The Bessel Polynomial Moments

1. Introduction

Since 1949 when the Bessel polynomials appeared, a hunt has been on to find a weight function ψ, defined on the real axis, with respect to which the polynomials would be orthogonal. Although at least two devices have been found which formally seem to fill the role of orthogonality generation, none was expressible in terms of integration on the real axis.

In a paper by Krall [1] it was shown that the Bessel polynomials $y_n(x,a,b)$ are orthogonal with respect to the distribution ψ given by

$$\langle f,\psi\rangle = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\alpha}^{\beta} f(x)\,\mathrm{Im}\{(-\frac{b}{x+i\,\varepsilon})\,{}_1F_1(a,-\frac{b}{x+i\,\varepsilon})\}\,d\,x\forall\alpha<0,\beta>0 \tag{1}$$

The key to showing that the above formula is valid is through the moments associated with the Bessel polynomials

$$\mu_n = \frac{(-b)^{n+1}}{(a)_n} \tag{2}$$

where $(a)_n = a\,(a+1)\,\ldots\,(a+n-1)\forall n=0,1,\ldots$ are the rising factorials, which were discovered by examining the differential equation satisfied by the Bessel polynomials. It is well known that, given the moments μ_n, there exists a function of a complex variable $f(z)$, called the Cauchy representation, given by

$$f(z) = \sum_{n=0}^{\infty}\frac{\mu_n}{z^{n+1}} \tag{3}$$

when the support of the weight function is compact, and in general satisfying

$$\lim_{z\to\infty} z^{k+1}\left[f(z)+\sum_{n=0}^{k}\frac{\mu_n}{z^{n+1}}\right] = -\mu_k\,\forall\delta<\arg\,z<\pi-\delta\exists\delta\in\left(0,\frac{\pi}{2}\right) \tag{4}$$

For the Bessel polynomials it is easy to see that

$$f(z) = \sum_{n=0}^{\infty}\frac{(-\frac{b}{z})^{n+1}}{(a)_n} = \frac{-b}{z}\cdot{}_1F_1(\;[1],[a]\;,-\frac{b}{z}) \tag{5}$$

By using the Stieltjes–Perron formula ([4] or in the distributional case [1]), the formula for ψ immediately follows. The fact that the interval $[\alpha,\beta]$ needs only to include the point of follows from the analyticity of $f(z)$ in any region excluding the origin.

2. Results

We state in summary

Theorem 1. The Bessel polynomials are mutually orthogonal with respect to ψ. That is,

$$\langle y_n, y_m, \psi \rangle = 0 \forall n \neq m$$

This, of course, follows from the Stieltjes–Perron formula, which shows that ψ generates the moments through $\langle x^n, \psi \rangle = \mu_n \forall n = 0, 1, \ldots$. A direct proof may be quickly given, however by noting that bracketed term

$$\psi(\beta) - \psi(\alpha) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\alpha}^{\beta} \sum_{n=0}^{\infty} \frac{(-b)^{n+1}}{(a)_n} \left[\sum_{j=0}^{[\frac{n}{2}]} \binom{n+1}{2j+1} \frac{n+1}{(2\,j+1)!} \frac{(-1)^{j+1} \varepsilon^{2j+1} x^{n-2j}}{(x^2+\varepsilon^2)^{n+1}} \right] d\,x$$

converges to

$$\lim_{\varepsilon \to 0} \frac{(-1)^n \frac{d^n}{d\,x^n} \left(\frac{\varepsilon}{x^2+\varepsilon^2} \right)}{n!} = \frac{(-1)^n \delta^{(n)}(x)}{n!} \tag{6}$$

as $\varepsilon \to 0$ where $\delta^{(n)}(x)$ is the nth distributional derivative of the Dirac delta function. Thus ψ has the distributional expansion

$$\psi = \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}(x)}{n!} \tag{7}$$

which is well known.

Theorem 2. For $n = 0, 1, \ldots,$

$$\langle y_n^2, \psi \rangle = \frac{b\,(-1)^{n+1} n!(2\,n+a-1)}{(a)_n} \tag{8}$$

This follows from using the Bessel polynomial's three term recurrence relation [2] and the fact that $\mu_0 = -b$.

Special cases. The instances where $a = b = 2$ and $a = b = 1$ are of historical importance.

Example 1. When $a = b = 2$, the moments are given by $\mu_n = (-2)^{n+1} (n+1)!$ and thus

$$\psi(\beta) - \psi(\alpha) = \lim_{\epsilon \to 0} \frac{-1}{\pi} \int_{\alpha}^{\beta} e^{-\frac{x}{x^2+\epsilon^2}} \sin\left[\frac{2\,\epsilon}{x^2+\epsilon^2} \right] d\,x \tag{9}$$

Then

$$\langle y_n, y_m, \psi \rangle = \begin{cases} 0 & \text{if } n \neq m \\ \frac{2(-1)^{n+1}}{2n+1} & \text{if } n = m \end{cases} \tag{10}$$

Example. When $a = b = 1$, the moments are given by

$$\mu_n = (-1)^{n+1} n! \tag{11}$$

and therefore

$$\psi(\beta) - \psi(\alpha) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_\alpha^\beta e^{-\frac{x}{x^2+\epsilon^2}} \left[\frac{\varepsilon \cos\left(\frac{\epsilon}{x^2+\epsilon^2}\right)}{x^2+\varepsilon^2} - \frac{x \sin\left(\frac{\epsilon}{x^2+\epsilon^2}\right)}{x^2+\epsilon^2} \right] dx \tag{12}$$

Then

$$\langle y_n y_m, \psi \rangle = \begin{cases} 0 & \text{if } n \neq m \\ \frac{(-1)^{n+1}}{2} & \text{if } n = m \end{cases} \tag{13}$$

Bibliography

[1] A.M. Krall. The besel polynomial moment problem. *Acta Mathematica Academiae Scientiarum Hungarica*, 38:105–107, 1981.