Unitary Time Changes of Stationary Gaussian Processes

by Stephen Crowley

August 12, 2025

Theorem 1. *[Gain Function for Unitary Time Changes] Let $X(t)$ be a zero-mean stationary Gaussian process and $\theta: \mathbb{R} \to \mathbb{R}$ be an absolutely continuous bijection with $\theta'(t) \neq 0$ almost everywhere. The unitary time change produces an oscillatory process $Z(t)$ with gain function*

$$A_t(\lambda) = \sqrt{|\theta'(t)|}\; e^{i\lambda(\theta(t) - t)}$$

Definition 2. *[Unitary Time Change Operator] Let $\theta: \mathbb{R} \to \mathbb{R}$ be an absolutely continuous bijection with $\theta'(t) \neq 0$ almost everywhere. The unitary time change operator U_θ on $L^2(\mathbb{R})$ is defined by*

$$(U_\theta f)(t) = \sqrt{|\theta'(t)|}\; f(\theta(t))$$

Lemma 3. *[Unitarity of Time Change Operator] The operator U_θ defined above is unitary on $L^2(\mathbb{R})$.*

Proof. For any $f \in L^2(\mathbb{R})$, compute

$$\|U_\theta f\|_2^2 = \int_{-\infty}^{\infty} |f(\theta(t))|^2 |\theta'(t)|\; dt \tag{1}$$

By the change of variables $s = \theta(t)$, we have $ds = \theta'(t)\; dt$, so

$$\|U_\theta f\|_2^2 = \int_{-\infty}^{\infty} |f(s)|^2\; ds = \|f\|_2^2 \tag{2}$$

Thus U_θ is an isometry. Since θ is a bijection, U_θ is surjective, hence unitary. □

Definition 4. *[Stationary Gaussian Process] A zero-mean stationary Gaussian process $X(t)$ has the spectral representation*

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\, d\,\Phi(\lambda)$$

where $\Phi(\lambda)$ is a complex-valued orthogonal increment process with $E[|d\,\Phi(\lambda)|^2] = d\,F(\lambda)$ for some finite measure F.

Definition 5. *[Oscillatory Process] An oscillatory process in the sense of Priestley is a process $Z(t)$ with the representation*

$$Z(t) = \int_{-\infty}^{\infty} \varphi_t(\lambda)\; d\,\Phi(\lambda)$$

where $\varphi_t(\lambda)$ is the oscillatory function and $\Phi(\lambda)$ is as in the stationary case. The gain function $A_t(\lambda)$ is defined by

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t}$$

so that

$$Z(t) = \int_{-\infty}^{\infty} A_t(\lambda)\, e^{i\lambda t}\; d\,\Phi(\lambda)$$

Start with the stationary process

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\; d\,\Phi(\lambda)$$

Apply the unitary time change operator to obtain

$$Z(t) = (U_\theta X)(t) = \sqrt{|\theta'(t)|}\; X(\theta(t))$$

Substituting the spectral representation:

$$Z(t) = \sqrt{|\theta'(t)|} \int_{-\infty}^{\infty} e^{i\lambda\theta(t)}\; d\,\Phi(\lambda) \tag{3}$$

$$= \int_{-\infty}^{\infty} \sqrt{|\theta'(t)|}\; e^{i\lambda\theta(t)}\; d\,\Phi(\lambda) \tag{4}$$

To express this in oscillatory form, factor out $e^{i\lambda t}$:

$$Z(t) = \int_{-\infty}^{\infty} \sqrt{|\theta'(t)|}\; e^{i\lambda(\theta(t) - t)}\; e^{i\lambda t}\; d\,\Phi(\lambda) \tag{5}$$

Comparing with the oscillatory representation $Z(t) = \int_{-\infty}^{\infty} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda)$, we identify the gain function:

$$A_t(\lambda) = \sqrt{|\theta'(t)|}\; e^{i\lambda(\theta(t)-t)}$$

The oscillatory function is therefore

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} = \sqrt{|\theta'(t)|}\; e^{i\lambda\theta(t)}$$

Theorem 6. *[Kernel Representation] The covariance kernel of the oscillatory process $Z(t)$ is given by*

$$K_Z(s,t) = \sqrt{|\theta'(s)||\,\theta'(t)|} \int_{-\infty}^{\infty} e^{i\lambda(\theta(s)-\theta(t))}\, d\,F(\lambda)$$

where $F(\lambda)$ is the spectral measure of the original stationary process.

Proof. The covariance is

$$\begin{aligned}
K_Z(s,t) =& E[Z(s)\overline{Z(t)}] && (6)\\
=& E\Big[\int_{-\infty}^{\infty} A_s(\lambda)\, e^{i\lambda s}\, d\,\Phi(\lambda) \int_{-\infty}^{\infty} \overline{A_t(\mu)\, e^{i\mu t}}\, d\overline{\Phi(\mu)}\Big] && (7)\\
=& \int_{-\infty}^{\infty} A_s(\lambda)\overline{A_t(\lambda)} e^{i\lambda(s-t)}\, d\,F(\lambda) && (8)
\end{aligned}$$

Substituting the gain function:

$$\begin{aligned}
K_Z(s,t) =& \int_{-\infty}^{\infty} \sqrt{|\theta'(s)|}\, e^{i\lambda(\theta(s)-s)}\, \sqrt{|\theta'(t)|}\, e^{-i\lambda(\theta(t)-t)}\, e^{i\lambda(s-t)}\, d\,F(\lambda) && (9)\\
=& \sqrt{|\theta'(s)||\,\theta'(t)|} \int_{-\infty}^{\infty} e^{i\lambda(\theta(s)-\theta(t))}\, d\,F(\lambda) && (10)
\end{aligned}$$

□