{Re}(s) > 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) = \prod_p \frac{1}{1 - p^{-s}}, \qquad (44.1)$$

where the product is extended over all prime numbers p. This product expansion was used already by Euler for special values of s. The absolute convergence of the product

$$\Pi\left(1 - \frac{1}{p^s}\right)$$

for $\mathrm{Re}(s) > 1$ is readily seen since

$$\sum_p \frac{1}{p^\sigma} < \sum \frac{1}{n^\sigma}, \qquad \sigma > 1 .$$

Since now in (44.1) all factors are different from 0 in the half-plane $\mathrm{Re}(s) > 1$, the absolute convergence entails

$$\zeta(s) \neq 0$$

for $\mathrm{Re}(s) > 1$.

Riemann's functional equation (40.6) shows therefore that in the half-plane $\mathrm{Re}(s) < 0$ the function $\zeta(s)$ can at most have zeros at the poles of $\Gamma(s/2)$, i.e. at 0 and at the negative even integers. Indeed we found in § 38 the "trivial zeros" $-2, -4, -6, \ldots$ of $\zeta(s)$ and observed also $\zeta(0) = -1/2$.

All other zeros of $\zeta(s)$ (and we shall show that such zeros exist) must therefore lie in the "critical strip" $0 \leqq \mathrm{Re}(s) \leqq 1$.

The functional equation shows immediately that with s also $1 - s$ must be a zero in the critical strip, in other words zeros must pairwise be symmetrical to the point $s = 1/2$. Since moreover the ζ-function is real on the real axis it has conjugate complex values in conjugate complex points and thus its zeros must also be symmetrical with respect to the real axis.

For a zero ϱ on the line $\mathrm{Re}(s) = 1/2$ these two symmetrics lead to the same further zero $\bar{\varrho} = 1 - \varrho$. If, however, there should exist in the critical strip a zero of real part different from $1/2$, this zero would appear together with the further zeros $\bar{\varrho}$, $1 - \varrho$, $1 - \bar{\varrho}$, which together with ϱ form the vertices of a rectangle. The location of the non-trivial zeros of $\zeta(s)$ is still an unsolved problem. Riemann conjectured that they all lie on the line $\operatorname{Re}(s) = 1/2$, and all actual computations[1] so far have indeed found zeros only on that line.

45. The borders of the critical strip are free of zeros of $\zeta(s)$

We shall discuss later the role that the location of the non-trivial zeros plays in the theory of the distribution of the prime numbers. Our first step in this direction is to show that $\zeta(s) \neq 0$ for $\operatorname{Re}(s) = 1$, which as we have just seen, implies also that no zeros lie on $\operatorname{Re}(s) = 0$.

We begin with the remark that

$$3 + 4\cos t + \cos 2t \geqq 0 \tag{45.1}$$

for all real t. Indeed

$$3 + 4\cos t + \cos 2t = 2 + 4\cos t + 2\cos^2 t = 2(1 + \cos t)^2 .$$

Now we have, for $\sigma > 1$,

$$\log|\zeta(\sigma + it)| = \operatorname{Re}\log\zeta(\sigma + it) = \operatorname{Re}\sum_{n=2}^{\infty} c_n n^{-\sigma - it} ,$$

where the c_n are non-negative coefficients which in consequence of (44.1) turn out to be

$$c_n = \begin{cases} 1/m & \text{if } n = p^m , \\ 0 & \text{otherwise} . \end{cases}$$

Using now

$$\log|\zeta(\sigma + it)| = \sum_{n=2}^{\infty} c_n n^{-\sigma}\cos(t\log n)$$

we obtain

$$\log|\zeta^3(\sigma)\,\zeta^4(\sigma + it)\,\zeta(\sigma + 2it)|$$

$$= \sum_{n=2}^{\infty} c_n n^{-\sigma}(3 + 4\cos(t\log n) + \cos(2t\log n)) \geqq 0$$

by virtue of (45.1) and $c_n \geqq 0$. We have thus

$$|\zeta^3(\sigma)\,\zeta^4(\sigma + it)\,\zeta(\sigma + 2it)| \geqq 1 \tag{45.2}$$

[1] See [20] for the first 1500 complex zeros and [34], p. 407 for the first 250000 zeros.

and consequently

$$((\sigma - 1)\,\zeta(\sigma)^3)\left|\frac{\zeta(\sigma + it)}{\sigma - 1}\right|^4 |\zeta(\sigma + 2it)| \geqq \frac{1}{\sigma - 1} \quad \text{for}\, \sigma > 1. \tag{45.3}$$

If now $\zeta(1 + it) = 0$ we would have

$$\lim_{\sigma \to 1} \frac{\zeta(\sigma + it)}{\sigma - 1} = \lim_{\sigma \to 1} \frac{\zeta(\sigma + it) - \zeta(1 + it)}{\sigma - 1} = \zeta'(1 + it)\,,$$

and the left-hand side of (45.3) would have the limit

$$1 \cdot |\zeta'(1 + it)|^4 \cdot |\zeta(1 + it)|\,,$$

whereas the right-hand member goes to ∞ as $\sigma \to 1$. This contradiction excludes $\zeta(1 + it) = 0$[1].

In virtue of the functional equation we have then also $\zeta(it) \neq 0$.

46. Preparation for the proof of the prime-number theorem

The significance of (44.1) obviously is that it gives two different expressions of the ζ-function, one which exhibits the prime numbers and another which can be studied without any reference to prime numbers, as we have done in Chapter 6.

In order to have the prime numbers appear in a more manageable form we take the logarithmic derivative of (44.1) for $\mathrm{Re}(s) > 1$

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \frac{p^{-s}\log p}{1 - p^{-s}} = -\sum_p \log p \cdot (p^{-s} + p^{-2s} + \cdots)$$

or

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}\,, \tag{46.1}$$

where we have introduced the symbol $\Lambda(n)$ with the following meaning

$$\Lambda(n) = \begin{cases} \log p & \text{for } n = p^k\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{46.2}$$

Our goal is to estimate the number $\pi(x)$ of prime numbers not exceeding x, which can be written briefly as

$$\pi(x) = \sum_{p \leqq x} 1\,. \tag{46.3}$$

---

[1] This arrangement of de la Vallée Poussin's classical proof is due to Ingham [26], p. 29.

We shall prove the *prime-number theorem*

$$\lim_{x\to\infty} \pi(x) \Big/ \frac{x}{\log x} = 1, \tag{46.4}$$

a theorem first conjectured by Gauss around 1793 and proved for the first time by Hadamard and de la Vallée Poussin in 1896 ([74], Ch. 3).

The formula (46.1), however, makes the sum

$$\psi(x) = \sum_{n\leqq x} \Lambda(n) \tag{46.5}$$

much more easily accessible than $\pi(x)$.

We show therefore first that for (46.4) it suffices to prove

$$\lim_{x\to\infty} \psi(x)/x = 1. \tag{46.6}$$

Indeed, let $\omega = \omega(x)$ be a certain monotone increasing function of x, for $x \geqq 2$, for which

$$0 < \omega(x) < x. \tag{46.7}$$

We keep $\omega(x)$ at our disposal for a later choice.

Then we have from the definitions (46.2), (46.3), (46.5)

$$\begin{aligned}
\frac{\psi(x)}{x} &= \frac{1}{x}\sum_{n\leqq x}\Lambda(n) = \frac{1}{x}\sum_{p\leqq x}\log p\left[\frac{\log x}{\log p}\right] \\
&\leqq \frac{1}{x}\sum_{p\leqq x}\log p\,\frac{\log x}{\log p} = \frac{\log x}{x}\pi(x) \\
&= \frac{\log x}{x}\pi((\omega(x)) + \frac{\log x}{x}(\pi(x) - \pi(\omega(x))) \\
&\leqq \frac{\log x}{x}\pi(\omega(x)) + \frac{\log x}{x}\sum_{\omega<p\leqq x}\frac{\log p}{\log \omega} \\
&\leqq \frac{\log x}{x}\omega(x) + \frac{\log x}{x}\,\frac{1}{\log\omega}\psi(x)
\end{aligned}$$

and thus

$$\frac{\psi(x)}{x} \leqq \pi(x) \Big/ \frac{x}{\log x} \leqq \frac{\log x}{x}\omega(x) + \frac{\log x}{\log\omega}\,\frac{\psi(x)}{x}. \tag{46.8}$$

If we can choose $\omega(x)$ as a monotonic increasing function, fulfilling (46.7) and also showing the properties

$$\frac{\log x}{x}\omega(x) \to 0, \qquad \frac{\log x}{\log \omega(x)} \to 1$$

as $x \to \infty$, then (46.8) shows that (46.6) implies (46.4). Now indeed

$$\omega(x) = \begin{cases} x/4\,, & 2 \leqq x \leqq e^2\,, \\ x/(\log x)^2\,, & e^2 \leqq x \end{cases}$$

is such a function. We need therefore only be concerned with (46.6).

Since (46.8) can be restated as

$$\frac{\pi(x)\log\omega(x)}{x} - \frac{\log\omega(x)}{x}\,\omega(x) \leqq \frac{\psi(x)}{x} \leqq \frac{\pi(x)\log x}{x}$$

we have actually proved that (46.4) and (46.6) are *equivalent.*

47. A lemma

We need the following

Lemma. *For any integer* $k \geqq 1$ *and for real* $y > 0$ *we have*

$$I_k(y) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{y^s}{s(s+1)\cdots(s+k)}\,ds = \begin{cases} 0 \textit{ for } 0 < y \leqq 1\,, \\ (1-1/y)^k/k! \textit{ for } 1 \leqq y\,. \end{cases} \tag{47.1}$$

Proof. The absolute convergence of the integral follows immediately from $|y^s| = y^2$.

We consider first the case $0 < y \leqq 1$. Let C_ω be the semicircle of radius ω about $s = 2$ in the half-plane $\mathrm{Re}(s) \geqq 2$. We have then

$$\int_{2-i\omega}^{2+i\omega} \frac{y^s}{s(s+1)\cdots(s+k)}\,ds = \int_{C_\omega}$$

and therefore, since $y^\sigma \leqq y^2$ for $\sigma \geqq 2$

$$\left|\int_{C_\omega}\right| \leqq \pi\omega\,\frac{y^2}{\omega^{k+1}} \to 0$$

as $\omega \to \infty$. This proves $I_k(y) = 0$ in the first case.

Now let $1 \leqq y$. Let C'_ω be the semicircle of radius ω about $s = 2$ in the half-plane $\mathrm{Re}(s) \leqq 2$. We take $\omega > k + 2$ so that C'_ω together with the diameter from $2 - i\omega$ to $2 + i\omega$ encloses the poles $0, -1, \ldots, -k$ of the integrand. The residue at the pole $s = -j$ is

$$\frac{y^{-j}}{(-1)^j j!\,(k-j)!}\,, \qquad j = 0, 1, \ldots, k\,.$$

We have then

$$\frac{1}{2\pi i}\int_{2-i\omega}^{2+i\omega} \frac{y^s}{s(s+1)\cdots(s+k)}\,ds = \frac{1}{2\pi i}\int_{C'_\omega} + \sum_{j=0}^{k}(-1)^j\,\frac{y^{-j}}{j!\,(k-j)!}\,.$$

Since

$$\left|\int_{C'_\omega}\right| \leqq \pi\omega \frac{y^2}{(\omega-k-2)^{k+1}} \to 0 \qquad \text{as } \omega \to \infty$$

we have in this case

$$I_k(y) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-y)^{-j} = \frac{1}{k!}\left(1-\frac{1}{y}\right)^k,$$

which concludes the proof. ▯

48. Expression of a function $\Psi(x)$ connected with $\psi(x)$ by means of an integral

We choose a real number $x > 1$ and consider

$$\Psi(x) = -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^s}{s(s+1)}\frac{\zeta'(s)}{\zeta(s)}\,ds$$

$$= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^s}{s(s+1)}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\,ds. \tag{48.1}$$

Because of absolute and uniform convergence of sum and integral the interchange of the two operations is permissible and we have

$$\Psi(x) = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\Lambda(n)\int_{2-i\infty}^{2+i\infty}\frac{\left(\frac{x}{n}\right)^s}{s(s+1)}\,ds = \sum_{n\leqq x}\Lambda(n)\left(1-\frac{n}{x}\right) \tag{48.2}$$

according to (47.1). Now we introduce in the integral of (48.1) the function

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}, \tag{48.3}$$

which is regular at $s = 1$, and since $\zeta(s) \neq 0$ for $\operatorname{Re}(s) \geqq 1$, also regular for $\operatorname{Re}(s) \geqq 1$. We can read off from (47.1)

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{x^s}{s(s+1)}\frac{ds}{s-1} = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{x^{s+1}}{(s+1)(s+2)s}\,ds$$

$$= \frac{x}{2}\left(1-\frac{1}{x}\right)^2,$$

where we only observed that the integral in (47.1) can be shifted from the abscissa 2 to the abscissa 1. We obtain thus from (48.1)

$$\Psi(x) = \frac{x}{2}\left(1 - \frac{1}{x}\right)^2 - R(x) \tag{48.4}$$

with

$$R(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^s}{s(s+1)} \left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right) ds\,. \tag{48.5}$$

49. Some estimates for $\zeta(s)$, $\zeta'(s)$, $1/\zeta(s)$

We are planning to shift the path of integration in (48.5) to the abscissa 1. This requires an estimate of $\zeta'(s)/\zeta(s)$ for $\mathrm{Re}(s) \geqq 1$.

Assume first $\mathrm{Re}(s) > 1$. Then the Euler-MacLaurin formula (7.2), (7.21) applied for $n > M$ (the case $M = 1$ was used in § 38) and $q = 1$ yields

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{M} \frac{1}{n^s} + \lim_{N\to\infty}\left\{\int_M^N x^{-s}\,dx + \frac{1}{2}(N^{-s} - M^{-s})\right.\\ &\qquad\left. - s\int_M^N B_1(x-[x])\,x^{-s-1}\,dx\right\}\\ &= \sum_{n=1}^{M} \frac{1}{n^s} + \frac{M^{1-s}}{s-1} - \frac{1}{2}M^{-s} - s\int_M^\infty B_1(x-[x])\,x^{-s-1}\,dx. \end{aligned} \tag{49.1}$$

The integral here is convergent for $\mathrm{Re}(s) > 0$.

Taking now

$$1 \leqq \sigma \leqq 2\,, \qquad |t| \geqq 1\,, \qquad M = [|t|] \tag{49.2}$$

we obtain

$$\begin{aligned}\zeta(s) &= O\left(\sum_{n=1}^{M} \frac{1}{n}\right) + O(|t|^{-\sigma}) + O(|t|\cdot|t|^{-\sigma})\\ &= O(\log|t|), \quad \text{as } |t| \to \infty\,. \end{aligned} \tag{49.3}$$

Differentiation of (49.1) gives

$$\begin{aligned}\zeta'(s) &= -\sum_{n=1}^{M} \frac{\log n}{n^s} - \frac{M^{1-s}\log M}{s-1} - \frac{M^{1-s}}{(s-1)^2} + \frac{1}{2}M^{-s}\log M\\ &\qquad - \int_M^\infty B_1(x-[x])\,x^{-s-1}\,dx + s\int_M^\infty B_1(x-[x])\,x^{-s-1}\log x\,dx. \end{aligned}$$

In the range (49.2) we obtain here

$$\zeta'(s) = O(\log^2 |t|) . \tag{49.4}$$

For the estimation of $\zeta(s)^{-1}$, $1 \leqq \operatorname{Re}(s) \leqq 2$, it is evident that the non-vanishing of $\zeta(s)$ for $s = 1 + it$ has to come into play, so that § 45 has to be used. From (45.2) we read off, for $\sigma > 1$,

$$\frac{1}{|\zeta(\sigma + it)|} \leqq \zeta(\sigma)^{3/4} |\zeta(\sigma + it)|^{1/4} \leqq B \log^{1/4} |t| \frac{1}{(\sigma - 1)^{3/4}} \tag{49.5}$$

because of the pole of $\zeta(s)$ at $s = 1$ and in view of (49.3).

Now

$$\zeta(1 + it) - \zeta(\sigma + it) = -\int_1^\sigma \zeta'(u + it)\, du$$
$$= O((\sigma - 1) \log^2 |t|) \tag{49.6}$$

and thus

$$|\zeta(1 + it)| \geqq \left| A(\sigma - 1) \log^2 |t| - \frac{1}{B} (\sigma - 1)^{3/4} \log^{-1/4} |t| \right|,$$

where we can assume $A \geqq 2$, $0 < 1/B \leqq 1$ from the meaning of the O-symbols in (49.5) and (49.6). If we choose $\sigma - 1 = \log^{-9} |t|$, $|t| \geqq e$, we have

$$|\zeta(1 + it)| \geqq \log^{-7} |t| .$$

Now we have, for $|t| \geqq e$:

$$|\zeta(\sigma + it)| \geqq |\zeta(1 + it)| - |\zeta(\sigma + it) - \zeta(1 + it)|$$
$$\geqq \log^{-7} |t| - C |\sigma - 1| \log^2 |t|$$

and for

$$0 \leqq \sigma - 1 \leqq \frac{1}{2C \log^9 |t|},$$
$$|\zeta(\sigma + it)| \geqq (1/2) \log^{-7} |t| . \tag{49.71}$$

For

$$\sigma - 1 \geqq \frac{1}{2C \log^9 t},$$

we have from (49.5)

$$\frac{1}{\zeta(\sigma + it)} \leqq B \log^{1/4} t (2C \log^9 |t|)^{3/4} \leqq C_1 \log^7 |t| . \tag{49.72}$$

Then (49.71) and (49.72) together show that for $1 \leqq \sigma \leqq 2$

$$\frac{1}{\zeta(s)} = O(\log^7 |t|) . \tag{49.8}$$

In the same σ-interval we have, in view of (49.4),

$$\frac{\zeta'(s)}{\zeta(s)} = O(\log^9 |t|) \,. \tag{49.9}$$

50. The prime-number theorem

This estimation shows that in (48.5) we can shift the path of integration to the abscissa 1, so that

$$R(x) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{x^s}{s(s+1)} \left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right) ds$$

$$= \frac{x}{2\pi} \int_{-\infty}^{\infty} \frac{e^{it\log x}}{(1+it)(2+it)} \left(\frac{\zeta'(1+it)}{\zeta(1+it)} + \frac{1}{it}\right) dt \,. \tag{50.1}$$

The last integral goes to 0 with $x \to \infty$ as the following lemma shows:

Lemma. *If $f(t)$ is continuous and*

$$\int_{-\infty}^{\infty} f(t)\, dt$$

absolutely convergent then

$$\lim_{\lambda\to\infty} \int_{-\infty}^{\infty} e^{it\lambda} f(t)\, dt = 0 \,.$$

Proof. We first choose ω so large, that to a given $\varepsilon > 0$ we have

$$\left|\int_{\omega}^{\infty}\right| + \left|\int_{-\infty}^{-\omega}\right| < \frac{\varepsilon}{2} \,. \tag{50.2}$$

Now

$$\int_{-\omega}^{\omega} e^{it\lambda} f(t)\, dt = \frac{1}{2}\left\{\int_{-\omega}^{\omega} e^{it\lambda} f(t)\, dt + \int_{-\omega-\pi/\lambda}^{\omega-\pi/\lambda} e^{i(t+\pi/\lambda)\lambda} f\left(t+\frac{\pi}{\lambda}\right) dt\right.$$

$$= \frac{1}{2}\int_{-\omega}^{\omega-\pi/\lambda} e^{it\lambda}\left(f(t) - f\left(t+\frac{\pi}{\lambda}\right)\right) dt$$

$$- \frac{1}{2}\int_{-\omega-\pi/\lambda}^{-\omega} e^{it\lambda} f\left(t+\frac{\pi}{\lambda}\right) dt + \frac{1}{2}\int_{\omega-\pi/\lambda}^{\omega} e^{it\lambda} f(t)\, dt \,,$$

$$\left|\int_{-\omega}^{\omega} e^{it\lambda} f(t)\, dt\right| \leqq \frac{1}{2}\int_{-\omega}^{\omega-\pi/\lambda} \left|f(t) - f\left(t+\frac{\pi}{\lambda}\right)\right| dt$$

$$+ \frac{1}{2}\int_{-\omega}^{-\omega+\pi/\lambda} \left|f(t)\right| dt + \frac{1}{2}\int_{\omega-\pi/\lambda}^{\omega} \left|f(t)\right| dt \,.$$

Now $f(t)$ is uniformly continuous and bounded in $|t| \leqq \omega$. Thus λ can be chosen so large that all 3 integrals in the right-hand member become each less than $\varepsilon/3$, and thus

$$\left| \int_{-\omega}^{\omega} e^{it\lambda} f(t)\, dt \right| < \frac{\varepsilon}{2}.$$

This together with (50.2) shows that for λ sufficiently large we have

$$\left| \int_{-\infty}^{\infty} e^{it\lambda} f(t)\, dt \right| < \varepsilon ,$$

which was to be proved. ▯

(This lemma is a special case of the Riemann-Lebesgue theorem; see Titchmarsh [73], Theorem 1 on p. 11.)

By means of this lemma we obtain from (50.1) and (48.4)

$$x\Psi(x) = \frac{x^2}{2} + o\,(x)^2 .$$

Now after (48.2)

$$x\Psi(x) = \sum_{n\leqq x} \Lambda(n)\,(x-n) = \sum_{n\leqq x} \Lambda(n) \int_{n}^{x} dv = \int_{1}^{x} \left(\sum_{n\leqq v} \Lambda(n) \right) dv$$

$$= \int_{1}^{x} \psi(v)\, dv$$

in view of definition (46.5), so that we have

$$\int_{1}^{x} \psi(v)\, dv \sim \frac{x^2}{2}. \tag{50.3}$$

In order to return to $\psi(x)$ we need the

Lemma. *If, for $v \geqq 1$, $g(v)$ is a monotone non-decreasing function such that*

$$\int_{1}^{x} g(v)\, dv \sim \frac{x^2}{2}$$

then

$$g(x) \sim x .$$

Proof. For any given $\delta > 0$ there exists an $x_0 = x_0(\delta)$ so that for $x \geqq x_0$

$$(1-\delta)\frac{x^2}{2} < \int_{1}^{x} g(v)\, dv < (1+\delta)\frac{x^2}{2}$$

and thus, for $\varepsilon > 0$

$$\begin{aligned}\int_x^{x(1+\varepsilon)} g(v)\,dv &= \int_1^{x(1+\varepsilon)} g(v)\,dv - \int_1^x g(v)\,dv \\ &< (1+\delta)\frac{x^2}{2}(1+\varepsilon)^2 - (1-\delta)\frac{x^2}{2} \\ &= \frac{x^2}{2}\left((1+\delta)(1+\varepsilon)^2 - 1 + \delta\right) \\ &= x^2\left(\varepsilon\left(1+\frac{\varepsilon}{2}\right)(1+\delta) + \delta\right).\end{aligned}$$

Because of monotonicity of $g(x)$ we have

$$\int_x^{x(1+\varepsilon)} g(v)\,dv \geqq x\varepsilon\, g(x)$$

so that

$$g(x) < x\left(\left(1+\frac{\varepsilon}{2}\right)(1+\delta) + \frac{\delta}{\varepsilon}\right)$$

and thus

$$\varlimsup_{x\to\infty} \frac{g(x)}{x} \leqq \left(1+\frac{\varepsilon}{2}\right)(1+\delta) + \frac{\delta}{\varepsilon}$$

which, by the choice

$$\varepsilon = \sqrt{\delta}, \quad \delta \to 0,$$

shows

$$\varlimsup_{x\to\infty} \frac{g(x)}{x} \leqq 1. \tag{50.4}$$

In a similar way we obtain from the consideration of $\int_{x(1-\varepsilon)}^{x} g(v)\,dv$ the inequality

$$g(x) > x\left(\left(1-\frac{\varepsilon}{2}\right)(1+\delta) - \frac{\delta}{\varepsilon}\right),$$

$$\varliminf_{x\to\infty} \frac{g(x)}{x} \geqq 1,$$

which with (50.4) proves the lemma. ▯

Now $\psi(x)$ is monotone, and the application of the Lemma on (50.3) with $g(x) = \psi(x)$ yields

$$\psi(x) \sim x\,, \tag{50.5}$$

which is, as we know, equivalent with the prime number theorem.

51. The error term in the prime-number theorem

Theorem. *If we set*

$$\psi(x) = x + r(x) \tag{51.1}$$

then we have proved

$$r(x) = o(x)\,.$$

Much more, however, is known about $r(x)$. De la Vallée Poussin already proved

$$r(x) = O\left(x e^{-\alpha\sqrt{\log x}}\right) \tag{51.2}$$

with a certain positive α. The basis for a proof of this and still sharper theorems is the non-vanishing of $\zeta(s)$ not only on $\sigma = 1$ but in a region to the left of that line. De la Vallée Poussin found $\zeta(s) \neq 0$ in a region

$$1 - \frac{1}{C \log|t|} \leqq \sigma\,, \qquad |t| \geqq 2$$

for a certain C, and this leads to (51.2). An account of this proof can be found in Landau's famous treatise on the distribution of prime numbers [32], vol. 2. An improvement was possible through Weyl's theory of exponential sums, by which Littlewood obtained

$$r(x) = O\left(x e^{-\alpha\sqrt{\log x \operatorname{loglog} x}}\right)\,.$$

See [33], vol. 2.

The best estimate known at present is

$$\begin{aligned} r(x) &= x \exp\left(-\alpha \frac{\log x^{4/7}}{\operatorname{loglog} x^{3/7}}\right) \\ &= x \exp\{-c(\Theta) \log^{\Theta} x\} \quad \text{for any } \Theta < 3/5 \end{aligned}$$

found independently by Vinogradov and Korobov in 1958 [23, 30].

It is not our purpose here to go into these theories, since they have found excellent presentations. Instead, we propose here to outline the importance of the Riemann hypothesis for the theory of the distribution of prime numbers.

We define Θ as the least upper bound of the zeros of $\zeta(s)$. Certainly $\Theta \leqq 1$. We have in our theorems not yet established he existence of a single zero in the critical strip. If there is none then we would have $\Theta = -2$. Any zero in the critical strip can have only a positive real part β, since $\operatorname{Re}(s) = 0$ is free of zeros. Because of the symmetry of the zeros discussed in § 44 there must then be another zero of real part $1 - \beta$, and Θ can thus not be less than 1/2. We have therefore a priori to reckon with the two possibilities:

(I) No zeros in the critical strip, $\Theta = -2$.

(II) Existence of zeros in the critical strip, $1/2 \leqq \Theta \leqq 1$.

Our aim is to rule out (I) and moreover to amplify (II) by establishing infinitely many zeros of $\zeta(s)$ in the critical strip. We shall get these results by investigating the connection of Θ with the order of the error term $r(x)$ in (51.1). Riemann's hypothesis, mentioned at the end of § 44, can simply be expressed as $\Theta = 1/2$.

52. Carathéodory's lemma

We need for the following the

Lemma. *Let $\varphi(z)$ be analytic and regular in $|z| < R$, and*

$$\operatorname{Re} \varphi(z) \geqq 0 .$$

Then we have

$$|\varphi(z) - \varphi(0)| \leqq \frac{2|z|}{R - |z|} \operatorname{Re} \varphi(0) ,$$

$$\left| \frac{\varphi^{(\nu)}(z)}{\nu!} \right| \leqq \frac{2R}{(R - |z|)^{\nu+1}} \operatorname{Re} \varphi(0) , \qquad \nu = 1, 2, \ldots .$$

Proof. We put

$$\varphi(z) = \sum_{n=0}^{\infty} c_n z^n , \qquad |z| < R$$

with

$$c_n = \frac{1}{2\pi i} \int_C \frac{\varphi(z)}{z^{n+1}} dz ,$$

where C can be taken as a circle about 0 of radius $r < R$. With $z = re^{i\vartheta}$ we get then

$$c_n = \frac{1}{2\pi r^n} \int_0^{2\pi} \varphi(re^{i\vartheta}) e^{-in\vartheta} d\vartheta , \qquad n \geqq 0 .$$

On the other hand we have, for $n \geqq 1$,

$$0 = \frac{1}{2\pi i} \int_C \varphi(z) z^{n-1} \, dz$$

and hence

$$0 = \frac{1}{2\pi r^n} \int_0^{2\pi} \varphi(r e^{i\vartheta}) e^{in\vartheta} \, d\vartheta \, , \qquad n \geqq 1 \, .$$

Splitting now $\varphi(z)$ into real and imaginary parts we put

$$\varphi(r e^{i\varphi}) = P + iQ$$

and have thus

$$c_n = \frac{1}{2\pi r^n} \int_0^{2\pi} (P + iQ) e^{-in\vartheta} \, d\vartheta \, , \qquad n \geqq 0 \, . \tag{52.1}$$

and

$$0 = \frac{1}{2\pi r^n} \int_0^{2\pi} (P + iQ) e^{in\vartheta} \, d\vartheta \, , \qquad n \geqq 1 \, .$$

The last equation can be replaced by

$$0 = \frac{1}{2\pi r^n} \int_0^{2\pi} (P - iQ) e^{-in\vartheta} \, d\vartheta \tag{52.2}$$

since a complex number vanishes together with its conjugate. From (51.1) and (51.2) we find by addition

$$c_n = \frac{1}{\pi r^n} \int_0^{2\pi} P e^{-in\vartheta} \, d\vartheta \, , \qquad n \geqq 1 \, ,$$

and from (52.1) alone

$$\operatorname{Re} c_0 = \frac{1}{2\pi} \int_0^{2\pi} P d\vartheta \, .$$

Since by hypothesis $P \geqq 0$ and thus $|P| = P$ we conclude

$$|c_n| \leqq \frac{1}{\pi r^n} \int_0^{2\pi} |P| \, d\vartheta = \frac{1}{\pi r^n} \int_0^{2\pi} P d\vartheta = \frac{2 \operatorname{Re} c_0}{r^n} \, .$$

This is valid for all $r < R$, so that $r \to R$ yields

$$|c_n| \leqq \frac{2 \operatorname{Re} c_0}{R^n} \, .$$

Therefore

$$|\varphi(z) - \varphi(0)| = \left|\sum_{n=1}^{\infty} c_n z^n\right| \leqq 2 \operatorname{Re} c_0 \sum_{n=1}^{\infty} \left(\frac{|z|}{R}\right)^n = \frac{2|z|}{R - |z|} \operatorname{Re} \varphi(0) . \tag{52.3}$$

Moreover, we have

$$|\varphi^{(\nu)}(z)| \leqq \sum_{n=\nu}^{\infty} n(n-1) \cdots (n-\nu+1) |c_n| |z|^{n-\nu}$$

$$\leqq \frac{2 \operatorname{Re} c_0 \nu!}{R^\nu} \sum_{n=\nu}^{\infty} \binom{n}{\nu} \left(\frac{|z|}{R}\right)^{n-\nu} = \frac{2 \operatorname{Re} c_0 \nu!}{R^\nu} \frac{1}{\left(1 - \frac{|z|}{R}\right)^{\nu+1}}$$

or

$$\left|\frac{\varphi^{(\nu)}(z)}{\nu!}\right| \leqq \frac{2R}{(R - |z|)^{\nu+1}} \operatorname{Re} \varphi(0) . \tag{52.4}$$

Formulae (52.3) and (52.4) contain the lemma. ▯

Corollary (Carathéodory's lemma). *If* $f(s)$ *is regular in* $|s - s_0| < R$ *and if there* $\operatorname{Re} f(s) \geqq M$ *then*

$$|f(s) - f(s_0)| \leqq \frac{2|s - s_0|}{R - |s - s_0|} \left(M - \operatorname{Re} f(s_0)\right),$$

$$|f'(s)| \leqq \frac{2R}{(R - |s - s_0|)^2} \left(M - \operatorname{Re} f(s_0)\right).$$

Indeed this corollary is obtained from the lemma by putting

$$\varphi(z) = M - f(z + s_0) .$$

53. Application of Carathéodory's lemma

We need an estimate of the function (48.3). We have

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = \frac{d}{ds}\left(\log\left(\zeta(s)(s-1)\right)\right),$$

where $\log\left(\zeta(s)(s-1)\right)$ is regular for $\operatorname{Re}(s) > \Theta$. Now from (42.7) and (43.2) we obtain

$$|\zeta(s)(s-1)| < C(|t| + 2)^4$$

and therefore

$$\operatorname{Re} \log\left(\zeta(s)(s-1)\right) = \log|\zeta(s)(s-1)| < C \log(|t| + 2) \tag{53.1}$$

with C in a new meaning[1]. This estimate permits the application of Carathéodory's lemma. We choose $s_0 = 2 + it_0$, $R = 2 - \Theta \leqq 4$, so that the circle $|s - s_0| < R$ remains in the region of regularity of $\log(\zeta(s)(s-1))$. We choose further a number $\Theta_1 > \Theta$ and consider s only in $|s - s_0| \leqq 2 - \Theta_1$. The corollary then yields

$$\left|\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right| \leqq \frac{4 - 2\Theta}{(\Theta_1 - \Theta)^2}\left(C\log(|t| + 6) - \log|\zeta(s_0)(s_0 - 1)|\right),$$

where the M of the corollary is taken from (53.1). Moreover we have

$$\begin{aligned} -\log|\zeta(s_0)(s_0-1)| &= \log\left|\sum_{n=1}^{\infty}\frac{1}{n^{2+t_0 i}}\right|^{-1} + \log|1 + it_0|^{-1} \\ &\leqq \log\frac{1}{1 - \sum_2^{\infty}\frac{1}{n^2}} = \log\frac{1}{2 - \frac{\pi^2}{6}} = \log\frac{6}{12 - \pi^2} \\ &< \log 3\,, \end{aligned}$$

and thus with suitable C for all $s = \sigma + t_0 i$, $\Theta_1 \leqq \sigma \leqq 2$

$$\left|\frac{\zeta'(\sigma + ti)}{\zeta(\sigma + ti)} + \frac{1}{\sigma + ti - 1}\right| \leqq C\frac{\log(|t| + 6)}{(\Theta_1 - \Theta)^2}\,, \tag{53.2}$$

where we have dropped the subscript of t_0. This estimate shows that we can shift the abscissa of the path of integration in (48.5) from 2 to Θ_1.

The cases I and II in § 51 differ slightly with respect to the occurrence of poles. This prompts us to make the following decision. We define

(I) $\Theta_1 = -1/2$ in the case of $\Theta = -2$,

(II) $\Theta_1 = \Theta + \varepsilon < 2$ in the case of $1/2 \leqq \Theta \leqq 1$,

with a number $\varepsilon > 0$ to be kept at our disposal (we shall make it depend on x).

54. The error term $r(x)$

We get from $\Psi(x)$ and $R(x)$ of (48.2), (48.4), and (48.5) to $\psi(x)$ and $r(x)$ as follows:

$$\begin{aligned} (x+1)\Psi(x+1) - x\Psi(x) &= \sum_{n \leqq x+1}\Lambda(n)(x + 1 - n) - \sum_{n \leqq x}\Lambda(n)(x - n) \\ &= \sum_{n \leqq x}\Lambda(n) + \Lambda([x] + 1)(x - [x]) \\ &= \psi(x) + \Lambda([x] + 1)(x - [x])\,. \end{aligned}$$

[1] We shall leave out the phrase "in a new meaning" in the sequel, keeping in mind that C is a constant, but not always the same.

On the other hand

$$(x+1)\,\Psi(x+1) - x\Psi(x) = \left\{\frac{(x+1)^2}{2} - (x+1) + \frac{1}{2}\right\}$$
$$- \left\{\frac{x^2}{2} - x + \frac{1}{2}\right\}$$
$$- (x+1)\,R(x+1) + xR(x)$$
$$= x - \frac{1}{2} - I(x)$$

with

$$I(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{(x+1)^{s+1} - x^{s+1}}{s(s+1)} \left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right) ds \tag{54.1}$$

so that in

$$\psi(x) = x + r(x) \tag{54.21}$$

we have

$$r(x) = -\frac{1}{2} - A([x]+1)\,(x-[x]) - I(x)\,. \tag{54.22}$$

We now shift the path of integration in $I(x)$ from 2 to Θ_1. In case I we pass in this process the pole at $s = 0$ with the residue

$$\frac{\zeta'(0)}{\zeta(0)} - 1\,,$$

whereas in case II we do not reach that pole. Thus we obtain

$$I(x) = A(\Theta) + \frac{1}{2\pi i} \int_{\Theta_1-i\infty}^{\Theta_1+i\infty} \frac{(x+1)^{s+1} - x^{s+1}}{s(s+1)} \left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right) ds \tag{54.31}$$

with

$$A(\Theta) = \begin{cases} \zeta'(0)/\zeta(0) - 1 & \text{for } \Theta = -2 \text{ (case I)}\,, \\ 0 & \text{for } 1/2 \leqq \Theta \leqq 1 \text{ (case II)}\,. \end{cases} \tag{54.32}$$

For the estimation of $I(x)$ we observe first

$$(x+1)^{s+1} - x^{s+1} = (s+1) \int_x^{x+1} u^s\, du$$

and therefore, for $\operatorname{Re}(s) = \Theta_1$,

$$\left|(x+1)^{s+1} - x^{s+1}\right| \leqq |s+1| \left|\int_x^{x+1} u^{\Theta_1}\, du\right| < 4\,|s+1|\,x^{\Theta_1}$$

for $1 \leqq x$, $-1/2 \leqq \Theta_1 < 2$.

This estimate cannot be used on the whole path of $I(x)$, since it would furnish a divergent integral. We need also the less precise appraisal

$$\left|(x+1)^{s+1} - x^{s+1}\right| \leqq (x+1)^{\Theta_1+1} + x^{\Theta_1+1} < 9\,x^{\Theta_1+1}.$$

Then we obtain, in view of (53.2),

$$\left|\int_{\Theta_1-i\infty}^{\Theta_1+i\infty} \frac{(x+1)^{s+1} - x^{s+1}}{s(s+1)} \left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right) ds\right|$$

$$< C\frac{x^{\Theta_1}}{(\Theta_1-\Theta)^2}\int_0^T \frac{\log(t+6)}{1/2+t}\,dt + C\frac{x^{\Theta_1+1}}{(\Theta_1-\Theta)^2}\int_T^\infty \frac{\log(t+6)}{t^2}\,dt,$$

where $T > 0$ can still be chosen.

Now

$$\int_0^T \frac{\log(t+6)}{1/2+t}\,dt < \log(T+6)\int_0^T \frac{dt}{1/2+t} \log(T+6)\log(2T+1)$$

$$< 2\log^2(T+6)$$

and

$$\int_T^\infty \frac{\log(t+6)}{t^2}\,dt = \left[-\frac{\log(t+6)}{t}\right]_T^\infty + \int_T^\infty \frac{dt}{t(t+6)}$$

$$< \frac{\log(T+6)}{T} + \frac{1}{T} < 2\,\frac{\log(T+6)}{T}.$$

Therefore

$$\left|\int_{\Theta_1-i\infty}^{\Theta_1+i\infty}\right| < C\frac{x^{\Theta_1}}{(\Theta_1-\Theta)^2}\log^2(T+6) + C\frac{x^{\Theta_1+1}}{(\Theta_1-\Theta)^2}\,\frac{\log(T+6)}{T}.$$

Here $T = x$ is an obvious choice because of the different powers of x in the two summands. We obtain then in (54.31)

$$I(x) = A(\Theta) + O\left(\frac{x^{\Theta_1}\log^2 x}{(\Theta_1-\Theta)^2}\right). \tag{54.4}$$

We return now to the distinction of the two cases. In case I we have $\Theta_1 - \Theta = 3/2$ and therefore

$$r(x) = \frac{1}{2} - \frac{\zeta'(0)}{\zeta(0)} - \Lambda([x]+1)(x-[x]) + O(x^{-1/2}\log^2 x). \quad (54.51)$$

In case II we have $\Theta_1 - \Theta = \varepsilon$. Here we choose

$$\varepsilon = \frac{1}{\log x},$$

which produces

$$x^{\Theta_1} = x^{\Theta+\varepsilon} = x^{\Theta} e^{\varepsilon \log x} = e\,x^{\Theta}$$

and therefore

$$r(x) = O(x^{\Theta}\log^4 x), \quad (54.52)$$

where two terms of (54.22) could be absorbed in the O-term.

Now formula (54.51) makes it evident that *case I cannot occur*. Let us take $x = m$ as an integer. Then we would have after (54.21)

$$\begin{aligned}\psi(m) - \psi(m-1) = \Lambda(m) &= 1 + r(m) - r(m-1)\\ &= 1 + O(m^{-1/2}\log^2 m)\end{aligned}$$

because of (54.51). But this is absurd, since for prime numbers $m = p$ we have $\Lambda(p) = \log p$, and

$$\log p = 1 + O(p^{-1/2}\log^2 p)$$

contains a contradiction.

We have thus proved that only case II is possible, in other words that $1/2 \leqq \Theta \leqq 1$. Therefore $\zeta(s)$ possesses zeros in the critical strip $0 < \mathrm{Re}(s) < 1$.

55. Existence of infinitely many non-trivial zeros

We can even go a step farther and show that $\zeta(s)$ must possess infinitely many zeros in the critical strip. Let us assume that $\zeta(s)$ has only the finitely many non-trivial zeros

$$\varrho_1, \varrho_2, \ldots, \varrho_N$$

with the respective multiplicities

$$\mu_1, \mu_2, \ldots, \mu_N.$$

If we split into real and imaginary parts

$$\varrho_k = \beta_k + i\gamma_k$$

we must have

$$0 < \beta_k < 1\,, \qquad k = 1, 2, \ldots, N\,, \tag{55.1}$$

since the boundaries of the critical strip are zero-free. Let G be so chosen that

$$|\gamma_k| \leqq G\,, \qquad k = 1, 2, \ldots, N\,.$$

We start now again with (54.1) and intend to shift the path of integration to $-1/2$. For this purpose we have to revise first an argument in the application of Carathéodory's lemma, since the function $\log(\zeta(s)\,(s-1))$ in this case is regular not everywhere for $\operatorname{Re}(s) > -2$, but only in that part for which $|t| > G$. The estimate (53.2) can then be obtained as before, for $|t| > G + 4$. We can therefore shift the path of integration to $\sigma = -1/2$, if we observe only the poles which we pass. Instead of (54.51) we obtain here thus

$$r(x) = \frac{1}{2} - \frac{\zeta'(0)}{\zeta(0)} - \Lambda([x]+1)\,(x - [x])$$

$$- \sum_{k=1}^{N} \frac{(x+1)^{\varrho_k+1} - x^{\varrho_k+1}}{\varrho_k\,(\varrho_k+1)}\,\mu_k + O(x^{-1/2}\log^2 x)\,.$$

In order to show that this is again impossible we form as before $\psi(m) - \psi(m-1)$ for a positive integer $m > 2$ and find

$$\Lambda(m) = 1 + \sum_{k=1}^{N} \frac{\mu_k}{\varrho_k} \int_m^{m+1} (u^{\varrho_k} - (u-1)^{\varrho_k})\,du + O(m^{-1/2}\log^2 m)$$

$$= 1 + \sum_{k=1}^{N} \mu_k \int_m^{m+1} \int_{u-1}^{u} v^{\varrho_k - 1}\,dv\,du + O(m^{-1/2}\log^2 m)\,.$$

Now

$$\left| \int_m^{m+1} \int_{u-1}^{u} v^{\varrho_k-1}\,dv\,du \right| \leqq \int_m^{m+1} \int_{u-1}^{u} v^{\beta_k-1}\,dv\,du \leqq (m-1)^{\beta_k - 1}\,,$$

so that we would have

$$\Lambda(m) = 1 + \sum_{k=1}^{N} O(m^{\beta_k - 1}) + O(m^{-1/2}\log^2 m)\,.$$

Because of (55.1) all the O-terms here tend to 0 as $m \to \infty$, which is in contradiction to the behavior of $\Lambda(n)$ as we have seen before.

This shows that the assumption of at most finitely many zeros in the critical strip is untenable, and we conclude that $\zeta(s)$ has infinitely many "non-trivial" zeros.

56. Additional remarks

The existence of infinitely many non-trivial zeros of $\zeta(s)$ is usually proved through the application of Hadamard's theory of entire functions to the function

$$s(s-1)\,\Phi(s)\,,$$

where $\Phi(s)$ is defined by (40.71), whereas our argument connects it with the existence of infinitely many primes. Of course Hadamard's theory yields much more than merely the existence of infinitely many non-trivial zeros. Riemann conjectured, and *v.* Mangoldt proved, that the number $N(T)$ of zeros $\varrho = \beta + i\gamma$ with $0 < \gamma \leqq T$ is

$$N(T) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1\right) + O(\log T)\,. \tag{56.1}$$

Hardy showed first in 1914 [16] that infinitely many of these zeros lie on the middle line $\sigma = 1/2$ of the critical strip. The best result known at present in this direction is A. Selberg's theorem that the number $N_0(T)$ of zeros on $\sigma = 1/2$ satisfies

$$N_0(T) > AT\log T$$

for a certain positive A. Compared with (56.1) Selberg's result states that among all non-trivial zeros those on the middle-line have a *positive density.*

It is also known that most non-trivial zeros lie on or near the line $\sigma = 1/2$. From a result of Selberg [68] follows indeed that the number of zeros $N_\varepsilon(T)$ for which $|\beta - 1/2| > \varepsilon$ is at most

$$N(T) = O(T^{1-3\varepsilon})\,.$$

We refer to the literature on Riemann's ζ-function, especially to Titchmarsh's book [67] for details and further information.

57. Dirichlet series and the best order of the error term in the prime-number theorem

The significance of (54.52) lies, of course, in the connection of Θ, the supremum of the real parts of the zeros of $\zeta(s)$, with the order of the error term in (54.52). Since at present we know only $1/2 \leqq \Theta \leqq 1$ we have through (54.52) not even proved (50.5), which, however, we could confirm by a different method, which did not require the crossing of $\sigma = 1$.

It might be asked whether by methods different from those which we employed possibly error terms could be obtained of lower order than x^{Θ}. We shall show that this is impossible.

As preparation we need two lemmas about Dirichlet series. The series for $\zeta(s)$ and for $\zeta'(s)/\zeta(s)$ are special instances of series of the form

$$\sum_{n=1}^{\infty} a_n n^{-s},$$

called Dirichlet series.

Lemma A. *If* $\sum_{n=1}^{\infty} a_n n^{-s}$ *is a Dirichlet series which is convergent for* $s = s_0$ *then it is also for all* s *with* $\operatorname{Re}(s) > \operatorname{Re}(s_0)$, *and it represents a regular analytic function in that half-plane.*

Proof. Take $\operatorname{Re}(s) > \operatorname{Re}(s_0)$. We have then

$$\sum_{n=M}^{N} \frac{a_n}{n^s} = \sum_{n=M}^{N} \frac{a_n}{n^{s_0}} \cdot \frac{1}{n^{s-s_0}} = \sum_{n=M}^{N} (l_n - l_{n-1}) \frac{1}{n^{s-s_0}},$$

where

$$l_m = \sum_{n=1}^{m} \frac{a_n}{n^{s_0}}.$$

The hypothesis of the lemma can be stated as

$$\lim_{m\to\infty} l_m = L. \tag{57.1}$$

By partial summation we get now

$$\sum_{n=M}^{N} \frac{a_n}{n^s} = \sum_{n=M}^{N-1} l_n \left(\frac{1}{n^{s-s_0}} - \frac{1}{(n+1)^{s-s_0}}\right) - \frac{l_{M-1}}{M^{s-s_0}} + \frac{l_N}{N^{s-s_0}}$$

$$= \sum_{n=M}^{N-1} l_n (s - s_0) \int_n^{n+1} \frac{du}{u^{s-s_0+1}} - \frac{l_{M-1}}{M^{s-s_0}} + \frac{l_N}{N^{s-s_0}}$$

and therefore

$$\left|\sum_{n=M}^{N} \frac{a_n}{n^s}\right| \leq \sum_{n=M}^{N-1} |l_n| |s - s_0| \int_n^{n+1} \frac{du}{u^{\sigma-\sigma_0+1}} + \frac{|l_{M-1}|}{M^{\sigma-\sigma_0}} + \frac{|l_N|}{N^{\sigma-\sigma_0}}.$$

Now (57.1) implies $|l_n| < C$, so that we have

$$\left|\sum_{n=M}^{N} \frac{a_n}{n^s}\right| \leqq C \left| |s - s_0| \int_M^N \frac{du}{u^{\sigma-\sigma_0+1}} + \frac{C}{M^{\sigma-\sigma_0}} + \frac{C}{N^{\sigma-\sigma_0}} \right.$$

$$< C \frac{|s - s_0|}{\sigma - \sigma_0} \frac{1}{M^{\sigma-\sigma_0}} + \frac{2C}{M^{\sigma-\sigma_0}} .$$

This estimate shows the convergence of the Dirichlet series for $\sigma > \sigma_0$ and its uniform convergence in every compact subset in that half-plane. This proves the lemma. ▯

Let us call β the infimum of all $\sigma_0 = \operatorname{Re}(s_0)$ for which the Dirichlet series $\sum a_n n^{-s_0}$ converges. Here $\beta = -\infty$ and $\beta = +\infty$ are admissible, the latter possibility meaning simply that the series does not converge for any s_0. A Dirichlet series converges in a half-plane, a fact which we state in the

Corollary. *A Dirichlet series is convergent in the half plane* $\operatorname{Re}(s) > \beta$.

In the proof of the preceding lemma only $l_n = O(1)$ was used instead of the full content of (57.1). This observation leads to a similar lemma.

Lemma B. *If* $A_m = \sum_{n=1}^{m} a_n = O(m^\kappa)$ *then the Dirichlet series* $\sum_{1}^{\infty} a_n n^{-s}$ *is convergent in the half-plane* $\operatorname{Re}(s) > \kappa$ *and represents there a regular analytic function.*

Proof. We apply again partial summation and get

$$\sum_{n=M}^{N} a_n n^{-s} = \sum_{n=M}^{N} (A_n - A_{n-1}) n^{-s}$$

$$= \sum_{n=M}^{N-1} A_n \left(n^{-s} - (n+1)^{-s}\right) - A_{M-1} M^{-s} + A_N N^{-s}$$

$$= \sum_{n=M}^{N-1} A_n s \int_n^{n+1} u^{-s-1}\, du - A_{M-1} M^{-s} + A_N N^{-s} .$$

Now assume $\sigma > \kappa$. Then we have

$$\left|\sum_{n=M}^{N} a_n n^{-s}\right| < C |s| \sum_{n=M}^{N-1} n^\kappa \int_n^{n+1} u^{-\sigma-1}\, du + C M^{\kappa-\sigma} + C N^{\kappa-\sigma}$$

$$< C\, |s| \sum_{n=M}^{\infty} n^{\kappa-\sigma-1} + 2 C M^{\kappa-\sigma} ,$$

which tends to 0 as $M \to \infty$ since $\kappa - \sigma < 0$. Uniformity of convergence in any compact subset of $\operatorname{Re}(s) > \kappa$ is also evident from this estimate. ▯

We return to the problem of the order of the error term in the prime number theorem. Let us assume that for a certain $\mu > 0$ the formula

$$\psi(x) = x + O(x^{\mu}) \tag{57.2}$$

is valid. We define then the function

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

with

$$a_n = \Lambda(n) - 1 .$$

Since here

$$A_m = \sum_{n=1}^{m} a_n = \psi(m) - m = O(m^{\mu})$$

Lemma B states that $f(s)$ is regular for $\mathrm{Re}(s) > \mu$.

Now, for $\mathrm{Re}(s) > 1$

$$f(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n) - 1}{n^s} = -\frac{\zeta'(s)}{\zeta(s)} - \zeta(s) ,$$

and through analytic continuation we have

$$f(s) = -\frac{\zeta'(s)}{\zeta(s)} - \zeta(s)$$

in the whole half-plane $\mathrm{Re}(s) > \mu$. Hence $\zeta'(s)/\zeta(s)$ has only the pole $s = 1$ in $\mathrm{Re}(s) > \mu$ (assuming $\mu < 1$, otherwise (57.2) is without interest). In particular $\zeta(s) \neq 0$ in $\mathrm{Re}(s) > \mu$. This means

$$\Theta \leqq \mu .$$

Since we have seen that $\Theta + \varepsilon$ is an admissible μ we have

$$\Theta = \inf \mu \tag{57.3}$$

for all such μ which make (57.2) a correct formula. An estimation $r(x) = O(x^{\mu})$ with $\mu < \Theta$ is impossible.

With the possible exception of the logarithmic factor (which can indeed be improved)[1] the formula (54.52) contains the best possible estimate.

[1] If $\Theta > 1/2$, then $r(x) = O(x^{\Theta})$ (see [14]); if $\Theta = 1/2$ then the best known result is $r(x) = O(x^{1/2} \log^2 x)$ (see [46]) and (without any assumption on Θ) $r(x) > C x^{1/2} \log\log\log x$ for some positive constant C and arbitrarily large values of x [37].

Chapter 8
The Eisenstein Series

58. Definition of the Eisenstein series and of $\wp(u)$

Let ω_1, ω_2 be two complex numbers, different from zero, such that the quotient $\tau = \omega_2/\omega_1$ is not real. The totality Ω of all complex numbers $m_1\omega_1 + m_2\omega_2$ with m_1, m_2 integers, forms a point-lattice in the complex plane.

The sums

$$G_r(\omega_1, \omega_2) = \sum_{m_1,m_2}{}' \frac{1}{(m_1\omega_1 + m_2\omega_2)^r} \tag{58.1}$$

are called *Eisenstein series* after the mathematician who first made use of them. The dash $'$ means here and in the sequel that in the summation (m_1, m_2) runs over all pairs of integers except the pair $m_1 = m_2 = 0$. If we restrict r to such integers for which the series is absolutely convergent then G_r depends only on the totality of the lattice and not on the order of summation. We need therefore the following

Lemma. *For $q > 2$ the series*

$$S = \sum_{m_1,m_2}{}' \frac{1}{|m_1\omega_1 + m_2\omega_2|^q} \tag{58.2}$$

is convergent.

Proof. The points ω_1, ω_2, $-\omega_1$, $-\omega_2$ are the vertices of a parallelogram P_1. Let d be the shortest distance between 0 and a side of P_1. We order now the points according to $n = |m_1| + |m_2|$:

$$S = \sum_{n=1}^{\infty} \sum_{|m_1|+|m_2|=n} \frac{1}{|m_1\omega_1 + m_2\omega_2|^q}.$$

The points

$$\Omega = m_1\omega_1 + m_2\omega_2 \tag{58.3}$$

for a fixed $n = |m_1| + |m_2|$ lie on a parallelogram P_n, which is the n times magnified P_1. Since $|m_1|$ can run through $0, 1, \ldots, n$, we have $4n$ points lying on P_n. Each point on P_n has at least the distance nd from 0, so that we have

$$S \leqq \sum_{n=1}^{\infty} \frac{4n}{(nd)^q} = \frac{4}{d^q} \sum_{n=1}^{\infty} \frac{1}{n^{q-1}}, \tag{58.31}$$

which proves the lemma. $\square$

The expansion (58.1) converges therefore absolutely for the integers $r \geqq 3$. But for odd integers r the sum G_r vanishes identically, since with each pair m_1, m_2 there appears also $-m_1$, $-m_2$ in the sum. We consider therefore the sum only for even integers $r = 2k \geqq 4$.

Before studying the $G_{2k}(\omega_1, \omega_2)$ any further, we introduce with Weierstrass the function

$$\wp(u) = \frac{1}{u^2} + \sum_{m_1, m_2}{}' \left\{ \frac{1}{(u - \Omega)^2} - \frac{1}{\Omega^2} \right\}, \tag{58.4}$$

where Ω is the abbreviation (58.3). This is a Mittag-Leffler expression of a meromorphic function in the complex u-plane, with poles of second order at the points Ω. In order to prove the absolute convergence of the series in (58.4) we observe

$$\frac{1}{(u-\Omega)^2} - \frac{1}{\Omega^2} = \int_0^u \frac{-2}{(v-\Omega)^3}\, dv ,$$

and therefore

$$\left| \frac{1}{(u-\Omega)^2} - \frac{1}{\Omega^2} \right| \leqq 2\,|u| \operatorname*{Max}_{|v| \leqq |u|} \frac{1}{|v - \Omega|^3} . \tag{58.5}$$

We take now $|u| \leqq R$ and consider only those terms in the sum for which $|\Omega| > 2R$. This means the omission of only a finite number of terms and has no influence on the question of convergence. For $|v| \leqq R$ and $|\Omega| > 2R$ we have

$$|v - \Omega| > \frac{|\Omega|}{2}$$

and thus

$$\sum_{|\Omega| > 2R} \left| \frac{1}{(u-\Omega)^2} - \frac{1}{\Omega^2} \right| < 16R \sum_{|\Omega| > 2R} \frac{1}{|\Omega|^3} ,$$

which shows the convergence, according to the lemma. Differentiation of (58.4) gives

$$\wp'(u) = -\frac{2}{u^3} - \sum_{m_1, m_2}{}' \frac{2}{(u - \Omega)^3} = -2 \sum_{\Omega} \frac{1}{(u-\Omega)^3} .$$

Now the lattice $\{\Omega\}$ goes over into itself by the process $\Omega \to \Omega - \omega_1$, $\Omega \to \Omega - \omega_2$. The last equation shows therefore

$$\wp'(u + \omega_1) = \wp'(u) , \quad \wp'(u + \omega_2) = \wp'(u) .$$

We conclude therefore

$$\wp(u+\omega_1)-\wp(u)=C_1\,,\quad \wp(u+\omega_2)-\wp(u)=C_2\,.$$

Now the constant C_1 can be found by the special choice $u=-\omega_1/2$:

$$C_1=\wp\left(\frac{\omega_1}{2}\right)-\wp\left(-\frac{\omega_1}{2}\right).$$

The definition (58.4) shows immediately that $\wp(u)$ is an even function since the lattice $\{\Omega\}$ is the same as $\{-\Omega\}$. Therefore we have $C_1=0$, and for the same reason $C_2=0$, and thus

$$\wp(u+\omega_1)=\wp(u)\,,\quad \wp(u+\omega_2)=\wp(u)\,.$$

The function $\wp(u)$ is *doubly periodic*, with the periods ω_1 and ω_2.

The parallelogram of vertices 0, ω_1, ω_2, $\omega_1+\omega_2$ is a *fundamental region* of $\wp(u)$, in which the function takes all values it can take at all, and the values at other points can be obtained from those in the fundamental parallelogram by adding suitable multiples of ω_1 and ω_2.

59. Expansion of $\wp(u)$ in a Laurent series

By means of

$$\frac{1}{(u-\Omega)^2}=\frac{1}{\Omega^2\left(1-\dfrac{u}{\Omega}\right)^2}=\frac{1}{\Omega^2}\left(1+2\frac{u}{\Omega}+3\frac{u^2}{\Omega^2}+\cdots\right)$$

for $|u|<|\Omega|$ we find

$$\wp(u)=\frac{1}{u^2}+\sum_{\Omega}{}'\left\{2\frac{u}{\Omega^3}+3\frac{u^2}{\Omega^4}+\cdots\right\},$$

valid for $|u|<\varrho$, where

$$\varrho=\min_{\Omega\neq 0}|\Omega|\,. \tag{59.1}$$

Interchanging of the two summations is permitted, at least for $|u|<d$, since $d<\varrho$ according to (58.31), and we obtain

$$\wp(u)=\frac{1}{u^2}+2u\sum_{\Omega}{}'\frac{1}{\Omega^3}+3u^2\sum_{\Omega}{}'\frac{1}{\Omega^4}+\cdots.$$

We make use of the definition (58.1) of the Eisenstein series, observing at the same time that $G_r = 0$ for odd r, and obtain

$$\wp(u) = \frac{1}{u^2} + 3G_4(\omega_1, \omega_2)u^2 + 5G_6(\omega_1, \omega_2)u^4 + \cdots \tag{59.2}$$

This expansion, as a Laurent series about the pole $u = 0$, is convergent to the nearest pole, i.e. for $|u| < \varrho$.

Differentiation of (59.2) leads to

$$\wp'(u) = -\frac{2}{u^3} + 2\cdot 3G_4(\omega_1, \omega_2)u + 4\cdot 5G_6(\omega_1, \omega_2)u^3 + \cdots .$$

We construct now a combination of $\wp(u)$ and $\wp'(u)$ which will have no pole at $u = 0$. First we have

$$\begin{aligned}\wp'(u)^2 - 4\wp(u)^3 &= \frac{4}{u^6} - \frac{24G_4}{u^2} - 80G_6 + O(u^2)\\ &\quad - 4\left\{\frac{1}{u^6} + \frac{9G_4}{u^2} + 15G_6 + O(u^2)\right\}\\ &= -\frac{60G_4}{u^2} - 140G_6 + O(u^2)\end{aligned}$$

and then

$$H(u) = \wp'(u)^2 - 4\wp(u)^3 + 60G_4\,\wp(u) + 140G_6 = O(u^2)\,. \tag{59.3}$$

Now $H(u)$ is regular at $u = 0$. Since $H(u)$ has still the two periods ω_1, ω_2 it is regular also at all $\Omega = m_1\omega_1 + m_2\omega_2$ and thus regular in the whole plane. After a theorem of Liouville $H(u)$ must then be a constant, since it is bounded in the compact fundamental parallelogram and thus in the whole plane. The constant can only be zero, since the power series on the right-hand side of (59.3) is 0 for $u = 0$. We have thus the Weierstrass differential equation

$$\wp'(u)^2 = 4\wp(u)^3 - 60G_4(\omega_1, \omega_2)\,\wp(u) - 140G_6(\omega_1, \omega_2)\,. \tag{59.4}$$

The coefficients here are usually denoted as

$$g_2(\omega_1, \omega_2) = 60G_4(\omega_1, \omega_2)\,, \quad g_3(\omega_1, \omega_2) = 140G_6(\omega_1, \omega_2)\,.$$

Through differentiation we obtain

$$2\wp'(u)\wp''(u) = \wp'(u)\left(12\wp(u)^2 - 60G_4\right)$$

and consequently

$$\wp''(u) = 6\wp(u)^2 - 30G_4\,. \tag{59.5}$$

Expressed in terms of the Laurent series (59.2) this means, after division by 6

$$\frac{1}{u^4} + 5G_4 + \sum_{k=2}^{\infty}\binom{2k-1}{3}G_{2k}u^{2k-4} = \left\{\frac{1}{u^2} + \sum_{l=2}^{\infty}(2l-1)G_{2l}u^{2l-2}\right\}^2,$$

and after some cancellations and multiplication by u^4,

$$\sum_{k=4}^{\infty}\frac{1}{3}(2k+1)(2k-1)(k-3)G_{2k}u^{2k}$$

$$= \sum_{k=4}^{\infty}u^{2k}\sum_{l=2}^{k-2}(2l-1)(2k-2l-1)G_{2l}G_{2(k-l)}\,.$$

A comparison of equal powers of u gives the sequence of relations

$$\frac{1}{3}(2k+1)(2k-1)(k-3)G_{2k}$$

$$= \sum_{l=2}^{k-2}(2l-1)(2k-2l-1)G_{2l}G_{2(k-l)}\,, \qquad k = 4, 5, \ldots. \tag{59.6}$$

This shows that G_{2k} is a polynomial in $G_4, G_6, \ldots, G_{2k-4}$ with rational coefficients, and by induction we infer the

Theorem. *The Eisenstein series $G_{2k}(\omega_1, \omega_2)$ are polynomials in G_4, G_6 with rational coefficients.*

The first few examples are, from (59.6),

$$G_8 = \frac{3}{7}G_4^2\,, \qquad G_{10} = \frac{5}{11}G_4G_6\,,$$

$$G_{12} = \frac{1}{11.13}(42G_4G_8 + 25G_6^2) = \frac{1}{11.13}(18G_4^3 + 25G_6^2)\,.$$

60. Lambert series

We write (58.1) for $r = 2k$ in the following form

$$G_{2k}(\omega_1, \omega_2) = 2\sum_{m_1=1}^{\infty}\frac{1}{(m_1\omega_1)^{2k}} + \sum_{m_2}{}'\sum_{m_1}\frac{1}{(m_1\omega_1 + m_2\omega_2)^{2k}}$$

$$= 2\sum_{m_1=1}^{\infty}\frac{1}{(m_1\omega_1)^{2k}} + 2\sum_{m_2=1}^{\infty}\sum_{m_1}\frac{1}{(m_1\omega_1 - m_2\omega_2)^{2k}}\,,$$

which results from the fact that $-m_1$ runs through all integers as m_1 does. We introduce now the notation, to be kept in the sequel,

$$\tau = \frac{\omega_2}{\omega_1} \tag{60.11}$$

and assume

$$\operatorname{Im}(\tau) > 0 . \tag{60.12}$$

This is possible since τ is not real, and can always be achieved by interchanging ω_1 and ω_2 if needed. Then we have

$$G_{2k}(\omega_1, \omega_2) = \frac{2}{\omega_1^{2k}} \left\{ \zeta(2k) + \sum_{m_2=1}^{\infty} \sum_{m_1} \frac{1}{(m_1 - m_2\tau)^{2k}} \right\}.$$

Now

$$\sum_{m_1=-\infty}^{\infty} \frac{1}{(m_1 - m_2\tau)^{2k}} = (-1)^k \sum_{m_1} \frac{1}{(m_2\tau/i + m_1 i)^{2k}} ,$$

which permits the application of Lipschitz's formula (37.1), since

$$\operatorname{Re}\left(m_2, \frac{\tau}{i}\right) > 0 .$$

We obtain then, if we also observe (9.1),

$$G_{2k}(\omega_1, \omega_2) = (-1)^k \frac{2(2\pi)^{2k}}{(2k-1)!\,\omega_1^{2k}}$$

$$\times \left\{ -\frac{B_{2k}}{4k} + \sum_{m_2=1}^{\infty} \sum_{m=1}^{\infty} m^{2k-1} e^{2\pi i m m_2 \tau} \right\}.$$

The double sum here is

$$S = \sum_{m_2=1}^{\infty} \sum_{m=1}^{\infty} m^{2k-1} x^{m_2 m}$$

with the abbreviation

$$x = e^{2\pi i \tau} , \tag{60.2}$$

where the series for $|x| < 1$ is absolutely convergent. We treat it in two ways. If we carry out the summation over m_2 we obtain

$$S = \sum_{m=1}^{\infty} m^{2k-1} \frac{x^m}{1-x^m} . \tag{60.3}$$

Or we can arrange the terms according to $n = m_2 m$ and have

$$S = \sum_{n=1}^{\infty} \sigma_{2k-1}(n)\, x^n , \tag{60.41}$$

where

$$\sigma_r(n) = \sum_{m|n}^{\infty} m^r. \tag{60.42}$$

is the sum of the r-th powers of the divisors of n. The series in (60.3) is called a Lambert series. This one is clearly convergent for $|x| < 1$. Through (60.41) we obtain

$$G_{2k}(\omega_1, \omega_2) = (-1)^k \frac{2\,(2\pi)^{2k}}{(2k-1)!\,\omega_1^{2k}} \left\{-\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n)\, x^n\right\}. \tag{60.5}$$

61. Some arithmetical consequences

Let us for the moment, and only in the present context, put[1]

$$\sigma_{2k-1}(0) = -\frac{B_{2k}}{4k}, \tag{61.1}$$

when the expression in { } of (60.5) becomes simply

$$\sum_{n=0}^{\infty} \sigma_{2k-1}(n)\, x^n .$$

Equation (59.6) appears then, by means of (60.5) and after some obvious simplification, as

$$\sum_{n=0}^{\infty} \sigma_{2k-1}(n)\, x^n = 6\,\frac{(2k-2)\,(2k-3)}{(2k+1)\,(k-3)} \sum_{l=2}^{k-2} \binom{2k-4}{2l-2}$$

$$\times \sum_{\mu=0}^{\infty} \sigma_{2l-1}(\mu)\, x^\mu \sum_{\nu=0}^{\infty} \sigma_{2(k-l)-1}(\nu)\, x^\nu ,$$

valid for $k \geqq 4$. Comparing coefficients we obtain

$$\sigma_{2k-1}(n) = 6\,\frac{(2k-2)\,(2k-3)}{(2k+1)\,(k-3)} \sum_{l=2}^{k-2} \binom{2k-4}{2l-2}$$

$$\times \sum_{\nu=0}^{\infty} \sigma_{2l-1}(\nu)\, \sigma_{2(k-l)-1}(n-\nu) . \tag{61.2}$$

Let us first take the case $n = 0$,

$$\sigma_{2k-1}(0) = 6\,\frac{(2k-2)\,(2k-3)}{(2k+1)\,(k-3)} \sum_{l=2}^{k-2} \binom{2k-4}{2l-2} \sigma_{2l-1}(0)\, \sigma_{2(k-l)-1}(0) ,$$

[1] $\sigma_r(0)$ is as such a meaningless symbol, since every integer divides 0.

which is after our convention (61.1) the following quadratic relation among Bernoulli numbers:

$$\frac{B_{2k}}{k} = 3\frac{(k-1)(2k-3)}{(2k+1)(k-3)} \sum_{l=2}^{k-2} \binom{2k-4}{2l-2} \frac{B_{2l}}{l} \frac{B_{2(k-l)}}{k-l}.$$

For $n > 0$ (61.2) takes the following form, $k \geqq 4$,

$$\sigma_{2k-1}(n) = 6\frac{(2k-2)(2k-3)}{(2k+1)(k-3)} \sum_{l=2}^{k-2} \binom{2k-4}{2l-2} \left\{-\frac{B_{2l}}{4l}\sigma_{2k-2l-1}(n) - \frac{B_{2(k-l)}}{4(k-l)}\sigma_{2l-1}(n) + \sum_{\nu=1}^{n-1} \sigma_{2l-1}(\nu)\,\sigma_{2(k-l)-1}(n-\nu)\right\},$$

of which the cases $k = 4$ and 5

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{\nu=1}^{n-1} \sigma_3(\nu)\,\sigma_3(n-\nu),$$

$$11\sigma_9(n) = 21\sigma_5(n) - 10\sigma_3(n) + 5040\sum_{\nu=1}^{n-1} \sigma_3(\nu)\,\sigma_5(n-\nu)$$

may be the most striking ones [59], pp. 136—162, esp. p. 146. The latter one has the curious consequence

$$\sigma_3(n) - \sigma_5(n) + 2\sum_{\nu=1}^{n-1} \sigma_3(\nu)\,\sigma_5(n-\nu) \equiv 0 \pmod{11}.$$

62. Modular forms

The functions $G_{2k}(\omega_1, \omega_2)$ are expressed by sums which are absolutely convergent. Any arrangement of the terms is permitted, and important is only that the sum is extended over all lattice points Ω. Now the point lattice $\{\Omega\}$ is determined by the generating vectors ω_1, ω_2, but the lattice in turn does not uniquely define its generating vectors.

Let ω_1', ω_2' be two periods in the lattice $\{\Omega\}$. We assume again

$$\operatorname{Im}\left(\frac{\omega_2'}{\omega_1'}\right) > 0. \tag{62.1}$$

Now the fact that ω_1', ω_2' are in $\{\Omega\}$ is expressed by

$$\omega_1' = a\omega_1 + b\omega_2, \quad \omega_2' = c\omega_1 + d\omega_2 \tag{62.2}$$

with certain integers a, b, c, d. Assume now that ω_1', ω_2' play also the role of generating vectors. Then all Ω would be expressible by them and in particular

$$\omega_1 = A\omega_1' + B\omega_2', \quad \omega_2 = C\omega_1' + D\omega_2',$$

hence

$$0 = (Aa + Bc - 1)\omega_1 + (Ab + Bd)\omega_2\,,$$

$$0 = (Ca + Dc)\omega_1 + (Cb + Dd - 1)\omega_2\,.$$

Since ω_1, ω_2 are linearly independent we conclude

$$Aa + Bc = 1\,, \quad Ab + Bd = 0\,,$$

$$Ca + Dc = 0\,, \quad Cb + Dd = 1$$

or, in short,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and consequently

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix}\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1\,.$$

All matrix elements and thus the determinants are integers, so that we can infer

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm 1\,,$$

and this is also sufficient for a pair of generating vectors. However, the condition (62.1) determines also the sign. We write

$$\tau = \frac{\omega_2}{\omega_1}\,, \quad \tau' = \frac{\omega_2'}{\omega_1'}$$

and have from (62.2)

$$\tau' = \frac{d\tau + c}{b\tau + a} = \frac{(d\tau + c)\,(b\bar{\tau} + a)}{|b\tau + a|^2} = \frac{db\tau\bar{\tau} + ac + ad\tau + bc\bar{\tau}}{|b\tau + a|^2}\,.$$

Therefore

$$\operatorname{Im}(\tau') = \frac{(ad - bc)\operatorname{Im}(\tau)}{|b\tau + a|^2}\,,$$

and in view of (60.12), (62.1) we have

$$ad - bc > 0\,.$$

A substitution (62.2) with integers a, b, c, d and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$ is called a *homogeneous modular substitution*. These substitutions, as is easily seen from the composition of their matrices, form a group, the *modular group*.

We have

$$G_{2k}(\omega_1, \omega_2) = G_{2k}(\omega_1', \omega_2') \tag{62.3}$$

since both expressions are the sum

$$\sum{}' \frac{1}{\Omega^{2k}}$$

over all lattice points $\Omega \neq 0$. An Eisenstein series $G_{2k}(\omega_1, \omega_2)$ is therefore an invariant of the (homogeneous) modular group. Moreover, the definition of G_{2k} shows

$$G_{2k}(\lambda\omega_1, \lambda\omega_2) = \lambda^{-2k} G_{2k}(\omega_1, \omega_2) . \tag{62.4}$$

A function with these two properties (viz.,invariance and homogeneity) is called a *modular form of dimension*[1] $-2k$.

63. Definition of $G_2(\omega_1, \omega_2)$

We know that the series

$$\sum_{m_1, m_2}{}' \frac{1}{(m_1\omega_1 + m_2\omega_2)^2}$$

is not absolutely convergent. Its value therefore depends on the order of summation. But under a modular substitution just any order of summation (e.g., first with respect to m_2, then m_1) would be thoroughly disturbed, and the above series would not lead to a useful definition.

There is one way out, which has been followed by Hurwitz [25], vol. 1, pp. 23 ff., to define G_2 by a prescribed order of summation and to investigate the change of the sum produced by a modular substitution. But in this way we obviously lose the fundamental invariance property (62.3).

We follow another way, which was suggested by Hecke in some of his papers [22], pp. 468 ff., although the analytical procedure is a little involved. The salient point is to introduce a summability factor and a limit process.

We introduce the auxiliary function

$$G(\omega_1, \omega_2; s) = \sum_{m_1, m_2}{}' \frac{1}{(m_1\omega_1 + m_2\omega_2)^2 \, |m_1\omega_1 + m_2\omega_2|^s} \tag{63.1}$$

for $s > 0$ and *define* then

$$G_2(\omega_1, \omega_2) = \lim_{s \to 0} G_2(\omega_1, \omega_2; s) . \tag{63.2}$$

Here, because of absolute convergence, with (62.2)

$$G_2(\omega_1, \omega_2; s) = G_2(\omega_1', \omega_2'; s)$$

[1] Now usually called *degree*. Ed.

is evident and therefore also

$$G_2(\omega_1, \omega_2) = G_2(\omega_1', \omega_2'), \tag{63.21}$$

provided the limit exists. Moreover, by the same reasoning,

$$G_2(\lambda\omega_1, \lambda\omega_2) = \lambda^{-2} G_2(\omega_1, \omega_2)$$

is clear. We have to realize, however, that $G_2(\omega_1, \omega_2; s)$ is not analytic in $\tau = \omega_2/\omega_1$ because of the appearance of $|m_1\omega_1 + m_2\omega_2|^s$.

We have from (63.1)

$$\omega_1^2 |\omega_1|^s G_2(\omega_1, \omega_2; s) = G_2(1, \tau; s) = 2 \sum_{m_1=1}^{\infty} \frac{1}{m_1^{2+s}}$$
$$+ 2 \sum_{m_2=1}^{\infty} \sum_{m_1} \frac{1}{(m_1 + m_2\tau)^2 |m_1 + m_2\tau|^s}, \tag{63.3}$$

where m_1 in the inner sum runs over all integers. In order to study this inner sum we introduce

$$\psi(z; s) = \sum_{m_1} \frac{1}{(m_1 + z)^2 |m_1 + z|^s}, \quad \operatorname{Im}(z) > 0. \tag{63.41}$$

Poisson's summation formula (Theorem A in § 35) is here applicable with the result

$$\psi(z; s) = \sum_{n=-\infty}^{\infty} A_n(z; s) \tag{63.42}$$

with

$$A_n(z; s) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i n v}}{(v + z)^2 |v + z|^s} dv = i \int_{-\infty}^{\infty} \frac{e^{2\pi n i t}}{(\zeta + it)^2 |\zeta + it|^s} i dt,$$

where we have set $v = -t$ and

$$-iz = \zeta, \quad \operatorname{Re}(\zeta) > 0. \tag{63.43}$$

Through the substitution $\zeta + it = w$ we obtain

$$A_n(z; s) = i e^{-2\pi n \zeta} \int_{\varrho - i\infty}^{\varrho + i\infty} \frac{e^{2\pi n w}}{w^2 |w|^s} dw,$$

where

$$\varrho = \operatorname{Re}(\zeta) > 0. \tag{63.44}$$

Now $|w|^2 = w\overline{w}$, and on the path of integration we have $\operatorname{Re}(w) = \varrho$ and thus $\overline{w} = 2\varrho - w$. This leads to

$$A_n(z; s) = i e^{-2\pi n\zeta} \int\limits_{\varrho-i\infty}^{\varrho+i\infty} \frac{e^{2\pi n w}}{w^{2+s/2}(2\varrho - w)^{s/2}}\, dw$$

$$= i\frac{e^{-2\pi n\zeta}}{\varrho^{s+1}} \int\limits_{1-i\infty}^{1+i\infty} \frac{e^{2\pi n\rho w}}{w^{2+s/2}(2 - w)^{s/2}}\, dw \tag{63.5}$$

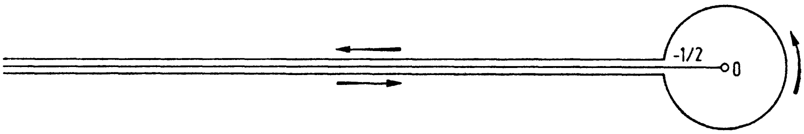
with w in a new meaning. This integral represents a function which is regular for $\operatorname{Re}(s) > -1$.

We now distinguish three cases.

(I) $n > 0$. In this case we can bend the path of integration into a loop around the negative real axis. (This really involves estimating the integrand on arcs of a large circle $|w| = R$). One obtains

$$A_n(z; s) = i\frac{e^{-2\pi n\zeta}}{\varrho^{s+1}} \int\limits_{-\infty}^{(0^+)} \frac{e^{2\pi n\rho w}}{w^{2+s/2}(2 - w)^{s/2}}\, dw\,, \tag{63.51}$$

which shows that $A_n(z; s)$ is actually an entire function of s (we have restricted s to positive real values).



If we specify the loop as the figure indicates we obtain the following estimate:

$$|A_n(z; s)| \leqq \frac{2e^{-2\pi n\rho}}{\varrho^{s+1}} \int\limits_{1/2}^{\infty} \frac{e^{-2\pi n\rho u}\, du}{u^{s/2+2}(2 + u)^{s/2}} + \frac{e^{-2\pi n\rho}}{\varrho^{s+1}} \frac{e^{\pi n\rho}}{\left(\frac{1}{2}\right)^{s/2+2}\left(\frac{3}{2}\right)^{s/2}}\pi$$

$$< \frac{Ce^{-3\pi n\rho}}{n\varrho^{s+1}} + \frac{Ce^{-\pi n\rho}}{\varrho^{s+1}}\left(\frac{4}{3}\right)^{s/2} < C\frac{e^{-\pi n\rho}}{\varrho^{s+1}}\left(\frac{4}{3}\right)^{s/2},$$

where C designates certain, not necessarily the same, constants.

(II) $n = 0$. We have

$$A_0(z; s) = \frac{i}{\varrho^{s+1}} \int\limits_{1-i\infty}^{1+i\infty} \frac{1}{w^{2+s/2}(2 - w)^{s/2}}\, dw$$

and thus

$$|A_0(2; s)| \leqq \frac{1}{\varrho^{s+1}} \int_{-\infty}^{\infty} \frac{dv}{(1+v^2)^{1+s/2}} \leqq \frac{\pi}{\varrho^{s+1}} \quad \text{for } 0 \leqq s\,.$$

(III) $n < 0$. In this case we bend the path into a loop around the real axis from $+2$ to $+\infty$, so that

$$A_n(z; s) = i\,\frac{e^{-2\pi n\zeta}}{\varrho^{s+1}} \int_{\infty}^{(2^-)} \frac{e^{-2\pi|n|\varrho w}\,dw}{w^{2+s/2}(2-w)^{s/2}}\,,$$

which is again an entire function of s. Here the w-plane is supposed to be cut from 2 to ∞ and also from 0 to $-\infty$, and $[w(2-w)]^{s/2}$ is that branch which is positive for $0 < w < 2$.

Specializing now the loop in the following way: w runs from ∞ to $5/2$ on the lower bank of the cut, then around 2 on a circle of radius $1/2$ in the negative sense and then from $5/2$ to ∞ on the upper bank of the cut, we obtain the estimate

$$|A_n(z; s)| \leqq 2\,\frac{e^{2\pi|n|\rho}}{\varrho^{1+s}} \int_{5/2}^{\infty} \frac{e^{-2\pi|n|\rho u}}{u^{2+s/2}(u-2)^{s/2}}\,du + \pi\,\frac{e^{2\pi|n|\rho}}{\varrho^{1+s}}$$
$$\times \frac{e^{-3\pi|n|\varrho}}{\left(\frac{3}{2}\right)^{2+s/2}\left(\frac{1}{2}\right)^{s/2}} < C\,\frac{e^{-\pi|n|\varrho}}{\varrho^{1+s}}\left(\frac{4}{3}\right)^{s/2}, \qquad s \geqq 0\,.$$

Putting together our results from (63.2) on we obtain

$$\begin{aligned}\omega_1^2 G_2(\omega_1, \omega_2) &= 2\zeta(2) + 2\lim_{s\to 0} \sum_{m_2=1}^{\infty} \psi(m_2\tau; s)\\ &= \frac{\pi^2}{3} + 2\lim_{s\to 0}\left\{\sum_{m_2=1}^{\infty}\sum_{n=1}^{\infty} A_n(m_2\tau; s) + \sum_{m_2=1}^{\infty} A_0(m_2\tau; s)\right.\\ &\qquad\left. + \sum_{m_2=1}^{\infty}\sum_{n=-1}^{-\infty} A_n(m_2\tau; s)\right\},\end{aligned}$$

provided the three sums converge separately. This they actually do, even absolutely for $s > 0$. Indeed, putting

$$\tau = \alpha + i\beta\,, \qquad \beta > 0$$

we have for $z = m_2\tau$ in view of (63.43), (63.44), that $\zeta = -m_2 i\tau$, $\varrho = m_2\beta$, and the sums in $\{\ \}$ are majorized by

$$C\sum_{m_2=1}^{\infty}\sum_{n=1}^{\infty} \frac{e^{-\pi n m_2\beta}}{(m_2\beta)^{s+1}}\left(\frac{4}{3}\right)^s + C\sum_{m_2=1}^{\infty} \frac{1}{(m_2\beta)^{s+1}}$$
$$+ C\sum_{m_2=1}^{\infty}\sum_{n=-1}^{-\infty} \frac{e^{-\pi|n|m_2\beta}}{(m_2\beta)^{s+1}}\left(\frac{4}{3}\right)^{s/2}.$$

The double sums are uniformly convergent in s for $0 \leqq s \leqq 1$. We can therefore perform the limit $s \to 0$ on these double sums separately and obtain thus

$$\omega_1^2 G_2(\omega_1, \omega_2) = \frac{\pi^2}{3} + 2 \sum_{m_2=1}^{\infty} \sum_{n=1}^{\infty} A_n(m_2\tau; 0)$$

$$+ 2 \sum_{m_2=1}^{\infty} \sum_{n=-1}^{-\infty} A_n(m_2\tau; 0) + 2 \lim_{s\to 0} \sum_{m_2=1}^{\infty} A_0(m_2\tau; s) .$$

We have seen that the $A_n(m_2\tau; s)$ for $n \neq 0$ are entire functions. The integral (63.5) represents them in its domain of convergence, which is the half-plane $\operatorname{Re}(s) > -1$. We have thus, for $n \neq 0$

$$A_n(m_2\tau; 0) = i \frac{e^{2\pi i n m_2 \tau}}{m_2\beta} \int_{1-i\infty}^{1+i\infty} \frac{e^{2\pi n m_2 \beta w}}{w^2} dw .$$

Now integrals of this sort have been investigated in the proof of § 37. We obtain here in the same manner

$$A_n(m_2\tau; 0) = \begin{cases} 0 & \text{for } n < 0 , \\ -(2\pi)^2 n e^{2\pi i n m_2 \tau} & \text{for } n \geqq 1 \end{cases}$$

and have thus

$$\omega_1^2 G_2(\omega_1, \omega_2) = \frac{\pi^2}{3} - 2(2\pi)^2 \sum_{m_2=1}^{\infty} \sum_{n=1}^{\infty} n e^{2\pi i n m_2 \tau}$$

$$+ 2 \lim_{s\to 0} \sum_{m_2=1}^{\infty} \frac{i}{(m_2\beta)^{s+1}} \int_{1-i\infty}^{1+i\infty} \frac{dw}{w^{2+s/2}(2-w)^{s/2}}$$

$$= \frac{\pi^2}{3} - 8\pi^2 \sum_{\nu=1}^{\infty} \sigma_1(\nu) e^{2\pi i \nu\tau} + S , \tag{63.6}$$

say. Here

$$S = \frac{2}{\beta} \lim_{s\to 0} \zeta(s+1)\, i \int_{1-i\infty}^{1+i\infty} \frac{dw}{w^{2+s/2}(2-w)^{s/2}}$$

$$= \frac{2}{\beta} \lim \left(\frac{1}{s} + O(1)\right) \frac{1}{i} I(s) . \tag{63.7}$$

As under (III) the path of integration of $I(s)$ can be bent into a loop around $(2, \infty)$:

$$\frac{1}{i} I(s) = \frac{1}{i} \int_{\infty}^{(2^-)} \frac{dw}{w^{2+s/2}(2-w)^{s/2}} .$$

We take here $0 < s \leqq 1$. The loop we specify as running on the lower bank of the cut along $(2, \infty)$ from ∞ to $2 + \varepsilon$, then in circle of radius ε around 2 in the negative sense and then on the upper bank from $2 + \varepsilon$ back to ∞. This gives, with the proper evaluation of the branches of $(2 - w)^{s/2}$

$$\frac{1}{i} I(s) = \frac{1}{i} \int_{\infty}^{2+\varepsilon} \frac{du}{u^{2+s/2}(u-2)^{s/2} e^{(s/2)\pi i}} + \frac{1}{i} \int_{|w-2|=\varepsilon}^{(-)} \frac{dw}{w^{2+s/2}(2-w)^{s/2}}$$

$$+ \frac{1}{i} \int_{2+\varepsilon}^{\infty} \frac{du}{u^{2+s/2}(u-2)^{s/2} e^{-(s/2)\pi i}} .$$

Here the second integral can be estimated as

$$\left| \int_{|w-2|=\varepsilon} \right| \leqq \frac{2\pi\varepsilon}{(2-\varepsilon)^{2+s/2}\varepsilon^{s/2}} \to 0 \text{ as } \varepsilon \to 0$$

for $0 < s \leqq 1$. The other integrals permit $\varepsilon \to 0$ with the result

$$\frac{1}{i} I(s) = 2 \sin \frac{\pi s}{2} \int_{2}^{\infty} \frac{du}{u^{2+s/2}(u-2)^{s/2}} = 2^{-s} \sin \frac{\pi s}{2} \int_{2}^{\infty} \frac{dx}{(1+x)^{2+s/2} x^{s/2}}$$

$$= 2^{-s} \sin \frac{\pi s}{2} \frac{\Gamma\left(1 - \frac{s}{2}\right) \Gamma(1+s)}{\Gamma\left(2 + \frac{s}{2}\right)}$$

after (28.62). This inserted in (63.7) yields

$$S = -\frac{2}{\beta} \frac{\pi}{2} \frac{\Gamma(1)\,\Gamma(1)}{\Gamma(2)} = -\frac{\pi}{\operatorname{Im}(\tau)} = \frac{-2\pi i}{\tau - \bar{\tau}}$$

and (63.6) goes over into

$$\omega_1^2 G_2(\omega_1, \omega_2) = \frac{\pi^2}{3} - 8\pi^2 \sum_{\nu=1}^{\infty} \sigma_1(\nu)\, e^{\pi i \nu \tau} - \frac{2\pi i}{\tau - \bar{\tau}}, \tag{63.81}$$

and with $x = e^{2\pi i \tau}$

$$G_2(\omega_1, \omega_2) = -\frac{2(2\pi)^2}{\omega_1^2} \left\{ -\frac{B_2}{4} + \sum_{n=1}^{\infty} \sigma_1(n)\, x^n \right\} - \frac{2\pi i}{\omega_1^2(\tau - \bar{\tau})}, \tag{63.82}$$

where $B_2 = 1/6$ has been observed. This formula is in complete analogy with (60.5), with the exception of the last summand. Our definition of $\omega_1^2 G_2(\omega_1, \omega_2)$ does not lead to an analytic function since a term in $\tau - \bar{\tau}$ appears.

64. The modular invariance of $G_2(\omega_1, \omega_2)$

The definition of $G_2(\omega_1, \omega_2)$ together with the proof of the existence of the limits ensures now (63.21), which we can write after (63.82) as

$$\frac{2\pi i}{\omega_1^2}\left\{\frac{1}{24} - \sum_{n=1}^{\infty} \frac{m\, e^{2\pi i n\tau}}{1 - e^{2\pi i n\tau}}\right\} + \frac{1}{2\omega_1^2(\tau - \bar{\tau})}$$

$$= \frac{2\pi i}{\omega_1'^2}\left\{\frac{1}{24} - \sum_{n=1}^{\infty} \frac{n\, e^{2\pi i n\tau'}}{1 - e^{2\pi i n\tau'}}\right\} + \frac{1}{2\omega_1'^2(\tau' - \bar{\tau}')}, \tag{64.1}$$

where we have introduced the Lambert series

$$\sum_{n=1}^{\infty} \frac{n\, x^n}{1 - x^n} = \sum_{n=1}^{\infty} \sigma_1(n)\, x^n$$

and where

$$\tau = \frac{\omega_2}{\omega_1}, \qquad \tau' = \frac{\omega_2'}{\omega_1'}.$$

We prefer from now on to write the homogeneous modular substitution as

$$\omega_2' = a\omega_2 + b\omega_1, \qquad \omega_1' = c\omega_2 + d\omega_1, \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1 \tag{64.21}$$

instead of (62.2). In this way we obtain

$$\tau' = \frac{a\tau + b}{c\tau + d}, \tag{64.22}$$

an inhomogeneous modular substitution. The inhomogeneous modular substitutions form again a group Γ, homomorphic but not isomorphic to the homogeneous group H. Indeed

$$\Gamma \cong H/E,$$

where E consists of the two elements

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

represent the same inhomogeneous modular substitution (64.22).

We have now from (64.21)

$$\frac{\omega_1'}{\omega_1} = c\tau + d,$$

so that (64.1) takes the form

$$2\pi i\left\{\frac{1}{24}-\sum_{n=1}^{\infty}\frac{n\,e^{2\pi i n\tau}}{1-e^{2\pi i n\tau}}\right\}=\frac{2\pi i}{(c\tau+d)^2}\left\{\frac{1}{24}-\sum_{n=1}^{\infty}\frac{n\,e^{2\pi i n\tau'}}{1-e^{2\pi i n\tau'}}\right\}$$
$$+\frac{1}{2}\left\{\frac{1}{(\tau'-\bar\tau')\,(c\tau+d)^2}-\frac{1}{\tau-\bar\tau}\right\}. \tag{64.3}$$

Now we observe

$$\frac{d\tau'}{d\tau}=\frac{a(c\tau+d)-c(a\tau+b)}{(c\tau+d)^2}=\frac{1}{(c\tau+d)^2}\,.$$

Moreover we have

$$\frac{1}{(\tau'-\bar\tau')\,(c\tau+d)^2}=\frac{1}{(c\tau+d)^2\left(\dfrac{a\tau+b}{c\tau+d}-\dfrac{a\bar\tau+b}{c\bar\tau+d}\right)}$$
$$=\frac{c\bar\tau+d}{(c\tau+d)\,(\tau-\bar\tau)}$$

and thus

$$\frac{1}{(\tau'-\bar\tau')\,(c\tau+d)^2}-\frac{1}{\tau-\bar\tau}=\frac{1}{\tau-\bar\tau}\left(\frac{c\bar\tau+d}{c\tau+d}-1\right)=-\frac{c}{c\tau+d}\,.$$

By virtue of these remarks we can replace (64.3) by

$$2\pi i\left\{\frac{1}{24}-\sum_{n=1}^{\infty}\frac{n\,e^{2\pi i n\tau}}{1-e^{2\pi i n\tau}}\right\}+\frac{1}{2}\,\frac{c}{c\tau+d}$$
$$=2\pi i\left\{\frac{1}{24}-\sum_{n=1}^{\infty}\frac{n\,e^{2\pi i n\tau'}}{1-e^{2\pi i n\tau'}}\right\}\frac{d\tau'}{d\tau} \tag{64.4}$$

as an expression of the invariance of $G_2(\omega_1, \omega_2)$ under modular transformations.

65. Dedekind function $\eta(\tau)$ and the discriminant $\Delta(\tau)$

If we integrate (64.4) with respect to τ, we obtain

$$\frac{\pi i\tau}{12}+\sum_{n=1}^{\infty}\log\left(1-e^{2\pi i n\tau}\right)+\frac{1}{2}\log\left(c\tau+d\right)+K_1$$
$$=\frac{\pi i\tau'}{12}+\sum_{n=1}^{\infty}\log\left(1-e^{2\pi i n\tau'}\right). \tag{65.1}$$

Here K_1 is a constant of integration, which can depend only on the modular substitution (64.22) after we have agreed that the logarithm appearing on both sides should be taken as the principal value

$$\log(1 - e^{2\pi i n\tau}) = -\sum_{k=1}^{\infty} \frac{1}{k} e^{2\pi i n k\tau}.$$

In order to deal always with the principal branch of the logarithm we choose for $c \neq 0$ always $c > 0$ and in (65.1) we replace $\log(c\tau + d)$ by $\log[-i(c\tau + d)]$ and K_1 by K, where now

$$\mathrm{Re}[-i(c\tau + d)] > 0,$$

$$\log[-i(c\tau + d)] = \log|c\tau + d| + \arg[-i(c\tau + d)],$$

$$|\arg(-i(c\tau + d))| < \pi/2,$$

and

$$K_1 = K - \pi i/4.$$

We define now, after Dedekind [6], the function

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau}). \tag{65.2}$$

Going over in (65.1) to exponentials we obtain

$$\eta(\tau') = C\sqrt{-i(c\tau + d)}\,\eta(\tau), \tag{65.31}$$

with a new constant C again depending only on a, b, c, d of (64.22). We notice that for $c = 0$ the term $c/(c\tau + d)$ does not appear in (64.4) and thus not the single logarithmic term on the left side of (65.1). In this case we have $\tau' = \tau + b$ and

$$\eta(\tau') = C_0 \eta(\tau).$$

However here the definition (65.2) gives directly

$$\eta(\tau + b) = e^{\pi i b/12} \eta(\tau), \tag{65.32}$$

i.e.

$$C_0 = e^{\pi i b/12}, \tag{65.33}$$

a 24th root of unity. We shall see that C is always a certain 24th root of unity.

In order to avoid the 24th root of unity we raise (65.31) and (65.32) to the 24th power. We have

$$\eta(\tau')^{24} = C^{24} (c\tau + d)^{12} \eta(\tau)^{24}. \tag{65.41}$$

For the case

(i) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

we have already found $C^{24} = C_0^{24} = 1$ through (65.33). The other simple case is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

or

(ii) $\quad \tau' = -\dfrac{1}{\tau}$.

We have here

$$\eta\left(-\frac{1}{\tau}\right)^{24} = C^{24}\tau^{12}\,\eta(\tau)^{24}. \tag{65.42}$$

In this case (ii) possesses the fixed point $\tau = i$, so that

$$\eta(i)^{24} = C^{24}\,\eta(i)^{24}.$$

Now the definition (65.2) shows that $\eta(\tau)$ does not vanish in the upper half-plane, so that $\eta(i) \neq 0$ and thus $C^{24} = 1$ also in case (ii).

We define now

$$\Delta(\omega_1, \omega_2) = \frac{(2\pi)^{12}}{\omega_1^{12}}\,\eta(\tau)^{24} = \frac{(2\pi)^{12}}{\omega_1^{12}}\,e^{2\pi i\tau}\prod_{n=1}^{\infty}(1 - e^{2\pi i n\tau})^{24} \tag{65.5}$$

(where the multiplicative constant $(2\pi)^{12}$ is introduced to simplify computations in the sequel).

Now $\Delta(\omega_1, \omega_2)$ has the following properties

(I) $\quad \Delta(\lambda\omega_1, \lambda\omega_2) = \lambda^{-12}\Delta(\omega_1, \omega_2)$,

(II) $\quad \Delta(\omega_1, \omega_2 + b\omega_1) = \Delta(\omega_1, \omega_2)$,

(III) $\Delta(\omega_2, -\omega_1) = \Delta(\omega_1, \omega_2)$.

(I) follows from the definition (65.5), (II) from (i) with $C^{24} = 1$, (III) from (ii) with $C^{24} = 1$. These properties together suffice to prove the

Theorem. $\Delta(\omega_1, \omega_2)$ *is a modular form of dimension* -12.

Proof. We have only to show that

$$\Delta(d\omega_1 + c\omega_2, b\omega_1 + a\omega_2) = \Delta(\omega_1, \omega_2) \tag{65.6}$$

for any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer elements and determinant 1.

Now for $c = 0$ this assertion is covered by (II). For $c \neq 0$ we can take $c > 0$ because after (I)

$$\Delta(\omega_1', \omega_2') = \Delta(-\omega_1', -\omega_2').$$

Then the case $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is stated in (III). It is well known that any modular substitution $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be composed of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Indeed, if $|a| < |c|$, we have

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$$

and thus

$$|c_1| = |a| < |c|,$$

which means a reduction of c.

If

$$|a| \geqq |c| \geqq 1$$

we can choose $m \neq 0$ so that

$$|a + mc| < |c|$$

and thus

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + mc & b + md \\ c & d \end{pmatrix}$$

and

$$|a_1| = |a + mc| < |c| \leqq |a|,$$

which means a reduction of a. In this way we arrive by iteration at either $a_l = 0$ or $c_l = 0$. In the latter case the end result is

$$\begin{pmatrix} 1 & b_l \\ 0 & 1 \end{pmatrix} = T S^{m_{l-1}} T \cdots \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which yields $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a product of powers of T and S. For our purpose we can write

$$T^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T \sim T^{-1}$$

because of (I).

If the chain ends with $a_l = 0$ we have

$$\begin{pmatrix} 0 & -1 \\ 1 & d_l \end{pmatrix} = P \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where P is a product of factors T and powers of S. Since

$$TS^{d_l} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d_l \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & d_l \end{pmatrix}$$

we have again expressed $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a product of S and T.

Since now the properties (I), (II), (III) show the invariance of Δ with respect to S and T we have proved the theorem. ▯

With

$$\omega_2' = a\omega_2 + b\omega_1 , \qquad \omega_1' = c\omega_2 + d\omega_1 ,$$

$$\tau' = \frac{\omega_2'}{\omega_1'} = \frac{a\tau + b}{c\tau + d} , \qquad \frac{\omega_1'}{\omega_1} = c\tau + d$$

we can state (65.6) explicitly as

$$\eta(\tau')^{24} = (c\tau + d)^{12} \eta(\tau)^{24} . \tag{65.7}$$

This shows that the constant $C = C(a, b, c, d)$ in (65.31) has the property

$$C^{24} = 1 ,$$

so that C is a certain 24th root of unity.

This constant $C(a, b, c, d)$ plays an important role in the theory of modular functions and its applications to additive number theory. We devote the next chapter to its determination.

Chapter 9

The Transformation of log $\eta(\tau)$ and the Theory of the Dedekind Sums

66. A formula of Iseki

The direct determination of the *multiplicative* constant $C = C(a, b, c, d)$ belongs to the theory of the ϑ-functions and Gaussian sums.

Since Dedekind, however, the task has been shifted to the determination of the *additive* constant K_1 of the equation (65.1), which we rewrite now as follows:

$$\log \eta(\tau') = K + \frac{1}{2} \log\left(\frac{c\tau + d}{i}\right) + \log \eta(\tau) , \qquad c > 0 . \tag{66.1}$$

Here we are taking into account the determination of the branch of the logarithm as explained after (65.1). Since we already know that in

(65.31) $|C| = 1$ we foresee that K must be purely imaginary, as it will indeed turn out.

There are a number of ways known to determine K; see [6; 72], vol. 3, pp. 282—303; 55; 52; 27. We prefer here that one by Iseki, which is based on a formula which is useful by itself. We begin with the

Lemma. *The sums defined by*

$$S^+(z;\alpha,\beta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{e^{2\pi i(n\alpha+m\beta)}}{m(mz+ni)}, \tag{66.21}$$

$$S^-(z;\alpha,\beta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{e^{2\pi i(n\alpha+m\beta)}}{m(mz-ni)} \tag{66.22}$$

for $\mathrm{Re}(z) > 0$, $0 < \alpha < 1$, $0 < \beta < 1$ *satisfy the relation*

$$S^+(z;\alpha,\beta) = S^-\left(\frac{1}{z};\beta,\alpha\right) + \frac{1}{i}\log(1-e^{2\pi i\alpha})\log(1-e^{2\pi i\beta}), \tag{66.23}$$

where log *designates again the principal branch of the logarithm.*

Proof. We prove first the convergence of the sums and begin with the inner sums. Put

$$S_P^Q = \sum_{n=P}^{Q} \frac{e^{2\pi in\alpha}}{mz+ni}.$$

With

$$s(n) = \sum_{\nu=0}^{n} e^{2\pi i\nu\alpha} = \frac{e^{2\pi i(n+1)\alpha}-1}{e^{2\pi i\alpha}-1}$$

and thus

$$|s(n)| \leq \frac{1}{\sin\pi\alpha} = A,$$

we have

$$S_P^Q = \sum_{n=P}^{Q} \frac{s(n)-s(n-1)}{mz+ni} = \sum_P^Q s(n)\left(\frac{1}{mz+ni} - \frac{1}{mz+(n+1)i}\right)$$
$$+ \frac{s(Q)}{mz+(Q+1)i} - \frac{s(P-1)}{mz+Pi},$$

$$|S_P^Q| \leq A\sum_P^Q \left|\frac{1}{mz+ni} - \frac{1}{mz+(n+1)i}\right| + \frac{A}{|mz+(Q+1)i|} + \frac{A}{|mz+Pi|}.$$

Now, with $z = x + iy$, $x > 0$, we have

$$\sum_P^Q \left| \frac{1}{mz + ni} - \frac{1}{mz + (n+1)i} \right| = \sum_P^Q \left| \int_n^{n+1} \frac{dt}{(mz + ti)^2} \right|$$

$$\leqq \int_P^{Q+1} \frac{dt}{(mx)^2 + (t + my)^2}$$

$$= \frac{1}{mx} \int_{\frac{P+my}{mx}}^{\frac{Q+1+my}{mx}} \frac{dt}{1+t^2} < \frac{\pi}{mx}$$

and thus

$$|S_P^Q| < \frac{C}{mx}.$$

Therefore

$$S^+(z; \alpha, \beta) = \sum_{m=1}^{\infty} \frac{e^{2\pi i m\beta}}{m} \sum_{n=1}^{\infty} \frac{e^{2\pi i n\alpha}}{mz + ni}$$

is majorized by

$$C \sum \frac{1}{m} \frac{1}{mx}$$

and thus convergent, and $S^+(z; \alpha, \beta)$ is analytic in the half-plane $\mathrm{Re}(z) > 0$. Similarly for $S^-(z; \alpha, \beta)$.

Let us now for the moment take $z = x$ real, $y = 0$. Then we have, for $1 \leqq P < Q$

$$|S_P^Q| \leqq \frac{A}{mx} \int_{\frac{P}{mx}}^{\infty} \frac{dt}{1+t^2} + \frac{2A}{|mx + Pi|} < \frac{C}{mx},$$

but also

$$|S_P^Q| < \frac{A}{mx}\left(\frac{\pi}{2} - \arctan \frac{P}{mx}\right) + \frac{2A}{|mx + Pi|}$$

$$= \frac{A}{mx} \arctan \frac{mx}{P} + \frac{2A}{|mx + Pi|} < \frac{C}{P}$$

so that

$$|S_P^Q| < C \min\left\{\frac{1}{mx}, \frac{1}{P}\right\} \leqq \frac{2C}{mx + P}. \tag{66.3}$$

We can now write (still maintaining $z = x > 0$)

$$S^+(z;\alpha,\beta) = \lim_{M\to\infty} \sum_{m=1}^{M} \frac{e^{2\pi i\beta m}}{m} \lim_{N\to\infty} S_1^N$$

$$= \lim_{M\to\infty} \sum_{m=1}^{M} \frac{e^{2\pi i\beta m}}{m} S_1^M + \lim_{M\to\infty} \sum_{m=1}^{M} \frac{e^{2\pi i\beta m}}{m} \lim_{M\to\infty} S_{M+1}^N, \tag{66.4}$$

where both sums are separately convergent after the previous test. We have now

$$\left| \sum_{m=1}^{M} \frac{e^{2\pi i\beta m}}{m} \lim_{N\to\infty} S_{M+1}^N \right| = \left| \sum_{m=1}^{M} \frac{e^{2\pi i\beta m}}{m} S_{M+1}^\infty \right|$$

and after (66.3)

$$\leqq 2C \sum_{m=1}^{M} \frac{1}{m(mx + M + 1)} \to 0 \text{ as } M \to \infty.$$

We infer therefore from (66.4)

$$S^+(z;\alpha,\beta) = \lim_{M\to\infty} \sum_{m=1}^{M} \sum_{n=1}^{M} \frac{e^{2\pi i(n\alpha+m\beta)}}{m(mz+ni)}. \tag{66.5}$$

A similar formula holds for S^-.

Now we have

$$\frac{1}{m(mz+ni)} = \frac{1}{mni} + \frac{1}{n(-mi + n/z)}$$

and thus

$$S^+(z;\alpha,\beta) = \lim_{M\to\infty} \sum_{m=1}^{M} \sum_{n=1}^{M} \frac{e^{2\pi i(n\alpha+m\beta)}}{n(n/z - mi)}$$

$$+ \lim_{M\to\infty} \frac{1}{i} \sum_{m=1}^{M} \sum_{n=1}^{M} \frac{e^{2\pi i(n\alpha+m\beta)}}{mn},$$

since the last limit clearly exists.

The formula for S^- corresponding to (66.5) yields then

$$S^+(z;\alpha,\beta) = S^-\left(\frac{1}{z};\beta,\alpha\right) + \frac{1}{i} \sum_{m=1}^{\infty} \frac{e^{2\pi i m\beta}}{m} \sum_{n=1}^{\infty} \frac{e^{2\pi i n\alpha}}{n},$$

which proves (66.23) first for $z = x > 0$. But since $S(z;\alpha,\beta)$ is analytic in $\mathrm{Re}(z) > 0$, as we have seen, the Lemma is fully proved through analytic continuation. $\square$

After this preparation we prove Iseki's formula contained in the

Theorem. *If* $\lambda(\alpha) = -\log(1 - e^{-2\pi\alpha})$ (*principal branch*) *and* $0 < \alpha < 1$, $0 < \beta < 1$, $\mathrm{Re}(z) > 0$ *then*

$$\sum_{q=0}^{\infty} \{\lambda((q+\alpha)z - i\beta) + \lambda((q+1-\alpha)z + i\beta)\} + \pi z B_2(\alpha)$$

$$= \sum_{q=0}^{\infty} \left\{\lambda\left((q+\beta)\frac{1}{z} + i\alpha\right) + \lambda\left((q+1-\beta)\frac{1}{z} - i\alpha\right)\right\}$$

$$+ \frac{\pi}{z} B_2(\beta) + 2\pi i\left(\alpha - \frac{1}{2}\right)\left(\beta - \frac{1}{2}\right), \tag{66.6}$$

where $B_2(x) = x^2 - x + 1/6$ *is the* 2nd *Bernoulli polynominal. The formula* (66.6) *holds also for* $\alpha = 0$ *or* 1, $0 < \beta < 1$ *and* $0 < \alpha < 1$, $\beta = 0$ *or* 1.

Proof. From Lipschitz's formula (37.1) we have for $s = 1$, $0 < \alpha < 1$, $m > 0$, $\mathrm{Re}(z) > 0$

$$2\pi \sum_{q=0}^{\infty} e^{-2\pi m(q+\alpha)z} = \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n\alpha}}{mz + ni}$$

$$= \sum_{n=1}^{\infty} \frac{e^{2\pi i n\alpha}}{mz + ni} + \sum_{n=1}^{\infty} \frac{e^{-2\pi i n\alpha}}{mz - ni} + \frac{1}{mz}.$$

Multiplying both sides by $(1/m)e^{2\pi i m\beta}$, $0 < \beta < 1$, and summing over m, we obtain

$$2\pi \sum_{m=1}^{\infty} \sum_{q=0}^{\infty} \frac{1}{m} e^{-2\pi m((q+\alpha)z - i\beta)}$$

$$= S^{+}(z; \alpha, \beta) + S^{-}(z; 1-\alpha, \beta) + \frac{1}{z} \sum_{m=1}^{\infty} \frac{e^{2\pi i m\beta}}{m^2}, \tag{66.7}$$

in the notation (66.21), (66.22). Let us split the sum on the left-hand side into two parts

$$\sum_{m=1}^{\infty} \sum_{q=1}^{\infty} + \sum_{\substack{m=1 \\ q=0}}^{\infty} = S_1 + S_2.$$

Here S_1 is absolutely and uniformly convergent for $0 \leqq \alpha$, all real β, and thus continuous in that range of α, β. For $0 < \alpha$, S_2 is continuous in β, and for $0 < \beta < 1$

$$S_2 = \sum_{m=1}^{\infty} \frac{e^{2\pi i m\beta}}{m} e^{-2\pi m\alpha z}$$

has a limit for $\alpha \to +0$ in virtue of Abel's limit theorem. The two first terms on the right of (66.7) are only defined for $0 < \alpha < 1$, $0 < \beta < 1$, which we still assume.

On the left-hand side of (66.7) the order of summation can be reversed since S_1 permits it:

$$2\pi \sum_{q=0}^{\infty} \lambda((q+\alpha)z - i\beta) = S^+(z;\alpha,\beta) + S^-(z;1-\alpha,\beta) + \frac{1}{z}\sum_{m=1}^{\infty}\frac{e^{2\pi m\beta}}{m^2}.$$

Replacing here α by $1-\alpha$, β by $1-\beta$ we obtain also

$$2\pi \sum_{q=0}^{\infty} \lambda((q+1-\alpha)z + i\beta) = S^+(z;1-\alpha,1-\beta) + S^-(z;\alpha,1-\beta) + \frac{1}{z}\sum_{m=1}^{\infty}\frac{e^{-2\pi i m\beta}}{m^2}.$$

Addition and observation of (8.62) yields

$$\begin{aligned} 2\pi \sum_{q=0}^{\infty} \lambda((q+\alpha)z - i\beta) &+ 2\pi \sum_{q=0}^{\infty} \lambda((q+1-\alpha)z + i\beta) - \frac{2\pi^2}{z} B_2(\beta) \\ &= S^+(z;\alpha,\beta) + S^-(z;1-\alpha,\beta) \\ &\quad + S^+(z;1-\alpha,1-\beta) + S^-(z;\alpha,1-\beta) \\ &= R(z;\alpha,\beta), \end{aligned} \tag{66.8}$$

say. By means of the Lemma we can transform the right-hand member into

$$\begin{aligned} R(z;\alpha,\beta) &= S^-\left(\frac{1}{z};\beta,\alpha\right) + S^+\left(\frac{1}{z};\beta,1-\alpha\right) \\ &\quad + S^-\left(\frac{1}{z};1-\beta,1-\alpha\right) + S^+\left(\frac{1}{z};1-\beta,\alpha\right) \\ &\quad - i\lambda(-i\alpha)\,\lambda(-i\beta) + i\lambda(-i\alpha)\,\lambda(i\beta) + i\lambda(i\alpha)\,\lambda(-i\beta) \\ &\quad - i\lambda(i\alpha)\,\lambda(i\beta) \\ &= R\left(\frac{1}{z};\beta,1-\alpha\right) \\ &\quad - i\{\lambda(-i\alpha) - \lambda(i\alpha)\}\,\{\lambda(-i\beta) - \lambda(i\beta)\} \\ &= R\left(\frac{1}{z};\beta,1-\alpha\right) + \frac{1}{i}\log\frac{1-e^{2\pi i\alpha}}{1-e^{-2\pi i\alpha}}\log\frac{1-e^{2\pi i\beta}}{1-e^{-2\pi i\beta}} \end{aligned}$$

and finally

$$R(z;\alpha,\beta) = R\left(\frac{1}{z};\beta,1-\alpha\right) + i(2\pi)^2\left(\frac{1}{2}-\alpha\right)\left(\frac{1}{2}-\beta\right). \quad (66.9)$$

If we express here now $R(1/z;\beta,1-\alpha)$ by means of (66.8), simply replacing z by $1/z$, α by β, β by $1-\alpha$, we obtain (66.6), having also observed (2.8).

The extension of (66.6) to the cases $\alpha = 0$ or 1 with $0<\beta<1$ follows from the continuity of all terms, as our discussion of the left-hand member in (66.7) shows. ▯

67. Application of Iseki's formula to the transformation of log $\eta(\tau)$

The modular transformation

$$\tau' = \frac{a\tau+b}{c\tau+d}$$

can be conveniently written as

$$(c\tau+d)(c\tau'-a) = -1\,,$$

so that if we put

$$c\tau+d = iz \quad (67.11)$$

we have

$$c\tau'-a = i/z\,. \quad (67.12)$$

Taking $c>0$ we have here

$$\mathrm{Re}(z)>0\,,\qquad \mathrm{Re}(1/z)>0\,.$$

The transformation formula (66.1) for $\eta(\tau)$ appears with the notation used in the theorem of § 66 as

$$\frac{\pi}{12c}\left(-\frac{1}{z}+ia\right)-\sum_{n=1}^{\infty}\lambda\left(\frac{n}{c}\left(\frac{1}{z}-ia\right)\right)$$
$$=K+\frac{1}{2}\log z+\frac{\pi}{12c}(-z-id)-\sum_{n=1}^{\infty}\lambda\left(\frac{n}{c}(z+id)\right). \quad (67.2)$$

Here K has to be determined. We treat first the case $c=1$, $a=d=0$. For this purpose we put $\beta=0$ in (66.6) and obtain, for $0<\alpha<1$

$$\sum_{q=0}^{\infty}\{\lambda((q+\alpha)z)+\lambda((q+1-\alpha)z)\}+\pi z B_2(\alpha)$$
$$=\sum_{q=0}^{\infty}\left\{\lambda\left(q\frac{1}{z}+i\alpha\right)+\lambda\left((q+1)\frac{1}{z}-i\alpha\right)\right\}+\frac{\pi}{6z}-\pi i\left(\alpha-\frac{1}{2}\right).$$

For $\alpha \to 0$ only the terms arising from $q = 0$ need some attention. We have

$$\lim_{\alpha\to+0} \left(\lambda(\alpha z) - \lambda(\alpha i)\right) = \lim_{\alpha\to+0} \log \frac{1-e^{-2\pi i\alpha}}{1-e^{-2\pi z\alpha}} = \log\frac{i}{z} = \frac{\pi i}{2} - \log z$$

and obtain thus

$$2\sum_{n=1}^{\infty} \lambda(nz) + \frac{\pi z}{6} + \frac{\pi i}{2} - \log z = 2\sum_{n=1}^{\infty} \lambda\left(\frac{n}{z}\right) + \frac{\pi}{6z} + \frac{\pi i}{2}, \quad (67.3)$$

which is (67.2) for $c = 1$, $a = d = 0$, with $K = 0$.

We consider now the case $c > 1$. Let μ be an integer in $1 \leqq \mu \leqq c-1$. We define

$$\mu^* = -d\mu - c\left[\frac{-d\mu}{c}\right]$$

so that $\mu^* \equiv -d\mu \pmod c$ and $1 \leqq \mu^* \leqq c-1$. We notice that μ^* runs in some order through 1 to $c-1$ as μ does. Now we put $\alpha = \mu/c$, $\beta = \mu^*/c$ in (66.6) and remark also that because of $ad - bc = 1$ we have $-d\alpha \equiv \beta \pmod 1$, $-a\beta \equiv \alpha \pmod 1$. We obtain then

$$\sum_{q=0}^{\infty}\left\{\lambda\left(\frac{1}{c}(cq+\mu)(z+id)\right) + \lambda\left(\frac{1}{c}(c(q+1)-\mu)(z+id)\right)\right\}$$
$$+ \pi z B_2\left(\frac{\mu}{i}\right)$$
$$= \sum_{q=0}^{\infty}\left\{\lambda\left(\frac{1}{c}(cq+\mu^*)\left(\frac{1}{z}-ia\right)\right.\right.$$
$$\left.\left. + \lambda\left(\frac{1}{c}(c(q+1)-\mu^*)\left(\frac{1}{z}-ia\right)\right)\right\}$$
$$+ \frac{\pi}{z} B_2\left(\frac{\mu^*}{c}\right) + 2\pi i\left(\frac{\mu}{c}-\frac{1}{2}\right)\left(\frac{\mu^*}{c}-\frac{1}{2}\right).$$

Here we sum over μ from 1 to $c-1$, observing (2.71), (2.72) and $B_2 = 1/6$, with the result

$$2\sum_{\substack{n=1\\ n\not\equiv 0 \pmod c}}^{\infty} \lambda\left(\frac{n}{c}(z+id)\right) + \pi z\frac{1-c}{6c}$$
$$= 2\sum_{\substack{n=1\\ n\not\equiv 0 \pmod c}}^{\infty} \lambda\left(\frac{n}{c}\left(\frac{1}{z}-ia\right)\right) + \frac{\pi}{z}\frac{1-c}{6c} + 2\pi i s(-d,c), \quad (67.4)$$

where

$$s(-d, c) = \sum_{\mu=1}^{c-1} \left(\frac{\mu}{c} - \frac{1}{2}\right)\left(\frac{\mu^*}{c} - \frac{1}{2}\right)$$

$$= \sum_{\mu=1}^{c-1} \left(\frac{\mu}{c} - \frac{1}{2}\right)\left(\frac{-d\mu}{c} - \left[\frac{-d\mu}{c}\right] - \frac{1}{2}\right). \tag{67.5}$$

In order to provide in (67.4) the summands $n \equiv 0 \pmod c$ we add (67.3) to (67.4) and obtain

$$2\sum_{n=1}^{\infty} \lambda\left(\frac{n}{c}(z + id)\right) + \frac{\pi z}{6c} - \log z$$

$$= 2\sum_{n=1}^{\infty} \lambda\left(\frac{n}{c}\left(\frac{1}{z} - ia\right)\right) + \frac{\pi}{6cz} + 2\pi i s(-d, c) .$$

This is (67.2) with

$$K = \pi i s(-d, c) + \frac{\pi i}{12c}(a + d)$$

and includes also the case $c = 1$, since $s(0, 1) = 0$. Remembering also the case $c = 0$, which can be read off directly from (65.32) we have the

Theorem. *The Dedekind function $\eta(\tau)$ fulfills the functional equation*

$$\log \eta\left(\frac{a\tau + b}{c\tau + d}\right) = \log \eta(\tau) + \begin{cases} \dfrac{\pi i}{12}\dfrac{b}{d}, \quad c = 0, \\ \dfrac{1}{2}\log\left(\dfrac{c\tau + d}{i}\right) + \pi i s(-d, c) \\ + \dfrac{\pi i}{12c}(a + d), \quad c > 0 . \end{cases} \tag{67.6}$$

68. The Dedekind sums

We now study the sums $s(-d, c)$ of (67.5) in more detail. For that purpose we introduce for real x the function

$$((x)) = \begin{cases} x - [x] - 1/2 & \text{for } x \text{ not integer}, \\ 0 & \text{for } x \text{ integer}. \end{cases} \tag{68.1}$$

This function is periodic with period 1 and odd:

$$((x + 1)) = ((x)), \quad ((-x)) = -((x)). \tag{68.2}$$

Let now h, k be two coprime integers. We define then the "Dedekind sum" $s(h, k)$ as

$$s(h, k) = \sum_{\mu \bmod k} \left(\left(\frac{\mu}{k}\right)\right) \left(\left(\frac{h\mu}{k}\right)\right), \tag{68.3}$$

which in virtue of the period 1 of $((x))$ does not depend on the representatives of the residue system modulo k chosen. We see at once

$$s(-h, k) = -s(h, k), \; s(h, -k) = s(h, k) . \tag{68.4}$$

Moreover, let h_1 be an integer such that $hh_1 \equiv 1 \pmod{k}$. Then $h_1\mu$ runs through a full residue system modulo k as μ does, so that

$$s(h, k) = \sum_{\mu \bmod k} \left(\left(\frac{h_1\mu}{k}\right)\right) \left(\left(\frac{hh_1\mu}{k}\right)\right) = \sum_{\mu \bmod k} \left(\left(\frac{h_1\mu}{k}\right)\right) \left(\left(\frac{\mu}{k}\right)\right) = s(h_1, k) . \tag{68.5}$$

Finally, since

$$\sum_{\mu \bmod k} \left(\left(\frac{\mu}{k}\right)\right) = 0 ,$$

we have also, for $k > 0$,

$$s(h, k) = \sum_{\mu=1}^{k} \left(\frac{\mu}{k} - \frac{1}{2}\right) \left(\left(\frac{h\mu}{k}\right)\right) = \sum_{\mu=1}^{k} \frac{\mu}{k} \left(\left(\frac{h\mu}{k}\right)\right). \tag{68.6}$$

We are now going to study (67.6) with respect to modular transformations. This must lead to certain arithmetical properties of the Dedekind sums, reflecting the functional properties of $\log \eta(\tau)$.

69. The formula of reciprocity of the Dedekind sums

In (67.6) we replace τ by $-1/\tau$ with the result

$$\log \eta\left(\frac{b\tau - a}{d\tau - c}\right) = \log \eta\left(-\frac{1}{\tau}\right) + \frac{1}{2} \log\left(\frac{d\tau - c}{i\tau}\right) - \pi i s(d, c)$$
$$+ \frac{\pi i}{12c}(a + d) .$$

The formula (67.6) furnishes also

$$\log \eta\left(-\frac{1}{\tau}\right) = \log \eta(\tau) + \frac{1}{2} \log \frac{\tau}{i}$$

so that we have, still with $c > 0$,

$$\log \eta\left(\frac{b\tau - a}{d\tau - c}\right) = \log \eta(\tau) + \frac{1}{2}\log(-d\tau + c) - \pi i s(d, c) + \frac{\pi i}{12c}(a + d), \tag{69.1}$$

where

$$\log(-d\tau + c) = \log\left(\frac{d\tau - c}{i\tau}\right) + \log\left(\frac{\tau}{i}\right)$$

and thus

$$|\mathrm{Im}\log(-d\tau + c)| < \pi \tag{69.2}$$

because of

$$\left|\mathrm{Im}\left(\log\frac{d\tau - c}{i\tau}\right)\right| < \frac{\pi}{2} \quad \text{and} \quad \left|\mathrm{Im}\log\frac{\tau}{i}\right| < \frac{\pi}{2}.$$

In case $d = 0$ and thus $bc = -1, c = 1, b = -1$ equation (69.1) furnishes only the first case of (67.6) in a different notation. We assume therefore $d \neq 0$.

Case I, $d > 0$. Here (67.6) gives

$$\log \eta\left(\frac{b\tau - a}{d\tau - c}\right) = \log \eta(\tau) + \frac{1}{2}\log\left(\frac{d\tau - c}{i}\right) - \pi i s(-c, d) + \frac{\pi i}{12d}(b - c), \tag{69.3}$$

with

$$\left|\mathrm{Im}\log\left(\frac{d\tau - c}{i}\right)\right| < \frac{\pi}{2}.$$

Subtracting this from (69.1) we obtain

$$0 = \frac{1}{2}\left\{\log(-d\tau + c) - \log\left(\frac{d\tau - c}{i}\right)\right\} - \pi i s(d, c) - \pi i s(c, d) + \frac{\pi i}{12c}(a + d) + \frac{\pi i}{12d}(c - b).$$

Now here

$$\log(-d\tau + c) - \log\left(\frac{d\tau - c}{i}\right) = \log\frac{1}{i},$$

where

$$\left|\mathrm{Im}\log\frac{1}{i}\right| < \frac{3\pi}{2}$$

by virtue of (69.2) and (69.3). We have to put therefore

$$\log\frac{1}{i} = -\frac{\pi i}{2}$$

and obtain

$$\begin{aligned} s(d, c) + s(c, d) &= -\frac{1}{4} + \frac{a+d}{12c} + \frac{c-b}{12d} \\ &= -\frac{1}{4} + \frac{d}{12c} + \frac{c}{12d} + \frac{1}{12cd}\,. \end{aligned} \tag{69.4}$$

This formula shows a "reciprocity", namely a simple connection between $s(d, c)$ and $s(c, d)$.

Case II, $d < 0$ yields by a similar treatment essentially the same as (69.4), namely

$$s(|d|, c) + s(c, |d|) = -\frac{1}{4} + \frac{|d|}{12c} + \frac{c}{12|d|} + \frac{1}{12c\,|d|}\,. \tag{69.5}$$

Now in (69.4) and (69.5) c and d can be any pair of coprime numbers, since to a given pair c, d with $(c, d) = 1$ always integers a, b can be found so that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1\,.$$

The modular properties of log $\eta(\tau)$ lead thus to the *reciprocity formula* for Dedekind sums

$$s(h, k) + s(k, h) = -\frac{1}{4} + \frac{h}{12k} + \frac{k}{12h} + \frac{1}{12hk} \tag{69.6}$$

for $(h, k) = 1$, $h > 0$, $k > 0$.

The reciprocity formula does not contain any reference to $\eta(\tau)$, it expresses a purely arithmetical property of the Dedekind sums. This property must admit a direct proof. A number of such proofs are known [2, 43, 47, 48, 52, 55]. We give a simple one in the next paragraph [56].

70. A direct proof of the reciprocity formula for Dedekind sums

We have, on the one hand,

$$\begin{aligned} \sum_{\mu=1}^{k}\left(\left(\frac{h\mu}{k}\right)\right)^2 = \sum_{\mu=1}^{k}\left(\left(\frac{\mu}{k}\right)\right)^2 &= \sum_{\mu=1}^{k-1}\left(\frac{\mu}{k} - \frac{1}{2}\right)^2 \\ &= \frac{1}{k^2}\sum_{\mu=1}^{k-1}\mu^2 - \frac{1}{k}\sum_{\mu=1}^{k-1}\mu + \frac{1}{4}\sum_{\mu=1}^{k-1}1\,. \end{aligned} \tag{70.1}$$

On the other hand we see

$$\sum_{\mu=1}^{k}\left(\left(\frac{h\mu}{k}\right)\right)^2 = \sum_{\mu=1}^{k-1}\left(\frac{h\mu}{k} - \left[\frac{h\mu}{k}\right] - \frac{1}{2}\right)^2$$

$$= 2h\sum_{\mu=1}^{k-1}\frac{\mu}{k}\left(\frac{h\mu}{k} - \left[\frac{h\mu}{k}\right] - \frac{1}{2}\right) + \sum_{\mu=1}^{k-1}\left[\frac{h\mu}{k}\right]$$

$$\times\left(\left[\frac{h\mu}{k}\right] + 1\right) - \frac{h^2}{k^2}\sum_{\mu=1}^{k-1}\mu^2 + \frac{1}{4}\sum_{\mu=1}^{k-1}1\,. \tag{70.2}$$

Comparing (70.1) and (70.2) and using (68.6) we get

$$2hs(h,k) + \sum_{\mu=1}^{k-1}\left[\frac{h\mu}{k}\right]\left(\left[\frac{h\mu}{k}\right] + 1\right)$$

$$= \frac{h^2+1}{k^2}\sum_{\mu=1}^{k-1}\mu^2 - \frac{1}{k}\sum_{\mu=1}^{k-1}\mu$$

$$= \frac{1}{6k}(h^2+1)(k-1)(2k-1) - \frac{k-1}{2}\,. \tag{70.3}$$

In the sum of the left-hand member we have

$$0 \leqq \left[\frac{h\mu}{k}\right] \leqq h-1\,.$$

We put now

$$\left[\frac{h\mu}{k}\right] = \nu - 1\,, \qquad \nu = 1, 2, \ldots, h \tag{70.4}$$

and determine the number N_ν of values μ which yield the same value of $\nu - 1$. Now (70.4) implies

$$\nu - 1 < \frac{h\mu}{k} < \nu\,,$$

where equality is excluded since $(h, k) = 1$ and $0 < \mu < k$. This we can rewrite as

$$\frac{k(\nu-1)}{h} < \mu < \frac{k\nu}{h}\,.$$

This means, for $1 \leqq \nu \leqq h - 1$,

$$\left[\frac{k(\nu-1)}{h}\right] + 1 \leqq \mu \leqq \left[\frac{k\nu}{h}\right]$$

and thus

$$N_\nu = \left[\frac{k\nu}{h}\right] - \left[\frac{k(\nu-1)}{h}\right], \qquad \nu = 1, \ldots, h-1\,.$$

For $\nu = h$, however $k\nu/h$ is an integer, and we have

$$N_h = k - \left[\frac{k(h-1)}{h}\right] - 1 .$$

We obtain therefore

$$\begin{aligned}
\sum_{\mu=1}^{k-1}\left[\frac{h\mu}{k}\right]\left(\left[\frac{h\mu}{k}\right]+1\right) &= \sum_{\nu=1}^{h} N_\nu(\nu-1)\nu \\
&= \sum_{\nu=1}^{h} \nu(\nu-1)\left(\left[\frac{k\nu}{h}\right]-\left[\frac{k(\nu-1)}{h}\right]\right) \\
&\quad - h(h-1) \\
&= \sum_{\nu=1}^{h-1}\left[\frac{k\nu}{h}\right]\big(\nu(\nu-1)-(\nu+1)\nu\big)+kh(h-1) \\
&\quad - h(h-1) \\
&= -2\sum_{\nu=1}^{h-1}\nu\left[\frac{k\nu}{h}\right] + (k-1)\,h(h-1) \\
&= 2h\sum_{\nu=1}^{h-1}\frac{\nu}{h}\left(\frac{k\nu}{h}-\left[\frac{k\nu}{h}\right]-\frac{1}{2}\right) \\
&\quad - \frac{2k}{h}\sum_{\nu=1}^{h-1}\nu^2 + \sum_{\nu=1}^{h-1}\nu + (k-1)\,h(h-1) \\
&= 2hs(k,h) - \frac{1}{3}k(h-1)(2h-1) \\
&\quad + \frac{1}{2}h(h-1) + (k-1)\,h(h-1) .
\end{aligned}$$

If this result is inserted in (70.3) we obtain after some straightforward calculations the reciprocity formula (69.6).

71. Composition of modular transformations of $\eta(\tau)$

If we introduce the notations

$$\operatorname{sign} x = \begin{cases} 0 & \text{for } x = 0, \\ \dfrac{x}{|x|} & \text{for } x \neq 0 \end{cases} \tag{71.1}$$

and

$$\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \dfrac{b}{d} & \text{for } c = 0, \\ \dfrac{a+d}{c} - 12 \operatorname{sign} c \cdot s(d, |c|) & \text{for } c \neq 0 \end{cases} \tag{71.21}$$

then the formula[1] (67.6) can be condensed into

$$\log \eta\left(\frac{a\tau+b}{c\tau+d}\right) = \log \eta(\tau) + \frac{1}{2}(\operatorname{sign} c)^2 \log \frac{|c|\tau + d \operatorname{sign} c}{i}$$

$$+ \frac{\pi i}{12}\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{71.22}$$

where the second term on the right is meant as 0 for $c = 0$. We are now going to investigate the composition of any two modular substitutions performed on $\eta(\tau)$, generalizing thus § 69.

Let

$$\tau' = \frac{a\tau+b}{c\tau+d}, \quad \tau'' = \frac{a'\tau'+b'}{c'\tau'+d'} \tag{71.31}$$

and thus

$$\tau'' = \frac{a''\tau+b''}{c''\tau+d''}$$

with

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{71.32}$$

Now, (71.22) gives immediately

$$\log \eta\left(\frac{a''\tau+b''}{c''\tau+d''}\right) = \log \eta(\tau) + \frac{1}{2}(\operatorname{sign} c'')^2 \log \frac{|c''|\tau + d'' \operatorname{sign} c''}{i}$$

$$+ \frac{\pi i}{12}\Phi\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}. \tag{71.4}$$

On the other hand, we obtain through the two steps (71.31)

$$\log \eta\left(\frac{a''\tau+b''}{c''\tau+d''}\right) = \log \eta(\tau) + \frac{1}{2}(\operatorname{sign} c)^2 \log\left(\frac{|c|\tau + d \operatorname{sign} c}{i}\right)$$

$$+ \frac{1}{2}(\operatorname{sign} c')^2 \log\left(\frac{|c'|\tau' + d' \operatorname{sign} c'}{i}\right)$$

$$+ \frac{\pi i}{12}\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \frac{\pi i}{12}\Phi\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

[1] The formula (67.6) is valid only for $c \geqq 0$. The analogous formula for $c < 0$ is obtained by applying (67.6) to the transformation $\dfrac{-a\tau - b}{-c\tau - d}$ in which $-c > 0$.

Comparing this with (71.4) we have

$$\frac{\pi i}{6}\left\{\Phi\begin{pmatrix}a & b\\ c & d\end{pmatrix}+\Phi\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}-\Phi\begin{pmatrix}a'' & b''\\ c'' & d''\end{pmatrix}\right\}$$

$$=(\operatorname{sign} c'')^2\log\left(\frac{|c''|\tau+d''\operatorname{sign} c''}{i}\right)-(\operatorname{sign} c')^2\log\left(\frac{|c'|\tau'+d'\operatorname{sign} c'}{i}\right)$$

$$-(\operatorname{sign} c)^2\log\left(\frac{|c|\tau+d\operatorname{sign} c}{i}\right). \tag{71.5}$$

Here the right-hand side, which we shall call S for short, must be independent of τ since the left-hand member is free of τ. We shall determine S and prove the

Theorem. *Under the composition* (71.32) *of modular substitutions the arithmetical function Φ fulfills the relation*

$$\Phi\begin{pmatrix}a & b\\ c & d\end{pmatrix}+\Phi\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}-\Phi\begin{pmatrix}a'' & b''\\ c'' & d''\end{pmatrix}=3\operatorname{sign}(cc'c''). \tag{71.6}$$

Proof. We have to prove

$$S=\frac{\pi i}{2}\operatorname{sign}(cc'c''). \tag{71.61}$$

We take first the case

$$\begin{pmatrix}a'' & b''\\ c'' & d''\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

so that

$$\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1}=\begin{pmatrix}d & -b\\ -c & a\end{pmatrix},$$

with $c>0$. Here the correctness of (71.6) can be read off directly from the definition (71.21) of Φ. But we can also show $S=0$, since we have here, with $c>0$,

$$S=-\log\left(\frac{c\tau'-a}{i}\right)-\log\left(\frac{c\tau+d}{i}\right).$$

Now

$$\frac{c\tau'-a}{i}\,\frac{c\tau+d}{i}=\frac{c\dfrac{a\tau+b}{c\tau+d}-a}{i}\,\frac{c\tau+d}{i}=1\,.$$

Since we know $|\operatorname{Im} S|<\pi$ we must choose $S=\log 1=0$, as stated. We have thus obtained

$$\Phi\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1}\right)=-\Phi\begin{pmatrix}a & b\\ c & d\end{pmatrix}, \tag{71.7}$$

first for $c \neq 0$. If of the numbers c, c', c'' two are zero then the third one vanishes also as (71.32) shows. In this case $S = 0$, and (71.6) is confirmed. This can, however, also be read off directly from the definition (71.21) of Φ since we deal here simply with

$$\begin{pmatrix} 1 & b'' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

i.e. $b'' = b' + b$.

Now let only one c vanish. We choose $c' = 0$ and therefore

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} 1 & b' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix}.$$

Then $S = 0$ because of $c' = c$, $d' = d$, confirming (71.6) also in this case. The other cases $c = 0$ and $c'' = 0$ can be reduced to this one by using inverse matrices in (71.32) and applying (71.7).

We are left with the principal case $cc'c'' \neq 0$. We can always choose $c > 0$, $c' > 0$ and have then to distinguish the cases $c'' > 0$, $c'' < 0$. According to (71.5), (71.61) we have to prove

$$S = \log\left(\frac{c''\tau + d''}{i}\right) - \log\left(\frac{c'\tau' + d'}{i}\right) - \log\left(\frac{c\tau + d}{i}\right)$$

$$= \frac{\pi i}{2} \text{ for } c' > 0\,, \tag{71.81}$$

$$S = \log\left(\frac{-c''\tau - d''}{i}\right) - \log\left(\frac{c'\tau' + d'}{i}\right) - \log\left(\frac{c\tau + d}{i}\right)$$

$$= -\frac{\pi i}{2} \text{ for } c'' < 0\,. \tag{71.82}$$

By substitution from (71.32) we obtain

$$\frac{c''\tau + d''}{i} \Big/ \left(\frac{c'\tau' + d'}{i}\,\frac{c\tau + d}{i}\right) = i\,.$$

Since S is the sum of 3 logarithms we know

$$|\mathrm{Im}\, S| < \frac{3\pi}{2},$$

so that we have

$$S = \log i = \frac{\pi i}{2}$$

in (71.81), and (71.82) is proved in the same way. This finishes the proof of the theorem. $\square$

We have proved (71.6) by function-theoretical means through the study of $\log \eta(\tau)$. However, with the definition (71.21), the relation (71.6) is a purely arithmetical one. It can indeed be proved arithmetically by means of the reciprocity formula of the Dedekind sums [55].

72. A group-theoretical remark

The equation (71.6) together with (71.7) shows that the modular substitutions $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for which

$$\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv 0 \pmod 3 \tag{72.1}$$

form a group. This group is described by the following

Theorem. *The group G of modular substitutions with property* (72.1) *are those whose matrices fulfill one of the congruences*[1]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \pmod 3 .$$

The group G is a normal subgroup of index 3 *in the full modular group* Γ.

Before entering into the proof we prepare two lemmata about Dedekind sums.

Lemma A. *The denominator of* $s(h, k)$ *is at most* $2k\,(3, k)$.

Proof. In view of (68.6) we have

$$\begin{aligned} s(h, k) &= \sum_{\mu=1}^{k-1} \frac{\mu}{k}\left(\frac{h\mu}{k} - \left[\frac{h\mu}{k}\right] - \frac{1}{2}\right) \\ &= \frac{h}{k^2}\sum_{\mu=1}^{k-1} \mu^2 - \frac{1}{k}\sum_{1}^{k-1} \mu\left[\frac{h\mu}{k}\right] - \frac{1}{2k}\sum_{\mu=1}^{k-1} \mu \\ &= \frac{h(k-1)(2k-1)}{6k} - \frac{1}{k}g - \frac{k-1}{4}, \quad g = \text{integer} \end{aligned}$$

so that the denominator can at most be $6k$. The previous line shows that the denominator is at most $2k^2$. Thus the denominator of $s(h, k)$ is at most

$$(6k, 2k^2) = 2k(3, k)\,,$$

as stated in the lemma. $\square$

---

[1] Because we are speaking here of the *inhomogeneous* modular group the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ are not distinguished.

Lemma B. *Let us put* $\vartheta = (3, k)$. *Then*

$$12hk\,s(h, k) \equiv h^2 + 1 \pmod{\vartheta k} . \tag{72.2}$$

Proof. Multiplication of (69.6) with $12hk$ yields

$$12hk\,s(h, k) + 12hk\,s(k, h) = -3hk + h^2 + k^2 + 1 . \tag{72.3}$$

Now, after Lemma A,

$$2h\,s(k, h) \text{ is an integer for } 3 \nmid h ,$$

$$6h\,s(k, h) \text{ is an integer for } 3 \mid h .$$

Since $(h, k) = 1$ we conclude

$$6hk\,s(k, h) \equiv 0 \pmod{\vartheta k} ,$$

and then read off (72.2) from (72.3). ▯

Corollary. *The function* Φ *assumes only integer values. Indeed, because of* $ad \equiv 1 \pmod c$, (72.2) *implies*

$$d(a + d) - 12dc\,s(d, c) \equiv 0 \pmod c$$

for $c > 0$ *and thus* $a + d - 12c\,s(d, c) \equiv 0 \pmod c$. *This consequence is true also for* $c < 0$, *as* (68.4) *shows.*

Now we return to the

Proof of the Theorem. For $c = 0$ we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

so that $\Phi \equiv 0 \pmod 3$ requires here simply $b \equiv 0$, in agreement with the first alternative of the theorem.

Assume now $c > 0$. According to (71.21) we have to find the necessary and sufficient conditions for

$$a + d - 12c\,s(d, c) \equiv 0 \pmod{3c} , \tag{72.4}$$

which, for $3 \mid c$, is equivalent with

$$d(a + d) - 12\,dc\,s(d, c) \equiv 0 \pmod{3c} .$$

Now Lemma B furnishes

$$d^2 + 1 - 12dc\,s(d, c) \equiv 0 \pmod{3c}$$

so that, as subtraction shows,

$$ad - 1 = bc \equiv 0 \pmod{3c}$$

or

$$3 \mid b$$

is necessary and sufficient in this case, which covers the first alternative of the theorem.

In case $3 \nmid c$ we need again (72.4). We have here now, by virtue of Lemma A

$$12c\,s(d, c) \equiv 0 \pmod 3 . \tag{72.51}$$

Lemma B furnishes

$$d(a + d) - 12dc\,s(d, c) \equiv 0 \pmod c$$

(because $ad \equiv 1 \pmod c$), and thus also

$$a + d - 12c\,s(d, c) \equiv 0 \pmod c . \tag{72.52}$$

Thus (72.51) and (72.52) show that

$$a + d \equiv 0 \pmod 3 \tag{72.6}$$

is here necessary and sufficient for (72.4).

Now (72.6) implies

$$-a^2 - bc \equiv 1 \pmod 3 ,$$

which for

$$a \equiv d \equiv 0 \pmod 3$$

has

$$b \equiv -c \pmod 3$$

as consequence (2nd alternative of the theorem).

For $3 \nmid a$ we see that necessarily

$$bc \equiv 1 \pmod 3$$

or

$$b \equiv c \pmod 3 ,$$

which covers the third and fourth alternatives. All these necessary conditions for (72.4) are obviously sufficient, and the first part of the theorem is proved. ▯

The index 3 follows from the fact that all modular substitutions can modulo 3 be classified as congruent to one of the following 12 matrices:

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \quad \begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix},$$

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}, \quad \begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}, \quad \begin{pmatrix}-1 & 0\\ 1 & -1\end{pmatrix},$$

$$\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}, \quad \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \quad \begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix},$$

of which the first four characterize the group G.

The group G is a *normal* subgroup of Γ, since for $M \in G$, $Q \in \Gamma$, $\Phi(QMQ^{-1}) \equiv 0$ by virtue of (71.6) and (71.7).

Remark. We mention here, without proof, that G has the independent generators

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix},$$

which fulfill the relations

$$T^2 = U_1^2 = U_2^2 = I\,,$$

I being the identity.

73. The Dedekind sums and the Jacobi residue symbol

The reciprocity of the Dedekind sums has something to do with the quadratic reciprocity law.

We take in this paragraph always k as *odd* and *positive*. Starting with (68.6) again we have

$$\begin{aligned} 12ks(h,k) &= 12\sum_{\mu=1}^{k-1}\mu\left(\frac{h\mu}{k} - \left[\frac{h\mu}{k}\right] - \frac{1}{2}\right) \\ &= 2h(k-1)(2k-1) - 12\sum_{\mu=1}^{k-1}\mu\left[\frac{h\mu}{k}\right] - 3k(k-1) \end{aligned}$$

and thus, in view of $4k \equiv 4 \pmod 8$, $k^2 \equiv 1 \pmod 8$,

$$\begin{aligned} 12ks(h,k) &\equiv -2h(k-1) + k(k-1) - 4\sum_{\substack{\mu=1 \\ \mu\ \mathrm{odd}}}^{k-1}\left[\frac{h\mu}{k}\right] \pmod 8 \\ &\equiv -2h(k-1) + 1 - k - 4\sum_{\mu=1}^{k-1}\left[\frac{h\mu}{k}\right] \\ &\quad + 4\sum_{\nu=1}^{(k-1)/2}\left[\frac{2h\nu}{k}\right] \pmod 8\,. \end{aligned}$$

Here the first sum is elementary:

$$\begin{aligned} \sum_{\mu=1}^{k-1}\left[\frac{h\mu}{k}\right] &= -\sum_{\mu=1}^{k-1}\left(\left(\frac{h\mu}{k}\right)\right) + \sum_{\mu=1}^{k-1}\frac{h\mu}{k} - \sum_{\mu=1}^{k-1}\frac{1}{2} \\ &= 0 + \frac{h(k-1)}{2} - \frac{k-1}{2} = \frac{1}{2}(h-1)(k-1)\ * \end{aligned}$$

---

* This sum is also obtainable as half the number of lattice points in the rectangle $0 < \mu < k$, $0 < \nu < h$.

so that we have

$$12ks(h, k) \equiv k - 1 + 4g_k(h) \pmod 8, \tag{73.1}$$

where we define

$$g_k(h) = \sum_{\nu=1}^{(k-1)/2} \left[\frac{2h\nu}{k}\right]. \tag{73.2}$$

Let us now also take h as *odd* and *positive*, $(h, k) = 1$. Then, analogous to (73.1) we have also

$$12hs(k, h) \equiv h - 1 + 4g_h(k) \pmod 8$$

and therefore

$$12hk\bigl(s(h, k) + s(k, h)\bigr) \equiv h(k-1) + k(h-1) + 4hg_k(h) + 4kg_h(k) \pmod 8.$$

Applying now the reciprocity formula of the Dedekind sums in the form (72.3) we arrive, after some simplifications, at

$$-hk + h + k - 1 \equiv 4\bigl(g_k(h) + g_h(k)\bigr) \pmod 8$$

or

$$g_k(h) + g_h(k) \equiv \frac{(h-1)(k-1)}{4} \pmod 2. \tag{73.3}$$

Let us now introduce the function

$$\chi_k(h) = (-1)^{g_k(h)}. \tag{73.4}$$

Then (73.3) is equivalent to

$$\chi_k(h)\,\chi_h(k) = (-1)^{((h-1)(k-1))/4}. \tag{73.5}$$

This formula suggests, of course, to identify $\chi_k(h)$ with the Jacobi symbol. This can be done in a few steps.

The definition (73.2) shows

$$g_k(1) = 0,$$

$$g_k(-1) = \sum_{\nu=1}^{(k-1)/2} \left[\frac{-2\nu}{k}\right] = \sum_{\nu=1}^{(k-1)/2} (-1) = -\frac{k-1}{2} \equiv \frac{k-1}{2} \pmod 2,$$

and for $h > 0$

$$g_k(-h) = \sum_{\nu=1}^{(k-1)/2} \left[\frac{-2h\nu}{k}\right]$$

$$= -g_k(h) - \frac{k-1}{2} \equiv g_k(h) + g_k(-1) \pmod 2$$

and furthermore, for $h_1 \equiv h_2 \pmod{k}$

$$g_k(h_1) \equiv g_k(h_2) \pmod{2} .$$

These statements can be translated by means of (73.4) into

$$\begin{aligned} \chi_k(h_1) &= \chi_k(h_2) \quad \text{for } h_1 \equiv h_2 \pmod{k} , \\ \chi_k(1) &= 1 , \\ \chi_k(-1) &= (-1)^{(k-1)/2} , \\ \chi_k(-h) &= \chi_k(h)\, \chi_k(-1) . \end{aligned} \tag{73.6}$$

Now the Legendre-Jacobi symbol satisfies in complete parallelism the relations, with h, k odd, positive:

$$\left(\frac{h}{k}\right)\left(\frac{k}{h}\right) = (-1)^{((h-1)\,(k-1))/4} ,$$

$$\left(\frac{h_1}{k}\right) = \left(\frac{h_2}{k}\right) \quad \text{for } h_1 \equiv h_2 \pmod{k} ,$$

$$\left(\frac{1}{k}\right) = 1 ,$$

$$\left(\frac{-1}{k}\right) = (-1)^{(k-1)/2} ,$$

$$\left(\frac{-h}{k}\right) = \left(\frac{h}{k}\right)\left(\frac{-1}{k}\right) .$$

But it is well-known and easily seen, that these relations by means ot a Euclidean algorithm define uniquely the symbol (h/k). Therefore, $\chi_k(h)$ satisfying the same relations must be identical with (h/k). We have proved the

Theorem. *The Legendre-Jacobi symbol for positive odd k is given by*

$$\left(\frac{h}{k}\right) = (-1)^{g_k(h)} \tag{73.7}$$

with

$$g_k(h) = \sum_{\nu=1}^{(k-1)/2} \left[\frac{2h\nu}{k}\right] .$$

Moreover, since we infer from (73.7)

$$\frac{1}{2}\left\{1 - \left(\frac{h}{k}\right)\right\} \equiv g_k(h) \pmod{2}$$

we can now restate (73.1):

Theorem [6], § 6. *The Dedekind sum satisfies the relation*

$$12ks(h,k) = k + 1 - 2\left(\frac{h}{k}\right) \pmod 8\,. \tag{73.8}$$

Remark. Under (73.6) one would expect to find also the "second supplementary theorem" (Ergänzungssatz)

$$\chi_k(2) = (-1)^{(k^2-1)/8}\,. \tag{73.9}$$

Although it is not needed here it is readily obtained. We have first

$$g_k(2) = \sum_{\nu=1}^{(k-1)/2} \left[\frac{4\nu}{k}\right].$$

Here we distinguish

$$\frac{4\nu}{k} = 0, \quad \text{i.e. } 0 < \frac{4\nu}{k} < 1 \quad \text{or} \quad 0 < \nu < \frac{k}{4}$$

and

$$\frac{4\nu}{k} = 1, \quad \text{i.e. } 1 < \frac{4\nu}{k} < 2 \quad \text{or} \quad \frac{k}{4} < \nu < \frac{k}{2}\,.$$

Hence

$$g_k(2) = \sum_{\nu=[k/4]+1}^{[k/2]} 1 = \left[\frac{k}{2}\right] - \left[\frac{k}{4}\right] = \left[\frac{k+1}{4}\right]$$

and thus

$$\chi_k(2) = (-1)^{[(k+1)/4]}\,.$$

But this is equivalent to (73.9) since the right-hand members of both equations have the period 8 and agree for $k = 1, 3, 5, 7$.

74. Again the transformation of $\eta(\tau)$

Let us return from $\log \eta(\tau)$ to $\eta(\tau)$. With $c > 0$ we infer from (71.22)

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \varepsilon(a, b, c, d) \sqrt{\frac{c\tau + d}{i}}\, \eta(\tau) \tag{74.11}$$

with

$$\varepsilon(a, b, c, d) = \exp\left(\frac{\pi i}{12}\, \Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right). \tag{74.12}$$

Since Φ is an integer (see corollary in § 72), ε appears here explicitly as a 24th root of unity as already stated in § 65 (ε called C there).

Now (71.22) carries more information than we need in (74.12) where Φ is only needed modulo 24. It should therefore be possible to replace the complicated exponent Φ in (74.12) by a simpler expression. The definition (71.21) shows that we have for this purpose to study $a + d - 12cs(d, c)$ modulo $24c$. We assume first that c is *odd* and *positive.*

From (73.1) we obtain

$$a + d - 12cs(d, c) \equiv a + d + 1 - c + 12cg_c(d) \pmod 8, \tag{74.21}$$

where we used $4 \equiv 12c \pmod 8$, c being odd. With $\vartheta = (3, c)$ we write (72.2) as

$$12dcs(d, c) \equiv d^2 + 1 \pmod{\vartheta c}. \tag{74.22}$$

Now $ad - 1 \equiv 0 \pmod c$ implies $(ad - 1)^2 \equiv 0 \pmod{\vartheta c}$ or

$$d(2a - a^2 d) \equiv 1 \pmod{\vartheta c}.$$

Multiplying (74.22) by $2a - a^2 d$ we obtain

$$12cs(d, c) \equiv d + 2a - a^2 d \pmod{\vartheta c},$$

which leads after some simplifications to

$$a + d - 12cs(d, c) \equiv abc \pmod{\vartheta c}. \tag{74.23}$$

Let us first take the case $3 \mid c$, or $\vartheta = 3$.

Since $c^2 \equiv 1 \pmod 8$ and $c^2 \equiv 0 \pmod{3c}$ we can conclude from (74.21) and (74.23)

$$a + d - 12cs(d, c) \equiv (a + d)\, c^2 + (1 - c)\, c^2 + abc(1 - c^2) + 12cg_c(d) \pmod{24c},$$

since this congruence is true modulo 8 and modulo $3c$ separately. Therefore we have

$$\Phi = \frac{a+d}{c} - 12s(d, c) \equiv (a + d)\, c + (1 - c)\, c + ab(1 - c^2) + 12g_c(d) \pmod{24}$$

and so for $3 \mid c$

$$\varepsilon = e^{(\pi i/12)\,\Phi} = \left(\frac{d}{c}\right) e^{(\pi i/12)\, c(1-c)}\, e^{(\pi i/12)\, (ab(1-c^2) + c(a+d))}, \tag{74.3}$$

where the Legendre-Jacobi symbol stems from (73.7).

Secondly, in the case $3 \nmid c$, (74.23) reads

$$a + d - 12cs(d, c) \equiv 0 \pmod{c}. \tag{74.41}$$

We can restate (74.21) as

$$a + d - 12cs(d, c) \equiv a + d + 3(c - 1) + 12cg_c(d) \pmod{8}$$

since $1 - c \equiv 3(c - 1) \pmod{8}$, c being odd.

Since we have

$$a + d - 12cs(d, c) \equiv a + d \pmod{3}$$

in view of Lemma A, § 72, we infer

$$a + d - 12cs(d, c) \equiv a + d + 3(c - 1) + 12cg_c(d) \pmod{24}, \tag{74.42}$$

and furthermore, because of $c^2 \equiv 1 \pmod{24}$, $c^2 \equiv 0 \pmod{c}$ from (74.41) and (74.42)

$$a + d - 12cs(d, c) \equiv c^2\big((a + d) + 3(c - 1)\big) + 12cg_c(d) \pmod{24c},$$

$$\Phi \equiv c(a + d) + 3c(c - 1) + 12g_c(d) \pmod{24}$$

and thus

$$\varepsilon = e^{(\pi i/12)\Phi} = \left(\frac{d}{c}\right) e^{(\pi i/4)\, c(c-1)}\, e^{(\pi i/12)\, c(a+d)}, \qquad 3 \nmid c \tag{74.5}$$

H. Weber [77], p. 126, (15) and Tannery and Molk [72], vol. 2, p. 267, XLVI (5) have a formula for ε (gained by other methods) which convers both cases $3 \mid c$ and $3 \nmid c$, c odd. It reads in our notation

$$\varepsilon = \varepsilon(a, b, c, d) = \left(\frac{d}{c}\right) i^{(1-c)/2}\, e^{(\pi i/12)\,(bd(1-c^2) + c(a+d))}. \tag{74.6}$$

It is easily seen that this is equivalent with our (74.3) and (74.5). (In the case $3 \mid c$ one has to observe that $ad \equiv 1 \pmod{3}$ and thus $a \equiv d \pmod{3}$. In the second case $c^2 - 1 \equiv 0 \pmod{24}$ is the clue.)

Let now finally c be *even*. Then d must be odd. From

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$$

we infer, after (71.6)

$$\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Phi\begin{pmatrix} b & -a \\ d & -c \end{pmatrix} + 3 \operatorname{sign} d,$$

c being positive as here throughout. Therefore

$$\varepsilon(a, b, c, d) = e^{\frac{\pi i}{12}\Phi\begin{pmatrix}a & b\\ c & d\end{pmatrix}} = e^{\frac{\pi i}{12}\Phi\begin{pmatrix}b & -a\\ d & -c\end{pmatrix}} e^{\frac{\pi i}{4}\operatorname{sign} d}$$

$$= \varepsilon(b, -a, d, -c)\, e^{\frac{\pi i}{4}\operatorname{sign} d}.$$

If now $d > 0$ then (74.6) furnishes

$$\varepsilon = \left(\frac{-c}{d}\right) e^{\pi i/4}\, i^{(1-d)/2}\, e^{\pi i/12}\, e^{(\pi i/12)\,(ac(1-d^2)+d(b-c))}$$

or

$$\varepsilon = \left(\frac{c}{d}\right) e^{(\pi i d/4)\,(ac(1-d^2)+d(b-c))}. \tag{74.7}$$

For $d < 0$ we have to use the fact

$$\Phi\begin{pmatrix}b & -a\\ d & -c\end{pmatrix} = \Phi\begin{pmatrix}-b & a\\ -d & c\end{pmatrix}.$$

Applying (74.6) again we obtain here after a short computation the same formula (74.7) if we agree to write

$$\left(\frac{c}{d}\right) = \left(\frac{c}{-d}\right), \tag{74.8}$$

a convention which is useful also in other contexts.

We have thus the

Theorem. *The function $\eta(\tau)$ satisfies the transformation equations*

$$\eta(\tau + b) = e^{\pi i b/12}\eta(\tau) \tag{74.91}$$

and, for $c > 0$,

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \varepsilon(a, b, c, d)\sqrt{\frac{c\tau + d}{i}}\,\eta(\tau) \tag{74.92}$$

with

$$\varepsilon(a, b, c, d) = \begin{cases} \left(\dfrac{d}{c}\right) i^{(1-c)/2}\, e^{(\pi i/12)\,(bd(1-c^2)+c(a+d))}, & c \text{ odd},\\[2ex] \left(\dfrac{c}{d}\right) e^{\pi i d/4}\, e^{(\pi i/12)\,(ac(1-d^2)+d(b-c))}, & d \text{ odd}, \end{cases} \tag{74.93}$$

where in case of negative d the Legendre-Jacobi symbol is explained by (74.8).

Chapter 10
The ϑ-functions

75. Introduction of the ϑ-functions

Whereas there exist doubly periodic *meromorphic* functions, as we have seen in § 57, entire doubly periodic functions can only be constants, since they are bounded in their fundamental parallelogram and thus in the whole plane.

In order to construct entire functions which have some resemblance to doubly periodic functions one has to relax the condition of periodicity somewhat. Let us take the two periods 1 and τ, where τ is non-real, and keep 1 as a true period, but let τ be only a "quasi-period", so that

$$\Theta(v+1) = \Theta(v)\,, \quad \Theta(v+\tau) = F\cdot\Theta(v)\,, \tag{75.1}$$

where $\Theta(v)$ is the entire function and F is a factor still to be determined. Now F must be periodic of period 1, since

$$\begin{aligned} F(v)\,\Theta(v) &= \Theta(v+\tau) = \Theta(v+\tau+1) = F(v+1)\,\Theta(v+1) \\ &= F(v+1)\,\Theta(v) \end{aligned}$$

so that $F(v) = F(v+1)$.

The simplest non-constant choice for F would be $F(v) = c e^{2\pi i v}$, $c \neq 0$. In order to conform to standard notation we prefer to choose

$$F(v) = c e^{-2\pi i v}\,,$$

where we shall dispose of c later.

Now since Θ has the period 1 it can be expanded in a Fourier series

$$\Theta(v) = \sum_{-\infty}^{\infty} A_n e^{2\pi i n v}$$

and we have then

$$F(v)\,\Theta(v) = \sum_{-\infty}^{\infty} A_n e^{2\pi i n (v+\tau)}$$

or

$$c e^{-2\pi i v} \sum_{-\infty}^{\infty} A_n e^{2\pi i n v} = \sum_{-\infty}^{\infty} A_n e^{2\pi i n \tau} e^{2\pi i n v}\,,$$

from which we take

$$A_{n+1} = c^{-1} A_n e^{2\pi i n \tau} \quad \text{for all } n\,.$$

Let us take $A_0 = 1$. Then we obtain $A_1 = c^{-1}$, $A_2 = c^{-2}e^{2\pi i\tau}$, and in general, for all integral n,

$$A_n = c^{-n}e^{\pi in(n-1)\tau} . \tag{75.2}$$

Let us put from now on

$$q = e^{\pi i\tau} \tag{75.3}$$

and choose $c = q^{-1}$. Then we have

$$A_n = e^{\pi in^2\tau} = q^{n^2}$$

and

$$\Theta(v) = \sum_{-\infty}^{\infty} q^{n^2}e^{2\pi inv} . \tag{75.4}$$

In order to have an entire function of v we need convergence for all v, which, because of the quadratic exponent of q is achieved through

$$|q| < 1 \tag{75.5}$$

or

$$\mathrm{Im}(\tau) > 0 . \tag{75.6}$$

The function $\Theta(v)$ satisfies now

$$\Theta(v+1) = \Theta(v) , \quad \Theta(v+\tau) = q^{-1}e^{2\pi iv}\,\Theta(v) . \tag{75.7}$$

The choice $c = q^{-1}$ is not essential, since for any other choice we would have obtained a related function. Indeed let us take $c = e^{2\pi i\lambda}$ with an arbitrary λ. We have then $A_2^* = e^{-2\pi in\lambda}q^{n(n-1)}$ and thus

$$\begin{aligned}\Theta^*(v) &= \sum_{n=-\infty}^{\infty} A_n^* e^{2\pi inv} = \sum_{n=-\infty}^{\infty} q^{n(n-1)}e^{2\pi in(v-\lambda)}\\ &= \sum_{n=-\infty}^{\infty} q^{n^2}e^{2\pi in(v-\lambda-\tau/2)}\\ &= \Theta(v-\lambda-\tau/2) .\end{aligned}$$

Nevertheless, it is useful to consider some other functions which are obtained from $\Theta(v)$ by a shift of the independent variable, in analogy to $\sin v$ and $\cos v$ for simply periodic functions.

76. Definition of the ϑ-functions

It is customary to consider the 4 following ϑ-functions which we write in the notation of Tannery and Molk [72], vol. 2, p. 252.

$$\begin{aligned}
\vartheta_1(v\,|\,\tau) &= -i\,\Sigma(-1)^n q^{(n+1/2)^2} e^{(2n+1)\pi i v}\,,\\
\vartheta_2(v\,|\,\tau) &= \Sigma q^{(n+1/2)^2} e^{(2n+1)\pi i v}\,,\\
\vartheta_3(v\,|\,\tau) &= \Sigma q^{n^2} e^{2\pi i n v}\,,\\
\vartheta_4(v\,|\,\tau) &= \Sigma(-1)^n q^{n^2} e^{2\pi i n v}\,,
\end{aligned}\tag{76.1}$$

where the summations are extended over n from $-\infty$ to ∞ and, as before, $q = e^{\pi i \tau}$, $\operatorname{Im}\tau > 0$. For every value of τ in this half-plane the functions are entire functions of v. The most noteworthy feature of the series is that the sequence of exponents of q forms an arithmetic progression of the second order. We shall sometimes, when there is no doubt about the period τ, simply write $\vartheta_\lambda(v)$ instead of $\vartheta_\lambda(v\,|\,\tau)$.

If the series are so rearranged that equal powers of q are combined into one term we obtain

$$\begin{aligned}
\vartheta_1(v\,|\,\tau) &= 2\sum_{m=0}^{\infty}(-1)^m q^{(m+1/2)^2}\sin(2m+1)\pi v\,,\\
\vartheta_2(v\,|\,\tau) &= 2\sum_{m=0}^{\infty} q^{(m+1/2)^2}\cos(2m+1)\pi v\,,\\
\vartheta_3(v\,|\,\tau) &= 1 + 2\sum_{m=1}^{\infty} q^{m^2}\cos 2m\pi v\,,\\
\vartheta_4(v\,|\,\tau) &= 1 + 2\sum_{m=1}^{\infty}(-1)^m q^{m^2}\cos 2m\pi v\,.
\end{aligned}\tag{76.2}$$

As (76.1) shows, $\vartheta_3(v\,|\,\tau)$ is our previous $\Theta(v)$. With respect to the (quasi) periods $1, \tau$ they behave as follows, as can be seen directly from (76.1):

$$\begin{aligned}
\vartheta_1(v+1) &= -\vartheta_1(v)\,, & \vartheta_1(v+\tau) &= -q^{-1}e^{-2\pi i v}\,\vartheta_1(v)\,,\\
\vartheta_2(v+1) &= -\vartheta_2(v)\,, & \vartheta_2(v+\tau) &= q^{-1}e^{-2\pi i v}\,\vartheta_2(v)\,,\\
\vartheta_3(v+1) &= \vartheta_3(v)\,, & \vartheta_3(v+\tau) &= q^{-1}e^{-2\pi i v}\,\vartheta_3(v)\,,\\
\vartheta_4(v+1) &= \vartheta_4(v)\,, & \vartheta_4(v+\tau) &= -q^{-1}e^{-2\pi i v}\,\vartheta_4(v)\,.
\end{aligned}\tag{76.3}$$

The four ϑ-functions are not essentially different, as we mentioned already in § 75. They can all be obtained from $\Theta(v) = \vartheta_3(v\,|\,\tau)$. The formulae (76.1) show e.g.

$$q^{1/4} e^{\pi i v}\,\vartheta_3\left(v + \frac{\tau}{2}\,\middle|\,\tau\right) = \sum_{n=-\infty}^{\infty} q^{n^2+n+1/4}\, e^{\pi i(2n+1)v} = \vartheta_2(v\,|\,\tau)$$

and similarly

$$\vartheta_3\left(v+\frac{1}{2}\middle|\tau\right)=\vartheta_4(v|\tau)\,,\quad \vartheta_2\left(v+\frac{1}{2}\middle|\tau\right)=-\vartheta_1(v|\tau)\,.$$

The following list shows all these relations, which can all be derived from the foregoing three together with (76.3), where $B = q^{-1/4}e^{-\pi i v}$.

	$v+\frac{1}{2}$	$v+\frac{\tau}{2}$	$v+\frac{1+\tau}{2}$
ϑ_1	$\vartheta_2(v)$	$iB\vartheta_4(v)$	$B\vartheta_3(v)$
ϑ_2	$-\vartheta_1(v)$	$B\vartheta_3(v)$	$-iB\vartheta_4(v)$
ϑ_3	$\vartheta_4(v)$	$B\vartheta_2(v)$	$iB\vartheta_1(v)$
ϑ_4	$\vartheta_3(v)$	$iB\vartheta_1(v)$	$B\vartheta_2(v)$

(76.4)

The series (76.1) are all of the form

$$\vartheta_{\mu\nu}(v|\tau)=\sum_{n=-\infty}^{\infty}(-1)^{\nu n}e^{\left(n+\frac{\mu}{2}\right)^2\pi i\tau}e^{2\pi i\left(n+\frac{\mu}{2}\right)v}, \tag{76.5}$$

with $\mu = 0$ or 1, $\nu = 0$ or 1[1]. From this it follows immediately that

$$\frac{\partial^2\vartheta_{\mu\nu}(v|\tau)}{\partial v^2}=4\pi i\,\frac{\partial\vartheta_{\mu\nu}(v|\tau)}{\partial\tau}\,. \tag{76.6}$$

This is a parabolic partial differential equation satisfied by all four ϑ-functions.

77. Zeros of the ϑ-functions

From (76.2) we read off

$$\vartheta_1(0|\tau)=0\,.$$

Therefore, in view of (76.3) we have also

$$\vartheta_1(m+n\tau|\tau)=0\,.$$

The zeros thus form a point lattice. We show that these are all the zeros of $\vartheta_1(v)$. It suffices to prove that the parallelogram P of vertices

$$-\frac{1}{2}-\frac{\tau}{2}\,,\quad \frac{1}{2}-\frac{\tau}{2}\,,\quad \frac{1}{2}+\frac{\tau}{2}\,,\quad -\frac{1}{2}+\frac{\tau}{2}$$

contains only the zero $v = 0$. For this purpose we compute

$$I=\frac{1}{2\pi i}\int_P\frac{\vartheta_1'(v|\tau)}{\vartheta_1(v|\tau)}\,dv\,,$$

[1] Some authors have used this notation with double subscripts, e.g., C. Hermite and H. Weber. However the notation adopted in (76.1) seems now to be prevalent.

where $\vartheta_1'(v\,|\,\tau)$ (and similarly for the other ϑ-functions) means the derivative with respect to the variable v. Assume first that there are no zeros on P. We have

$$I = \frac{1}{2\pi i}\int_{-1/2-\tau/2}^{1/2-\tau/2}\left\{\frac{d}{dv}\log\vartheta_1(v\,|\,\tau) - \frac{d}{dv}\log\vartheta_1(v+\tau\,|\,\tau)\right\}dv$$

$$+\frac{1}{2\pi i}\int_{-1/2+\tau/2}^{-1/2-\tau/2}\left\{\frac{d}{dv}\log\vartheta_1(v\,|\,\tau) - \frac{d}{dv}\log\vartheta_1(v+1\,|\,\tau)\right\}dv$$

$$=\frac{1}{2\pi i}\int_{-1/2-\tau/2}^{1/2-\tau/2}\frac{d}{dv}\log\frac{\vartheta_1(v\,|\,\tau)}{\vartheta_1(v+\tau\,|\,\tau)}\,dv$$

$$+\frac{1}{2\pi i}\int_{-1/2+\tau/2}^{-1/2-\tau/2}\frac{d}{dv}\log\frac{\vartheta_1(v\,|\,\tau)}{\vartheta_1(v+1\,|\,\tau)}\,dv\,.$$

The table (76.3) now shows

$$\log\frac{\vartheta_1(v\,|\,\tau)}{\vartheta_1(v+\tau\,|\,\tau)} = \pi i + \pi i\tau + 2\pi i v + 2\pi i k\,,$$

$$\log\frac{\vartheta_1(v\,|\,\tau)}{\vartheta_1(v+1\,|\,\tau)} = \pi i + 2\pi i l\,, \qquad k, l \text{ integers}\,,$$

and thus

$$I = \frac{1}{2\pi i}\int_{-1/2-\tau/2}^{1/2-\tau/2} 2\pi i\,dv = 1\,.$$

This means that $\vartheta_1(v\,|\,\tau)$ has only one simple zero in the parallelogram P, namely $v=0$. All zeros of $\vartheta_1(v\,|\,\tau)$ are then comprised in

$$\vartheta_1(m+n\tau\,|\,\tau) = 0\,, \tag{77.1}$$

and the table (76.4) shows also

$$\vartheta_2\left(m+\frac{1}{2}+n\tau\,\middle|\,\tau\right) = 0\,, \tag{77.2}$$

$$\vartheta_3\left(m+\frac{1}{2}+\left(n+\frac{1}{2}\right)\tau\,\middle|\,\tau\right) = 0\,, \tag{77.3}$$

$$\vartheta_4\left(m+\left(n+\frac{1}{2}\right)\tau\,\middle|\,\tau\right) = 0\,, \qquad n, m = 0, \pm 1, \pm 2, \ldots\,. \tag{77.4}$$

A simple argument by deforming P slightly will show that our initial assumption was correct; namely, that there are no zeros on P.

78. Product expansions of the ϑ-functions

The ϑ-functions, as the series (76.2) show, can be looked upon as functions of $z = e^{2\pi i v}$. As such we shall write them also as

$$\vartheta_\lambda(v\,|\,\tau) = \vartheta_\lambda(z; q)\,, \qquad z = e^{2\pi i v}\,, \qquad q = e^{\pi i \tau}\,.$$

We can now use our knowledge of the zeros to construct infinite products for the ϑ-functions.

We start with $\vartheta_3(z; q)$. Its zeros are, after (77.3),

$$z = \exp\left\{2\pi i\left(n + \frac{1}{2} + \left(m + \frac{1}{2}\right)\tau\right)\right\}$$
$$= -q^{2m+1}, \quad m = 0, \pm 1, \pm 2, \ldots .$$

Since $\sum\limits_{n=1}^{\infty} |q|^{2n-1}$ converges, the product

$$F(z) = \prod_{m=1}^{\infty} (1 + q^{2m-1} z)\,(1 + q^{2m-1} z^{-1})$$

converges and defines $F(z)$ as a function of z, regular at all z with the exception $z = 0$, and whose zeros coincide with those of $\vartheta_3(z; q)$. Moreover $F(z) = F(e^{2\pi i v}) = f(v)$ has the same quasi-periodic properties as $\vartheta_3(v\,|\,\tau)$. Obviously $f(v + 1) = f(v)$ and

$$f(v + \tau) = \prod_{m=1}^{\infty} (1 + q^{2m-1} e^{2\pi i v} e^{2\pi i \tau})\,(1 + q^{2m-1} e^{-2\pi i v} e^{-2\pi i \tau})$$
$$= \prod_{m=1}^{\infty} (1 + q^{2m+1} e^{2\pi i v})\,(1 + q^{2m-3} e^{-2\pi i v})$$
$$= \frac{1 + q^{-1} e^{-2\pi i v}}{1 + q e^{2\pi i v}} f(v) = q^{-1} e^{-2\pi i v} f(v)\,.$$

Hence $\vartheta_3(v\,|\,\tau)/f(v) = g(v)$ has the properties

$$g(v + 1) = g(v)\,, \qquad g(v + \tau) = g(v)\,,$$

is a doubly periodic function without singularities, and hence must be a constant (i.e. independent of v).

We can therefore write

$$\vartheta_3(z; q) = \sum_{n=-\infty}^{\infty} q^{n^2} z^n = T(q) \prod_{m=1}^{\infty} (1 + q^{2m-1} z)\,(1 + q^{2m-1} z^{-1})\,, \quad (78.1)$$

where $T(q)$ is free of v.

In order to determine $T(q)$ we apply a device due to Gauss. Putting $z = -1$ and then $z = i$ we obtain

$$T(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \Big/ \prod_{m=1}^{\infty} (1 - q^{2m-1})^2 \tag{78.2}$$

and

$$T(q) = \sum_{n=-\infty}^{\infty} i^n q^{n^2} \Big/ \prod_{m=1}^{\infty} (1 + i q^{2m-1})(1 - i q^{2m-1})$$

$$= \sum_{n=-\infty}^{\infty} (-1)^l q^{(2l)^2} \Big/ \prod_{m=1}^{\infty} (1 + q^{4m-2}).$$

We find on the one hand

$$\frac{T(q)}{\prod\limits_{m=1}^{\infty} (1 - q^{2m})} = \frac{\sum\limits_{n=-\infty}^{\infty} (-1)^n q^{n^2}}{\prod\limits_{m=1}^{\infty} (1 - q^{2m}) \prod\limits_{m=1}^{\infty} (1 - q^{2m-1})^2}$$

$$= \frac{\sum\limits_{n=-\infty}^{\infty} (-1)^n q^{n^2}}{\prod\limits_{m=1}^{\infty} (1 - q^{m}) \prod\limits_{m=1}^{\infty} (1 - q^{2m-1})},$$

and on the other hand

$$\frac{T(q)}{\prod\limits_{m=1}^{\infty} (1 - q^{2m})} = \frac{\sum\limits_{n=-\infty}^{\infty} (-1)^n q^{4n^2}}{\prod\limits_{m=1}^{\infty} (1 - q^{2m}) \prod\limits_{m=1}^{\infty} (1 + q^{4m-2})}$$

$$= \frac{\sum\limits_{n=-\infty}^{\infty} (-1)^n q^{4n^2}}{\prod\limits_{m=1}^{\infty} (1 - q^{4m}) \prod\limits_{m=1}^{\infty} (1 - q^{4m-2}) \prod\limits_{m=1}^{\infty} (1 + q^{4m-2})}$$

$$= \frac{\sum\limits_{n=-\infty}^{\infty} (-1)^n q^{4n^2}}{\prod\limits_{m=1}^{\infty} (1 - q^{4m}) \prod\limits_{m=1}^{\infty} (1 - q^{8m-4})}.$$

The two equations show that $G(q) = T(q) \Big/ \prod\limits_{m=1}^{\infty} (1 - q^{2m})$ remains unchanged if q is replaced by q^4.

Therefore

$$G(q) = G(q^4) = G(q^{16}) = \cdots = G(q^{4^k}) = \cdots ,$$

and since $G(q)$ is continuous in q, and $q^{4^k} \to 0$ as $k \to \infty$, we have

$$G(q) = G(0) ,$$

$$\frac{T(q)}{\prod\limits_{m=1}^{\infty}(1 - q^{2m})} = T(0) = 1 , \quad T(q) = \prod_{m=1}^{\infty}(1 - q^{2m}) .$$

This inserted in (78.1) gives the important result

$$\vartheta_3(z; q) = \vartheta_3(v\,|\,\tau) = \prod_{m=1}^{\infty}(1 - q^{2m})\,(1 + q^{2m-1}e^{2\pi iv})\,(1 + q^{2m-1}e^{-2\pi iv}) . \tag{78.31}$$

From this formula we derive, by means of (76.4),

$$\vartheta_1(v\,|\,\tau) = -iq^{1/4}e^{\pi iv}\,\vartheta_3\left(v + \frac{1+\tau}{2}\right)$$

$$= -iq^{1/4}e^{\pi iv}\prod_{m=1}^{\infty}(1 - q^{2m})\,(1 - q^{2m}e^{2\pi iv})\,(1 - q^{2m-2}e^{-2\pi iv}) ,$$

$$\vartheta_1(v\,|\,\tau) = 2q^{1/4}\sin\pi v\prod_{m=1}^{\infty}(1 - q^{2m})\,(1 - q^{2m}e^{2\pi iv})\,(1 - q^{2m}e^{-2\pi iv}) . \tag{78.32}$$

The formulae (76.4) furnish then the remaining two expressions

$$\vartheta_2(v\,|\,\tau) = 2q^{1/4}\cos\pi v\prod_{m=1}^{\infty}(1 - q^{2m})\,(1 + q^{2m}e^{2\pi iv})\,(1 + q^{2m}e^{-2\pi iv}) , \tag{78.33}$$

$$\vartheta_4(v\,|\,\tau) = \prod_{m=1}^{\infty}(1 - q^{2m})\,(1 - q^{2m-1}e^{2\pi iv})\,(1 - q^{2m-1}e^{-2\pi iv}) . \tag{78.34}$$

It is useful to have also the values of the ϑ-functions for $v = 0$, which we denote by

$$\vartheta_\lambda = \vartheta_\lambda(0\,|\,\tau) , \quad \lambda = 1, 2, 3, 4 .$$

Since $\vartheta_1(0\,|\,\tau) = 0$ it is more interesting to consider

$$\vartheta_1' = \frac{d}{dv}\vartheta_1(v\,|\,\tau)_{v=0} .$$

From (78.31) to (78.34) we obtain

$$\vartheta_1' = 2\pi q^{1/4} \prod_{m=1}^{\infty} (1 - q^{2m})^3 ,$$

$$\vartheta_2 = 2q^{1/4} \prod_{m=1}^{\infty} (1 - q^{2m}) (1 + q^{2m})^2 ,$$

$$\vartheta_3 = \prod_{m=1}^{\infty} (1 - q^{2m}) (1 + q^{2m-1})^2 ,$$

$$\vartheta_4 = \prod_{m=1}^{\infty} (1 - q^{2m}) (1 - q^{2m-1})^2 . \tag{78.4}$$

An easy deduction from the first of these equations is a famous identity of Jacobi. We find ϑ_1' as a series from (76.2) and obtain thus

$$\prod_{m=1}^{\infty} (1 - q^{2m})^3 = \sum_{n=0}^{\infty} (-1)^n (2n + 1) q^{n(n+1)} . \tag{78.51}$$

On both sides of this equation we have power series in q^2, convergent for $|q| < 1$. Putting $q^2 = x$ we can write Jacobi's identity as

$$\prod_{m=1}^{\infty} (1 - x^m)^3 = \sum_{n=0}^{\infty} (-1)^n (2n + 1) x^{n(n+1)/2} , \tag{78.52}$$

where the exponents of the power series on the right are the "triangular numbers".

We observe also

$$\pi\vartheta_2\vartheta_3\vartheta_4 = \vartheta_1' \prod_{m=1}^{\infty} \left((1 + q^{2m}) (1 + q^{2m-1}) (1 - q^{2m-1})\right)^2$$

$$= \vartheta_1' \prod_{m=1}^{\infty} \left((1 + q^{2m}) (1 - q^{4m-2})\right)^2 = \vartheta_1' H(q)^2 ,$$

say. But

$$H(q) = \prod_{m=1}^{\infty} (1 + q^{2m}) (1 - q^{4m-2})$$

$$= \prod_{m=1}^{\infty} (1 + q^{4m}) (1 + q^{4m-2}) (1 - q^{4m-2})$$

$$= \prod_{m=1}^{\infty} (1 + q^{4m}) (1 - q^{8m-4}) ,$$

or

$$H(q) = H(q^2) .$$

By an argument analogous to that applied above to $G(q)$ we conclude

$$H(q) = H(0) = 1$$

and obtain thus

$$\pi\vartheta_2\vartheta_3\vartheta_4 = \vartheta_1' . \tag{78.6}$$

Finally we prove an identity due to Euler. From (78.34) we obtain

$$\vartheta_4\left(\frac{\tau}{6}\middle|\tau\right) = \prod_{m=1}^{\infty}(1 - q^{2m})(1 - q^{2m-1}e^{\pi i\tau/3})(1 - q^{2m-1}e^{-\pi i\tau/3})$$

$$= \prod_{m=1}^{\infty}(1 - q^{2m})(1 - q^{2m-2/3})(1 - q^{2m-4/3})$$

$$= \prod_{\mu=1}^{\infty}(1 - q^{2\mu/3}) .$$

On the other hand (76.1) yields

$$\vartheta_4\left(\frac{\tau}{6}\middle|\tau\right) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{\pi i\tau n/3} = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/3} .$$

If we put here and in the previous equation

$$q^{2/3} = e^{2\pi i\tau/3} = x$$

and replace n by $-\lambda$ we obtain Euler's "pentagonal number theorem"

$$\prod_{m=1}^{\infty}(1 - x^m) = \sum_{\lambda=-\infty}^{\infty}(-1)\, x^{\lambda(3\lambda-1)/2} , \tag{78.7}$$

so called since the exponents here for $\lambda = 1, 2, 3, \ldots$, are the sequence 1, 5, 12, ... of the pentagonal numbers. We shall return to (78.7) and its arithmetical significance later (§§ 98, 99).

79. Transformation of the ϑ-functions

The zeros of $\vartheta_1(v|\tau)$ form a lattice

$$\Lambda = \{m + n\tau\} ,$$

where m and n run through all integers. If

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

is a modular transformation then we know that

$$c\tau + d , \quad a\tau + b$$

generate the same lattice Λ. The lattice

$$\Lambda' = \left\{m + n\,\frac{a\tau + b}{c\tau + d}\right\}$$

is obtained from the lattice Λ by the multiplier $(c\tau + d)^{-1}$. Therefore

$$\vartheta_1\left(\frac{v}{c\tau + d}\,\middle|\,\frac{a\tau + b}{c\tau + d}\right) \tag{79.1}$$

has the same zeros as $\vartheta_1(v\,|\,\tau)$. The theory of (linear) transformations of the ϑ-functions has the purpose of establishing relations between those two functions. We have now always to write the period explicitly.

We start with the two simple and basic cases $\tau' = \tau + 1$ and $\tau' = -1/\tau$. In the first case we have $c\tau + d = 1$, and the function to be considered is $\vartheta_1(v\,|\,\tau + 1)$. We consider therefore

$$G(v) = \frac{\vartheta_1(v\,|\,\tau + 1)}{\vartheta_1(v\,|\,\tau)},$$

which is an entire function without zeros. Now

$$G(v + 1) = \frac{\vartheta_1(v + 1\,|\,\tau + 1)}{\vartheta_1(v + 1\,|\,\tau)} = G(v)$$

according to the table (76.3), where the period, although not explicitly mentioned, is supposed to be τ.

We have also

$$G(v + \tau) = \frac{\vartheta_1(v + \tau\,|\,\tau + 1)}{\vartheta_1(v + \tau\,|\,\tau)} = -\frac{\vartheta_1(v + \tau + 1\,|\,\tau + 1)}{\vartheta_1(v + \tau\,|\,\tau)}$$

$$-\frac{e^{-\pi i(\tau+1)}\,e^{-2\pi i v}\vartheta_1(v\,|\,\tau+1)}{e^{-\pi i\tau}\,e^{-2\pi i v}\vartheta_1(v\,|\,\tau)} = G(v)\,.$$

Thus $G(v)$ is doubly periodic with the periods 1 and τ and must therefore be a constant:

$$\vartheta_1(v\,|\,\tau + 1) = C\vartheta_1(v\,|\,\tau)\,.$$

In order to obtain the constant C it is only necessary to replace the variable v by a specific value. Now $v = 0$ would lead to 0 on both sides of the equation. But differentiation with respect to v yields

$$\vartheta_1'(v\,|\,\tau + 1) = C\vartheta_1'(v\,|\,\tau)\,.$$

Using now the first formula of (78.4) we see

$$2\pi e^{\pi i(\tau+1)/4}\prod_{m=1}^{\infty}(1 - q_1^{2m})^3 = C\cdot 2\pi e^{\pi i\tau/4}\prod_{m=1}^{\infty}(1 - q^{2m})^3$$

with

$$q_1 = e^{\pi i(\tau+1)} = -q\,,$$

so that we obtain

$$C = e^{\pi i/4}$$

and hence

$$\vartheta_1(v\,|\,\tau + 1) = e^{\pi i/4}\,\vartheta_1(v\,|\,\tau)\,. \tag{79.2}$$

As next case we consider $\tau' = -1/\tau$; that is

$$H(v) = \frac{\vartheta_1\left(\dfrac{v}{\tau}\,\middle|\,-\dfrac{1}{\tau}\right)}{\vartheta_1(v\,|\,\tau)}\,,$$

which we can write as

$$H(v) = C\,e^{h(v)}\,, \tag{79.3}$$

since it is an entire function without zeros. We need first the formula

$$\vartheta_1(v - \tau\,|\,\tau) = -e^{-\pi i\tau}e^{2\pi iv}\,\vartheta_1(v\,|\,\tau)\,, \tag{79.4}$$

which is directly obtained from (76.3). Now

$$H(v + 1) = \frac{\vartheta_1\left(\dfrac{v}{\tau} + \dfrac{1}{\tau}\,\middle|\,-\dfrac{1}{\tau}\right)}{\vartheta_1(v + 1\,|\,\tau)}\,.$$

Applying here (79.4) to the numerator and (76.3) to the denominator we obtain

$$H(v + 1) = \frac{-e^{\pi i/\tau}e^{2\pi iv/\tau}\vartheta_1\left(\dfrac{v}{\tau}\,\middle|\,-\dfrac{1}{\tau}\right)}{-\vartheta_1(v\,|\,\tau)} = e^{\pi i/\tau + 2\pi iv/\tau}H(v)\,.$$

Similarly we find that

$$H(v + \tau) = e^{\pi i\tau + 2\pi iv}\,H(v)\,.$$

The last two equations show that $h(v)$ in (79.3), apart from possible multiples of $2\pi i$, has to fulfill the difference equations

$$h(v + 1) - h(v) = \frac{2\pi i}{\tau}v + \frac{\pi i}{\tau}\,,$$

$$h(v + \tau) - h(v) = 2\pi iv + \pi i\tau\,. \tag{79.5}$$

These expressions are linear in v, which suggests as solution a quadratic polynomial

$$h(v) = Av^2 + Bv\,,$$

so that

$$h(v+1) - h(v) = 2Av + A + B\,,$$

$$h(v+\tau) - h(v) = 2A\tau v + A\tau^2 + B\tau\,.$$

We see that both equations (79.5) are satisfied by

$$A = \frac{\pi i}{\tau}\,, \qquad B = 0\,.$$

The function

$$H^*(v) = e^{-\pi i v^2/\tau} H(v)$$

has therefore the properties

$$H^*(v+1) = H^*(v)\,, \quad H^*(v+\tau) = H^*(v)\,,$$

is an entire doubly periodic function, and therefore a constant; in other words,

$$\vartheta_1\left(\frac{v}{\tau}\,\middle|\,-\frac{1}{\tau}\right) = C e^{\pi i v^2/\tau}\vartheta_1(v\,|\,\tau)\,.$$

In order to compute C we differentiate with respect to v and then put $v = 0$, which yields

$$\vartheta_1'\left(0\,\middle|\,-\frac{1}{\tau}\right) = C\tau\vartheta_1'(0\,|\,\tau)\,. \tag{79.6}$$

Now (78.7) shows

$$\vartheta_1'(0\,|\,\tau) = 2\pi e^{\pi i\tau/4}\prod_{m=1}^{\infty}(1 - e^{2\pi i m\tau})^3 = 2\pi\eta(\tau)^3$$

according to Dedekind's definition (65.2), so that (79.6) reads

$$\eta\left(-\frac{1}{\tau}\right)^3 = C\tau\,\eta(\tau)^3\,.$$

Now (74.92) in our case

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

gives

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\,\eta(\tau)\,,$$

so that we have

$$\left(\sqrt{\frac{\tau}{i}}\right)^3 = C\tau$$

and hence

$$C = -i\sqrt{\frac{\tau}{i}}\,,$$

which leads to the result

$$\vartheta_1\left(\frac{v}{\tau}\,\middle|\,-\frac{1}{\tau}\right) = -i\sqrt{\frac{\tau}{i}}\,e^{\pi i v^2/\tau}\vartheta_1(v\,|\,\tau)\,, \tag{79.7}$$

with the principal value of the square root. The transformation formulae for $\vartheta_2(v)$, $\vartheta_3(v)$, $\vartheta_4(v)$ are now quickly obtained through the application of table (76.4). Equation (79.2) yields

$$\begin{aligned} \vartheta_2(v\,|\,\tau+1) &= e^{\pi i/4}\vartheta_2(v\,|\,\tau)\,, \\ \vartheta_3(v\,|\,\tau+1) &= \vartheta_4(v\,|\,\tau)\,, \\ \vartheta_4(v\,|\,\tau+1) &= \vartheta_3(v\,|\,\tau)\,, \end{aligned} \tag{79.8}$$

and from (79.7) follows

$$\begin{aligned} \vartheta_2\left(\frac{v}{\tau}\,\middle|\,-\frac{1}{\tau}\right) &= \sqrt{\frac{\tau}{i}}\,e^{\pi i v^2/\tau}\vartheta_4(v\,|\,\tau)\,, \\ \vartheta_3\left(\frac{v}{\tau}\,\middle|\,-\frac{1}{\tau}\right) &= \sqrt{\frac{\tau}{i}}\,e^{\pi i v^2/\tau}\vartheta_3(v\,|\,\tau)\,, \\ \vartheta_4\left(\frac{v}{\tau}\,\middle|\,-\frac{1}{\tau}\right) &= \sqrt{\frac{\tau}{i}}\,e^{\pi i v^2/\tau}\vartheta_2(v\,|\,\tau)\,. \end{aligned} \tag{79.9}$$

80. Transformation of $\vartheta_1(v\,|\,\tau)$, continued

The effect of the general modular transformation

$$\tau' = \frac{a\tau+b}{c\tau+d} \tag{80.1}$$

on $\vartheta_1(v\,|\,\tau)$ can be treated in a similar way by means of Theorem § 74. We can assume $c > 0$, since for $c = 0$, $a = d = 1$ we have immediately from (79.2)

$$\vartheta_1(v\,|\,\tau+b) = e^{\pi i b/4}\vartheta_1(v\,|\,\tau)\,. \tag{80.2}$$

We need also the formula

$$\vartheta_1(v+k+l\tau) = (-1)^{k+l}e^{-\pi i l^2\tau}e^{-\pi i l v}\,\vartheta_1(v\,|\,\tau)\,, \tag{80.31}$$

obtained from the first line of (76.3) by a simple process of iteration, with k, l integers.

We observe further as consequences of (80.1) the useful identities

$$(c\tau + d)\,(c\tau' - a) = -1\,, \tag{80.32}$$

and

$$\frac{1}{c\tau + d} = -c\tau' + a\,, \qquad \frac{\tau}{c\tau + d} = d\tau' - b\,. \tag{80.33}$$

Now, according to the remarks about (79.1), the quotient

$$F(v) = \frac{\vartheta_1\left(\dfrac{v}{c\tau + d}\,\middle|\,\tau'\right)}{\vartheta_1(v|\tau)}$$

is an entire function without zeros. We have, using (80.33),

$$F(v+1) = \frac{\vartheta_1\left(\dfrac{v}{c\tau + d} + \dfrac{1}{c\tau + d}\,\middle|\,\tau'\right)}{\vartheta_1(v + 1|\tau)} = \frac{\vartheta_1\left(\dfrac{v}{c\tau + d} + a - c\tau'|\tau'\right)}{\vartheta_1(v + 1|\tau)}$$

$$= (-1)^{a-c+1} e^{-\pi i c^2\tau'} e^{\frac{2\pi i c v}{c\tau+d}} F(v)\,,$$

and since

$$1 + a - c - ac = (1 + a)\,(1 - c)$$

is even (a and c cannot both be even), we can write

$$F(v+1) = e^{-\pi i c(c\tau' - a)}\, e^{\frac{2\pi i c}{c\tau+d}v} F(v) = e^{\frac{\pi i c}{c\tau+d}}\, e^{\frac{2\pi i c}{c\tau+dv}v} F(v)\,. \tag{80.4}$$

Similarly we obtain, again by (80.33) and (80.31),

$$F(v+\tau) = \frac{\vartheta_1\left(\dfrac{v}{c\tau + d} + \dfrac{\tau}{c\tau + d}\,\middle|\,\tau'\right)}{\vartheta_1(v + \tau|\tau)} = \frac{\vartheta_1\left(\dfrac{v}{c\tau + d} - b + d\tau'|\tau'\right)}{\vartheta_1(v + \tau|\tau)}$$

$$= (-1)^{-b+d+1} e^{-\pi i d^2\tau'} e^{\frac{-2\pi i d}{c\tau+d}v + 2\pi i v} e^{\pi i \tau} F(v)\,.$$

Now $1 - b + d - bd = (1 - b)\,(1 + d)$ being even we have

$$(-1)^{-b+d+1} = (-1)^{bd} = e^{\pi i b d}\,,$$

and thus

$$F(v+\tau) = e^{-\pi i (d(d\tau' - b) - \tau)}\, e^{\frac{-2\pi i v}{c\tau+d}(d - (c\tau + d))} F(v)\,.$$

According to (80.33) we have

$$(d\tau' - b)\,(c\tau + d) = \tau$$

and thus

$$d(d\tau' - b) - \tau = -c\tau(d\tau' - b)\,,$$

so that

$$F(v + \tau) = e^{\pi i c\tau(d\tau'-b)}\,e^{\frac{2\pi i}{c\tau+d}c\tau v}\,F(v)\,. \tag{80.5}$$

In the processes $v \to v + 1$, $v \to v + \tau$, $F(v)$ in both cases picks up an exponential factor which is linear in v. We write, as in the preceding paragraph,

$$F(v) = C\,e^{f(v)}\,,$$

which is possible since $F(v)$ is free of zeros, and try $f(v)$ as a quadratic polynomial $f(v) = A v^2 + Bv$, which then in view of (80.4), (80.5) has to fulfill

$$f(v + 1) - f(v) = 2Av + A + B = \frac{2\pi i c}{c\tau + d}\,v + \frac{\pi i c}{c\tau + d}\,, \tag{80.61}$$

$$\begin{aligned} f(v + \tau) - f(v) &= 2Av\tau + A\tau^2 + B\tau \\ &= \frac{2\pi i c}{c\tau + d}\,\tau v + \pi i c\tau(d\tau' - b). \end{aligned} \tag{80.62}$$

The terms linear in v yield here

$$A = \frac{\pi i c}{c\tau + d}\,.$$

Then (80.61) gives $B = 0$, which fits also in (80.62), since

$$\frac{\tau}{c\tau + d} = d\tau' - b$$

after (80.33). Therefore

$$F^*(v) = e^{\frac{-\pi i c}{c\tau+d}v^2}\,F(v)$$

fulfills the relations

$$F^*(v + 1) = F^*(v)\,,\quad F^*(v + \tau) = F^*(v)\,.$$

This means that $F^*(v)$ as a doubly periodic entire function has to be a constant, and we conclude

$$\vartheta_1\left(\frac{v}{c\tau + d}\,\middle|\,\tau'\right) = C\,e^{\frac{\pi i c}{c\tau+d}v^2}\,\vartheta_1(v\,|\,\tau)\,. \tag{80.7}$$

For the determination of the constant C we proceed as before:

$$\vartheta_1'(0|\tau') = (c\tau + d)\, C\vartheta_1'(0|\tau)\,,$$

or in view of (78.7),

$$\eta^3(\tau') = (c\tau + d)\, C\eta^3(\tau)\,.$$

Now the theorem in § 74 shows

$$\eta^3(\tau) = \varepsilon(a, b, c, d)^3 \frac{c\tau + d}{i} \sqrt{\frac{c\tau + d}{i}}\, \eta^3(\tau)\,,$$

where $\varepsilon(a, b, c, d)$ is given in (74.93), so that we obtain

$$C = -i\,\varepsilon(a, b, c, d)^3 \sqrt{\frac{c\tau + d}{i}}\,.$$

The formula (74.93) shows that ε^3 permits some simplifications, since $1 - c^2$ and $1 - d^2$, respectively, are divisible by 8. Observing also (80.2), we obtain the

Theorem. *The function $\vartheta_1(v|\tau)$ satisfies the following relations under the modular transformation*

$$\tau' = \frac{a\tau + b}{c\tau + d}\,.$$

(1) *For $c = 0$ we have*

$$\vartheta_1(v|\tau + b) = e^{\pi i b/4}\, \vartheta_1(v|\tau)\,; \tag{80.2}$$

(2) *for $c > 0$ we have*

$$\vartheta_1\left(\frac{v}{c\tau + d}\,\middle|\,\tau'\right) = \varepsilon_1(a, b, c, d) \sqrt{\frac{c\tau + d}{i}}\, e^{\pi i c v^2/(c\tau + d)}\, \vartheta_1(v|\tau) \tag{80.8}$$

with

$$\varepsilon_1(a, b, c, d) = -i\,\varepsilon(a, b, c, d)^3 = \begin{cases} \left(\dfrac{d}{c}\right) i^{(c-3)/2}\, e^{(\pi i/4)\, c(a+d)}\,, & c \text{ odd}\,, \\[2ex] \left(\dfrac{c}{d}\right) e^{\pi i/4}\, i^{(1-d)/2}\, e^{(\pi i/4)\, d(b-c)}\,, & d \text{ odd}. \end{cases}$$

81. Transformation of $\vartheta_2(v|\tau)$, $\vartheta_3(v|\tau)$, $\vartheta_4(v|\tau)$

By means of this theorem and the table (76.4) we can easily write down the transformation formulae for the other three ϑ-functions. First we obtain, for $c = 0$:

$$\vartheta_2(v|\tau + b) = e^{\pi i b/4}\vartheta_2(v|\tau) , \tag{81.11}$$

$$\vartheta_3(v|\tau + b) = \begin{cases} \vartheta_3(v|\tau), & b \text{ even} , \\ \vartheta_4(v|\tau) , & b \text{ odd} , \end{cases} \tag{81.12}$$

$$\vartheta_4(v|\tau + b) = \begin{cases} \vartheta_4(v|\tau) , & b \text{ even} , \\ \vartheta_3(v|\tau) , & b \text{ odd} . \end{cases} \tag{81.13}$$

For the study of the general modular transformation it is advisable to return to the notation (76.5) with double subscripts:

$$\vartheta_{\mu,\nu}(v|\tau) = \sum_{n=-\infty}^{\infty} (-1)^{\nu n} e^{(n+\mu/2)^2 \pi i \tau} e^{2\pi i (n+\mu/2) v} . \tag{81.2}$$

We have

$$\vartheta_1(v) = \frac{1}{i}\vartheta_{11}(v) , \quad \vartheta_2(v) = \vartheta_{10}(v) ,$$

$$\vartheta_3(v) = \vartheta_{00}(v) , \quad \vartheta_4(v) = \vartheta_{01}(v) . \tag{81.21}$$

If we do not restrict the subscripts μ, ν to the values 0 and 1 but admit any integers then we observe

$$\vartheta_{\mu,\nu+2}(v|\tau) = (-1)^{\nu}\vartheta_{\mu,\nu}(v|\tau) ,$$

$$\vartheta_{\mu,\nu+2}(v|\tau) = \vartheta_{\mu,\nu+2} . \tag{81.31}$$

We read off from (81.2) after some simple calculations

$$\vartheta_{\mu\nu}\left(v + \frac{r + s\tau}{2}\middle|\tau\right) = i^{\mu r} e^{-\pi i s^2 \tau/4} e^{-\pi i s v}\vartheta_{\mu+s,\nu+r}(v|\tau) , \tag{81.32}$$

a formula which with (81.21), (81.31) includes all the formulae (76.3), (76.4), (80.31) as special cases.

We consider now

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

with $c > 0$, and start from (80.8), where we replace on both sides $i\vartheta_1$ by ϑ_{11}. Substituting $v + 1/2$ for v we have

$$\vartheta_{11}\left(\frac{v + \frac{1}{2}}{c\tau + d}\middle|\tau'\right) = \varepsilon_1(a, b, c, d)\sqrt{\frac{c\tau + d}{i}}\, e^{\frac{\pi i c}{c\tau + d}\left(v + \frac{1}{2}\right)^2}\vartheta_{11}\left(v + \frac{1}{2}\tau\right)$$

and in view of (80.33), (81.32) and (81.21)

$$i^a e^{\frac{-\pi i\tau' c^2}{4}} e^{\frac{\pi i c v}{c\tau+d}} \vartheta_{1-c,1+a}\left(\frac{v}{c\tau+d}\,\middle|\,\tau'\right)$$
$$= \varepsilon_1 \sqrt{\frac{c\tau+d}{i}}\, e^{\frac{\pi i c}{c\tau+d}\left(v+\frac{1}{2}\right)^2}\, i\vartheta_2(v\,|\,\tau)\,,$$

which after some simplifications leads to

$$\vartheta_{1-c,1+a}\left(\frac{v}{c\tau+d}\,\middle|\,\tau'\right) = i^{1-a}\, e^{(\pi i/4)ac}\, \varepsilon_1 \sqrt{\frac{c\tau+d}{i}}\, e^{\pi i c v^2/(c\tau+d)}\, \vartheta_2(v\,|\,\tau)\,. \tag{81.4}$$

In the same way we obtain from (80.8) by the substitutions $v \to v + \tau/2$ and $v \to v + (1+\tau)/2$ respectively

$$\vartheta_{1+d,1-b}\left(\frac{v}{c\tau+d}\,\middle|\,\tau'\right) = -i^b\, e^{\pi i b d/4}\, \varepsilon_1 \sqrt{\frac{c\tau+d}{i}}\, e^{\pi i c v^2/(c\tau+d)}\, \vartheta_4(v\,|\,\tau) \tag{81.5}$$

and

$$\vartheta_{1+d-c,1-b+a}\left(\frac{v}{c\tau+d}\,\middle|\,\tau'\right)$$
$$= i^{1+b-a}\, e^{\pi i/4}\, e^{(\pi i/4)(c-d)(a-b)}\, \varepsilon_1 \sqrt{\frac{c\tau+d}{i}}\, e^{\pi i c v^2/(c\tau+d)}\, \vartheta_3(v\,|\,\tau)\,. \tag{81.6}$$

The last three formulae would be more useful if they would give the transformations of ϑ_2, ϑ_3, ϑ_4, in other words if they would run in the opposite direction. This can easily be achieved by introducing $v' = v/(c\tau+d)$, τ' as the independent variables and expressing v and τ through them. This requires only some straightforward calculation, at the end of which we drop the dashes of v' and τ'. Then we get the following

Theorem. *For $c = 0$ the transformation of ϑ_2, ϑ_3, ϑ_4 is expressed by the formulae* (81.11) *to* (81.13). *For $c > 0$ we have*

$$\vartheta_2\left(\frac{v}{c\tau+d}\,\middle|\,\frac{a\tau+b}{c\tau+d}\right)$$
$$= i^{3-d}\, e^{(\pi i/4)dc}\, \varepsilon_2 \sqrt{\frac{c\tau+d}{i}}\, e^{\pi i c v^2/(c\tau+d)}\, \vartheta_{1-c,1-d}(v\,|\,\tau)\,,$$

$$\vartheta_3\left(\frac{v}{c\tau+d}\,\middle|\,\frac{a\tau+b}{c\tau+d}\right)$$
$$= i^{3-b-d} e^{(-\pi i/4)} e^{(\pi i/4)(c+a)(d+b)}\, \varepsilon_2 \sqrt{\frac{c\tau+d}{i}}\, e^{\pi i c v^2/(c\tau+d)} \vartheta_{1-a-c,1-b-d}(v\,|\,\tau),$$

$$\vartheta_4\left(\frac{v}{c\tau+d}\,\middle|\,\frac{a\tau+b}{c\tau+d}\right)$$
$$= i^{2-b}\, e^{\pi i a b/4}\, \varepsilon_2 \sqrt{\frac{c\tau+d}{i}}\, e^{\pi i c v^2/(c\tau+d)}\, \vartheta_{1-a,1-b}(v\,|\,\tau)$$

with

$$\varepsilon_2 = \varepsilon_2(a, b, c, d) = \varepsilon_1(-d, b, c, -a)^{-1}$$

$$= \begin{cases} \left(\dfrac{a}{c}\right) i^{(c+1)/2}\, e^{(\pi i/4)\, c(a+d)}\,, & c \text{ odd}\,, \\ \left(\dfrac{c}{a}\right) e^{(-\pi i/4)}\, i^{-(a+1)/2}\, e^{(\pi i/4)\, a(b-c)}\,, & a \text{ odd}\,, \end{cases}$$

where for $a < 0$

$$\left(\frac{c}{a}\right) = \left(\frac{c}{-a}\right).$$

Chapter 11

Elliptic Functions and their Applications to Number Theory

82. Construction of elliptic functions from the ϑ-functions

The ϑ-functions take up a factor if the argument v is replaced by $v + \tau$ (see § 75). By forming quotients of ϑ-functions we can abolish this factor. Let us consider the functions

$$f_\alpha(v\,|\,\tau) = \frac{\vartheta_1'}{\vartheta_\alpha}\,\frac{\vartheta_\alpha(v\,|\,\tau)}{\vartheta_1(v\,|\,\tau)}\,, \qquad \alpha = 2, 3, 4\,. \tag{82.1}$$

These functions have a pole of first order at $v = 0$; the constant factor $\vartheta_1'/\vartheta_\alpha$ is so chosen that the residue of $f_\alpha(v/\tau)$ at $v = 0$ is just 1. Now, consulting the list (76.3) we see that

$$\begin{aligned} f_2(v+1\,|\,\tau) &= f_2(v\,|\,\tau)\,, & f_2(v+\tau\,|\,\tau) &= -f_2(v\,|\,\tau)\,, \\ f_3(v+1\,|\,\tau) &= -f_3(v\,|\,\tau)\,, & f_3(v+\tau\,|\,\tau) &= -f_3(v\,|\,\tau)\,, \\ f_4(v+1\,|\,\tau) &= -f_4(v\,|\,\tau)\,, & f_4(v+\tau\,|\,\tau) &= f_4(v\,|\,\tau)\,. \end{aligned} \tag{82.2}$$

The functions go over into themselves, aside from a $\pm$ sign. Thus their squares $f_\alpha^2(v\,|\,\tau)$ are doubly periodic meromorphic functions, in other words elliptic functions of periods $1, \tau$.

They all have poles of 2nd order at $v = 0$. They are even functions, so that their residue is 0 at these poles. Their Laurent expansions begin in all cases with $1/v^2$. Therefore any difference

$$f_\alpha^2(v\,|\,\tau) - f_\beta^2(v\,|\,\tau)$$

is doubly periodic, free of poles, and thus a constant. Let us put

$$f_\alpha^2(v\,|\,\tau) - f_\beta^2(v\,|\,\tau) = C_{\alpha\beta}$$

and determine this constant; which we can do by putting v equal to a special value. Observing that $\vartheta_\alpha(v)$ is even, $\alpha = 2, 3, 4$ and $\vartheta_1(v)$ odd we have

$$f_\alpha^2(v|\tau) = \left(\frac{\vartheta_1'}{\vartheta_\alpha}\,\frac{\vartheta_\alpha + \vartheta_\alpha''\dfrac{v^2}{2} + \cdots}{\vartheta_1' v + \vartheta_1'''\dfrac{v^3}{6} - \cdots}\right)^2 = \frac{1}{v^2}\,\frac{1 + \dfrac{\vartheta_\alpha''}{\vartheta_\alpha} v^2 + \cdots}{1 + \dfrac{\vartheta_1'''}{\vartheta_1'}\dfrac{v^2}{3} + \cdots}$$

$$= \frac{1}{v^2}\left(1 + \left(\frac{\vartheta_\alpha''}{\vartheta_\alpha} - \frac{1}{3}\frac{\vartheta_1'''}{\vartheta_1'}\right) v^2 + \cdots\right) \tag{82.3}$$

and thus

$$C_{\alpha\beta} = f_\alpha^2(v|\tau) - f_\beta^2(v|\tau) = \frac{\vartheta_\alpha''}{\vartheta_\alpha} - \frac{\vartheta_\beta''}{\vartheta_\beta},$$

where the dash $'$ refers to differentiation with respect to v, e.g.,

$$\vartheta_\alpha'' = \frac{d^2}{dv^2}\vartheta_\alpha(v|\tau)\,|_{v=0}\,.$$

In view of (76.6) we have

$$C_{\alpha\beta} = 4\pi i\left(\frac{\dfrac{\partial}{\partial\tau}\vartheta_\alpha}{\vartheta_\alpha} - \frac{\dfrac{\partial}{\partial\tau}\vartheta_\beta}{\vartheta_\beta}\right) = 4\pi i\frac{\partial}{\partial\tau}\log\frac{\vartheta_\alpha(0|\tau)}{\vartheta_\beta(0|\tau)}\,.$$

83. Sums of four squares

For an arithmetical application we are in particular interested in

$$C_{42} = f_4^2(v|\tau) - f_2^2(v|\tau) = 4\pi i\frac{\partial}{\partial\tau}\log\frac{\vartheta_4(0|\tau)}{\vartheta_2(0|\tau)}\,. \tag{83.1}$$

We have obtained C_{42} by putting $v = 0$; we can choose other values. With $v = 1/2$ we have

$$C_{42} = f_4^2\left(\frac{1}{2}\middle|\tau\right) = \left(\frac{\vartheta_1'}{\vartheta_4}\,\frac{\vartheta_4\left(\frac{1}{2}\middle|\tau\right)}{\vartheta_1\left(\frac{1}{2}\middle|\tau\right)}\right)^2 = \left(\frac{\vartheta_1'\vartheta_3}{\vartheta_4\vartheta_2}\right)^2 = \pi^2\vartheta_3^4\,, \tag{83.2}$$

where we have used (78.6). And finally for $v = (1+\tau)/2$ we obtain

$$C_{42} = f_4^2\left(\frac{1+\tau}{2}\middle|\tau\right) - f_2^2\left(\frac{1+\tau}{2}\middle|\tau\right) = \left(\frac{\vartheta_1'}{\vartheta_4}\,\frac{\vartheta_2}{\vartheta_3}\right)^2 + \left(\frac{\vartheta_1'}{\vartheta_2}\,\frac{\vartheta_4}{\vartheta_3}\right)^2$$

$$= \pi^2\vartheta_2^4 + \pi^2\vartheta_4^4\,.$$

This equation together with (83.2) gives the important result

$$\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4\,. \tag{83.3}$$

It is now convenient to go over to the notation

$$\vartheta_\alpha(0\,|\,\tau) = \vartheta_\alpha(0, q)$$

mentioned in § 78. With $q = e^{\pi i \tau}$ we have

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial q} \cdot \frac{dq}{d\tau} = \pi i q \frac{\partial}{\partial q}$$

and thus instead of (83.1)

$$C_{42} = -\,4\pi^2 q \frac{\partial}{\partial q} \log \frac{\vartheta_4(0, q)}{\vartheta_2(0, q)}\,.$$

Now after (78.4)

$$\frac{\vartheta_2(0, q)}{\vartheta_4(0, q)} = 2q^{1/4} \frac{\prod\limits_m (1 + q^{2m})^2}{\prod\limits_m (1 - q^{2m-1})^2} = 2q^{1/4} \frac{\prod\limits_m (1 - q^{4m})^2}{\prod\limits_m (1 - q^{m})^2}$$

and hence

$$C_{42} = 4\pi^2 q \left\{ \frac{1}{4q} - 8 \sum_{m=1}^{\infty} \frac{m q^{4m-1}}{1 - q^{4m}} + 2 \sum_{m=1}^{\infty} \frac{m q^{m-1}}{1 - q^m} \right\}. \tag{83.4}$$

Observing now, after (76.1),

$$\vartheta_3 = \sum_{n=-\infty}^{\infty} q^{n^2}$$

and taking into account (83.2) we have

$$\left(\sum_{-\infty}^{\infty} q^{n^2} \right)^4 = 1 + 8 \sum_{m=1}^{\infty} \frac{m q^m}{1 - q^m} - 8 \sum_{m=1}^{\infty} \frac{4m q^{4m}}{1 - q^{4m}}$$

$$= 1 + 8 \sum_{\substack{m \geqq 1 \\ m \not\equiv 0 (\mathrm{mod}\, 4)}} \frac{m q^m}{1 - q^m} = 1 + 8 \sum_{\substack{m=1 \\ m \not\equiv 0 (\mathrm{mod}\, 4)}}^{\infty} \sum_{k=1}^{\infty} m q^{km}\,,$$

$$\left(\sum_{-\infty}^{\infty} q^{n^2} \right)^4 = 1 + 8 \sum_{r=1}^{\infty} q^r \sum_{\substack{m | r \\ 4 \nmid m}} m \;{}^{1}. \tag{83.5}$$

This identity permits a number-theoretical interpretation. On both sides we have power series in q, convergent for $|q| < 1$. The coefficients must agree. If we write the left-hand member as

$$\sum_{\substack{n_1, n_2, n_3, n_4 \\ -\infty}}^{\infty} q^{n_1^2 + n_2^2 + n_3^2 + n_4^2} = \sum_{n=0}^{\infty} A_4(n) q^n$$

[1] This identity was discovered by Jacobi on April 24th, 1828.

then $A_4(n)$ gives the number of representations of n as a sum of 4 squares, where representations are counted separately if they differ in the arrangement of the summands, and also $(n_i)^2$ and $(-n_i)^2$ have to be counted as different summands. Then our previous equation states

$$A_4(n) = 8 \sum_{\substack{d|n \\ 4\nmid d}} d\,. \tag{83.6}$$

If n is odd then we have on the right 8 times the sum of the odd divisors of n. If n is even then

$$\sum_{\substack{d|n \\ 4\nmid d}} d = \sum_{\substack{d|n \\ d \text{ odd}}} d + 2 \sum_{\substack{d|n \\ d \text{ odd}}} d = 3 \sum_{\substack{d|n \\ d \text{ odd}}} d\,.$$

We have thus the

Theorem (Jacobi). *The number $A_4(n)$ of representations of a natural number n as a sum of squares is* 8 *times the sum of its odd divisors if n is odd, and* 24 *times the sum of its odd divisors if n is even.*

This implies Lagrange's less specific theorem that any positive integer is the sum of at most 4 squares[1].

84. Sums of two squares

With a little more effort we can derive also a formula for ϑ_3^2. The function $f_4(v\,|\,\tau)$ has the periods 2 and τ as (82.2) shows. It has simple poles at the lattice points $v \equiv 0$ (modd $1, \tau$) and in particular the residues

$$+1 \text{ at } v \equiv 0 \text{ (modd } 2, \tau)\,, \qquad -1 \text{ at } v \equiv 1 \text{ (modd } 2, \tau)\,.$$

On the other hand the function

$$\Phi(v\,|\,\tau) = \frac{\vartheta_1'(v\,|\,\tau)}{\vartheta_1(v\,|\,\tau)} = \frac{d}{dv} \log \vartheta_1(v\,|\,\tau) \tag{84.1}$$

has simple poles of residues $+1$ at all points $v \equiv 0$ (modd $1, \tau$). It cannot be an elliptic function, since the sum of the residues of an elliptic function must be 0 in a fundamental region. Indeed we see from (76.3)

$$\Phi(v+1\,|\,\tau) = \Phi(v\,|\,\tau)\,, \quad \Phi(v+\tau\,|\,\tau) = -2\pi i + \Phi(v\,|\,\tau)\,. \tag{84.2}$$

Now the function

$$\frac{1}{2}\,\Phi\left(\frac{v}{2}\,\middle|\,\frac{\tau}{2}\right)$$

[1] For a direct proof of Lagrange's theorem see, e.g., [33], vol. 1, pp. 107–9.

has poles of residues 1 at $v \equiv 0$ (modd 2, τ) and

$$\frac{1}{2}\,\Phi\left(\frac{v}{2}\middle|\frac{\tau}{2}\right) - \frac{1}{2}\,\Phi\left(\frac{v+1}{2}\middle|\frac{\tau}{2}\right)$$

has the periods 2, τ as (84.2) shows, and has obviously the residues

$$+1 \text{ at } v \equiv 0 \text{ (modd } 2, \tau) , \qquad -1 \text{ at } v \equiv 1 \text{ (modd } 2, \tau) ,$$

and agrees thus in its poles with $f_4(v\,|\,\tau)$. Therefore

$$f_4(v\,|\,\tau) - \frac{1}{2}\,\Phi\left(\frac{v}{2}\middle|\frac{\tau}{2}\right) + \frac{1}{2}\,\Phi\left(\frac{v+1}{2}\middle|\frac{\tau}{2}\right) = C\,, \tag{84.31}$$

since it is a doubly periodic function without singularities. In order to determine C we realize that the difference

$$f_4(v\,|\,\tau) - \frac{1}{2}\,\Phi\left(\frac{v}{2}\middle|\frac{\tau}{2}\right)$$

is regular at $v = 0$, and being an odd function, vanishes at $v = 0$. Therefore

$$C = \frac{1}{2}\,\Phi\left(\frac{1}{2}\middle|\frac{\tau}{2}\right) = \frac{1}{2}\,\frac{\vartheta_1'\left(\frac{1}{2}\middle|\frac{\tau}{2}\right)}{\vartheta_1\left(\frac{1}{2}\middle|\frac{\tau}{2}\right)} = \frac{1}{2}\,\frac{\vartheta_2'\left(0\middle|\frac{\tau}{2}\right)}{\vartheta_2\left(0\middle|\frac{\tau}{2}\right)} = 0\,, \tag{84.32}$$

since $\vartheta_2(v\,|\,\tau)$ is an even function of v. We have thus gained the equality

$$\frac{\vartheta_1'}{\vartheta_4}\,\frac{\vartheta_4\,(v\,|\,\tau)}{\vartheta_1\,(v\,|\,\tau)} = \frac{1}{2}\,\Phi\left(\frac{v}{2}\middle|\frac{\tau}{2}\right) - \frac{1}{2}\,\Phi\left(\frac{v+1}{2}\middle|\frac{\tau}{2}\right), \tag{84.4}$$

with Φ defined in (84.1).

The left-hand member of (84.3) becomes for $v = 1/2$

$$\frac{\vartheta_1'}{\vartheta_4}\,\frac{\vartheta_4\,(1/2\,|\,\tau)}{\vartheta_1\,(1/2\,|\,\tau)} = \frac{\vartheta_1'\,\vartheta_3}{\vartheta_4\,\vartheta_2} = \pi\vartheta_3^2 \tag{84.5}$$

after (78.6). On the other hand

$$\vartheta_1\left(v\middle|\frac{\tau}{2}\right) = 2q^{1/8}\sin\pi v \prod_{m=1}^{\infty}(1-q^m)\,(1-q^m e^{2\pi i v})\,(1-q^m e^{-2\pi i v})\,,$$

where we have replaced in (78.32) $q = e^{\pi i \tau}$ by $q^{1/2} = e^{\pi i \tau/2}$. By logarithmic differentiation we obtain

$$\begin{aligned}\Phi\left(v\middle|\frac{\tau}{2}\right) &= \frac{\vartheta_1'(v\,|\,\tau/2)}{\vartheta_1(v\,|\,\tau/2)}\\ &= \pi\cot\pi v + 2\pi i \sum_{m=1}^{\infty}\left(\frac{q^m e^{2\pi i v}}{1-q^m e^{2\pi i v}} + \frac{q^m e^{-2\pi i v}}{1-q^m e^{-2\pi i v}}\right)\end{aligned}$$

and thus

$$\Phi\left(\frac{1}{4}\middle|\frac{\tau}{2}\right) = \pi + 2\pi i \sum_{m=1}^{\infty}\left(\frac{-q^m i}{1-q^m i} + \frac{-q^m i}{1+q^m i}\right)$$

$$= \pi + 4\pi \sum_{m=1}^{\infty} \frac{q^m}{1+q^{2m}}.$$

Similarly,

$$\Phi\left(\frac{3}{4}\middle|\frac{\tau}{2}\right) = -\pi - 4\pi \sum_{m=1}^{\infty} \frac{q^m}{1+q^{2m}}.$$

These two results together with (84.4) and (84.5) yield now

$$\vartheta_3^2 = 1 + 4\sum_{m=1}^{\infty}\frac{q^m}{1+q^{2m}} = 1 + 4\sum_{m=1}^{\infty}\frac{q^m - q^{3m}}{1-q^{4m}}$$

$$= 1 + 4\sum_{m=1}^{\infty}\sum_{k=0}^{\infty} q^{(4k+1)m} - 4\sum_{m=1}^{\infty}\sum_{k=0}^{\infty} q^{(4k+3)m}$$

$$= 1 + 4\sum_{n=1}^{\infty} q^n \sum_{4k+1|n} 1 - 4\sum_{n=3}^{\infty} q^n \sum_{4k+3|n} 1,$$

or finally,

$$\left(\sum_{-\infty}^{\infty} q^{n^2}\right)^2 = 1 + 4\sum_{n=1}^{\infty}\sigma_0^{(1)}(n)\, q^n - 4\sum_{n=3}^{\infty}\sigma_0^{(3)}(n)\, q^n \tag{84.6}$$

where

$$\sigma_0^{(1)}(n) = \sum_{\substack{d|n\\ d\equiv 1(4)}} 1, \quad \sigma_0^{(3)}(n) = \sum_{\substack{d|n\\ d\equiv 3(4)}} 1.$$

We have here the

Theorem. *The number $A_2(n)$ of representations of n as a sum of 2 squares is*

$$A_2(n) = 4\left(\sigma_0^{(1)}(n) - \sigma_0^{(3)}(n)\right). \tag{84.7}$$

Corollary 1. $\sigma_0^{(1)}(n) \geqq \sigma_0^{(3)}(n)$ *for all* $n \geqq 1$,

and further

Corollary 2. *For a prime number* $p \equiv 1 \pmod 4$

$$A_2(p) = 8. \tag{84.8}$$

In other words, such a p is the sum of two squares. This is Fermat's famous theorem. Moreover (84.8) shows that for $p \equiv 1 \pmod 4$ the decomposition

$$p = a^2 + b^2$$

is essentially unique, if we disregard the order of the summands and the signs in $\pm a$, $\pm b$, which together would account for the 8 possibilities stated in (84.8).

It is noteworthy, that $A_4(n)$ in (83.5) and $A_2(n)$ in (84.7) appear as simple divisor functions of n. This feature will appear also in the representation of n by 6 and 8 squares, but will be modified for higher even numbers of square summands.

85. Lambert series for $f_\alpha(v)$

From (84.31), (84.32) we take

$$f_4(v\,|\,\tau) = \frac{1}{2}\,\Phi\left(\frac{v}{2}\,\middle|\,\frac{\tau}{2}\right) - \frac{1}{2}\,\Phi\left(\frac{v+1}{2}\,\middle|\,\frac{\tau}{2}\right) \tag{85.1}$$

with

$$\Phi\left(v\,\middle|\,\frac{\tau}{2}\right) = \frac{\vartheta_1'\left(v\,\middle|\,\frac{\tau}{2}\right)}{\vartheta_1\left(v\,\middle|\,\frac{\tau}{2}\right)} = \frac{d}{dv}\log\vartheta_1\left(v\,\middle|\,\frac{\tau}{2}\right)$$

$$= \pi\cot\pi v + 2\pi i\sum_{m=1}^{\infty}\left(\frac{-q^m e^{2\pi iv}}{1-q^m e^{2\pi iv}} + \frac{q^m e^{-2\pi iv}}{1-q^m e^{-2\pi iv}}\right),$$

where in (78.32) we had to replace $q = e^{\pi i\tau}$ by $q^{1/2} = e^{\pi i\tau/2}$. The above series is convergent in the whole v-plane, leaving the poles aside. Let us, however, now assume

$$|q e^{\pm 2\pi iv}| < 1$$

or

$$\operatorname{Im}\left(-\frac{\tau}{2}\right) < \operatorname{Im}(v) < \operatorname{Im}\frac{\tau}{2}. \tag{85.2}$$

Then we can expand each term of the Lambert series into a geometric series and obtain

$$\Phi\left(v\,\middle|\,\frac{\tau}{2}\right) = \pi\cot\pi v - 2\pi i\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left(q^{km}e^{2\pi ikv} - q^{km}e^{-2\pi ikv}\right)$$

$$= \pi\cot\pi v + 4\pi\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} q^{km}\sin 2\pi kv$$

$$= \pi\cot\pi v + 4\pi\sum_{k=1}^{\infty}\frac{q^k}{1-q^k}\sin 2\pi kv\,, \tag{85.3}$$

where the series is convergent in the strip (85.2) which includes only the poles at $v =$ integer, produced by the cotangent term. The insertion of

(85.3) into (85.1) yields, after some simplifications,

$$f_4(v|\tau) = \frac{\pi}{\sin \pi v} + 4\pi \sum_{j=1}^{\infty} \frac{q^{2j-1}}{1 - q^{2j-1}} \sin (2j - 1)\pi v . \tag{85.4}$$

Exercises. Show in analogy to § 84 that

$$f_2(v|\tau) = \Phi(v|2\tau) - \Phi(v + \tau|2\tau) + C = \frac{\vartheta_1'(v|2\tau)}{\vartheta_2(v|2\tau)} - \frac{\vartheta_4'(v|2\tau)}{\vartheta_4(v|2\tau)}, \tag{85.5}$$

and

$$\begin{aligned} f_3(v|\tau) &= \frac{1}{2}\Phi\left(\frac{v}{2}\middle|\frac{1+\tau}{2}\right) - \frac{1}{2}\Phi\left(\frac{v+1}{2}\middle|\frac{1+\tau}{2}\right) + C \\ &= \frac{1}{2}\,\frac{\vartheta_1'\left(\frac{v}{2}\middle\|\frac{1+\tau}{2}\right)}{\vartheta_1\left(\frac{v}{2}\middle|\frac{1+\tau}{2}\right)} - \frac{1}{2}\,\frac{\vartheta_2'\left(\frac{v}{2}\middle|\frac{1+\tau}{2}\right)}{\vartheta_2\left(\frac{v}{2}\middle|\frac{1+\tau}{2}\right)} . \end{aligned} \tag{85.6}$$

Derive then the expansions

$$f_2(v|\tau) = \pi \cot \pi v - 4\pi \sum_{k=1}^{\infty} \frac{q^{2k}}{1 + q^{2k}} \sin 2\pi k v \tag{85.7}$$

and

$$f_3(v|\tau) = \frac{\pi}{\sin \pi v} - 4\pi \sum_{j=1}^{\infty} \frac{q^{2j-1}}{1 + q^{2j-1}} \sin (2j - 1)\pi v . \tag{85.8}$$

Remark. The functions $f_\alpha(v|\tau)$ have a close relation to Jacobi's elliptic functions. If we put

$$\tau = i\frac{K'}{K}$$

we have[1]

$$f_2(v|\tau) = 2K\frac{cn 2Kv}{sn 2Kv},$$

$$f_3(v|\tau) = 2K\frac{dn 2Kv}{sn 2Kv},$$

$$f_4(v|\tau) = 2K\frac{1}{sn 2Kv},$$

and the formulae (85.4) and (85.8) correspond to the formulae § 39, (18) and (22) of Jacobi's "Fundamenta nova theoriae functionum ellipticarum" [28]. Our formula (85.7) has no exact counterpart there.

Exercises (continued). Use (85.7) to derive

$$if_2\left(v + \frac{\tau}{2}\middle|\tau\right) = \pi + 4\pi \sum_{k=1}^{\infty} \frac{q^k}{1 + q^{2k}} \cos 2k\pi v \tag{85.9}$$

(which is Jacobi's formula (25) in [28], § 39).

[1] See for comparison formulae LXXI, p. 284 in [72], vol. 2, together with formulae XXXVII, (1), (2), p. 258.

Prepare also the other formulae for

$$f_\alpha\left(v+\frac{\tau}{2}\middle|\tau\right),\quad f_\alpha\left(v+\frac{1}{2}\middle|\tau\right),\quad f_\alpha\left(v+\frac{1+\tau}{2}\middle|\tau\right),\quad \alpha=2,3,4.$$

86. Lambert series for $f_\alpha^2(v)$

The functions $f_\alpha^2(v\,|\,\tau)$ are doubly periodic with periods $1, \tau$ and poles of second order at the points $v \equiv 0$ (modd $1, \tau$). Their Laurent expansions at $v = 0$ begin with $1/v^2$. They have thus the same pole terms as $\wp(v; 1, \tau)$ and therefore differ from this function and among themselves only by a constant.

Now $f_4\left(\frac{\tau}{2}\middle|\tau\right) = 0$ and therefore we have

$$\begin{aligned}
f_4^2(v\,|\,\tau) &= \wp(v; 1, \tau) - \wp\left(\frac{\tau}{2}; 1, \tau\right)\\
&= \sum_{m_1, m_2 \atop -\infty}^{\infty}\left\{\frac{1}{(v+m_1+m_2\tau)^2} - \frac{1}{(\tau/2+m_1+m_2\tau)^2}\right\}\\
&= \sum_{m_1}\frac{1}{(v+m_1)^2} + \sum_{m_1}\sum_{m_2>0}\left\{\frac{1}{(v+m_1+m_2\tau)^2} - \frac{1}{(m_1+(m_2-\frac{1}{2})\tau)^2}\right\}\\
&\quad + \sum_{m_1}\sum_{m_2<0}\left\{\frac{1}{(v+m_1+m_2\tau)^2} - \frac{1}{(m_1+(m_2+\frac{1}{2})\tau)^2}\right\},
\end{aligned} \tag{86.1}$$

where we made a permissible change in the summands for $m_2 > 0$ in order to accommodate the term $1/(m_1 + \tau/2)^2$. We have thus

$$\begin{aligned}
f_4^2(v\,|\,\tau) &= \frac{\pi^2}{(\sin\pi v)^2} + \sum_{m_2>0}\left\{\sum_{m_1}\frac{1}{(v+m_1+m_2\tau)^2} - \sum_{m_1}\frac{1}{(m_1+(m_2-\frac{1}{2})\tau)^2}\right\}\\
&\quad + \sum_{m_2>0}\left\{\sum_{m_1}\left\{\frac{1}{(v+m_1-m_2\tau)^2} - \sum_{m_1}\frac{1}{(m_1-(m_2-\frac{1}{2})\tau)^2}\right\}\right.\\
&= \frac{\pi^2}{(\sin\pi v)^2} + \sum_{m_2>0}\left\{\sum_{m_1}\left\{\frac{-1}{\left(\dfrac{v+m_2\tau}{i}+m_1 i\right)^2}\right.\right.\\
&\qquad\qquad\left. + \sum_{m_1}\frac{1}{\left(\dfrac{(m_2-\frac{1}{2})\,\tau}{i}+m_1 i\right)^2}\right\}\\
&\quad + \sum_{m_2>0}\left\{\sum_{m_1}\left\{\frac{-1}{\left(\dfrac{-v+m_2\tau}{i}+m_1 i\right)^2} + \sum_{m_1}\frac{1}{\left(\dfrac{(m_2-\frac{1}{2})\tau}{i}+m_1 i\right)^2}\right\}\right.,
\end{aligned}$$

where in the first double sum we have replaced m_1 by $-m_1$, which does not matter since m_1 runs over all integers. We assume now

$$\operatorname{Im}(v+\tau) > 0\,, \quad \operatorname{Im}(-v+\tau) > 0\,,$$

in other words

$$-\operatorname{Im}\tau < \operatorname{Im} v < \operatorname{Im}\tau$$

and can then apply Lipschitz's formula (37.1) with the result

$$\begin{aligned}
f_4^2(v\,|\,\tau) = {} & \frac{\pi^2}{(\sin\pi v)^2} \\
& + \sum_{m_2>0} \frac{(2\pi)^2}{\Gamma(2)} \left\{ -\sum_{m=1}^{\infty} m\, e^{2\pi i(v+m_2\tau)m} + \sum_{m=1}^{\infty} m\, e^{2\pi i(m_2-1/2)\tau m} \right\} \\
& + \sum_{m_2>0} \frac{(2\pi)^2}{\Gamma(2)} \left\{ -\sum_{m=1}^{\infty} m\, e^{2\pi i(-v+m_2\tau)m} + \sum_{m=1}^{\infty} m\, e^{2\pi i(m_2-1/2)\tau m} \right\}.
\end{aligned}$$

Here the four double sums can be separated since they converge absolutely, and we obtain with $q = e^{\pi i\tau}$

$$\begin{aligned}
f_4^2(v\,|\,\tau) & = \frac{\pi^2}{\sin^2\pi v} + 8\pi^2 \sum_{m_2=1}^{\infty} \sum_{m=1}^{\infty} m q^{(2m_2-1)m} \\
& \quad - 8\pi^2 \sum_{m_2=1}^{\infty} \sum_{m=1}^{\infty} m q^{2m_2 m} \cos 2\pi m v\,, \\
f_4^2(v\,|\,\tau) & = \frac{\pi^2}{\sin^2\pi v} + 8\pi^2 \sum_{m_2=1}^{\infty} \frac{q^{2m_2-1}}{(1-q^{2m_2-1})^2} \\
& \quad - 8\pi^2 \sum_{m=1}^{\infty} \frac{m q^{2m}}{1-q^{2m}} \cos 2\pi m v\,.^{1}
\end{aligned} \tag{86.2}$$

We need later also an expansion for $f_4^2(v+\tau/2\,|\,\tau)$.
From (86.2) we obtain

$$\begin{aligned}
f_4^2\left(v+\frac{\tau}{2}\,\middle|\,\tau\right) = A & - \frac{4\pi^2}{(q e^{\pi i v}\, i\, q^{-1} e^{-\pi i v})^2} \\
& - 4\pi^2 \sum_{m=1}^{\infty} \frac{m q^{2m}}{1-q^{2m}} (q^m e^{2\pi i m v} + q^{-m} e^{-2\pi i m v})
\end{aligned}$$

with

$$A = 8\pi^2 \sum_{k=1}^{\infty} \frac{q^{2k-1}}{(1-q^{2k-1})^2} = 8\pi^2 \sum_{m=1}^{\infty} \frac{m q^m}{1-q^{2m}}\,. \tag{86.3}$$

[1] Jacobi [28a], vol. 1, § 42 (2), together with the two formulas for A, § 41, p. 166. Jacobi had no expansion into partial fractions at his disposal. He developed instead in § 41 a method by which $f_4^2(v\,|\,\tau)$ can be obtained by squaring directly the series for $f_4(v\,|\,\tau)$.

This leads to

$$f_4^2\left(v+\frac{\tau}{2}\middle|\tau\right)=A-4\pi^2\sum_{m=1}^{\infty} m\,q^m\,e^{2\pi imv}$$

$$-4\pi^2\sum_{m=1}^{\infty}\left(\frac{m\,q^{3m}}{1-q^{2m}}\,e^{2\pi imv}+\frac{m\,q^m}{1-q^{2m}}\,e^{-2\pi imv}\right),$$

$$f_4^2\left(v+\frac{\tau}{2}\middle|\tau\right)=A-8\pi^2\sum_{m=1}^{\infty}\frac{m\,q^m}{1-q^{2m}}\cos 2\pi mv\,.^1 \tag{86.4}$$

The expansions of $f_\alpha^2(v\,|\,\tau)$, $\alpha=2,3,4$, can differ only by additive constants. In particular we have from (86.2), (86.3)

$$f_4^2(v\,|\,\tau)-f_2^2(v\,|\,\tau)=f_4^2\left(\frac{1}{2}\middle|\tau\right)$$

$$=\pi^2\left(1+8\sum_{m=1}^{\infty}\frac{m\,q^m}{1-q^{2m}}\right)-8\pi^2\sum_{m=1}^{\infty}\frac{(-1)^m m\,q^{2m}}{1-q^{2m}}$$

$$=\pi^2\left(1+8\sum_{m=1}^{\infty}\frac{m\,q^m}{1+(-1)^m q^m}\right) \tag{86.5}$$

and from (86.4)

$$f_4^2(v\,|\,\tau)-f_3^2(v\,|\,\tau)=f_4^2\left(\frac{1+\tau}{2}\middle|\tau\right)=A-8\pi^2\sum_{m=1}^{\infty}\frac{(-1)^m q^m}{1-q^{2m}}$$

$$=16\pi^2\sum_{k=1}^{\infty}\frac{(2k-1)\,q^{2k-1}}{1-q^{4k-2}}\,. \tag{86.6}$$

87. Some addition formulae for ϑ-functions

Let us consider the function

$$G_\alpha(v)=\vartheta_1(v+c)\,\vartheta_\alpha(v-c)\,,\quad \alpha=2,3,4\,, \tag{87.1}$$

where c is a parameter. We obtain from (76.3)

$$G_\alpha(v+1)=\varepsilon\,G(v) \tag{87.21}$$

with

$$\varepsilon=\begin{cases}\;\;1 & \text{for } \alpha=2\,,\\ -1 & \text{for } \alpha=3,4\end{cases} \tag{87.22}$$

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[1] Jacobi [28a], vol.1 § 41(1), p. 166.

and

$$G_\alpha(v+\tau) = \eta q^{-2} e^{-4\pi i v} G_\alpha(v) \tag{87.23}$$

with

$$\eta = \begin{cases} 1 & \text{for } \alpha = 4\,, \\ -1 & \text{for } \alpha = 2, 3\,. \end{cases} \tag{87.24}$$

Let α, β, γ designate the three different numbers 2, 3, 4 in some order.

The function

$$H_\alpha(v) = \vartheta_\beta(v)\,\vartheta_\gamma(v) \tag{87.3}$$

behaves like $G_\alpha(v)$:

$$H_\alpha(v+1) = \varepsilon H_\alpha(v)\,, \qquad H_\alpha(v+\tau) = \eta H_\alpha(v)\,,$$

where ε, η are the same as in (87.22), (87.24).

The functions

$$\frac{\vartheta_1(v)\,\vartheta_\alpha(v)}{\vartheta_1(-c)\,\vartheta_\alpha(-c)\,\vartheta_1(v+c)\,\vartheta_\alpha(v-c)} \text{ and } \frac{\vartheta_\beta(v)\,\vartheta_\gamma(v)}{\vartheta_\beta(-c)\,\vartheta_\gamma(-c)\,\vartheta_1(v+c)\,\vartheta_\alpha(v-c)}$$

are therefore elliptic functions with the periods 1 and τ; moreover the constant factors in the denominators are so chosen that the functions have the same residue at $v = -c$. They differ therefore only by a constant, so that

$$\frac{\vartheta_1(v)\,\vartheta_\alpha(v)}{\vartheta_1(c)\,\vartheta_\alpha(c)} + \frac{\vartheta_\beta(v)\,\vartheta_\gamma(v)}{\vartheta_\beta(c)\,\vartheta_\gamma(c)} = C\vartheta_1(v+c)\,\vartheta_\alpha(v-c)\,.$$

For $v = 0$ we obtain

$$\frac{\vartheta_\beta\vartheta_\gamma}{\vartheta_\beta(c)\,\vartheta_\gamma(c)} = C\vartheta_1(c)\,\vartheta_\alpha(c)\,,$$

from which we infer

$$\vartheta_\beta\vartheta_\gamma\vartheta_1(v+c)\,\vartheta_\alpha(v-c) = \vartheta_\beta(c)\,\vartheta_\gamma(c)\,\vartheta_1(v)\,\vartheta'_\alpha(v) + \vartheta_1(c)\,\vartheta_\alpha(c)\,\vartheta_\beta(v)\,\vartheta_\gamma(v)\,. \tag{87.4}$$

By replacing v by $v + 1/2$, $v + \tau/2$, $v + (1+\tau)/2$ we could derive other formulae from this one, which are however of no importance for our present purpose.

88. Formulae of differentiation

Differentiating (87.4) with respect to c and setting then $c = 0$ we obtain

$$\vartheta_\beta \vartheta_\gamma \left(\vartheta_1'(v)\, \vartheta_\alpha(v) - \vartheta_1(v)\, \vartheta_\alpha'(v) \right) = \vartheta_1' \vartheta_\alpha \vartheta_\beta(v)\, \vartheta_\gamma(v)\,,$$

where we have taken note of the fact that $\vartheta_\alpha' = 0$, $\alpha = 2, 3, 4$ since the $\vartheta_\alpha(v)$ are even. We rewrite the equation as

$$\frac{\vartheta_1'}{\vartheta_\alpha} \frac{d}{dv} \frac{\vartheta_\alpha(v)}{\vartheta_1(v)} = - \frac{\vartheta_1'^2}{\vartheta_\beta \vartheta_\gamma} \frac{\vartheta_\beta(v)\, \vartheta_\gamma(v)}{\vartheta_1^2(v)}\,,$$

which yields the important formula

$$f_\alpha'(v) = -f_\beta(v) f_\gamma(v), \qquad \alpha = 2, 3, 4. \tag{88.1}$$

This formula makes it possible to express all derivatives of $f_\alpha(v)$ as polynomials in the f_α, $\alpha = 2, 3, 4$.

In the final step we differentiate both sides of (88.1),

$$f_\alpha''(v) = -f_\beta'(v) f_\gamma(v) - f_\beta(v) f_\beta'(v),$$

and apply now (88.1) itself on the right-hand member with the result

$$f_\alpha''(v) = f_\alpha(v) \left(f_\beta^2(v) + f_\gamma^2(v) \right). \tag{88.2}$$

Progressing in the same way we obtain further

$$f_\alpha'''(v) = -f_\beta(v) f_\gamma(v) \left(4 f_\alpha^2(v) + f_\beta^2(v) + f_\gamma^2(v) \right), \tag{88.3}$$

$$\begin{aligned} f_\alpha^{\mathrm{IV}}(v) = {} & f_\alpha(v) \left(f_\beta^2(v) + f_\gamma^2(v) \right) \left(4 f_\alpha^2(v) + f_\beta^2(v) + f_\gamma^2(v) \right) \\ & + 12 f_\alpha(v) f_\beta^2(v) f_\gamma^2(v). \end{aligned} \tag{88.4}$$

We obtain also from (88.1)

$$\left(f_\alpha^2(v) \right)' = -2 f_\alpha(v) f_\beta(v) f_\gamma(v) = -2 f_2(v) f_3(v) f_4(v), \tag{88.5}$$

which is the same for all α, according to § 86, where we have found that $f_\alpha^2(v)$ differs from $\wp(v; 1, \tau)$ only by a constant, so that

$$\left(f_\alpha^2(v) \right)' = \wp'(v; 1, \tau), \quad \alpha = 2, 3, 4.$$

Proceeding from (88.5) we find also

$$\left(f_\alpha^2(v) \right)'' = 2 \left(f_2^2(v) f_3^2(v) + f_3^2(v) f_4^2(v) + f_4^2(v) f_2^2(v) \right), \tag{88.6}$$

$$\left(f_\alpha^2(v) \right)''' = -8 f_2(v) f_3(v) f_4(v) \left(f_2^2(v) + f_3^2(v) + f_4^2(v) \right), \tag{88.7}$$

$$\begin{aligned} \left(f_\alpha^2(v) \right)^{\mathrm{IV}} = {} & 8 \left(f_2^2(v) f_3^2(v) + f_3^2(v) f_4^2(v) + f_4^2(v) f_2^2(v) \right) \\ & \times \left(f_2^2(v) + f_3^2(v) + f_4^2(v) \right) + 48 f_2^2(v) f_3^2(v) f_4^2(v). \end{aligned} \tag{88.8}$$

This list could easily be extended, but the formulae become more and more complicated.

89. Even powers of ϑ_3 expressed by derivatives of $f_\alpha(v)$ and $f_\alpha^2(v)$

By means of the definition (82.1), by the table (76.4), and the identity (78.6) we obtain the following list of values:

$$f_2\left(\frac{1}{2}\right)=0\,,\qquad f_3\left(\frac{1}{2}\right)=\pi\vartheta_4^2\,,\qquad f_4\left(\frac{1}{2}\right)=\pi\vartheta_3^2\,,$$
$$f_2\left(\frac{\tau}{2}\right)=-i\pi\vartheta_3^2\,,\quad f_3\left(\frac{\tau}{2}\right)=-i\pi\vartheta_2^2\,,\quad f_4\left(\frac{\tau}{2}\right)=0\,,$$
$$f_2\left(\frac{1+\tau}{2}\right)=-i\pi\vartheta_4^2\,,\ f_3\left(\frac{1+\tau}{2}\right)=0\,,\qquad f_4\left(\frac{1+\tau}{2}\right)=\pi\vartheta_2^2\,. \tag{89.1}$$

The ϑ-values appear here all in the second power. Thus all derivatives of the f will lead to expressions which contain the ϑ-values only in *even* powers. By means of the formulae (88.1) to (88.8) we could now complete a list of the values of the first four derivatives of $f_\alpha(v)$ and $f_\alpha^2(v)$ for the values $v=1/2, \tau/2, (1+\tau)/2$. There is no point in reproducing the whole list here. We pick out only the following eight equations which will turn out to be useful for the purpose of expressing even powers of ϑ_3.

$$f_4''\left(\frac{1}{2}\right)=\pi^3\,\vartheta_3^2\,\vartheta_4^4\,,\qquad f_2''\left(\frac{\tau}{2}\right)=i\pi^3\,\vartheta_3^2\,\vartheta_2^4\,,$$
$$f_4^{\mathrm{IV}}\left(\frac{1}{2}\right)=\pi^5\,\vartheta_3^2\,\vartheta_4^4\,(4\vartheta_3^4+\vartheta_4^4)\,,\quad f_2^{\mathrm{IV}}\left(\frac{\tau}{2}\right)=-i\pi^5\,\vartheta_3^2\,\vartheta_2^4(4\vartheta_3^4+\vartheta_2^4)\,,$$
$$(f_\alpha^2(v))''_{v=1/2}=2\pi^4\,\vartheta_3^4\,\vartheta_4^4\,,\qquad (f_\alpha^2(v))''_{v=\tau/2}=2\pi^4\vartheta_2^4\vartheta_3^4\,,$$
$$(f_\alpha^2(v))^{\mathrm{IV}}_{v=1/2}=8\pi^6\,\vartheta_3^4\,\vartheta_4^4\,(\vartheta_3^4+\vartheta_4^4)\,,$$
$$(f_\alpha^2(v))^{\mathrm{IV}}_{v=\tau/2}=-8\pi^6\,\vartheta_2^4\,\vartheta_3^4\,(\vartheta_2^4+\vartheta_3^4)\,. \tag{89.2}$$

From (89.1) and (89.2) we obtain the following equations:

$$\pi\vartheta_3^2=f_4\left(\frac{1}{2}\right), \tag{89.3}$$

$$\pi^2\vartheta_3^4=f_4^2\left(\frac{1}{2}\right), \tag{89.4}$$

$$\pi^3\vartheta_3^6=f_4''\left(\frac{1}{2}\right)-if_2''\left(\frac{\tau}{2}\right), \tag{89.5}$$

$$2\pi^4\vartheta_3^8=(f_\alpha(v))''_{v=1/2}+(f_\alpha^2(v))''_{v=\tau/2}\,, \tag{89.6}$$

$$5\pi^5\vartheta_3^{10}=f_4^{\mathrm{IV}}\left(\frac{1}{2}\right)+if_2^{\mathrm{IV}}\left(\frac{\tau}{2}\right)+2\pi^5\,\vartheta_3^2\,\vartheta_2^4\,\vartheta_4^4\,, \tag{89.7}$$

$$16\pi^6\vartheta_3^{12}=(f_\alpha^2(v))^{\mathrm{IV}}_{v=1/2}-(f_\alpha(v))^{\mathrm{IV}}_{v=\tau/2}+16\pi^2\,\vartheta_1'^4\,. \tag{89.8}$$

Here we have made use of the relations (78.6) and (83.3).

90. Lambert series for the even powers of ϑ_3

In order to make use of the equations (89.3) to (89.8) we compile a list of derivatives. From (85.4) we obtain by differentiation

$$f_4''(v\,|\,\tau) = \frac{\pi^3}{\sin \pi v}\,(2\cot^2 \pi v + 1)$$

$$- 4\pi^3 \sum_{m=1}^{\infty} \frac{(2m-1)^2 q^{2m-1}}{1 - q^{2m-1}} \sin (2m-1)\,\pi v\,,$$

$$f_4^{\mathrm{IV}}(v\,|\,\tau) = \frac{\pi^5}{\sin \pi v}\,(24\cot^4 \pi v + 28\cot^2 \pi v + 5)$$

$$+ 4\pi^5 \sum_{m=1}^{\infty} \frac{(2m-1)^4 q^{2m-1}}{1 - q^{2m-1}} \sin (2m-1)\,\pi v\,,$$

and from (85.9)

$$i f_2''\left(v + \frac{\tau}{2}\middle|\,\tau\right) = -\,16\pi^3 \sum_{m=1}^{\infty} \frac{m^2 q^m}{1 + q^{2m}} \cos 2m\pi v\,,$$

$$i f_2^{\mathrm{IV}}\left(v + \frac{\tau}{2}\middle|\,\tau\right) = 64\pi^5 \sum_{m=1}^{\infty} \frac{m^4 q^m}{1 + q^{2m}} \cos 2m\pi v\,.$$

Furthermore (86.2) yields

$$\frac{d^2}{dv^2} f_\alpha^2\,(v\,|\,\tau) = \frac{\pi^4}{\sin^2 \pi v}\,(6\cot^2 \pi v + 2)$$

$$+ 32\pi^4 \sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1 - q^{2m}} \cos 2\pi m v\,,$$

$$\frac{d^4}{dv^4} f_\alpha^2\,(v\,|\,\tau) = \frac{\pi^6}{\sin^2 \pi v}\,(120\cot^4 \pi v + 120\cot^2 \pi v + 16)$$

$$- 128\pi^6 \sum_{m=1}^{\infty} \frac{m^5 q^{2m}}{1 - q^{2m}} \cos 2\pi m v\,,$$

and (86.4) leads to

$$\frac{d^2}{dv^2} f^2\left(v + \frac{\tau}{2}\middle|\,\tau\right) = 32\pi^4 \sum_{m=1}^{\infty} \frac{m^3 q^m}{1 - q^{2m}} \cos 2\pi m v\,,$$

$$\frac{d^4}{dv^4} f^2\left(v + \frac{\tau}{2}\middle|\,\tau\right) = -128\pi^6 \sum_{m=1}^{\infty} \frac{m^5 q^m}{1 - q^{2m}} \cos 2\pi m v\,.$$

These results together with (85.4), (86.2) we insert in (89.3) to (89.8) with the following results:

$$\vartheta_3^2 = 1 + 4\sum_{m=1}^{\infty}(-1)^{m-1}\frac{q^{2m-1}}{1-q^{2m-1}}, \tag{90.1}$$

$$\vartheta_3^4 = 1 + 8\sum_{m=1}^{\infty}\frac{mq^m}{1-q^{2m}} - 8\sum_{m=1}^{\infty}(-1)^m\frac{mq^{2m}}{1-q^{2m}}, \tag{90.2}$$

$$\vartheta_3^6 = 1 + 16\sum_{m=1}^{\infty}\frac{m^2q^m}{1+q^{2m}} - 4\sum_{m=1}^{\infty}(-1)^{m-1}\frac{(2m-1)^2q^{2m-1}}{1-q^{2m-1}}, \tag{90.3}$$

$$\vartheta_3^8 = 1 + 16\sum_{m=1}^{\infty}(-1)^m\frac{m^3q^{2m}}{1-q^{2m}} + 16\sum_{m=1}^{\infty}\frac{m^3q^m}{1-q^{2m}}, \tag{90.4}$$

$$\vartheta_3^{10} = 1 + \frac{4}{5}\left\{\sum_{m=1}^{\infty}\frac{2m^4q^m}{1+q^{2m}} + \sum_{m=1}^{\infty}(-1)^{m-1}\frac{(2m-1)^4q^{2m-1}}{1-q^{2m-1}}\right\} + \frac{2}{5}\vartheta_3^2\vartheta_2^4\vartheta_4^4, \tag{90.5}$$

$$\vartheta_3^{12} = 1 + 8\sum_{m=1}^{\infty}\frac{m^5q^m}{1-q^{2m}} - 8\sum_{m=1}^{\infty}(-1)^m\frac{m^5q^{2m}}{1-q^{2m}} + \frac{1}{\pi^4}\vartheta_4'^4. \tag{90.6}$$

91. Sums of an even number of squares

By developing the foregoing Lambert series into power series we obtain expressions for the number $A_r(n)$ of representations of n as a sum of r squares, for the cases $r = 2, 4, 6, 8, 10, 12$.

The formulae (90.1) and (90.2) yield again the theorems of § 84 and § 83, respectively. We go through the other formulae one by one.

From (90.3) we derive

$$\vartheta_3^6 = 1 - 4\sum_{m\equiv 1(4)} m^2\sum_{k=1}^{\infty} q^{mk} + 4\sum_{m\equiv 3(4)} m^2\sum_{k=1}^{\infty} q^{mk} + 16\sum_{m=1}^{\infty} m^2\sum_{k=1}^{\infty}(-1)^{k-1}q^{m(2k-1)}$$

or

$$\vartheta_3^6 = 1 - 4\sum_{n=1}^{\infty}q^n\left(\sum_{\substack{d\mid n\\ d\equiv 1(4)}} d^2 - \sum_{\substack{d\mid n\\ d\equiv 3(4)}} d^2\right) + 16\sum_{n=1}^{\infty}q^n\left(\sum_{\substack{d\mid n\\ \delta\equiv 1(4)}} d^2 - \sum_{\substack{d\mid n\\ \delta\equiv 3(4)}} d^2\right), \tag{91.1}$$

where here and in the sequel δ is the complementary divisor to d, i.e. $d\delta = n$.

From (90.4) we get

$$\vartheta_3^8 = 1 + 16 \sum_{m\,\text{even}} m^3 \frac{q^m(1+q^m)}{1-q^{2m}} + 16 \sum_{m\,\text{odd}} m^3 \frac{q^m(1-q^m)}{1-q^{2m}}$$

$$= 1 + 16 \sum_{m\,\text{even}} \frac{m^3 q^m}{1-q^m} + 16 \sum_{m\,\text{odd}} \frac{m^3 q^m}{1+q^m}$$

$$= 1 + 16 \sum_{m\,\text{even}} \sum_k m^3 q^{mk} + 16 \sum_{m\,\text{odd}} \sum_k (-1)^{k-1} m^3 q^{mk}$$

$$= 1 + 16 \sum_m \sum_k (-1)^{m(k-1)} m^3 q^{mk} ,$$

$$\vartheta_3^8 = 1 + 16 \sum_{n=1}^{\infty} q^n \sum_{d|n} (-1)^{n-d} d^3 . \tag{91.2}$$

For the treatment of (90.51) we introduce the arithmetical function $\varrho(n)$ by the definition

$$\vartheta_3^2 \vartheta_2^4 \vartheta_4^4 = 16 \sum_{n=1}^{\infty} \varrho(n) q^n , \tag{91.31}$$

in which, as the formulae (78.4) show, $\varrho(n)$ is an integer. Then we derive from (90.5)

$$\vartheta_3^{10} = 1 + \frac{4}{5} \sum_m \sum_k (2m)^4 (-1)^{k-1} q^{(2k-1)m}$$

$$+ \frac{4}{5} \sum_m \sum_k (2m-1)^4 (-1)^{m-1} q^{(2m-1)k} + \frac{32}{5} \sum_{m=1}^{\infty} \varrho(n) q^n$$

and consequently

$$\vartheta_3^{10} = 1 + \frac{4}{5} \sum_{n=1}^{\infty} q^n \left\{ \left(\sum_{d\equiv 1(4)} d^4 - \sum_{d\equiv 3(4)} d^4 \right) + 16 \left(\sum_{\delta\equiv 1(4)} d^4 - \sum_{\delta\equiv 3(4)} d^4 \right) \right\}$$

$$+ \frac{32}{5} \sum_1^{\infty} \varrho(n) q^n . \tag{91.32}$$

Finally, for the discussion of (90.6) we introduce

$$\left(\frac{\vartheta_1'}{\pi} \right)^4 = 16 q \prod_1^{\infty} (1-q^{2m})^{12} = 16 \sum_1^{\infty} \omega(n) q^n , \tag{91.41}$$

where evidently $\omega(n) = 0$ for n even. We obtain then

$$\vartheta_3^{12} = 1 + 8\sum_m \frac{m^5 q^m (1-(-q)^m)}{1-q^{2m}} + \left(\frac{\vartheta_1'}{\pi}\right)^4$$

$$= 1 + 8\sum_m \frac{m^5 q^m}{1+(-q)^m} + \left(\frac{\vartheta_1'}{\pi}\right)^4$$

$$= 1 + 8\sum_m\sum_k m^5\,(-1)^{(m+1)(k-1)} q^{mk} + \left(\frac{\vartheta_1'}{\pi}\right)^4,$$

$$\vartheta_3^{12} = 1 + 8\sum_{n=1}^{\infty}(-1)^{n-1}q^n\sum_{d|n}(-1)^{d+\delta}d^5 + 16\sum_{n=1}^{\infty}\omega(n)\,q^n\,. \tag{91.42}$$

Comparison of coefficients on both sides of (91.1), (91.2) (91.32), (91.42) yields now the following theorem, which supplements the theorems of § 83 and § 84:

Theorem. *The number $A_r(n)$ of representations of a natural number n as a sum of r squares is given for $r = 6, 8, 10, 12$ by the formulae:*

$$A_6(n) = 16\left(\sum_{\delta\equiv 1(4)} d^2 - \sum_{\delta\equiv 3(4)} d^2\right) - 4\left(\sum_{d\equiv 1(4)} d^2 - \sum_{d\equiv 3(4)} d^2\right), \tag{91.5}$$

$$A_8(n) = 16\sum_d (-1)^{n-d} d^3\,, \tag{91.6}$$

$$A_{10}(n) = \frac{4}{5}\left\{16\left(\sum_{\delta\equiv 1(4)} d^4 - \sum_{\delta\equiv 3(4)} d^4\right) + \left(\sum_{d\equiv 1(4)} d^4 - \sum_{d\equiv 3(4)} d^4\right)\right\}$$

$$+\frac{32}{5}\,\varrho(n)\,, \tag{91.7}$$

$$A_{12}(n) = 8\,(-1)^{n-1}\sum_d (-1)^{d+\delta} d^5 + 16\omega(n)\,. \tag{91.8}$$

Here d runs through all divisors of n, δ is the complementary divisor to d, $d\delta = n$, and $\varrho_{10}(n)$ and $\omega(n)$ are defined by (91.31), (91.41).

92. Discussion of the foregoing results

All these expressions contain sums of divisors, and in the cases $r = 2, 4, 6, 8$ the function $A_r(n)$ is expressed solely by such sums of divisors. For $r = 10$ and 12 we have the further terms $(32/5)\,\varrho(n)$ and $16\omega(n)$. However we can show that the divisor sums form the "principal term" of the expressions, in the sense that they determine asymptotically the behavior of $A_{10}(n)$ and $A_{12}(n)$. The terms $(32/5)\,\varrho(n)$ and $16\omega(n)$

appear then as a sort of "remainder term" of lower order of magnitude. Indeed, let us designate the divisor sums as $C_r(n)$ so that we have in particular

$$A_{10}(n) = C_{10}(n) + \frac{32}{5}\varrho(n)\,, \tag{92.1}$$

$$A_{12}(n) = C_{12}(n) + 16\omega(n)\,. \tag{92.2}$$

In $C_r(n)$ we introduce the complementary divisor δ as summation letter, and have then

$$C_{10}(n) = \frac{64}{5} n^4 \sum_{\delta | n} \frac{a(\delta)}{\delta^4}\,,$$

where $a(1) = 1$, otherwise $a(\delta) = 0, \pm 1, \pm 1/16$, according to the residues of d and δ modulo 4 and similarly

$$C_{12}(n) = 8n^5 \sum_{\delta | n} \frac{b(\delta)}{\delta^5}\,,$$

with $b(1) = 1$, otherwise $b(\delta) = \pm 1$. Therefore

$$1 - \sum_{\delta=2}^{\infty} \frac{1}{\delta^4} < \frac{5C_{10}(n)}{64n^4} < 1 + \sum_{\delta=2}^{\infty} \frac{1}{\delta^4} = \zeta(4)\,,$$

$$\left| \frac{5C_{10}(n)}{64n^4} - 1 \right| < \zeta(4) - 1 < \frac{1}{9}\,,$$

and similarly

$$\left| \frac{C_{12}(n)}{8n^5} - 1 \right| < \zeta(5) - 1 = 0.0369\ldots$$

so that in any case

$$C_{10}(n) > K_1 n^4\,, \qquad C_{12}(n) > K_2 n^5$$

with certain $K_j > 0$. We have to prove that $\varrho(n)$ and $\omega(n)$ are of lower order than n^4, n^5 respectively. We begin with the latter, by majorizing the coefficients in (91.41), using the symbol $\ll$ between two power series in order to express that the coefficients of the power series on the right are non-negative and at least equal to the absolute value of the corresponding coefficients on the left. In this meaning we observe after (78.4) and (78.51)

$$\begin{aligned}\left(\frac{\vartheta_1'}{\pi}\right)^4 &\ll 16q\left(\sum_{n=0}^{\infty}(2n+1)q^{n(n+1)}\right)^4 \\ &= 16q \sum_{n_1}\sum_{n_2}\sum_{n_3}\sum_{n_4}(2n_1+1)(2n_2+1)(2n_3+1)(2n_4+1) \\ &\quad \times q^{n_1(n_1+1)+n_2(n_2+1)+n_3(n_3+1)+n_4(n_4+1)}\,,\end{aligned}$$

or

$$\left(\frac{\vartheta_1'}{\pi}\right)^4 \ll 16q \sum_0^\infty a(n) q^n$$

with

$$a(n) = \sum_{n_1, n_2, n_3, n_4} (2n_1 + 1)(2n_2 + 1)(2n_3 + 1)(2n_4 + 1),$$

summed over all those n_1, n_2, n_3, n_4, $0 \leqq n_j$ for which

$$\sum_{j=1}^4 n_j(n_j + 1) = n$$

or

$$\sum_{j=1}^4 (2n_j + 1)^2 = 4n + 4.$$

Now

$$(2n_1 + 1)(2n_2 + 1) \leqq \frac{1}{2}\left((2n_1 + 1)^2 + (2n_2 + 1)^2\right) \leqq 2n + 2,$$

$$(2n_3 + 1)(2n_4 + 1) \leqq \frac{1}{2}\left((2n_3 + 1)^2 + (2n_4 + 1)^2\right) \leqq 2n + 2,$$

so that according to Cauchy's inequality of the geometric and arithmetic means we have now

$$(2n_1 + 1)(2n_2 + 1)(2n_3 + 1)(2n_4 + 1)$$
$$\leqq \left(\frac{(2n_1 + 1)^2 + (2n_2 + 1)^2 + (2n_3 + 1)^2 + (2n_4 + 1)^2}{4}\right)^2 = (n + 1)^2$$

and thus

$$a(n) \leqq (n + 1)^2 \sum_{n_1, n_2, n_3, n_4} 1 \leqq (n + 1)^2 A_4(4n + 4).$$

Let us assume for a moment the

Lemma.

$$A_4(n) = O\left(n \log(n + 1)\right).$$

Then we arrive at

$$a(n) = O(n^3 \log n)$$

so that

$$\sum_1^\infty \omega(n) q^n \ll C \sum n^3 \log(n + 1) q^n,$$

in other words

$$\omega(n) = O\big(n^3 \log(n+1)\big), \tag{92.3}$$

which is indeed of lower order of magnitude than $C_{12}(n)$.

Proof of the lemma.

$$A_4(n) \leqq 24 \sum_{\substack{d|n \\ d\,\mathrm{odd}}} d \leqq 24n \sum_{\delta|n} \frac{1}{\delta} \leqq 24n \sum_{\nu \leqq n} \frac{1}{\nu} = O\big(n \log(n+1)\big). \quad \square$$

For the study of $\varrho(n)$ we rewrite (91.31) as

$$16 \sum_{n=1}^{\infty} \varrho(n) q^n = \left(\frac{\vartheta_1'}{\pi}\right)^2 \vartheta_2^2 \vartheta_4^2 .$$

Now, after (78.51) and (76.1) we have

$$\begin{aligned}\left(\frac{\vartheta_1'}{\pi}\right)^2 \vartheta_2^2 \vartheta_4^2 &\leqq 4q\left(\sum_{n=0}^{\infty}(2n+1)\,q^{n(n+1)}\right)^2\left(\sum_{-\infty}^{\infty} q^{n(n+1)}\right)^2\left(\sum_{-\infty}^{\infty} q^{n^2}\right)^2 \\ &= 4q \sum_{n_1,\dots,n_6} (2n_1+1)(2n_2+1) \\ &\qquad \times q^{n_1(n_1+1)+n_2(n_2+1)+n_3(n_3+1)+n_4(n_4+1)+n_5^2+n_6^2},\end{aligned}$$

where $n_1 \geqq 0$, $n_2 \geqq 0$, n_3 to n_6 any integers.

We rewrite this as

$$16 \sum \varrho(n) q^n \ll 4q \sum_{0}^{\infty} b(n) q^n ,$$

where

$$b(n) = \sum_{n_1,\dots,n_6} (2n_1+1)(2n_2+1), \tag{92.4}$$

where the summation runs over all those $n_1, n_2, \dots, n_6$ for which

$$n_1(n_1+1) + n_2(n_2+1) + n_3(n_3+1) + n_4(n_4+1) + n_5^2 + n_6^2 = n$$

or

$$\begin{aligned}(2n_1+1)^2 + (2n_2+1)^2 + (2n_3+1)^2 + (2n_4+1)^2 + (2n_5)^2& \\ + (2n_6)^2 = 4n+4 .&\end{aligned} \tag{92.5}$$

We have thus

$$(2n_1+1)(2n_2+1) \ll 2n+2 ,$$

and the number of solutions of (92.5) is after (91.5) at most

$$A_6(4n+4) = O(n^2) .$$

This permits us to conclude from (92.4)

$$b(n) = O(n^3)$$

so that we obtain

$$\varrho(n) = O(n^3)\,, \tag{92.6}$$

again of lower order than $C_{10}(n)$.

93. Further discussion of $\varrho(n)$

Glaisher [12][1] found an expression for $\varrho(n)$ that refers also to divisors, however to *complex* divisors of n. His formula is

$$\varrho(n) = \frac{1}{4} \sum_{a^2+b^2=n} (a+bi)^4\,. \tag{93.1}$$

For the proof of this equation we need the following

Lemma.

$$\frac{\vartheta_3^{\mathrm{IV}}}{\vartheta_3} - 3\left(\frac{\vartheta_3''}{\vartheta_3}\right)^2 = 2\pi^4\vartheta_2^4\vartheta_4^4\,. \tag{93.21}$$

We postpone the proof to the end of this section.

We can write (93.21) also as

$$\vartheta_3^{\mathrm{IV}}\vartheta_3 - 3\vartheta_3''^2 = 2\pi^4\vartheta_3^2\vartheta_2^4\vartheta_4^4\,, \tag{93.22}$$

where the right-hand member is, except for a numerical factor, the generating function for $\varrho(n)$. We have

$$\vartheta_3 = \sum_{m=-\infty}^{\infty} q^{m^2}$$

and obtain from (76.1)

$$\vartheta_3^{\mathrm{IV}} = 16\pi^4 \sum_{n=-\infty}^{\infty} n^4 q^{n^2}$$

so that

$$\vartheta_3\vartheta_3^{\mathrm{IV}} = 16\pi^4 \sum_{m,n} n^4 q^{m^2+n^2}\,.$$

Similarly we infer from

$$\vartheta_3'' = -4\pi^2 \sum_{-\infty}^{\infty} m^2 q^{m^2}\,, \qquad (\vartheta_3'')^2 = 16\pi^4 \sum_{m,n} m^2n^2 q^{m^2+n^2}\,;$$

[1] Our $\varrho(n)$ is there called $X_4(n)$. See also [75].

after (93.22) we have thus

$$\vartheta_3^2\vartheta_2^4\vartheta_4^4 = 8\sum_{m,n}(m^4 - 3m^2n^2)q^{m^2+n^2} = 8\sum_{N=0}^{\infty} q^N \sum_{m^2+n^2=N} (m^4 - 3m^2n^2).$$

On the other hand we find

$$(m + in)^4 = m^4 + 4im^3n - 6m^2n^2 - 4imn^3 + n^4$$

and have thus

$$\begin{aligned}&\sum_{m,n}(m + in)^4 q^{m^2+n^2}\\ &= \sum_{m,n}(m^4 + 4im^3n - 6m^2n^2 - 4imn^3 + n^4)q^{m^2+n^2}\\ &= \sum_{m,n}(m^4 - 6m^2n^2 + n^4)q^{m^2+n^2} = 2\sum(m^4 - 3m^2n^2)q^{m^2+n^2},\end{aligned}$$

the latter because of the symmetry in m, n. We can thus write

$$\vartheta_3^2\vartheta_2^4\vartheta_4^4 = 4\sum_0^{\infty} q^N \sum_{m^2+n^2=N} (m + ni)^4.$$

Comparison of this equation with the definition (91.31) proves (93.1).

This result helps to improve the estimate (92.6) considerably. Indeed we have

$$\varrho(n) \leqq \frac{1}{4}\sum_{a^2+b^2=n} |a + bi|^4 = \frac{1}{4}\sum_{a^2+b^2=n} |a^2 + b^2|^2 = \frac{1}{4}n^2 \sum_{a^2+b^2=n} 1.$$

Now after the theorem of § 84 we see

$$\sum_{a^2+b^2=n} 1 = A_2(n) \leqq 4\sigma_0(n).$$

It is known[1] that $\sigma_0(n) = O(n^{\epsilon})$, so that we obtain now

$$\varrho(n) = O(n^{2+\epsilon}), \tag{93.3}$$

instead of the less sharp estimate (92.6).

We return now to the

Proof of the lemma. For the function

$$\Phi(v\,|\,\tau) = \frac{\vartheta_1'(v\,|\,\tau)}{\vartheta_1(v\,|\,\tau)},$$

defined in (84.1) we have the relations (84.2). Therefore

$$\Phi'(v + 1\,|\,\tau) = \Phi'(v + \tau\,|\,\tau) = \Phi'(v\,|\,\tau).$$

[1] See, e.g. [33], p. 250, Satz 260.

Therefore $\Phi'(v)$ is an elliptic function with periods 1 and τ. If we write explicitly

$$\Phi'(v) = \frac{\vartheta_1''(v)}{\vartheta_1(v)} - \left(\frac{\vartheta_1'(v)}{\vartheta_1(v)}\right)^2$$

we see that it has poles of the second order and the points $v \equiv 0 \pmod{1,\tau}$. The Laurent expansion begins with $-1/v^2$. Therefore

$$\frac{\vartheta_1''(v)}{\vartheta_1(v)} - \left(\frac{\vartheta_1'(v)}{\vartheta_1(v)}\right)^2 + f_\alpha^2(v) = C_\alpha\,, \qquad \alpha = 2, 3, 4 \tag{93.4}$$

is a constant, since an elliptic function cannot have a single pole of order 1 per period parallelogram. For the determination of C_α we rewrite the last equation in the form

$$\vartheta_1''(v)\,\vartheta_1(v) - \vartheta_1'(v)^2 + \frac{\vartheta_1'^2}{\vartheta_\alpha^2}\vartheta_\alpha^2(v) = C_\alpha \vartheta_1^2(v)$$

according to the definition (82.1) of $f_\alpha(v)$. Expansion into power series yields

$$\begin{aligned}
&\left(\vartheta_1''' v + \vartheta_1^{\mathrm{V}} \frac{v^3}{3!} + \cdots\right)\left(\vartheta_1' v + \vartheta_1''' \frac{v^3}{3!} + \cdots\right) \\
&\quad - \left(\vartheta_1' + \vartheta_1''' \frac{v^2}{2!} + \vartheta_1 \frac{v^4}{4!} + \cdots\right)^2 \\
&\quad + \vartheta_1'^2 \left(1 + \frac{\vartheta_\alpha''}{\vartheta_\alpha}\frac{v^2}{2!} + \frac{\vartheta_\alpha^{\mathrm{IV}}}{\vartheta_\alpha}\frac{v^4}{4!} + \cdots\right)^2 \\
&= C_\alpha \left(\vartheta_1' v + \vartheta_1''' \frac{v^3}{3!} + \cdots\right)^2.
\end{aligned} \tag{93.41}$$

The constant terms here cancel. Comparison of the coefficients of v^2 yields

$$C_\alpha = \frac{\vartheta_\alpha''}{\vartheta_\alpha}\,, \qquad \alpha = 2, 3, 4\,. \tag{93.5}$$

If on the other hand we put $v = 1/2$ in (93.4) we obtain

$$\frac{\vartheta_2''}{\vartheta_2} + f_\alpha^2\left(\frac{1}{2}\right) = C_\alpha\,,$$

which in view of (78.6) and (93.5) leads for $\alpha = 3$ and 4 to

$$\frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_3''}{\vartheta_3} = -\pi^2 \vartheta_4^4\,, \tag{93.61}$$

$$\frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_4''}{\vartheta_4} = -\pi^2 \vartheta_3^4\,, \tag{93.62}$$

to which we can add, because of (83.3),

$$\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_4''}{\vartheta_4} = -\pi^2 \vartheta_2^4 . \tag{93.63}$$

If we compare coefficients of v^4 in (93.41) we get

$$\frac{\vartheta_1^{\mathrm{V}}}{\vartheta_1'} - \left(\frac{\vartheta_1'''}{\vartheta_1}\right)^2 + 3\left(\frac{\vartheta_\alpha''}{\vartheta_\alpha}\right)^2 + \frac{\vartheta_\alpha^{\mathrm{IV}}}{\vartheta_\alpha} - 4\frac{\vartheta_\alpha''}{\vartheta_\alpha}\frac{\vartheta_1'''}{\vartheta_1'} = 0 . \tag{93.7}$$

Logarithmic differentiation of (78.6) with respect to τ yields

$$\frac{\frac{\partial}{\partial\tau}\vartheta_1'(0\,|\,\tau)}{\vartheta_1'(0\,|\,\tau)} = \frac{\frac{\partial}{\partial\tau}\vartheta_2(0\,|\,\tau)}{\vartheta_2(0\,|\,\tau)} + \frac{\frac{\partial}{\partial\tau}\vartheta_3(0\,|\,\tau)}{\vartheta_3(0\,|\,\tau)} + \frac{\frac{\partial}{\partial\tau}\vartheta_4(0\,|\,\tau)}{\vartheta_4(0\,|\,\tau)} ,$$

and a further differentiation with respect to τ leads to

$$\begin{aligned}
&\frac{\frac{\partial^2}{\partial\tau^2}\vartheta_1'(0\,|\,\tau)}{\vartheta_1'(0\,|\,\tau)} - \left(\frac{\frac{\partial}{\partial\tau}\vartheta_1'(0\,|\,\tau)}{\vartheta_1'(0\,|\,\tau)}\right)^2 \\
&= \frac{\frac{\partial^2}{\partial\tau^2}\vartheta_2(0\,|\,\tau)}{\vartheta_2(0\,|\,\tau)} + \frac{\frac{\partial^2}{\partial\tau^2}\vartheta_3(0\,|\,\tau)}{\vartheta_3(0\,|\,\tau)} + \frac{\frac{\partial^2}{\partial\tau^2}\vartheta_4(0\,|\,\tau)}{\vartheta_4(0\,|\,\tau)} \\
&\quad - \left(\frac{\frac{\partial}{\partial\tau}\vartheta_2(0\,|\,\tau)}{\vartheta_2(0\,|\,\tau)}\right)^2 - \left(\frac{\frac{\partial}{\partial\tau}\vartheta_3(0\,|\,\tau)}{\vartheta_3(0\,|\,\tau)}\right)^2 - \left(\frac{\frac{\partial}{\partial\tau}\vartheta_4(0\,|\,\tau)}{\vartheta_4(0\,|\,\tau)}\right)^2 .
\end{aligned}$$

These last two equations can be written much more simply if we consider the partial differential equation (76.6), which is valid for all four ϑ-functions. We get thus

$$\frac{\vartheta_1}{\vartheta_1'} = \frac{\vartheta_2''}{\vartheta_2} + \frac{\vartheta_3''}{\vartheta_3} + \frac{\partial_4''}{\partial_4} \tag{93.81}$$

and

$$\frac{\vartheta_1^{\mathrm{V}}}{\vartheta_1'} - \left(\frac{\vartheta'''}{\vartheta_1}\right)^2 = \frac{\vartheta_2^{\mathrm{IV}}}{\vartheta_2} + \frac{\vartheta_3^{\mathrm{IV}}}{\vartheta_3} + \frac{\vartheta_4^{\mathrm{IV}}}{\vartheta_4} - \left(\frac{\vartheta_2''}{\vartheta_2}\right)^2 - \left(\frac{\vartheta_3''}{\vartheta_\alpha}\right)^2 - \left(\frac{\vartheta_4''}{\vartheta_4}\right)^2 , \tag{93.82}$$

where the dashes and the Roman numerals as superscripts indicate differentiations with respect to v and the ϑ-symbols written without an argument mean here (as always) the case $v = 0$.

We use now (93.7) to eliminate ϑ_1^{V} and obtain

$$\begin{aligned}
2\frac{\vartheta_\alpha^{\mathrm{IV}}}{\vartheta_\alpha} + \frac{\vartheta_\beta^{\mathrm{IV}}}{\vartheta_\beta} + \frac{\vartheta_\gamma^{\mathrm{IV}}}{\vartheta_\gamma} + 2\left(\frac{\vartheta_\alpha''}{\vartheta_\alpha}\right)^2 - \left(\frac{\vartheta_\beta''}{\vartheta_\beta}\right)^2 - \left(\frac{\vartheta_\gamma''}{\vartheta_\gamma}\right)^2& \\
- 4\frac{\vartheta_1'''}{\vartheta_1'}\frac{\vartheta_\alpha''}{\vartheta_\alpha} = 0 .&
\end{aligned} \tag{α}$$

Here α, β, γ designate the 3 different indices 2, 3, 4 in some order. Equation (α) gives preference to the index α over β, γ. We can write down in analogy equations (β) and (γ), by permuting α, β, γ. We then combine the equations according to the scheme $3(\alpha) - (\beta) - (\gamma)$, with the result

$$4\frac{\vartheta_\alpha^{\mathrm{IV}}}{\vartheta_\alpha} + 8\left(\frac{\vartheta_\alpha''}{\vartheta_\alpha}\right)^2 - 4\left(\frac{\vartheta_\beta''}{\vartheta_\beta}\right)^2 - 4\left(\frac{\vartheta_\gamma''}{\vartheta_\gamma}\right)^2$$
$$- 4\frac{\vartheta_1'''}{\vartheta_1'}\left(3\frac{\vartheta_\alpha''}{\vartheta_\alpha} - \frac{\vartheta_\beta''}{\vartheta_\beta} - \frac{\vartheta_\gamma''}{\vartheta_\gamma}\right) = 0\,.$$

Cancelling here the factor 4 and inserting $\vartheta_1'''\vartheta_1'$ from (93.81) we obtain

$$\frac{\vartheta_\alpha^{\mathrm{IV}}}{\vartheta_\alpha} + 2\left(\frac{\vartheta_\alpha''}{\vartheta_\alpha}\right)^2 - \left(\frac{\vartheta_\beta''}{\vartheta_\beta}\right)^2 - \left(\frac{\vartheta_\gamma''}{\vartheta_\gamma}\right)^2$$
$$- \left(\frac{\vartheta_\alpha''}{\vartheta_\alpha} + \frac{\vartheta_\beta''}{\vartheta_\beta} + \frac{\vartheta_\gamma''}{\vartheta_\gamma}\right)\left(3\frac{\vartheta_\alpha''}{\vartheta_\alpha} - \frac{\vartheta_\beta''}{\vartheta_\beta} - \frac{\vartheta_\gamma''}{\vartheta_\gamma}\right) = 0\,.$$

Through rearrangements of the terms this goes over into

$$\frac{\vartheta_\alpha^{\mathrm{IV}}}{\vartheta_\alpha} - 3\left(\frac{\vartheta_\alpha''}{\vartheta_\alpha}\right)^2 + 2\left(\frac{\vartheta_\alpha''}{\vartheta_\alpha} - \frac{\vartheta_\beta''}{\vartheta_\beta}\right)\left(\frac{\vartheta_\alpha''}{\vartheta_\alpha} - \frac{\vartheta_\gamma''}{\vartheta_\gamma}\right) = 0\,, \quad \alpha = 2, 3, 4\,. \tag{93.9}$$

For $\alpha = 3$ and because of (93.61), (93.63) this proves the statement (93.21) of the lemma. []

We have at the same time proved

$$\frac{\vartheta_2^{\mathrm{IV}}}{\vartheta_2} - 3\left(\frac{\vartheta_2''}{\vartheta_2}\right)^2 = -2\pi^4\vartheta_4^4\vartheta_3^4\,, \qquad \frac{\vartheta_4^{\mathrm{IV}}}{\vartheta_4} - 3\left(\frac{\vartheta_4''}{\vartheta_4}\right)^2 = -2\pi^4\vartheta_2^4\vartheta_3^4\,.$$

Concluding remarks. It may be of interest to mention that the arithmetical function $\omega(n)$ defined in (91.41) and appearing as remainder term in (92.2) can be expressed by means of *quaternions* [60], esp. pp. 302, 303. An argument similar to that of this section shows that

$$\omega(n) = \frac{1}{8}\Sigma(x + i_1 y + i_2 z + i_3 w)^4\,,$$

where $x + i_1 y + i_2 z + i_3 w$ are quaternions with integer components x, y, z, w and the sum is extended over all quaternions with norm $N(x + i_1 y + i_2 z + i_3 w) = n$.

We have obtained expressions for the number of representations of a number n as a sum of 2, 4, 6, 8, 10, 12 squares. With more labor the list of formulae could be extended. Glaisher [3] has given formulae also for 14, 16 and 18 squares. However, a different method which we shall develop later will lead to a more systematic treatment of these questions, will yield more results and will open a deeper insight into the nature of these and similar representations.

III. Formal Power Series

Chapter 12

Formal Power Series and the Theory of Partitions

94. Introduction and definitions

In the previous chapter we have used power series in order to obtain results in additive number theory. The method consisted in comparing coefficients in two different expansions of the same function and interpreting the coefficients in an arithmetical way.

This method goes back to Euler, who in his "Introductio in Analysis Infinitorum" (1742) has a chapter captioned "De partitione numerorum", in which he uses power series for purposes of additive number theory.

Euler never mentions in these developments the question of convergence of the power series. He was quite right in this respect, since his arguments are quite unaffected by problems of convergence. His developments are purely formal and are concerned actually only with *formal power series.*

A formal power series is an expression

$$A = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n ,$$

in which the symbol x represents an indeterminate, whose different powers serve only to keep the coefficients a_n apart. The indeterminate x can never be replaced by a special value. The coefficients a_n are taken from a commutative ring with unit element (written simply a 1) and without divisors of zero. In special cases the coefficients may have to belong to a field.

Formal power series themselves form also a commutative ring with unit element and without divisors of zero, in short a "domain" of integrity $\mathfrak{D}$, under the following definitions:

(I) Addition: To

$$A = a_0 + a_1 x + a_2 x^2 + \cdots$$

and

$$B = b_0 + b_1 x + b_2 x^2 + \cdots$$

we define the *sum* $C = A + B$,

$$C = c_0 + c_1 x + c_2 x^2 + \cdots$$

with

$$c_n = a_n + b_n\,, \qquad n = 0, 1, 2, \ldots .$$

(II) Multiplication: We define the *product* $D = A \cdot B$ as

$$D = d_0 + d_1 x + d_2 x^2 + \cdots$$

with

$$d_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0\,, \qquad n = 0, 1, 2, \ldots .$$

It is easily seen that the commutative, associative, and distributive laws are fulfilled for these two operations. The formal power series therefore form a ring. This ring has the unit element

$$I = 1 + 0x + 0x^2 + \cdots$$

and the zero element

$$0 = 0 + 0x + 0x^2 + \cdots .$$

We shall designate these two elements simply as 1 and 0.

The ring of formal power series (under our assumptions concerning the realm of coefficients) has no divisor of zero. Indeed assume that neither A nor B is 0, and consider

$$A \cdot B = (a_0 + a_1 x + a_2 x^2 + \cdots)\,(b_0 + b_1 x + b_2 x^2 + \cdots)\,.$$

Now let a_k be the first non-vanishing coefficient of A and b_l that of B. Then in

$$A \cdot B = C = c_0 + c_1 x + c_2 x^2 + \cdots$$

we find

$$c_{k+l} = a_0 b_{k+l} + \cdots + a_k b_l + \cdots + a_{k+l} b_0\,,$$

where in the sum all terms except $a_k b_l$ must vanish,

$$c_{k+l} = c_k b_l \neq 0$$

since the coefficient ring has no zero divisors. This shows that $A\,B \neq 0$. Therefore the *law of cancellation* holds:

If $A \neq 0$ and $A\,B = A\,C$ then $B = C$.

(Equality of two power series means equality of their corresponding coefficients).

Under certain conditions division is possible within the domain $\mathfrak{D}$. A series

$$D = 1 + a_1x + a_2x^2 + \cdots$$

with leading coefficient 1 possesses a unique inverse[1]

$$D^{-1} = 1 + b_1x + b_2x^2 + \cdots ,$$

where $b_1, b_2, \ldots$ can be computed from

$$(1 + a_1x + a_2x^2 + \cdots)\,(1 + b_1x + b_2x^2 + \cdots) = 1$$

or

$$a_1 + b_1 = 0\,,$$

$$a_2 + a_1b_1 + b_2 = 0\,,$$

$$a_3 + a_2b_1 + a_1b_2 + b_3 = 0 \text{ etc.}$$

Evidently the b_n belong to the ring of the a_n.

In this manner we have

$$(1 - x)^{-1} = 1 + x + x^2 + \cdots . \tag{94.1}$$

Indeed

$$(1 - x)\,(1 + x + x^2 + \cdots) = 1 \tag{94.2}$$

since

$$(1 - x)\,(1 + b_1x + b_2x^2 + \cdots) = 1$$

demands

$$-1 + b_1 = 0\,, \qquad -b_{n-1} + b_n = 0\,,$$

so that

$$b_n = 1 \quad \text{for } n = 1, 2, \ldots .$$

It is easily seen that, if D_1, D_2 have leading coefficients 1, then

$$D_1^{-1} \cdot D_2^{-1} = (D_1D_2)^{-1} .$$

These algebraic rules governing the ring $\mathfrak{D}$ need one further extension, namely addition and multiplication of infinitely many formal power series under certain conditions.

We admit infinite sums of formal power series

$$A_1 + A_2 + A_3 + \cdots$$

[1] In the algebra of the ring of formal power series such a series D would be called a "unit".

only under one condition, namely that for any given N only finitely many of the summands contain terms x^n with $n < N$, or in other words that that infinite sum is "modulo x^N" congruent to a polynomial. The equation

$$A_1 + A_2 + A_3 + \cdots = S$$

means then that the congruence

$$A_1 + A_2 + A_3 + \cdots \equiv S \pmod{x^N}$$

is satisfied for any N. It is clear that this condition can only be satisfied if

$$A_k = a_{n_k}^{(k)} x^{n_k} + \cdots$$

and $n_k \to \infty$ as $k \to \infty$, in other words that the summands A_k begin for high enough k with terms of order x^N or higher, for any given N.
Similarly

$$B_1 \cdot B_2 \cdot B_3 \cdot \cdots = M$$

means

$$B_1 \cdot B_2 \cdot B_3 \cdot \cdots \equiv M \pmod{x^N}$$

for any natural number N. For each x^N only finitely many factors B_k can contribute to terms in M lower than x^N. Therefore for sufficiently high k we must have

$$B_k = 1 + b_{n_k}^{(k)} x^{n_k} + \cdots$$

with $n_k \to \infty$ as $k \to \infty$.

With these definitions of infinite sums and products the laws of addition and multiplication are maintained, since modulo x^N we have only finite sums and products of polynomials (i.e. formal power series with only finitely many non-vanishing coefficients) before us. Rearrangements of the terms of infinite sums and infinite products of formal power series are covered by the commutative laws of addition and multiplication.

This reduction of equations between formal power series to congruences modulo x^N for arbitrary N justifies immediately also the following

Rule. *In an equation the indeterminate x can be replaced by any polynomial without a constant term.*

Indeed since modulo x^N we have only to deal with polynomials and since the n-th power of a polynomial without constant term cannot have lower terms than x^n, the substitution can be reduced to the case of polynomials, for which it is trivial.

The theory of formal power series will lead to new proofs for some identities which have been derived in previous chapters (§§ 84, 85, 78); it will, however, also appear as the most adequate tool for the proof of certain peculiar theorems of additive number theory which are not easily accessible by other methods.

95. Some elementary identities

Let

$$G = 1 + x + x^2 + \cdots$$

and

$$P = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)\ldots.$$

We are going to prove

$$G = P. \tag{95.1}$$

For this purpose we consider

$$(1 - x)P = (1 - x)(1 + x)(1 + x^2)(1 + x^4)(1 + x^8)\ldots.$$

Multiplying the first 2 factors we obtain

$$(1 - x)P = (1 - x^2)(1 + x^2)(1 + x^4)(1 + x^8)\ldots,$$

and continuing this way:

$$\begin{aligned}(1 - x)P &= (1 - x^4)(1 + x^4)(1 + x^8)\ldots\\ &= \cdots\\ &= (1 - x^{2^k})(1 + x^{2^k})(1 + x^{2^{k+1}})\ldots,\end{aligned}$$

which shows that for every non-negative integer k,

$$(1 - x)P \equiv 1 \pmod{x^{2^k}},$$

and according to our definition of an infinite product,

$$(1 - x)P = 1.$$

In view of (94.2) we have therefore

$$(1 - x)P = (1 - x)G,$$

and after the law of concellation,

$$P = G.$$

A comparison of the coefficients of x^n on both sides of (95.1) gives the well-known

Theorem. *Any natural number n can in only one way*[1] *be expressed as a sum of different powers of* 2 (Binary expansion).

In the identity (95.1), explicitly written

$$1 + x + x^2 + x^3 + \cdots = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)\ldots,$$

we replace successively x by $x^3, x^5, \ldots, x^m, \ldots$, where m is any positive odd number:

$$\begin{aligned} 1 + x^3 + x^6 + x^9 + \cdots &= (1 + x^3)(1 + x^6)(1 + x^{12})\ldots, \\ 1 + x^5 + x^{10} + x^{15} + \cdots &= (1 + x^5)(1 + x^{10})(1 + x^{20})\ldots, \\ 1 + x^m + x^{2m} + x^{3m} + \cdots &= (1 + x^m)(1 + x^{2m})(1 + x^{4m})\ldots. \end{aligned}$$

We multiply all these equations with each other. This is permissible since any power x^n appears at most in finitely many of the power series and all of these power series begin with 1. On the right-hand side appear all expressions $1 + x^n$ and each just once, for all natural numbers n, since each n has a unique factorization $n = 2^k m$, with m odd.

We obtain thus, with rearrangement on the right side,

$$\prod_{\substack{m=1 \\ m \text{ odd}}}^{\infty} (1 + x^m + x^{2m} + x^{3m} + \cdots) = \prod_{n=1}^{\infty} (1 + x^n) = \sum_{r=0}^{\infty} a_r x^r, \tag{95.2}$$

say. Here a_r has two interpretations: (1) a_r is the number of times that the exponent r of x^r can be obtained additively from the *different* summands n in the second product, (2) a_r is the number of times that r can be expressed as a sum of multiples of odd numbers m in the first product, or in other words as a sum of odd numbers with repetition permitted. Rearrangements of the summands are to be disregarded, since each factor stems exactly from one factor of the products in (95.2), whereas, e.g., in (85.3) a square exponent could appear in any of the four factors, so that there rearrangements of the summands correspond to different terms in the product expansion.

A decomposition of a number n into summands without regard to their order is called a *partition* of n; the summands are then sometimes called parts.

In this terminology we express the statements derived from (95.2) in the following

Theorem (Euler). *A number can as often be partitioned into different summands (parts) as it can be partitioned into odd summands (parts) with repetition permitted.*

[1] Rearrangements of summands disregarded.

Example. The number 8 can be partitioned into *different parts* as follows

$$8, \quad 1+7, \quad 2+6, \quad 3+5, \quad 1+2+5, \quad 1+3+4$$

and into *odd parts* with repetitions:

$$1+7, \quad 3+5, \quad 1+1+1+5, \quad 1+1+3+3,$$

$$1+1+1+1+1+3, \quad 1+1+1+1+1+1+1+1,$$

in both cases 6 times.

The derivation of (95.2) can be shortened. We write it after (94.1) as

$$\prod_{m \text{ odd}} \frac{1}{1-x^m} = \prod_{n=1}^{\infty} (1+x^n) \,,$$

which is equivalent to

$$\prod_{n=1}^{\infty} (1+x^n) \prod_{n=1}^{\infty} (1-x^{2n-1}) = 1 \,.$$

Let us name the left-hand member $Q(x)$. Then

$$\begin{aligned} Q(x) &= \Pi(1+x^{2n}) \Pi(1+x^{2n-1}) \Pi(1-x^{2n-1}) \\ &= \Pi(1+x^{2n}) \Pi(1-x^{4n-2}) \end{aligned}$$

so that

$$Q(x) = Q(x^2) \,.$$

But the definition of $Q(x)$ shows that it is a power series beginning with 1:

$$Q(x) = 1 + a_k x^k + a_{k+1} x^{k+1} + \cdots,$$

where a_k is the first non-vanishing coefficient. But then

$$Q(x^2) = 1 + a_k x^{2k} + a_{k+1} x^{2(k+1)} + \cdots$$

so that from

$$Q(x) \equiv 1 \pmod{x^k}$$

follows

$$Q(x) \equiv 1 \pmod{x^{2k}} \,.$$

Through iteration we infer that

$$Q(x) \equiv 1 \pmod{x^N}$$

for arbitrarily high N, or in other words $Q(x) = 1$, as we had to prove.

96. Partitions with restricted size or number of parts

In order to distinguish partitions with respect to the number of parts, Euler introduced a parameter z in the power series. This means in our terminology that the coefficients of the power series are now taken from the ring of polynomials in z with integer coefficients.

We begin with the example

$$(1 + zx)(1 + zx^2)(1 + zx^3)\cdots = 1 + zx + zx^2 + (z + z^2)x^3 + \cdots . \tag{96.1}$$

On both sides we have formal power series in x. On the right hand we separate the power series in x according to powers of z and obtain

$$1 + zA_1(x) + z^2A_2(x) + \cdots = 1 + \sum_{k=1}^{\infty} z^k A_k(x) , \tag{96.2}$$

say. The $A_k(x)$ are themselves power series in x, and we have an infinite sum of them. But this is an admissible addition since $A_k(x)$ is accompanied by z^k and can thus only be obtained by multiplying k terms of the sort zx^l in (96.1) so that $A_k(x)$ begins with $x^{1+2+\cdots+k} = x^{k(k+1)/2}$.

To make our argument stringent and to get at the same time expressions for $A_k(x)$ we consider the power series modulo x^N. We put

$$P_N(x; z) = \prod_{m=1}^{N}(1 + zx^m)$$

and have clearly

$$P_N(x; z) \equiv \prod_{m=1}^{\infty}(1 + zx^m) \pmod{x^{N+1}} . \tag{96.3}$$

Now $P_N(x; z)$ is a polynomial in x and z and its terms can be arranged in any order. We put

$$P_N(x; z) = 1 + zA_1^{(N)}(x) + z^2A_2^{(N)}(x) + \cdots + z^N A_N^{(N)}(x) , \tag{96.4}$$

where the $A_k^{(N)}(x)$ are polynomials in x. We replace here z by xz and have on the one hand

$$P_N(x; xz) = \prod_{m=2}^{N+1}(1 + zx^m) = \frac{1 + zx^{N+1}}{1 + zx} P_N(x; z)$$

and on the other hand

$$P_N(x; xz) = 1 + xzA_1^{(N)}(x) + x^2z^2A_2^{(N)}(x) + \cdots + x^N z^N A_N^{(N)}(x)$$

so that we obtain the equation

$$\begin{aligned}&(1 + zx^{N+1})\left(1 + zA_1^{(N)}(x) + \cdots + z^N A_N^{(N)}(x)\right)\\ &= (1 + zx)\left(1 + xzA_1^{(N)}(x) + \cdots + x^N z^N A_N^{(N)}(x)\right).\end{aligned}$$

Comparing here on both sides the terms with z^k we obtain

$$A_k^{(N)}(x) + x^{N+1} A_{k-1}^{(N)}(x) = x^k A_k^{(N)}(x) + x^k A_{k-1}^{(N)}(x)$$

with $A_0(x) = 1$, so that

$$A_k^{(N)}(x) = \frac{x^k - x^{N+1}}{1 - x^k} A_{k-1}^{(N)}(x)$$

and consequently

$$A_k^{(N)}(x) \equiv \frac{x^k}{1 - x^k} A_{k-1}^{(N)}(x) \pmod{x^{N+1}} .$$

This yields by induction

$$A_k^{(N)}(x) \equiv \frac{x^{k(k+1)/2}}{(1-x)(1-x^2)\cdots(1-x^k)} \pmod{x^{N+1}} .$$

This congruence together with (96.3), (96.4) furnishes

$$\prod_{m=1}^{\infty} (1 + z x^m) \equiv 1 + \sum_{k=1}^{\infty} \frac{z^k x^{k(k+1)/2}}{(1-x)(1-x^2)\cdots(1-x^k)} \pmod{x^{N+1}}$$

for any N and therefore finally the identity

$$\prod_{m=1}^{\infty} (1 + z x^m) = 1 + \sum_{k=1}^{\infty} z^k A_k(x) \tag{96.5}$$

with

$$A_k(x) = \frac{x^{k(k+1)/2}}{(1-x)(1-x^2)\cdots(1-x^k)} . \tag{96.6}$$

The coefficient $a_n^{(k)}$ of $z^k x^n$ in the power series expansion of the left-hand side of (96.5) gives the number of ways to write n as a sum

$$n = m_1 + m_2 + \cdots + m_k , \quad 0 < m_1 < m_2 < \cdots < m_k , \tag{96.7}$$

in other words it is the number of partitions of n into k different parts. On the right-hand side it is the coefficient of x^n in $A_k(x)$ or of $x^{n-(k(k+1)/2)}$ in

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^k)} = \sum_{n_1=0}^{\infty} x^{n_1} \sum_{n_2=0}^{\infty} x^{2n_2} \cdots \sum_{n_k=0}^{\infty} x^{kn_k} ,$$

i.e. the number of solutions of the diophantine equation

$$n - \big(k(k+1)/2\big) = n_1 + 2n_2 + \cdots + kn_k , \quad n_j \geqq 0$$

or the number of partitions of partitions of $n - \big(k(k+1)/2\big)$ into parts not exceeding k, repetition of parts admissible. We have thus the

Theorem. *The number n has as many paritition*s *into k different parts as $n - (k(k+1)/2)$ has into parts not exceeding k.*

Suppose we have a partition (96.7) of n into k different parts. If we add $0, 1, \ldots, k-1$ respectively to the parts we obtain from (96.7)

$$n + (k(k-1)/2) = m_1 + (m_2+1) + (m_3+2) + \cdots + (m_k + k-1)$$
$$= m_1' + m_2' + \cdots + m_k',$$

say, where

$$m_{j+1}' - m_j' \geqq 2, \quad 0 < m_1.$$

Thus, the number of partitions of N into k parts which differ by at least 2 is equal to the number of partitions of $N - (k(k-1)/2)$ into parts which differ by at least 1. Therefore, since $k(k+1)/2 + k(k-1)/2 = k^2$, it follows from (96.6) that the coefficient of x^n in

$$A_k^*(x) = \frac{x^{k^2}}{(1-x)\cdots(1-x^k)} \tag{96.8}$$

gives the number of partitions of n into k parts which differ by at least 2.

If starting from the above partition, we add successively $1, 2, \ldots, k$ to the parts we obtain

$$n + (k(k+1)/2) = (m_1+1) + (m_2+2) + \cdots + (m_k+k)$$
$$= m_1'' + m_2'' + \cdots + m_k'',$$

where again

$$m_{j+1}'' - m_j'' \geqq 2$$

but also

$$m_1'' > m_1 > 0,$$

or

$$m_1'' > 2.$$

We observe therefore that the coefficient of x^n in

$$A_k^{**}(x) = \frac{x^{k(k+1)}}{(1-x)\cdots(1-x^k)} \tag{96.9}$$

gives the number of partitions of n into k parts, differing at least by 2, with the further condition that the smallest part is at least 2.

97. Some similar theorems

In a completely analogous manner the following identity can be proved

$$\frac{1}{\prod_{m=1}^{\infty}(1-zx^m)} = 1 + \sum_{k=1}^{\infty} z^k B_k(x) \tag{97.1}$$

with

$$B_k(x) = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}\,. \tag{97.2}$$

On both sides of (97.1) we have formal power series in x. The coefficient of $z^k x^n$ on the left side gives the number of partitions of n into k parts. On the right it represents the coefficient of x^n in $B_k(x)$. This coefficient gives the number of solutions of

$$n - k = n_1 + 2n_2 + \cdots + kn_k\,, \qquad n_j \geqq 0 \tag{97.3}$$

or in other words the number of partitions of $n - k$ with parts not exceeding k. We have thus the

Theorem A. *The number of partitions of n into k parts is equal to the number of partitions of $n - k$ into parts not exceeding k.*

If we rewrite (97.3) as

$$n = n_1 + 2n_2 + \cdots + (k-1)n_{k-1} + k(n_k + 1)\,, \qquad n_j \geqq 0$$

we have before us a partition of n in which the greatest part is k, since $n_k + 1 > 0$. We can therefore reformulate the previous theorem in a more useful manner as

Theorem B. *The number of partitions of n into k parts is equal to the number of partitions of n with greatest part k.*

In this form the theorem is easily demonstrated by referring to an array of dots. We represent each of the k parts by a line of dots, the parts being arranged in non-increasing order

$$\begin{array}{l} \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \cdot \\ \cdot \end{array}$$

This array now read columnwise gives a partition of the same number n with the greatest part k. (Two such partitions are sometimes called

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}
C_0	**1**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
C_1	1	**1**	1	1	1	1	1	1	1	1	1	1	1	1	1	1
			1	1	2	2	3	3	4	4	5	5	6	6	7	7
C_2	1	1	**2**	2	3	3	4	4	5	5	6	6	7	7	8	8
				1	1	2	3	4	5	7	8	10	12	14	16	19
C_3	1	1	2	**3**	4	5	7	8	10	12	14	16	19	21	24	27
					1	1	2	3	5	6	9	11	15	18	23	27
C_4	1	1	2	3	**5**	6	9	11	15	18	23	27	34	39	47	54
						1	1	2	3	5	7	10	13	18	23	40
C_5	1	1	2	3	5	**7**	10	13	18	23	30	37	47	57	70	84
							1	1	2	3	5	7	11	14	20	26
C_6	1	1	2	3	5	7	**11**	14	20	26	35	44	58	71	90	110
								1	1	2	3	5	7	11	15	21
C_7	1	1	2	3	5	7	11	**15**	21	28	38	49	65	82	105	131
									1	1	2	3	5	7	11	15
C_8	1	1	2	3	5	7	11	15	**22**	29	40	52	70	89	116	146
										1	1	2	3	5	7	11
C_9	1	1	2	3	5	7	11	15	22	**30**	41	54	73	94	123	157

conjugate.) This figure shows $n = 20$, $k = 6$, $20 = 7 + 4 + 4 + 3 + 1 + 1 = 6 + 4 + 4 + 3 + 1 + 1 + 1$. For such graphical proof of the theorems of § 96 see, e.g., [38], vol. 2, Ch. 1, § VII.

Another example, of which we shall later make an important application, is the identity

$$\prod_{r \text{ odd}} (1 + z x^r) = 1 + \sum_{k=1}^{\infty} z^k C_k(x) \tag{97.4}$$

with

$$C_k(x) = \frac{x^{k^2}}{(1 - x^2)(1 - x^4) \cdots (1 - x^{2k})}. \tag{97.5}$$

This identity, proved in a similar way, can be translated into the

Theorem C. *The number of partitions of n into k different odd parts is equal to the number of partitions of $(n - k^2)/2$ into parts not exceeding k (evidently $n - k^2$ is even).*

x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}	x^{24}	x^{25}	x^{26}	x^{27}	x^{28}
0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1
8	8	9	9	10	10	11	11	12	12	13	13	14
9	9	10	10	11	11	12	12	13	13	14	14	15
21	24	27	30	33	37	40	44	48	52	56	61	65
30	33	37	40	44	48	52	56	61	65	70	75	80
34	39	47	54	64	72	84	94	108	120	136	150	169
64	72	84	94	108	120	136	150	169	185	206	225	249
37	47	57	70	84	101	119	141	164	192	221	255	291
101	119	141	164	192	221	255	291	333	377	427	480	540
35	44	58	71	90	110	136	163	199	235	282	331	391
136	163	199	235	282	331	391	454	532	612	709	811	931
28	38	49	65	82	105	131	164	201	248	300	364	436
164	201	248	300	364	436	522	618	733	860	1009	1175	1367
22	29	40	52	70	89	116	146	186	230	288	352	434
186	230	288	352	434	525	638	764	919	1090	1297	1527	1801
15	22	30	41	54	73	94	123	157	201	252	318	393
201	252	318	393	488	598	732	887	1076	1291	1549	1845	2194

In (96.6), (97.2), (97.5) the essential task in computing the number of partitions will be to compute $p_k(n)$ in (97.6)

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^k)} = \sum_{n=0}^{\infty} p_k(n)\,x^n\,, \qquad p_0 = 1\,. \tag{97.6}$$

This problem can in principle be solved by decomposing algebraically the rational function on the left into a sum of partial fractions, e.g.,

$$\frac{1}{(1-x)(1-x^2)} = \frac{1/2}{(1-x)^2} + \frac{1/4}{1-x} + \frac{1/4}{1+x}\,,$$

$$\frac{1}{(1-x)(1-x^2)(1-x^3)} = \frac{1/6}{(1-x)^3} + \frac{1/4}{(1-x)^2} + \frac{17/12}{1-x} + \frac{1/8}{1+x}$$

$$+ \frac{1/9}{1-x/\varrho} + \frac{1/9}{1-x/\bar{\varrho}}$$

with $\varrho^2 + \varrho + 1 = 0$, $\varrho^3 = 1$. These fractions can then be expanded in power series:

$$\frac{1}{(1-x)(1-x^2)} = \frac{1}{2}\sum_{n=0}^{\infty}(n+1)x^n + \frac{1}{2}\sum_{m=0}^{\infty}x^{2m},$$

$$\frac{1}{(1-x)(1-x^2)(1-x^3)} = \frac{1}{6}\sum_{n=0}^{\infty}\binom{n+2}{2}x^n + \frac{1}{4}\sum_{n=0}^{\infty}(n+1)x^n$$

$$+\frac{1}{4}\sum_{m=0}^{\infty}x^{2m} + \frac{1}{3}\sum_{m=0}^{\infty}x^{3m}.$$

From these expansions one can read off the coefficients $p_2(n)$, $p_3(n)$ in (97.6). After some simplifications one gets the results:

The number of partitions of n into parts not exceeding 2 is $p_2(n) = [n/2] + 1$; into parts not exceeding 3, is $p_3(n) = [n(n+6)/12] + 1$.

Formulae of a similar nature can in principle be given for each k; however they get more and more involved for increasing k, and no general rule covering all k is known. Investigations of this kind have in particular been carried out by J. J. Sylvester [70, 71].

Euler proceeded in a different way. From (97.6) one can read off the identity

$$\sum_{n=0}^{\infty}p_{k-1}(n)\,x^n = (1-x^k)\sum_{n=0}^{\infty}p_k(n)\,x^n,$$

which yields the formula of recursion

$$p_k(n) = p_{k-1}(n) + p_k(n-k), \tag{97.7}$$

where $p_k(n)$ is the number of partitions of n into parts not exceeding k; in the formula we have

$$p_k(0) = 1 \quad \text{and} \quad p_k(l) = 0 \quad \text{for } l < 0,$$

also

$$p_1(n) = 1.$$

Formula (97.7) can be used very effectively for the tabulation of $p_k(n)$. The appended table is computed row-wise. The numbers $p_k(n)$ in the k-th row are found as follows: the first k entries agree with $p_{k-1}(n)$, according to (97.7). These are then used to furnish k of the numbers $p_k(n-k)$, written above the line k. Then the addition indicated in (97.7) is carried out and yields k more values of $p_k(n)$. The vertical bars indicate these batches of k values each so computed (below the k-th line) and then skifted by k (above the k-th line).

Euler[1] quotes the value $p_7(22) = 522$, found also in our table, and $p_7(43) = 8946$ beyond its limits. According to (96.6) and the theorem in § 96 on the one hand and (97.2), Theorem A in § 97 on the other hand, these numbers show that the number of partitions of 50 into 7 different parts is 522, the number of partitions of 50 into 7 parts with permissible repetition is 8946.

98. Unrestricted partitions

No partition of n can contain a part exceeding n itself. Thus in our notation $p_n(n)$ counts partitions without any restriction as to the number of the parts or their properties (size, parity, etc.). We write $p(n)$ instead of $p_n(n)$. In our table these numbers appear in front of the broken line.

The generating series for $p(n)$ is

$$\frac{1}{\prod\limits_{m=1}^{\infty}(1-x^m)} = \sum_{n=0}^{\infty} p(n)\, x^n\,, \qquad p(0) = 1\,. \tag{98.1}$$

Indeed

$$\frac{1}{\prod\limits_{m=1}^{\infty}(1-x^m)} = \sum_0^{\infty} x^{n_1} \sum_0^{\infty} x^{2n_2} \sum_0^{\infty} x^{3n_3} \ldots \sum_0^{\infty} x^{mn_m} \ldots\,,$$

which shows that the coefficient $p(n)$ of this formal power series is the number of solutions of

$$n = n_1 + 2n_2 + 3n_3 + \cdots + mn_m + \cdots, \qquad n_j \geqq 0\,.$$

This is the number of unrestricted partitions of n, where n_1 counts the number of parts equal to $1, \ldots, n_m$ the number of parts equal to m.

The whole theory of unrestricted partitions now rests on the expansion of the denominator of the left member of (98.1). Euler found by actual computation up to x^{50}

$$\prod_{m=1}^{\infty}(1-x^m) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + + - - \cdots \tag{98.2}$$

and observed that only the coefficients 0 and ± 1 appeared in the power series. He came to a conjecture about the actually appearing exponents by considering their differences:

$$\begin{array}{ccccccccccccccccc} 0 & & 1 & & 2 & & 5 & & 7 & & 12 & & 15 & & 22 & & 26\,, \\ & 1 & & 1 & & 3 & & 2 & & 5 & & 3 & & 7 & & 4\,. & \end{array}$$

[1] Euler had been asked by Philipp Naude of the Academy of Berlin about the number of partitions of 50 into 7 parts. It seems to be this question which started Euler on his investigations in "De partitione numerorum".

The sequence of differences can be separated into a sequence of the odd numbers alternating with the sequence of all natural numbers. A better way to study the sequence of exponents is to take them alternatingly and write them in two directions, increasing both from the center 0, and forming then the two first differences

$$\begin{array}{cccccccccc} \cdots 26 & 15 & 7 & 2 & 0 & 1 & 5 & 12 & 22 & \cdots, \\ \cdots & -11 & -8 & -5 & -2 & 1 & 4 & 7 & 10 & \cdots, \\ \cdots & & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \cdots, \end{array}$$

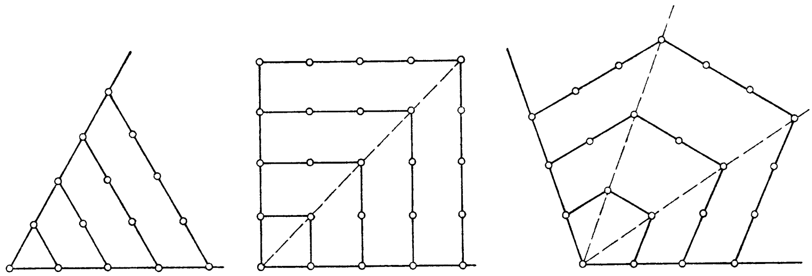
The exponents form thus an arithmetic progression of the second order. It is well known that the general term ω_λ of such a progression can be expressed as a quadratic polynomial in the index

$$\omega_\lambda = a\lambda^2 + b\lambda + c\,.$$

With $\omega_0 = 0$, $\omega_1 = 1$, $\omega_2 = 5$ we find here

$$\omega_\lambda = \frac{\lambda(3\lambda - 1)}{2}, \qquad \lambda = 0, \pm 1, \pm 2, \pm 3, \ldots. \tag{98.3}$$

These numbers are called (for positive index) the *pentagonal numbers*, which have played some role in early mathematics as generalizations of trigonal numbers, and square numbers, indicated by the following diagrams:



We have thus what is called the "pentagonal numbers theorem", which Euler first conjectured in 1742 and finally proved in 1750.

Theorem A.

$$\prod_{m=1}^{\infty}(1 - x^m) = \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda x^{\lambda(3\lambda-1)/2}\,. \tag{98.4}$$

This theorem we have already proved in the theory of the ϑ-function as formula (78.7). However, it seems desirable to prove it in the present context within the theory of formal power series. We use Euler's proof.

Proof. We break up the infinite product (98.2) by using the distributive law

$$\begin{aligned} P &= (1-x)(1-x^2)(1-x^3)(1-x^4)\cdots \\ &= 1 - x - (1-x)x^2 - (1-x)(1-x^2)x^3 \\ &\quad - (1-x)(1-x^2)(1-x^3)x^4 - \cdots . \end{aligned} \tag{98.5}$$

This is an infinite series of formal power series (actually polynomials), which is admissible since modulo x^N only finitely many come into play.

Opening the first parenthesis of every product we distribute with a shift each product over two lines

$$\begin{aligned} 1 - x - x^2 - (1-x^2)x^3 - (1-x^2)(1-x^3)x^4 - \cdots \\ + x^3 + (1-x^2)x^4 + \cdots , \end{aligned}$$

which we recombine, considering common factors, into

$$1 - x - x^2 + x^5 + (1-x^2)x^7 + (1-x^2)(1-x^3)x^9 + \ldots . \tag{98.6}$$

We open again the first parenthesis of each term, distribute with a shift over two lines and recombine:

$$\begin{aligned} 1 - x - x^2 + x^5 + x^7 + (1-x^3)x^9 + (1-x^3)(1-x^4)x^{11} + \cdots \\ - x^9 - (1-x^3)x^{11} - \cdots \\ = 1 - x - x^2 + x^5 + x^7 - x^{12} - (1-x^3)x^{15} \\ - (1-x^3)(1-x^4)x^{18} - \cdots . \end{aligned} \tag{98.7}$$

We have obtained in (98.6) and (98.7) some single powers of x with exponents which are indeed pentagonal numbers, and furthermore infinitely many terms still containing parentheses as factors. In order to make the induction complete let us assume that we have reached the stage

$$\begin{aligned} P &= \sum_{\lambda=-k+1}^{k} (-1)^\lambda x^{\omega_\lambda} + (-1)^k x^{\omega_k + k} \\ &\quad \times \{(1-x^k) + (1-x^k)(1-x^{k+1})x^k \\ &\quad + (1-x^k)(1-x^{k+1})(1-x^{k+2})x^{2k} + \cdots\} \end{aligned} \tag{98.8}$$

of which (98.5), (98.6), (98.7) are the examples $k = 1, 2, 3$. Here we open, as before, the first parenthesis in each term and distribute the result in two infinite sums:

$$P = \sum_{\lambda=-k+1}^{k} (-1)^{\lambda} x^{\omega_\lambda} + (-1)^k x^{\omega_k + k}$$
$$\times \{1 + (1 - x^{k+1}) x^k + (1 - x^{k+1}) (1 - x^{k+2}) x^{2k} + \cdots$$
$$- x^k - (1 - x^{k+1}) x^{2k} - \cdots\}.$$

If we combine the two series we obtain

$$P = \sum_{\lambda=-k+1}^{k} (-1)^{\lambda} x^{\omega_\lambda} + (-1)^k x^{\omega_k + k} - (-1)^{k+1} (-1)^{\omega_k + 3k + 1}$$
$$+ (-1)^{k+1} x^{\omega_k + 4k + 2} \{1 + (1 - x^{k+1}) x^{k+1}$$
$$+ (1 - x^{k+1}) \quad (1 - x^{k+2}) x^{2k+2} + \cdots\}.$$

Now after (98.3)

$$\omega_k + k = \frac{k(3k+1)}{2} = \omega_{-k}, \quad \omega_k + 3k + 1 = \frac{(k+1)(3k+2)}{2} = \omega_{k+1}$$

so that we can write

$$P = \sum_{\lambda=-k}^{k+1} (-1)^{\lambda} x^{\omega_\lambda} + (-1)^{k+1} x^{\omega_{k+1} + k + 1}$$
$$\times \{1 + (1 - x^{k+1}) x^{k+1} + (1 - x^{k+1}) (1 - x^{k+2}) x^{2k+2} + \cdots\},$$

a formula which emerges from (98.8) by the substitution of $k + 1$ for k. We see thus that

$$P \equiv \sum_{\lambda=-\infty}^{\infty} (-1)^{\lambda} x^{\omega_\lambda} \pmod{x^{\omega_{k+1}}}$$

for any k. This proves the theorem. ▯

If we write

$$\sum_{m=1}^{\infty} (1 - x^m) = \sum_{n=0}^{\infty} d_n x^n$$

then we can interpret d_n as the difference of the number of ways to express n as a sum of an even number of different parts and as a sum of an odd number of different parts. This difference d_n is mostly 0. It is $(-1)^\lambda$ if n is a pentagonal number ω_λ. We obtain thus the

Theorem B. *A number n which is not a pentagonal number can be partitioned as often into an even number of different parts as into an odd number of different parts. If however $n = \omega_\lambda = \lambda(3\lambda - 1)/2$ then the excess of the first sort over the second sort of partition is $(-1)^\lambda$.*

This theorem is obviously equivalent with Theorem A. It was proved in a direct and elementary way by Fabian Franklin [9][1].

If we insert now (98.4) into (98.1) we obtain the equation

$$1 = \sum_{\lambda=-\infty}^{\infty} (-1)^{\lambda} x^{\lambda(3\lambda-1)/2} \sum_{n=0}^{\infty} p(n)\, x^n ,$$

from which we derive, besides $p(0) = 1$, considering the coefficient of x^n,

$$0 = \sum_{0 \leqq \omega_\lambda \leqq n} (-1)^{\lambda} p(n - \omega_\lambda) ,$$

ω_λ again the abbreviation (98.3) for the pentagonal number. If we transpose the term with $\lambda = 0$ we have the recursion formula

$$p(n) = \sum_{0 < \omega_\lambda \leqq n} (-1)^{\lambda-1} p(n - \omega_\lambda) . \tag{98.9}$$

This formula has been used to compute $p(n)$ up to $n = 600$[2].

Let us carry out a few steps.

$$\begin{aligned}
p(0) &= 1 ,\\
p(1) &= p(1-1) = 1 ,\\
p(2) &= p(2-1) + p(2-2) = 1 + 1 = 2 ,\\
p(3) &= p(3-1) + p(3-2) = 2 + 1 = 3 ,\\
p(4) &= p(4-1) + p(4-2) = 3 + 2 = 5 ,\\
p(5) &= p(5-1) + p(5-2) - p(5-5) = 5 + 3 - 1 = 7 ,\\
p(6) &= p(6-1) + p(6-2) - p(6-5) = 7 + 5 - 1 = 11 ,\\
p(7) &= p(7-1) + p(7-2) - p(7-5) - p(7-7)\\
&= 11 + 7 - 2 - 1 = 15 .
\end{aligned}$$

The equations get longer and longer, but not exceedingly so. We have on the right-hand side as many terms as there are 0 with $\lambda < \omega_\lambda \leqq n$. A short computation shows that the right member of (98.9) contains about $(2/3)\sqrt{6n}$ terms. For $n = 600$ this will be 40 terms. By the way, $p(600)$ is a number greater than $4 \cdot 5 \cdot 10^{23}$.

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[1] See, e.g. [59], p. 308.

[2] For the values up to $n = 200$ by MacMahon, see [19], pp. 285—287; for $n = 201$ to 600 by Hansraj Gupta [15].

99. Formal differentiation and its application

Before drawing further consequences from Euler's pentagonal numbers theorem we have to introduce one more operation on formal power series.

If

$$A = a_0 + a_1 x + a_2 x^2 + \cdots$$

is a formal power series then we define its *derivative* as

$$A' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots,$$

which is also a formal power series[1]. This operation is evidently linear:

$$(A + B)' = A' + B', \qquad (cA)' = cA',$$

where c is a factor taken from the ring of the coefficients of A. Moreover we have, typical for the operation of differentiation

$$(A \cdot B)' = A' \cdot B + A \cdot B'. \tag{99.1}$$

This is obviously true if one of the power series reduces to a constant term alone. It is true further in the special case $A = x^m$, $B = x^n$, for then we have after the definition

$$A' = m x^{m-1}, \qquad B' = n x^{n-1},$$

$$(AB)' = (x^{m+n})' = (m+n)\, x^{m+n-1},$$

to be compared with

$$A'B + AB' = m x^{m-1} x^n + x^m n x^{n-1} = (m+n)\, x^{m+n-1}.$$

Because of linearity the rule (99.1) is also valid for polynomials in x, and then for formal power series in general, since modulo x^N they reduce to polynomials.

Suppose now that A and B have absolute coefficients 1. They possess then reciprocals, and we can replace (99.1) by

$$\frac{(AB)'}{AB} = \frac{A'}{A} + \frac{B'}{B},$$

which we call the rule of logarithmic differentiation. This rule can, of course, be extended to any finite number of factors, as

$$\frac{(ABC)'}{ABC} = \frac{A'}{A} + \frac{(BC)'}{BC} = \frac{A'}{A} + \frac{B'}{B} + \frac{C'}{C}$$

[1] Differentiation is defined as a formal process, performed on the coefficients of the power series; no limit process is involved.

shows; in general

$$\frac{\left(\prod_{k=1}^{K} A_k\right)'}{\prod_{k=1}^{K} A_k} = \sum_{k=1}^{K} \frac{A_k'}{A_k}. \tag{99.2}$$

If the *infinite* product $\prod_{k=1}^{\infty} A_k$ is admissible then $\sum_{k=1}^{\infty} \frac{A_k'}{A_k}$ is also admissible, since in the sequence

$$A_k = 1 + a_{n_k}^{(k)} x^{n_k} + \cdots$$

$n_k \to \infty$ as $k \to \infty$ and thus

$$\sum_{k=1}^{\infty} \frac{A_k'}{A_k} = \sum_{k=1}^{\infty} n_k a_n^{(k)} x^{n_k - 1} + \cdots$$

is an admissible infinite sum. The validity of

$$\frac{\left(\prod_{k=1}^{\infty} A_k\right)'}{\prod_{k=1}^{\infty} A_k} = \sum_{k=1}^{\infty} \frac{A_k'}{A_k} \tag{99.3}$$

is then inferred modulo x^N for arbitrary N from (99.2). We apply now logarithmic differentiation first to (98.4) with the result

$$\sum_{m=1}^{\infty} \frac{-m x^{m-1}}{1 - x^m} = \frac{\sum_{\lambda} (-1)^{\lambda} \omega_{\lambda} x^{\omega_\lambda - 1}}{\sum_{\lambda} (-1)^{\lambda} x^{\omega_\lambda}},$$

where ω_λ is the pentagonal number (98.3). Changing the sign and multiplying both sides by x we have

$$\sum_{m=1}^{\infty} \frac{m x^m}{1 - x^m} = \frac{\sum_{\lambda} (-1)^{\lambda-1} \omega_{\lambda} x^{\omega_\lambda}}{\sum_{\lambda} (-1)^{\lambda} x^{\omega_\lambda}}. \tag{99.4}$$

On the left side we have a *Lambert series*. Such series we have already studied analytically in §§ 84, 85. The formal treatment is similar, ignoring only questions of convergence:

$$\sum_{m=1}^{\infty} \frac{m x^m}{1 - x^m} = \sum_{m=1}^{\infty} m \sum_{k=1}^{\infty} x^{km} = \sum_{r=1}^{\infty} x^r \sum_{mk=r} m = \sum_{r=1}^{\infty} \sigma_1(r) x^r \tag{99.5}$$

with

$$\sigma_1(r) = \sum_{d|r} d, \tag{99.6}$$

as usual.

If we use this result in (99.4) we obtain

$$\sum_{r=1}^{\infty}\sigma_1(r)\,x^r \sum_{\lambda}(-1)^{\lambda}x^{\omega_\lambda} = \sum_{\lambda}(-1)^{\lambda-1}\omega_\lambda x^{\omega_\lambda}\,.$$

Comparing the coefficients of x^n on both sides we get

$$\sum_{0\leqq\omega_\lambda\leqq n-1}(-1)^{\lambda}\sigma_1(n-\omega_\lambda) = \begin{cases}(-1)^{\lambda-1}\,\omega_\lambda & \text{if } n=\omega_\lambda\,,\\ 0 & \text{otherwise}\,,\end{cases}$$

and isolating the term with $\sigma_1(n)$,

$$\sigma_1(n) = \sum_{0<\omega_\lambda\leqq n-1}(-1)^{\lambda-1}\sigma_1(n-\omega_\lambda) + \begin{cases}(-1)^{\lambda-1}\omega_\lambda & \text{for } n=\omega_\lambda\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

In order to simplify the notation let us introduce, *only for the present purpose*, the definition

$$\sigma_1(n-n) = n \tag{99.7}$$

($\sigma_1(0)$ being meaningless in the definition (99.6)). Then we can simply write

$$\sigma_1(n) = \sum_{0<\omega_\lambda\leqq n}(-1)^{\lambda-1}\sigma_1(n-\omega_\lambda)\,, \tag{99.8}$$

where the definition (99.7) can only come into action if n is a pentagonal number.

This recurrence relation is formally the same as (98.9) for $p(n)$. The difference lies only in the definitions of $p(n-n)=p(0)=1$ and $\sigma_1(n-n)=n$.

Numerically $p(n)$ and $\sigma_1(n)$ behave quite differently. We shall see later (Chapter 14) that $p(n)$ increases very fast with n, whereas

$$\sigma_1(n) = \sum_{d\mid n} d = \sum_{\delta\mid n}\frac{n}{\delta} = 0\left(n\sum_{\delta\leqq n}\frac{1}{\delta}\right) = O(n\log n)\,,$$

and $\sigma_1(n) = n+1$ infinitely often, viz. if n is a prime number. It is remarkable that (99.8) is a purely additive relation, whereas the definition of $\sigma_1(n)$ is based on the multiplicative properties of n.

There exist more relations between $p(n)$ and $\sigma_1(n)$. Logarithmic differentiation of (98.1) and subsequent multiplication by x yields

$$\sum_{m=1}^{\infty}\frac{m\,x^m}{1-x^m} = \frac{\sum n\,p(n)\,x^n}{\sum p(n)\,x^n}\,.$$

The left-hand member has been expressed as a power series in (99.5). We have thus

$$\sum_{r=1}^{\infty}\sigma_1(r)\,x^r\cdot\sum_{s=0}^{\infty}p(s)\,x^s=\sum_{n=0}^{\infty}np(n)\,x^n\,,$$

from which we infer by comparison of coefficients,

$$\sum_{r=1}^{n}\sigma_1(r)\,p(n-r)=np(n)\,.$$

This formula can also be proved in a direct and elementary manner. We leave this to the reader with the hint that the following lemma should first be proved:

Lemma. *The number of times a part h appears in all partitions of n together is*

$$\sum_{1\leqq l\leqq n/h}p(n-lh)\,.$$

Finally we rewrite (99.4) as

$$\sum_{n=1}^{\infty}\sigma_1(n)\,x^n=\frac{\sum_{\lambda}(-1)^{\lambda-1}\omega_\lambda x^{\omega\lambda}}{\prod_m(1-x^m)}=\sum_{\lambda}(-1)^{\lambda-1}\omega_\lambda x^{\omega\lambda}\cdot\sum_{l=0}^{\infty}p(l)\,x^l\,,$$

from which we deduce

$$\sigma_1(n)=\sum_{0\leqq\omega_\lambda\leqq n}(-1)^{\lambda-1}\omega_\lambda\,p(n-\omega_\lambda)\,, \tag{99.9}$$

a formula which might be of use as a check for tables of partitions.

100. Jacobi's triple product

If in (78.31) we replace q by x and $e^{2\pi iv}$ by z we have

$$\sum_{n=-\infty}^{\infty}z^n x^{n^2}=\prod_{m=1}^{\infty}(1-x^{2m})\,(1+zx^{2m-1})\,(1+z^{-1}x^{2m-1})\,. \tag{100.1}$$

On both sides we have here a formal power series in x with coefficients from the ring $R(z,z^{-1})$. Our present task is to give for (100.1) a proof within the theory of formal power series.

We introduce

$$\Phi_N(x;z)=\prod_{m=1}^{N}(1-x^{2m})\,(1+zx^{2m-1})\,(1+z^{-1}x^{2m-1}) \tag{100.2}$$

and observe that

$$\Phi_N(x;z) \equiv \Phi_\infty(x;z) \pmod{x^{2N}}, \tag{100.3}$$

where Φ_∞ is the infinite sum of the left member of (100.1).

Now $\Phi_N(x;z)$ is a polynomial in x, z, z^{-1} which can be arranged in any manner. We write

$$\begin{aligned}\Phi_N(x;z) = C_0(x) + (z+z^{-1})\,C_1(x) + z^2 + z^{-2}\,C_2(x) + \cdots \\ + (z^N + z^{-N})\,C_N(x)\,,\end{aligned} \tag{100.4}$$

where we have made use of the symmetry in z and z^{-1} which Φ_N exhibits. The $C_r(x)$ are polynomials, which depend also on N. We can connect them with each other by a recursion formula in the following way: we consider

$$\Phi_N(x;zx^2) = \prod_{m=1}^{N}(1-x^{2m})\,(1+zx^{2m+1})\,(1+z^{-1}x^{2m-3})\,,$$

and find through comparison with (100.2)

$$\begin{aligned}&\Phi_N(x;zx^2)\,(1+zx)\,(1+z^{-1}x^{2N-1})\\ &= \Phi_N(x;z)\,(1+z^{-1}x^{-1})\,(1+zx^{2N+1})\,.\end{aligned}$$

Multiplying both sides by zx and cancelling the factor $(1+zx)$ we obtain

$$\Phi_N(x;zx^2)\,(zx+x^{2N}) = \Phi_N(x;z)\,(1+zx^{2N+1})\,,$$

which applied to (100.4) yields

$$\begin{aligned}&(zx+x^{2N})\\ &\times\{C_0(x) + (zx^2+z^{-1}x^{-2})\,C_1(x) + \cdots + (z^Nx^{2N}+z^{-N}x^{-2N})\,C_N(x)\}\\ &= (1+zx^{2N+1})\,\{C_0(x) + (z+z^{-1})\,C_1(x)\\ &\quad + \cdots + (z^N+z^{-N})\,C(x)\}.\end{aligned}$$

Here we have on both sides polynomials in x, z, z^{-1}. We compare the terms of equal powers of z, say of z^k, on both sides with the result

$$C_{k-1}(x)\,x^{2k-1} + C_k(x)\,x^{2N+2k} = C_k(x) + C_{k-1}(x)\,x^{2N+1}$$

so that

$$C_{k-1}(x)\,x^{2k-1}(1-x^{2N-2k+2}) = C_k(x)\,(1-x^{2N+2k})\,. \tag{100.5}$$

We need now a starting value of $C_k(x)$, for which we take not $C_0(x)$, which is too complicated, but $C_N(x)$. Indeed we get the power z^N in (100.2) only by choosing from all factors always $z\,x^{2m-1}$ and leaving out completely any factor $z^{-1}x^{2m-1}$, so that we have

$$C_N(x) = \prod_{m=1}^{N}(1 - x^{2m})\cdot x^{1+3+\cdots+(2N-1)} = x^{N^2}\prod_{m=1}^{N}(1 - x^{2m}) .$$

Then (100.5) shows, for $k = N$

$$C_{N-1}(x)\,x^{2N-1}(1 - x^2) = x^{N^2}\prod_{m=1}^{N}(1 - x^{2m})\cdot(1 - x^{4N})$$

or

$$C_{N-1}(x) = x^{(N-1)^2}\prod_{m=2}^{N}(1 - x^{2m})\cdot(1 - x^{4N}) ,$$

and then for $k = N - 1$

$$C_{N-2}(x)\,x^{2N-3}(1 - x^4) = x^{(N-1)^2}\prod_{m=2}^{N}(1 - x^{2m})\cdot(1 - x^{4N})\,(1 - x^{4N-2})$$

or

$$C_{N-2}(x) = x^{(N-2)^2}\prod_{m=3}^{N}(1 - x^{2m})\cdot\prod_{m=1}^{2}(1 - x^{4N-2m+2}) .$$

Suppose we have already reached

$$C_r(x) = x^{r^2}\prod_{m=N-r+1}^{N}(1 - x^{2m})\cdot\prod_{m=1}^{N-r}(1 - x^{4N-2m+2}) . \tag{100.6}$$

Then we infer from (100.5)

$$C_{r-1}(x)\,x^{2r-1}(1 - x^{2N-2r+2}) = x^{r^2}\prod_{m=N-r+1}^{N}(1 - x^{2m})\cdot\prod_{m=1}^{N-r+1}(1 - x^{4N-2m+2})$$

or

$$C_{r-1}(x) = x^{(r-1)^2}\prod_{m=N-r+2}^{N}(1 - x^{2m})\cdot\prod_{m=1}^{N-r+1}(1 - x^{4N-2m+2}) ,$$

which completes the induction.

Now (100.6) shows

$$C_0(x) = \prod_{m=1}^{N}(1 - x^{4N-2m+2}) = 1 - x^{2N+2} + \cdots ,$$

and for $r \geqq 1$,

$$C_r(x) = x^{r^2}(1 - x^{2N-2r+2}) \ldots$$

so that

$$C_0(x) \equiv 1 \pmod{x^{2N}}$$

and

$$C_r(x) \equiv x^{r^2} \pmod{x^{2N}}$$

the latter since $r^2 > 2r - 2$. Returning to (100.4) we have now obtained:

$$\begin{aligned}\Phi_N(x;z) \equiv 1 &+ (z + z^{-1})\,x + (z^2 + z^{-2})\,x^{2^2} + \cdots \\ &+ (z^N + z^{-N})\,x^{N^2} \pmod{x^{2N}}\end{aligned}$$

and thus

$$\Phi_N(x;z) \equiv \sum_{n=-\infty}^{\infty} z^n x^{n^2} \pmod{x^{2N}}\,.$$

In view of (100.3) this implies

$$\prod_{m=1}^{\infty}(1 - x^{2m})\,(1 + zx^{2m-1})\,(1 + z^{-1}x^{2m-1}) \equiv \sum_{n=-\infty}^{\infty} z^n x^{n^2} \pmod{x^{2N}} \tag{100.7}$$

for any N. This proves the formula (100.1) in the sense of formal power series.

101. Another proof of the pentagonal numbers theorem

In (100.7) actually two polynomials are compared, namely the assemblage of those terms in which x reaches only powers lower than the $2N$-th. We can write this explicitly as

$$\prod_{m=1}^{N}(1 - x^{2m})\,(1 + zx^{2m-1})\,(1 + z^{-1}x^{2m-1}) \equiv \sum_{n=-[\sqrt{2N}]}^{[\sqrt{2N}]} z^n x^{n^2} \pmod{x^{2N}}\,. \tag{101.1}$$

We replace here x by y^3 which changes nothing essential in the congruence:

$$\prod_{m=1}^{N}(1 - y^{6m})\,(1 + zy^{6m-3})\,(1 + z^{-1}y^{6m-3}) \equiv \sum_{n=-[\sqrt{2N}]}^{[\sqrt{2N}]} z^n y^{3n^2} \pmod{y^{6N}}\,. \tag{101.2}$$

In these polynomials now we make the substitution

$$z = -y\,. \tag{101.3}$$

We know that terms from both sides of (101.2), if they contain powers of y lower than y^{6N}, must agree. They will also agree after the substitution (101.3). But through the appearance of z^{-1} in (101.2) the substitution (101.3) may lower some powers of y, but by not more than N on the left, and at most by $\left[\sqrt{2N}\right] \leqq N$ on the right side. So, in order to avoid comparison of such terms in (101.2) whose agreement after the substitution (101.3) is no longer guaranteed, we have to omit terms higher than y^{6N-N}; in other words we must consider the congruence modulo y^{5N}. The left-hand side can be written more concisely

$$\prod_{r=1}^{3N}(1-y^{2r}).$$

We can moreover introduce on both sides terms whose powers exceed y^{5N}:

$$\prod_{r=1}^{\infty}(1-y^{2r}) \equiv \sum_{-\infty}^{\infty}(-1)^n y^{n(3n+1)} \pmod{y^{5N}}.$$

Since this is valid for any N we have the identity between formal power series

$$\prod_{r=1}^{\infty}(1-y^{2r}) = \sum_{-\infty}^{\infty}(-1)^n y^{n(3n+1)}.$$

On both sides appear only even powers of y. With the substitution $y^2 = x$ and replacement of n by $-\lambda$ we have again Euler's theorem of § 98.

102. A Jacobi formula

Let us make in (100.1) the substitution

$$z \to -zx \tag{102.1}$$

so that we obtain

$$\sum_{n=-\infty}^{\infty}(-1)^n z^n x^{n(n+1)} = \prod_{m=1}^{\infty}(1-x^{2m})(1-zx^{2m})(1-z^{-1}x^{2m-2}). \tag{102.2}$$

This can be justified in the same manner as the substitution in the previous paragraph. Indeed, if we carry out (102.1) in the congruence (101.1) the congruence will still remain valid modulo x^N, since the N terms $z^{-1}x^{2m-1}$ can at most lower the exponents of x by N. From the congruence we can then go over to the identity (102.2). We rearrange the terms on both sides

$$\sum_{n=0}^{\infty}(-1)^n x^{n(n+1)}(z^n - z^{-n-1})$$

$$= (1-z^{-1})\prod_{m=1}^{\infty}(1-x^{2m})(1-zx^{2m})(1-z^{-1}x^{2m})$$

or

$$(1 - z^{-1}) \prod_{m=1}^{\infty} (1 - x^{2m}) (1 - z x^{2m}) (1 - z^{-1} x^{2m})$$
$$= \sum_{n=0}^{\infty} (-1)^n x^{n(n+1)} z^n (1 - z^{-1}) (1 + z^{-1} + z^{-2} + \cdots + z^{-2n}). \qquad (102.3)$$

We have on both sides formal power series in x with coefficients from the ring $R(z, z^{-1})$. This ring, as is easily seen, contains no divisors of zero. Therefore we can cancel on both sides of (102.3) the factor $(1 - z^{-1})$, which is not zero in the ring, and obtain the identity

$$\prod_{m=1}^{\infty} (1 - x^{2m}) (1 - z x^{2m}) (1 - z^{-1} x^{2m})$$
$$= \sum_{n=0}^{\infty} (-1)^n x^{n(n+1)} z^n (1 + z^{-1} + z^{-2} + \cdots + z^{-2n}).$$

Here we make the substitution $z = 1$. This is allowed, as we realize if we replace the equation by a congruence modulo x^N. Then we compare on both sides polynomials in x, z, z^{-1}, whose terms agree up to those with x^{N-1}, and they will still agree after the substitution $z = 1$. (The only thing important in the theory of formal power series in x is never to substitute a value for the indeterminate x itself.) We obtain thus, replacing at the same time the indeterminate x^2 by another indeterminate y,

$$\prod_{m=1}^{\infty} (1 - y^m)^3 = \sum_{n=0}^{\infty} (-1)^n (2n + 1) y^{n(n+1)/2},$$

which is again Jacobi's identity (78.52), but this time proved in a purely formal way.

103. An identity of Euler

We rewrite (100.1) as

$$\prod_{m=1}^{\infty} (1 + z x^{2m-1}) \cdot \prod_{l=1}^{\infty} (1 + z^{-1} x^{2l-1}) = \prod_{m=1}^{\infty} (1 - x^{2m})^{-1} \cdot \sum_{n=-\infty}^{\infty} z^n x^{n^2}. \qquad (103.1)$$

The left-hand side, which we call L, can be put into another form through the use of (97.4), (97.5)

$$L = \left(1 + \sum_{j=1}^{\infty} z^j C_j(x)\right) \left(1 + \sum_{k=1}^{\infty} z^{-k} C_k(x)\right). \qquad (103.2)$$

Now (103.1), (103.2) present power series in x. We can break them up into infinite sums of power series in x, if we assemble those terms with equal powers of z into a power series each. This is possible since on the right as well as on the left z^k is the factor of a power series beginning with x^{k^2}. This permits addition of infinitely many such power series. Let us single out in (103.1), (103.2) that power series which has z^0 as factor:

$$1 + \sum_{k=1}^{\infty} C_k^2(x) = \prod_{m=1}^{\infty} (1 - x^{2m})^{-1} .$$

We now use the formula (97.5) for $C_k(x)$, replace throughout x^2 by x, observe (98.1) and obtain thus

$$\sum_{r=0}^{\infty} p(n)\, x^n = 1 + \frac{x}{(1-x)^2} + \frac{x^4}{(1-x)^2\,(1-x^2)^2} + \frac{x^9}{(1-x)^2\,(1-x^2)^2\,(1-x^3)^2} + \cdots . \tag{103.3}$$

This formula of Euler [7] has played an important role in Hardy's and Ramanujan's investigations about the asymptotic growth of $p(n)$ as $n \to \infty$ [59], pp. 279, 286, 287.

We obtain similar formulae by considering other powers of z, say z^l, where we can without loss of generality assume $l > 0$ because of the symmetry of (103.1) in z and z^{-1}. We find

$$\sum_{k=0}^{\infty} C_k(x)\, C_{k+l}(x) = x^{l^2} \prod_{m=1}^{\infty} (1 - x^{2m})^{-1} , \qquad C_0(x) = 1 .$$

If we insert here (97.5), divide both sides by x^2 and replace x^2 by x we obtain

$$\sum_{n=0}^{\infty} p(n)\, x^n = \sum_{k=0}^{\infty} \frac{x^{k(k+l)}}{(1-x^2)^2 \cdots (1-x^k)^2 \cdot (1-x^{k+1}) \cdots (1-x^{k+l})} ,$$

which includes (103.3) as the case $l = 0$.

Chapter 13

Ramanujan's Congruences and Identities

104. Some divisibility properties of $p(n)$

By inspecting a list of values of $p(n)$ for n up to 200, Ramanujan came to the conjecture that

$$p(5n + 4) \equiv 0 \pmod 5 , \tag{104.1}$$

$$p(7n + 5) \equiv 0 \pmod 7 ,$$
$$p(11n + 6) \equiv 0 \pmod{11} . \tag{104.2}$$

Ramanujan proved these congruences and also similar ones for the moduli 5^2, 7^2, 11^2, and came then to the general conjecture that if $r = 5^\alpha 7^\beta 11^\gamma$, and $24\lambda \equiv 1 \pmod r$ then

$$p(nr + \lambda) \equiv 0 \pmod r .$$

In this form the statement is not true. Indeed S. Chowla observed, using a table of partitions prepared by Gupta that

$$p(243) \equiv 245 \not\equiv 0 \pmod{7^3}$$

but

$$p(243) \equiv 0 \pmod{7^2} .$$

Here $\lambda = 243$ and $24\lambda = 24 \cdot 243 \equiv 1 \pmod{7^3}$.

It is now known that if

$$24\lambda \equiv 1 \pmod{5^\alpha 7^{2\beta-1} 11^\gamma}$$

then

$$p(5^\alpha 7^{2\beta-1} 11^\gamma n + \lambda) \equiv 0 \bmod (5^\alpha 7^\beta 11^\gamma) ,$$

and numerous new congruences to such moduli as 13, 17, 19 have been discovered (cf. p. 314(2)).

All these investigations are based on the theory of modular forms. The cases of 5, 7 and their powers turn out to be of a much simpler nature than the modulus 11 and its powers.

In the present context we give, following Ramanujan, the proofs for (104.1), (104.2) by means of formal power series. In order to prove (104.1) we consider the series

$$\sum_{l=1}^{\infty} p(l-1)\, x^l = \frac{x}{\prod\limits_{m=1}^{\infty} (1 - x^m)} . \tag{104.3}$$

We have to show that the coefficients of x^l with $l \equiv 0 \pmod 5$ are divisible by 5.

We begin with a preliminary remark. If in the series $\sum\limits_1^\infty a_n x^n$ the coefficients $a_{5k} \equiv 0 \pmod 5$ then in

$$\sum_1^\infty c_n x^n = \sum_1^\infty a_n x^n (1 + b_1 x^5 + b_2 x^{10} + \cdots) ,$$

with b_k integers, we have also $c_{5k} \equiv 0 \pmod 5$ since

$$c_n = a_n + a_{n-5} b_1 + a_{n-10} b_2 + \cdots . \tag{104.4}$$

Conversely, if $c_{5k} \equiv 0 \pmod 5$ then we have also $a_{5k} \equiv 0 \pmod 5$, since (104.4) shows $a_5 \equiv c_5 \equiv 0 \pmod 5$ and by induction in general $a_{5k} \equiv 0 \pmod 5$.

For a prime p we observe now that in

$$(1 + x)^p = 1 + \binom{p}{1} x + \binom{p}{2} x^2 + \cdots + \binom{p}{p} x^p$$

all binomial coefficients on the right side with the exception of the first and last are divisible by p. We write this as

$$(1 + x)^p \equiv 1 + x^p \pmod p\,,$$

which means that the difference of the two sides of the congruence contains only coefficients divisible by p. We have thus

$$(1 - x)^5 \equiv 1 - x^5 \pmod 5\,,$$

and therefore also

$$\prod_{m=1}^{\infty} (1 - x^m)^5 \equiv \prod_{m=1}^{\infty} (1 - x^{5m}) \pmod 5\,.$$

Returning now to (104.3) we multiply both sides by $\prod_{m=1}^{\infty}(1 - x^{5m})$, which is a series that contains only powers x^{5k},

$$\sum_{l=1}^{\infty} p(l-1)\, x^l \cdot \prod_{m=1}^{\infty}(1 - x^{5m}) = \frac{x \prod_{m=1}^{\infty}(1 - x^{5m})}{\prod_{m=1}^{\infty}(1 - x^m)}$$

$$\equiv \frac{x}{\prod_{m=1}^{\infty}(1 - x^m)} \prod_{m=1}^{\infty}(1 - x^m)^5 \pmod 5 \equiv x \prod_{m=1}^{\infty}(1 - x^m)^4$$

$$\equiv x \prod_{m=1}^{\infty}(1 - x^m) \prod_{\mu=1}^{\infty}(1 - x^{\mu})^3 \pmod 5\,, \tag{104.5}$$

where we want to investigate the coefficients of the powers x^l, with $5 \mid l$. Now

$$x \prod_{m=1}^{\infty}(1 - x^m) \prod_{\mu=1}^{\infty}(1 - x^{\mu})^3$$

$$= x \sum_{\lambda=-\infty}^{\infty} (-1)^{\lambda} x^{\lambda(3\lambda-1)/2} \sum_{k=0}^{\infty} (-1)^k (2k+1)\, x^{k(k+1)/2}\,,$$

and the typical term is here

$$(-1)^{\lambda+k} (2k+1)\, x^{1+\{\lambda(3\lambda-1)+k(k+1)\}/2}\,. \tag{104.6}$$

We are looking for exponents divisible by 5:

$$1+\frac{1}{2}\lambda(3\lambda-1)+\frac{1}{2}k(k+1)\equiv 0 \pmod 5,$$

or what is the same

$$2(\lambda-1)^2+(2k+1)^2\equiv 0 \pmod 5.$$

This is of the form

$$2A^2+B^2\equiv 0 \pmod 5.$$

Now the only possibilities are

$$B^2\equiv 0,1,4 \pmod 5, \quad 2A^2\equiv 0,2,3 \pmod 5,$$

and the only combinations $\equiv 0 \pmod 5$ are

$$A\equiv B\equiv 0 \pmod 5.$$

This means $2k+1\equiv 0 \pmod 5$, which shows that the coefficients in (104.6) and therefore in (104.5) belonging to x^l, $5\,|\,l$, are themselves divisible by 5.

Remembering our preliminary remark we infer that also $p(l-1)\equiv 0 \pmod 5$ for $5\,|\,l$ or $p(5n+4)\equiv 0 \pmod 5$, as we had to prove.

We write down the similar proof for $p(7n+5)\equiv 0$ without much explanation.

$$\sum_{l=2}^{\infty} p(l-2)\,x^l=\frac{x^2}{\prod_{m=1}^{\infty}(1-x^m)},$$

$$\sum_{l=2}^{\infty} p(l-2)\,x^l\prod_{m=1}^{\infty}(1-x^{7m})=\frac{x^2}{\prod_{m=1}^{\infty}(1-x^m)}\prod_{m=1}^{\infty}(1-x^{7m})$$

$$\equiv\frac{x^2}{\prod_{m=1}^{\infty}(1-x^m)}\prod_{m=1}^{\infty}(1-x^m)^7 \pmod 7$$

$$\equiv x^2\prod_{m=1}^{\infty}(1-x^m)^6 \pmod 7.$$

Here

$$x^2\prod_{m=1}^{\infty}(1-x^m)^6=x^2\sum_{j=0}^{\infty}(-1)^j(2j+1)\,x^{j(j+1)/2}$$

$$\times\sum_{k=0}^{\infty}(-1)^k(2k+1)\,x^{k(k+1)/2}.$$

The typical term of this product is

$$(-1)^{j+k}(2j+1)(2k+1)\,x^{2+\{j(j+1)+k(k+1)\}/2}\,. \tag{104.7}$$

Looking here for exponents divisible by 7 we need

$$2+\frac{1}{2}j(j+1)+\frac{1}{2}k(k+1)\equiv 0 \pmod 7$$

or

$$(2j+1)^2+(2k+1)^2\equiv 0 \pmod 7.$$

For an expression A^2+B^2 we have only the possibilities

$$A\equiv 0,1,2,4, \qquad B\equiv 0,1,2,4 \pmod 7\,,$$

among which the only combination $A^2+B^2\equiv 0 \pmod 7$ is $A\equiv B \equiv 0 \pmod 7$. The proof is now completed in the same way.

105. Two Ramanujan identities

The congruence properties (104.1) and (104.2) are obvious consequences of the following identities, also discovered by Ramanujan,

$$\sum_{n=0}^{\infty} p(5n+4)\,x^n = 5\,\frac{\varphi(x^5)^5}{\varphi(x)^6}\,, \tag{105.1}$$

$$\sum_{n=0}^{\infty} p(7n+5)\,x^n = 7\,\frac{\varphi(x^7)^3}{\varphi(x)^4}+49\,x\,\frac{\varphi(x^7)^7}{\varphi(x)^8}\,, \tag{105.2}$$

where we use the abbreviation

$$\varphi(x)=\prod_{m=1}^{\infty}(1-x^m)\,. \tag{105.3}$$

These identities belong to the theory of modular functions and have been proved within the framework of that theory [3, 63, 42, 57, 53,]. Recently, however O. Kolberg [29] has given a proof using formal power series. We shall give here only that of (105.1), that of (105.2) being of the same nature with a few more complicated details.

Let us introduce, besides (105.3), the following abbreviations

$$G_s(x)=\sum_{\lambda(3\lambda-1)/2\equiv s(5)}(-1)^{\lambda}x^{\lambda(3\lambda-1)/2}\,, \tag{105.4}$$

$$H_s(x)=\sum_{\substack{n(n+1)/2\equiv s(5)\\ n\geqq 0}}(-1)^n(2n+1)\,x^{n(n+1)/2}\,, \tag{105.5}$$

$$P_s(x)=\sum_{n=0}^{\infty}p(5n+s)\,x^{5n+s}\,. \tag{105.6}$$

We have then

$$\varphi(x) = \sum_{s=0}^{4} G_s(x), \quad \varphi^3(x) = \sum_{s=0}^{4} H_s(x) ,$$

$$\sum_{s=0}^{4} P_s(x) = \sum_{n=0}^{\infty} p(n)\, x^n = \frac{1}{\varphi(x)}$$

so that

$$\sum_{s=0}^{4} G_s(x) \cdot \sum_{t=0}^{4} P_t(x) = 1 .$$

On the left-hand side we combine the 25 products $G_s P_t$ according to the residue class of $s + t$ modulo 5:

$$\sum_{a=0}^{4} \sum_{s+t\equiv a(5)} G_s P_t = 1 . \tag{105.7}$$

The power series

$$S_a(x) = \sum_{s+t\equiv a(5)} G_s(x)\, P_t(x)$$

contains only powers x^n with $n \equiv a \pmod 5$ because of the definitions (105.4), (105.6). The five series $S_a(x)$ can therefore not interfere with each other in (105.7). The power series "1" on the right side can thus only be produced as S_0, and the others are 0. In this way we obtain five linear equations for $P_0, \ldots, P_4$:

$$\begin{aligned}
G_0 P_0 + G_4 P_1 + G_3 P_2 + G_2 P_3 + G_1 P_4 &= 1 ,\\
G_1 P_0 + G_0 P_1 + G_4 P_2 + G_3 P_3 + G_2 P_4 &= 0 ,\\
G_2 P_0 + G_1 P_1 + G_0 P_2 + G_4 P_3 + G_3 P_4 &= 0 ,\\
G_3 G_0 + G_2 P_1 + G_1 P_2 + G_0 P_3 + G_4 P_4 &= 0 ,\\
G_4 P_0 + G_3 P_1 + G_2 P_2 + G_1 P_3 + G_0 P_4 &= 0 .
\end{aligned} \tag{105.8}$$

Determinants of formal power series are again contained in the ring of formal power series. The determinant D of the system (105.8) with the unknowns $P_0, P_1, \ldots, P_4$ is

$$D = |G_{h-j}| , \quad 0 \leqq h , j \leqq 4 ,$$

where the indices $h - j$ are to be taken modulo 5. This determinant is cyclic. In order to evaluate it we have to enlarge the ring of coefficients, which so far was the ring $[1]$ of integers, to the ring $[\xi]$, where ξ is a primitive fifth root of unity. The evaluation of cyclic determinants is well-known. We carry out the computation of D explicitly to make sure that we remain always within the domain of admissible operations. We multiply D by the Vandermonde determinant

$$V = |\xi^{jk}|, \quad 0 \leqq j, k \leqq 4,$$

$$D \cdot V = |G_{h-j}| \cdot |\xi^{jk}| = \left| \sum_j G_{h-j} \xi^{jk} \right| .$$

Here the general element is

$$a_{hk} = \sum_{j=0}^{4} G_{h-j} \xi^{jk} = \xi^{hk} \sum_{j=0}^{4} G_{h-j}\, \xi^{(j-h)k} = \xi^{hk} \sum_{s=0}^{4} G_s \xi^{-sk} ,$$

the latter because of the period 5 of G_s. Now

$$G_s(\xi^{-k} x) = \xi^{-sk} G_s(x)$$

since $G_s(x)$ contains only powers x^n with $n \equiv s \pmod 5$. This makes

$$a_{hk} = \xi^{hk} \sum_{s=0}^{4} G_s(\xi^{-k} x) = \xi^{hk} \varphi(\xi^{-k} x) .$$

We have thus

$$D \cdot V = |a_{hk}| = \varphi(x)\, \varphi(\xi^{-1} x)\, \varphi(\xi^{-2} x)\, \varphi(\xi^{-3} x)\, \varphi(\xi^{-4} x)\, |\xi^{hk}|$$
$$= \prod_{s=0}^{4} \varphi(\xi^{-s} x) \cdot V ,$$

and since $V \neq 0$ we find by cancellation

$$D = \prod_{s=0}^{4} \varphi(\xi^{-s} x) = \prod_{s=0}^{4} \prod_{m=1}^{\infty} (1 - \xi^{-sm} x^m) = \prod_{m=1}^{\infty} \prod_{s=0}^{4} (1 - \xi^{-sm} x^m) .$$

For the evaluation of the inner product we distinguish two cases:

(1) $m \equiv 0 \pmod 5$

$$\prod_{s=0}^{4} = (1 - x^m)^5 ,$$

(2) $m \not\equiv 0 \pmod 5$

$$\prod_{s=0}^{4} (1 - \xi^{-sm} x^m) = 1 - x^{5m} ,$$

since ξ^{-sm} runs here through all roots of $y^5 - 1 = 0$.

Thus finally

$$D = \prod_{m=1}^{\infty}(1 - x^{5m})^5 \cdot \prod_{m \not\equiv 0(5)} (1 - x^{5m})$$

so that

$$D = \frac{\varphi(x^5)^6}{\varphi(x^{25})} . \tag{105.9}$$

We see that D is a formal power series beginning with 1. It can therefore be used as a divisor.

We can now solve the linear equations for P_s, in particular for P_4, in which we are interested:

$$P_4 = \frac{D_4}{D} \tag{105.91}$$

with

$$D_4 = \begin{vmatrix} G_1 & G_0 & G_4 & G_3 \\ G_2 & G_1 & G_0 & G_4 \\ G_3 & G_2 & G_1 & G_0 \\ G_4 & G_3 & G_2 & G_1 \end{vmatrix} .$$

106. Relations between the G_s, H_s and φ

In order to compute D_4 we have to go back to the definitions of the G_s and H_s.

The condition of summation in (105.4) can be written as

$$(6\lambda - 1)^2 \equiv 24s + 1 \equiv -s + 1 \pmod 5 . \tag{106.1}$$

This can be fulfilled by some λ only if

$$\left(\frac{1-s}{5}\right) = +1 \text{ or } 0 .$$

If therefore $s = 3, 4$ there is no λ which fulfills (106.1), and we have

$$G_3(x) = G_4(x) = 0 . \tag{106.2}$$

The case $s = 1$ gives

$$6\lambda \equiv 1 \pmod 5$$

or

$$\lambda \equiv 1 \pmod 5$$

so that

$$G_1(x) = \sum_{l=-\infty}^{\infty} (-1)^{5l+1} x^{(5l+1)(15l+2)/2} = -x \sum_{l=-\infty}^{\infty} (-1)^l x^{25(l(3l+1)/2)}$$

$$= -x\varphi(x^{25}) . \qquad (106.3)$$

Through (106.2) the determinant D_4 simplifies very much, and we obtain, observing also (106.3)

$$D_4 = x^4\varphi(x^{25})^4 - 3\,G_0 G_2 \cdot x^2 \varphi(x^{25})^2 + G_0^2 G_2^2 . \qquad (106.4)$$

Let us now in the equation

$$\varphi(x^3) = (G_0 + G_1 + G_2)^3$$

take on both sides only those terms x^n in which $n \equiv 2 \pmod 5$. In view of the definitions (105.4), (105.5) we obtain

$$H_2(x) = 3\,G_0^2\, G_2 + 3\,G_0\, G_1^2 . \qquad (106.5)$$

The conditions of summation in (105.5) can be written

$$(2n + 1)^2 \equiv 3s + 1 \pmod 5 .$$

Therefore $H_s = 0$ if $\left(\frac{3s+1}{5}\right) = -1$. This takes place for $s = 2, 4$:

$$H_2(x) = H_4(x) = 0 .$$

Thus (106.5) furnishes

$$G_0(G_0G_2 + G_1^2) = 0 ,$$

and since

$$G_0 = 1 + x^5 - x^{15} \cdots \neq 0$$

we conclude

$$G_0 G_2 = -G_1^2 = -x^2\varphi(x^{25})^2 .$$

This inserted in (106.4) gives

$$D_4 = 5\,x^4\varphi(x^{25})^4$$

so that in view of (105.6), (105.9), (105.91),

$$\sum_{n=0}^{\infty} p(5n + 4)\, x^{5n+4} = 5\,x^4 \frac{\varphi(x^{25})^5}{\varphi(x^5)^6} .$$

Cancelling x^4 on both sides and replacing x^5 by x we obtain Ramanujan's identity (105.1)

O. Kolberg obtains by this method also some new identities, e.g.,

$$\sum_0^\infty p(5n+1)\,x^n \sum_0^\infty p(5n+2)\,x^n = 2\,\frac{\varphi(x^5)^4}{\varphi(x)^6} + 25\,x\,\frac{\varphi(x^5)^{10}}{\varphi(x)^{12}}$$

and

$$\sum_0^\infty p(5n)\,x^n \sum_0^\infty p(5n+3)\,x^n = 3\,\frac{\varphi(x^5)^4}{\varphi(x)^6} + 25\,x\,\frac{\varphi(x^5)^{10}}{\varphi(x)^{12}}\,.$$

The identity concerning $p(7n+5)$ can be proved in complete analogy. The computations are more laborious since the determinants will be of order 7 and 6.

107. The Rogers-Ramanujan identities. Introductory remarks

We are going to discuss and prove now the famous identities

$$1 + \sum_{k=1}^\infty \frac{x^{k^2}}{(1-x)(1-x^2)\cdots(1-x^k)} = \frac{1}{\prod\limits_{m=1}^\infty (1-x^{5m-4})(1-x^{5m-1})}\,, \tag{107.1}$$

$$1 + \sum_{k=1}^\infty \frac{x^{k(k+1)}}{(1-x)(1-x^2)\cdots(1-x^k)} = \frac{1}{\prod\limits_{m=1}^\infty (1-x^{5m-3})(1-x^{5m-2})}\,. \tag{107.2}$$

These identities have an interesting history. They are mentioned in a letter of S. Ramanujan written from India in February 1913 to G. H. Hardy in Cambridge [59], p. 344. No proof was known to either of these mathematicians, and they were printed without a proof in MacMahon's Combinatory Analysis [38], vol. 2, pp. 33, 35. Actually, they had already been found, proved and published by G. L. Rogers in 1894 [62], pp. 328/9. However they were overlooked at that time, since they appear as corollaries of some more general formulae with many parameters. In 1917 Ramanujan, looking through old numbers of the Proceedings found there Roger's proof. Then Rogers, referring to his older proof gave a new condensed version of it [61], pp. 315–317.

Independent of the English mathematicians the German mathematician I. Schur had rediscovered these identities and published two different proofs of them in 1917 [66]. Schur's proofs are not the shortest among those known. They have, however, the distinction that they give some deeper arithmetical insight into the structure of these formulae. In particular, Schur employs the important "Gaussian polynomials" in his proof, of which I shall give here another version.

108. Arithmetical statement of the identities

Both sides of the identities are formal power series. If we write the right-hand side of (107.1) as $\sum_{0}^{\infty} q_1(n)\, x^n$, then $q_1(n)$ is evidently the number of partitions of n into parts which are congruent to ± 2 modulo 5.

The terms of the series in the left members of (107.1), (107.2) are known to us from (96.8), (96.9) where their arithmetical meaning is also described. A power x^n appears on the left of (107.1) only finitely often. Its coefficient, let us call it $r_1(n)$, is then after § 96 the number of partitions into parts differing by at least 2. The coefficient $r_2(n)$ of x^n in the formal power series for the left-hand members of (107.2) is the number of partitions of n into parts differing by at least 2, with the proviso that the smallest part must be at least 2.

The identities (107.1), (107.2) mean therefore the following, in the language of additive number theory:

Theorems (Rogers-Ramanujan-Schur).

(I) *A number n can be partitioned as often into parts* $\equiv \pm 1 \pmod 5$ *as it can be partitioned into parts differing by at least* 2.

(II) *A number n can be partitioned as often into parts* $\equiv \pm 2 \pmod 5$ *as it can be partitioned into parts differing by at least* 2, *all parts being greater than* 1.

We write out now the two formal power series mentioned above for the left-hand members of the identities[1]

$$D_1(x) = 1 + \sum_{n=1}^{\infty} r_1(n)\, x^n \tag{108.1}$$

and

$$D_2(x) = 1 + \sum_{n=1}^{\infty} r_2(n)\, x^n\,, \tag{108.2}$$

where $r_1(n)$ is the number of partitions of n

$$n = l_1 + l_2 + \cdots, \quad 0 < l_1\,, \quad l_{j+1} - l_j \geqq 2\,. \tag{108.31}$$

The definition of $r_2(n)$ agrees with that of $r_1(n)$ but has the additional condition

$$1 < l_1\,. \tag{108.32}$$

[1] I. Schur at this point introduces infinite determinants, which have no other purpose than to lead to a recurrence formula.

As in previous discussions of formal power series we "approximate" $D_1(x)$ and $D_2(x)$ modulo x^{N+1} by polynomials. We define

$$D_1^{(N)}(x) = 1 + \sum_{n\geqq 1} r_1^{(N)}(n)\, x^n\,, \tag{108.41}$$

where $r_1^{(N)}(n)$ is the number of partitions of n

$$n = l_1 + l_2 + \cdots, \quad l_{j+1} - l_j \geqq 2\,, \quad 0 < l_j \leqq N\,, \tag{108.42}$$

limiting the size of the admissible parts. Clearly since this limitation does not interfere with the partitions of $n \leqq N$ we have

$$D_1^{(N)}(x) \equiv D_1(x) \text{ (modulo } x^{N+1})\,. \tag{108.5}$$

We define similarly $D_2^{(N)}(x)$, by forbidding only the part 1:

$$1 < l_j \leqq N\,,$$

and we have

$$D_2^{(N)}(x) \equiv D_2(x) \text{ (modulo } x^{N+1})\,. \tag{108.6}$$

Because of the conditions imposed on the parts l_j, $D_1^{(N)}(x)$ and $D_2^{(N)}(x)$ are polynomials whose degree does not exceed $(N+1)^2/4$.

Before we continue it might be advisable to write down a few $D_\mu^{(N)}(x)$, $\mu = 1, 2$, for low N, just by trying out all admissible partitions. We obtain, e.g.,

$$\begin{aligned} D_1^{(1)}(x) &= 1 + x\,, \\ D_1^{(2)}(x) &= 1 + x + x^2\,, \\ D_1^{(3)}(x) &= 1 + x + x^2 + x^3 + x^4 \end{aligned} \tag{108.71}$$

and

$$\begin{aligned} D_2^{(1)}(x) &= 1\,, \\ D_2^{(2)}(x) &= 1 + x^2\,, \\ D_2^{(3)}(x) &= 1 + x^2 + x^3\,. \end{aligned} \tag{108.72}$$

If we now incorporate the definition of $r_1^{(N)}(n)$ in the formula for $D_1^{(N)}(x)$, we can write

$$D_1^{(N)}(x) = 1 + \sum_{l_1, l_2, \ldots} x^{l_1 + l_2 + \cdots}\,,$$

where $l_1, l_2, \ldots$ run through all systems.

$$0 < l_j \leqq N\,, \quad l_{j+1} - l_j \geqq 2\,.$$

Now let us separate those partitions with greatest part equal to N from those with all parts $l_j < N$, so that we have

$$D_1^{(N)}(x) = 1 + \sum_{0<l_j\leqq N-1} x^{l_1+l_2+\cdots} + x^N \left\{1 + \sum_{0<l_j\leqq N-2} x^{l_1+l_2+\cdots}\right\}. \tag{108.8}$$

Here we have in the second sum already isolated the greatest part N and realize that, if there are any other parts at all, they must differ from N by at least 2. From (108.8) we now read off the recursion formula

$$D_1^{(N)}(x) = D_1^{(N-1)}(x) + x^N D_1^{(N-2)}(x)\,.$$

If we replace in the foregoing identities the condition $0 < l_j$ by $1 < l_j$ every argument goes through as before, and we obtain simply

$$D_2^{(N)}(x) = D_2^{(N-1)}\,(x) + x^N D_2^{(N-2)}\,(x)\,.$$

Our examples (108.71), (108.72) fulfill these recursion formulae. We add, for the sake of uniformity, the definitions

$$D_\mu^{(0)}(x) = 1\,, \qquad \mu = 1, 2\,, \tag{108.73}$$

which also fit into the formula

$$D_\mu^{(N)}(x) = D_\mu^{(N-1)}(x) + x^N D_\mu^{(N-2)}(x)\,,\ \mu = 1, 2, \quad N \geqq 2\,. \tag{108.9}$$

This formula together with the initial conditions (108.73) and the first entries of (108.71), (108.72) defines now recursively the polynomials $D_\mu^{(N)}$ and determines thus the formal power series $D_1(x)$ and $D_2(x)$ in view of (108.5), (108.6).

109. Reformulation of the problem

The theorem which we have to prove, given in § 107, can now be written

$$D_1(x) = \frac{1}{\prod_1^\infty (1 - x^{5m-4})\,(1 - x^{5m-1})}\,, \tag{109.1}$$

$$D_2(x) = \frac{1}{\prod_1^\infty (1 - x^{5m-3})\,(1 - x^{5m-2})}\,. \tag{109.2}$$

However, we prefer to put those assertions first into another, equivalent form.

We have

$$\frac{1}{\prod_1^\infty (1 - x^{5m-4})(1 - x^{5m-1})} = \frac{\prod_1^\infty (1 - x^{5m})\,(1 - x^{5m-1})\,(1 - x^{5m-3})}{\prod_1^\infty (1 - x^m)}\,.$$

In the numerator we use Jacobi's triple product formula (100.1), in which we replace first x by x^5 and then z by $-x$. The first of these substitutions is only a change of name. The permissibility of the second substitution can be justified in a manner analogous to the discussion in §§ 100, 101.

We have then

$$\prod_1^\infty (1 - x^{10m})(1 - x^{10m-4})(1 - x^{10m-6}) = \sum_{-\infty}^{\infty} (-1)^n x^{n(5n+1)},$$

and substituting x for x^2,

$$\prod_1^\infty (1 - x^{5m})(1 - x^{5m-2})(1 - x^{5m-3}) = \sum_{-\infty}^{\infty} (-1)^n x^{n(5n+1)/2}.$$

Therefore the statement (109.1) can be replaced by the equivalent one

$$D_1(x) = \frac{\sum\limits_{-\infty}^{\infty} (-1)^n x^{n(5n+1)/2}}{\prod\limits_1^\infty (1 - x^m)}. \tag{109.3}$$

By a similar argument, replacing x by x^5 and then z by $-x^3$ in Jacobi's formula, we arrive at

$$D_2(x) = \frac{\sum\limits_{-\infty}^{\infty} (-1)^n x^{(5n+3)/2}}{\prod\limits_1^\infty (1 - x^m)}. \tag{109.4}$$

We are going to prove these equations. They are equivalent to the identities (107.1), (107.2) and to the theorem in § 108.

110. The Gaussian polynomials

I. Schur's method consists now in giving solutions of the difference equation (108.9) for polynomials and adjusting the solutions to fit the initial conditions (108.71), (108.72), (108.73). The tools for this investigation are the Gaussian polynomials, which come to mind since they show a recurrence relation reminiscent of (108.9).

Gauss defines the following rational functions of the indeterminate x

$$\begin{bmatrix} k \\ l \end{bmatrix} = \frac{(1 - x^k)(1 - x^{k-1}) \cdots (1 - x^{k-l+1})}{(1 - x)(1 - x^2) \cdots (1 - x^l)}, \quad k \geqq 0, \quad l > 0. \tag{110.1}$$

They will turn out to be polynomials. In particular (110.1) implies

$$\begin{bmatrix} 0 \\ l \end{bmatrix} = 0\,, \quad \begin{bmatrix} k \\ l \end{bmatrix} = 0 \quad \text{for } l > k\,. \tag{110.2}$$

We extend immediately the definition (110.1) by setting

$$\begin{bmatrix} k \\ 0 \end{bmatrix} = 1\,, \quad \text{including} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1\,, \tag{110.3}$$

$$\begin{bmatrix} k \\ -1 \end{bmatrix} = \begin{bmatrix} k \\ -2 \end{bmatrix} = \cdots = 0\,.$$

However, we always keep the "numerator" $k \geqq 0$.
Formula (110.1) can be written

$$\begin{bmatrix} k \\ l \end{bmatrix} = \frac{(1-x^k)(1-x^{k-1})\cdots(1-x^{k-l+1})(1-x^{k-l})\cdots(1-x)}{(1-x)(1-x^2)\cdots(1-x^l)(1-x)\cdots(1-x^{k-l})}\,,$$

which shows the symmetry

$$\begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} k \\ k-l \end{bmatrix}. \tag{110.4}$$

This symmetry remains in force also under the extensions (110.3), as comparison with (110.2) shows.
Now we have

$$\begin{bmatrix} k \\ l \end{bmatrix} = \frac{1-x^k}{1-x^l}\begin{bmatrix} k-1 \\ l-1 \end{bmatrix} = \begin{bmatrix} k-1 \\ l-1 \end{bmatrix} + \frac{x^l-x^k}{1-x^l}\begin{bmatrix} k-1 \\ l-1 \end{bmatrix}$$

$$= \begin{bmatrix} k-1 \\ l-1 \end{bmatrix} + x^l\,\frac{1-x^{k-l}}{1-x^l}\begin{bmatrix} k-1 \\ l-1 \end{bmatrix},$$

and finally

$$\begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} k-1 \\ l-1 \end{bmatrix} + x^l\begin{bmatrix} k-1 \\ l \end{bmatrix}, \quad k \geqq 1\,. \tag{110.5}$$

We ascertain again that also the supplementary definitions (110.3) satisfy this relation.
By means of the symmetry (110.4) we infer also

$$\begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} k \\ k-l \end{bmatrix} = \begin{bmatrix} k-1 \\ k-l-1 \end{bmatrix} + x^{k-l}\begin{bmatrix} k-1 \\ k-l \end{bmatrix}$$

so that

$$\begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} k-1 \\ l \end{bmatrix} + x^{k-l}\begin{bmatrix} k-1 \\ l-1 \end{bmatrix}, \quad k \geqq 1\,. \tag{110.6}$$

Again the supplementary definitions (110.4) are also satisfied.

Starting now with

$$\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = 1\,, \quad \begin{bmatrix}1\\l\end{bmatrix} = 0 \text{ for } l > 1$$

we see that the relations (110.5) or (110.6) inductively define $\begin{bmatrix}k\\l\end{bmatrix}$ as polynomials in x for $k > 1$, e.g.,

$$\begin{bmatrix}2\\1\end{bmatrix} = 1 + x\,, \quad \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix} = 1 + x + x^2\,.$$

The Gaussian polynomials show in their symmetry and their recursion formulae some resemblance with the binomial coefficients. This is the reason that I. Schur chose the notation with brackets for them, a notation which differs from that of Gauss.

(It is evident that

$$\begin{bmatrix}k\\l\end{bmatrix} = \binom{k}{l} \text{ for } x = 1\,,$$

but we shall in the sequal always retain x as an indeterminate.) The resemblence with binomial coefficients is also shown if we compare

$$(1 + y)^k = \sum_{k=0}^{k} \binom{k}{l} y^l$$

with the identity

$$(1 + y)\,(1 + xy) \ldots (1 + x^{k-1}y) = \sum_{l=0}^{k} \begin{bmatrix}k\\l\end{bmatrix} x^{l(l-1)/2} y^l\,,$$

a proof of which we leave to the reader.

111. Schur's functions

Now Schur succeeds in building up out of the Gaussian polynomials other polynomials, which fulfill the recursive relation (108.9). He begins with the rational functions

$$F^{(\varepsilon)}(k, l) = \begin{bmatrix}k\\l\end{bmatrix} - x^{k-2l+2+\varepsilon} \begin{bmatrix}k\\l-2-\varepsilon\end{bmatrix}, \quad \varepsilon = 0, 1\,, \tag{111.1}$$

where $k \geqq 0$, as always in Gaussian polynomials, but l any integer. These rational functions do not have to be polynomials, but the lowest exponent of x is in any case $\geqq -k - 2 - \varepsilon$. The Schur functions vanish identically for $l < 0$ and for $l > k + 2 + \varepsilon$.

Following I. Schur we prove for these functions the

Lemma.

$$F^{(0)}(k,l) = F^{(1)}(k-1,l) + x^{k-1}F^{(0)}(k-2,l-1)\,. \tag{111.2}$$

Proof.

$$\begin{aligned}
&F^{(0)}(k,l) - F^{(1)}(k-1,l)\\
&= \begin{bmatrix} k \\ l \end{bmatrix} - x^{k-2l+2}\begin{bmatrix} k \\ l-2 \end{bmatrix} - \begin{bmatrix} k-1 \\ l \end{bmatrix} + x^{k-2l+2}\begin{bmatrix} k-1 \\ l-3 \end{bmatrix}\\
&= x^{-k-l}\left\{\begin{bmatrix} k-1 \\ l-1 \end{bmatrix} - \begin{bmatrix} k-1 \\ l-2 \end{bmatrix}\right\}\\
&= x^{k-l}\left\{\begin{bmatrix} k-2 \\ l-2 \end{bmatrix} + x^{l-1}\begin{bmatrix} k-2 \\ l-1 \end{bmatrix} - \begin{bmatrix} k-2 \\ l-2 \end{bmatrix} - x^{k-l+1}\begin{bmatrix} k-2 \\ l-3 \end{bmatrix}\right\}\\
&= x^{k-1}\left\{\begin{bmatrix} k-2 \\ l-1 \end{bmatrix} - x^{k-2l+2}\begin{bmatrix} k-2 \\ l-3 \end{bmatrix}\right\}\\
&= x^{k-1}F^{(0)}(k-2,l-1)\,,
\end{aligned}$$

where (110.5), (110.6) have been used repeatedly.

Let now $n \geqq 0$ and in the following

$$\varepsilon_n \equiv n \pmod 2\,, \qquad \varepsilon_n = 0, 1\,, \tag{111.31}$$

which can also be expressed as

$$\varepsilon_n = \left[\frac{n+1}{2}\right] - \left[\frac{n}{2}\right]. \tag{111.32}$$

With certain integers α, β to be kept at our disposal, we introduce then

$$F_n = F_n(x,\alpha) = F^{(\varepsilon_n)}\left(n+1\,,\ \left[\frac{n+1}{2}\right] - \alpha\right), \tag{111.4}$$

$$G_n = G_n(x,\beta) = F^{(1-\varepsilon_n)}\left(n+1\,,\ \left[\frac{n}{2}\right] - \beta\right). \tag{111.5}$$

A special case of (111.2) is the following

$$\begin{aligned}
F^{(0)}\left(n+1, \left[\frac{n+1}{2}\right] - \alpha\right) &= F^{(1)}\left(n, \left[\frac{n+1}{2}\right] - \alpha\right)\\
&\quad + x^n F^{(0)}\left(n-1, \left[\frac{n-1}{2}\right] - \alpha\right),
\end{aligned}$$

which for n even can, by means of (111.4), be written simply as

$$F_n = F_{n-1} + x^n F_{n-2}\,, \qquad n \text{ even}, \tag{111.6}$$

since $\left[\frac{n+1}{2}\right] = \left[\frac{n}{2}\right]$.

Similarly

$$F^{(0)}\left(n+1, \left[\frac{n}{2}\right]-\beta\right) = F^{(1)}\left(n, \left[\frac{n}{2}\right]-\beta\right) + x^n F^{(0)}\left(n-1, \left[\frac{n-2}{2}\right]-\beta\right)$$

can in virtue of (111.5) for n *odd* be read as

$$G_n = G_{n-1} + x^n G_{n-2}, \quad n \text{ odd}, \tag{111.7}$$

since here $\left[\frac{n}{2}\right] = \left[\frac{n-1}{2}\right]$. We notice that the F_n satisfy (108.9) for $n = N$ even, and G_n satisfy it for $n = N$ odd.

We calculate now F_n from the definitions (111.4) and (111.1)

$$F_n = \begin{bmatrix} n+1 \\ \left[\frac{n+1}{2}\right]-\alpha \end{bmatrix} - x^{2\alpha+3} \begin{bmatrix} n+1 \\ \left[\frac{n}{2}\right]-\alpha-2 \end{bmatrix}, \tag{111.8}$$

where use has been made of (111.32) and of $n = \left[\frac{n}{2}\right] + \left[\frac{n+1}{2}\right]$. Similarly G_n is obtained from (111.5) and (111.1) as

$$G_n = \begin{bmatrix} n+1 \\ \left[\frac{n}{2}\right]-\beta \end{bmatrix} - x^{2\beta+4} \begin{bmatrix} n+1 \\ \left[\frac{n+1}{2}\right]-\beta-3 \end{bmatrix}. \tag{111.9}$$

It should be emphasized that (111.8), (111.9) are valid for any n, whereas (111.6) and (111.7) are true only for the specified parities of n[1]. ▯

112. Linear combinations of Schur's functions

Let us now have a sequence of integers α_λ, $-\infty < \lambda < \infty$, with the properties

$$\alpha_\lambda \to \infty \quad \text{as } \lambda \to \infty, \qquad \alpha_\lambda \to -\infty \quad \text{as } \lambda \to -\infty. \tag{112.1}$$

Monotonicity of the sequence is not assumed, although it will appear in our result later.

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[1] The use of [ ] in two different meanings in (111.8), (111.9), namely for the Gaussian polynomials as well as for the greatest integer not exceeding the argument, should not cause any confusion.

Moreover let a sequence of polynomials $v_\lambda = v_\lambda(x)$ be given. Then the linear combination

$$R_n(x) = \sum_{\lambda=-\infty}^{\infty} v_\lambda F_n(x, \alpha_\lambda)$$

contains only finitely many non-vanishing terms in virtue of (112.1). It represents therefore a rational function.

For *n even* the relation (111.6) will carry over to the R_n since it is fulfilled for each single term:

$$R_n(x) = R_{n-1}(x) + x^n R_{n-2}(x)\,, \qquad n \text{ even}\,. \tag{112.2}$$

Now the F_n are differences, as (111.8) shows. We have indeed

$$R_n(x) = \sum_{\lambda=-\infty}^{\infty} v_\lambda \left\{ \left[{n+1 \atop \left[\frac{n+1}{2}\right] - \alpha_\lambda} \right] - x^{2\alpha_\lambda+3} \left[{n+1 \atop \left[\frac{n}{2}\right] - \alpha_\lambda - 2} \right] \right\}. \tag{112.3}$$

We regroup now the terms in R_n in such a way that we combine each subtrahend with the minuend of the subsequent parenthesis:

$$R_n(x) = -\sum_{\lambda} \left\{ v_\lambda x^{2\alpha_\lambda+3} \left[{n+1 \atop \left[\frac{n}{2}\right] - \alpha_\lambda - 2} \right] \right.$$
$$\left. - v_{\lambda+1} \left[{n+1 \atop \left[\frac{n+1}{2}\right] - \alpha_{\lambda+1}} \right] \right\}. \tag{112.4}$$

If we determine the α_λ and v_λ in such a manner that the expression in $\{\ \}$ in (112.4) becomes a $G_n(x, \beta)$, aside from a possible factor independent of n, then (111.7) would come into force and (112.2) would be proved also for n odd.

This purpose requires, as a comparison of the "denominators" in the symbol for the Gaussian polynomials in (112.4) and (111.9) shows, that

$$\beta = \alpha_\lambda + 2\,, \qquad \beta + 3 = \alpha_{\lambda+1} \tag{112.5}$$

and thus

$$\alpha_{\lambda+1} = \alpha_\lambda + 5$$

and therefore

$$\alpha_\lambda = \alpha_0 + 5\lambda\,.$$

This is valid also for negative λ, with a certain α_0, still to be determined. Furthermore we need

$$v_{\lambda+1} = v_\lambda x^{2\alpha_\lambda+3+2\beta+4}\,.$$

Now (112.5) implies

$$2\beta + 3 = \alpha_\lambda + 2 + \alpha_{\lambda+1} = 2\alpha_\lambda + 7$$

so that

$$v_{\lambda+1} = v_\lambda x^{4\alpha_\lambda+11} = v_\lambda x^{4\alpha_0+20\lambda+11}\,.$$

This formula yields by induction

$$v_\lambda = v_0 x^{\lambda(10\lambda+4\lambda_0+1)}\,.$$

We achieved thus the desired determination. It is clear that conversely such α_λ, v_λ will indeed make $R_n(x)$ in (112.4) a sum of $G_n(x,\beta)$, so that we are now certain that with these α_λ, v_λ (112.2) is fulfilled also for n odd.

If we insert the results for α_λ and v_λ in (112.3) we obtain

$$R_n(x) = v_0 \sum_{\lambda=-\infty}^{\infty} \left\{ x^{\lambda(10\lambda+4\alpha_0+1)} \begin{bmatrix} n+1 \\ \left[\frac{n+1}{2}\right] - \alpha_0 - 5\lambda \end{bmatrix} - x^{\lambda(10\lambda+4\alpha_0+1)+2\alpha_0+10\lambda+3} \begin{bmatrix} n+1 \\ \left[\frac{n}{2}\right] - \alpha_0 - 5\lambda - 2 \end{bmatrix} \right\}. \tag{112.6}$$

Now

$$\lambda(10\lambda + 4\alpha_0 + 1) = \frac{2\lambda}{2}(5\cdot 2\lambda + 4\alpha_0 + 1)$$

and

$$\lambda(10\lambda + 4\alpha_0 + 1) + 2\alpha_0 + 10\lambda + 3$$

$$= \frac{2\lambda+1}{2}(5(2\lambda+1) + 4\alpha_0 + 1).$$

Also in the minuend

$$\left[\frac{n+1}{2}\right] - 5\lambda = \left[\frac{n+1-5\cdot 2\lambda}{2}\right]$$

and, in the subtrahend

$$\left[\frac{n}{2}\right] - 5\lambda - 2 = \left[\frac{n+1-5(2\lambda+1)}{2}\right].$$

If we replace therefore 2λ in the minuend and $2\lambda+1$ in the subtrahend by μ, then $R(x)$ appears as a sum of single terms with alternating signs

$$R_n(x) = v_0 \sum_{\mu=-\infty}^{\infty} (-1)^{\mu(\mu/2)(5\mu+4\alpha_0+1)} \left[\begin{matrix} n+1 \\ \left[\frac{n+1-5\mu}{2}\right] - \alpha_0 \end{matrix}\right]. \qquad (112.7)$$

(Schur credits this formula to Frobenius.)

The sum $R_n(x)$ represents always polynomials which satisfy, for given v_0 and α_0, the recursion formula

$$R_n(x) = R_{n-1}(x) + x^n R_{n-2}(\alpha),$$

i.e. the formula (108.9) for $D_\mu^{(n)}(x)$. The available constants v_0 and α_0 have to be adjusted so that the initial conditions (108.73) and (108.71), (108.72) for $n = 0, 1$ are fulfilled. The factor v_0 must be ± 1 in order to permit $R_0(x)$ to be 1.

For $v_0 = 1$, $\alpha_0 = 0$, formula (112.6) yields

$$R_0(x) = 1, \qquad R_1(x) = 1 + x.$$

This means that for these values of v_0 and α_0, $R_n(x)$ fulfills the initial conditions of $D_1^{(r)}(x)$, so that for all positive N

$$D_1^{(N)}(x) = R_N(x) = \sum_{\mu=-\infty}^{\infty} (-1)\, x^{\mu(\mu/2)(5\mu+1)} \left[\begin{matrix} N+1 \\ \left[\frac{N+1-5\mu}{2}\right] \end{matrix}\right]. \qquad (112.8)$$

For $v_0 = 1$, $\alpha_0 = -1$ formula (112.6) leads to

$$R_0(x) = 1, \qquad R_1(x) = 1.$$

In this case $R_n(x)$ fulfills the initial conditions for $D_2^{(n)}(x)$, so that for all positive N we have

$$D_2^{(N)}(x) = R_N(x) = \sum_{\mu=-\infty}^{\infty} (-1)^\mu x^{(\mu/2)(5\mu-3)} \left[\begin{matrix} N+1 \\ \left[\frac{N+1-5\mu}{2}\right] + 1 \end{matrix}\right]. \qquad (112.9)$$

113. Determination of $D_1(x)$ and $D_2(x)$

In the numerator of

$$\begin{bmatrix} N+1 \\ \left[\frac{N+1-5\mu}{2}\right] \end{bmatrix}$$

the powers of x go down to $\left[\frac{N+5\mu}{2}\right]+1$. We have

$$\frac{\mu}{2}(5\mu+1)+\left[\frac{N+5\mu}{2}\right]+1 \geqq \frac{|\mu|\,(5\,|\mu|-1)}{2}+\frac{N-5\,|\mu|}{2}$$

$$\geqq \frac{5\,|\mu|\,(|\mu|-2)}{2}+\frac{N}{2}>\left[\frac{N}{2}\right]-3$$

and therefore, for any μ,

$$x^{\mu(5\mu+1)/2}\begin{bmatrix} N+1 \\ \left[\frac{N+1-5\mu}{2}\right] \end{bmatrix} \equiv \frac{x^{\mu(5\mu+1)/2}}{(1-x)\,(1-x^2)\cdots(1-x^{[(N+1-5\mu)/2]})}$$

$$\equiv \frac{x^{\mu(5\mu+1)/2}}{\prod\limits_{m=1}^{\infty}(1-x^m)} \pmod{x^{[N/2]-3}}\,.$$

Consequently

$$D_1^{(N)}(x) \equiv \sum_{\mu=-\infty}^{\infty}(-1)^{\mu}\,x^{\mu(5\mu+1)/2}\cdot\frac{1}{\prod\limits_{m=1}^{\infty}(1-x^m)} \pmod{x^{[N/2]-3}}\,.$$

On the other hand, according to (108.5),

$$D_1^{(N)}(x) \equiv D_1(x) \pmod{x^{[N/2]-3}}\,.$$

Therefore

$$D_1(x) \equiv \sum_{\mu=-\infty}^{\infty}(-1)^{\mu}\,x^{\mu(5\mu+1)/2}\cdot\frac{1}{\prod\limits_{1}^{\infty}(1-x^m)} \pmod{x^{[N/2]-3}}\,.$$

This means that these two power series agree up to any given power of x. In other words, (109.13) is proved. A completely analogous argument for $D_2(x)$ shows that (109.4) is true. This finishes the proof of the identities (107.1), (107.2) and the equivalent theorem in § 107.

114. A digression, concerning a further proof of the pentagonal number theorem

I. Schur mentions also the formula

$$\begin{bmatrix} k \\ l \end{bmatrix} - x^{k-2l+1} \begin{bmatrix} k \\ l-1 \end{bmatrix} = \begin{bmatrix} k-1 \\ l \end{bmatrix} - x^{k-2l+1} \begin{bmatrix} k-1 \\ l-2 \end{bmatrix}. \tag{114.1}$$

The proof is immediate: the difference of the two members of (114.1) is

$$\begin{bmatrix} k \\ l \end{bmatrix} - \begin{bmatrix} k-l \\ l \end{bmatrix} - x^{k-2l+1} \left(\begin{bmatrix} k \\ l-1 \end{bmatrix} - \begin{bmatrix} k-1 \\ l-2 \end{bmatrix} \right)$$

$$= x^{k-l} \begin{bmatrix} k-1 \\ l-1 \end{bmatrix} - x^{k-2l+1} x^{l-1} \begin{bmatrix} k-1 \\ l-1 \end{bmatrix} = 0,$$

after (110.6) and (110.5). In a new proof of the pentagonal numbers theorem formula (114.1) will play a role similar to (111.1).

Let us define now two sets of rational functions

$$H_n = H_n(x, j) = \begin{bmatrix} n+1 \\ \left[\frac{n+1}{2}\right] - \gamma \end{bmatrix} - x^{2+2\gamma} \begin{bmatrix} n+1 \\ \left[\frac{n}{2}\right] - 1 - \gamma \end{bmatrix} \tag{114.2}$$

and

$$J_n = J_n(x, \gamma) = \begin{bmatrix} n+1 \\ \left[\frac{n}{2}\right] - \gamma \end{bmatrix} - x^{3+2\gamma} \begin{bmatrix} n+1 \\ \left[\frac{n+1}{2}\right] - 2 - \gamma \end{bmatrix} \tag{114.3}$$

with arbitrary integers γ. Now if n is *even* we have

$$\left[\frac{n+1}{2}\right] = \frac{n}{2} = \left[\frac{n-1}{2}\right] + 1,$$

and (114.1) furnishes then

$$H_n = \begin{bmatrix} n \\ \left[\frac{n+1}{2}\right] - \gamma \end{bmatrix} - x^{2+2\gamma} \begin{bmatrix} n \\ \left[\frac{n}{2}\right] - 2 - \gamma \end{bmatrix}$$

$$= \begin{bmatrix} n \\ \left[\frac{n}{2}\right] - \gamma \end{bmatrix} - x^{2+2\gamma} \begin{bmatrix} n \\ \left[\frac{n-1}{2}\right] - 1 - \gamma \end{bmatrix}$$

or

$$H_n = H_{n-1}, \qquad n \text{ even}. \tag{114.4}$$

Similarly, for n *odd*,

$$\left[\frac{n+1}{2}\right] = \left[\frac{n}{2}\right] + 1 = \left[\frac{n-1}{2}\right] + 1$$

and, from (114.3) and (114.1)

$$J_n = \begin{bmatrix} n \\ \left[\frac{n}{2}\right] - \gamma \end{bmatrix} - x^{3+2\gamma} \begin{bmatrix} n \\ \left[\frac{n+1}{2}\right] - 3 - \gamma \end{bmatrix}$$

$$= \begin{bmatrix} n \\ \left[\frac{n-1}{2}\right] - \gamma \end{bmatrix} - x^{3+2\gamma} \begin{bmatrix} n \\ \left[\frac{n}{2}\right] - 2 - \gamma \end{bmatrix}$$

or

$$J_n = J_{n-1}\,, \quad n \text{ odd.} \tag{114.5}$$

Let now $\{\gamma_\lambda\}$ be a sequence of integers

$$\ldots \gamma_{-2}, \gamma_{-1}, \gamma_0, \gamma_1, \gamma_2 \ldots$$

with

$$\gamma_\lambda \to \infty \text{ as } \lambda \to \infty\,, \quad \gamma_\lambda \to -\infty \text{ as } \lambda \to -\infty\,. \tag{114.6}$$

We form moreover with certain polynomials t_λ the expression

$$S_n = \sum_{-\infty}^{\infty} t_\lambda H_n(x, \gamma_\lambda)\,,$$

which, in virtue of (114.6), has only finitely many non-vanishing terms. Moreover, we know from (114.4)

$$S_n = S_{n-1} \text{ for } n \text{ even}\,.$$

Now H_n is a difference, so that

$$S_n = \sum_{-\infty}^{\infty} t_\lambda \left\{ \begin{bmatrix} n+1 \\ \left[\frac{n+1}{2}\right] - \gamma_\lambda \end{bmatrix} - x^{2+2\gamma_\lambda} \begin{bmatrix} n+1 \\ \left[\frac{n}{2}\right] - 1 - \gamma_\lambda \end{bmatrix} \right\}. \tag{114.7}$$

We use this fact, as in § 112, to regroup the terms in S_n, taking always a subtrahend together with the following minuend:

$$S_n = -\sum_{-\infty}^{\infty} \left\{ t_\lambda\, x^{2+2\gamma_\lambda} \begin{bmatrix} n+1 \\ \left[\frac{n}{2}\right] - 1 - \gamma_\lambda \end{bmatrix} \right.$$

$$\left. - t_{\lambda+1} \begin{bmatrix} n+1 \\ \left[\frac{n+1}{2}\right] - \gamma_\lambda + 1 \end{bmatrix} \right\}.$$

If we succeed in giving the expression in { } the shape of a J_n, then because of (114.5) we shall have proved $S_n = S_{n-1}$ also for n *odd*.

Now this requires after (114.3) first

$$-1 - \gamma_\lambda + \gamma_{\lambda+1} = 2\,, \qquad \gamma_{\lambda+1} = \gamma_\lambda + 3\lambda$$

and thus

$$\gamma_\lambda = \gamma_0 + 3\lambda\,,$$

for positive as well as negative λ. We also need

$$t_{\lambda+1} = t_\lambda x^{2+2\gamma_\lambda+2+\gamma_\lambda+\gamma_{\lambda+1}}$$

or

$$t_{\lambda+1} = t_\lambda x^{4\gamma_0+7+12\lambda}$$

and thus

$$t_\lambda = t_0 x^{\lambda(6\lambda+4\gamma_0+1)}\,.$$

The requirements can be fulfilled as we see, so that S_n appears as a sum of $J(n, \gamma_\lambda)$ and therefore satisfies

$$S_n = S_{n-1} \tag{114.8}$$

for all $x \geqq 1$.

Inserting our results in (114.7) we obtain

$$S_n = t_0 \sum_{\lambda=-\infty}^{\infty} x^{\lambda(6\lambda+4\gamma_0+1)} \left\{ \begin{bmatrix} n+1 \\ \left[\frac{n+1}{2}\right] - \gamma_0 - 3\lambda \end{bmatrix} \right.$$
$$\left. - x^{2+2\gamma_0+6\lambda} \begin{bmatrix} n+1 \\ \left[\frac{n}{2}\right] - 1 - \gamma_0 - 3\lambda \end{bmatrix} \right\}.$$

Now we have

$$\lambda(6\lambda + 4\gamma_0 + 1) = \frac{2\lambda}{2}(3\cdot 2\lambda + 4\gamma_0 + 1)$$

and

$$\lambda(6\lambda + 4\gamma_0 + 1) + 2 + 2\gamma_0 + 6\lambda$$
$$= \frac{2\lambda+1}{2}\left(3(2\lambda+1) + 4\gamma_0 + 1\right).$$

Moreover

$$\left[\frac{n+1}{2}\right] - \gamma_0 - 3 = \left[\frac{n+1-3\cdot 2\lambda}{2}\right] - \gamma_0$$

and

$$\left[\frac{n}{2}\right] - 1 - \gamma_0 - 3\lambda = \left[\frac{n-2-3\cdot 2\lambda}{2}\right] - \gamma_0$$
$$= \left[\frac{n+1-3(2\lambda+1)}{2}\right] - \gamma_0 \,.$$

If we replace therefore 2λ und $2\lambda+1$ by even and odd μ respectively, we can write the series for S_n as a sum of single terms with alternating signs:

$$S_n = t_0 \sum_{-\infty}^{\infty} (-1)^\mu x^{\mu(3\mu+4\gamma_0+1)/2} \left[\begin{matrix} n+1 \\ \left[\frac{n+1-3\mu}{2}\right] - \gamma_0 \end{matrix}\right].$$

Now (114.8) implies

$$S_n = S_0 \,,$$

and for $\gamma_0 = 0$ we have

$$S_0 = t_0 \sum_{-\infty}^{\infty} (-1)^\mu x^{\mu(3\mu+1)/2} \left[\begin{matrix} 1 \\ \left[\frac{1-3\mu}{2}\right] \end{matrix}\right] = t_0$$

so that

$$1 = \sum_{-\infty}^{\infty} (-1)^\mu x^{\mu(3\mu+1)/2} \left[\begin{matrix} n+1 \\ \left[\frac{n+1-3\mu}{2}\right] \end{matrix}\right]$$

for any $n \geqq 0$.

In the numerator of the Gaussian polynomial the powers of x go down to at most $\left[\frac{n+3\mu}{2}\right] + 1$ (if the polynomial does not vanish). Now

$$\left[\frac{n+3\mu}{2}\right] + 1 + \frac{\mu}{2}(3\mu+1) \geqq \frac{n+3\mu+\mu(3\mu+1)}{2}$$
$$= \frac{n}{2} + \frac{\mu(3\mu+4)}{2} \geqq \frac{n-1}{2}.$$

Therefore

$$1 \equiv \sum_{-\infty}^{\infty} (-1)^\mu x^{\mu(3\mu+1)/2} \frac{1}{(1-x)(\cdots(1-x^{[n+1-3\mu/2]})} \quad (\mathrm{mod}\ x^{[(n-1)/2]})$$

and therefore also,

$$1 \equiv \sum_{-\infty}^{\infty} (-1)^\mu x^{\mu(3\mu+1)/2} \frac{1}{\prod_1^\infty (1-x^m)} \quad (\text{mod } x^{[(n-1)/2]}) , \tag{114.9}$$

since we have here only introduced terms of a power higher than

$$\frac{\mu}{2}(3\mu+1) + \left[\frac{n+1-3\mu}{2}\right] \geqq \frac{n}{2} + \frac{\mu(3\mu-2)}{2} \geqq \frac{n}{2} .$$

But the congruence (114.9) shows that two formal power series agree to any power of x and are therefore identical. Multiplying through by the infinite product we obtain thus

$$\prod_1^\infty (1-x^m) = \sum_{-\infty}^{\infty} (-1)^\mu x^{\mu(3\mu+1)/2} ,$$

the pentagonal number theorem.

115. A further remark

The recursion formulae (108.9) together with the initial conditions (108.73), (108.71), (108.72) show that $D_1^{(n)}(x)$ and $D_2^{(n)}(x)$ are nothing else than the numerators and denominators of the convergents of the formal continued fraction

$$K(x) = 1 + \cfrac{x}{1 + \cfrac{x^2}{1 + \cfrac{x^3}{1 + \dots}}} .$$

The first few convergents are

$$1 , \quad \frac{1+x}{1} , \quad \frac{1+x+x^2}{1+x^2} , \dots ,$$

which agree with

$$\frac{D_1^{(0)}(x)}{D_2^{(0)}(x)} , \quad \frac{D_1^{(1)}(x)}{D_2^{(1)}(x)} , \quad \frac{D_1^{(2}(x)}{D_2^{(2)}(x)} , \dots .$$

We have thus, again in the sense of formal power series,

$$D(x) = \frac{D_1(x)}{D_2(x)} = \frac{\prod_1^\infty (1-x^{5m-3})(1-x^{5m-2})}{\prod_1^\infty (1-x^{5m-4})(1-x^{5m-1})} = \frac{\sum_\mu^\infty (-1)^\mu x^{\mu(5\mu-1)/2}}{\sum_\mu^\infty (-1)^\mu x^{\mu(5\mu-3)/2}} .$$

(Ramanujan, taking series not in a formal sense, but as *convergent* for $|x| < 1$, computed $K(e^{-2\pi})$, $K(e^{-\pi})$, $K(e^{-2\pi\sqrt{5}})$, from these identities.)

IV. The Circle Method

Chapter 14

Analytic Theory of Partitions

Preliminary remarks. The chapters of Part III dealt only with formal power series in which the indeterminate could not be replaced.[1] If we now, however, replace the indeterminate x by a complex variable x, the formal power series become power series in the ordinary sense to which the concept of convergence, totally absent in Part III, applies. The convergent power series are analytic functions, and the formal identities become equations between analytic functions. This step opens the whole store of analytic tools for the treatment of arithmetical problems in additive number theory.

We have so far by means of formal power series obtained only relations between different values of $p(n)$ and some congruence properties of $p(n)$, but could determine $p(n)$ itself only by recursion. This situation is changed through the application of the theory of analytic functions, as Hardy and Ramanujan showed in their epoch-making paper of 1917 [18]; [59], pp. 276—309.

116. A Cauchy integral and a special path of integration

If in the identity

$$f(x) = \sum_{n=0}^{\infty} p(n)\, x^n = \prod_{m=1}^{\infty} (1 - x^m)^{-1},$$

we consider x as a complex variable, the right-hand member shows that the infinite product and thus also the infinite series are convergent for $|x| < 1$.

Thus if we substitute

$$x = e^{2\pi i \tau}, \qquad \operatorname{Im} \tau > 0$$

we remain in the domain of convergence and can apply Cauchy's integral with the result

$$p(n) = \int_{\tau_0}^{\tau_0+1} f(e^{2\pi i \tau})\, e^{-2\pi i n \tau}\, d\tau, \tag{116.1}$$

[1] The exceptions in §§ 101, 102 needed specific arguments for justification.

where τ_0 is an arbitrary point in the upper half-plane, and any path from τ_0 to $\tau_0 + 1$ in the upper half-plane is permissible.

The formula (116.1) gains significance through a comparison of (98.1) and (65.2) which shows that

$$f(e^{2\pi i\tau}) = e^{\pi i\tau/12}\,\eta(\tau)^{-1}\,. \tag{116.2}$$

This makes available our theory of the function $\gamma(\tau)$ in Chapter 9. It is worth while to note that

$$f(e^{2\pi i\tau}) = \sum_{n=0}^{\infty} p(n)\, e^{2\pi i n\tau}$$

has no terms for $n < 0$. The integral in (116.1) makes sense also for $n < 0$ and has then the trivial value zero. We use this fact as a definition

$$p(n) = 0 \quad \text{for } n < 0\,. \tag{116.3}$$

In (116.1) we choose now a path the construction of which is based on the properties of the Farey series. These properties are well known [18], pp. 87/8, [25], vol. 2, pp. 136—156, [49], pp. 7—10, 41, 42, so that we can just state them:

(1) The ordered sequence of all reduced proper fractions

$$\frac{0}{1}\,,\ \frac{1}{N}\,,\ \frac{1}{N-1}\,,\ldots,\ \frac{N-1}{N}\,,\ \frac{1}{1}\,,$$

i.e. $0 \leqq h/k \leqq 1$ with $0 \leqq h \leqq k$, $(h, k) = 1$, $k \leqq N$ is the "Farey sequence of order N".

(2) Two adjacent fractions $h_1/k_1 < h_2/k_2$ have the difference

$$\frac{h_2}{k_2} - \frac{h_1}{k_1} = \frac{1}{k_1 k_2}\,.$$

(3) The "mediant" $(h_1 + h_2)/(k_1 + k_2)$ of the two fractions h_1/k_1, h_2/k_2 satisfies

$$\frac{h_1}{k_1} < \frac{h_1 + h_2}{k_1 + k_2} < \frac{h_2}{k_2}\,,$$

is also a reduced fraction and belongs to a Farey sequence of higher order than N. Clearly therefore,

$$\left|\frac{h_1 + h_2}{k_1 + k_2} - \frac{h_1}{k_1}\right| = \frac{1}{k_1(k_1 + k_2)}$$

and thus

$$\frac{1}{2Nk_1} \leqq \left|\frac{h_1 + h_2}{k_1 + k_2} - \frac{h_1}{k_1}\right| < \frac{1}{Nk_1}\,.$$

(4) Starting from the Farey sequence of order 1: 0/1, 1/1, all Farey sequences of higher order can be obtained by successive inclusion of mediants.

(5) To each h/k there belongs a "Ford circle" $C(h, k)$

$$\left|\tau - \left(\frac{h}{k} + \frac{i}{2k^2}\right)\right| = \frac{1}{2k^2}$$

in the upper half-plane, tangent to the real axis at $\tau = h/k$.

(6) Two Ford circles $C(h, k)$ and $C(l, m)$ do not intersect. They are tangent to each other if and only if they belong to fractions h/k, l/m which are adjacent in some Farey sequence.

(7) If $h_1/k_1 < h/k < h_2/k_2$ are three adjacent fractions in a Farey sequence, then $C(h, k)$ touches $C(h_1, k_1)$ and $C(h_2, k_2)$ respectively in the points

$$\frac{h}{k} + \zeta'_{hk} \quad \text{and} \quad \frac{h}{k} + \zeta''_{hk}$$

where

$$\zeta'_{hk} = -\frac{k_1}{k(k^2 + k_1^2)} + \frac{i}{k^2 + k_1^2},$$

$$\zeta''_{hk} = \frac{k_2}{k(k^2 + k_2^2)} + \frac{i}{k^2 + k_2^2}. \tag{116.4}$$

A further remark explains the configuration of the set of all Ford circles:

(8) The set of all Ford circles at all rational points h/k, $-\infty < h/k < \infty$ is the set of the images of the line $\tau = x + i$, x real, under all modular transformations.

Indeed the modular transformation

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

can be written

$$c\tau' - a = \frac{-1}{c\tau + d}.$$

Putting here $\tau = x + i$ we obtain

$$c\tau' - a - \frac{i}{2c} = \frac{-i}{2c} \frac{c(x - i) + d}{c(x + i) + d}$$

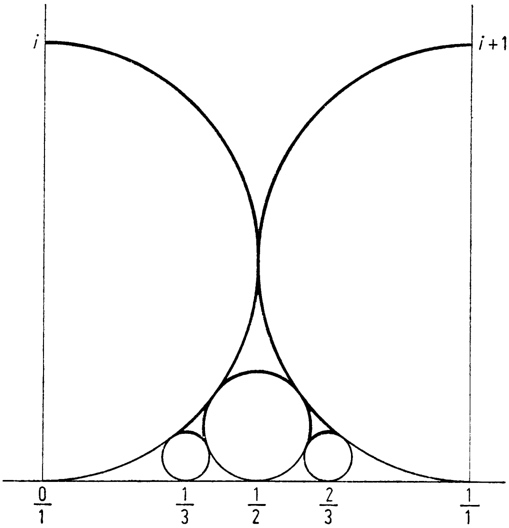
and thus

$$\left|c\tau' - a - \frac{i}{2c}\right| = \frac{1}{2c}$$

or

$$\left|\tau' - \left(\frac{a}{c} + \frac{i}{2c^2}\right)\right| = \frac{1}{2c^2}.$$

Now, to each reduced fraction a/c there exist infinitely many b, d which fulfill $ad - bc = 1$, which proves statement 8.



Let now a natural number N be given. We construct the Ford circles belonging to the Farey sequence of order N. On each circle $C(h, k)$ we choose that arc γ_{hk} which connects the tangency points $h/k + \zeta'_{hk}$ and $h/k + \zeta''_{kh}$ and which does not touch the real axis (the "upper" arc). The circles $C(0, 1)$ and $C(1, 1)$ are congruent. Because of the periodicity of $f(e^{2\pi i\tau})$ we choose one half of each, from i to $0/1 + \zeta''_{01}$ on $C(0, 1)$ and from $i+1$ to $1/1 + \zeta'_{11}$ on $C(1,1)$, and call γ_{01} the union of the two arcs. These arcs are attached to each other at the points of contact of Ford circles. They together form the path of integration in (116.1)

117. An expression for $p(n)$

We continue now from (116.1),

$$p(n) = \int_i^{i+1} f(e^{2\pi i\tau})\, e^{-2\pi in\tau}\, d\tau = \sum_{0 \leqq h < k \leqq N}{}' \int_{\gamma_{hk}} f(e^{2\pi i\tau})\, e^{-2\pi in\tau}\, d\tau\,,$$

where the dash in the summation means that $(h, k) = 1$.

We have in the integrals,

$$p(n) = \sum_{0 \leqq h < k \leqq N}' \int_{\zeta'_{hk}}^{\zeta''_{hk}} f(e^{2\pi i(h/k+\zeta)})\, e^{-2\pi i n(h/k+\zeta)}\, d\zeta$$

with ζ'_{hk}, ζ''_{hk} from (116.4). In the integral belonging to h/k the variable ζ runs on an arc of the circle $|\zeta - i/(2k^2)| = 1/(2k^2)$. We introduce in each integral the new variable z by $\zeta = iz/k^2$ and obtain

$$p(n) = \sum_{0 \leqq h < k \leqq N}' \frac{i}{k^2} e^{-2\pi i n h/k} \int_{z'_{hk}}^{z''_{hk}} f(e^{2\pi i h/k - 2\pi z/k^2})\, e^{2\pi n z/k^2} dz\,, \tag{117.1}$$

where z runs on the circle

$$\left| z - \frac{1}{2} \right| = \frac{1}{2} \tag{117.2}$$

on that arc between $z'_{hk} = (1/i)\, k^2 \zeta'_{hk}$ and $z''_{hk} = (1/i)\, k^2 \zeta''_{hk}$, which does not touch the imaginary axis. Thus always, $\operatorname{Re} z > 0$. By virtue of (116.4) we have

$$z'_{hk} = \frac{k^2}{k^2 + k_1^2} + i \frac{k k_1}{k^2 + k_1^2}\,, \quad z''_{hk} = \frac{k^2}{k^2 + k_2^2} - i \frac{k^2}{k^2 + k_2^2}\,. \tag{117.3}$$

The variable z runs in the negative direction on the circle (117.2).

118. Application of the transformation formula for $\eta(\tau)$

For the integral in (117.1) we use (116.2) with

$$\tau = \frac{h}{k} + \frac{iz}{k}$$

and have

$$f(e^{2\pi i h/k - 2\pi z/k}) = e^{\pi i h/12k - \pi z/12k}\, \eta\left(\frac{h}{k} + \frac{iz}{k}\right)^{-1}. \tag{118.1}$$

We determine h' so that

$$hh' \equiv -1 \pmod{k}\,.$$

Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} h & -\dfrac{hh'+1}{k} \\ k & -h' \end{pmatrix} \tag{118.2}$$

is the matrix of a modular substitution, and we have, as a short computation shows,

$$\frac{a\left(\frac{h'}{k}+\frac{i}{kz}\right)+b}{c\left(\frac{h'}{k}+\frac{i}{kz}\right)+d}=\frac{h}{k}+\frac{iz}{k}.$$

The application of (74.11) to (118.1) yields then

$$f\left(e^{2\pi ih/k-2\pi z/k}\right)=e^{\pi ih/12k-\pi z/12k}\,\varepsilon(a,b,c,d)^{-1}\sqrt{z}\,\eta\left(\frac{h'}{k}+\frac{i}{zk}\right)^{-1}, \tag{118.3}$$

where according to (74.12) and (71.21)

$$\varepsilon(a,b,c,d)=\exp\left\{\pi i\frac{h-h'}{12k}-\pi i s(h,k)\right\}. \tag{118.4}$$

Here $s(-h',k)=s(h,k)$ has been taken into account (see (68.5)). Thus we have

$$f\left(e^{2\pi ih/k-2\pi z/k}\right)=\omega_{hk}\,e^{(\pi/12k)(1/z-z)}\sqrt{z}\,f\left(e^{2\pi ih'/k-2\pi/kz}\right) \tag{118.41}$$

with

$$\omega_{hk}=e^{\pi is(h,k)}. \tag{118.42}$$

We replace z by z/k and get

$$f\left(e^{2\pi ih/k-2\pi z/k^2}\right)=\frac{\omega_{hk}}{\sqrt{k}}\,\Psi_k(z)\,f\left(e^{2\pi ih'/k-2\pi/z}\right) \tag{118.5}$$

with the abbreviation

$$\Psi_k(z)=\sqrt{z}\;e^{\pi/12z-\pi z/12k^2}. \tag{118.6}$$

The application of (118.5) changes (117.1) into

$$p(n)=\sum_{0\leq h<k\leq N}{}'\frac{i}{k^{5/2}}\,\omega_{hk}\,e^{-2\pi ihn/k}\int_{z'_{hk}}^{z''_{hk}}\Psi_k(z)\,f\left(e^{2\pi ih'/k-2\pi/z}\right)e^{2\pi nz/k^2}\,dz.$$

Now we remember

$$f(x)=1+p(1)\,x+p(2)\,x^2+\cdots.$$

In our case

$$x=e^{2\pi ih'/k-2\pi/z}. \tag{118.7}$$

We plan to make Re z small, so that Re $1/z$ will become large and thus $|x|$ small. Therefore, for the subsequent estimates, we compare $f(x)$ with 1. For this purpose we write

$$p(n) = \sum_{0\leqq h<k\leqq N}{}' \frac{i}{k^{5/2}} \omega_{hk} e^{-2\pi i hn/k} \int_{z'_{hk}}^{z''_{hk}} \Psi_k(z) e^{2\pi nz/k^2} dz$$

$$+ \sum_{0\leqq h<k\leqq N}{}' \frac{i}{k^{5/2}} \omega_{hk} e^{-2\pi i hn/k} \int_{z'_{hk}}^{z''_{hk}} \Psi_k(z) \{f(x) - 1\} e^{2\pi nz/k^2} dz ,$$

where x is given by (118.7).

119. Estimates and evaluations

We consider now the integrals

$$I_{hk} = \int_{z'_{hk}}^{z''_{hk}} \Psi_k(z) e^{2\pi nz/k^2} dz \tag{119.1}$$

and

$$I^*_{hk} = \int_{z'_{hk}}^{z''_{hk}} \Psi_k(z) \{f(e^{2\pi i h'/k - 2\pi/z}) - 1\} e^{2\pi nz/k^2} dz \tag{119.2}$$

separately, beginning with (119.2). The variable z is running on the circle (117.2), avoiding the point $z = 0$. In and on the circle we have therefore

$$0 < \mathrm{Re}\, z \leqq 1 , \ \mathrm{Re}\, \frac{1}{z} \geqq 1 . \tag{119.3}$$

Thus

$$\left|\Psi_k(z) \{f(e^{2\pi i h'/k - 2\pi/z}) - 1\} e^{2\pi nz/k^2}\right|$$

$$= \left| z^{1/2} e^{\pi/12z - \pi z/12k^2} \sum_{m=1}^{\infty} p(m)\, e^{(2\pi i h'/k - 2\pi/z)m}\, e^{2\pi nz/k^2} \right|$$

$$\leqq |z|^{1/2} e^{(2\pi n/k^2)\mathrm{Re}\, z} \sum_{m=1}^{\infty} p(m)\, e^{-2\pi(m-(1/24))\mathrm{Re}\,(1/z)}$$

$$\leqq |z|^{1/2} e^{(2\pi n/k^2)\mathrm{Re}\, z} \sum_{m=1}^{\infty} p(m)\, e^{-2\pi(m-(1/24))}$$

$$= C e^{(2\pi n/k^2)\mathrm{Re}\, z} |z|^{1/2} . \tag{119.4}$$

The integrand in I^*_{hk} is regular inside the circle (117.2). We integrate therefore from z'_{hk} to z''_{hk} along a chord of the circle. On this chord we have

$$|z| \leqq \max\left(|z'_{hk}|\,,\ |z''_{hk}|\right),$$

and according to (117.3)

$$|z'_{hk}|^2 \leqq \frac{k^4 + k^2 k_1^2}{(k^2 + k_1^2)^2} = \frac{k^2}{k^2 + k_1^2} \leqq \frac{2k^2}{(k + k_1)^2} < \frac{2k^2}{N^2}.$$

The same estimate holds for $|z''_{hk}|^2$. This furnishes

$$|z| \leqq \frac{\sqrt{2}\,k}{N} \tag{119.5}$$

on the chord $z'_{hk} z''_{hk}$. Similarly for the real parts

$$0 < \operatorname{Re} z \leqq \max\left(\frac{k^2}{k^2 + k_1^2}\,,\ \frac{k^2}{k^2 + k_2^2}\right)$$

and thus

$$0 < \operatorname{Re} z < \frac{2k^2}{N^2} \tag{119.6}$$

on the chord. The length of the chord, i.e. the length of the path of integration, is $\leqq |z'_{hk}| + |z''_{hk}| < 2\sqrt{2}k/N$.

Using (119.4) we obtain thus

$$|I^*_{hk}| < C\,\frac{k^{3/2}}{N^{3/2}}\,e^{4\pi|n|/N^2} = O\left(k^{3/2} N^{-3/2}\right)$$

for any fixed $n \gtreqless 0$.

120. Continuation of estimates and evaluations. The final formula for $p(n)$

If we extend the integration in I_{hk} along the whole circle K given by (117.2) we have

$$I_{hk} = \int_{K^{(-)}} \Psi_k(z)\, e^{2\pi n z/k^2}\, dz - \int_0^{z'_{hk}} - \int_{z''_{hk}}^{0}, \tag{120.1}$$

where the rotation $K^{(-)}$ is meant to indicate that the circle is travelled in the negative sense. The latter two integrals have to be understood as improper integrals with limits of integration not equal to zero but tending to zero.

On K we have

$$\operatorname{Re}\frac{1}{z} = 1\,, \qquad 0 < \operatorname{Re}(z) \leqq 1\,. \tag{120.2}$$

The length of the arc from 0 to z'_{hk} is less than

$$\frac{\pi}{2}\left|z'_{hk}\right| < \frac{\pi}{2}\frac{\sqrt{2}\,k}{N},$$

and also, on that arc, we have

$$|z| < \frac{\sqrt{2}\,k}{N}.$$

Moreover, (119.6) and (119.3) are valid so that

$$\left|\int_0^{z'_{hk}} \Psi_k(z)\, z^{2\pi n z/k^2}\, dz\right| = O\left(k^{3/2} N^{-3/2} e^{4\pi|n|/N^2}\right) \tag{120.3}$$

and similarly for the last integral in (120.1). These estimates are the same as (119.7) for I^*_{hk}. We have now (118.8) in the notation (119.1), (119.2) as

$$p(n) = \sum_{0\le h<k\le N}{}' \frac{i}{k^{5/2}}\,\omega_{hk}\, e^{-2\pi i h n/k}\left\{I_{hk} + I^*_{hk}\right\}.$$

Collecting now (119.7), (120.1) and (120.3) we obtain

$$p(n) = \sum_{0\le h<k\le N}{}' \frac{i}{k^{5/2}}\,\omega_{hk}\, e^{-2\pi i h n/k}\int_{K^{(-)}} \Psi_k(z)\, e^{2\pi n z/k^2}\, dz + R,$$

where

$$R = O\Bigl(\sum_{0\le h<k\le N} k^{-1} N^{-3/2}\Bigr) e^{4\pi|n|/N^2} = O\Bigl(\sum_{0<k\le N} e^{4\pi|n|/N^2} N^{-3/2}\Bigr)$$
$$= O\left(e^{4\pi|n|/N^2} N^{-1/2}\right)$$

and thus

$$p(n) = i\sum_{k=1}^{N} \frac{1}{k^{5/2}} A_k(n) \int_{K^{(-)}} \Psi_k(z)\, e^{2\pi n z/k^2}\, dz + O\left(e^{4\pi|n|/N^2} N^{-1/2}\right)$$

with

$$A_k(n) = \sum_{h \bmod k}{}' \omega_{hk}\, e^{-2\pi i h n/k}. \tag{120.5}$$

This summation over h could not be carried out earlier since in the previous steps the paths of the integrals depended on h.

The left-hand side *does not depend on* N. We can therefore let N tend to ∞ and must obtain a *convergent series*, since the remainder term goes to 0:

$$p(n) = i\sum_{k=1}^{\infty} \frac{1}{k^{5/2}} A_k(n) \int_{K^{(-)}} \Psi_k(z)\, e^{2\pi n z/k^2}\, dz. \tag{120.6}$$

This is an exact formula for $p(n)$. Incidentally, the absolute convergence of (120.6) is evident, since after (118.6) and (120.2) we find that

$$\left| \int_{K^{(-)}} \right| \leqq 2\pi e^{\pi/12} e^{2\pi|n|} ,$$

and from (120.5) we obtain the trivial estimate

$$|A_k(n)| \leqq k ,$$

so that the sum in (120.6) is majorized by

$$2\pi e^{\pi/12} e^{2\pi|n|} \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} .$$

We have now to compute the integral in (120.6). If we put

$$w = \frac{1}{z}, \qquad dz = -\frac{dw}{w^2}$$

we obtain

$$p(n) = \frac{1}{i} \sum_{k=1}^{\infty} \frac{1}{k^{5/2}} A_k(n) \int_{1-i\infty}^{1+i\infty} w^{-\frac{5}{2}} e^{\frac{\pi w}{12} + \frac{2\pi}{k^2}\left(n - \frac{1}{24}\right)\frac{1}{w}} dw ,$$

and further, through $\pi w/12 = t$,

$$p(n) = \frac{1}{i} \left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} k^{-5/2} A_k(n) \int_{c-i\infty}^{c+i\infty} t^{-\frac{5}{2}} e^{t + \frac{\pi^2}{6k^2 t}\left(n - \frac{1}{24}\right)} dt$$

with $c > 0$. The integral here is known in terms of Bessel functions. Indeed, from (25.3) we infer, if we take into account also the remark about bending the path of integration,

$$p(n) = 2\pi \left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} k^{-5/2} A_k(n)$$
$$\times \left(\frac{\pi}{2k} \sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)^{-3/2} I_{3/2}\left(\frac{\pi}{k} \sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right). \tag{120.7}$$

This formula can be simplified through some lemmas about Bessel functions. We treat together the two sorts of Bessel functions defined by (25.1) and (25.3) by writing, with $\varepsilon = \pm 1$,

$$J_\nu^{(\varepsilon)}(z) = \sum_{n=0}^{\infty} \varepsilon^n \frac{\left(\frac{z}{2}\right)^{2n+\nu}}{n!\,\Gamma(\nu + n + 1)} , \tag{120.8}$$

so that

$$J_\nu^{(-1)}(z) = J_\nu(z) , \qquad J_\nu^{(1)}(z) = I_\nu(z) . \tag{120.9}$$

Lemma I.

$$\frac{d}{dz}\left(z^{\nu} J_{\nu}^{(\varepsilon)}(z)\right) = z^{\nu} J_{\nu-1}^{(\varepsilon)}(z)\,,$$

$$\frac{d}{dz}\left(z^{-\nu} J_{\nu}^{(\varepsilon)}(z)\right) = \varepsilon z^{-\nu} J_{\nu+1}^{(\varepsilon)}(z)\,.$$

The proofs, consisting only of straight applications of (120.8), can be omitted here. The case $\nu = 1/2$ leads to elementary functions as expressed by

Lemma II.

$$J_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin z\,, \qquad I_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sinh z\,.$$

These formulae can be read off directly from (120.8) and (120.9). With the abbreviation

$$\lambda_n = \sqrt{n - \frac{1}{24}}\,, \qquad K = \frac{\pi}{k}\sqrt{\frac{2}{3}}$$

we have therefore, after application of Lemmas I and II,

$$\frac{d}{dn}\,\frac{\sinh K\lambda_n}{\lambda_n} = \frac{\sqrt{\pi}\,K^3}{2^3}\,\frac{I_{3/2}(K\lambda_n)}{\left(\frac{K}{2}\lambda_n\right)^{3/2}}\,.$$

If we insert this result in (120.7) we obtain

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} k^{1/2} A_k(n)\,\frac{d}{dn}\,\frac{\sinh \frac{\pi}{k}\sqrt{\frac{2}{3}}\,\lambda_n}{\lambda_n}\,, \tag{120.10}$$

our final formula for $p(n)$.

We can again check easily that the infinite series is absolutely convergent. Indeed, with $C = \pi\sqrt{2/3}$ we have

$$\frac{\sinh \frac{C}{k}\lambda_n}{\lambda_n} = \frac{C}{k} + \frac{1}{6}\left(\frac{C}{k}\right)^3 \lambda_n^2 + \cdots,$$

$$\frac{d}{dn}\,\frac{\sinh \frac{C}{k}\lambda_n}{\lambda_n} = \frac{1}{6}\left(\frac{C}{k}\right)^3 + \cdots = O(k^{-3})\,,$$

$$\sum_{k=1}^{\infty} k^{1/2} A_k(n)\,\frac{d}{dn}\,\frac{\sinh \frac{C}{k}\lambda_n}{\lambda_n} = O\left(\sum_{k=1}^{\infty} k^{-3/2}\right).$$

121. A partial sum with error term

Using the explicit evaluations of the previous section one can rewrite (120.10) as

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{N} k^{1/2} A_k(n) \frac{d}{dn} \frac{\sinh \frac{\pi}{k}\sqrt{\frac{2}{3}}\,\lambda_n}{\lambda_n} + O(e^{4\pi n/N^2} N^{-1/2})\,.$$

If we choose here, as Hardy and Ramanujan do, $N = [\alpha n^{1/2}]$, with fixed α, then

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\alpha n^{1/2}} k^{1/2} A_k(n) \frac{d}{dn} \frac{\sinh \frac{\pi}{k}\sqrt{\frac{2}{3}}\,\lambda_n}{\lambda_n} + O(n^{-1/4})\,.$$

This is not quite yet the Hardy-Ramanujan formula. We have

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

and thus

$$p(n) = \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{\alpha n^{1/2}} k^{1/2} A_k(n) \frac{d}{dn} \frac{\exp \frac{\pi}{k}\sqrt{\frac{2}{3}}\,\lambda_n}{\lambda_n} + R + O(n^{-1/4})$$

with

$$R = O\left(\sum_{k=1}^{\alpha n^{1/2}} k^{3/2} \frac{d}{dn} \frac{\exp\left(-\frac{\pi}{k}\sqrt{\frac{2}{3}}\,\lambda_n\right)}{\lambda_n}\right).$$

Now with $C = \pi\sqrt{2/3}$ for abbreviation,

$$\frac{d}{dn}\frac{e^{-(C/k)\lambda_n}}{\lambda_n} = -e^{-(C/k)\lambda_n}\left(\frac{C}{2k\lambda_n^2} + \frac{1}{2\lambda_n^3}\right)$$

so that

$$\begin{aligned} R &= O\left(\sum_{k=1}^{\alpha n^{1/2}} k^{3/2} \frac{1}{\lambda_n^2}\left(\frac{1}{k} + \frac{1}{\lambda_n}\right)\right) \\ &= O\left(n^{-1}\sum_{1}^{\alpha n^{1/2}} k^{1/2}\right) + O\left(n^{-3/2}\sum_{1}^{\alpha n^{1/2}} k^{3/2}\right) \\ &= O(n^{-1}\cdot n^{3/4}) + O(n^{-3/2}\cdot n^{5/4}) = O(n^{-1/4})\,, \end{aligned}$$

and the Hardy-Ramanujan formula with error term $O(n^{-1/4})$ emerges.

For numerical computation by means of (120.10) an estimation of the error term after N terms is necessary, with a known constant implied by the symbol $O(n^{-1/4})$. For this purpose it would be possible to follow the estimation in §§ 119, 120 in detail. A better way is to go to the convergent series (120.10).

We write

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{N} k^{1/2} A_k(n) \frac{d}{dn} \frac{\sinh \frac{C}{k} \lambda_n}{\lambda_n} + R_N , \tag{121.1}$$

where

$$R_N = \frac{1}{\pi\sqrt{2}} \sum_{k=N+1}^{\infty} k^{1/2} A_k(n) \frac{d}{dn} \frac{\sinh \frac{C}{k} \lambda_n}{\lambda_n} \tag{121.2}$$

with $C = \pi\sqrt{2/3}$.

We are going to use now

$$|A_k(n)| < 2k^{3/4} , \tag{121.3}$$

which we shall prove later. Then

$$\begin{aligned}
|R_N| &< \frac{\sqrt{2}}{\pi} \sum_{N+1}^{\infty} k^{5/4} \left| \frac{d}{dn} \frac{\sinh \frac{C}{k} \lambda_n}{\lambda_n} \right| \\
&= \frac{\sqrt{2}}{\pi} \sum_{N+1}^{\infty} k^{5/4} \frac{d}{dn} \sum_{\nu=0}^{\infty} \frac{\left(\frac{C}{k}\right)^{2\nu+1} \left(n - \frac{1}{24}\right)^{\nu}}{(2\nu+1)!} \\
&= \frac{\sqrt{2}}{\pi} \sum_{N+1}^{\infty} k^{5/4} \sum_{\nu=1}^{\infty} \frac{\nu}{(2\nu+1)!} \left(\frac{C}{k}\right)^{2\nu+1} \lambda_n^{2\nu-2} \\
&< \frac{\sqrt{2}}{\pi} \sum_{\nu=1}^{\infty} \frac{\nu}{(2\nu+1)!} C^{2\nu+1} \lambda_n^{2\nu-2} \int_N^{\infty} \frac{dk}{k^{2\nu-1/4}} \\
&= \frac{\sqrt{2}}{\pi} \sum_{\nu=1}^{\infty} \frac{\nu}{(2\nu+1)!\,(2\nu-5/4)} C^{2\nu+1} \frac{\lambda_n^{2\nu-2}}{N^{2\nu-5/4}} \\
&= \frac{\sqrt{2}}{\pi} \frac{N^{9/4}}{\lambda_n^3} \sum_{\nu=1}^{\infty} \frac{\nu}{(2\nu+1)!\,(2\nu-5/4)} \left(\frac{C\lambda_n}{N}\right)^{2\nu+1} \\
&< \frac{\sqrt{2}}{\pi} \frac{N^{9/4}}{\lambda_n^3} \left\{ \frac{2}{9} \left(\frac{C\lambda_n}{N}\right)^3 + \frac{8}{11} \sum_{\nu=2}^{\infty} \frac{(C\lambda_n/N)^{2\nu+1}}{(2\nu+1)!} \right\} \\
&= \frac{\sqrt{2}}{\pi} \frac{N^{9/4}}{\lambda_n^3} \left\{ \frac{2}{9} \left(\frac{C\lambda_n}{N}\right)^3 + \frac{8}{11} \left(\sinh \frac{C\lambda_n}{N} - \frac{C\lambda_n}{N} - \frac{1}{6} \left(\frac{C\lambda_n}{N}\right)^3 \right) \right\} \\
&= \frac{\sqrt{2}}{\pi} \frac{N^{9/4}}{\lambda_n^3} \left\{ \frac{10}{99} \left(\frac{C\lambda_n}{N}\right)^3 + \frac{8}{11} \left(\sinh \frac{C\lambda_n}{N} - \frac{C\lambda_n}{N} \right) \right\}.
\end{aligned}$$

The last product increases with λ_n/N, and therefore, since $\lambda_n < n^{1/2}$,

$$R_N < \frac{\sqrt{2}}{\pi}\left\{\frac{10}{99} C^3 N^{-3/4} + \frac{8}{11} N^{-3/4}\left(\frac{N}{\sqrt{n}}\right)^3 \left(\sinh\frac{C\sqrt{n}}{N} - \frac{C\sqrt{n}}{N}\right)\right\}.$$

We replace also

$$\frac{\sinh \frac{C}{k}\lambda_n}{\lambda_n} \text{ by } \frac{1}{2}\frac{\exp\frac{C}{k}\lambda_n}{\lambda_n}$$

in the finite sum of (121.1), producing thereby an additional error

$$R_N^* = \frac{1}{2\pi\sqrt{2}}\sum_{k=1}^{N} k^{1/2} A_k(n)\frac{d}{dn}\frac{\exp\left(-\frac{C}{k}\lambda_n\right)}{\lambda_n}.$$

By virtue of (121.3) we obtain thus the estimate

$$\begin{aligned}|R_N^*| &< \frac{1}{\pi\sqrt{2}}\sum_{k=1}^{N} k^{5/4}\left(\frac{\exp\left(-\frac{C}{k}\lambda_n\right)}{2\lambda_n^2}\frac{C}{k} + \frac{\exp\left(-\frac{C}{k}\lambda_n\right)}{2\lambda_n^3}\right) \\ &< \frac{C}{2\pi\sqrt{2}\,\lambda_n^2}\exp\left(-\frac{C}{N}\lambda_n\right) \\ &\quad\times\int_0^{N+1} k^{1/4}\,dk + \frac{1}{2\pi\sqrt{2}\,\lambda_n^3}\exp\left(-\frac{C}{N}\lambda_n\right)\int_0^{N+1} k^{5/4}\,dk \\ &< \frac{2\exp\left(-\frac{C\sqrt{n-1}}{N}\right)}{(n-1)}(N+1)^{5/4}\left(\frac{1}{5\sqrt{3}} + \frac{1}{9\pi\sqrt{2}}\frac{N+1}{\sqrt{n-1}}\right).\end{aligned}$$

The final formula with remainder term is therefore

$$p(n) = \frac{1}{2\pi\sqrt{2}}\sum_{k=1}^{N} k^{1/2} A_k(n)\frac{d}{dn}\frac{\exp\frac{C}{k}\lambda_n}{\lambda_n} + \vartheta N^{-3/4}\{S_1 + S_2\} \tag{121.4}$$

with $|\vartheta| < 1$ and

$$S_1 = \frac{10\sqrt{2}}{99\pi}C^3 + \frac{8\sqrt{2}}{11\pi}\left(\frac{N}{\sqrt{n}}\right)^3\left(\sinh\frac{C\sqrt{n}}{N} - \frac{C\sqrt{n}}{N}\right),$$

$$S_2 = 2\exp\left(-\frac{C\sqrt{n-1}}{N}\right)\cdot\frac{(N+1)^2}{n-1}\left(\frac{1}{5\sqrt{3}} + \frac{1}{9\pi\sqrt{2}}\frac{N+1}{\sqrt{n-1}}\right).$$

It is clear that for $\sqrt{n}/N$ bounded, S_1 and S_2 will remain bounded, and the error term will be of order $O(N^{-3/4})$ or $O(n^{-3/8})$. The error will thus become less than $1/2$ for suitably large n and N. On the other hand, $p(n)$ is an integer, so that in the formula (121.4) $p(n)$ appears as the nearest integer to the sum $\sum_{k=1}^{N}$.

Let us take in particular

$$N = \left[2\sqrt{n}/3\right] \tag{121.5}$$

and consider only $n \geqq 24^2 = 576$, since $p(n)$ is tabulated for $n \leqq 600$ (MacMahon [38], Gupta [15]). We have under these conditions

$$\frac{3}{2} \leqq \frac{\sqrt{n}}{N} \leqq \frac{8}{5}, \quad \frac{\sqrt{n-1}}{N} \geqq \frac{15}{48}\sqrt{23}, \quad \frac{N+1}{\sqrt{n-1}} \leqq \frac{17}{5\sqrt{23}}.$$

We observe also that $x^{-3}(\sinh x - x)$ is monotone increasing with x.

Numerical computations give

$$S_1 < 3.3077, \qquad S_2 < 0.0029,$$

and we see that the error term in (121.4) under these conditions is in absolute value less than

$$N^{-3/4} \cdot 3.3106 < 1/2 \quad \text{for } n \geqq 576, \tag{121.6}$$

which suffices to compute $p(n)$ exactly by (121.4) if N is chosen according to (121.5).

Closer inspection of S_1 and S_2 shows that even $N = \left[\frac{8C}{3}\frac{\sqrt{n}}{\log n}\right]$ terms in (121.4) will make the error term tend to zero for increasing n.

We use now (121.4) also for a rather rough asymptotic determination of $p(n)$ by fixing $N = 2$. We obtain after a short computation, putting the summand for $k = 2$ into the error term,

$$p(n) = \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\frac{\exp C\lambda_n}{\lambda_n} + O\left(\frac{\exp\frac{C}{2}\lambda_n}{n}\right),$$

and still less precisely,

$$p(n) \sim \frac{1}{4\sqrt{3}}\frac{e^{C\lambda_n}}{n}, \quad \lambda_n = \sqrt{n - \frac{1}{24}}, \quad C = \pi\sqrt{\frac{2}{3}}.$$

122. Discussion of the sums $A_k(n)$. A new expression for ω_{hk}

For the actual computation of $p(n)$ by means of the formula (121.4), the $A_k(n)$ are of utmost importance. They are defined through (120.5) and are in principle known, since the ω_{hk} are determined by (118.41), where the Dedekind sums can be computed by means of formulae in §§ 68—70. However, this would be a rather laborious computation. D. H. Lehmer [35] has given a factorization of the $A_k(n)$ into rather simple factors, which make computations feasible. Later A. Selberg gave a new formula for the $A_k(n)$ from which Lehmer's results can then be obtained easily (Whiteman [78], Rademacher [51]).

Our discussion of the ω_{hk}, following Dedekind, is based on the product expansion of $\eta(\tau)$, whereas Selberg's formula has its origin in the infinite series for $\eta(\tau)$.

Using (118.1), (118.3), and (118.42) we have

$$\eta\left(\frac{h'}{k}+\frac{i}{kz}\right)=\omega_{hk}\, e^{(\pi i/12k)\,(h'-h)}\sqrt{z}\;\eta\left(\frac{h}{k}+\frac{iz}{k}\right), \tag{122.1}$$

with $(h,k)=1$, $hh'\equiv -1 \pmod k$. We know also, e.g. from (118.42) or from Theorem § 65, that ω_{hk} is a root of unity, so that

$$\omega_{hk}^{-1}=\overline{\omega_{hk}}\,. \tag{122.2}$$

Now we have, after Euler's pentagonal number formula,

$$\eta(\tau)=e^{\pi i\tau/12}\prod_1^\infty(1-e^{2\pi i m\tau})=e^{\pi i\tau/12}\sum_{\lambda=-\infty}^{\infty}(-1)^\lambda e^{\pi i\lambda(3\lambda-1)\tau}$$

$$=\sum_{\lambda=-\infty}^{\infty}(-1)^\lambda e^{3\pi i(\lambda-1/6)^2\tau}.$$

We put here $\tau=h/k+iz/k$, $\lambda=2kq+j$, $\operatorname{Re} z>0$, and obtain

$$\eta\left(\frac{h}{k}+\frac{iz}{k}\right)=\sum_{q=-\infty}^{\infty}\sum_{j=0}^{2k-1}(-1)^j e^{\frac{3\pi i h}{k}\left(2kq+j-\frac16\right)^2} e^{\frac{-3\pi z}{k}\left(2kq+j-\frac16\right)^2}$$

$$=\sum_{j=0}^{2k-1}(-1)^j e^{\frac{3\pi i h}{k}\left(j-\frac16\right)^2}\sum_{q=-\infty}^{\infty} e^{-\frac{3\pi z}{k}\left(2kq+j-\frac16\right)^2}. \tag{122.3}$$

We use now (36.2) on the inner sum here with

$$t=12kz\,,\qquad \alpha=\frac{1}{2k}\left(j-\frac16\right)$$

with the result

$$\sum_q e^{-\frac{3\pi z}{k}\left(2kq+j-\frac16\right)^2}=\frac{1}{2\sqrt{3kz}}\sum_{m=-\infty}^{\infty} e^{\frac{-\pi m^2}{12kz}+\frac{\pi i m}{k}\left(j-\frac16\right)}$$

so that (122.3) goes over into

$$\eta\left(\frac{h}{k}+\frac{iz}{k}\right)=\frac{1}{2\sqrt{3kz}}\sum_{j=0}^{2k-1}(-1)^j e^{\frac{3\pi ih}{k}\left(j-\frac{1}{6}\right)^2}\sum_{m=-\infty}^{\infty} e^{\frac{-\pi m^2}{12kz}+\frac{\pi im}{k}\left(j-\frac{1}{6}\right)}. \tag{122.4}$$

We apply now (122.4) on the right-hand side of (122.1) and (122.3) on the left-hand side; in the latter application we have, of course, to change in (122.3) h into h' and z into $1/z$. After some cancellation we obtain thus

$$\sum_{j=0}^{2k-1}(-1)^j e^{\frac{\pi ih'}{k}j(3j-1)}\sum_{q=-\infty}^{\infty} e^{-\frac{3\pi}{kz}\left(2kq+j-\frac{1}{6}\right)^2}$$
$$=\omega_{hk}\frac{1}{2\sqrt{3k}}\sum_{j=0}^{2k-1}(-1)^j e^{\frac{\pi ih}{k}j(3j-1)}\sum_{m=-\infty}^{\infty} e^{\frac{-\pi m^2}{12kz}+\frac{\pi im}{k}\left(j-\frac{1}{6}\right)}.$$

Here we have in both sides power series in $Z=e^{-\pi/12kz}$. The first power of Z appears on both sides, namely for $q=j=0$ on the left and $m=\pm 1$ on the right. Since the coefficients must agree, we infer

$$1=\omega_{hk}\frac{1}{2\sqrt{3k}}\sum_{j=0}^{2k-1}(-1)^j e^{\frac{\pi ih}{k}j(3j-1)}\left(e^{\frac{\pi i}{k}\left(j-\frac{1}{6}\right)}+e^{\frac{-\pi i}{k}\left(j-\frac{1}{6}\right)}\right),$$

$$\omega_{hk}=\frac{1}{2\sqrt{3k}}\sum_{j \bmod 2k}(-1)^j e^{-\frac{\pi ih}{k}j(3j-1)}\left(e^{\frac{\pi i}{k}\left(j-\frac{1}{6}\right)}+e^{-\frac{\pi i}{k}\left(j-\frac{1}{6}\right)}\right). \tag{122.5}$$

123. A lemma by Whiteman and the Selberg sum

It is not obvious that the right-hand member of (122.5) represents a root of unity. To show this one would have to use the theory of Gaussian sums (cf. [8]).

Of course, in the definition of ω_{hk} it is always assumed that $(h,k)=1$. However, the right-hand member of (122.5) also makes sense for $(h,k)\neq 1$. This case is described by the

Lemma (Whiteman [78]). *If* $(h,k)=d>1$, *then*

$$\Omega_{hk}=\sum_{j \bmod 2k}(-1)^j e^{-\frac{\pi ih}{k}j(3j-1)}e^{\pm\frac{\pi ij}{k}}=0.$$

Proof. We put

$$h=Hd\,,\quad k=Kd\,,\quad (H,K)=1$$

and

$$j=2Kl+r\,,\quad 0\leqq l<d\,,\quad 0\leqq r<2K\,.$$

Thus

$$\Omega_{hk} = \sum_{l=0}^{d-1} \sum_{r=0}^{2K-1} (-1)^r e^{-\frac{\pi i H}{K} r(3r-1)} e^{\pm \frac{\pi i}{Kd}(2lK+r)}$$

$$= \sum_{r=0}^{2K-1} (-1)^r e^{-\frac{\pi i H}{K} r(3r-1)} e^{\pm \frac{\pi i r}{Kd}} \sum_{l=0}^{d-1} e^{\pm \frac{2\pi i l}{d}} = 0$$

since the inner sum vanishes. ▯

Thus the sum in the right member of (122.5) represents ω_{hk} when $(h, k) = 1$ and 0 otherwise. If we therefore replace ω_{hk} by this sum in (120.5), we can sum over a *full* residue system modulo k and have thus

$$A_k(n) = \frac{1}{2\sqrt{3k}} \Bigg\{ \sum_{h \bmod k} e^{\frac{-2\pi i h n}{k}} \sum_{j \bmod 2k} (-1)^j e^{-\frac{\pi i h}{k} j(3j-1) + \frac{\pi i}{k}\left(j-\frac{1}{6}\right)}$$

$$+ \sum_{h \bmod k} e^{-\frac{2\pi i h n}{k}} \sum_{j \bmod 2k} (-1)^j e^{-\frac{\pi i h}{k} j(3j-1) - \frac{\pi i}{k}\left(j-\frac{1}{6}\right)} \Bigg\}$$

$$= \frac{1}{2\sqrt{3k}} \Bigg\{ e^{\frac{-\pi i}{6k}} \sum_{j \bmod 2k} (-1)^j e^{\frac{\pi i j}{k}} \sum_{h \bmod k} e^{-\frac{2\pi i h}{k}\left(n + \frac{j(3j-1)}{2}\right)}$$

$$+ e^{\frac{\pi i}{6k}} \sum_{j \bmod 2k} (-1)^j e^{\frac{-\pi i j}{k}} \sum_{h \bmod k} e^{-\frac{2\pi i h}{k}\left(n + \frac{j(3j-1)}{2}\right)} \Bigg\}$$

$$= \frac{\sqrt{k}}{2\sqrt{3}} \Bigg\{ e^{\frac{-\pi i}{6k}} \sum_{\substack{j \bmod 2k \\ \frac{j(3j-1)}{2} \equiv -n \pmod{k}}} (-1)^j e^{\frac{\pi i j}{k}}$$

$$+ e^{\frac{\pi i}{6k}} \sum_{\substack{j \bmod 2k \\ \frac{j(3j-1)}{2} \equiv -n \pmod{k}}} (-1)^j e^{-\frac{\pi i j}{k}} \Bigg\}, \tag{123.1}$$

and finally

$$A_k(n) = \sqrt{\frac{k}{3}} \sum_{\substack{j \bmod 2k \\ \frac{j(3j-1)}{2} \equiv -n \pmod{k}}} (-1)^j \cos \frac{(6j-1)\pi}{6k}. \tag{123.2}$$

This is the formula of A. Selberg (Whiteman [78]). It makes it evident that the $A_k(n)$ are real numbers. It is a little more convenient for the discussion of the $A_k(n)$ to write the sum in a slightly different way. The congruence condition

$$\frac{1}{2} j(3j-1) \equiv -n \pmod{k}$$

is equivalent with

$$(6j-1)^2 \equiv -24n+1 \pmod{24k}\,.$$

Let us put here and in the sequel

$$\nu = -24n+1\,. \tag{123.3}$$

Then (123.1) goes over into

$$A_k(n) = \frac{1}{2}\sqrt{\frac{k}{3}}\left\{\sum_{\substack{j \bmod 2k\\ (6j-1)^2\equiv\nu(24k)}} (-1)^j e^{\frac{\pi i}{6k}(6j-1)} + \sum_{\substack{j \bmod 2k\\ (6j+1)^2\equiv\nu(24k)}} (-1)^j e^{\frac{\pi i}{6k}(6j+1)}\right\},$$

where we have in the second sum j replaced by $2k-j$, so that, in brief,

$$A_k(n) = \frac{1}{2}\sqrt{\frac{k}{3}} \sum_{\substack{6j\pm1 \pmod{12k}\\ (6j\pm1)^2\equiv\nu \pmod{24k}}} (-1)^j e^{\frac{\pi i}{6k}(6j\pm1)}\,.$$

Let us put

$$6j\pm1 = l$$

so that always $(l, 6) = 1$. From l we return to j by

$$j = \frac{l\mp1}{6} \quad\text{or}\quad j = \left\{\frac{l}{6}\right\},$$

where by $\{x\}$ we mean the nearest integer to x. In this notation we obtain

$$A_k(n) = \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{\substack{l \bmod 24k\\ l^2\equiv\nu \pmod{24k}}} (-1)^{\left\{\frac{l}{6}\right\}} e^{\frac{\pi i l}{6k}}\,. \tag{123.4}$$

The condition $(l, 6) = 1$ is here already implied by $l^2 \equiv \nu \pmod{24k}$ in view of the definition (123.3). We shall from now on write

$$B_k(\nu) = A_k(n)\,, \tag{123.5}$$

since the sum in (123.4) contains explicitly only the variable ν. We remember that always

$$\nu \equiv 1 \pmod{24}\,.$$

124. Different cases of $B_k(\nu)$ according to k

The function $(-1)^{\{l/6\}}$, $(l, 6) = 1$, is evidently periodic of period 12. We have the list

l	1	5	7	11
$(-1)^{\{l/6\}}$	$+1$	-1	-1	$+1$

We compare this with the following Legendre-Jacobi symbols for the same arguments 1, 5, 7, 11:

$(l/3)$	$+1$	-1	$+1$	-1
$(-1/l)$	$+1$	$+1$	-1	-1

These tables show that

$$(-1)^{\{l/6\}} = \left(\frac{l}{3}\right)\left(\frac{-1}{l}\right) = \left(\frac{3}{l}\right) \tag{124.1}$$

so that we can write

$$\begin{aligned} B_k(\nu) &= \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{l^2 \equiv \nu \pmod{24k}} \left(\frac{l}{3}\right)\left(\frac{-1}{l}\right) e^{\frac{\pi i l}{6k}} \\ &= \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{l^2 \equiv \nu \pmod{24k}} \left(\frac{3}{l}\right) e^{\frac{\pi i l}{6k}} . \end{aligned} \tag{124.2}$$

We define now

$$d = (24, k^3) , \tag{124.21}$$

which clearly takes only the four values 1, 3, 8, 24. We write also

$$e = \frac{24}{d} , \qquad e = 24, 8, 3, 1 ,$$

so that

$$(e, d) = (e, k) = 1 .$$

We utilize the numbers d and e to break the condition of summation in the right-hand member of (124.2) into the two conditions

$$l^2 \equiv \nu \pmod{dk} ,$$

$$l^2 \equiv \nu \equiv 1 \pmod{e} ,$$

where in turn the second condition can be replaced by $(l, e) = 1$. If we put now $l = er + dkj$, then the condition "l modulo $24k$" is equivalent to "r modulo dk" and "j modulo e" so that

$$B_k(\nu) = \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{(er)^2 \equiv \nu \bmod dk} \sum_{\substack{j \bmod e \\ (j,e)=1}} \left(\frac{er+dkj}{3}\right)\left(\frac{-1}{er+dkj}\right) e^{\frac{\pi i(er+dkj)}{6k}}$$

$$= \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{(er)^2 \equiv \nu(dk)} e^{\frac{\pi i e r}{6k}} S_k(r)\,,$$

where

$$S_k(r) = \sum_{j \bmod e}{}' \left(\frac{er+dkj}{3}\right)\left(\frac{-1}{er+dkj}\right) e^{\frac{\pi i d j}{6}} = \sum_{j \bmod e}{}' \left(\frac{3}{er+dkj}\right) e^{\frac{\pi i d j}{6}},$$

where the dash (′) indicates $(j, e) = 1$.

We consider now the four cases $d = 1, 3, 8, 24$ separately.

(I) $d = 1$, $e = 24$, i.e. $(k, 6) = 1$. Here we have

$$S_k(r) = \sum_{j \bmod 24}{}' \left(\frac{3}{kj}\right) e^{\frac{\pi i j}{6}} = \left(\frac{3}{k}\right) \sum_{j \bmod 24}{}' \left(\frac{3}{j}\right) e^{\frac{\pi i j}{6}}$$

$$= 2\left(\frac{3}{k}\right) \sum_{j \bmod 12}{}' (-1)^{\left\{\frac{j}{6}\right\}} e^{\frac{\pi i j}{6}}$$

$$= 2\left(\frac{3}{k}\right)\left((-1)^0 e^{\frac{\pi i}{6}} + (-1)^1 e^{\frac{5\pi i}{6}} + (-1)^1 e^{\frac{7\pi i}{6}} + (-1)^2 e^{\frac{11\pi i}{6}}\right)$$

$$= 4\left(\frac{3}{k}\right)\left(\cos\frac{\pi}{6} - \cos\frac{5\pi}{6}\right) = 4\left(\frac{3}{k}\right)\sqrt{3}$$

so that we have now

$$B_k(\nu) = \left(\frac{3}{k}\right)\sqrt{k} \sum_{(24r)^2 \equiv \nu(k)} e^{\frac{4\pi i r}{k}}, \quad (k, 6) = 1\,. \tag{124.3}$$

(II) $d = 3$, $e = 8$, i.e. $(k, 6) = 3$.

$$S_k(r) = \sum_{j \bmod 8}{}' \left(\frac{8r}{3}\right)\left(\frac{-1}{3kj}\right) e^{\frac{\pi i j}{2}} = \left(\frac{r}{3}\right)\left(\frac{-1}{k}\right) \sum_{j \bmod 8}{}' \left(\frac{-1}{j}\right) i^j$$

$$= \left(\frac{r}{3}\right)\left(\frac{-1}{k}\right)(i - i^3 + i^5 - i^7) = 4\left(\frac{r}{3}\right)\left(\frac{-1}{k}\right) i\,,$$

which yields

$$B_k(\nu) = i\left(\frac{-1}{k}\right)\sqrt{\frac{k}{3}} \sum_{(8r)^2\equiv\nu(3k)} \left(\frac{r}{3}\right) e^{\frac{4\pi i r}{3k}}, \quad (k, 6) = 3. \qquad (124.4)$$

(III) $d = 8$, $e = 3$, i.e. $(k, 6) = 2$.

$$S_k(r) = \sum_{j \bmod 3}{}' \left(\frac{2kj}{3}\right)\left(\frac{-1}{3r}\right) e^{\frac{4\pi i j}{3}} = \left(\frac{k}{3}\right)\left(\frac{-1}{r}\right)\left(e^{\frac{4\pi i}{3}} - e^{\frac{8\pi i}{3}}\right)$$

$$= -i\left(\frac{k}{3}\right)\left(\frac{-1}{r}\right)\sqrt{3},$$

$$B_k(\nu) = \frac{1}{4i}\left(\frac{k}{3}\right)\sqrt{k} \sum_{(3r)^2\equiv\nu(8k)} \left(\frac{-1}{r}\right) e^{\frac{\pi i r}{2k}}, \quad (k, 6) = 2. \qquad (124.5)$$

(IV) Finally for $d = 24$, $e = 1$, i.e. $6\,|\,k$, the sum S_k reduces to a single term

$$S_k(r) = \left(\frac{r}{3}\right)\left(\frac{-1}{r}\right) = \left(\frac{3}{r}\right)$$

with the consequence

$$B_k(\nu) = \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{r^2\equiv\nu(24k)} \left(\frac{3}{r}\right) e^{\frac{\pi i r}{6k}}, \quad 6\,|\,k, \qquad (124.6)$$

where nothing is gained over (124.2).

125. Multiplicativity of $B_k(\nu)$

We are going to establish now a rule

$$B_k(\nu) = B_{k_1k_2}(\nu) = B_{k_1}(\nu_1)\cdot B_{k_2}(\nu_2)$$

for $(k_1, k_2) = 1$ and suitable ν_1, ν_2. Two cases have to be considered.

(A) At least one of the k_1, k_2 is prime to 6, say $(k_1, 6) = 1$.

(B) Both k_1 and k_2 have a factor in common with 6, in which case we may assume $2\,|\,k_1$, $3\,|\,k_2$.

In case (A) we use (124.3) for $B_{k_1}(\nu_1)$ and the general equation (124.2) for $B_{k_2}(\nu_2)$, so that we have

$$B_{k_1}(\nu_1)\cdot B_{k_2}(\nu_2) = \frac{1}{4}\sqrt{\frac{k_1k_2}{3}} \sum_{(24r)^2\equiv\nu_1(k_1)} \sum_{l^2\equiv\nu_2(24k_2)} \left(\frac{3}{k_1\, l}\right) e^{\frac{\pi i(24k_2r + k_1l)}{6k_1k_2}}. \qquad (125.1)$$

If we put

$$t \equiv 24k_2 r + k_1 l \pmod{24k_1 k_2}$$

we have

$$t^2 \equiv 24^2 k_2^2 r^2 + k_1^2 l^2 \pmod{24k_1 k_2}\,.$$

We define now a number ν satisfying

$$\nu \equiv k_2^2 \nu_1 \pmod{k_1}\,, \qquad \nu \equiv k_1^2 \nu_2 \pmod{24k_2}\,. \tag{125.2}$$

Then the conditions of summation in (125.1) can be contracted into

$$t^2 \equiv \nu \pmod{24k_1 k_2}\,.$$

Moreover

$$\left(\frac{3}{k_1 l}\right) = \left(\frac{3}{24k_2 r + k_1 l}\right) = \left(\frac{3}{t}\right),$$

so that (125.1) goes over into

$$B_{k_1}(\nu_1)\cdot B_{k_2}(\nu_2) = \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{t^2\equiv\nu(24k)} \left(\frac{3}{t}\right) e^{\frac{\pi i t}{6k}} = B_k(\nu) \tag{125.3}$$

with $k = k_1 k_2$. This equation can also be read in the opposite sense: if we start with $B_k(\nu)$ and break k into coprime factors, $k = k_1 k_2$ with $(k_1, 6) = 1$, we can determine ν_1 and ν_2 by (125.2) so that (125.3) holds. This formula gives thus a factorization of $B_k(\nu)$.

In case (B), $2|k_1$, $3|k_2$, we apply (124.5) to $B_{k_1}(\nu_1)$ and (124.4) to $B_{k_2}(\nu_2)$, obtaining thus

$$B_{k_1}(\nu_1)\cdot B_{k_2}(\nu_2)$$
$$= \frac{1}{4}\sqrt{\frac{k_1 k_2}{3}} \sum_{(3r)^2\equiv\nu_1(8k_1)} \sum_{(8l)^2\equiv\nu_2(3k_2)} \left(\frac{-1}{k_2 r}\right)\left(\frac{k_1 l}{3}\right) e^{\frac{\pi i(3k_2 r + 8k_1 l)}{6k_1 k_2}}\,.$$

Let us put

$$t \equiv 3k_2 r + 8k_1 l \pmod{24k_1 k_2}\,.$$

Then each pair r, l in the ranges modulo $8k_1$ and $3k_2$ respectively determines one t modulo $24k_1 k_2$ and conversely. From

$$t^2 \equiv (3k_2 r)^2 + (8k_1 l)^2 \pmod{24k_1 k_2}$$

we infer that the summation conditions of the above formula are equivalent with

$$t^2 \equiv \nu \pmod{24k_1 k_2}$$

if and only if

$$k_2^2\nu_1 \equiv \nu \pmod{8k_1}\,, \qquad k_1^2\nu_2 \equiv \nu \pmod{3k_2}\,. \tag{125.4}$$

Moreover

$$\left(\frac{3}{t}\right) = \left(\frac{-1}{t}\right)\left(\frac{t}{3}\right) = \left(\frac{-1}{3k_2r + 8k_1l}\right)\left(\frac{3k_2r + 8k_1l}{3}\right) = \left(\frac{-1}{k_2r}\right)\left(\frac{k_1l}{3}\right)$$

so that

$$B_{k_1}(\nu)\cdot B_{k_2}(\nu) = \frac{1}{4}\sqrt{\frac{k}{3}}\sum_{t^2\equiv\nu \pmod{24k}}\left(\frac{3}{t}\right)e^{\frac{\pi i t}{6k}} = B_k(\nu) \tag{125.5}$$

with $k = k_1k_2$ under the conditions (125.4) for ν.

The definition (124.2) shows that $B_k(\nu)$ depends on ν only modulo $24k$. However, the equations (124.3), (124.4), and (124.5) give the more precise information that $B_k(\nu)$ depends on ν only modulo dk, where d is defined through (124.21). In view of this fact we can contract the statements (125.3), (125.5) for the cases (A) and (B) respectively into the following.

Theorem. *Let* $k = k_1k_2$, $(k_1, k_2) = 1$ *and* d, d_1, d_2 *be defined by* $d = (24, k^3)$, $d_1 = (24, k_1^3)$, $d_2 = (24, k_2^3)$ *(so that* $d = d_1d_2$*). Let moreover*

$$\nu \equiv \nu_1 \equiv \nu_2 \equiv 1 \pmod{24}\,.$$

Then

$$B_{k_1}(\nu_1)\cdot B_{k_2}(\nu_2) = B_k(\nu) \tag{125.6}$$

if

$$\nu \equiv k_2^2\nu_1 \pmod{d_1k_1}\,, \qquad \nu \equiv k_1^2\nu_2 \pmod{d_2k_2}\,.$$

For the application of this theorem to $A_k(n)$ we remember that $A_k(n) = B_k(1 - 24n)$.

126. Evaluation of $B_k(\nu)$ for a prime power

Let $k = p^\lambda$, where we assume first $p > 3$, so that $(p, 6) = 1$. We apply (124.3). If, first, $(\nu/p) = -1$, then the condition of summation cannot be fulfilled for any r, so that we find

$$B_{p^\lambda}(\nu) = 0\,. \tag{126.1}$$

If, secondly, $(\nu/p) = 1$, then

$$x^2 \equiv \nu \pmod{p^\lambda} \tag{126.2}$$

has exactly 2 solutions, as can be seen by induction with respect to λ.

Indeed, suppose

$$x_0^2 \equiv \nu \pmod{p^\lambda}$$

and assume

$$(x_0 + p^\lambda y)^2 \equiv \nu \pmod{p^{\lambda+1}}\,.$$

This yields for y the condition

$$x_0^2 + 2x_0 p^\lambda y \equiv \nu \pmod{p^{\lambda+1}}$$

or

$$\frac{x_0^2 - \nu}{p^\lambda} + 2x_0 y \equiv 0 \pmod{p}\,,$$

which gives just one solution for y. Since the hypothesis of the existence of exactly 2 solutions x is fulfilled for $\lambda = 1$, it follows now for all p^λ.

Thus

$$(24r)^2 \equiv \nu \pmod{p^\lambda}$$

has two solutions, which we may write as $\pm r$. In this case we obtain thus from (124.3)

$$B_{p^\lambda}(\nu) = 2\left(\frac{3}{p}\right)^\lambda p^{\lambda/2} \cos\frac{4\pi r}{p^\lambda}\,.$$

Thirdly we may have $(\nu/p) = 0$, i.e. $p \mid \nu$. For $\lambda = 1$

$$(24r)^2 \equiv 0 \pmod{p}$$

has only the solution $r = 0$, so that

$$B_p(\nu) = \left(\frac{3}{p}\right) p^{1/2}\,.$$

For $\lambda \geqq 2$ the congruence

$$(24r)^2 \equiv \nu \pmod{p^\lambda} \tag{126.21}$$

may not have any solution[1], in which case

$$B_{p^\lambda}(\nu) = 0\,. \tag{126.22}$$

If it has a solution r, then $p \mid r$, so that we have $r = pr_1$. But then

$$p(r_1 + jp^{\lambda-2})\,,\ j = 0, 1, \ldots, p-1$$

[1] E.g., in the case that ν is divisible only by the first power of p.

are all also solutions of (126.21), and the exponential sum in (124.3) becomes

$$e^{\frac{4\pi i r_1}{p^{\lambda-1}}} \sum_{j=0}^{p-1} e^{\frac{4\pi i j}{p}} = 0\,,$$

which again leads to (126.22).

Summarizing we state the

Theorem. *For* $k = p^\lambda$, $\lambda \geqq 1$, $p > 3$ *the following possibilities appear*

$$A_k(n) = B_{p^\lambda}(\nu) = \begin{cases} 0 & \textit{for} \left(\dfrac{\nu}{p}\right) = -1\,, \\ 2\left(\dfrac{3}{p}\right)^\lambda p^{\lambda/2} \cos \dfrac{4\pi r}{p^\lambda} & \textit{for} \left(\dfrac{\nu}{p}\right) = +1\,, \\ \left(\dfrac{3}{p}\right) p^{1/2}\,, & \textit{for}\ p \mid \nu\,,\ \lambda = 1\,, \\ 0\,, & \textit{for}\ p \mid \nu\,,\ \lambda > 1\,. \end{cases} \qquad (126.3)$$

Here r is a solution of $(24r)^2 \equiv \nu \pmod{p^\lambda}$.

There remain the cases $k = 3^\lambda$, $k = 2^\lambda$ to be investigated. We take first $k = 3^\lambda$. Since by definition $\nu \equiv 1 \pmod 3$, we have always $(\nu/p) = 1$, and the arguments connected with $(\nu/p) = 1$ carry over completely. That means we have always exactly two solutions of $(8r)^2 \equiv \nu \pmod{3^\lambda}$, which we can write as $\pm r$ with $(\pm r/3) = \pm 1$. Since we have here $(k, 6) = 3$, we employ (124.4) with the result

$$B_{3^\lambda}(\nu) = i(-1)^\lambda 3^{\frac{\lambda-1}{2}} \left(e^{\frac{4\pi i r}{3^{\lambda+1}}} - e^{-\frac{4\pi i r}{3^{\lambda+1}}}\right),$$

which yields the

Theorem.

$$A_{3^\lambda}(n) = B_{3^\lambda}(\nu) = 2(-1)^{\lambda+1} 3^{\frac{\lambda-1}{2}} \sin \frac{4\pi r}{3^{\lambda+1}} \qquad (126.4)$$

with

$$(8r)^2 \equiv \nu \pmod{3^\lambda}\,.$$

Finally, with $k = 2$, we have $(k, 6) = 2$ and have to consider, according to (124.5) the solutions of

$$(3r)^2 \equiv \nu \pmod{2^{\lambda+3}}\,. \qquad (126.5)$$

Now

$$x^2 \equiv \nu \pmod{2^{\lambda+3}}$$

has exactly four solutions. Indeed,

$$x^2 \equiv \nu \pmod{2^3}$$

has the solutions $x = 1, 3, 5, 7$, since by definition $\nu \equiv 1 \pmod 8$. We put then with an unknown y

$$(x + 4y)^2 \equiv \nu \pmod{2^4}$$

or

$$\frac{x^2 - \nu}{8} + xy \equiv 0 \pmod 2 .$$

This shows that to each x modulo 2^3 there is exactly one y mod 2, so that we obtain one solution $x' = x + 4y$ modulo 2^4. This argument can be repeated indefinitely, and in particular up to the modulus $2^{\lambda+3}$.

There are thus exactly 4 solutions r of (126.5). If r is one of them, then all four can be written as

$$\pm (r + h \cdot 2^{\lambda+2}) , \qquad h = 0, 1 .$$

This yields after (124.5) the expression

$$B_{2^\lambda}(\nu) = \frac{1}{4i} (-1)^\lambda 2^{\lambda/2} (-1)^{\frac{r-1}{2}}$$

$$\times \left\{ e^{\frac{\pi i r}{2^{\lambda+1}}} - e^{-\frac{\pi i r}{2^{\lambda+1}}} + e^{\frac{\pi i (r+2^{\lambda+2})}{2^{\lambda+2}}} - e^{-\frac{\pi i (r+2^{\lambda+2})}{2^{\lambda+1}}} \right\}$$

and thus the

Theorem. *For $k = 2^\lambda$ we have*

$$B_{2^\lambda}(\nu) = (-1)^{\lambda + \frac{r+1}{2}} 2^{\lambda/2} \sin \frac{\pi r}{2^{\lambda+1}}$$

with

$$(3r)^2 \equiv \nu \pmod{2^{\lambda+3}} .$$

We are now in a position to compute easily $A_k(n)$ for given k and n. As an example we choose

$$A_{15}(7) = B_{15}(1 - 24 \cdot 7) = B_{15}(-167) = B_5(\nu_1) \cdot B_3(\nu_2)$$

with

$$\nu_1 \equiv \nu_2 \equiv 1 \pmod{24}$$

and

$$3^2 \nu_1 \equiv -167 \pmod 5, \qquad 5^2 \nu_2 \equiv -167 \pmod{24 \cdot 3} ,$$

which have solutions

$$\nu_1 = -23,\ \nu_2 = 25\ .$$

Thus

$$A_{15}(7) = B_5(-23)\cdot B_3(25)\ .$$

Now

$$\left(\frac{-23}{5}\right) = -1\ ,$$

and therefore after (126.3), $B_5(-23) = 0$, so that

$$A_{15}(7) = 0\ .$$

This example corrects a statement about A_{15} in Ramanujan's Collected Papers [59], p. 307.

127. Estimations of $A_k(n)$

The results (126.3), (126.4) and (126.6) show that

$$\begin{aligned} &|A_{p^\lambda}(n)| < 2p^{\lambda/2}\ , \quad p > 3\ ,\\ &|A_{3^\lambda}(n)| < \frac{2}{\sqrt{3}}\, 3^{\lambda/2}\ ,\\ &|A_{2^\lambda}(n)| < 2^{\lambda/2}\ , \end{aligned} \tag{127.1}$$

from which we conclude

$$|A_k(n)| < k^{1/2} \prod_{p|k} 2 \leqq k^{1/2}\,\sigma_0(k)\ , \tag{127.2}$$

since

$$\sigma_0(k) = \sigma_0(p_1^{\lambda_1}\cdots p_r^{\lambda_r}) = (\lambda_1+1)\ldots(\lambda_r+1) \geqq 2^r\ .$$

Now it can be proven quite simply [19], Th. 315, p. 260, that $\sigma_0(k) = O(k^\varepsilon)$, which yields then

$$A_k(n) = O(k^{1/2+\varepsilon})\ . \tag{127.3}$$

If we choose a special value for $\varepsilon > 0$, we can easily supply a numerical constant in this estimate. We choose $\varepsilon = 1/4$.

Theorem. *For all k and n we have*

$$|A_k(n)| < 2k^{3/4}\ . \tag{127.4}$$

Proof. Let us have

$$k = 2^\alpha 3^\beta p_1^{\lambda_1} \dots p_r^{\lambda_r}$$

with

$$\alpha \geqq 0\,, \quad \beta \geqq 0\,, \quad \lambda_1, \lambda_2 \dots \lambda_r > 0\,.$$

Then, in view of (127.1) we have to prove only

$$\frac{2}{\sqrt{3}} \cdot 2^r < 2 \cdot 3^{\beta/4} p_1^{\lambda_1/4} \dots p_r^{\lambda_r/4} \text{ for } \beta > 0$$

and

$$2^r < 2 \cdot p_1^{\lambda_1/4} \dots p_r^{\lambda_r/4} \qquad \text{for } \beta = 0\,.$$

Since $2/\sqrt{3} < 3^{1/4}$, it suffices to verify only the latter inequality, and that one only for the critical case $\lambda_1 = \dots = \lambda_r = 1$, i.e.

$$\frac{2}{p_1^{1/4}} \cdot \frac{2}{p_2^{1/4}} \cdots \frac{2}{p_r^{1/4}} < 2.$$

Here, primes for which

$$\frac{2}{p^{1/4}} \leqq 1\,, \text{ i.e. } p \geqq 16$$

can be dismissed, and we have to verify only the case of the 4 primes $3 < p < 16$, i.e. $2^4 < 2(5 \cdot 7 \cdot 11 \cdot 13)^{1/4}$, which comes down to

$$2^{12} = 4096 < 5005 = 5 \cdot 7 \cdot 11 \cdot 13\,.$$

This proves (127.4) of the theorem, to which we have already referred in formula (121.3). ▯

128. The generating function $f(x)$ for $p(n)$

The explicit value for $p(n)$ in (121.1) can now be used for a new study of the generating function

$$f(x) = \prod_{m=1}^{\infty} \frac{1}{1 - x^m} = \sum_{n=0}^{\infty} p(n)\, x^n\,. \tag{128.0}$$

For brevity we write (120.10) as

$$p(n) = 2\pi \left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} A_k(n)\, k^{-5/2} L_{3/2}\left(\left(\frac{\pi}{12k}\right)^2 (24n - 1)\right) \tag{128.1}$$

with

$$L_{3/2}(y) = y^{-3/4} I_{3/2}(2\sqrt{y}) = \sum_{q=0}^{\infty} \frac{y^q}{q!\,\Gamma(3/2+q+1)}\,. \tag{128.11}$$

We remember, incidentally, from (116.3), that (120.10) remains also true for $n \leqq 0$, giving $p(0) = 1$ and $p(n) = 0$ für $n < 0$. Putting now (128.1) into the series for $f(x)$ we obtain

$$f(x) = 2\pi\left(\frac{\pi}{12}\right)^{3/2} \sum_{n=0}^{\infty} x^n \sum_{k=1}^{\infty} A_k(n)\, k^{-5/2} L_{3/2}\left(\left(\frac{\pi}{12k}\right)^2 (24n-1)\right). \tag{128.2}$$

Now

$$\left| L_{3/2}\left(\left(\frac{\pi}{12k}\right)^2 (24n-1)\right)\right| < \sum_{q=0}^{\infty} \frac{\left(\frac{\pi}{k\sqrt{6}}\right)^{2q}\left(|n| + \frac{1}{24}\right)^q}{q!\,\Gamma(1/2)\cdot 1/2\cdot 3/2 \cdots (q+3/2)}$$

$$= \frac{2}{\sqrt{\pi}} \sum_{q=0}^{\infty} \frac{\left(\frac{C}{k}\right)^{2q}\left(|n| + \frac{1}{24}\right)^q}{(2q+1)!\,(2q+3)}$$

$$< \frac{1}{\sqrt{\pi}} \sum_{p=0}^{\infty} \frac{\left(\frac{C}{k}\right)^p (\sqrt{|n|}+1)^p}{p!}$$

$$\leqq \frac{1}{\sqrt{\pi}}\, e^{C\sqrt{|n|+1}}\,,$$

where as before $C = \pi\sqrt{2/3}$. This estimate is valid also for $n < 0$, although we do not need this at this moment. Therefore, the series (128.2) with

$$A_k(n) = \sum_{h \bmod k}{}' \omega_{hk}\, e^{-\frac{2\pi i h n}{k}}$$

is majorized by

$$K \sum_{n=0}^{\infty} x^n \sum_{k=1}^{\infty} k^{-3/2}\, e^{C\sqrt{|n|+1}}\,,$$

which converges absolutely for $|x| < 1$. Thus the summations in n and k

can be interchanged:

$$f(x)=2\pi\left(\frac{\pi}{12}\right)^{3/2}\sum_{k=1}^{\infty}k^{-5/2}\sum_{n=0}^{\infty}x^n\sum_{h \bmod k}{}'\,\omega_{hk}\,e^{-\frac{2\pi i h n}{k}}L_{3/2}\left(\left(\frac{\pi}{12k}\right)^2(24n-1)\right)$$

$$=2\pi\left(\frac{\pi}{12}\right)^{3/2}\sum_{k=1}^{\infty}k^{-5/2}\sum_{h \bmod k}{}'\,\omega_{hk}\,\Phi_k\left(xe^{-\frac{2\pi i h}{k}}\right) \qquad (128.31)$$

with

$$\Phi_k(z)=\sum_{n=0}^{\infty}L_{3/2}\left(\frac{\pi^2}{6k^2}(n-\alpha)\right)z^n\,, \qquad |z|<1\,, \qquad \alpha=\frac{1}{24}\,. \quad (128.32)$$

129. Discussion of $\Phi_k(z)$

We shall prove that $\Phi_k(z)$ exists in the whole plane, with the exception of $z=1$, as an entire function of $1/(z-1)$. For this assertion we could quote a theorem of Wigert [81], $L_{3/2}(w)$ being an entire function of order less than 1, namely of order 1/2. We shall, however, give the explicit Laurent expansion of $\Phi_k(z)$ about $z=1$. For a further purpose we need also the expansion at $z=\infty$.

We have

$$\Phi_k(z)=\sum_{n=0}^{\infty}z^n\sum_{q=0}^{\infty}\frac{\left(\frac{\pi}{k\sqrt{6}}\right)^{2q}}{q!\,\Gamma\left(\frac{5}{2}+q\right)}(n-\alpha)^q$$

$$=\sum_{q=0}^{\infty}\frac{\left(\frac{\pi}{k\sqrt{6}}\right)^{2q}}{q!\,\Gamma\left(\frac{5}{2}+q\right)}\psi_q(z,\alpha)\,, \qquad (129.1)$$

where

$$\psi_q(z,\alpha)=\sum_{n=0}^{\infty}(n-\alpha)^q z^n\,, \qquad |z|<1\,. \qquad (129.11)$$

The function $\psi_q(z,\alpha)$ will turn out to be a polynomial of degree $q+1$ in $1/(z-1)$. Indeed,

$$\psi_0(z,\alpha)=\sum_{n=0}^{\infty}z^n=-\frac{1}{z-1}\,. \qquad (129.2)$$

Since

$$z\psi_q'(z,\alpha)=\sum_{n=0}^{\infty}n(n-\alpha)^q z^n\,,$$

we find

$$\Psi_{q+1}(z,\alpha) = \sum_{n=0}^{\infty} (n-\alpha)^{q+1} z^n = \bigl(1+(z-1)\bigr)\,\psi_q'(z,\alpha) - \alpha\psi_q(z,\alpha)$$

which together with (129.2) proves our assertion through induction. Thus the function

$$\Psi_q(\mu,\alpha) = (-1)^{q+1}\,\psi_q\left(\frac{\mu+1}{\mu},\alpha\right) \tag{129.3}$$

obtained by the substitution

$$z = \frac{\mu+1}{\mu}, \qquad \mu = \frac{1}{z-1}$$

is a polynomial of degree $r+1$ in μ.

Theorem.[1] *The generating function of the* $\psi_q(z,\alpha)$ *is*

$$\sum_{q=0}^{\infty} \psi_q(z,\alpha)\frac{w^q}{q!} = \frac{e^{-\alpha w}}{1-ze^w}. \tag{129.4}$$

It follows that

$$(-1)^q\,\psi_q\left(\frac{1}{z},\alpha\right) = \alpha^q - \psi_q(z,-\alpha) \tag{129.5}$$

and for

$$|1-z| \geqq \delta, \qquad 0<\delta<1,$$

$$|\psi_q(z)| < K_\delta \cdot \left(\frac{3}{\delta}\right)^q q!\,.$$

Proof. We have

$$\sum_{q=0}^{\infty} \psi_q(z,\alpha)\frac{w^q}{q!} = \sum_{q=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n-\alpha)^q w^q}{q!} z^n = \sum_{n=0}^{\infty} e^{(n-\alpha)w} z^n = \frac{e^{-\alpha w}}{1-ze^w},$$

which is (129.4). Replacing here z by $1/z$ we obtain

$$\sum_{q=0}^{\infty} \psi_q\left(\frac{1}{z},\alpha\right)\frac{w^q}{q!} = \frac{ze^{-\alpha w}}{z-e^w} = \frac{-ze^{-(\alpha+1)w}}{1-ze^{-w}} = e^{-\alpha w} - \frac{e^{-\alpha w}}{1-ze^{-w}}.$$

Comparison with (129.4) shows

$$\sum_{q=0}^{\infty} \psi_q\left(\frac{1}{z},\alpha\right)\frac{w^q}{q!} = \sum_{q=0}^{\infty} \frac{(-\alpha w)^q}{q!} - \sum_{q=0}^{\infty} \psi_q(z,-\alpha)\frac{(-w)^q}{q!}.$$

---

[1] The use of the generating function for the discussion of the $\psi_q(z,\alpha)$ I owe to a a remark by Professor C. Pisot.

The coefficient of w^q on both sides gives then (129.5). Now the expression on the right side of (129.4) exhibits a meromorphic function in w with the simple poles at $w = \log 1/z$ (for $z = 0$ it is an entire function). The power series in w converges, therefore, certainly for $|w| < |\log z|$, where the smallest possible value of $|\log z|$ is taken. In particular for

$$|z - 1| \geqq \delta, \quad 0 < \delta \leqq \frac{1}{2} \tag{129.6}$$

we have for any determination of $\log z$

$$|\log z| > \frac{\delta}{2} \tag{129.61}$$

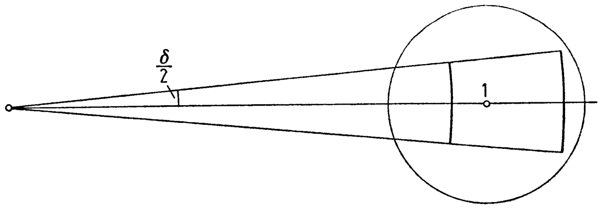
since, as can be seen easily, the circle

$$|z - 1| < \delta \leqq \frac{1}{2}$$

contains the domain

$$1 - \frac{\delta}{2} < |z| < \frac{1}{1 - \frac{\delta}{2}}, \qquad |\arg z| < \frac{\delta}{2}$$

outside of which (129.61) is fulfilled.



The power series

$$\sum_{q=0}^{\infty} \psi_q(z, \alpha) \frac{w^q}{q!}$$

converges, therefore, certainly for $|w| < \delta/2$ if z lies in the domain (129.6). This has as a consequence

$$\limsup_{q \to \infty} \left| \frac{\varphi_q(z, \alpha)}{q!} \right|^{1/q} \leqq \frac{2}{\delta}$$

and thus

$$|\psi_q(z, \alpha)| < K \left(\frac{3}{\delta}\right)^q \cdot q!$$

with a suitable constant K. This concludes the proof of the theorem. ▯

The series in (129.1) is therefore majorized by

$$K_q \cdot \sum_{q=0}^{\infty} \frac{1}{\Gamma(5/2+q)} \left(\frac{\pi}{k\sqrt{6}}\right)^{2q} \left(\frac{3}{\delta}\right)^q, \tag{129.62}$$

which means that it converges uniformly outside each circle $|z-1| = \delta$. In other words, $\Phi_k(z)$ has only the point $z = 1$ as an essential singularity. The functions $\psi_q(z, \alpha)$ all vanish at $z = \infty$ and so does $\Phi_k(z)$, which is regular at $z = \infty$.

For the explicit expansion of $\Phi_k(z)$ about the point $z = 1$, we utilize (129.3), (129.4) and prove the

Theorem. *With the designation*

$$\Delta_\alpha f(\alpha) = f(\alpha+1) - f(\alpha),$$

$$\Delta_\alpha^j f(\alpha) = \sum_{h=0}^{j} (-1)^h \binom{j}{h} f(\alpha + j - h)$$

(the j-fold application of Δ_α on $f(\alpha)$) and

$$\Delta_\alpha^0 f(\alpha) = f(\alpha),$$

we have

$$\Psi_q(\mu, \alpha) = \sum_{j=0}^{q} \Delta_\alpha^j [(\alpha+1)^q]\, \mu^{j+1}. \tag{129.7}$$

Proof. We use again the method of generating series. From the definition (129.3) and from (129.4) follows

$$\sum_{q=0}^{\infty} \Psi_q(\mu, \alpha) \frac{w^q}{q!} = -\sum_{q=0}^{\infty} \psi_q\left(\frac{\mu+1}{\mu}, \alpha\right) \frac{(-w)^q}{q!}$$
$$= -\frac{\mu e^{\alpha w}}{\mu - (\mu+1)e^{-w}} = \frac{\mu e^{(\alpha+1)w}}{1 - \mu(e^w - 1)}.$$

On the other hand, we have the generating function for $\Delta_\alpha^j[(\alpha+1)^q]$:

$$\sum_{q=0}^{\infty} \Delta_\alpha^j [(\alpha+1)^q] \frac{w^q}{q!} = \sum_{q=0}^{\infty} \sum_{h=0}^{j} (-1)^h \binom{j}{h} \frac{(\alpha+1+j-h)^q w^q}{q!}$$
$$= \sum_{h=0}^{j} (-1)^h \binom{j}{h} e^{(\alpha+1+j-h)w} = e^{(\alpha+1)w}(e^w - 1)^j.$$

We multiply both sides by μ^{j+1}, and under the assumption $|\mu(e^w - 1)| < 1$, sum over j

$$\sum_{j=0}^{\infty} \mu^{j+1} \sum_{q=0}^{\infty} \Delta_\alpha^j [(\alpha+1)^q] \frac{w^q}{q!} = \frac{\mu e^{(\alpha+1)w}}{1 - \mu(e^w - 1)} = \sum_{q=0}^{\infty} \Psi_q(\mu, \alpha) \frac{w^q}{q!}.$$

Now the left-hand member can be written

$$\sum_{j=0}^{\infty} \mu^{j+1} \sum_{q=j}^{\infty} \Delta_{\alpha}^{j}[(\alpha+1)^q] \frac{w^q}{q!}$$

since $\Delta_{\alpha}^{j}[(\alpha+1)^q] = 0$ for $q < j$. Interchange of the summations in j and q leads to

$$\sum_{q=0}^{\infty} \sum_{j=0}^{\infty} \Delta_{\alpha}^{j}[(\alpha+1)^q]\, \mu^{j+1} \frac{w^q}{q!} = \sum_{q=0}^{\infty} \Psi_q(\mu, \alpha) \frac{w^q}{q!},$$

which implies the statement (129.7) of the theorem. The just-performed interchange of the order of summations can be justified by the majorization

$$|\Delta_{\alpha}^{j}[(\alpha+1)^q]| \leqq q(q-1)\dots(q-j+1)\,(|\alpha|+j+1)^{q-j}, \tag{129.8}$$

which follows from a j-fold application of the mean-value theorem.

We have now available three expansions of $\Phi_k(z)$, viz. about $z = 0$, $z = \infty$, and $z = 1$.

(I) Equation (128.32) gives $\Phi_k(z)$ as a power series in z:

$$\Phi_k(z) = \sum_{n=0}^{\infty} L_{3/2}\left(\frac{\pi^2}{6k}(n-\alpha)\right) z^n, \qquad |z| < 1. \tag{129.91}$$

(II) Using (129.5), after replacing α by $-\alpha$, we obtain from (129.1) and (129.11)

$$\begin{aligned}
\Phi_k(z) &= \sum_{q=0}^{\infty} \frac{\left(\dfrac{\pi^2}{6k^2}\right)^q}{q!\,\Gamma\left(\dfrac{5}{2}+q\right)} \{(-\alpha)^q - (-1)^q \psi_q(1/z, -\alpha)\} \\
&= -\sum_{q=0}^{\infty} \frac{\left(-\dfrac{\pi^2}{6k^2}\right)^q}{q!\,\Gamma\left(\dfrac{5}{2}+q\right)} \sum_{n=1}^{\infty} (n+\alpha)^q z^{-n} \\
&= -\sum_{n=1}^{\infty} z^{-n} \sum_{q=0}^{\infty} \frac{\left(-\dfrac{\pi^2}{6k^2}(n+\alpha)\right)^q}{q!\,\Gamma\left(\dfrac{5}{2}+q\right)},
\end{aligned}$$

$$\Phi_k(z) = -\sum_{n=1}^{\infty} L_{3/2}\left(-\frac{\pi^2}{6k^2}(n+\alpha)\right) z^{-n}, \qquad |z| > 1, \tag{129.92}$$

the power series expansion about $z = \infty$.

(III) Equations (129.3) and (129.7) applied to (129.1) give

$$\Phi_k(z) = -\sum_{q=0}^{\infty} \frac{\left(-\frac{\pi^2}{6k^2}\right)^q}{q!\,\Gamma\left(\frac{5}{2}+q\right)} \Psi_q\left(\frac{1}{z-1}\right)$$

$$= -\sum_{q=0}^{\infty} \frac{\left(-\frac{\pi^2}{6k^2}\right)^q}{q!\,\Gamma\left(\frac{5}{2}+q\right)} \sum_{j=0}^{q} \Delta_\alpha^j[(\alpha+1)^q] \frac{1}{(z-1)^{j+1}}$$

$$= -\sum_{j=0}^{\infty} \frac{1}{(z-1)^{j+1}} \sum_{q=j}^{\infty} \frac{\left(\frac{-\pi^2}{6k^2}\right)^q}{q!\,\Gamma\left(\frac{5}{2}+q\right)} \Delta_\alpha^j[(\alpha+1)^q],$$

where the estimate (129.8) permits the interchange of the summations for large enough $|z-1|$, as a short computation shows. The inner sum over q can now be extended to $0 < q \leqq j$, and the difference symbol be pulled out of the infinite sum:

$$\Phi_k(z) = -\sum_{j=0}^{\infty} \frac{1}{(z-1)^{j+1}} \Delta_\alpha^j \sum_{q=0}^{\infty} \frac{\left(-\frac{\pi^2}{6k^2}(\alpha+1)\right)^q}{q!\,\Gamma\left(\frac{5}{2}+q\right)}$$

or

$$\Phi_k(z) = -\sum_{j=0}^{\infty} (z-1)^{-j-1} \Delta_\alpha^j L_{3/2}\left(-\frac{\pi^2}{6k^2}(\alpha+1)\right). \qquad (129.93)$$

This equation is at first valid only for large enough $|z-1|$, as we observed, but since $\Phi_k(z)$ has only the point $z = 1$ as singularity, it gives the Laurent expansion about $z = 1$ in the whole z-plane with the exception of $z = 1$. This expansion is the common analytic continuation of (129.91) and (129.92). ▯

130. Decomposition of $f(x)$ into partial fractions

We utilize now our formulae for $\Phi_k(z)$ for a further, more detailed expression of (128.31). For all x with $|x| < 1-\delta$, as well as $|x| > 1+\delta$ we have

$$\left| x e^{-\frac{2\pi i h}{k}} - 1 \right| > \delta .$$

In these cases the estimation (129.62) applies

$$\left|\Phi_k\left(xe^{-\frac{2\pi ih}{k}}\right)\right| < K_\delta \sum_{q=0}^{\infty} \frac{1}{\Gamma\left(\frac{5}{2}+q\right)} \left(\frac{\pi^2}{6k^2}\cdot\frac{3}{\delta}\right)^q$$

$$\leqq K_\delta \sum_{q=0}^{\infty} \frac{1}{\Gamma\left(\frac{5}{2}+q\right)} \left(\frac{\pi^2}{2\delta}\right)^q = H(\delta) \text{ for } k = 1, 2, 3, \ldots$$

The series (128.31) is therefore majorized by

$$2\pi\left(\frac{\pi}{12}\right)^{3/2} H(\delta) \sum_{k=1}^{\infty} k^{-3/2}$$

and converges uniformly in $|x| < 1 - \delta$ and $|x| > 1 + \delta$. Since δ is arbitrary, the series obtained by inserting (129.93) in the right member of (128.31), namely,

$$F(x) = -2\pi\left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} k^{-5/2} \sum_{h \bmod k}' \omega_{hk} \sum_{j=0}^{\infty} \Delta_\alpha^j \left[L_{3/2}\left(-\frac{\pi^2}{6k^2}(\alpha+1)\right)\right]$$

$$\times\left(xe^{-\frac{2\pi ih}{k}} - 1\right)^{-j-1}, \quad \alpha = \frac{1}{24} \tag{130.1}$$

represents an analytic function for $|x| < 1$ as well as for $|x| > 1$. For $|x| < 1$ we know from our deduction that

$$F(x) = f(x)\,.$$

The circumference $|x| = 1$ is a natural boundary for $f(x)$, since (118.41) shows that $f(x)$ goes to infinity near all roots of unity, which is seen by letting z go through positive values to $+0$. We have to determine $F(x)$ for $|x| > 1$. In this case we apply (129.92) and obtain

$$F(x) = -2\pi\left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} k^{-5/2} \sum_{h \bmod k}' \omega_{hk} \sum_{n=1}^{\infty} L_{3/2}\left(-\frac{\pi^2}{6k^2}(n+\alpha)\right) x^{-n} e^{\frac{2\pi ihn}{k}}$$

$$= -2\pi\left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} k^{-5/2} \sum_{n=1}^{\infty} L_{3/2}\left(-\frac{\pi^2}{6k^2}(n+\alpha)\right) A_k(-n)\, x^{-n} \tag{130.2}$$

with $A_k(-n)$ in the meaning of (120.5). Here the coefficient of x^{-n}, $n = 1, 2, \ldots$, is

$$-2\pi\left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} k^{-5/2} A_k(-n)\, L_{3/2}\left(\left(\frac{\pi}{12k}\right)^2(-24n-1)\right),$$

which is $-p(-n)$ after (128.1). But we know by (116.3) that $p(-n) = 0$

for $n = 1, 2, 3, \ldots$ Therefore, the power series (130.2) has only vanishing coefficients, and we conclude

$$F(x) = 0 \text{ for } |x| > 1 .$$

The series (130.1) defines thus two completely different analytic functions for $|x| < 1$ and $|x| > 1$:

$$F(x) = \begin{cases} \sum\limits_{n=0}^{\infty} p(n) x^n , & |x| < 1 , \\ 0 , & |x| > 1 . \end{cases} \tag{130.3}$$

We can look upon (130.1) as partial fraction decomposition around the singularities

$$x = e^{\frac{2\pi i h}{k}} ,$$

which, however, in distinction from more elementary cases, lie everywhere dense on the line $|x| = 1$.

There is a much simpler expansion into partial fractions which behave like $F(x)$ in (130.3). That is the expansion

$$\lim_{N\to\infty} \frac{1}{(1-x)(1-x^2)\cdots(1-x^N)} = \frac{1}{\prod\limits_{m=1}^{\infty}(1-x^m)} .$$

Indeed, this expression fulfills, as we know from the start of these discussions, the first line of (130.3). For $|x| > 1$ we see that

$$\frac{1}{\prod\limits_{m=1}^{N}(1-x^m)} = (-1)^N \frac{x^{-N(N+1)/2}}{\prod\limits_{m=1}^{N}(1-x^{-m})} \to 0, \tag{130.4}$$

which corresponds to the second statement in (130.3).

Let us write the unique algebraic partial fraction decomposition of (130.4) as

$$\frac{1}{\prod\limits_{m=1}^{N}(1-x^m)} = \sum_{0\leq h<k\leq N}' \sum_{l=1}^{[N/k]} \frac{C_{hkl}(N)}{\left(x - e^{\frac{2\pi i h}{k}}\right)^l} . \tag{130.5}$$

The $C_{hkl}(N)$ can be obtained algebraically as expressions containing roots of unity, although the actual computation becomes soon very cumbersome with increasing N. No explicit formula for $C_{hkl}(N)$ is known, not even for the simplest case $h = 0, k = 1, l = 1$, and variable N.

I conjecture now that the partial fraction decomposition (130.5) converges termwise to the expansion (130.1). More explicitly, I propose the

Conjecture.

$$\lim_{N\to\infty} C_{hkl}(N)$$

exists and is equal to

$$C_{hkl}(\infty) = -2\pi\left(\frac{\pi}{12}\right)^{3/2} \frac{\omega_{hk} e^{\frac{2\pi i h l}{k}}}{k^{5/2}} \Delta_\alpha^{l-1} L_{3/2}\left(-\frac{\pi^2}{6k^2}(\alpha+1)\right),$$

$$\alpha = \frac{1}{24}. \tag{130.6}$$

These numbers can easily be computed, since they reduce to trigonometric functions. Some simple calculations starting from (128.11) show that

$$L_{3/2}(-y^2) = -\frac{1}{2\sqrt{\pi}\, y}\frac{d}{dy}\left(\frac{\sin 2y}{y}\right) = -\frac{1}{2\sqrt{\pi}\, y^2}\left(2\cos 2y - \frac{\sin 2y}{y}\right).$$

The following table contain some values for the C_{hkl}, the algebraic ones from (130.5) as well as the transcendental ones from (130.6):

N	C_{011}	C_{012}	C_{121}
1	-1	0	0
2	$-\frac{1}{4} = -0.25$	$\frac{1}{2} = 0.5$	$\frac{1}{4} = 0.25$
3	$-\frac{17}{72} = -0.23611\ldots$	$\frac{1}{4} = 0.25$	$\frac{1}{8} = 0.125$
4	$-\frac{17}{72} = -0.23611\ldots$	$\frac{59}{288} = 0.204861\ldots$	$\frac{1}{8} = 0.125$
5	$-\frac{20\,591}{86\,400} = -0.23832\ldots$	$\frac{3}{16} = 0.1875$	$\frac{13}{128} = 0.1015625$

$$C_{011}(\infty) = -\frac{6}{25}\left(1 + \frac{2\sqrt{3}}{5\pi}\right) = -0.273339\ldots,$$

$$C_{012}(\infty) = \frac{24}{25\cdot 49}\left(6 + \frac{109\sqrt{3}}{35\pi}\right) = 0.15119\ldots,$$

$$C_{121}(\infty) = -\frac{\sqrt{6}}{25}\left(\cos\frac{5\pi}{12} - \frac{12}{5\pi}\sin\frac{5\pi}{12}\right) = 0.046941\,.$$

These values may lend some support to the conjecture.

Chapter 15

Application of the Circle Method to Modular Forms of Positive Dimension

131. Generalized modular forms

The circle method, which led to an explicit formula for the Fourier coefficients $p(n)$ of $e^{-\pi i\tau/12}\eta(\tau)^{-1}$, can be generalized to deal with general modular forms of positive dimension.

We have already encountered examples of modular forms, especially the Eisenstein series $C_{2k}(w_1, w_2)$ of dimension $-2k$ and $\Delta(w_1, w_2)$ of dimension -12. We repeat here the definition of a modular form.

Definition. *A modular form is an analytic function $F(w_1, w_2)$ of two variables w_1, w_2 defined for*

$$\operatorname{Im}\frac{w_2}{w_1} > 0$$

and possessing the two properties:

(1) *Homogeneity of dimension r*

$$F(\lambda w_1, \lambda w_2) = \lambda^r F(w_1, w_2).$$

(2) *Modular invariance*

$$F(dw_1 + cw_2, bw_1 + aw_2) = F(w_1, w_2),$$

where $ad - bc = 1$.

Application of property (1), at first only for integer r, permits us to write (2) in the inhomogeneous manner

$$F\left(1, \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{-r} F(1, \tau) \tag{131.1}$$

with

$$\tau = \frac{w_2}{w_1}, \quad \operatorname{Im}\tau > 0.$$

From now on we write simply $F(\tau)$ instead of $F(1, \tau)$ and shall speak of $F(\tau)$, although written in the inhomogeneous manner,

Definition. *A function $F(\tau)$, analytic for* $\operatorname{Im}\tau > 0$, *with at most poles as singularities, is called a modular form of dimension r if it satisfies*

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{-r} F(\tau).$$

A modular form of dimension 0 *is simply called a modular function. A modular form which is regular for* $\operatorname{Im}\tau > 0$ *we call an entire modular form.*

We wish now to introduce also modular forms of any real dimension.

In order to overcome difficulties involved in the determination of $(c\tau + d)^r$ for non-integer r, we introduce a factor ε of absolute value 1 and generalize (131.1) to

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = \varepsilon \left(\frac{c\tau + d}{i}\right)^{-r} F(\tau)\,, \tag{131.2}$$

where we take always $c > 0$. Here the multiplier $\varepsilon = \varepsilon(a, b, c, d)$ depends only on the parameters of the modular transformation, and $\left(\frac{1}{i}(c\tau + d)\right)^{-r}$ is taken in the meaning of its principal value:

$$\left(\frac{c\tau + d}{i}\right)^{-r} = |c\tau + d|^{-r} \exp\left(-ir \arg \frac{c\tau + d}{i}\right)$$

with the determination

$$\frac{-\pi}{2} < \arg \frac{c\tau + d}{i} < \frac{\pi}{2}\,.$$

Only this provision makes ε unique. The case $c = 0$, i.e. $\tau' = \tau + b$ is treated separately:

$$F(\tau + b) = \varepsilon_b F(\tau)\,,$$

and in particular

$$F(\tau + 1) = \varepsilon_1 F(\tau) = e^{2\pi i\alpha} F(\tau)\,, \tag{131.3}$$

where α can be restricted to

$$0 \leqq \alpha < 1\,.$$

Since all modular substitutions are a product of the substitutions S and T (see proof of theorem in § 65), it follows that $\varepsilon(a, b, c, d)$ for any modular substitution $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is determined (if at all) by

$$\varepsilon_1 = \varepsilon(1, 1, 0, 1) = e^{2\pi i\alpha}$$

and

$$\varepsilon_0 = \varepsilon(0, -1, 1, 0)\,.$$

We have from (131.3)

$$e^{-2\pi i\alpha(\tau+1)} F(\tau + 1) = e^{-2\pi i\alpha\tau} F(\tau)\,.$$

From now on we shall discuss only *entire* modular forms, i.e. those which are regular-analytic in the upper τ-half-plane. For such functions the last equation shows the validity of a Fourier expansion

$$e^{-2\pi i\alpha\tau}F(\tau) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n\tau}. \tag{131.4}$$

We introduce one further restriction: we consider only modular forms $F(\tau)$ for which the expansion (131.3) contains only finitely many terms for negative n:

$$F(\tau) = e^{2\pi i\alpha\tau} \sum_{n=-\nu}^{\infty} a_n e^{2\pi i n\tau}. \tag{131.41}$$

Here

$$P(e^{2\pi i\tau}) = \sum_{n=-\mu}^{-1} a_n e^{2\pi i n\tau}$$

is the principal part of the singularity of $e^{-2\pi i\alpha\tau}F(\tau)$ expressed in the variable $x = e^{2\pi i\tau}$ at $x = 0$ or $\tau = i\infty$.

For clarifying our notation we compare more examples: For

$$\eta(\tau) = e^{\frac{\pi i\tau}{12}} \prod_{m=1}^{\infty} (1 - e^{2\pi i m\tau})$$

we proved

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon(a, b, c, d)\sqrt{\frac{c\tau+d}{i}}\,\eta(\tau).$$

Thus $\eta(\tau)$ is a modular form of dimension $-1/2$. Moreover,

$$\eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau),$$

which makes $\alpha = 1/24$ in this case.

It also follows again (see theorem in § 65) that

$$\Delta(\tau) = \eta(\tau)^{24}$$

is a modular form of dimension -12, with $\alpha = 0$.

It is also clear that $\eta(\tau)^{-1}$ is a modular form of dimension $+1/2$. However, with our determination $0 \leqq \alpha < 1$ we have

$$\eta(\tau+1)^{-1} = e^{\frac{-\pi i}{12}}\eta(\tau)^{-1} = e^{\frac{23\pi i}{12}}\eta(\tau)^{-1},$$

which gives here $\alpha = 23/24$.

It is convenient to put $x = e^{2\pi i\tau}$ and

$$e^{-2\pi i\alpha\tau}F(\tau) = f(x) = \sum_{n=-\mu}^{\infty} a_n x^n . \tag{131.6}$$

Our aim is to determine the coefficients $a_0, a_1, \ldots$ by means of the coefficients $a_{-\mu}, a_{-\mu+1}, \ldots, a_{-1}$ from (131.5).

In the case of $\eta(\tau)^{-1}$, which we discussed in the previous chapter, we have

$$\begin{aligned} e^{-2\pi i\frac{23}{24}\tau}\eta(\tau)^{-1} = f(x) &= x^{-1}\Pi(1-x^m)^{-1} \\ &= x^{-1}\{1 + p(1)x + p(2)x^2 + \cdots\} \\ &= x^{-1} + a_0 + a_1 x + \cdots , \end{aligned}$$

which makes in our present notation

$$a_n = p(n+1) .$$

The $f(x)$ here and in Chapter 14 differ thus by a factor x^{-1}. This change of notation is motivated by the purpose of emphasizing the behavior of the function under investigation at $\tau = i\infty$.

132. Computation of the coefficients of the modular form

In order to compute the coefficients a_n of $f(x)$ in (131.4) we can verbatim copy the procedure of §§ 116, 117, and 118. We have thus

$$\begin{aligned} a_n = \int_{P_N} f(e^{2\pi i n\tau})\, e^{-2\pi i n\tau} d\tau &= \sum_{0\le h<k\le N}{}' \int_{\gamma_{hk}} f(e^{2\pi i n\tau})\, e^{-2\pi i n\tau} d\tau \\ &= \sum_{0\le h<k\le N}{}' \frac{i}{k^2} e^{-\frac{2\pi i n h}{k}} \int_{z'_h}^{z''_{hk}} f\left(e^{\frac{2\pi i h}{k} - \frac{2\pi z}{k^2}}\right) e^{\frac{2\pi n z}{k^2}} dz , \end{aligned} \tag{132.1}$$

where z runs on an arc of the circle (117.2), whose ends z'_{hk} and z''_{hk} are given by (117.3).

We introduce now the transformation formula (131.2) for

$$F(\tau) = e^{2\pi i\alpha\tau} f(e^{2\pi i\tau}) .$$

We write, as in § 118

$$\tau = \frac{h + iz}{k} , \qquad \tau' = \frac{h' + i/z}{k}$$

and have

$$F\left(\frac{h' + i/z}{k}\right) = \varepsilon(h, k)\; z^{-r} F\left(\frac{h + iz}{k}\right), \tag{132.2}$$

where $\varepsilon(h, k)$ is an abbreviated notation for the root of unity

$$\varepsilon(a, b, c, d) = \varepsilon\left(h, -\frac{hh' + 1}{k}, k, -h'\right).$$

Replacing z by z/k we obtain then from (132.2)

$$f\left(e^{\frac{2\pi ih}{k} - \frac{2\pi z}{k^2}}\right) = \omega_{hk}\left(\frac{z}{k}\right)^r \Psi_k(z)\, f\left(e^{\frac{2\pi ih'}{k} - \frac{2\pi}{z}}\right)$$

with

$$\omega_{hk} = \varepsilon^{-1}(h, k)\, e^{\frac{2\pi i\alpha}{k}(h'-h)} \tag{132.3}$$

and

$$\Psi_k(z) = e^{2\pi\alpha\left(\frac{z}{k^2} - \frac{1}{z}\right)}, \tag{132.4}$$

and this, inserted in (132.1), leads to

$$a_n = i \sum_{0 \leq h < k \leq N}{}' \frac{1}{k^{2+r}}\, \omega_{hk}\, e^{-\frac{2\pi inh}{k}} \int_{z'_{hk}}^{z''_{hk}} z^r \Psi_k(z)\, f\left(e^{\frac{2\pi ih'}{k} - \frac{2\pi}{z}}\right) e^{\frac{2\pi nz}{k^2}}\, dz\,.$$

Now we remember

$$f(x) = P(x) + \sum_{\nu=0}^{\infty} a_\nu x^\nu\,,$$

where the power series converges for $|x| < 1$. We write

$$\begin{aligned} a_n = i \sum_{0 \leq h < k \leq N}{}' \frac{\omega_{hk}}{k^{2+r}}\, e^{\frac{-2\pi inh}{k}} \int_{z'_{hk}}^{z''_{hk}} z^r \Psi_k(z)\, P\left(e^{\frac{2\pi ih'}{k} - \frac{2\pi}{z}}\right) e^{\frac{2\pi nz}{k^2}}\, dz \\ + i \sum_{0 \leq h < k \leq N}{}' \frac{\omega_{hk}}{k^{2+r}}\, e^{\frac{-2\pi inh}{k}} \int_{z'_{hk}}^{z''_{hk}} z^r e^{\frac{2\pi z(n+\alpha)}{k^2}}\, e^{\frac{-2\pi\alpha}{z}} \sum_{\nu=0}^{\infty} a_\nu e^{\left(\frac{2\pi ih'}{k} - \frac{2\pi}{z}\right)\nu}\, dz\,, \end{aligned} \tag{132.5}$$

and for abbreviation we write

$$a_n = S_1 + S_2\,.$$

133. Estimations

We estimate here S_2, in which we can integrate along the chord from z'_{hk} to t''_{hk} in the interior of the circle (117.2). On the chord $|z|$ is estimated by (119.5), and $\operatorname{Re} z$ by (119.6) and $\operatorname{Re} 1/z$ by (119.3). Moreover, the length of the chord is less than $2\sqrt{2}k/N$. This yields

$$\left| \int_{z'_{hk}}^{z''_{hk}} z^r e^{\frac{2\pi z(n+\alpha)}{k^2}} e^{-\frac{2\pi\alpha}{z}} \sum_{\nu=0}^{\infty} a_\nu e^{\left(\frac{2\pi i h'}{k} - \frac{2\pi}{z}\right)\nu} dz \right|$$

$$\leqq \frac{2\sqrt{2}k}{N} \left(\frac{\sqrt{2}k}{N}\right)^r e^{\frac{4\pi(|n|+1)}{N^2}} \sum_{\nu=1}^{\infty} |a_\nu| e^{-2\pi\nu} = C\left(\frac{k}{N}\right)^{r+1} e^{\frac{4\pi(|n|+1)}{N^2}},$$

and thus

$$|S_2| \leqq C e^{\frac{4\pi(|n|+1)}{N^2}} \frac{1}{N^{r+1}} \sum_{0\leqq h<k\leqq N}' \frac{1}{k} = C e^{\frac{4\pi(|n|+1)}{N^2}} N^{-r}$$

$$= O(N^{-r}) \tag{133.1}$$

for every fixed n.

The integration in S_1 we extend now over the whole circumference of the circle K of (117.2) so that we can write

$$\int_{z'_{hk}}^{z''_{hk}} z^r \Psi_k(z)\, P\left(e^{\frac{2\pi i h'}{k} - \frac{2\pi}{z}}\right) e^{\frac{2\pi n z}{k^2}} dz = \int_{K^{(-)}} - \int_0^{z'_{hk}} - \int_{z''_{hk}}^{0}.$$

The notation $K^{(-)}$ is meant to emphasize that the circle K is traversed in the negative direction from 0 to 0. All three integrals are improper integrals at the ends 0. Now on the circle K we have $\operatorname{Re}(1/z) = 1$, and the length of the arc on K from 0 to z'_{hk} is

$$\leqq \frac{\pi}{2}|z'_{hk}| \leqq \frac{\pi}{\sqrt{2}} \frac{k}{N}.$$

This leads to the estimate

$$\left| \int_0^{z'_{hk}} z^r \Psi_k(z) P\left(e^{\frac{2\pi i h'}{k} - \frac{2\pi}{z}}\right) e^{\frac{2\pi n z}{k^2}} dz \right|$$

$$\leqq \frac{\pi}{\sqrt{2}} \frac{k}{N} \left(\frac{\sqrt{2}k}{N}\right)^r e^{\frac{4\pi\alpha}{N^2} - 2\pi\alpha} \sum_{\nu=1}^{\mu} |a_{-\nu}| e^{2\pi\nu} e^{\frac{4\pi|n|}{N^2}} < C\left(\frac{k}{N}\right)^{r+1}$$

for any fixed n, and hence

$$\left| \sum_{0 \leq h < k \leq N}' \frac{\omega_{hk}}{k^{2+r}} e^{\frac{-2\pi i h n}{k}} \int_0^{z'_{hk}} \right| \leq \frac{C}{N^{r+1}} \sum_{0 \leq h < k \leq N}' \frac{1}{k} \leq \frac{C}{N^r}.$$

This is the same order as that in (133.1). The same result is, of course, valid for the sum with the integral from z''_{hk} to 0.

We insert all these estimations in (132.5) with the result

$$a_n = i \sum_{0 \leq h < k \leq N} \frac{\omega_{hk}}{k^{2+r}} e^{\frac{-2\pi i n h}{k}} \int_{K^{(-)}} z^r \Psi_k(z)\, P\left(e^{\frac{2\pi i h'}{k} - \frac{2\pi}{z}}\right) e^{\frac{2\pi n z}{k^2}} dz$$

$$+ O(N^{-r}),$$

which through the definition of $P(x)$ given in (131.5) goes over into

$$a_n = \sum_{\nu=1}^{\mu} a_{-\nu} \sum_{k=1}^{N} \frac{i}{k^{2+r}} A_k(n, \nu) \int_{K^{(-)}} z^r e^{\frac{2\pi z}{k^2}(n+\alpha) + \frac{2\pi}{z}(\nu-\alpha)} dz + O(N^{-r})$$

with

$$A_k(n, \nu) = \sum_{h \bmod k}' \omega_{hk} e^{-\frac{2\pi i}{k}(nh + \nu h')}. \tag{133.2}$$

Up to this point the dimension r of $F(\tau)$ was irrelevant. We assume now that $r > 0$. If under this circumstance we let N go to ∞, then the error term in the foregoing equation tends to 0. This means that the main term must have a limit for $N \to \infty$. But in

$$a_n = \lim_{N \to \infty} \sum_{\nu=1}^{\mu} a_\nu \sum_{k=1}^{N} \cdots$$

we can actually put the limit sign under the finite sum over ν, since the resulting μ infinite series converge separately. Indeed, for fixed N the integrals over $K^{(-)}$ remain bounded for all k, as (120.2) shows, and the $A_k(n, \nu)$, being sums of $\varphi(k)$ complex numbers of absolute value 1, are in absolute value at most k. Thus

$$\left| \frac{A_k(n, \nu)}{k^{2+\epsilon}} \int_{K^{(-)}} \right| \leq \frac{C}{k^{1+\epsilon}},$$

which guarantees the convergence of the sum over k. We have therefore reached the result

$$a_n = \sum_{\nu=1}^{\mu} a_{-\nu} \sum_{k=1}^{\infty} \frac{i}{k^{2+r}} A_k(n, \nu) \int_{K^{(-)}} z^r e^{\frac{2\pi z}{k^2}(n+\alpha) + \frac{2\pi}{z}(\nu-\alpha)} dz. \tag{133.3}$$

134. The final formula for the coefficients

In this formula we put (as before in § 120)

$$w = \frac{1}{z}$$

and obtain

$$i \int_{K^{(-)}} = \frac{1}{i} \int_{1-i\infty}^{1+i\infty} w^{-r-2} e^{2\pi(\nu-\alpha)w + \frac{2\pi(n+\alpha)}{k^2 w}} dw\,, \tag{134.1}$$

where in our case $\nu - \alpha > 0$. Let us put, in order to simplify notations,

$$L_\lambda(y) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} w^{-\lambda-1} e^{w+y/w}\, dw = \sum_{q=0}^{\infty} \frac{y^q}{q!\,\Gamma(1+\lambda+q)}\,, \tag{134.2}$$

the latter obtained by expanding $\exp(y/w)$ into the power series and applying Hankel's formula (cf. [79], § 12.22, p. 245). In this notation the Bessel function of order λ appears as

$$J_\lambda(z) = \left(\frac{z}{2}\right)^\lambda L_\lambda\left(-\frac{z^2}{4}\right) \tag{134.3}$$

and the "hyperbolic Bessel function" as

$$I_\lambda(z) = \left(\frac{z}{2}\right)^\lambda L_\lambda\left(\frac{z^2}{4}\right). \tag{134.4}$$

Equation (134.1) yields then

$$i \int_{K^{(-)}} = 2\pi\big(2\pi(\nu-\alpha)\big)^{r+1} L_{r+1}\left(\frac{4\pi^2}{k^2}(\nu-\alpha)(n+\alpha)\right).$$

Hence, it follows from (133.3) that

$$a_n = (2\pi)^{r+2} \sum_{\nu=1}^{\mu} a_{-\nu} (\nu-\alpha)^{r+1} \sum_{k=1}^{\infty} \frac{A_k(n,\nu)}{k^{r+2}} \times L_{r+1}\left(\frac{4\pi^2}{k^2}(\nu-\alpha)(n+\alpha)\right). \tag{134.5}$$

This formula is valid for all integers n, positive, negative, and zero.

In particular we know from (131.41) that

$$a_n = 0 \quad \text{for} \quad n < -\mu \tag{134.6}$$

and have, in terms of Bessel functions,

$$a_n = 2\pi(n+\alpha)^{-\frac{r+1}{2}} \sum_{\nu=1}^{\mu} a_{-\nu}(\nu-\alpha)^{\frac{r+1}{2}} \sum_{k=1}^{\infty} \frac{A_k(n,\nu)}{k}$$

$$\times\, I_{r+1}\left(\frac{4\pi}{k}\sqrt{(\nu-\alpha)(n+\alpha)}\right), \quad n \geqq 0\,,$$

$$a_n = 2\pi(|n|-\alpha)^{-\frac{r+1}{2}} \sum_{\nu=1}^{\mu} a_{-\nu}(\nu-\alpha)^{\frac{r+1}{2}} \sum_{k=1}^{\infty} \frac{A_k(n,\nu)}{k}$$

$$\times\, J_{r+1}\left(\frac{4\pi}{k}\sqrt{(\nu-\alpha)(|n|-\alpha)}\right), \quad n \leqq -1\,. \tag{134.7}$$

In the sequel we shall however prefer the notation (134.5). For the case $\mu = 1, r = 1/2, \alpha = 23/24$, the first formula in (134.7) agrees with (121.1) for $p(n)$ with the exception that the n at present appears as $(n+1)$ in the former expression, for the reason explained in § 131.

If we introduce

$$c_{n\nu} = (2\pi)^{r+2}(\nu-\alpha)^{r+1} \sum_{k=1}^{\infty} \frac{A_k(n,\nu)}{k^{r+2}} L_{r+1}\left(\frac{4\pi^2}{k^2}(\nu-\alpha)(n+\alpha)\right), \tag{134.8}$$

then (134.5) can be written as

$$a_n = \sum_{\nu=1}^{\mu} c_{n\nu} a_{-\nu}\,.$$

The coefficients $a_{-\nu}$, $\nu = 1, \ldots, \mu$ of the principal part determine therefore uniquely all coefficients of the modular form $F(\tau)$ with given positive dimension r and multiplier system $\{\varepsilon(a, b, c, d)\}$. This, of course, does not say that to any arbitrary assignment there must exist a modular form of dimension r and given multiplier system.

135. The series for the modular form $F(\tau)$

Just as in § 128 we use now the explicit formula of the coefficients (134.5) to study the series for $F(\tau)$. In order to avoid inconvenient negative powers we prefer to discuss, instead of (131.4), rather

$$e^{2\pi i(\mu-\alpha)\tau} F(\tau) = x^{\mu} f(x) = \sum_{n=-\mu}^{\infty} a_n x^{n+\mu}$$

with $x = e^{2\pi i\tau}$. We obtain by means of (134.5)

$$x^\mu f(x) = (2\pi)^{r+2} \sum_{\nu=1}^{\mu} a_{-\nu}(\nu-\alpha)^{r+1} \sum_{k=1}^{\infty} \frac{1}{k^{r+2}} \sum_{h \bmod k}{}' \omega_{hk} e^{\frac{2\pi i}{k}(\mu h - \nu h')}$$
$$\times \sum_{q=0}^{\infty} \frac{\left(\frac{4\pi^2}{k^2}(\nu-\alpha)^q\right)}{q!\,\Gamma(r+2+q)} \psi_q\left(x e^{-\frac{2\pi i h}{k}}\right) \tag{135.1}$$

with

$$\psi_q(z) = \sum_{n=-\mu}^{\infty} (n+\alpha)^q z^{n+\mu} = \sum_{m=0}^{\infty} (m-\mu+\alpha)^q z^m . \tag{135.2}$$

But this is the function $\psi_q(z, \mu-\alpha)$ defined in (129.11), i.e. with $\mu-\alpha$ instead of α. We know from § 129 that $\psi_q(z)$ is a polynomial of degree $q+1$ in $1/(z-1)$.

We put, as we did earlier,

$$(-1)^{q+1}\psi_q\left(1+\frac{1}{\mu}\right) = \Psi_q(\mu)$$

and have here as in (129.7)

$$\Psi_q(\mu) = \sum_{j=0}^{q} \Delta_\mu^{(j)} (\mu-\alpha+1)^q \mu^{j+1} . \tag{135.3}$$

In (135.1) we combine

$$\Phi_{k,\nu} = \sum_{q=0}^{\infty} \frac{\left(\frac{4\pi^2(\nu-\alpha)}{k^2}\right)^q}{q!\,\Gamma(r+2+q)} \psi_q(z)$$
$$= \sum_{q=0}^{\infty} (-1)^{q+1} \frac{\left(\frac{4\pi^2(\nu-\alpha)}{k^2}\right)^q}{q!\,\Gamma(r+2+q)} \sum_{j=0}^{q} \Delta_\mu^{(j)} (\mu-\alpha+1)^q \frac{1}{(z-1)^{j+1}}$$

and find, as in § 129, that $\Phi_{k,\nu}(z)$ is an entire function of $1/(z-1)$. We obtain thus

$$x^\mu f(x) = (2\pi)^{r+2} \sum_{\nu=1}^{\mu} a_{-\nu}(\nu-\alpha)^{r+1} \sum_{k=1}^{\infty} \frac{1}{k^{r+2}}$$
$$\times \sum_{h \bmod k}{}' \omega_{hk} e^{\frac{2\pi i}{k}(h\mu - h'\nu)} \Phi_{k,\nu}\left(x e^{-\frac{2\pi i h}{k}}\right). \tag{135.4}$$

The individual summand has here an essential singularity at $x = e^{2\pi i h/k}$. The singularities together occupy all the roots of unity on the unit circle.

The expansion (135.4), like (130.1) converges for $|x| < 1$ as well as for $|x| > 1$ to two analytic functions, which are separated by the natural boundary $|x| = 1$. However, $f(x)$ is defined only for $|x| < 1$.

In order to study the series (135.4) outside the unit circle we have to expand the rational functions $\psi_q(z)$ about $z = \infty$. We have as before

$$\psi_0(z) = \frac{z^{-1}}{z^{-1} - 1} = -\sum_{m=1}^{\infty} z^{-m}, \qquad |z| > 1$$

and using (129.5), with $\alpha \to \mu - \alpha$, we get

$$\psi_q(z) = (-1)^{q+1} \sum_{m=1}^{\infty} (m + \mu - \alpha)^q z^{-m}, \qquad |z| > 1,$$

which leads to

$$\Phi_{k,\nu}(z) = \sum_{q=0}^{\infty} (-1)^{q+1} \frac{\left(\frac{4\pi^2(\nu - \alpha)}{k^2}\right)^q}{q!\,\Gamma(r + 2 + q)} \sum_{m=1}^{\infty} (m + \mu - \alpha)^q z^{-m}.$$

Then (135.4) after rearrangement goes over into

$$-(2\pi)^{r+2} \sum_{m=1}^{\infty} x^{-m} \sum_{\nu=1}^{\mu} a_{-\nu} (\nu - \alpha)^{r+1} \sum_{k=1}^{\infty} \frac{1}{k^{r+2}} \sum_{h \bmod k}' \omega_{hk} e^{\frac{2\pi i}{k}(h\mu - h'\nu + hm)}$$

$$\times \sum_{q=0}^{\infty} \frac{\left(-\frac{4\pi^2(\nu - \alpha)(m + \mu - \alpha)}{k^2}\right)^q}{q!\,\Gamma(r + 2 + q)}$$

$$= -(2\pi)^{r+2} \sum_{m=1}^{\infty} x^{-m} \sum_{\nu=1}^{\mu} a_{-\nu} (\nu - \alpha)^{r+1} \sum_{k=1}^{\infty} \frac{A_k(-(m + \mu), \nu)}{k^{r+2}}$$

$$\times L_{r+1}\left(\frac{4\pi^2}{k^2}(-\mu - m + \alpha)(\nu - \alpha)\right)$$

with the abbreviation (133.2), (134.2).

Now, however, formula (134.5) is applicable and shows the above expression equals

$$-(2\pi)^{r+2} \sum_{m=1}^{\infty} a_{-m-\mu} x^{-m} = 0$$

after (134.6).

In other words, the analytic function which the convergent series (135.4) represents in the domain $|x| > 1$ vanishes identically.

Editor's Notes

1. (p. 29) For a history of the formula see Whittaker-Watson [79], p. 127 and the references cited there.
2. (p. 238) The above congruence was proved by G. N. Watson for all powers of 5 and 7 (J. für Math. **179**)) 97–128 (1938)), by J. Lehner for 11^{α}, $\alpha \leq 3$ (Amer. J. Math. **65,** 492–520 (1943),; Proc. Amer. Math. Soc. **1,** 172–181, (1950), and by A. O. L. Atkin for all $\alpha \geq 1$ (Glasgow Math. J. **8,** 14–32 (1967)). For congruences to other moduli see M. Newman, Trans. Amer. Math. Soc. **97,** 225–236 (1960), and A. O. L. Atkin and J. N. O'Brien, Trans. Amer. Math. Soc. **126,** 442–459 (1967).

Bibliography

1. Appel, P.: Annales de l'École Normale (2) **10,** 119–145 (1880).
2. Carlitz, L.: The reciprocity theorem of Dedekin sums. Pacific J. of Math. **3,** 513–522 (1953).
3. Darling, H. B. C.: Proofs of certain identities and congruences enunciated by S. Ramanujan. Proc. London Math. Soc. (2) **19,** 350–372 (1921).
4. Davenport, H.: Multiplicative number theory, Chicago: Markham Publ. Co. 1967.
5. Dedekind, R.: Gesammelte mathematische Werke, Braunschweig: 1930–1932.
6. Dedekind, R.: Schreiben an Herrn Borchardt über die Theorie der elliptischen Modulfunktionen. J. reine angew. Math. **83,** 265–292 (1877).
7. Euler, L.: Introductio in analysin infinitorum, vol. 1 (1748), Marcum-Michaelem Bousquet, Lausannae, Ch. 16. De Partitione Numerorum, in Euler, L.: Opera Omnia, vol. 8, Leipzig: Teubner, 1922, pp. 313–338.
8. Fischer, W.: On Dedekind's function $\eta(\tau)$. Pacific J. of Math. **1,** 83–95 (1951).
9. Franklin, F.: Comptes Rendus de l'Académie des Sciences **92,** 448–450 (1881).
10. Frobenius, G.: Über die Bernoullischen Zahlen und die Eulerschen Polynome. Sitzungsberichte Berliner Akademie der Wissenschaften 1910, 809–847.
11. Gauss, K. F.: Werke, 6 vols., Göttingen, Königliche Gesellschaft der Wissenschaften, 1870–1880.
12. Glaisher, J. W. L.: On elliptic-function expansions in which the coefficients are powers of the complex numbers having n as norm. Quarterly J. of Math. **39,** 266–200, 1908.
13. Glaisher, J. W. L.: On the numbers of representations of a number as a sum of $2r$ squares, where $2r$ does not exceed eighteen. Proc. London Math. Soc. (2) **5,** 479–490 (1907).
14. Grosswald, E.: Sur l'ordre de grandeur des différences $\varphi(x) - x$ et $\pi(x) - \mathrm{eli}\, x$. C. R. Acad. Sc. Paris **260,** 3813–3816 (1965).
15. Gupta, H.: A table of partitions. Proc. Lond. Math. Soc. **39,** 142–149 (1935); **42,** 546–549 (1937).
16. Hardy, G. H.: Sur les zéros de le function $\zeta(s)$ de Riemann. Comptes rendus (Paris) **158,** 1012–1014 (1914).
17. Hardy, G. H., Ingham, A. E., Polya, G.: Theorems concerning the mean values of analytic functions. Proc. Roy. Soc. London **113,** 542–569 (1927).
18. Hardy, G. H., Ramanujan, S.: Asymptotic formulae in combinatory analysis. Proc. London Math. Soc. (2) **17,** 75–115 (1918).
19. Hardy, G. H., Wright, E. M.: An introduction to the theory of numbers, 4th ed., Oxford: Clarendon Press 1960.
20. Haselgrove, C. B., Miller, J. C. P.: Tables of the Riemann zeta function (Roy. Soc. Math. Tables, vol. 8), Cambridge: Cambridge Press 1960.
21. Hecke, E.: Analytische Funktionen und algebraische Zahlen I. Hamburger Abhandlungen **1,** 102–126 (1922).
22. Hecke, E.: Mathematische Werke, 2nd ed., Göttingen: Vandenhoeck & Ruprecht 1970.
23. Hua, L.-K.: Die Abschätzung von Exponentialsummen und ihre Anwendung in der Zahlentheorie. Enzyklopädie der mathematischen Wissenschaften, Bd. I, 2, Heft 13, Teil I, Leipzig: Teubner 1959.

24. Hurwitz, A.: Grundlehren einer independenten Theorie der elliptischen Modulfunktionen und Theorie der Multiplicator Gleichungen erster Stufe. Math. Ann. **18,** 528–592 (1881).
25. Hurwitz, A.: Mathematische Werke, 2 vols., Basel: Birkhauser 1932–1933.
26. Ingham, A. E.: The Distribution of Prime Numbers, Cambridge: Cambridge Press 1932.
27. Iseki, S.: The transformation formula for the Dedekind modular function and related functional equations. Duke Math. J. **24,** 653–662 (1957).
28. Jacobi, C. G. J.: Fundamenta nova theoriae functionum ellipticarum, Regiomonti, fratrum Bornträger, 1829. Reprinted in [28a] below, vol. 1, pp. 49–239.
28a. Jacobi: Gesammelte Werke, Berlin: Reimer 1881–1891.
29. Kolberg, O.: Some identities involving the partition function. Math. Scandinavica **5,** 77–92 (1957).
30. Korobov, N. M.: Estimates of trigonometric sums and their applications. Uspehi Mat. Nauk **13,** No. 4 (82), 185–192 (1958).
31. Landau, E.: Elementary numbery theory, New York: Chelsea 1958.
32. Landau, E.: Handbuch der Lehre von der Verteilung der Primzahlen, 2d ed., New York: Chelsea Publ. Co. 1953.
33. Landau, E.: Vorlesungen über Zahlentheorie, 3 vols., New York: Chelsea Publ. Co. 1946–1947.
34. Lehman, R. S.: On the difference $\pi(x) - \mathrm{li}\, x$. Acta Arithmetica **11,** 397–410 (1956).
35. Lehmer, D. H.: On the series for the partition function. Trans. Amer. Math. Soc. **43,** 271–295 (1938).
36. Lipschitz, R.: Untersuchung der Eigenschaften einer Gattung von unendlichen Reihen. J. reine angew. Math. **105,** 127–156 (1889).
37. Littlewood, J. E.: Sur la distribution des nombres premiers. C. R. Acad. Sc. Paris **158,** 1869–1872 (1914).
38. MacMahon, P. A.: Combinatory Analysis, 2 vols., Cambridge: Cambridge Press 1915–1916.
39. Meissner, W.: Tafel der Γ-Funktion mit komplexem Argumentbereich. Deutsche Math. **4,** 535–555 (1939).
40. Mellin, H.: Abriß einer einheitlichen Theorie der Gamma- und der hypergeometrischen Funktionen. Math. Ann. **68,** 305–337 (1910).
41. Mordell, L. J.: Note on certain modular relations considered by Messrs. Ramanujan, Darling, and Rogers. Proc. London Math. Soc. (2) **20,** 408–416 (1922).
42. Mordell, L. J.: Some applications of Fourier series in the analytic theory of numbers. Proc. Cambridge Philos. Soc. **24,** 585–596 (1928).
43. Mordell, L. J.: The reciprocity formula for Dedekind sums. Amer. J. Math. **73,** 593–598 (1951).
44. Nielsen, N.: Traité Elémentaire des Nombres de Bernoulli, Paris: Gauthier-Villars 1923.
45. Phragmén, E., Lindelöf, E.: Sur une extension d'un principe classique de l'analyse. Acta Math. **31,** 381–406 (1908).
46. Prachar, K.: Primzahlverteilung, Berlin/Göttingen/Heidelberg: Springer 1957.
47. Rademacher, H.: Die Reziprozitätsformel für Dedekindsche Summen. Acta Sci. Math. (Szeged) **12** (B), 57–60 (1950).
48. Rademacher, H.: Eine arithmetische Summenformel. Monatshefte für Math. u. Physik **39,** 221–228 (1932).
49. Rademacher, H.: Lectures on elementary number theory, New York: Blaisdell 1964.
50. Rademacher, H.: On the Phragmén-Lindelöf theorem and some applications. Math. Z. **72,** 192–204 (1959).

51. Rademacher, H.: On the Selberg formula for $A_k(n)$. J. Indian Math. Soc. (N.S.) **21** (1957), 41–55 (1958).
52. Rademacher, H.: On the transformation of log $\eta(\tau)$. J. Indian Math. Soc. **19** 25–30 (1955).
53. Rademacher, H.: The Ramanujan identities under modular substitutions. Trans. Amer. Math. Soc. **51**, 609–636 (1942).
54. Rademacher, H.: Zur additiven Primzahltheorie algebraischer Zahlkörper III. Math. Z. **27**, 321–426 (1927).
55. Rademacher, H.: Zur Theorie der Modulfunktionen. Crelle **167**, 312–226 (1931).
56. Rademacher, H., Whiteman, A.: Theorems on Dedekind sums. American J. Math. **63**, 377–407 (1941).
57. Rademacher, H., Zuckerman, H. S.: A new proof of two of Ramanujani's identities. Ann. of Math. **40**, 473–489 (1939).
58. Radó, T.: Subharmonic Functions (Ergebnisse der Mathematik, vol. 5/1), Berlin: Springer 1937.
59. Ramanujan, S.: Collected Papers, Cambridge: Cambridge Press 1927.
60. Rankin, R. A.: A certain class of multiplicative functions. Duke Math. J. **13**, 281–306 (1946).
61. Rogers, G. L.: On two theorems of Combinatory Analysis and some allied identities. Proc. London Math. Soc. (2) **16**, 315–336 (1917).
62. Rogers, G. L.: Second memoir on the expansion of certain infinite products. Proc. London Math. Soc. (1) **25**, 318–343 (1894).
63. Rogers, L. J.: On a type of modular relation. Proc. London Math. Soc. (2), 387–397 (1921).
64. Saalschütz, L.: Vorlesungen über die Bernoullischen Zahlen, ihren Zusammenhang mit den Secanten-Coefficienten und ihre wichtigeren Anwendungen, Berlin: Springer 1893.
65. Schnee, W.: Die Funktionalgleichung der Zetafunktion und der Dirichletschen Reihen mit periodischen Koeffizienten. Math. Z. **31**, 378–390 (1930).
66. Schur, I.: Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 302–321 (1917).
67. Selberg, A.: Contributions to the theory of the Riemann zeta-function. Arch. for Math. og Naturv. B, **48**, No. 5 (1946).
68. Selberg, A.: On the zeros of Riemannis zeta-function on the critical line. Arch. for Math. og Naturv. B, **45**, 101–114 (1942).
69. Siegel, C. L.: Über Riemannis Nachlaß zur analytischen Zahlentheorie. Quellen u. Studien f. Geschichte d. Math., Astr. u. Phys., Abt. B, **2**, 45–80 (1932).
70. Sylvester, J. J.: On a discovery in the theory of partitions. Quarterly J. of Math. **1**, 81–84 (1857); On the partition of numbers, *ibid.*, 141–152.
71. Sylverster, J. J.: Outlines of seven lectures on the partitions of numbers. Proc. London Math. Soc. **27**, 33–96 (1896).
72. Tannery, J., Molk, J.: Eléments de la Théorie des Fonctions Elliptiques, 4 vols., Paris: Gauthiers-Villars 1896.
73. Titchmarsh, E. C.: Introduction to the theory of Fourier integrals, Oxford: Clarendon Press 1937.
74. Titchmarsh, E. C.: The theory of the Riemann Zeta-function, Oxford: Clarendon Press 1951.
75. Walton, J. B.: Theta series in the Gaussian field. Duke Math. J. **16**, 479–491 (1949).
76. Watson, G. N.: A treatise on the theory of Bessel functions, 2nd ed., Cambridge: Cambridge University Press 1962.

77. Weber, H.: Lehrbuch der Algebra, vol. 3: Elliptische Funktionen und algebraische Zahlen, Braunschweig: Vieweg 1908.
78. Whiteman, A. L.: A sum connected with the series for the partition function. Pacific J. Math. **6,** 159–176 (1956).
79. Whittacker, E. M., Watson, G. N.: A course of modern analysis, 4th ed., Cambridge: Cambridge Press 1940.
80. Widder, D. V.: Functions harmonic in a strip. Proc. Amer. Math. Soc. **12,** 67–72 (1961).
81. Wigert, S.: Sur un théorème concernant les fonctions entières. Ark. für Mat. Astron. och Fys. **11,** Nr. 21, 5 pp. (1916).

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