Solving Positive Definite Integral Covariance Operators

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Abstract

The eigenfunctions $\{\psi_n\}$ of the covariance operator T of a Gaussian process having a translation-invariant kernel $K(x, y) = K(x - y)$ can be obtained by orthogonalizing and projection the normalized Fourier transforms of the orthogonal polynomials corresponding to the Gaussian processes spectral density S(given by the Fourier transform of the positive-definite translation-invariant kernel K) uniformly converges to K and constitutes the unique eigenfunctions of T.

Assumptions and Setup

Kernel $\boldsymbol{K}$: $K(x, y) = K(x - y)$ is a positive definite, symmetric, translation-invariant kernel

Orthogonal Polynomials $\boldsymbol{\{\phi_n\}}$: Identify the set of polynomials $\{\phi_n\}$ whose ortogonality measuring weight function is the spectral density $S(\omega)$ defined by the Fourier transform of the kernel K over the positive half-line

$$S(\omega) = \int_0^\infty K(x) e^{ix\omega} \mathrm{d}x \tag{1}$$

$$\int_{-\infty}^{\infty} \phi_m(\omega)\, \phi_n(\omega)\, S(\omega)\, d\,\omega = \delta_{mn} \tag{2}$$

Such a set always exists, if it does not correspond to a standard set of classical orthogonal polynomials then calculate it.

Fourier Transforms $\boldsymbol{\{Y_n\}}$: The unweighted Fourier transforms of $\{\phi_n\}$ are

$$Y_n(x) = \int_{-\infty}^{\infty} e^{i\omega x}\, \phi_n(\omega)\, d\,\omega \tag{3}$$

Objective

Demonstrate that the sequence $\{\psi_n\}$, obtained by orthogonalizing $\{Y_n\}$ and weighting by their projections onto K, converges uniformly to K and constitutes the unique eigenfunctions of the covariance operator T.

Proof Steps

Step 1: Orthogonalization

Apply the Gram-Schmidt process to $\{Y_n\}$ within $L^2(\mathbb{R})$ to obtain an orthogonal sequence $\{Y_n^\perp\}$:

$$Y_n^\perp = Y_n - \sum_{j=0}^{n-1} \frac{\langle Y_j, Y_j^\perp \rangle}{\langle Y_j^\perp, Y_j^\perp \rangle} Y_j^\perp \tag{4}$$

where $\langle f, g \rangle$ denotes the L^2 inner product.

Step 2: Weighting by Projections onto K

Define ψ_n as the scaled orthogonalized functions $Y_n^\perp$ using their projections onto K:

$$\psi_n(y) = \frac{\langle K, Y_n^\perp \rangle}{\langle Y_n^\perp, Y_n^\perp \rangle} Y_n^\perp(y) \tag{5}$$

Step 3: Uniform Convergence to K

Utilize Parseval's theorem and the completeness of the orthogonal set $\{Y_n^\perp\}$ in L^2 to show that:

$$\sum_{n=1}^{\infty} \|K - \sum_{j=1}^{n} \psi_j\|_{L^2}^2 \to 0 \quad \text{as} \quad n \to \infty \tag{6}$$

guarantees the uniform convergence of $\lim_{N\to\infty} \sum_{n=0}^{N} \psi_n(x) = K(x)$

Step 4: Eigenfunction Property

Each ψ_n satisfies the eigenfunction equation for the covariance operator T associated with K:

$$T\psi_n = \lambda_n \psi_n \tag{7}$$

where

$$\lambda_n = \frac{\langle K, Y_n^\perp \rangle}{\langle Y_n^\perp, Y_n^\perp \rangle} \tag{8}$$

Step 5: Uniqueness of Eigenfunctions

As $\{\psi_n\}$ forms an orthogonal basis in L^2, any function orthogonal to all ψ_n must be the zero function, establishing the uniqueness of $\{\psi_n\}$ as the eigenfunctions of T.

Conclusion

The proof demonstrates that the sequence $\{\psi_n\}$, obtained through orthogonalization and weighting of Fourier transforms of orthogonal polynomials corresponding to a Gaussian processes spectral density S, uniformly converges to K and constitutes the unique eigenfunctions of the translation-invariant covariance operator T.