1 Stationary Dilations

Definition 1. *Let* $(\Omega, \mathcal{F}, P)$ *and* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ *be probability spaces. We say that* $(\Omega, \mathcal{F}, P)$ *is a factor of* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ *if there exists a measurable surjective map* $\phi: \tilde{\Omega} \to \Omega$ *such that:*

1. *For all* $A \in \mathcal{F}$, $\phi^{-1}(A) \in \tilde{\mathcal{F}}$
2. *For all* $A \in \mathcal{F}$, $P(A) = \tilde{P}(\phi^{-1}(A))$

In other words, $(\Omega, \mathcal{F}, P)$ *can be obtained from* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ *by projecting the larger space onto the smaller one while preserving the probability measure structure.*

Remark 2. In the context of stationary dilations, this means that the original non-stationary process $\{X_t\}$ can be recovered from the stationary dilation $\{Y_t\}$ through a measurable projection that preserves the probabilistic structure of the original process.

Definition 3. (Stationary Dilation) *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space and let* $\{X_t\}_{t \in \mathbb{R}_+}$ *be a nonstationary stochastic process. A stationary dilation of* $\{X_t\}$ *is a stationary process* $\{Y_t\}_{t \in \mathbb{R}_+}$ *defined on a larger probability space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ *such that:*

1. $(\Omega, \mathcal{F}, P)$ *is a factor of* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$
2. *There exists a measurable projection operator* Π *such that:*

$$X_t = \Pi\, Y_t \quad \forall t \in \mathbb{R}_+ \tag{1}$$

Theorem 4. (Representation of Nonstationary Processes) *For a continuous-time nonstationary process* $\{X_t\}_{t \in \mathbb{R}_+}$, *its stationary dilation exists which has sample paths* $t \mapsto X_t(\omega)$ *which are continuous with probability one when* X_t:

- *is uniformly continuous in probability over compact intervals:*

$$\lim_{s \to t} P\left(|X_s - X_t| > \epsilon\right) = 0 \quad \forall \epsilon > 0, t \in [0, T], T > 0 \tag{2}$$

- *has finite second moments:*

$$\mathbb{E}[|X_t|^2] < \infty \quad \forall t \in \mathbb{R}_+ \tag{3}$$

- *has an integral representation of the form:*

$$X_t = \int_0^t \eta(s)\, d\, s \tag{4}$$

where $\eta(t)$ is a measurable random function that is stationary in the wide sense (with $\int_0^t \mathbb{E}[|\eta(s)|^2]\, d\,s < \infty$ for all t)

- *and has a covariance operator*

$$R(t,s) = \mathbb{E}\,[X_t\, X_s] \tag{5}$$

which is symmetric $(R(t,s) = R(s,t))$, positive definite and continuous

Under these conditions, there exists a representation:

$$X_t = M(t) \cdot S_t \tag{6}$$

where:

- *$M(t)$ is a continuous deterministic modulation function*
- *$\{S_t\}_{t \in \mathbb{R}_+}$ is a stationary process*

This representation can be obtained through the stationary dilation by choosing:

$$Y_t = \begin{pmatrix} M(t) \\ S_t \end{pmatrix} \tag{7}$$

with the projection operator Π defined as:

$$\Pi\, Y_t = M(t) \cdot S_t \tag{8}$$

Proposition 5. (Properties of Dilation) *The stationary dilation satisfies:*

1. *Preservation of moments:*

$$\mathbb{E}[|X_t|^p] \leq \mathbb{E}[|Y_t|^p] \quad \forall p \geq 1 \tag{9}$$

2. *Minimal extension: Among all stationary processes that dilate X_t, there exists a minimal one (unique up to isomorphism) in terms of the probability space dimension*

Corollary 6. *For any nonstationary process satisfying the above conditions, the stationary dilation provides a canonical factorization into deterministic time-varying components and stationary stochastic components.*