Band-Limited White Noise: Mathematical Formulation and Properties

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1 Fundamental Definitions

Definition 1. *[Band-Limited White Noise]A zero-mean Gaussian stochastic process* $\{W_B(t), t \in \mathbb{R}\}$ *is called band-limited white noise with bandwidth* $B > 0$ *if its power spectral density is given by*

$$S_{W_B}(\omega) = \begin{cases} \frac{N_0}{2}, & |\omega| \leq B \\ 0, & |\omega| > B \end{cases} \tag{1}$$

where $N_0 > 0$ *is the spectral level parameter.*

Definition 2. *[Sinc Function]The sinc function is defined as*

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(x)}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \tag{2}$$

2 Spectral and Covariance Properties

Theorem 3. *[Autocovariance Function]The autocovariance function of band-limited white noise* $W_B(t)$ *is given by*

$$R_{W_B}(\tau) = \frac{N_0 B}{2\pi} \operatorname{sinc}(B\tau) \tag{3}$$

Proof. By the Wiener-Khintchine theorem, the autocovariance function is the inverse Fourier transform of the power spectral density:

$$
\begin{aligned}
R_{W_B}(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{W_B}(\omega)\, e^{i\omega\tau}\, d\omega && (4)\\
&= \frac{1}{2\pi} \int_{-B}^{B} \frac{N_0}{2}\, e^{i\omega\tau}\, d\omega && (5)\\
&= \frac{N_0}{4\pi} \int_{-B}^{B} e^{i\omega\tau}\, d\omega && (6)
\end{aligned}
$$

For $\tau \neq 0$:

$$
\begin{aligned}
R_{W_B}(\tau) &= \frac{N_0}{4\pi} \left[\frac{e^{i\omega\tau}}{i\tau} \right]_{-B}^{B} && (7)\\
&= \frac{N_0}{4\pi i\tau} (e^{iB\tau} - e^{-iB\tau}) && (8)\\
&= \frac{N_0}{4\pi i\tau} \cdot 2i \sin(B\tau) && (9)\\
&= \frac{N_0}{2\pi\tau} \sin(B\tau) && (10)\\
&= \frac{N_0 B}{2\pi} \frac{\sin(B\tau)}{B\tau} && (11)\\
&= \frac{N_0 B}{2\pi} \mathrm{sinc}(B\tau) && (12)
\end{aligned}
$$

For $\tau = 0$:

$$
\begin{aligned}
R_{W_B}(0) &= \frac{N_0}{4\pi} \int_{-B}^{B} d\omega && (13)\\
&= \frac{N_0}{4\pi} \cdot 2B && (14)\\
&= \frac{N_0 B}{2\pi} && (15)
\end{aligned}
$$

Since $\mathrm{sinc}(0) = 1$, we have $R_{W_B}(0) = \frac{N_0 B}{2\pi} \mathrm{sinc}(0) = \frac{N_0 B}{2\pi}$.

Therefore, equation (3) holds for all $\tau \in \mathbb{R}$. □

Theorem 4. *[Variance and Power] The variance of band-limited white noise $W_B(t)$ is*

$$
Var[W_B(t)] = R_{W_B}(0) = \frac{N_0 B}{2\pi} \qquad (16)
$$

Proof. This follows directly from Theorem 3 by setting $\tau = 0$. □

3 Construction and Filtering Properties

Theorem 5. *[Filter Construction]Let $W(t)$ be ideal white noise with power spectral density $S_W(\omega) = N_0/2$ for all $\omega \in \mathbb{R}$. Let $H(\omega)$ be the frequency response of an ideal low-pass filter:*

$$H(\omega) = \begin{cases} 1, & |\omega| \leq B \\ 0, & |\omega| > B \end{cases} \tag{17}$$

Then the output process $Y(t) = (H * W)(t)$ is band-limited white noise with bandwidth B.

Proof. The power spectral density of the output process is given by

$$\begin{aligned} S_Y(\omega) &= |H(\omega)|^2 S_W(\omega) && (18) \\ &= |H(\omega)|^2 \frac{N_0}{2} && (19) \end{aligned}$$

For $|\omega| \leq B$: $H(\omega) = 1$, so $S_Y(\omega) = \frac{N_0}{2}$.

For $|\omega| > B$: $H(\omega) = 0$, so $S_Y(\omega) = 0$.

Therefore:

$$S_Y(\omega) = \begin{cases} \frac{N_0}{2}, & |\omega| \leq B \\ 0, & |\omega| > B \end{cases} \tag{20}$$

This matches the definition of band-limited white noise in Definition 1. □

Theorem 6. *[Impulse Response]The impulse response of the ideal low-pass filter in Theorem 5 is*

$$h(t) = \frac{B}{\pi} \operatorname{sinc}(B t) \tag{21}$$

Proof. The impulse response is the inverse Fourier transform of the frequency response:

$$\begin{aligned} h(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} d\omega && (22) \\ &= \frac{1}{2\pi} \int_{-B}^{B} e^{i\omega t} d\omega && (23) \end{aligned}$$

For $t \neq 0$:

$$h(t) = \frac{1}{2\pi}\left[\frac{e^{i\omega t}}{i t}\right]_{-B}^{B} \tag{24}$$

$$= \frac{1}{2\pi i t}\left(e^{iBt} - e^{-iBt}\right) \tag{25}$$

$$= \frac{1}{2\pi i t} \cdot 2 i \sin(B t) \tag{26}$$

$$= \frac{\sin(B t)}{\pi t} \tag{27}$$

$$= \frac{B}{\pi}\frac{\sin(B t)}{B t} \tag{28}$$

$$= \frac{B}{\pi}\mathrm{sinc}(B t) \tag{29}$$

For $t = 0$:

$$h(0) = \frac{1}{2\pi}\int_{-B}^{B} d\omega \tag{30}$$

$$= \frac{1}{2\pi} \cdot 2 B \tag{31}$$

$$= \frac{B}{\pi} \tag{32}$$

Since $\mathrm{sinc}(0) = 1$, we have $h(0) = \frac{B}{\pi}\mathrm{sinc}(0) = \frac{B}{\pi}$.

Therefore, equation (21) holds for all $t \in \mathbb{R}$. □

4 Spectral Containment Properties

Theorem 7. *[Spectral Support]The band-limited white noise process $W_B(t)$ has spectral support contained in the interval $[-B, B]$.*

Proof. By Definition 1, the power spectral density $S_{W_B}(\omega) = 0$ for $|\omega| > B$. Since the power spectral density completely characterizes the second-order properties of a Gaussian process, all spectral content is contained within $[-B, B]$. □

Corollary 8. *[Sampling Theorem Applicability]The band-limited white noise process $W_B(t)$ satisfies the conditions for the sampling theorem with Nyquist rate $2B$.*

Proof. This follows immediately from Theorem 7 and the classical sampling theorem for band-limited signals. □