Upper Bounds on Covering Numbers for Stationary Covariance Kernels and Operators

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Consider a stationary covariance kernel K with orthonormal expansion on $L^2(0,\infty)$:

$$K(x) = \sum_{i=0}^{\infty} \langle K, \psi_i \rangle \, \psi_i(x) \tag{1}$$

The associated covariance operator T is self-adjoint due to stationarity:

$$T(f)(y) = \int_0^{\infty} K\,(x - y)\, f(x)\, d\, x \tag{2}$$

Define:

- For kernel K:

$$N_K(\varepsilon) = \min\{n \colon \|K - K_n\|_{\infty} \leq \varepsilon\} \tag{3}$$

  where

$$K_n(x) = \sum_{i=0}^{n} \langle K, \psi_i \rangle \, \psi_i(x) \tag{4}$$

  and

$$\|K - K_n\|_{\infty} = \max_{x \in [0,\infty)} |K(x) - K_n(x)| \tag{5}$$

- For operator T:

$$N_T(\varepsilon) = \min\{n \colon \|T - T_n\| \leq \varepsilon\} \tag{6}$$

  where T_n is the operator with kernel K_n and $\|T\|$ is the operator norm

Define:

$$n(\varepsilon) = \max\{k \colon |\langle K, \psi_k \rangle| > \sqrt{\varepsilon}\}$$

Note: The stationarity of K ensures that T is self-adjoint, which guarantees real eigenvalues and orthogonal eigenfunctions.

Theorem 1

For stationary kernel K,

$$N_K(\varepsilon) = N_T(\varepsilon) \leq n(\varepsilon) \tag{7}$$

Proof. 1. Kernel bound $N_K(\varepsilon) \leq n(\varepsilon)$:

$$|K(x) - K_{n(\varepsilon)}(x)| = |\sum_{i>n(\varepsilon)} \langle K, \psi_i \rangle \, \psi_i(x)| \tag{8}$$

By Cauchy-Schwarz:

$$\leq \sqrt{\sum_{i>n(\varepsilon)} |\langle K, \psi_i \rangle|^2} \sqrt{\sum_{i>n(\varepsilon)} |\psi_i(x)|^2} \tag{9}$$

Since $\{\psi_i\}$ is orthonormal, $|\psi_i(x)| \leq 1$ and $\sum |\psi_i(x)|^2 \leq 1$ By definition of $n(\varepsilon)$, $|\langle K, \psi_i \rangle| \leq \sqrt{\varepsilon}$ for $i > n(\varepsilon)$ Therefore:

$$\|K - K_{n(\varepsilon)}\|_\infty \leq \varepsilon \tag{10}$$

2. Operator bound $\|T - T_n\| \leq \|K - K_n\|_\infty$: For any f with $\|f\| \leq 1$:

$$\|T(f) - T_n(f)\| \leq \|K - K_n\|_\infty \|f\| \leq \|K - K_n\|_\infty \tag{11}$$

3. Reverse inequality

$$\|T - T_n\| \geq \|K - K_n\|_\infty \tag{12}$$

: Using stationarity, for any x_0, construct f_{x_0} as follows:

$$f_{x_0}(x) = \frac{\phi\left(\frac{x - x_0}{\delta}\right)}{\sqrt{\delta}} \tag{13}$$

where: - ϕ is a smooth bump function with support in [-1,1] and $\|\phi\|_{L^2} = 1$ - $\delta > 0$ is chosen small enough

Then by stationarity of K:

$$\lim_{\delta \to 0} (T - T_n)(f_{x_0})(x_0) = (K - K_n)(x_0) \tag{14}$$

Since $\|f_{x_0}\| = 1$ by construction, this shows:

$$\|T - T_n\| \geq \|K - K_n\|_\infty \tag{15}$$

Therefore

$$\|T - T_n\| = \|K - K_n\|_\infty \tag{16}$$

establishing

$$N_K(\varepsilon) = N_T(\varepsilon) \tag{17}$$

The stationarity of K is essential for the construction in step 3, as it ensures translation invariance and allows the limit argument to work. □