Error Bounds for Asymptotic Expansions, with an Application to Cylinder Functions of Large Argument

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1 Introduction

In 1886, Poincaré [poincare1886] introduced the notion of an asymptotic expansion

$$f(x) \sim t_0 + \frac{t_1}{x} + \frac{t_2}{x^2} + \cdots \tag{1}$$

of an arbitrary function $f(x)$. According to his definition the coefficients t_s are independent of x, and

$$f(x) = \sum_{s=0}^{m-1} \frac{t_s}{x^s} + \epsilon_m(x) \tag{2}$$

where, for each m, $\epsilon_m(x) = o(1/x^{m-1})$ as $x \to \infty$. This concept admitted a new class of divergent series expansions to be useful in analysis, enabling them to be manipulated in much the same way as convergent power series. In turn, this has led to the development of a new calculus, later called "pure asymptotics" by van der Corput [?]. A feature of this development has been the generalization of the original definition of Poincaré. Schmidt [schmidt] showed that the restrictive assumption in Poincaré's definition is not necessary. More recently, Erdélyi [?] has extended the concept still further, and, with Wyman, applied it to the asymptotic evaluation of certain integrals [?]. Erdélyi's generalization is given in [?] below.

For some time, however, many numerical mathematicians have been aware that in quite another way, the Poincaré definition is not restrictive enough. To understand this point of view, consider for example the well-known asymptotic expansion for the Hankel function $H_\nu^{(1)}(z)$ for large $|z|$ and fixed ν, given by

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{i\phi} \sum_{s=0}^{\infty} \frac{i^s a_s}{z^s} \tag{3}$$

in which

$$\phi(z) = z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi \tag{4}$$

and

$$a_s = \frac{(\frac{1}{2} - \nu^2)(\frac{9}{4} - \nu^2)\cdots((\frac{s}{2})^2 - \nu^2)}{s!8^s} \tag{5}$$

This expansion holds in Poincaré's sense [?] when $-\pi < \arg z < 2\pi$; in fact for arbitrary values of a positive number δ, it is uniformly valid in the sector $-\pi + \delta \leq \arg z \leq 2\pi - \delta$ in the accepted sense that if the series is truncated after m terms, then the constant implied in the error term $O(|z|^{-m})$ can be assigned independently of $\arg z$.

Now it will be seen later [?] that when z is large and if $\arg z$ is not too close to the constant in the uniform error bound depends on z and becomes increasingly large as $\delta \to 0$. As a consequence, the concept of "uniform validity" can be quite misleading in applications: an unsuspecting computer evaluating the series (3) by the usual computational procedure of truncation at the smallest term may obtain inaccurate results in the sectors $-\pi < \arg z < -\pi + \delta$ and $2\pi - \delta < \arg z < 2\pi$, grossly so in the neighbourhoods of $\arg z = -\pi$ and 2π.

This situation, although perhaps not widely appreciated, is really a fairly surprising, because (3) is known to break down completely on crossing the boundaries $\arg z = -\pi$ and 2π. It is more natural [?] to expect this failure to be gradual than abrupt as the boundaries are approached. The behaviour of an ordinary Taylor-series expansion of an analytic function is somewhat similar, inasmuch as the circle of convergence is approached. The analogy is not complete, however. A computer is warned of the inaccuracy of a truncated Taylor series near the boundary of its region of validity by a diminution in the rate of convergence. No similar warning is available for an asymptotic expansion.

Some way of excluding the direct use of an asymptotic expansion near the boundaries of its region of validity is therefore desirable, and it is in this sense that the Poincaré definition is insufficiently restrictive enough. To understand this point of view, the restrictive definition should be modified. The specific problem of Hankel and Bessel functions of large arguments and derive some new results in connection with the expansion (3).

2 Complete asymptotic expansions

Practical dangers attending the use of asymptotic expansions have been stressed previously by Miller [?] [?]. In the expansions for the Weber functions given in these references, Miller distinguishes between regions of validity in the sense of Poincaré and the more restrictive "complete sense of Watson." Essentially, the difference is that in the former sense of contributions of an exponentially small character are neglected (as they may be, according to the definition), whereas in the latter sense they are retained if they have numerical significance.

For example, if z is not an odd integer, then the definition, whereas in the complete sense they are retained if they have numerical significance.

For example, if z is not an odd integer then according to the definition (2), the quadrant $\frac{1}{2}\pi < \arg z \leq \pi$; this is demonstrated in [?] below. To achieve complete validity in the quadrant $-\pi < \arg z \leq 2\pi$ it is necessary to add the series

$$(1 + e^{-2\nu i\pi})\sqrt{\frac{2}{\pi z}}\, e^{-i\phi} \sum_{s=0}^{\infty} (-i)^s \frac{a_s}{z^s} \tag{6}$$

to the right of (3), whereas of course in the Poincaré sense (6) is negligible compared with the right of (3) when $\frac{\pi}{2} < \arg z < 2\pi$. Similarly, to achieve complete validity in the quadrant $-\pi < \arg z < -\frac{\pi}{2}$, (6) is subtracted from the right of (3).

By introducing exponentially small contributions of this type the numerical difficulties can be overcome satisfactorily in many cases, and in other cases, the precise is difficult to justify mathematically, however, without an investigation of the remainder term of the given in [?] below. There is no readily applicable general definition of complete validity available, nor is it easy to find what "numerical significance" is too vague a criterion by itself. A drawback in practice of Watson's theory of the uniqueness of asymptotic expansions [?] is the need to assess properties of the remainder term which are not immediately available in many applications. Furthermore, in cases where these properties are known it quite possible that the statistic bound for the remainder term is also known, thereby obviating the need for the theory for numerical purposes. This is certainly true of the example given in Watson's paper [?].

The difficulty of recognizing when to include the numerical contribution of an exponentially small term in an asymptotic expansion is illustrated by the following example, which arose some years ago in computations at the National Physical Laboratory, Teddington.

Let

$$I(n) = \int_0^{\pi} \frac{\cos n\,t}{t^2+1}\,d\,t \tag{7}$$

By repeated integration by parts, one readily shows that for large positive integer values of n, $I(n)$ has the Poincaré expansion

$$I(n) \sim (-1)^{n-1}\left(\frac{\lambda_1}{n} - \frac{\lambda_2}{n^2} + \frac{\lambda_3}{n^3} - \cdots\right) \tag{8}$$

in which the coefficients λ_s are given by

$$\lambda_s = (n^2+1)^{-2s}\, p_{2s-1}(n) \tag{9}$$

the $p_s(t)$ being polynomials in t of degree s, defined recursively by $p_0(t) = 1$, and

$$p_s(t) = 2\,a\,t \cdot p_{s-1}(t) - (t^2+1)\,p'_{s-1}(t) \quad (s = 1, 2, \ldots). \tag{10}$$

Explicit expressions for the first six polynomials are

$$p_0(t) = 1 \quad p_1(t) = 2\,t \quad p_2(t) = 2\,(3\,t^2 - 1) \quad p_3(t) = 24\,(t^3 - t) \tag{11}$$

$$p_4(t) = 24\,(5\,t^4 - 10\,t^2 + 1) \quad p_5(t) = 240\,(3\,t^5 - 10\,t^3 + 3\,t) \tag{12}$$

and on numerical evaluation, one obtains to five decimals

$$\lambda_1 = 0.05318 \quad \lambda_2 = 0.04791 \quad \lambda_3 = 0.08985 \tag{13}$$

Thus for $n = 10$, the series (8) gives

$$I(10) \approx -(0.0005318 - 0.0000048 + 0.0000001 - \cdots) = -0.0005271. \tag{14}$$

This answer is quite incorrect however, because direct numerical quadrature of the expression (7) yields, to seven decimals,

$$I(10) = -0.0004558 \tag{15}$$

The inclusion of additional terms in the expansion would not help matters, and a partial explanation of the discrepancy is as follows. One may write

$$I(n) = \int_0^{\infty} \frac{\cos n\,t}{t^2+1}\,d\,t - \int_{\pi}^{\infty} \frac{\cos n\,t}{t^2+1}\,d\,t \tag{16}$$

The first of these integrals equals $\frac{\pi}{2}\,e^{-n}$; the second may again be expanded by repeated partial integration. In this way, one finds that

$$I(n) \sim \frac{1}{2}\,\pi\,e^{-n} + (-1)^{n-1}\left(\frac{\lambda_1}{n^2} - \frac{\lambda_2}{n^4} + \frac{\lambda_3}{n^6} - \cdots\right) \tag{17}$$

where the λ_s are the same as in (9). From this result one obtains the correct numerical value at $n = 10$, because

$$\frac{1}{2}\,\pi\,e^{-10} = 0.0000713 \tag{18}$$

which is exactly the discrepancy between the values (14) and (15).

An alternative way of deriving (17) is to apply the Residue theorem and Watson's lemma [?], page 236, to the contour integral

$$\int_C \frac{e^{int}}{t^2+1}\,d\,t \tag{19}$$

where C is the rectangle having vertices $\pm\pi$ and $\pm\pi + i$.

In the sense of Miller and Watson, (17) is apparently a complete asymptotic expansion for positive integer n, whereas (8) is incomplete. There is, however, no self-evident conclusive mathematical reason why this should be so, in fact (17) was obtained by less obvious procedures. It is even possible that (17) is itself incomplete, for without further investigation one does not know whether or not there have been neglected other exponential terms, for example e^{-2n}, which make numerically significant contributions for smaller values of n.

3 The need for error bounds

The difficulty illustrated by the examples of the previous sections is linked to a fundamental weakness of the Poincaré definition: it provides no direct answer to the question "What is the precise relation between an asymptotic expansion and the function from which it is derived?" Thus strictly speaking there is no connection between pure asymptotics and applied mathematics, except in the limit $|z| = \infty$. To establish a connection, two courses are open. Either one can seek upper bounds for the differences between the partial sums of an asymptotic expansion and the function from which it was obtained, or one can endeavour to transform the expansion into a convergent form, as, for example, in [reference17] and [reference19]. In the present paper the authors confine themselves to the former possibility.

Although the theory of pure asymptotics has been extensively developed and applied, the corresponding theory of error bounds has been comparatively neglected. The literature on this aspect consists mainly of scattered results applicable to special functions.

The few theorems of a general nature which have been discovered [reference3], [reference5], are concerned with asymptotic expansions of integral representations with real variables. (The example of (7), incidentally, is not covered by these theorems.) A possible reason for this neglect is the belief [reference1], [reference17], that when error bounds are needed they can be obtained merely by retracing the steps of the asymptotic proof. This is frequently a difficult and tedious undertaking, and the bounds it yields are often quite unrealistic. There is no readily applicable general theorem, and the writer's experience with expansions arising from differential equations indicates that it may often be necessary to develop entirely new proofs of the theorems of pure asymptotics before attempting to follow through explicit treatment of the error terms.

In this connection, attention may be drawn to the suggestion of Wyman [wyman] that the main direction in which the modern theory of asymptotics will move is towards the use of more general concepts in the theory of pure asymptotics. The importance of investigating such generalizations is indisputable, but perhaps there is a need to stress that the bridging of a gap between pure and applied mathematics in this branch of analysis by the development of satisfactory theories of error bounds is also of importance. Moreover, such theories may sometimes provide an alternative way of overcoming one of the difficulties which has helped stimulate the recent further generalizations [reference6] in the definition of an asymptotic expansion: the need to avoid narrow concepts concerning both the choice of asymptotic variable for a given expansion, and the nature of the uniformity of the expansion with respect to other variables.

This observation may be illustrated briefly by the following example. In [reference17], (?), Erdélyi and Wyman have published a generalized series expansion in terms of Airy functions for the Hankel function $H_\nu^{(1)}(x)$ when ν and x are real and positive, having a "scale"

$$\tau^{-1}\,(2\,m/3) \quad \text{as} \quad \tau \to \infty, \quad \text{where} \tag{20}$$

$$\tau = |\frac{1}{4}\,(\nu^2 - x^2)|^{1/2} + (\frac{1}{2}\,\nu)^{3/2} \tag{21}$$

This means that for each fixed integer m, the $(m+1)$th partial sum of the series differs from $H_\nu^{(1)}(x)$ by $o(\tau^{-1-(2m/3)})$ as $\tau \to \infty$. Thus these authors have succeeded in describing the behaviour of $H_\nu^{(1)}(x)$ when either x or ν is large by means of a single asymptotic expansion. Other investigators, using Poincaré expansions, have had to distinguish between the two cases. Recently however, the present writer [?] has derived sharp error bounds for the most powerful of the existing Poincaré-type expansions for $H_\nu^{(1)}(x)$ for large x, namely the uniform expansion in terms of Airy functions. From these bounds it can be seen that although the expansion was derived on the assumption that ν is large, it also has an asymptotic property for large x. Indeed, without going into detailed proof, it can be stated that the uniform expansion in terms of Airy functions for large x is also a generalized expansion, in the sense of Erdélyi, this time with respect to the scale $|(\nu\, t)^{-m(x-\nu)}|$ as $x \to \infty$, for any non-negative number δ. This is, in fact, a considerably more powerful scale than that of the new expansion.

(Notwithstanding the generally and greater power of the Airy function expansion, the Poincaré-type expansion of $H_\nu^{(1)}(x)$ for fixed ν and large x remains important, owing to its simplicity, and a further study of it is made in (?).)

4 Nature of the error bounds

In seeking bounds for the error term of the partial expansion of the form (2), what kind of success can be hoped for? If t_m is non-zero, then on replacing x by the complex variable z, the bound $|\epsilon_m(z)| \sim |t_m\, z^{-m}|$. Hence the most that can be established, in general, is that $|\epsilon_m(z)|$ is bounded by the modulus of the first (non-vanishing) neglected term of the series. This bound cannot apply when $|\arg t_{m+1}/(z\, t_m)| < \frac{1}{2}\pi$, however, for the modulus of the right side of the equation

$$\epsilon_m(z) = t_m\, z^{-m} + t_{m+1}\, z^{-m-1} + o(|z|^{-m-1}) \tag{22}$$

would exceed that of its first term for all sufficiently large $|z|$. In particular, this happens when t_m and t_{m+1} are real and of the same sign, and z is real and positive.

A modest error bound which would always be feasible is a multiple (>1) of the modulus of the first non-vanishing neglected term. This multiple itself could depend on z, and then ideally it would tend to unity as $|z| \to \infty$. The last condition is not essential from the standpoint of most applications however, and a bound of this kind if likely to be quite satisfactory with any value of $|z|$ not too large. Perhaps this can be appreciated best by observing that for a specified precision in (z), the difference between having $|p_m\, t_m\, z^{-m}|$ and $|t_m\, z^{-m}|$ as bounds for $\epsilon_m(z)$ only affects the minimum allowable value of $|z|$ by the factor $(p_m)^{1/m}$. Certainly, for example, if $p_m < 10$ there are few situations in which such a reduction in the region of applicability is likely to be of importance. From this point of view, the expenditure of heavy analytical effort to achieve a slight reduction in the value of p_m is unjustified, except possibly in the case of the dominant term ($m = 1$) of the expansion.

5 Hankel functions of large argument

The following theorem is obtained from Theorem 7 of [reference12] by taking the parameter u occurring there to be unity, and making minor changes:

Theorem 1. *Let $f(z)$ be regular in a simply-connected complex domain D, and a sequence of functions $A_s(z)$ be defined by $A_0(z)=1$ and*

$$A_{s+1}(z)=-\frac{1}{2}A_s'(z)+\frac{1}{4}\int f(z)\,A_s(z)\,d\,z \quad (s=0,1,\dots) \tag{23}$$

Then the differential equation

$$\frac{d^2\,w}{d\,z^2}=\{1+f(z)\}\,w \tag{24}$$

has a solution $w_m(z)$, depending on an arbitrary point a of D and an arbitrary positive integer m, such that

$$w_m(z)=e^z\left[\sum_{s=0}^{m-1}\frac{A_s(z)+\epsilon_m(z)}{z^s}\right] \tag{25}$$

and

$$w_m'(z)=e^z\left[\sum_{s=0}^{m-1}\{A_s(z)+A_s'(z)\}+\eta_m(z)\right] \tag{26}$$

where

A form of error bound which has emerged from recent investigations by the writer [reference12], [reference13] of the asymptotic solutions of certain second-order differential equations with respect to a parameter effectively consists of the variation, that is, the integral of the modulus of the derivative, of the first neglected term of the series taken over a suitable contour. More recent work, in preparation for publication, has shown that bounds of this type are also applicable, in certain cases, to asymptotic expansions of solutions for differential equations without a parameter $z\to\infty$ and (1.01), and from the observations made in (?) might expect that as z approaches the boundaries of the regions of validity, the contour of integration would naturally be subject to certain restrictions, and from the observations made in (?) we might expect that the bound it decreases in $\Re z$.

In the remaining part of this paper, it is shown that this variational form of error bound is applicable to the standard cylinder functions of large argument, and that it does indeed have the feature just described.

$$|\epsilon_m(z)|,|\eta_m(z)|\leq 2\exp\{2|u|\cdot(A_s)_P\}\cdot(A_s)_P\cdot\epsilon_m\cdot H(a) \tag{27}$$

In this result the symbol $V_{a,z}(A_s)$ denotes the variation of the function A_s over a path P connecting a and z, given by

$$V_{a,z}(A_s)=\int_P|A_s'(t)|\,d\,t \tag{28}$$

similarly for $V_{a,z}(A'_s)$. The region $H(a)$ is the subset of D comprising those points for which there exists a path P such that:

1. P lies entirely in D;
2. P consists of a finite number of Jordan arcs, each having a parametric equation of the form $t = t(\tau)$ with $t'(\tau)$ continuous and $t'(\tau)$ non-vanishing;
3. $\Re t$ is monotonic non-decreasing as t traverses P from a to z.

The point a, incidentally, may be the point at infinity on a straight line t lying in D; in this event one supposes that P coincides with t for all sufficiently large $|t|$.

The original purpose of this theorem was to provide asymptotic developments of solutions of (24), complete with error bounds, when $f(z)$ depends on a large parameter u. Suppose, however, that the parameter is absent, and

$$f(z) \sim k\, z^{-1-\sigma} \quad \text{as} \quad |z| \to \infty \tag{29}$$

where k, σ are constants and $\sigma > 0$. Then for large $|z|$, (25) and (26) are generalized asymptotic developments, complete with error bounds. For if the limits of integration on the right of (25) are taken to be $a = \infty$ and z, it readily follows by induction that $A_s(z) = O(|z|^{-s\sigma_1})$ $(s = 1, 2, \dots)$, where $\sigma_1 = \min(\sigma, 1)$. Moreover, from (28), with $a = \infty$, one derives $\epsilon_m(z), \eta_m(z) = O(|z|^{-m\sigma_1})$.

Thus in the sense of Erdélyi [?], equations (25) and (26) are generalized asymptotic expansions with respect to the scale $\{|z|^{-s\sigma_1}\}$ as $|z| \to \infty$.

The expansion (25) is generally less convenient than the usual Thomé asymptotic expansions in descending powers of z [reference5], [reference2], because of the need to evaluate the functions $A_s(z)$ for $s \geq 1$. The case of Bessel's differential equation is special, however, in that the two forms of expansion become the same with a suitable choice of $f(z)$.

Set

$$f(z) = (\nu^2 - \frac{1}{4}) / z^2 \tag{30}$$

where ν is a constant. The solution of equation (24) is then given by $w = z^{1/2}\, C_\nu(\pm i\, z)$, where C_ν denotes the general cylinder function of order ν. Applying the theorem with $a = -\infty$, and replacing z by $i\, z$, one constructs a solution

$$w_m = e^{iz} \left\{ \sum_{s=0}^{m-1} \frac{a_s}{z^s} + \epsilon_m \right\} \tag{31}$$

in which a_s is defined by (4) and

$$|\epsilon_m| \leq 2 \exp\{|\nu^2 - \frac{1}{4}|\, V_{i\infty,z}(\tau^{-m})\}\, V_{i\infty,z}\,(a_m\, \tau^{-m}) \tag{32}$$

The path of variation is subject to the condition that $\Im t$ is monotonic, and this restricts z to the region $-\pi < \arg z < 2\pi$.

Clearly

$$w_m = A\, z^{1/2}\, H_\nu^{(1)}(z) + B\, z^{1/2}\, H_\nu^{(2)}(z), \tag{33}$$

where A, B are independent of z. Letting $z \to i\,\infty$, one sees that $C=0$ and $e^{i\nu\pi} = -1$. Using the known asymptotic forms of the Hankel functions [reference18], one deduces that $B=0$ and $A = (\frac{1}{2}\, i)^{1/2}\, e^{i(\frac{1}{2}\nu + \frac{1}{4})\pi}$. Thus one derives the main result of this section:

$$H_\nu^{(1)}(z) = \left(\frac{2}{\pi\, z}\right)^{1/2} e^{i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} \left\{ \sum_{s=0}^{m-1} \frac{i^s\, a_s}{z^s} + \epsilon_m \right\} \tag{34}$$

when $-\pi < \arg z < 2\,\pi$, where ϵ_m is subject to (32). The bound is now proceed to an evaluation of this bound.

6 Evaluation of the variations

The problem discussed in this section is the choice of the path P connecting $i\,\infty$ and z to minimize the quantity

$$V_{i\infty,z}(\tau^{-m}) = m \int_P |\tau^{-m-1}|\, d\,t| \quad (m \geq 1) \tag{35}$$

One writes $\theta = \arg z - \frac{1}{2}\pi$, and considers in turn the cases $|\theta| \leq \frac{1}{2}\pi$, $\frac{1}{2}\pi < |\theta| \leq \pi$, $\pi < |\theta| < \frac{3}{2}\pi$.

1. $|\theta| \leq \frac{1}{2}\pi$. Consider the path which is indicated on Figure ? when θ is positive and is its image in the imaginary axis when θ is negative. It comprises part of the imaginary axis, a circular arc of radius R centred at the origin, where $R > |z|$ is arbitrary, and the straight line with parametric equation

$$t = z + \tau\, e^{i(\theta + \frac{1}{2}\pi)} \quad (0 \leq \tau \leq R - |z|) \tag{36}$$

As $R \to \infty$ the contributions to the variation from the imaginary axis and the circular arc both vanish, and one obtains

$$\lim_{R\to\infty} V_{i\infty,z}(\tau^{-m}) = \int_0^\infty \frac{m}{|z + \tau\, e^{i(\theta + \frac{1}{2}\pi)}|^{m+1}} d\,t = \int_0^\infty \frac{m}{|z|^m} d\,t \tag{37}$$

Since this actually equals the modulus of the difference between the values of τ^{-m} at the extremities of the path, no other path can yield a smaller variation.

2. $\frac{1}{2}\pi < |\theta| \leq \pi$. Consider the path indicated in Figure ?. Again, as the radius R of the circular arc tends to infinity the contributions from this arc and the imaginary axis both vanish, and one obtains

$$V_{i\infty,z}(\tau^{-m}) = \int_0^\infty \frac{m\,dt}{|z-\tau|^{m+1}} = \int_0^\infty \frac{m\,dt}{(\tau+|z|)^2+y^2}\cdot\frac{1}{|\tau|^{m+1}} \tag{38}$$

where x and y denote the real and imaginary parts of z, respectively.

That this path minimizes the variation can be seen as follows. Let one travel a prescribed distance τ along any admissible path from z, arriving at t_0 say. For the path of Figure ? one has $t_0 = z - \tau$, and for any other path t lies within or on the circle centred at z and passing through t_0; see Figure ?. Clearly $|t| > |t_0|$ only if t lies within the shaded lune bounded by this circle and the circular arc centred at the origin and passing through t_0. No path can be admitted to this lune however, because $\Im t < \Im z$ in its interior. Hence $|t| \leq |t_0|$, which leads to the stated result of this paragraph.

The integral (38) can be evaluated in terms of elementary functions for all integer values of m; for example

$$V_{i\infty,z}(\tau^{-1}) = \left|\frac{1}{2} - \tan^{-1}\frac{|z|}{y}\right| \quad (y \neq 0), \tag{39}$$

$$V_{i\infty,z}(\tau^{-1}) = \frac{1}{|x|} \quad (y = 0). \tag{40}$$

To avoid unnecessary complication however, the bound is established by the slightly weaker form

$$V_{i\infty,z}(\tau^{-m}) \leq \int_0^\infty \frac{m\,d\tau}{(\tau^2+x^2+y^2)^{1/2m+1/2}} = \frac{\chi(m)}{|z|^m}, \tag{41}$$

in which

$$\chi(m) = \sqrt{\pi}\,\frac{\Gamma\left(\frac{1}{2}m+1\right)}{\Gamma\left(\frac{1}{2}m+\frac{1}{2}\right)}. \tag{42}$$

This bound is in fact attained when $x = 0$.

3. $\pi < |\theta| < \frac{3}{2}\pi$. The minimizing path is indicated on Figure ?. To prove this assertion, let any other path intersect the negative imaginary axis at the point $i\,\mu$. If $\kappa \neq y$ the result follows immediately from (ii), hence one supposes that $\kappa = y > 0$. On travelling a distance τ from $i\,\mu$ towards z one arrives at a point t somewhere within or on the smaller circle of Figure ?, whereas on the minimizing path one arrives at $t_0 = i\,\mu + \tau$. Again one has $|t| \leq |t_0|$ except within the inadmissible lune.

   On letting $R \to \infty$, one obtains

$$V_{i\infty,z}(\tau^{-m}) = \int_y^\infty \frac{m\,d\tau}{|m+1|} = \int_0^\infty \frac{m\,d\tau}{|i\,y+\tau|^{m+1}} + \int_0^\infty \frac{m\,d\tau}{|i\,y+\tau|^{m+1}} \tag{43}$$

The last of these integrals equals $\chi(m)|y|^{-m}$ and the one before it is bounded by this quantity. Therefore

$$V_{i\infty,z}(\tau^{-m}) < 2\,\chi(m)\,|\Im z|^{-m} \tag{44}$$

One observes that as $\arg z$ approaches either of its extreme values $-\pi$ and $2\,\pi$, $V_{i\infty,z}(\tau^{-m})$ becomes increasingly large, as anticipated in §§1 and 4 above.

7 Collected results for cylinder functions

On combining the analysis of the last two sections and extending it by means of (26) to the derivative $\frac{d}{dz} H_\nu^{(1)}(z)$, one has the following results, in which ν is an unrestricted real or complex number.

$$H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{i\phi} \left\{ \sum_{s=0}^{m-1} \frac{i^s\, a_s}{z^s} + \epsilon_m \right\} \tag{45}$$

$$\frac{d}{dz} H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} i\, e^{i\phi} \left\{ \sum_{s=0}^{m-1} i^s \frac{b_s}{z^s} + i^m \frac{(b_m - a_m)}{z^m} + \eta_m + \frac{1}{2\,z}\,\epsilon_m \right\} \tag{46}$$

where $\phi = z - \frac{1}{2}\,\nu\,\pi - \frac{1}{4}\,\pi$

$$a_s = \frac{(\frac{1}{4}\,\nu^2 - \frac{1}{4})\,(\frac{1}{4}\,\nu^2 - \frac{9}{4}) \cdots (\frac{1}{4}\,\nu^2 - (2\,s-1)^2)}{8^s\, s!} \qquad b_s = \frac{4\,\nu^2 + 4\,s^2 - 1}{8}\, a_s \tag{47}$$

$$|\epsilon_m|, |\eta_m| \le 2|a_m| \exp\{|\nu^2 - \frac{1}{4}|\, V_{i\infty,z}(\tau^{-1})\}\, V_{i\infty,z}(\tau^{-m}) \tag{48}$$

and

$$V_{i\infty,z}(\tau^{-m}) \le \begin{cases} |z|^{-m} & (0 \le \arg z \le \pi) \\ \chi(m)|z|^{-m} & (-\frac{1}{2}\,\pi < \arg z < 0 \text{ or } \pi < \arg z \le \frac{3}{2}\,\pi) \\ 2\,\chi(m)\,|\Im z|^{-m} & (-\pi < \arg z < -\frac{1}{2}\,\pi \text{ or } \frac{3}{2}\,\pi < \arg z < 2\,\pi) \end{cases} \tag{49}$$

In (45) and (46) the branch of $z^{1/2}$ is $\exp(\frac{1}{2}\ln|z| + \frac{1}{2}\,i \arg z)$, and in (49) $\chi(m)$ is defined by (42). The values of the first ten $\chi(m)$ to two decimal places are as follows:

m	$\chi(m)$	m	$\chi(m)$
1	1.57	6	3.20
2	2.00	7	3.44
3	2.36	8	3.66
4	2.67	9	3.87
5	2.95	10	4.06

For large m, $\chi(m) \sim \sqrt{\frac{1}{2} m\,\pi}$.

The corresponding results for $H_\nu^{(2)}(z)$ and its derivative are obtained by changing the sign of i throughout, and replacing the z-exponential by its respective conjugates.

In applying these results, the bounds for the quantity $V_{i\infty,z}(\tau^{-1})$ appearing in (48) are obtained by setting $m=1$ and $\chi(1)=\frac{\pi}{2}$ in (49). It should be observed that for all values of z in the region $-\frac{\pi}{2} \leq \arg z \leq \frac{3\,\pi}{2}$ for which the expansions (45) and (46) are computationally useful, the factor $\exp\{|\nu^2-\frac{1}{4}|\,V_{i\infty,z}(\tau^{-1})\}$ is approximately unity, because a necessary condition that $|a_{s+1}^m|$ be small compared with the leading term 1 of each series is that $|z|>|\nu^2-\frac{1}{4}|$.

For other ranges of $\arg z$ use may be made of the connection formula [reference18], §3.62,

$$H_\nu^{(1)}(z\,e^{n\pi i}) = \sin(1-n)\,\nu\pi\,\cos e\,c\,\nu\pi\,H_\nu^{(1)}(z) + \sin n\,\pi\,\cos e\,c\,\nu\,\pi\,H_\nu^{(1)}(z\,e^{n\pi}), \tag{50}$$

in which n is a positive or negative integer. In the application of this formula, $\arg z$ can always be taken in the range $(-\frac{\pi}{2}, \frac{3\,\pi}{2})$, which means that the use of (45) and (46) can be confined to the sector $-\frac{\pi}{2} \leq \arg z \leq \frac{3\,\pi}{2}$. Thus the direct use of (45) and (46) in the sectors $-\pi < \arg z < -\frac{\pi}{2}$ and $\frac{3\,\pi}{2} < \arg z < 2\,\pi$ is fraught with the danger of a large error term; it can be avoided altogether. In effect, the more accurate connection-formula procedure improves the accuracy of (45) and (46) in these two sectors by adding in appropriate contributions of an exponentially small nature, and accordingly provides a rigorous justification, in the present example, of Miller's use of complete asymptotic expansions [reference2].

Corresponding results for the functions $J_\nu(z)$ and $Y_\nu(z)$ are immediately deducible from those for the Hankel functions by means of the relations

$$J_\nu(z) = \frac{1}{2}\,(H_\nu^{(1)}(z) + H_\nu^{(2)}(z)), \quad Y_\nu(z) = \frac{1}{2\,i}\,(H_\nu^{(2)}(z) - H_\nu^{(1)}(z)). \tag{51}$$

One finds that

$$J_\nu(z) = \left(\frac{2}{\pi\,z}\right)^{1/2} \left\{\cos\phi \sum_{s=0}^{m-1} (-1)^s \frac{a_{2s}}{z^{2s}} - \sin\phi \sum_{s=0}^{m-1} (-1)^s \frac{a_{2s+1}}{z^{2s+1}} + \epsilon_{2m}\right\} \tag{52}$$

$$= \left(\frac{2}{\pi\,z}\right)^{1/2} \left\{\cos\phi \sum_{s=0}^{m} (-1)^s \frac{a_{2s}}{z^{2s}} - \sin\phi \sum_{s=0}^{m-1} (-1)^s \frac{a_{2s+1}}{z^{2s+1}} + \epsilon_{2m+1}\right\}, \tag{53}$$

and

$$Y_\nu(z) = \left(\frac{2}{\pi\,z}\right)^{1/2} \left\{\sin\phi \sum_{s=0}^{m-1} (-1)^s \frac{a_{2s}}{z^{2s}} + \cos\phi \sum_{s=0}^{m-1} (-1)^s \frac{a_{2s+1}}{z^{2s+1}} + \beta_{2m}\right\} \tag{54}$$

$$= \left(\frac{2}{\pi\,z}\right)^{1/2} \left\{\sin\phi \sum_{s=0}^{m} (-1)^s \frac{a_{2s}}{z^{2s}} + \cos\phi \sum_{s=0}^{m-1} (-1)^s \frac{a_{2s+1}}{z^{2s+1}} + \beta_{2m+1}\right\}, \tag{55}$$

where

$$|\alpha_m|, |\beta_m| \leq (|e^{\nu\pi i/2}| + |e^{-\nu\pi i/2}|) \exp\left\{\frac{|\nu^2 - \frac{1}{4}|}{2} \frac{|a_m|}{z^m}\right\} \quad (\arg z = 0), \tag{56}$$

$$|\alpha_m|, |\beta_m| \leq (|e^{\pi|\nu|/2}| + \chi(m)|e^{\pi|\nu|/2}|) \exp\left\{\frac{\pi\,|\nu^2 - \frac{1}{4}|}{2|z|}\right\}\frac{|a_m|}{|z|^m} \quad (0 < |\arg z| \leq \frac{1}{2}\pi), \tag{57}$$

$$|\alpha_m|, |\beta_m| \leq (|e^{\pi|\nu|}| + 2\,\chi(m)|e^{\pi|\nu|}|) \exp\left\{\frac{\pi\,|\nu^2 - \frac{1}{4}|}{|\Im z|}\right\}\frac{|a_m|}{|\Re z|^m} \quad (\frac{1}{2}\pi < |\arg z| < \pi). \tag{58}$$

The upper or lower signs are taken in (57) and (58), according as $\arg z$ is positive or negative. Again, one sees that to achieve maximum accuracy the use of (52) and (54) should be confined to the half-plane $|\arg z| \leq \frac{1}{2}\pi$ and connection formulae used elsewhere.

Next, the modified Bessel functions are considered:

$$K_\nu(z) = \frac{\pi}{2}\frac{i^{\nu+1}\,H_\nu^{(1)}(i\,z)}{i^{1/2}}, \quad I_\nu(z) = \frac{1}{2}\,i^{-\nu-1}\,(H_\nu^{(1)}(i\,z) + H_\nu^{(2)}(i\,z)). \tag{59}$$

For the former, one derives immediately from (45)

$$K_\nu(z) = \left(\frac{\pi}{2\,z}\right)^{1/2} e^{-z}\left\{\sum_{s=0}^{m-1}\frac{a_s}{z^s} + \gamma_m\right\}, \tag{60}$$

where

$$|\gamma_m| \leq \begin{cases} 2\exp\left\{-\frac{|\nu^2 - \frac{1}{4}|}{|z|}\right\}\frac{|a_m|}{|z|^m} & (|\arg z| \leq \frac{3}{2}\pi), \\ 2\,\chi(m)\exp\left\{\frac{\pi\,|\nu^2 - \frac{1}{4}|}{2|z|}\right\}\frac{|a_m|}{|z|^m} & (\frac{1}{2}\pi < |\arg z| \leq \pi), \\ 4\,\chi(m)\exp\left\{\frac{\pi\,|\nu^2 - \frac{1}{4}|}{|\Re z|}\right\}\frac{|a_m|}{|\Re z|^m} & (\pi < |\arg z| < \frac{3}{2}\pi). \end{cases} \tag{61}$$

For the latter, one finds that

$$I_\nu(z) = \frac{e^z}{(2\,\pi\,z)^{1/2}}\left\{\sum_{s=0}^{m-1}(-1)^s\frac{a_s}{z^s} + \delta_m\right\} - i\,e^{-\nu\pi i}\,\frac{e^{-z}}{(2\,\pi\,z)^{1/2}}\left\{\sum_{s=0}^{m-1}\frac{a_s}{z^s} + \gamma_m\right\} \quad (-\frac{3}{2}\pi < \arg z < \frac{1}{2}\pi). \tag{62}$$

Here γ_m is the same as in (60) and is therefore bounded by (61); δ_m also is subject to (61) except that the applicable regions are changed to

$-\frac{3}{2}\pi \leq \arg z \leq -\frac{1}{2}\pi,\ -\frac{1}{2}\pi < \arg z \leq 0,\ 0 < \arg z < \frac{1}{2}\pi,$

respectively.

Again, the use of (60) and (62) should be confined to the regions $|\arg z| \leq \pi$ and $-\pi \leq \arg z \leq 0$, respectively, and connection formulae used elsewhere. In particular, by using the relation $I_\nu(z) = e^{-\nu\pi i}\, I_\nu\,(z\, e^{\pi i})$ one deduces from (62) its conjugate form applicable to the region $0 \leq \arg z \leq \pi$.

Finally, one observes that by setting $\nu = \frac{1}{2}$ and $\frac{3}{2}$ one may deduce error bounds for the asymptotic expansions of the Airy functions and their derivatives, but these shall not be recorded here.

The above bounds are by no means the first which have been given for the remainder terms in the Hankel expansions of Watson [?], pages 205–220, describes in detail researches of Hankel, Stieltjes and himself for real ν and positive z, and of Weber and Schläfli for complex ν and z. Subsequently Schläfli's results have been extended by Döring [?] and Meijer [?]. Quite recently, Döring [?] has critically examined the bounds of Schläfli and Meijer in the case of real ν and complex z, and made simplifications to make them more readily computable.

Except for Weber each author derives his results from integral representations of the Hankel functions. Weber uses a defining differential equation in a way which bears some resemblance to the theorem of §5 above, but it is more complicated.

It is not claimed that the present bounds are superior to previous results in every respect. Indeed, for certain combinations of ν and z, particularly real ν, some of the earlier results are sharper. However, although it must be added that in regions where the expansions are meaningfully the sharpening seldom exceeds a factor of 2 (compare the remarks made in the second paragraph of §4), one does claim, however, that for the general combination of complex values of ν and z, the present results are considerably simpler than the aggregate of earlier results, and furthermore they are completely realistic for all combinations of the variables. Of the earlier results, the most complete are those of Meijer. They are more complicated than (45) to (49), involving the solution of a transcendental equation in some regions. Moreover, they break down for complex values of ν arbitrarily close to, though not lying on, the lines $\Re\,\nu = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots$.

8 Summary

In the first part of this paper the authors discussed, in general terms, practical difficulties in the use of asymptotic expansions, particularly in the vicinity of the boundaries of their regions of validity in the complex plane. It was pointed out that the use of "complete asymptotic expansions" in the sense of Watson and Miller, though often expedient in practice, is difficult to place on a firm mathematical foundation. It was indicated, however, that the practical difficulties could be overcome by the development of satisfactory theories of error bounds. It was also suggested that such theories might provide an alternative or supplementary mathematical tool to generalizations of Poincaré's definition.

In the second part of the paper new error bounds were derived for the well-known Hankel asymptotic expansions for cylinder functions of large complex argument and given real or complex order. These bounds were obtained by application of the asymptotic theory of ordinary differential equations. A characteristic feature of their evaluation was the minimization of the variation of the first neglected term of the series over a prescribed type of contour in the complex plane; the bounds appear to be the first ones for the Hankel expansions which are completely satisfactory for all combinations of the variables. They are well adapted to the control of accuracy in the construction of general-purpose automatic computing routines for the cylinder functions.

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The author acknowledges helpful criticisms of this paper by Mr. G. F. Miller and Dr. J. C. P. Miller.