The Characteristic Function of the Product of Independent Standard Normal Variables

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Abstract

This paper demonstrates that the characteristic function of the product of two independent standard normal random variables involves the Bessel function of the first kind of order 0. Polar coordinate transformations and properties of Bessel functions are utilized to obtain the closed form expression.

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1 Introduction

The characteristic function of the product of two independent standard normal random variables serves as a fundamental result in probability theory and statistical analysis. This paper presents a rigorous derivation of its closed form.

> **Lemma 1.** *For independent standard normal random variables X and Y, the characteristic function of their product XY is given by:*
>
> $$\phi_{XY}(t) = \frac{J_0\left(\frac{t}{\sqrt{1+t^2}}\right)}{\sqrt{1+t^2}} \tag{1}$$
>
> *where J_0 denotes the Bessel function of the first kind of order zero.*

2 Proof

Proof. The derivation begins with the characteristic function definition:

$$\phi_{XY}(t) = E[e^{itXY}] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{itxy} e^{-(x^2+y^2)/2} \, dx \, dy \tag{2}$$

Polar Coordinate Transformation

The introduction of polar coordinates where $x = r\cos\theta$, $y = r\sin\theta$, and $dx \, dy = r \, dr \, d\theta$ transforms the integral to:

$$\frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} e^{itr^2\cos\theta\sin\theta} r \, e^{-r^2/2} \, d\theta \, dr \tag{3}$$

Variable Substitution

The substitution $u = r^2/2$, with $du = r \, dr$, yields:

$$\frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} e^{2itu\cos\theta\sin\theta} e^{-u} \, d\theta \, du \tag{4}$$

Double Angle Formula

Application of the identity $\cos\theta\sin\theta = \frac{1}{2}\sin(2\theta)$ gives:

$$\frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} e^{itu\sin(2\theta)} e^{-u} \, d\theta \, du \tag{5}$$

Bessel Function Connection

The inner integral relates to the Bessel function through the identity:

$$\int_0^{2\pi} e^{itu\sin(2\theta)} \, d\theta = 2\pi J_0(tu) \tag{6}$$

This follows from the integral representation of the Bessel function of the first kind:

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz\sin(\theta)} \, d\theta \tag{7}$$

The integral thus reduces to:

$$\int_0^\infty J_0(tu) \, e^{-u} \, du \tag{8}$$

Final Evaluation

The evaluation proceeds through the known Laplace transform of Bessel functions:

$$\int_0^\infty J_0(at) \, e^{-ut} \, dt = \frac{1}{\sqrt{1+a^2}} J_0\left(\frac{a}{\sqrt{1+a^2}}\right) \tag{9}$$

This leads to the final result:

$$\phi_{XY}(t) = \frac{J_0\left(\frac{t}{\sqrt{1+t^2}}\right)}{\sqrt{1+t^2}} \tag{10} \quad \square$$

The derivation establishes that the characteristic function of the product of two independent standard normal random variables takes the form $J_0(t/\sqrt{1+t^2})/\sqrt{1+t^2}$. The proof relies on coordinate transformation, properties of Bessel functions, and integral transforms. This result holds significance in various applications of probability theory and statistical analysis.

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The introduction of polar coordinates where $x = r\cos\theta$, $y = r\sin\theta$, and $dx\,dy = r\,dr\,d\theta$ transforms the integral to:

$$\frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} e^{itr^2\cos\theta\sin\theta} r \, e^{-r^2/2} \, d\theta \, dr \tag{12}$$

Variable Substitution

The substitution $u = r^2/2$, with $du = r\,dr$, yields:

$$\frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} e^{2itu\cos\theta\sin\theta} e^{-u} \, d\theta \, du \tag{13}$$

Double Angle Formula

Application of the identity $\cos\theta\sin\theta = \frac{1}{2}\sin(2\theta)$ gives:

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} e^{itu\sin(2\theta)}\,e^{-u}\,d\theta\,du \tag{14}$$

Bessel Function Connection

The inner integral relates to the Bessel function through the identity:

$$\int_0^{2\pi} e^{itu\sin(2\theta)}\,d\theta = 2\pi\,J_0(tu) \tag{15}$$

This follows from the integral representation of the Bessel function of the first kind:

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The integral thus reduces to:

$$\int_0^\infty J_0(tu)\,e^{-u}\,du \tag{17}$$

Final Evaluation

The evaluation proceeds through the known Laplace transform of Bessel functions:

$$\int_0^\infty J_0(at)\,e^{-ut}\,dt = \frac{1}{\sqrt{1+a^2}}\,J_0\!\left(\frac{a}{\sqrt{1+a^2}}\right) \tag{18}$$

This leads to the final result:

$$\phi_{XY}(t) = \frac{J_0\!\left(\dfrac{t}{\sqrt{1+t^2}}\right)}{\sqrt{1+t^2}} \tag{19} \;\square$$

3 Zero Counting Function via Regularized Transform for Gaussian Processes on $[0,\infty)$

3.1 Natural Framework and Preliminaries

Let $\{X_t\}_{t\in[0,\infty)}$ be a real-valued centered Gaussian process whose covariance operator K is compact relative to the canonical metric it induces. This compactness is characterized by the finiteness of Dudley's metric entropy integral:

$$\int_0^1 \sqrt{\log N(\varepsilon, B_T, d)}\,d\varepsilon < \infty \tag{20}$$

where $N(\varepsilon, B_T, d)$ is the covering number - the minimal number of ε-balls needed to cover any bounded set $B_T = [0, T]$ under the canonical metric:

$$d(s,t) = \sqrt{K(s,s) + K(t,t) - 2K(s,t)} \tag{21}$$

3.2 Note on Compactness Verification

While the covering number $N(\varepsilon, B_T, d)$ represents the exact supremum over all errors of finite rank approximations, its direct computation is typically infeasible. However, the upper bound:

$$N(\varepsilon, B_T, d) \leq \min\{n \in \mathbb{N} : \lambda_n < \varepsilon^2\} \tag{22}$$

is sufficient to prove compactness.

Importantly, one need not verify compactness a priori. The very existence of an orthogonal polynomial system for the spectral density implies compactness of the corresponding kernel operator (Rao, M.M., Stochastic Processes: Inference Theory). Thus, the success of this expansion method itself confirms compactness - if the kernel were not compact, no such orthogonal system would exist.

The compactness of K ensures the existence of a countable orthonormal basis $\{\phi_n\}$ with corresponding eigenvalues $\{\lambda_n\}$. Importantly, K is not required to be trace class.

3.3 Bidirectional Representations

Given this spectral decomposition, the process admits two equivalent representations:

Given a path X_t, its projection coefficients are:

$$Z_n = \frac{\int_0^\infty X_t\, \phi_n(t)\, dt}{\sqrt{\lambda_n}} \tag{23}$$

Conversely, given the projection coefficients $\{Z_n\}$, the path is reconstructed as:

$$X_t = \sum_{n=0}^\infty Z_n \sqrt{\lambda_n}\, \phi_n(t) \tag{24}$$

3.4 Simplicity of Zeros

For such a Gaussian process, the impossibility of simultaneous vanishing of the process and its mean square derivative follows from the properties of joint normal distributions. At any point t, consider (X_t, X_t'), where X_t' is the mean square derivative. These form a bivariate normal distribution with covariance matrix:

$$\Sigma = \begin{pmatrix} K(t,t) & \partial_2 K(t,t) \\ \partial_2 K(t,t) & -\partial_1 \partial_2 K(t,t) \end{pmatrix} \tag{25}$$

where ∂_i denotes partial derivative with respect to the ith argument.

The probability of both vanishing simultaneously is:

$$P(X_t = 0, X_t' = 0) = \frac{e^{-\frac{(0,0)\Sigma^{-1}(0,0)^T}{2}}}{2\pi\sqrt{\det(\Sigma)}} = 0 \tag{26}$$

since the determinant of Σ is strictly positive due to the non-degeneracy of the process.

3.5 Explicit Emergence of the Bessel Function

The joint distribution of (X_t, X_t') at any point t has density:

$$p(x, y) = \frac{1}{2\pi\sqrt{\det(\Sigma)}} e^{-\frac{1}{2}(x,y)\Sigma^{-1}(x,y)^T} \tag{27}$$

Transforming to polar coordinates (r, θ) with $x = r\cos\theta$, $y = r\sin\theta$:

$$\begin{aligned} p(r, \theta) &= \frac{re^{-\frac{r^2}{2}(\cos\theta,\sin\theta)\Sigma^{-1}(\cos\theta,\sin\theta)^T}}{2\pi\sqrt{\det(\Sigma)}} \\ &= \frac{re^{-\frac{r^2}{2}Q(\theta)}}{2\pi\sqrt{\det(\Sigma)}} \end{aligned} \tag{28}$$

where $Q(\theta)$ is the quadratic form in the exponential.

Integrating over θ yields:

$$\begin{aligned} \int_0^{2\pi} p(r, \theta)\, d\theta &= \frac{r\int_0^{2\pi} e^{-\frac{r^2}{2}Q(\theta)}\, d\theta}{2\pi\sqrt{\det(\Sigma)}} \\ &= r\, J_0\left(r\sqrt{\det(\Sigma)}\right) \end{aligned} \tag{29}$$

This natural emergence of J_0 from the joint distribution structure directly connects to our counting function representation.

3.6 Zero Counting Function Development

The expected zero counting function takes the form:

$$\begin{aligned} N(T) &= \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_0^T \int_{-\infty}^{\infty} J_0(\epsilon r)|r| \exp\left(-ir\sum_{n=0}^{\infty} Z_n\sqrt{\lambda_n}\,\phi_n(t)\right) dr\, dt \\ &= \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_0^T \mathcal{H}_{0, r \to \epsilon}\left[\exp\left(-ir\sum_{n=0}^{\infty} Z_n\sqrt{\lambda_n}\,\phi_n(t)\right)\right] dt \end{aligned} \tag{30}$$

where $\mathcal{H}_{0,r\to s}[f] = \int_0^\infty r\, f(r)\, J_0(s\,r)\, d\,r$ is the Hankel transform of order zero.

By Fubini's theorem:

$$
\begin{aligned}
N(T) &= \lim_{\epsilon\to 0} \frac{1}{2\,\pi}\, \mathcal{H}_{0,r\to\epsilon}\left[\exp\left(-i\,r \sum_{n=0}^{\infty} Z_n\,\sqrt{\lambda_n} \int_0^T \phi_n(t)\,d\,t\right)\right] \\
&= \lim_{\epsilon\to 0} \frac{1}{2\,\pi} \int_{-\infty}^{\infty} J_0(\epsilon\,r)|r|\exp\left(-i\,r \sum_{n=0}^{\infty} Z_n\,\sqrt{\lambda_n} \int_0^T \phi_n(t)\,d\,t\right) d\,r
\end{aligned}
\tag{31}
$$

Using the integral representation of J_0:

$$
\begin{aligned}
N(T) &= \lim_{\epsilon\to 0} \frac{1}{2\,\pi} \int_{-\infty}^{\infty} |r|\left(\frac{1}{2\,\pi}\int_0^{2\pi} e^{i\epsilon r\cos\theta}\,d\,\theta\right)\exp\left(-i\,r \sum_{n=0}^{\infty} Z_n\,\sqrt{\lambda_n} \int_0^T \phi_n(t)\,d\,t\right) d\,r \\
&= \lim_{\epsilon\to 0} \frac{1}{2\,\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} |r|\,e^{i r(\epsilon\cos\theta - \sum_{n=0}^{\infty} Z_n\sqrt{\lambda_n}\int_0^T \phi_n(t)dt)}\,d\,r\,d\,\theta
\end{aligned}
\tag{32}
$$

The inner integral evaluates to:

$$
\int_{-\infty}^{\infty} |r|\,e^{i r(\epsilon\cos\theta - \sum_{n=0}^{\infty} Z_n\sqrt{\lambda_n}\int_0^T \phi_n(t)dt)}\,d\,r = \frac{2}{(\epsilon\cos\theta - \sum_{n=0}^{\infty} Z_n\,\sqrt{\lambda_n}\int_0^T \phi_n(t)\,d\,t)^2}
\tag{33}
$$

Therefore:

$$
\begin{aligned}
N(T) &= \lim_{\epsilon\to 0} \frac{1}{\pi} \int_0^{2\pi} \frac{1}{(\epsilon\cos\theta - \sum_{n=0}^{\infty} Z_n\,\sqrt{\lambda_n}\int_0^T \phi_n(t)\,d\,t)^2}\,d\,\theta \\
&= \lim_{\epsilon\to 0} \frac{1}{2\,\pi} \frac{1}{((\sum_{n=0}^{\infty} Z_n\,\sqrt{\lambda_n}\int_0^T \phi_n(t)\,d\,t)^2 + \epsilon^2)^{3/2}}
\end{aligned}
\tag{34}
$$

Commentary on Mathematical Structure

The emergence of J_0 through the Hankel transform is not merely computational convenience but reflects the fundamental structure of Gaussian processes. The regularization parameter ϵ provides the necessary resolution while preserving the natural symmetries of the process. This framework reveals why the counting function takes this particular form and provides a natural setting for understanding both its average behavior and fluctuations.

The simplicity of zeros follows directly from the Gaussian process properties, as the path and its mean square derivative cannot simultaneously vanish. This elementary fact, combined with the natural emergence of J_0 from joint distributions, provides a complete and elegant description of the zero counting function.