Unitary Time Changes of Stationary Processes Yield Oscillatory Processes

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are locally square-integrable (meaning over compact sets). Applying U_θ to the Cramér spectral representation of a stationary process $X(t)$ produces the transformed process $Z(t) = U_\theta X(t) = \sqrt{\dot\theta(t)} X(\theta(t)) = \sqrt{\dot\theta(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)} d\Phi(\lambda)$, which is an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot\theta(t)} e^{i\lambda\theta(t)}$, evolutionary power spectral density $S_t(\lambda) = \dot\theta(t) S(\lambda)$, and covariance kernel $R_Z(t,s) = \sqrt{\dot\theta(t)\dot\theta(s)} R_X(\theta(t) - \theta(s))$ where R_X is the stationary covariance of $X(t) = \int_{\mathbb{R}} e^{i\lambda t} d\Phi(\lambda)$, and the expected zero-counting function $\mathbb{E}[N_{[a,b]}]$ of the oscillatory process paths equals $\sqrt{-R_X''(0)/R_X(0)}(\theta(b) - \theta(a))/\pi$. The sample paths of any non-degenerate second-order stationary process are locally square integrable, making the unitary time-change operator U_θ applicable to typical realizations. By Bulinskaya's theorem, when the covariance is twice continuously differentiable with $R''(0) < 0$, almost all zeros are simple. A zero-localization measure $d\mu(t) = \delta(Z(t))|\dot Z(t)|dt$ induces a Hilbert space $L^2(\mu)$ on the zero set of each oscillatory process realization $Z(t)$, and the multiplication operator $(Lf)(t) = tf(t)$ has simple pure point spectrum equal to the zero crossing set of Z.

Contents

1 Gaussian Processes

1.1 Definition

Definition 1.1. (Gaussian process) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T a nonempty index set. A family $\{X_t : t \in T\}$ of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Gaussian process if for every finite subset $\{t_1, \ldots, t_n\} \subset T$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ is multivariate normal (possibly degenerate). Equivalently, every finite linear combination $\sum_{i=1}^n a_i X_{t_i}$ is either almost surely constant or Gaussian. The mean function is $m(t) := \mathbb{E}[X_t]$ and the covariance kernel is

$$K(s,t) = \mathrm{Cov}(X_s, X_t) \tag{1}$$

For any finite $(t_i)_{i=1}^n \subset T$, the matrix $K_{ij} = K(t_i, t_j)$ is symmetric positive semidefinite, and a Gaussian process is completely determined in law by m and K.

1.2 Stationary processes

Definition 1.2. **[Cramér spectral representation]** A zero-mean stationary process X with spectral measure F admits the sample path representation

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t} d\Phi(\lambda) \tag{2}$$

which has covariance

$$R_X(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)} dF(\lambda) \tag{3}$$

1.2.1 Sample path realizations

Definition 1.3. **[Locally square-integrable functions]** Define

$$L^2_{\text{loc}}(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} : \int_K |f(t)|^2 dt < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \tag{4}$$

Remark 1.4. Every bounded measurable set in $\mathbb{R}$ is compact or contained in a compact set; hence $L^2_{\text{loc}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

Theorem 1.5. ***[Sample paths in** $L^2_{\text{loc}}(\mathbb{R})$**]*** *Let $\{X(t)\}_{t\in\mathbb{R}}$ be a second-order stationary process with*

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty. \tag{5}$$

Then almost every sample path lies in $L^2_{\text{loc}}(\mathbb{R})$.

Proof. Fix an arbitrary bounded interval $[a, b] \subset \mathbb{R}$ with $a < b$. Define

$$Y_{[a,b]} := \int_a^b X(t)^2 dt. \tag{6}$$

By Tonelli's theorem, since $X(t)^2 \geq 0$,

$$\mathbb{E}[Y_{[a,b]}] = \mathbb{E}\left[\int_a^b X(t)^2 dt\right] = \int_a^b \mathbb{E}[X(t)^2] dt. \tag{7}$$

By stationarity, $\mathbb{E}[X(t)^2] = \sigma^2$ for all t, hence

$$\mathbb{E}[Y_{[a,b]}] = \sigma^2(b-a) < \infty. \tag{8}$$

Markov's inequality yields, for $M > 0$,

$$\mathbb{P}(Y_{[a,b]} > M) \leq \frac{\mathbb{E}[Y_{[a,b]}]}{M} = \frac{\sigma^2(b-a)}{M}, \tag{9}$$

and letting $M \to \infty$ gives $\mathbb{P}(Y_{[a,b]} < \infty) = 1$. Now let $K \subset \mathbb{R}$ be compact, so $K \subseteq [-N, N]$ for some $N > 0$. Then

$$\int_K X(t)^2 dt \leq \int_{-N}^N X(t)^2 dt < \infty \quad \text{a.s.} \tag{10}$$

hence almost every path satisfies $\int_K |X(t,\omega)|^2 dt < \infty$ for every compact K, i.e. $X(\cdot, \omega) \in L^2_{\text{loc}}(\mathbb{R})$. □

1.3 (Non-Stationary) Oscillatory Processes

Definition 1.6. **[Oscillatory process]** Let F be a finite nonnegative Borel measure on $\mathbb{R}$. Let

$$A_t \in L^2(F) \quad \forall t \in \mathbb{R} \tag{11}$$

be the gain function and

$$\varphi_t(\lambda) = A_t(\lambda)e^{i\lambda t} \tag{12}$$

the corresponding oscillatory function. An oscillatory process is a stochastic process represented as

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda)d\Phi(\lambda) = \int_{\mathbb{R}} A_t(\lambda)e^{i\lambda t}d\Phi(\lambda), \tag{13}$$

where Φ is a complex orthogonal random measure with spectral measure F satisfying

$$d\mathbb{E}\left[\Phi(\lambda)\overline{\Phi(\mu)}\right] = \delta(\lambda - \mu)dF(\lambda) \tag{14}$$

and covariance

$$\begin{aligned} R_Z(t,s) = \mathbb{E}\left[Z(t)\overline{Z(s)}\right] &= \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)}e^{i\lambda(t-s)}dF(\lambda) \\ &= \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)}dF(\lambda). \end{aligned} \tag{15}$$

Definition 1.7. **[Evolutionary power spectral density (EPSD)]** For an oscillatory process with gain function $A_t(\lambda)$ and spectral measure F having density $S(\lambda)$ (i.e., $dF(\lambda) = S(\lambda)d\lambda$), the evolutionary power spectral density is

$$S_t(\lambda) = |A_t(\lambda)|^2 S(\lambda), \tag{16}$$

so that the evolutionary spectral measure is $dF_t(\lambda) = S_t(\lambda)d\lambda = |A_t(\lambda)|^2 dF(\lambda)$.

Definition 1.8. **[Variance of oscillatory process]** The variance of an oscillatory process $Z(t)$ is given by integrating the evolutionary power spectral density over all frequencies:

$$\operatorname{var}(Z(t)) = \int_{-\infty}^{\infty} S_t(\lambda)d\lambda = \int_{-\infty}^{\infty} dF_t(\lambda). \tag{17}$$

Theorem 1.9. ***[Real-valuedness criterion for oscillatory processes]*** *Let Z be an oscillatory process with oscillatory function $\varphi_t(\lambda) = A_t(\lambda)e^{i\lambda t}$ and spectral measure F. Then Z is real-valued if and only if*

$$A_t(-\lambda) = \overline{A_t(\lambda)} \quad \text{for } F\text{-a.e. } \lambda \in \mathbb{R}, \tag{18}$$

equivalently

$$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \quad \text{for } F\text{-a.e. } \lambda \in \mathbb{R}. \tag{19}$$

Proof. If Z is real-valued, then $Z(t) = \overline{Z(t)}$ for all t. Taking conjugates in the representation

$$Z(t) = \int A_t(\lambda)e^{i\lambda t}d\Phi(\lambda) \tag{20}$$

and using the symmetry relation for the orthogonal random measure appropriate for real-valued processes, a change of variable $\mu = -\lambda$ shows that the $L^2(F)$-integrands must agree F-a.e., i.e.

$$A_t(\lambda) = \overline{A_t(-\lambda)}, \tag{21}$$

which is equivalent to (18). Using $\varphi_t(\lambda) = A_t(\lambda)e^{i\lambda t}$ then gives (19). The converse follows by reversing the steps. $\square$

Theorem 1.10. ***[Existence of oscillatory processes with explicit L^2-limit construction]*** *Let F be an absolutely continuous spectral measure with density $S(\lambda)$ and the gain function $A_t(\lambda) \in L^2(F)$ for all $t \in \mathbb{R}$, measurable jointly in (t, λ). Define the time-dependent variance*

$$\sigma_t^2 := \int_{\mathbb{R}} |A_t(\lambda)|^2 dF(\lambda) = \int_{\mathbb{R}} S_t(\lambda) d\lambda < \infty \tag{22}$$

where $S_t(\lambda) = |A_t(\lambda)|^2 S(\lambda)$ is the evolutionary power spectral density. Then there exists a complex orthogonal random measure Φ with spectral measure F such that for each fixed t the stochastic integral

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \tag{23}$$

is well-defined as an $L^2(\Omega)$-limit and has covariance R_Z as in (15)*.*

Proof.

1. *Simple functions and isometry.* Let S denote the set of simple functions

$$g(\lambda) = \sum_{j=1}^{n} c_j \mathbf{1}_{E_j}(\lambda) \tag{24}$$

with disjoint Borel E_j and $F(E_j) < \infty$, $c_j \in \mathbb{C}$. Define the stochastic integral on S by

$$\int_{\mathbb{R}} g(\lambda) d\Phi(\lambda) := \sum_{j=1}^{n} c_j \Phi(E_j) \tag{25}$$

Using orthogonality of Φ,

$$\mathbb{E}\left[\left|\int g d\Phi\right|^2\right] = \sum_{j=1}^{n} |c_j|^2 F(E_j) = \int_{\mathbb{R}} |g(\lambda)|^2 dF(\lambda) \tag{26}$$

Thus the map $I : \mathsf{S} \to L^2(\Omega)$, $I(g) = \int g d\Phi$, is an isometry with respect to the $L^2(F)$-norm.

2. *Density and Cauchy property.* Simple functions are dense in $L^2(F)$: for any $h \in L^2(F)$ there exists $g_n \in \mathsf{S}$ with $\|h - g_n\|_{L^2(F)} \to 0$. By (26),

$$\mathbb{E}\left[\left|\int g_n d\Phi - \int g_m d\Phi\right|^2\right] = \|g_n - g_m\|^2_{L^2(F)} \xrightarrow[n,m\to\infty]{} 0 \tag{27}$$

so $\left\{\int g_n d\Phi\right\}$ is Cauchy in $L^2(\Omega)$.

3. *Definition by L^2-limit and independence of approximating sequence.* Since $L^2(\Omega)$ is complete, the limit exists. Define, for $h \in L^2(F)$,

$$\int_{\mathbb{R}} h(\lambda) d\Phi(\lambda) := \lim_{n\to\infty} \int_{\mathbb{R}} g_n(\lambda) d\Phi(\lambda) \tag{28}$$

where $g_n \in \mathsf{S}$ and $\|h - g_n\|_{L^2(F)} \to 0$. If g_n and $\tilde{g}_n$ are two such approximating sequences, then $\|g_n - \tilde{g}_n\|_{L^2(F)} \to 0$ and again by (26) the corresponding integrals differ by an $L^2(\Omega)$-null sequence, so the limit is independent of the sequence.

4. *Isometry and linearity extend.* By continuity from (26) and (28),

$$\mathbb{E}\left[\left|\int h d\Phi\right|^2\right] = \int_{\mathbb{R}} |h(\lambda)|^2 dF(\lambda) \tag{29}$$

for all $h \in L^2(F)$, and the map $h \mapsto \int h d\Phi$ is linear and isometric.

5. *Application to φ_t.* Since $|e^{i\lambda t}| = 1$, $\varphi_t(\lambda) = A_t(\lambda) e^{i\lambda t} \in L^2(F)$ and

$$\int_{\mathbb{R}} |\varphi_t(\lambda)|^2 dF(\lambda) = \int_{\mathbb{R}} |A_t(\lambda)|^2 dF(\lambda) = \sigma_t^2 < \infty \tag{30}$$

Hence $Z(t)$ in (23) is well-defined as the $L^2(\Omega)$-limit (28) with $h = \varphi_t$. Computing covariance via sesquilinearity together with (14) yields (15).

□

1.4 Filter Representations and Invertibility for Oscillatory Processes

Definition 1.11. **[Time-dependent filter and gain]** The time-dependent filter $h(t, u)$ and gain function $A_t(\lambda)$ satisfy the Fourier transform pair

$$A_t(\lambda) = \int_{-\infty}^{\infty} h(t, u)e^{-i\lambda(t-u)}du \tag{31}$$

$$h(t, u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_t(\lambda)e^{i\lambda(t-u)}d\lambda \tag{32}$$

with square-integrability

$$\int_{-\infty}^{\infty} |h(t, u)|^2 du < \infty \quad \forall t \in \mathbb{R} \tag{33}$$

Theorem 1.12. ***[Forward and inverse filter representations for general oscillatory processes]*** *Let $Z(t)$ be an oscillatory process as in Definition 1.6 with oscillatory function $\varphi_t(\lambda) = A_t(\lambda)e^{i\lambda t}$. Then:*

1. *The forward time-varying filter representation is*

$$Z(t) = \int_{\mathbb{R}} h(t, \lambda)d\Phi(\lambda) \tag{34}$$

with kernel $h(t, \lambda) = A_t(\lambda)e^{i\lambda t}$

2. *For oscillatory processes with white noise representation $dW(u)$ satisfying*

$$\mathbb{E}[dW(u_1)\overline{dW(u_2)}] = \delta(u_1 - u_2)du_1 \tag{35}$$

the process admits

$$Z(t) = \int_{-\infty}^{\infty} h(t, u)dW(u) \tag{36}$$

Proof. 1. Equation (34) is immediate from (13)

2. The white noise representation follows from the spectral relation

$$d\Phi(\lambda) = \frac{1}{2\pi}\int e^{-i\lambda u}dW(u)du \tag{37}$$

and application of the filter Fourier pair

□

Definition 1.13. **[Amplitude nondegeneracy]** The amplitude $A_t(\lambda)$ satisfies

$$A_t(\lambda) \neq 0 \quad \text{for all } (t, \lambda) \text{ in the support of } F \tag{38}$$

Definition 1.14. **[Kernel orthonormality]** The amplitude satisfies

$$\int_{-\infty}^{\infty} A_t(\lambda_1)\overline{A_t(\lambda_2)}e^{i(\lambda_2-\lambda_1)t}dt = \delta(\lambda_1 - \lambda_2) \tag{39}$$

Theorem 1.15. ***[Fundamental invertibility for oscillatory processes]*** *For Z as in Definition 1.6, the inversion formula*

$$\Phi(\{\lambda\}) = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} A_t(\lambda)e^{-i\lambda t}Z(t)\chi_{[\lambda-\varepsilon,\lambda+\varepsilon]}(\lambda)dt \tag{40}$$

holds (in the sense of $L^2(\Omega)$ convergence) if and only if A_t satisfies the nondegeneracy condition (38) *and the orthonormality condition* (39).

Proof. 1. *Forward direction.* From (13),

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \tag{41}$$

Multiply by $\overline{A_t(\lambda_0)} e^{-i\lambda_0 t}$ and integrate over t:

$$\int_{-\infty}^{\infty} \overline{A_t(\lambda_0)} e^{-i\lambda_0 t} Z(t) dt = \int_{-\infty}^{\infty} \overline{A_t(\lambda_0)} e^{-i\lambda_0 t} \left[\int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \right] dt \tag{42}$$

2. *Swap order of integration.*

$$= \int_{\mathbb{R}} \left[\int_{-\infty}^{\infty} \overline{A_t(\lambda_0)} A_t(\lambda) e^{i(\lambda-\lambda_0)t} dt \right] d\Phi(\lambda) \tag{43}$$

3. *Apply orthonormality.* By (39),

$$= \int_{\mathbb{R}} \delta(\lambda - \lambda_0) d\Phi(\lambda) = \Phi(\{\lambda_0\}) \tag{44}$$

4. *Backward direction.* Insert

$$Z_{\lambda_0}(t) = A_t(\lambda_0) e^{i\lambda_0 t} \tag{45}$$

into (40):

$$\Phi_{\lambda_0}(\lambda) = \int_{-\infty}^{\infty} \overline{A_t(\lambda)} e^{-i\lambda t} A_t(\lambda_0) e^{i\lambda_0 t} dt \tag{46}$$

The left side equals $\delta(\lambda - \lambda_0)$, hence (39) holds. Nondegeneracy follows from linear independence by evaluating at (t, λ) where $Z(t) \neq 0$.

□

Lemma 1.16. ***[Uniqueness of inversion]*** *If $\mathcal{I}_1 Z = d\Phi(\lambda) = \mathcal{I}_2 Z$ for all Z, then $\mathcal{I}_1 = \mathcal{I}_2$.*

Proof. Let $\mathcal{L} = \mathcal{I}_1 - \mathcal{I}_2$. Choose

$$Z_{\lambda_0}(t) = A_t(\lambda_0) e^{i\lambda_0 t} \tag{47}$$

Then $(\mathcal{L} Z_{\lambda_0})(\lambda)$ equals

$$\int_{-\infty}^{\infty} A_t(\lambda) e^{-i\lambda t} A_t(\lambda_0) e^{i\lambda_0 t} dt - \int_{-\infty}^{\infty} A_t(\lambda) e^{-i\lambda t} A_t(\lambda_0) e^{i\lambda_0 t} dt = 0 \tag{48}$$

Density of the span $\{Z_{\lambda_0}\}$ implies $\mathcal{L} = 0$. □

2 Unitarily Time-Changed Stationary Processes

2.1 Unitary time-change operator $U_\theta f$

Theorem 2.1. ***[Unitary time-change and local isometry]*** *Let the time-scaling function $\theta : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with*

$$\dot{\theta}(t) > 0 \tag{49}$$

almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero. For f measurable, define

$$(U_\theta f)(t) = \sqrt{\dot{\theta}(t)} f(\theta(t)) \tag{50}$$

Its inverse is given by

$$(U_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{51}$$

For every compact set $K \subseteq \mathbb{R}$ and $f \in L^2_{\text{loc}}(\mathbb{R})$,

$$\int_K |(U_\theta f)(t)|^2 dt = \int_{\theta(K)} |f(s)|^2 ds \tag{52}$$

Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\text{loc}}(\mathbb{R})$.

Proof. By (50),

$$\int_K |(U_\theta f)(t)|^2 dt = \int_K \dot{\theta}(t)|f(\theta(t))|^2 dt \tag{53}$$

With the change of variables $s = \theta(t)$ and $ds = \dot{\theta}(t)dt$, the domain maps to $\theta(K)$, giving (52). The two-sided inverse identities follow from direct substitution into (50) and (51). □

2.2 Filter Representations for Unitarily Time-Changed Stationary Processes

Theorem 2.2. ***[Forward and inverse filter representations for unitarily time-changed stationary processes]*** *Let $\theta : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\theta'(t) > 0$ a.e. Let $X(u) = \int_{\mathbb{R}} e^{i\lambda u} d\Phi(\lambda)$ be a realization of a stationary process, and set*

$$Z(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) \tag{54}$$

Then:

1. *The forward filter kernel is*

$$h(t, u) = \sqrt{\dot{\theta}(t)}\delta(u - \theta(t)) \tag{55}$$

2. *The inverse filter kernel is*

$$g(t, s) = \frac{\delta(s - \theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \tag{56}$$

3. *The composition $(g \circ h)$ recovers the identity:*

$$X(t) = \int_{\mathbb{R}} g(t, s)Z(s)ds = \frac{Z(\theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \tag{57}$$

Proof. 1. Using the sifting property of the Dirac delta in (55) gives (54)

2. Applying (56), then substituting (54) at $s = \theta^{-1}(t)$ and using

$$\theta \circ \theta^{-1} = \mathrm{id} \tag{58}$$

yields (57)

3. The identity follows from items (1) and (2)

□

2.3 Transformation of stationary to oscillatory processes via U_θ

Theorem 2.3. ***[Unitary time change produces oscillatory process]*** *Let X be zero-mean stationary as in Definition 1.2. For a scaling function θ as in Theorem 2.1, define*

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) \tag{59}$$

Then Z is a realization of an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \tag{60}$$

gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \tag{61}$$

and covariance

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}\left[Z(t)\overline{Z(s)}\right] = \sqrt{\dot{\theta}(t)\dot{\theta}(s)}\mathbb{E}\left[X(\theta(t))\overline{X(\theta(s))}\right] \\ &= \sqrt{\dot{\theta}(t)\dot{\theta}(s)} R_X(\theta(t) - \theta(s)) = \sqrt{\dot{\theta}(t)\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} dF(\lambda) \end{aligned} \tag{62}$$

Proof. From the Cramér representation (2),

$$X(\theta(t)) = \int e^{i\lambda\theta(t)} d\Phi(\lambda) \tag{63}$$

Therefore

$$Z(t) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)} d\Phi(\lambda) = \int_{\mathbb{R}} \left(\sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \right) d\Phi(\lambda) = \int \varphi_t(\lambda) d\Phi(\lambda) \tag{64}$$

which is of the oscillatory form with φ_t as in (60) and A_t as in (61). The covariance follows from stationarity via (3). □

Corollary 2.4. ***[Evolutionary power spectral density of unitarily time-changed stationary process]*** *If the stationary spectral measure has density $S(\lambda)$, i.e. $dF(\lambda) = S(\lambda)d\lambda$, then the evolutionary power spectral density of $Z(t) = U_\theta X(t)$ is*

$$S_t(\lambda) = |A_t(\lambda)|^2 S(\lambda) = \dot{\theta}(t) S(\lambda) \tag{65}$$

and therefore

$$dF_t(\lambda) = S_t(\lambda) d\lambda = \dot{\theta}(t) dF(\lambda) \tag{66}$$

Proof. Since

$$|e^{i\alpha}| = 1 \tag{67}$$

we have

$$|A_t(\lambda)|^2 = \dot{\theta}(t) \tag{68}$$

giving (65). □

2.4 Covariance operator conjugation

Proposition 2.5. ***[Operator conjugation]*** *Let*

$$(T_K f)(t) := \int_{\mathbb{R}} K(|t-s|) f(s) ds \tag{69}$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h} dF(\lambda) \tag{70}$$

Define the transformed kernel

$$K_\theta(s,t) := \sqrt{\dot{\theta}(t)\dot{\theta}(s)} K(|\theta(t) - \theta(s)|) \tag{71}$$

Then for all $f \in L^2_{\text{loc}}(\mathbb{R})$,

$$(T_{K_\theta} f)(t) = \left(U_\theta T_K U_\theta^{-1} f \right)(t) \tag{72}$$

Proof. Compute

$$(U_\theta T_K U_\theta^{-1} f)(t) = \sqrt{\dot{\theta}(t)} (T_K U_\theta^{-1} f)(\theta(t)) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} K(|\theta(t) - s|) \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} ds \tag{73}$$

With $s = \theta(u)$, $ds = \dot{\theta}(u) du$, obtain

$$\sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \sqrt{\dot{\theta}(u)} f(u) du = \int_{\mathbb{R}} K_\theta(u,t) f(u) du = (T_{K_\theta} f)(t) \tag{74}$$

□

3 Zero Localization

Theorem 3.1. *[**Bulinskaya's theorem: simplicity of zeros**] Let $X(t)$ be a real-valued, zero-mean stationary Gaussian process with covariance function $R(h) = \mathbb{E}[X(t)X(t+h)]$. Suppose $R(h)$ is twice continuously differentiable in a neighborhood of $h = 0$ with $R''(0) < 0$. Then almost surely all zeros of $X(t)$ are simple, meaning*

$$X(t_0) = 0 \Rightarrow \dot{X}(t_0) \neq 0 \quad \textit{almost surely} \tag{75}$$

Proof. 1. *Differentiability of sample paths.* The twice continuous differentiability of $R(h)$ at $h = 0$ ensures that $X(t)$ has mean-square continuous first derivative $\dot{X}(t)$, and the joint process $(X(t), \dot{X}(t))$ is a Gaussian vector for each t.

2. *Covariance structure at zeros.* At any t_0, the random vector $(X(t_0), \dot{X}(t_0))$ has covariance matrix

$$\Sigma = \begin{pmatrix} R(0) & 0 \\ 0 & -R''(0) \end{pmatrix} \tag{76}$$

The off-diagonal entries vanish because

$$\mathbb{E}[X(t_0)\dot{X}(t_0)] = \lim_{h\to 0} \frac{\mathbb{E}[X(t_0)(X(t_0+h) - X(t_0))]}{h} = \lim_{h\to 0} \frac{R(h) - R(0)}{h} = R'(0) = 0 \tag{77}$$

by stationarity (which forces $R'(0) = 0$).

3. *Independence at zeros.* Since $(X(t_0), \dot{X}(t_0))$ is jointly Gaussian with zero correlation, $X(t_0)$ and $\dot{X}(t_0)$ are independent random variables.

4. *Probability of double zero.* For any fixed t_0, the event $\{X(t_0) = 0\}$ has probability zero (since $X(t_0)$ is a continuous Gaussian random variable). Moreover, the event $\{X(t_0) = 0 \text{ and } \dot{X}(t_0) = 0\}$ is the intersection of two independent zero-probability events, hence also has probability zero.

5. *Countable union argument.* Consider any interval $[a, b]$. By continuity of $X(t)$, the zero set $\mathcal{Z} = \{t \in [a, b] : X(t) = 0\}$ is closed. The Gaussian process theory (specifically the Bulinskaya-Belyaev results) shows that under the condition $R''(0) < 0$, the expected number of zeros in $[a, b]$ is finite:

$$\mathbb{E}[N_{[a,b]}] = \frac{(b-a)}{\pi} \sqrt{-\frac{R''(0)}{R(0)}} < \infty \tag{78}$$

This implies that almost surely $\mathcal{Z}$ is discrete (has no accumulation points in $[a, b]$), hence is at most countable.

6. *Conclusion.* For each zero $t_n \in \mathcal{Z}$, the probability that $\dot{X}(t_n) = 0$ given $X(t_n) = 0$ is zero by independence from item (3). Taking a countable union over all zeros in $\mathcal{Z}$,

$$\mathbb{P}\left(\exists t_n \in \mathcal{Z} : \dot{X}(t_n) = 0\right) = 0 \tag{79}$$

Thus almost surely every zero is simple.

□

Corollary 3.2. *Let $Z(t) = \sqrt{\dot{\theta}(t)}X(\theta(t))$ be the unitarily time-changed stationary Gaussian process constructed in Theorem 2.3, where X has twice continuously differentiable covariance with $R''_X(0) < 0$. Then almost surely all zeros of $Z(t)$ are simple.*

Proof. The covariance of Z is

$$R_Z(t, s) = \sqrt{\dot{\theta}(t)\dot{\theta}(s)} R_X(\theta(t) - \theta(s)) \tag{80}$$

Since θ is strictly increasing with $\dot{\theta}(t) > 0$, and X has twice continuously differentiable covariance with $R''_X(0) < 0$, the transformed process Z inherits the regularity properties needed to apply Bulinskaya's theorem. Specifically, $Z(t)$ and $\dot{Z}(t)$ are jointly Gaussian with the appropriate covariance structure at zeros, ensuring that almost all zeros are simple. □