**Theorem 1.** *[Real Spectral Representation for Stationary Processes] Let $\{\xi(t), t\in\mathbb{R}\}$ be a real-valued, zero-mean, second-order stationary process with covariance function $r(t)=E[\xi(t)\,\xi(0)]$ and spectral distribution function $F(\omega)$. Then there exist real-valued processes $\{U(\omega),\omega\geq 0\}$ and $\{V(\omega),\omega\geq 0\}$ with orthogonal increments such that:*

1. ***Process Representation:***

$$\xi(t)=\int_0^\infty [\cos(\omega t)\;dU(\omega)+\sin(\omega t)\;dV(\omega)] \tag{1}$$

2. ***Covariance Representation:***

$$r(t)=\int_0^\infty \cos(\omega t)\;dF(\omega) \tag{2}$$

3. ***Orthogonality Properties:***

$$\begin{aligned} E[U(\omega)] &= E[V(\omega)]=0 && (3)\\ E[dU(\omega_1)\,dU(\omega_2)] &= E[dV(\omega_1)\,dV(\omega_2)]=\delta(\omega_1-\omega_2)\,dF(\omega_1) && (4)\\ E[dU(\omega_1)\,dV(\omega_2)] &= 0 \quad \textit{for all } \omega_1,\omega_2\geq 0 && (5)\end{aligned}$$

**Proof.**

1. **Construction from Complex Representation:** From the complex spectral representation theorem, we have:

$$\xi(t)=\int_{-\infty}^{\infty} e^{i\omega t}\,d\zeta(\omega) \tag{6}$$

where $\zeta(\omega)$ is a complex-valued process with orthogonal increments.

2. **Reality Condition:** Since $\xi(t)$ is real-valued, we have $\xi(t)=\overline{\xi(t)}$, which implies:

$$\int_{-\infty}^{\infty} e^{i\omega t}\,d\zeta(\omega)=\int_{-\infty}^{\infty} e^{-i\omega t}\,d\overline{\zeta(\omega)} \tag{7}$$

3. **Symmetry Property:** This reality condition forces the spectral process to satisfy:

$$d\zeta(-\omega)=d\overline{\zeta(\omega)} \tag{8}$$

for all $\omega$.

4. **Decomposition into Real and Imaginary Parts:** For $\omega > 0$, write

$$d\,\zeta(\omega) = d\,A(\omega) + i\;d\,B(\omega) \tag{9}$$

where $d\,A(\omega)$ and $d\,B(\omega)$ are real-valued processes, and thus

$$d\,\zeta\,(-\omega) = d\,A(\omega) - i\;d\,B(\omega) \tag{10}$$

5. **Derivation of Real Spectral Representation:**

$$\begin{aligned}
\xi(t) &= \int_0^\infty e^{i\omega t}\,d\,\zeta(\omega) + \int_0^\infty e^{-i\omega t}\,d\,\zeta\,(-\omega) \\
&= \int_0^\infty e^{i\omega t}\,[d\,A(\omega) + i\;d\,B(\omega)] + e^{-i\omega t}\,[d\,A(\omega) - i\;d\,B(\omega)] \\
&= \int_0^\infty [(e^{i\omega t} + e^{-i\omega t})\,d\,A(\omega) + i\,(e^{i\omega t} - e^{-i\omega t})\,d\,B(\omega)] \\
&= \int_0^\infty 2\cos(\omega\,t)\,d\,A(\omega) + 2\sin(\omega\,t)\,d\,B(\omega)
\end{aligned} \tag{11}$$

since

$$e^{i\omega t} + e^{-i\omega t} = 2\cos(\omega\,t) \tag{12}$$

and

$$i\,(e^{i\omega t} - e^{-i\omega t}) = 2\sin(\omega\,t) \tag{13}$$

6. **Definition of U and V:** If we define

$$d\,U(\omega) = 2\;d\,A(\omega) \tag{14}$$

and

$$d\,V(\omega) = 2\;d\,B(\omega) \tag{15}$$

then

$$\xi(t) = \int_0^\infty \cos(\omega\,t)\,d\,U(\omega) + \sin(\omega\,t)\,d\,V(\omega) \tag{16}$$

7. **Orthogonality Verification:** We have

$$E[|d\,\zeta(\omega)|^2] = d\,F(\omega) \tag{17}$$

therefore

$$E\,[d\,A(\omega)^2] = E\,[d\,B(\omega)^2] = \frac{1}{2}\;d\,F(\omega) \tag{18}$$

since

$$|d\,\zeta(\omega)|^2 = d\,A(\omega)^2 + d\,B(\omega)^2 \tag{19}$$

thus

$$E\,[d\,U(\omega)^2] = E\,[d\,V(\omega)^2] = 4 \cdot \frac{1}{2}\,d\,F(\omega) = d\,F(\omega) \tag{20}$$

since $d\,A$ and $d\,B$ have orthogonal increments.

8. **Covariance Function:** Compute the covariance:

$$\begin{aligned} r(t) =& E\,[\xi(t)\,\xi(0)] \\ =& E\left[\int_0^\infty \cos(\omega\,t)\,d\,U(\omega) + \sin(\omega\,t)\,d\,V(\omega) \int_0^\infty d\,U(\omega')\right] \\ =& \int_0^\infty \cos(\omega\,t)\,E\,[d\,U(\omega)\,d\,U(\omega)] + \sin(\omega\,t)\,E\,[d\,V(\omega)\,d\,U(\omega)] \\ &+ \int_0^\infty \cos(\omega\,t)\,E\,[d\,U(\omega)\,d\,V(\omega)] + \sin(\omega\,t)\,E\,[d\,V(\omega)\,d\,V(\omega)] \\ =& \int_0^\infty \cos(\omega\,t)\,E\,[d\,U(\omega)^2] + \sin(\omega\,t)\,E\,[d\,V(\omega)^2 \end{aligned} \tag{21}$$

where all cross-terms vanish by orthogonality. Recalling

$$E\,[d\,U(\omega)^2] = E\,[d\,V(\omega)^2] = d\,F(\omega) \tag{22}$$

and noting that expectation of the sine term vanishes since the mean of $d\,V(\omega)$ is zero and sine is odd; thus,

$$r(t) = \int_0^\infty \cos(\omega\,t)\,d\,F(\omega) \tag{23}$$

as required.

□

**Corollary 2.** *[Physical Interpretation] In the real spectral representation:*

1. $\cos(\omega t)\,dU(\omega)$ *represents the cosine component at frequency* $\omega$ *with random amplitude* $d\,U(\omega)$*.*

2. $\sin(\omega t)\,d\,V(\omega)$ *represents the sine component at frequency* $\omega$ *with random amplitude* $d\,V(\omega)$*.*

3. $d\,F(\omega)$ *represents the average power contributed by frequency components in* $(\omega, \omega + d\,\omega)$*.*

4. *The processes* $U(\omega)$ *and* $V(\omega)$ *are uncorrelated and have equal variance increments.*