Definition 1. **(Convex Function)** *A function* $f\colon \mathbb{R}\to\mathbb{R}$ *is convex if for any* $x, y \in \mathbb{R}$ *and* $t \in [0,1]$*:*

$$f\,(t\,x + (1-t)\,y) \leq t\,f(x) + (1-t)\,f(y)$$

Definition 2. **(Convex Functional)** *A functional* $J\colon X \to \mathbb{R}$ *on a function space* X *is convex if for any* $f(x), g(x) \in X$ *and* $t \in [0,1]$*:*

$$J\,(t\cdot f(x) + (1-t)\cdot g(x)) \leq t\cdot J(f(x)) + (1-t)\cdot J(g(x))$$

Definition 3. **(Monotonically Increasing Function)** *A function* $f\colon \mathbb{R}\to\mathbb{R}$ *is monotonically increasing if:*

$$x < y \Longrightarrow f(x) < f(y) \quad \forall x, y \in \mathbb{R}$$

Definition 4. **(Monotonically Decreasing Function)** *A function* $f\colon \mathbb{R}\to\mathbb{R}$ *is monotonically decreasing if:*

$$x < y \Longrightarrow f(x) > f(y) \quad \forall x, y \in \mathbb{R}$$

Definition 5. **(Non-decreasing Function)** *A function* $f\colon \mathbb{R}\to\mathbb{R}$ *is non-decreasing if:*

$$x \leq y \Longrightarrow f(x) \leq f(y) \quad \forall x, y \in \mathbb{R}$$

Definition 6. **(Non-increasing Function)** *A function* $f\colon \mathbb{R}\to\mathbb{R}$ *is non-increasing if:*

$$x \leq y \Longrightarrow f(x) \geq f(y) \quad \forall x, y \in \mathbb{R}$$

Theorem 7. **(Convexity and Monotonicity)** *If a function* $f\colon \mathbb{R}\to\mathbb{R}$ *is twice differentiable and* $\frac{d^2 f}{d\,x^2}(x) > 0$ *for all* x*, then* f *is convex and monotonically increasing.*

Remark 8. A function that is not monotonic may have points where a line cannot be drawn through its graph without intersecting it at more than two points, violating the definition of convexity.

Theorem 9. **(Jensen's Inequality)** *For a convex function* $f\colon \mathbb{R}\to\mathbb{R}$*, real numbers* $x_1, \ldots, x_n$*, and non-negative weights* $\alpha_1, \ldots, \alpha_n$ *with* $\sum_{i=1}^{n} \alpha_i = 1$*:*

$$f\left(\sum_{i=1}^{n} \alpha_i\,x_i\right) \leq \sum_{i=1}^{n} \alpha_i\,f(x_i) \tag{1}$$

Equality holds if and only if all x_i are equal or f is linear.

Corollary 10. *For a convex function f and a random variable X:*

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)] \tag{2}$$

Theorem 11. (Convexity and Local Minima) *If $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, then any local minimum of f is also a global minimum.*

Theorem 12. (Karush-Kuhn-Tucker (KKT) Conditions) *For a convex optimization problem:*

$$\min_{x \in \mathbb{R}^n} f(x) \text{ subject to } g_i(x) \leq 0, h_j(x) = 0 \tag{3}$$

where f and g_i are convex and h_j are affine, the KKT conditions are necessary and sufficient for optimality.

Definition 13. (Affine Function) *A function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is affine if it can be expressed as:*

$$f(x) = A\,x + b \tag{4}$$

where A is an $m \times n$ matrix and b is an m-dimensional vector.

Theorem 14. (Properties of Affine Functions)

1. *Every affine function is both convex and concave.*
2. *The composition of an affine function with a convex function is convex.*
3. *The sum of an affine function and a convex function is convex.*

Theorem 15. (Affine Hull) *The affine hull of a set $S \subset \mathbb{R}^n$ is the smallest affine set containing S:*

$$\mathit{aff}(S) = \left\{ \sum_{i=1}^{k} \alpha_i\, x_i : x_i \in S, \sum_{i=1}^{k} \alpha_i = 1 \right\} \tag{5}$$

Theorem 16. (Affine Independence) *Vectors $\{v_n\}_{n=1}^{k}$ are affinely independent if and only if $\{v_n - v_1\}_{n=1}^{k}$ are linearly independent.*