# The Covariance of Ergodic Stationary Processes

by Stephen Crowley

**Definition 1.** *A stochastic process* $\xi(t)$, $t \in \mathbb{R}$, *is called* strictly stationary *if for all* $t_1, t_2, \ldots, t_n$ *and all* $\tau \in \mathbb{R}$,

$$(\xi(t_1+\tau), \ldots, \xi(t_n+\tau)) \overset{d}{=} (\xi(t_1), \ldots, \xi(t_n)) \tag{1}$$

*A strictly stationary process is called* ergodic *if every invariant event under the temporal shift transformation has probability zero or one.*

**Theorem 2.** **(Exact Covariance Function from a Single Sample Path)** *Let* $\xi(t)$ *be a real-valued, zero-mean, strictly stationary, and ergodic process with* $\mathbb{E}[\xi^2(0)] < \infty$. *Let* $x(t)$ *be a realization of* $\xi(t)$. *Then for every fixed* $\tau \in \mathbb{R}$,

$$r(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\, dt \tag{2}$$

*almost surely, where* $r(\tau) = \mathbb{E}[\xi(0)\, \xi(\tau)]$ *is the covariance function.*

**Proof. Step 1: Establish integrability conditions.**
Since $\xi(t)$ is strictly stationary, $\mathbb{E}[\xi^2(t)] = \mathbb{E}[\xi^2(0)] < \infty$ for all $t \in \mathbb{R}$. For any fixed $\tau \in \mathbb{R}$, the Cauchy-Schwarz inequality yields

$$\mathbb{E}[|\xi(0)\, \xi(\tau)|] \leq \sqrt{\mathbb{E}[\xi^2(0)] \cdot \mathbb{E}[\xi^2(\tau)]} = \sqrt{\mathbb{E}[\xi^2(0)] \cdot \mathbb{E}[\xi^2(0)]} = \mathbb{E}[\xi^2(0)] < \infty \tag{3}$$

Therefore, the random variable $\xi(0)\, \xi(\tau)$ is integrable.

**Step 2: Define the measurable function and shift operator.**
Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(s) = \xi(s)\, \xi(s+\tau)$ for fixed $\tau$. Let $T_h$ denote the shift operator defined by $(T_h\, \xi)(t) = \xi(t+h)$ for $h \in \mathbb{R}$. The strict stationarity condition implies that the measure induced by $\xi$ is invariant under $T_h$ for all $h$.

**Step 3: Verify ergodicity conditions.**
Since $\xi(t)$ is ergodic, the shift-invariant $\sigma$-algebra has trivial tail structure: every shift-invariant event has probability 0 or 1. This ensures that the conditions of the Birkhoff-Khinchin ergodic theorem are satisfied for the dynamical system $(\Omega, \mathcal{F}, P, T_h)$ where $\Omega$ is the sample space of the process.

**Step 4: Apply the Birkhoff-Khinchin ergodic theorem.**
For the integrable function $f(s) = \xi(s)\,\xi\,(s+\tau)$, the ergodic theorem states that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(s)\ d s = \mathbb{E}[f(0)] \tag{4}$$

almost surely with respect to the probability measure of the process. Substituting our function:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \xi(s)\,\xi\,(s+\tau)\ d s = \mathbb{E}\,[\xi(0)\,\xi(\tau)] \tag{5}$$

almost surely.

**Step 5: Connect to sample path realization.**
For any particular realization $x(t) = \xi(t,\omega)$ where $\omega$ belongs to the set of full measure on which the ergodic theorem holds, we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(s)\,x\,(s+\tau)\ d s = \mathbb{E}\,[\xi(0)\,\xi(\tau)] \tag{6}$$

**Step 6: Establish covariance function equality.**
By definition of the covariance function for a zero-mean process:

$$\begin{aligned} r(\tau) &= \mathrm{Cov}(\xi(0), \xi(\tau)) \\ &= \mathbb{E}\,[\xi(0)\,\xi(\tau)] - \mathbb{E}[\xi(0)]\,\mathbb{E}[\xi(\tau)] \\ &= \mathbb{E}\,[\xi(0)\,\xi(\tau)] - 0\cdot 0 \\ &= \mathbb{E}\,[\xi(0)\,\xi(\tau)] \end{aligned} \tag{7}$$

**Step 7: Conclude the main result.**
Combining Steps 5 and 6:

$$r(\tau) = \mathbb{E}\,[\xi(0)\,\xi(\tau)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,x\,(t+\tau)\ d t \tag{8}$$

almost surely. The exceptional set (where this equality fails) has probability zero by the ergodic theorem. $\square$

**Remark 3.** The almost sure convergence implies that for any specific realization drawn from the process, the temporal average will equal the ensemble covariance function, provided the realization belongs to the set of full measure guaranteed by ergodicity.