Injective Measure-Preserving Time-Changes of Stationary Processes are Oscillatory

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Oscillatory Processes and Normalized Injective Time-Changes

Definition 1. *[Oscillatory Process] A complex-valued second-order stochastic process $\{X_t\}_{t \in I}$ is said to be oscillatory if there exists a family of functions $\phi_t(\omega)$ and a complex orthogonal increment process $Z(\omega)$ with $E\,|d\,Z(\omega)|^2 = d\,\mu(\omega)$ such that*

$$X_t = \int_{-\infty}^{\infty} \phi_t(\omega)\ d\,Z(\omega)\,,$$

where $\phi_t(\omega) = A_t(\omega)\,e^{i\omega t}$ and $A_t(\omega)$ is quadratically integrable with respect to $d\,\mu$.

Definition 2. *[Stationary Process] A second-order process $\{S_t\}_{t \in J}$ is stationary if it admits the spectral representation*

$$S_t = \int_{-\infty}^{\infty} e^{i\omega t}\ d\,Z(\omega)$$

for some orthogonal increment process $Z(\omega)$ with $E\,|d\,Z(\omega)|^2 = d\,\mu(\omega)$.

Theorem 3. *[Time-Varying Filter for Injective Time-Change] Let S_t be a stationary process and $\theta: \mathbb{R} \to \mathbb{R}$ be smooth and strictly increasing with $\theta'(t) > 0$. To achieve the transformation*

$$X_t = \sqrt{\theta'(t)}\ S_{\theta(t)}$$

via convolution $X_t = \int_{-\infty}^{\infty} S_{t-u}\,h_t(u)\,d\,u$, the time-varying impulse response must be

$$h_t(u) = \sqrt{\theta'(t)}\ \delta\left(u - (t - \theta(t))\right).$$

Proof. For the convolution to yield $X_t = \sqrt{\theta'(t)}\ S_{\theta(t)}$, the argument of S in the integrand must equal $\theta(t)$ when the delta function is activated. This requires:

$$t - u = \theta(t)$$

Solving for u:

$$u = t - \theta(t)$$

Therefore:

$$h_t(u) = \sqrt{\theta'(t)}\, \delta\, (u - (t - \theta(t)))$$

Verification by direct computation:

$$\begin{aligned}
X_t &= \int_{-\infty}^{\infty} S_{t-u}\, h_t(u)\, d\,u && (1)\\
&= \int_{-\infty}^{\infty} S_{t-u}\, \sqrt{\theta'(t)}\, \delta\, (u - (t - \theta(t)))\, d\,u && (2)\\
&= \sqrt{\theta'(t)}\, S_{t-(t-\theta(t))} && (3)\\
&= \sqrt{\theta'(t)}\, S_{\theta(t)} && (4)
\end{aligned}$$

□

Theorem 4. *[Oscillatory Representation of Injective Time-Change] The process $X_t = \sqrt{\theta'(t)}\, S_{\theta(t)}$ admits the oscillatory representation*

$$X_t = \int_{-\infty}^{\infty} \phi_t(\omega)\, d\,Z(\omega)$$

where

$$\phi_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega\theta(t)}.$$

Proof. Starting from the spectral representation of S_t:

$$\begin{aligned}
X_t &= \sqrt{\theta'(t)}\, S_{\theta(t)} && (5)\\
&= \sqrt{\theta'(t)} \int_{-\infty}^{\infty} e^{i\omega\theta(t)}\, d\,Z(\omega) && (6)\\
&= \int_{-\infty}^{\infty} \sqrt{\theta'(t)}\, e^{i\omega\theta(t)}\, d\,Z(\omega) && (7)
\end{aligned}$$

Thus $\phi_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega\theta(t)}$. □

Corollary 5. *[Envelope in Standard Form] The oscillatory functions can be written in the standard form $\phi_t(\omega) = A_t(\omega)\, e^{i\omega t}$ where*

$$A_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega(\theta(t)-t)}.$$

Proof. Factor the exponential:

$$\phi_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega\theta(t)} \tag{8}$$
$$= \sqrt{\theta'(t)}\, e^{i\omega(\theta(t)-t)}\, e^{i\omega t} \tag{9}$$
$$= A_t(\omega)\, e^{i\omega t} \tag{10}$$

where $A_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega(\theta(t)-t)}$. $\square$

Theorem 6. *[Evolutionary Power Spectrum] For the oscillatory process $X_t = \sqrt{\theta'(t)}\, S_{\theta(t)}$ with envelope $A_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega(\theta(t)-t)}$, the evolutionary power spectrum at time t is*

$$d\,F_t(\omega) = |A_t(\omega)|^2\, d\,\mu(\omega) = \theta'(t)\, d\,\mu(\omega).$$

Proof. The evolutionary power spectrum is defined as $d\,F_t(\omega) = |A_t(\omega)|^2\, d\,\mu(\omega)$. Computing the magnitude squared:

$$|A_t(\omega)|^2 = \left|\sqrt{\theta'(t)}\, e^{i\omega(\theta(t)-t)}\right|^2 \tag{11}$$
$$= \theta'(t)|e^{i\omega(\theta(t)-t)}|^2 \tag{12}$$
$$= \theta'(t) \cdot 1 \tag{13}$$
$$= \theta'(t) \tag{14}$$

Therefore $d\,F_t(\omega) = \theta'(t)\, d\,\mu(\omega)$. $\square$

Theorem 7. *[L^2-Norm Preservation] The transformation $S_t \mapsto \sqrt{\theta'(t)}\, S_{\theta(t)}$ preserves the L^2-norm in the sense that*

$$\int_I E|X_t|^2\, d\,t = \int_J E|S_s|^2\, d\,s$$

where I is the domain of t and $J = \theta(I)$.

Proof. Using the change of variables $s = \theta(t)$, so $d\,s = \theta'(t)\, d\,t$:

$$\int_I E|X_t|^2\, d\,t = \int_I E\left|\sqrt{\theta'(t)}\, S_{\theta(t)}\right|^2 d\,t \tag{15}$$
$$= \int_I \theta'(t)\, E|S_{\theta(t)}|^2\, d\,t \tag{16}$$
$$= \int_J E|S_s|^2\, d\,s \tag{17}$$

$\square$