The Characteristic Function of the Product of Independent Standard Normal Variables

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Abstract

This paper demonstrates that the characteristic function of the product of two independent standard normal random variables involves the Bessel function of the first kind of order 0. Polar coordinate transformations and properties of Bessel functions are utilized to obtain the closed form expression.

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1 Introduction

The characteristic function of the product of two independent standard normal random variables serves as a fundamental result in probability theory and statistical analysis. This paper presents a rigorous derivation of its closed form.

2 Main Result

Theorem 1. *For independent standard normal random variables X and Y, the characteristic function of their product XY is given by:*

$$\phi_{XY}(t) = \frac{J_0\left(\frac{t}{\sqrt{1+t^2}}\right)}{\sqrt{1+t^2}} \tag{1}$$

where J_0 denotes the Bessel function of the first kind of order zero.

3 Proof

Proof. The derivation begins with the characteristic function definition:

$$\phi_{XY}(t) = E[e^{itXY}] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{itxy}\, e^{-(x^2+y^2)/2}\, dx\, dy \tag{2}$$

Polar Coordinate Transformation

The introduction of polar coordinates where $x = r\cos\theta$, $y = r\sin\theta$, and $dx\,dy = r\,dr\,d\theta$ transforms the integral to:

$$\frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} e^{itr^2\cos\theta\sin\theta}\, r\, e^{-r^2/2}\, d\theta\, dr \tag{3}$$

Variable Substitution

The substitution $u = r^2/2$, with $du = r\,dr$, yields:

$$\frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} e^{2itu\cos\theta\sin\theta}\, e^{-u}\, d\theta\, du \tag{4}$$

Double Angle Formula

Application of the identity $\cos\theta\sin\theta = \frac{1}{2}\sin(2\theta)$ gives:

$$\frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} e^{itu\sin(2\theta)}\, e^{-u}\, d\theta\, du \tag{5}$$

Bessel Function Connection

The inner integral relates to the Bessel function through the identity:

$$\int_0^{2\pi} e^{itu\sin(2\theta)}\, d\theta = 2\pi J_0(tu) \tag{6}$$

This follows from the integral representation of the Bessel function of the first kind:

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin(\theta)}\, d\theta \tag{7}$$

The integral thus reduces to:

$$\int_0^{\infty} J_0(tu)\, e^{-u}\, du \tag{8}$$

Final Evaluation

The evaluation proceeds through the known Laplace transform of Bessel functions:

$$\int_0^\infty J_0(a\,t)\,e^{-ut}\,d\,t = \frac{1}{\sqrt{1+a^2}}\,J_0\!\left(\frac{a}{\sqrt{1+a^2}}\right) \tag{9}$$

This leads to the final result:

$$\phi_{XY}(t) = \frac{J_0\!\left(\frac{t}{\sqrt{1+t^2}}\right)}{\sqrt{1+t^2}} \tag{10}$$ □

4 Conclusion

The derivation establishes that the characteristic function of the product of two independent standard normal random variables takes the form $J_0(t/\sqrt{1+t^2})/\sqrt{1+t^2}$. The proof relies on coordinate transformation, properties of Bessel functions, and integral transforms. This result holds significance in various applications of probability theory and statistical analysis.