Proposition 1

Let $K\colon T\times T\to\mathbb{C}$ be a covariance function such that the associated RKHS $\mathcal{H}_K$ is separable where $T\subset\mathbb{R}$. Then there exists a family of vector functions

$$\Psi(t,x)=(\psi_n(t,x),n\geq 1)\forall t\in T \tag{1}$$

and a Borel measure μ on T such that $\psi_n(t,x)\in L^2(T,\mu)$ in terms of which K is representable as:

$$K(s,t)=\int_T\sum_{n=1}^{\infty}\psi_n(s,x)\overline{\psi_n(t,x)}d\,\mu(x) \tag{2}$$

The vector functions $\Psi(s,.),s\in T$ and the measure μ may not be unique, but all such $(\Psi,.),.)$ determine K and its reproducing kernel Hilbert space (RKHS) H_K uniquely and the cardinality of the components determining K remains the same. [1,]

Remark 2. 1. If $\Psi(t,.)$ is a scalar, then we have

$$K(s,t)=\int_T\Psi(s,x)\overline{\Psi(t,x)}d\,\mu(x) \tag{3}$$

which includes the tri-diagonal triangular covariance with μ absolutely continuous relative to the Lebesgue measure.

2. The following notational simplification of (25) can be made. Let $n=R\times Z_+=S\otimes P$, where P is the power set of integers Z, and let P = u @ o where o is the counting measure. Then

$$\Psi(t,n)=(\psi_n(t,x),n\in Z) \tag{4}$$

Hence

$$|\Psi^*(t)|_{L^2}^2=\int_T|\psi_n(t,x)|^2\,d\,\mu(x) \tag{5}$$

This content is adapted from MM Rao's book, *Stochastic Processes: Inference Theory*. Propisition 8

Bibliography

[1] Malempati M. Rao. *Stochastic Processes: Inference Theory*. Springer Monographs in Mathematics. Springer, 2nd edition, 2014.