Unitary Bijections From Strictly Increasing Functions On The Real Line

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Abstract

This paper establishes a comprehensive theory of unitary change-of-variables operators on L^2 spaces, encompassing both the general framework for $L^2(\mathbb{R})$ and specialized results for measure-preserving transformations on unbounded domains. The investigation begins with the characterization of when weighted composition operators $(U f)(x) = f(T(x)) \cdot w(x)$ achieve unitarity, requiring measurable bijections modulo null sets, mutual absolute continuity of measures, and specific weight functions involving Radon-Nikodym derivatives. For differentiable transformations, this reduces to the condition $|w(x)|^2 = |T'(x)|$. The analysis then specializes to C^1 bijective transformations $g: I \rightarrow J$ between unbounded intervals with positive derivative almost everywhere, where L^2 norm preservation under Lebesgue measure is achieved through the unitary change of variables operator $T_g f = f(g(y))\sqrt{g'(y)}$. The framework is further extended to arbitrary σ-finite measures μ and ν, where the scaling factor becomes the square root of the Radon-Nikodym derivative $\sqrt{\frac{d\,(\mu \circ g^{-1})}{d\,\mu}(y)}$. The necessity of these specific scaling factors is rigorously established through variational arguments in all settings. These findings provide a unified theoretical foundation bridging the change-of-variables formula in real analysis with the unitary structure of L^2 spaces over general measure spaces, with applications in ergodic theory, functional analysis, and measure theory.

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1 Introduction

This paper presents a comprehensive theory of unitary change-of-variables operators on L^2 spaces, establishing the fundamental relationship between unitary bijections and measure-preserving transformations in both general and specialized settings. The investigation begins with the general framework for weighted composition operators on $L^2(\mathbb{R})$, then specializes to measure-preserving transformations on unbounded domains, extending from classical Lebesgue measure to general σ-finite measures.

2 General Framework: Unitary Change-of-Variables Operators

Definition 1. *A* change-of-variables operator *on $L^2(\mathbb{R},\mu)$ where μ is Lebesgue measure is a bounded linear operator $U\colon L^2(\mathbb{R})\to L^2(\mathbb{R})$ of the form*

$$(Uf)(x)=f(T(x))\cdot w(x) \tag{1}$$

for some measurable map $T\colon\mathbb{R}\to\mathbb{R}$ and measurable weight function $w\colon\mathbb{R}\to\mathbb{C}$ with $|w(x)|>0$ almost everywhere.

Theorem 2. *Let U be a change-of-variables operator as in Definition 1. Then U is unitary if and only if the following conditions hold:*

1. *$T\colon\mathbb{R}\to\mathbb{R}$ is a measurable bijection modulo null sets;*
2. *$\mu\circ T^{-1}\ll\mu$ and $\mu\ll\mu\circ T^{-1}$ (mutual absolute continuity);*
3. *$|w(x)|^2=\frac{d(\mu\circ T)}{d\mu}(x)$ almost everywhere;*
4. *$w(x)=\sqrt{\frac{d(\mu\circ T)}{d\mu}(x)}\cdot e^{i\theta(x)}$ for some measurable phase function $\theta\colon\mathbb{R}\to\mathbb{R}$.*

Furthermore, if T is differentiable almost everywhere with $T'(x)\neq 0$ a.e., then condition (3) becomes

$$|w(x)|^2=|T'(x)| \tag{2}$$

Proof. The proof proceeds by establishing necessity and sufficiency separately.

Necessity: Assume U is unitary. Since U is an isometry, for any $f\in L^2(\mathbb{R})$,

$$\|Uf\|_2^2=\|f\|_2^2 \tag{3}$$

Computing the left side:

$$\|Uf\|_2^2 = \int_{\mathbb{R}} |f(T(x))|^2 |w(x)|^2 \, d\mu(x) \tag{4}$$

Define the measure ν by $d\nu = |w|^2 d\mu$. By the change-of-variables formula for the push-forward measure,

$$\int_{\mathbb{R}} |f(T(x))|^2 |w(x)|^2 \, d\mu(x) = \int_{\mathbb{R}} |f(y)|^2 \, d(T_* \nu)(y) \tag{5}$$

where

$$(T_* \nu)(A) = \nu(T^{-1}(A)) \tag{6}$$

for measurable sets A. From equation (3), we require

$$\int_{\mathbb{R}} |f(y)|^2 \, d(T_* \nu)(y) = \int_{\mathbb{R}} |f(y)|^2 \, d\mu(y) \tag{7}$$

for all $f \in L^2(\mathbb{R})$.

This implies $T_* \nu = \mu$ as measures. Therefore, for any measurable set A,

$$\mu(A) = \nu(T^{-1}(A)) = \int_{T^{-1}(A)} |w(x)|^2 \, d\mu(x) \tag{8}$$

For U to be surjective (hence unitary rather than merely isometric), T must be invertible modulo null sets. This requires both directions of absolute continuity in condition (2).

By the Radon-Nikodym theorem, since $\mu \circ T^{-1} \ll \mu$, there exists $\rho \geq 0$ such that

$$\rho(y) = \frac{d(\mu \circ T^{-1})}{d\mu}(y) \tag{9}$$

The standard change-of-variables identity gives, for nonnegative measurable g,

$$\int_{\mathbb{R}} g(T(x)) \, d\mu(x) = \int_{\mathbb{R}} g(y) \, \rho(y) \, d\mu(y) \tag{10}$$

Comparing with the isometry requirement from equation (7), we obtain

$$\int_{\mathbb{R}} g(T(x)) |w(x)|^2 \, d\mu(x) = \int_{\mathbb{R}} g(y) \, d\mu(y) \tag{11}$$

This requires

$$|w(x)|^2 = \rho(T(x))^{-1} \tag{12}$$

almost everywhere. By the chain rule for Radon-Nikodym derivatives,

$$|w(x)|^2 = \frac{d(\mu \circ T)}{d\mu}(x) \tag{13}$$

The phase freedom in condition (4) follows from the fact that only $|w|^2$ is determined by the isometry condition.

Sufficiency: Conversely, assume conditions (1)-(4) hold. Define U as in Definition 1 with the specified T and w. The computation above shows that U is isometric. Since T is bijective modulo null sets with mutual absolute continuity, the operator U^* exists and is given by

$$(U^* g)(x) = g(T^{-1}(x)) \cdot \overline{w(T^{-1}(x))} \cdot \sqrt{\frac{d(\mu \circ T^{-1})}{d\mu}(x)} \cdot e^{-i\theta(T^{-1}(x))} \tag{14}$$

Direct computation verifies $UU^* = U^* U = I$, establishing unitarity.

The final statement regarding differentiable T follows from the fact that for such maps,

$$\frac{d(\mu \circ T)}{d\mu}(x) = |T'(x)| \tag{15}$$

by the classical change-of-variables theorem. $\square$

Lemma 3. *If $T: \mathbb{R} \to \mathbb{R}$ is a measurable bijection that is differentiable almost everywhere, then T is either almost everywhere monotone increasing or almost everywhere monotone decreasing.*

Proof. Since T is a bijection of $\mathbb{R}$, the intermediate value theorem and injectivity require that T cannot change monotonicity on any interval where it is continuous. As T is differentiable almost everywhere, it is continuous almost everywhere, and the set where T' exists has full measure. On this set, T' cannot change sign without violating the bijection property, hence $T'(x) \geq 0$ almost everywhere or $T'(x) \leq 0$ almost everywhere. $\square$

3 Bijective Transformations on Unbounded Domains

Theorem 4. (Bijectivity of Strictly Increasing Functions on Unbounded Domains) *Let $g: I \to \mathbb{R}$ be a strictly increasing function where $I \subseteq \mathbb{R}$ is an unbounded interval. Then g is bijective onto its range $J = g(I)$, and J is also an unbounded interval.*

Proof. Since g is strictly increasing, injectivity is immediate. For any $x_1, x_2 \in I$ with $x_1 < x_2$, one has $g(x_1) < g(x_2)$.

For surjectivity onto $J = g(I)$, let $y \in J$. By definition, there exists $x \in I$ such that $g(x) = y$. The uniqueness of such x follows from injectivity.

To establish that J is unbounded, consider two cases:

1. If $I = (a, \infty)$ or $I = [a, \infty)$ for some $a \in \mathbb{R}$, then as $x \to \infty$, since g is strictly increasing, either $g(x) \to \infty$ or $g(x)$ approaches some finite supremum. If the latter held, then by the intermediate value theorem and strict monotonicity, g would map (a, ∞) to some bounded interval, contradicting the strict increase property over an unbounded domain.

2. If $I = (-\infty, b)$ or $I = (-\infty, b]$, a similar argument shows J extends to $-\infty$.

3. If $I = \mathbb{R}$, then J must be unbounded in both directions.

Therefore, $g: I \to J$ is bijective with both I and J unbounded intervals. ☐

Theorem 5. (Differentiable Bijections with Positive Derivative) *Let $g: I \to J$ be a C^1 bijection between unbounded intervals $I, J \subseteq \mathbb{R}$ such that $g'(y) > 0$ for all $y \in I$ except possibly on a set of measure zero. Then g is a well-defined change of variables for Lebesgue integration.*

Proof. The condition $g'(y) > 0$ almost everywhere ensures that g is locally invertible almost everywhere. Since g is already assumed bijective and C^1, the standard change of variables formula applies:

$$\begin{aligned} \int_J f(x)\, dx &= \int_I f(g(y))|g'(y)|\, dy \\ &= \int_I f(g(y))\, g'(y)\, dy \end{aligned} \tag{16}$$

where the last equality uses $g'(y) > 0$ almost everywhere. The points where $g'(y) = 0$ form a set of measure zero and do not affect the integral. ☐

4 L^2 Norm Preservation Under Lebesgue Measure

Definition 6. (Unitary Change of Variables Operator) *Let $g: I \to J$ be a C^1 bijection between unbounded intervals with $g'(y) > 0$ almost everywhere. For $f \in L^2(J, dx)$, define the unitary change of variables operator T_g by:*

$$(T_g f)(y) = f(g(y))\sqrt{g'(y)} \tag{17}$$

Theorem 7. (L^2 Norm Preservation for Unbounded Domains) *Under the conditions of Definition 6, the operator $T_g: L^2(J, dx) \to L^2(I, dy)$ is an isometric isomorphism. Specifically:*

$$\|T_g f\|_{L^2(I, dy)} = \|f\|_{L^2(J, dx)} \tag{18}$$

Proof. For $f \in L^2(J, dx)$, compute directly:

$$\|T_g f\|_{L^2(I,dy)}^2 = \int_I |f(g(y))\sqrt{g'(y)}|^2 \, dy \tag{19}$$

$$= \int_I |f(g(y))|^2 \, g'(y) \, dy \tag{20}$$

By the change of variables formula from Theorem 5 with $x = g(y)$:

$$\int_I |f(g(y))|^2 g'(y) \, dy = \int_J |f(x)|^2 \, dx = \|f\|_{L^2(J,dx)}^2 \tag{21}$$

Since both I and J are unbounded, the change of variables is justified by approximating with bounded subintervals and applying the monotone convergence theorem.

Therefore:

$$\|T_g f\|_{L^2(I,dy)} = \|f\|_{L^2(J,dx)} \tag{22}$$

The fact that $T_g f \in L^2(I, dy)$ follows immediately from equation (22) and the assumption $f \in L^2(J, dx)$. □

Theorem 8. (Necessity of Square Root Unitary Transformation) *Let $g: I \to J$ be as in Theorem 7. If $\phi: I \to \mathbb{R}^+$ is any measurable function such that $f(g(y))\,\phi(y) \in L^2(I, dy)$ and*

$$\|f(g(\cdot))\,\phi(\cdot)\|_{L^2(I,dy)} = \|f\|_{L^2(J,dx)} \tag{23}$$

for all $f \in L^2(J, dx)$, then $\phi(y) = \sqrt{g'(y)}$ almost everywhere.

Proof. From the norm condition in equation (23):

$$\int_I |f(g(y))|^2 \phi(y)^2 \, dy = \int_J |f(x)|^2 \, dx \tag{24}$$

Using the change of variables $x = g(y)$ on the right side:

$$\int_I |f(g(y))|^2 \phi(y)^2 \, dy = \int_I |f(g(y))|^2 \, g'(y) \, dy \tag{25}$$

This gives:

$$\int_I |f(g(y))|^2 \, (\phi(y)^2 - g'(y)) \, dy = 0 \tag{26}$$

Since this holds for all $f \in L^2(J, d\,x)$ and the composition $f(g(\cdot))$ generates a dense subspace of $L^2(I, g'(y)\; d\,y)$, the fundamental lemma of calculus of variations implies:

$$\phi(y)^2 = g'(y) \text{almost everywhere} \tag{27}$$

Taking $\phi(y) > 0$, one obtains $\phi(y) = \sqrt{g'(y)}$ almost everywhere. □

5 Extension to General σ-Finite Measures

Theorem 9. (Extension to General Measures) *Let μ and ν be σ-finite measures on I and J respectively, and let $g: I \to J$ be a measurable bijection. If $\nu = \mu \circ g^{-1}$ (i.e., $\nu(E) = \mu(g^{-1}(E))$ for all measurable $E \subseteq J$), then for $f \in L^2(J, d\,\nu)$:*

$$\tilde{f}(y) = f(g(y)) \sqrt{\frac{d\,(\mu \circ g^{-1})}{d\,\mu}(y)} \tag{28}$$

satisfies $\|\tilde{f}\|_{L^2(I,d\mu)} = \|f\|_{L^2(J,d\nu)}$, where $\frac{d\,(\mu \circ g^{-1})}{d\,\mu}$ is the Radon-Nikodym derivative.

Proof. When μ and ν are both Lebesgue measure and g is differentiable, the Radon-Nikodym derivative is $|g'(y)|$, reducing to Theorem 7.

For the general case, compute:

$$\begin{aligned} \|\tilde{f}\|^2_{L^2(I,d\mu)} &= \int_I |f(g(y))|^2 \frac{d\,(\mu \circ g^{-1})}{d\,\mu}(y)\; d\,\mu(y) && (29) \\ &= \int_I |f(g(y))|^2\; d\,(\mu \circ g^{-1})(y) && (30) \end{aligned}$$

By the definition of the pushforward measure $\mu \circ g^{-1}$ and since $\nu = \mu \circ g^{-1}$:

$$\int_I |f(g(y))|^2 d\,(\mu \circ g^{-1})(y) = \int_J |f(x)|^2\; d\,\nu(x) = \|f\|^2_{L^2(J,d\nu)} \tag{31}$$

The change of variables follows from the same argument using the definition of the push-forward measure. Therefore:

$$\|\tilde{f}\|_{L^2(I,d\mu)} = \|f\|_{L^2(J,d\nu)} \tag{32}$$ □

6 Conclusion

The results establish a comprehensive theory of unitary change-of-variables operators on L^2 spaces. The general framework shows that unitarity requires measurable bijections modulo null sets, mutual absolute continuity, and weight functions given by square roots of Radon-Nikodym derivatives. For L^2 norm preservation under measurable bijections, the scaling factor $\sqrt{g'}$ for Lebesgue measure generalizes to $\sqrt{\frac{d\,(\mu \circ g^{-1})}{d\,\mu}}$ for arbitrary σ-finite measures. These factors are both necessary and sufficient for isometry, linking the change-of-variables formula to unitary structure on L^2 spaces over arbitrary measure spaces.

Bibliography

[petersen1989ergodic] K. Petersen, *Ergodic Theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1989.

[halmos1956lectures] P. R. Halmos, *Lectures on Ergodic Theory*, Chelsea Publishing Company, 1956.

[walters1982introduction] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics, Springer-Verlag, 1982.

[reed1980functional] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, 1980.