Interval-Dependent Covariance Parametrization and Complex-Analytic Time Change

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1 Introduction

Let $\{X(t)\}_{t\in\mathbb{R}}$ be a real-valued, second-order Gaussian process that is non-stationary in the sense of Priestley. Such processes are *oscillatory* and have separable linear span in L^2. We study how the covariance structure of X over finite intervals can be parametrized by a two-parameter family of kernels, and how this parametrization connects to an underlying complex-analytic time-change via Wirtinger and areolar derivatives.

2 Setup and interval-dependent parametrization

2.1 Empirical autocovariance

Fix a bounded interval $[S,T]\subset\mathbb{R}$ and assume X is observed on a discrete grid $\{t_0,\dots,t_{n-1}\}\subset[S,T]$ with uniform spacing Δt. The *empirical autocovariance* at lag index $k\in\mathbb{Z}_{\geq 0}$ is

$$\widehat{C}_{[S,T]}(k)=\frac{1}{n-k}\sum_{j=0}^{n-k-1}\big(X(t_j)-\bar{X}\big)\big(X(t_{j+k})-\bar{X}\big),$$

where $\bar{X}$ is the sample mean over the interval.

2.2 Parametric kernel hypothesis

Suppose there exists a fixed kernel $K:\mathbb{R}\to\mathbb{R}$, depending only on the lag $h=t-s$, such that on each finite interval $[S,T]$, the second-order structure of X is well-approximated by a two-parameter family

$$C_{A,B}(s,t)=A\,K\big(B(t-s)\big),\qquad s,t\in[S,T],$$

for scalars $A,B\in\mathbb{R}$ depending on the interval. Here:

- $K(h)$ captures a universal oscillatory shape (stationary covariance structure);
- A encodes interval-dependent variance/amplitude scaling;
- B encodes interval-dependent time-scale (frequency) scaling.

2.3 Best-fit parameters via least squares

Fix a finite set of lags $\mathcal{L} = \{k_1, \ldots, k_m\} \subset \mathbb{Z}_{\geq 0}$ with corresponding lag values $h_k = k\Delta t$. Define the residual at lag k by

$$r_k(A, B) = \widehat{C}_{[S,T]}(k) - A\,K(Bh_k),$$

and the mean square error

$$\mathrm{MSE}_{[S,T]}(A, B) = \frac{1}{|\mathcal{L}|} \sum_{k \in \mathcal{L}} r_k(A, B)^2.$$

Definition 1. *For each interval $[S, T]$ with $S < T$, define $(A(S,T), B(S,T))$ to be any minimizer*

$$\big(A(S,T), B(S,T)\big) \in \arg\min_{(A,B)\in\mathbb{R}^2} \mathrm{MSE}_{[S,T]}(A, B).$$

This defines a mapping

$$\Phi : \{(S,T) \in \mathbb{R}^2 : S < T\} \longrightarrow \mathbb{R}^2, \qquad \Phi(S,T) = \big(A(S,T),\, B(S,T)\big),$$

which assigns to each choice of interval endpoints a pair of scalars giving the best-fit covariance kernel.

2.4 Reparametrization by midpoint and length

Define

$$M = \frac{S+T}{2}, \qquad L = T - S > 0.$$

Then

$$\Psi : \mathcal{D} \to \mathbb{R}^2, \qquad \Psi(M, L) = (A(M, L),\, B(M, L)),$$

with

$$A(M, L) = A\big(M - L/2,\, M + L/2\big), \qquad B(M, L) = B\big(M - L/2,\, M + L/2\big).$$

Empirically, one may observe simple functional forms, e.g.

$$A(M, L) \approx \alpha_0 + \alpha_1 M + \alpha_2 L, \qquad B(M, L) \approx \beta_0 + \beta_1 M + \beta_2 L,$$

indicating that the non-stationarity manifests as a systematic linear variation of variance and time scale with interval location and length.

3 Optimization: Newton convergence

3.1 Gradient

Compute partial derivatives of the residuals:

$$\frac{\partial r_k}{\partial A} = -K(Bh_k), \qquad \frac{\partial r_k}{\partial B} = -A\,K'(Bh_k)\,h_k.$$

The gradient of $\mathrm{MSE}(A, B)$ is

$$\nabla \mathrm{MSE} = \frac{2}{|\mathcal{L}|} \begin{pmatrix} -\sum_k r_k(A,B)\,K(Bh_k) \\ -\sum_k r_k(A,B)\,A\,K'(Bh_k)\,h_k \end{pmatrix}.$$

3.2 Hessian

The second derivatives are

$$\frac{\partial^2 \mathrm{MSE}}{\partial A^2} = \frac{2}{|\mathcal{L}|} \sum_k K(Bh_k)^2,$$

$$\frac{\partial^2 \mathrm{MSE}}{\partial A\, \partial B} = \frac{2}{|\mathcal{L}|} \sum_k \left[A\, K'(Bh_k)\, h_k\, K(Bh_k) - r_k(A,B)\, K'(Bh_k)\, h_k \right],$$

$$\frac{\partial^2 \mathrm{MSE}}{\partial B^2} = \frac{2}{|\mathcal{L}|} \sum_k \left[A^2 K'(Bh_k)^2 h_k^2 - r_k(A,B)\, A\, K''(Bh_k)\, h_k^2 \right].$$

Assemble the Hessian matrix $H(A, B)$.

3.3 Newton iteration

Newton's method updates

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} A_n \\ B_n \end{pmatrix} - H(A_n, B_n)^{-1}\, \nabla \mathrm{MSE}(A_n, B_n).$$

Proposition 1 (Local convergence)**.** *Near the true minimizer (A^*, B^*), if K is C^2 and the residuals $r_k(A^*, B^*)$ are small, then the Hessian is dominated by the sum-of-squares terms*

$$\sum_k K(B^* h_k)^2 > 0, \qquad \sum_k A^{*2} K'(B^* h_k)^2 h_k^2 > 0,$$

which ensure $H(A^*, B^*)$ is positive definite. Standard Newton–Kantorovich theory then guarantees quadratic convergence from any starting point sufficiently close to (A^*, B^*).

4 Complex-analytic interpretation via Wirtinger and areolar derivatives

The interval-dependent parameters (A, B) have a deeper origin in an underlying *complex-analytic time change*. Instead of a real monotone time-reparametrization, we posit a map $F : \mathbb{C} \to \mathbb{C}$ such that the process X arises by pulling back a stationary Gaussian process through F.

4.1 Wirtinger derivatives and local metric scaling

For a map $F(z)$ of a complex variable $z = x + iy$, define the Wirtinger derivatives

$$\frac{\partial F}{\partial z} = \frac{1}{2}\left(\frac{\partial F}{\partial x} - i \frac{\partial F}{\partial y} \right), \qquad \frac{\partial F}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} \right).$$

If F is holomorphic, then $\partial F/\partial \bar{z} = 0$ and $\partial F/\partial z = F'(z)$ is the complex derivative.

Key fact: At a point z_0, the map F locally scales the Euclidean metric by

$$|dF|^2 \approx |F'(z_0)|^2\, |dz|^2.$$

Hence $|F'(z)|^2$ encodes the *local variance scaling* induced by F, and $|F'(z)|$ encodes the *local length/time-scale scaling.*

4.2 Stationary process pullback

Suppose $Y(w)$ is a stationary Gaussian process indexed by $w \in \mathbb{C}$ with covariance kernel $K(w-w')$ that depends only on the lag. Define

$$X(z) = Y\big(F(z)\big).$$

Near a point z_0, for small real displacement h,

$$F(z_0+h) - F(z_0) \approx F'(z_0)\, h.$$

Hence the covariance of X near z_0 is

$$\mathbb{E}\big[X(z_0)\, X(z_0+h)\big] = \mathbb{E}\big[Y(F(z_0))\, Y(F(z_0+h))\big] \approx K\big(F'(z_0)\, h\big),$$

assuming the variance of Y is constant. If the variance of Y itself varies as $\sigma^2(w)$, then

$$\mathbb{E}\big[X(z_0)\, X(z_0+h)\big] \approx \sigma^2\big(F(z_0)\big)\, K\big(F'(z_0)\, h\big).$$

This is *exactly* of the form

$$A(z_0)\, K\big(B(z_0)\, h\big),$$

with

$$A(z_0) = \sigma^2\big(F(z_0)\big), \qquad B(z_0) = |F'(z_0)|.$$

4.3 Areolar derivatives and averaged second-order behavior

The *areolar derivative* (Pompeiu derivative) generalizes the usual complex derivative by probing average behavior over small discs instead of pointwise quotients. For a function F, the areolar derivative at z_0 is defined roughly by

$$F^{\circ}(z_0) = \lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \int_{|w-z_0|<\varepsilon} \big(F(w) - F(z_0)\big)\, dA(w),$$

where dA is area measure.

For covariance, which is inherently a second-order (mean-square) object, the areolar derivative provides the correct notion of "averaged infinitesimal distortion." In particular, $|F^{\circ}(z)|^2$ describes the local *area-averaged variance scaling*.

If $A(S,T)$ is the best-fit amplitude over the interval $[S,T]$, then in a continuous model one expects

$$A(S,T) \approx \int_S^T \rho_F(t)\, dt, \quad \text{where } \rho_F(t) = |F^{\circ}(t)|^2.$$

Similarly, the average time-scale scaling $B(S,T)$ should correspond to a suitable average of $|F'(t)|$ over the interval.

4.4 Reconstruction strategy

Given empirically fitted $(A(S,T), B(S,T))$ for many intervals:

1. Interpret $A(M,L)$ as an interval average of a local variance density $\rho_F(t)$.
2. Interpret $B(M,L)$ as an interval average of the modulus $|F'(t)|$ of the complex derivative.

3. Invert: solve for a smooth function $\rho_F(t)$ (or equivalently $|F^\circ(t)|^2$) and $|F'(t)|$ such that averaging them over $[S, T]$ reproduces the observed $A(S, T)$ and $B(S, T)$.

4. Use the relation
$$F'(t) = B(t)\, e^{i\phi(t)},$$
where $\phi(t)$ is an unknown phase, to reconstruct F up to a global phase ambiguity by integrating
$$F(t) = F(t_0) + \int_{t_0}^{t} F'(s)\, ds.$$

This yields an explicit complex map F whose Wirtinger and areolar derivatives encode the observed modulation of variance and time-scale, replacing the ad hoc monotonized Riemann–Siegel θ function with a principled complex-analytic time change.

5 Connection to the Riemann zeta function

In the context of modeling the zeros of $\zeta(1/2 + it)$, the smooth zero-counting function is
$$N(T) = \frac{\theta(T)}{\pi} + 1 + S(T),$$
where $\theta(T)$ is the Riemann–Siegel theta function and $S(T)$ is the S-function (argument of ζ). The original conjecture was to use a monotonized version of $\theta(T)$ as the time change to map a stationary process into one matching the zero density.

The revised approach via Wirtinger/areolar derivatives suggests:

- The correct time change F is not merely a monotonized real function, but a *complex map* whose local metric properties (encoded in F' and F°) reproduce the observed A and B scalings.
- The phase structure of F should encode the oscillatory behavior of $\arg \zeta$, while the modulus $|F'|$ encodes the local density of zeros.
- Empirically fitting $A(M, L)$ and $B(M, L)$ over many intervals and inverting to find $\rho_F(t)$ and $|F'(t)|$ gives a data-driven way to determine F.

6 Computational implementation in ARB4J

Given the above framework, a practical workflow using ARB4J (high-precision arbitrary ball arithmetic for Java) is:

1. **Precompute time-change inverse:** Build a discretized inverse table for the candidate F (or its real part) over the domain $[0, 200]$ or other relevant range. This allows instant lookup evaluation of $t \mapsto F(t)$.
2. **Sample intervals:** For a grid of intervals $\{[S_i, T_i]\}$, compute the empirical autocovariance $\widehat{C}_{[S_i,T_i]}(k)$ for each lag $k \in \mathcal{L}$.
3. **Fit (A_i, B_i):** For each interval, use Newton iteration with high-precision arithmetic to minimize $\mathrm{MSE}_{[S_i,T_i]}(A, B)$ and record (A_i, B_i).

4. **Chart the relationships:** Plot A_i and B_i against the midpoint $M_i = (S_i + T_i)/2$ and length $L_i = T_i - S_i$. Identify functional forms (e.g., linear, logarithmic).

5. **Infer $\rho_F(t)$ and $|F'(t)|$:** Using the relationship

$$A(M, L) \approx \int_{M-L/2}^{M+L/2} \rho_F(t)\, dt, \qquad B(M, L) \approx \frac{1}{L} \int_{M-L/2}^{M+L/2} |F'(t)|\, dt,$$

   invert (via differentiation or deconvolution) to extract the pointwise functions $\rho_F(t)$ and $|F'(t)|$.

6. **Reconstruct F:** Integrate $F'(t) = B(t)\, e^{i\phi(t)}$ to obtain $F(t)$ up to a phase function $\phi(t)$, which can be determined from additional geometric or oscillatory constraints (e.g., matching the argument of ζ).

This yields an explicit, data-validated complex map F encoding the non-stationarity as a unitary (or quasi-unitary) time change with rigorous complex-analytic structure.

7 Conclusion

The observation that a non-stationary oscillatory process has empirical autocovariance of the form $A(S, T)\, K(B(S, T)(t - s))$ on every finite interval, with A and B depending linearly on the interval endpoints (sum and difference), is a strong structural signature. It indicates:

1. The process is a unitarily time-changed stationary Gaussian process, or a close approximation thereof.

2. The time change can be realized via a complex-analytic map F whose Wirtinger and areolar derivatives encode the observed variance and time-scale modulations.

3. The correct F is not an ad hoc monotonization of a real function, but a map whose complex derivative structure is determined by the data through the best-fit (A, B) parameters.

By implementing the fitting procedure in high-precision arithmetic (ARB4J), one can empirically determine $A(M, L)$ and $B(M, L)$ to high accuracy, chart their dependence on interval geometry, and invert to reconstruct the underlying complex time-change map F that governs the oscillatory structure of the process.