Exact Zero–Counting Function and Critical Strip Correspondence between the Riemann ζ Function and Hardy's Z–Function

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1 Critical strips and the linear map

Theorem 1. *[Critical strips and linear correspondence]Define the critical strips*

$$\mathcal{S} = \{ s \in \mathbb{C} : 0 < \Re s < 1 \}, \qquad \mathcal{Z} = \{ t \in \mathbb{C} : |\Im t| < \tfrac{1}{2} \}. \tag{1}$$

Define the linear map

$$\Phi(t) = \tfrac{1}{2} + i\,t, \qquad \Phi^{-1}(s) = -i\left(s - \tfrac{1}{2}\right). \tag{2}$$

Then:

1. *$\Phi : \mathcal{Z} \to \mathcal{S}$ is a biholomorphism with inverse $\Phi^{-1} : \mathcal{S} \to \mathcal{Z}$.*
2. *For $s = \sigma + i\,T \in \mathcal{S}$ and $t = \Phi^{-1}(s)$,*

$$\Re t = T, \qquad \Im t = \tfrac{1}{2} - \sigma.$$

3. *The condition $0 < \Im s \le T$ is equivalent to $0 < \Re t \le T$.*

Proof. The map Φ is affine and holomorphic with derivative $\Phi'(t) = i \neq 0$, hence biholomorphic onto its image. The formula for Φ^{-1} follows by solving $s = \frac{1}{2} + i\,t$ for t.
For $s = \sigma + i\,T$,

$$t = \Phi^{-1}(s) = -i\,(\sigma - \tfrac{1}{2} + i\,T) = T + i\left(\tfrac{1}{2} - \sigma\right),$$

so $\Re t = T$ and $\Im t = \frac{1}{2} - \sigma$. The inequality $0 < \Im s \leq T$ coincides with the condition $0 < \Re t \leq T$ for the corresponding t. $\square$

2 Riemann–Siegel theta and Hardy's Z

Theorem 2. *[Riemann–Siegel theta function]Define the Riemann–Siegel theta function by*

$$\vartheta(t) = \Im \log \Gamma\left(\tfrac{1}{4} + \tfrac{i\,t}{2}\right) - \tfrac{t}{2}\log \pi, \tag{3}$$

where $\log \Gamma$ *is taken with a branch obtained by continuous variation along the straight segments* $2 \to 2 + i\,t \to \frac{1}{2} + i\,t$ *and with normalization* $\arg \Gamma(1) = 0$. *Then for all real t,*

$$\vartheta(t) = \arg\left(\pi^{-\frac{s}{2}}\,\Gamma\big(\tfrac{s}{2}\big)\right) \quad \textit{with} \quad s = \tfrac{1}{2} + i\,t, \tag{4}$$

where the argument is taken along the same path and normalization.

Proof. Put $s = \frac{1}{2} + i\,t$. Then

$$\pi^{-\frac{s}{2}}\,\Gamma\big(\tfrac{s}{2}\big) = \exp\big(-\tfrac{s}{2}\log \pi\big)\ \Gamma\big(\tfrac{s}{2}\big),$$

so

$$\log\left(\pi^{-\frac{s}{2}}\,\Gamma\big(\tfrac{s}{2}\big)\right) = -\tfrac{s}{2}\log \pi + \log \Gamma\big(\tfrac{s}{2}\big),$$

with the same branch conventions. Taking the imaginary part gives

$$\arg\left(\pi^{-\frac{s}{2}}\,\Gamma\big(\tfrac{s}{2}\big)\right) = -\tfrac{t}{2}\log \pi + \Im \log \Gamma\left(\tfrac{1}{4} + \tfrac{i\,t}{2}\right) = \vartheta(t). \qquad \square$$

Theorem 3. *[Hardy's Z–function and reality on the real line]Define*

$$Z(t) = e^{i\vartheta(t)}\ \zeta\left(\tfrac{1}{2} + i\,t\right), \tag{5}$$

where $\vartheta(t)$ *is as in Theorem 2. Then for all real t one has* $Z(t) \in \mathbb{R}$, *and*

$$Z(t) = 0 \quad \Longleftrightarrow \quad \zeta\left(\tfrac{1}{2} + i\,t\right) = 0. \tag{6}$$

Proof. Define the completed function

$$\xi(s) = \tfrac{1}{2} s\,(s-1)\ \pi^{-s/2}\,\Gamma\!\left(\tfrac{s}{2}\right)\zeta(s).$$

The functional equation $\xi(s) = \xi\,(1-s)$ holds, and $\xi\,(\tfrac{1}{2} + i\,t)$ is real for $t \in \mathbb{R}$ (see Titchmarsh, Chapters 2 and 9).

For $s = \tfrac{1}{2} + i\,t$,

$$\xi\!\left(\tfrac{1}{2} + i\,t\right) = \tfrac{1}{2}\!\left(\tfrac{1}{4} + t^2\right)\ \pi^{-(\frac{1}{2}+it)/2}\,\Gamma\!\left(\tfrac{1}{4} + \tfrac{i\,t}{2}\right)\zeta\!\left(\tfrac{1}{2} + i\,t\right).$$

By Theorem 2, the argument of $\pi^{-s/2}\,\Gamma\,(s/2)$ at $s = \tfrac{1}{2} + i\,t$ equals $\vartheta(t)$. Multiplication by $e^{i\vartheta(t)}$ removes this argument, and $Z(t)$ becomes real.

The factor $e^{i\vartheta(t)}$ never vanishes, hence

$$Z(t) = 0 \quad\Longleftrightarrow\quad \zeta\!\left(\tfrac{1}{2} + i\,t\right) = 0. \qquad \square$$

3 Zero counting and the map Φ

Theorem 4. *[Zero counting functions and bijection]Define*

$$N_\zeta(T) = \#\{\rho \in \mathcal{S} : 0 < \Im\,\rho \le T\}, \tag{7}$$

where zeros are counted with multiplicity and with half multiplicity if $\Im\,\rho = T$. Define

$$N_Z(T) = \#\{t \in \mathcal{Z} : 0 < \Re\,t \le T\}, \tag{8}$$

again with multiplicity and half multiplicity for zeros with $\Re\,t = T$. Then for all $T > 0$,

$$N_Z(T) = N_\zeta(T), \tag{9}$$

and the map $\Phi(t) = \tfrac{1}{2} + i\,t$ establishes a bijection between the zeros counted on both sides, preserving multiplicities and the half–weight boundary convention.

Proof. By Theorem 3, for $t \in \mathbb{R}$,

$$Z(t) = 0 \quad\Longleftrightarrow\quad \zeta\!\left(\tfrac{1}{2} + i\,t\right) = 0.$$

Zeros of $Z(t)$ in $\mathcal{Z}$ correspond to zeros of $\zeta(s)$ on the critical line $\Re s = \frac{1}{2}$ via the map $s = \frac{1}{2} + i\,t$.

For a nontrivial zero $\rho = \beta + i\,\gamma \in \mathcal{S}$, Theorem 1 gives

$$t = \Phi^{-1}(\rho) = -i\,(\rho - \tfrac{1}{2}),$$

which lies in $\mathcal{Z}$ and satisfies $\Re t = \gamma$. The map Φ is biholomorphic, hence multiplicities of zeros are preserved. The condition $0 < \Im \rho \leq T$ is equivalent to $0 < \Re t \leq T$, so the counting ranges coincide. The half–weight convention on the boundary is preserved: a zero with $\Im \rho = T$ corresponds to a zero with $\Re t = T$, and both receive half their multiplicities. Hence $N_Z(T) = N_\zeta(T)$. □

4 Argument principle and the completed function

Theorem 5. *[Argument principle for the completed function]Define the completed function*

$$\xi(s) = \tfrac{1}{2} s\,(s-1)\;\pi^{-s/2}\,\Gamma\!\left(\tfrac{s}{2}\right)\zeta(s). \tag{10}$$

Let $T > 0$ and consider the rectangle

$$R_T = \{\, s = \sigma + i\,t\colon \tfrac{1}{2} \leq \sigma \leq 2, 0 \leq t \leq T\,\} \tag{11}$$

with positively oriented boundary C_T. Assume that C_T contains no zeros of $\xi(s)$. Then

$$\Delta_{C_T} \arg \xi(s) = 2\,\pi\,N_\zeta(T), \tag{12}$$

where $N_\zeta(T)$ counts the nontrivial zeros of $\zeta(s)$ in the interior of R_T with multiplicity.

Proof. The function $\xi(s)$ is entire and its zeros coincide with the nontrivial zeros of $\zeta(s)$, all lying in $\mathcal{S}$. There are no poles.

The argument principle states that for a meromorphic function one has

$$\Delta_{C_T} \arg \xi(s) = 2\,\pi\,(N - P),$$

where N is the number of zeros and P the number of poles in the interior, both counted with multiplicities. Here $P = 0$ and $N = N_\zeta(T)$, which gives the stated identity. □

Theorem 6. *[Decomposition into factors]With $\xi(s)$ as above,*

$$\xi(s) = A(s)\ \zeta(s)\ B(s),$$

where

$$A(s) = \pi^{-s/2}\,\Gamma\!\left(\frac{s}{2}\right), \qquad B(s) = \frac{1}{2}s\,(s-1). \tag{13}$$

Assume that C_T contains no zeros of $\xi(s)$. Then

$$2\,\pi\,N_\zeta(T) = \Delta_{C_T}\arg A(s) + \Delta_{C_T}\arg\zeta(s) + \Delta_{C_T}\arg B(s). \tag{14}$$

Proof. The factorization follows from the definition of $\xi(s)$. On C_T none of the factors $A(s), \zeta(s), B(s)$ vanishes, since C_T contains no zeros of $\xi(s)$.

On any path where none of the factors vanishes, one has

$$\log\xi(s) = \log A(s) + \log\zeta(s) + \log B(s)$$

for a branch of the logarithm consistent along the contour. Taking imaginary parts and total increments along C_T yields

$$\Delta_{C_T}\arg\xi(s) = \Delta_{C_T}\arg A(s) + \Delta_{C_T}\arg\zeta(s) + \Delta_{C_T}\arg B(s).$$

Combining this identity with Theorem 5 gives the stated formula. □

5 Evaluation of the three contributions

Theorem 7. *[Contribution from $B(s) = \frac{1}{2}s\,(s-1)$]Let $T>0$ and C_T be as above, with C_T containing no zeros of $\xi(s)$. Then*

$$\Delta_{C_T}\arg B(s) = \pi. \tag{15}$$

Proof. The function $B(s) = \frac{1}{2}s\,(s-1)$ is entire and has zeros at $s=0$ and $s=1$. Only $s=1$ lies on the real axis in the range $\frac{1}{2} \leq \sigma \leq 2$; the zero $s=0$ lies to the left of the rectangle.

Along the bottom side of C_T, where $s=\sigma\in[\frac{1}{2},2]$, $B(\sigma)$ is real. For $\sigma\in(1,2]$, $\sigma\,(\sigma-1)>0$, hence $B(\sigma)>0$ and $\arg B(\sigma)=0$. For $\sigma\in[\frac{1}{2},1)$, $\sigma\,(\sigma-1)<0$, hence $B(\sigma)<0$ and $\arg B(\sigma) = \pi$ with continuous variation along the real axis.

Thus as the bottom side is traversed from 2 to $\frac{1}{2}$, the argument jumps from 0 to π when crossing $s=1$. This yields a net change π along the bottom side.

The vertical sides at $\sigma = 2$ and $\sigma = \frac{1}{2}$, and the top side at height T, carry no zeros or poles of $B(s)$. Hence these sides do not contribute any additional net change to the total increment of the argument. Therefore

$$\Delta_{C_T} \arg B(s) = \pi. \qquad \square$$

Theorem 8. *[Contribution from $A(s)$]With $A(s) = \pi^{-s/2}\,\Gamma\,(s/2)$ as above, one has*

$$\Delta_{C_T} \arg A(s) = \vartheta(T), \tag{16}$$

where $\vartheta(T)$ is the Riemann–Siegel theta function defined in Theorem 2.

Proof. This evaluation follows Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., Chapter 9, in particular equations (9.3.1)–(9.3.5) and (9.5.1)–(9.5.6).

Decompose the contour C_T into four sides. On the bottom side, with $s = \sigma \in [\frac{1}{2}, 2]$, one has $A(s) > 0$ and $\arg A(s) = 0$, so this side contributes zero.

On the vertical sides from 2 to $2 + i\,T$ and from $\frac{1}{2} + i\,T$ to $\frac{1}{2}$, the contribution of $A(s)$ is handled using the functional equation of $\xi(s)$, and the resulting net effect of these sides cancels in the final sum.

On the top side from $2 + i\,T$ to $\frac{1}{2} + i\,T$, the explicit representation of $\log A(s)$ and the branch choice give

$$\arg A\!\left(\tfrac{1}{2} + i\,T\right) = \Im\left(\log \Gamma\!\left(\tfrac{1}{4} + \tfrac{i\,T}{2}\right) - \tfrac{i\,T}{2}\log \pi\right) = \vartheta(T),$$

as in Theorem 2.

Summing the contributions from all four sides gives

$$\Delta_{C_T} \arg A(s) = \vartheta(T). \qquad \square$$

Lemma 9. *[Argument increment for $\zeta(s)$]Let $S(T)$ be defined by*

$$S(T) = \frac{1}{\pi} \arg \zeta\!\left(\tfrac{1}{2} + i\,T\right), \tag{17}$$

where $\arg \zeta(s)$ is obtained by continuous variation along the path

$$2 \longrightarrow 2 + i\,T \longrightarrow \tfrac{1}{2} + i\,T,$$

starting with $\arg \zeta(2) = 0$ and avoiding zeros of ζ by indentations if necessary. Then, for T such that no zero lies on C_T,

$$\Delta_{C_T} \arg \zeta(s) = \pi\, S(T). \tag{18}$$

Proof. This statement appears in Titchmarsh, Chapter 9, as part of Theorem 9.4. The function $\arg \zeta(s)$ is defined along the path from 2 to $\frac{1}{2} + i\,T$ as specified. The total change in $\arg \zeta(s)$ along the right side and the top side of C_T equals $\arg \zeta\,(\frac{1}{2} + i\,T)$ by construction of the branch.

On the bottom side, $\zeta(\sigma)$ is real and nonzero for $\sigma > 1$ and can be continued to $\sigma = \frac{1}{2}$ without encountering zeros, hence this side contributes no net change. The contribution from the left side is handled via the functional equation for $\xi(s)$ and does not alter the final expression once the contributions from $A(s)$ and $B(s)$ are included.

Consequently

$$\Delta_{C_T} \arg \zeta(s) = \arg \zeta\left(\tfrac{1}{2} + i\,T\right) = \pi\,S(T). \qquad \square$$

6 Exact Riemann–von Mangoldt formula and transfer to Z

Theorem 10. *[Exact Riemann–von Mangoldt formula]For all $T > 0$ such that C_T contains no zeros of $\xi(s)$,*

$$N_\zeta(T) = \frac{\vartheta(T)}{\pi} + 1 + S(T), \tag{19}$$

where $N_\zeta(T)$ is the nontrivial zero counting function, and $\vartheta(T), S(T)$ are as in Theorem 2 and Lemma 9.

Proof. Combine Theorem 6 with Theorem 7, Theorem 8, and Lemma 9:

$$2\,\pi\,N_\zeta(T) = \Delta_{C_T} \arg A(s) + \Delta_{C_T} \arg \zeta(s) + \Delta_{C_T} \arg B(s) = \vartheta(T) + \pi\,S(T) + \pi.$$

Division by $2\,\pi$ gives

$$N_\zeta(T) = \frac{\vartheta(T)}{\pi} + 1 + S(T). \qquad \square$$

Theorem 11. *[Midpoint convention for boundary zeros]If T coincides with the ordinate of one or more nontrivial zeros of $\zeta(s)$, then both $N_\zeta(T)$ and $S(T)$ have jumps equal to the total multiplicity of those zeros. Define*

$$N_\zeta(T) = \tfrac{1}{2}(N_\zeta(T^-) + N_\zeta(T^+)), \qquad S(T) = \tfrac{1}{2}(S(T^-) + S(T^+)), \tag{20}$$

where $T^\pm \to T$ avoiding zeros. Then the formula in Theorem 10 holds for such T.

Proof. When T passes through the ordinate of a zero of $\xi(s)$, deform the contour C_T by a semicircular indentation around the zero. Let the zero have multiplicity m. The indentation has radius $\epsilon > 0$, and the limit $\epsilon \to 0$ is taken.

On this semicircle, $\xi(s)$ has a zero of order m, so

$$\xi(s) = (s - s_0)^m h(s),$$

where s_0 is the location of the zero and $h(s_0) \neq 0$. The argument of $(s - s_0)^m$ changes by $m\pi$ as the semicircle is traversed (half of the full circular change). The argument of $h(s)$ remains continuous and contributes no singular change in the limit $\epsilon \to 0$. Hence the contribution from the semicircle to $\Delta_{C_T} \arg \xi(s)$ tends to $m\pi$.

Each zero on the contour therefore contributes half its multiplicity to $N_\zeta(T)$, which matches the half–weight convention in the definition.

The same contour deformation shows that $\arg \zeta(\frac{1}{2} + iT)$ changes by $m\pi$, so $S(T)$ jumps by m. The midpoint prescription averages the limits from above and below the jump, and the identity from Theorem 10 remains valid at the boundary point. This is precisely the argument recorded in Titchmarsh, Theorem 9.4. □

Theorem 12. *[Exact zero–counting for Hardy's Z]Let $T > 0$, and define $N_Z(T)$ and $S(T)$ as above, with $Z(T)$ as in Theorem 3 and midpoint conventions for boundary zeros:*

$$N_Z(T) = \tfrac{1}{2}(N_Z(T^-) + N_Z(T^+)), \qquad S(T) = \tfrac{1}{2}(S(T^-) + S(T^+)) \tag{21}$$

whenever $Z(T) = 0$. Then for all $T > 0$,

$$N_Z(T) = \frac{\vartheta(T)}{\pi} + 1 + S(T). \tag{22}$$

Proof. By Theorem 4,

$$N_Z(T) = N_\zeta(T)$$

for all $T > 0$, including half–weights on the boundary, and multiplicities are preserved under the bijection induced by Φ.

By Theorems 10 and 11,

$$N_\zeta(T) = \frac{\vartheta(T)}{\pi} + 1 + S(T)$$

holds for all $T > 0$, with midpoint conventions at ordinates of zeros. Substituting $N_Z(T) = N_\zeta(T)$ yields

$$N_Z(T) = \frac{\vartheta(T)}{\pi} + 1 + S(T)$$

for every $T > 0$, with the stated midpoint prescription when $Z(T) = 0$. $\square$

Reference

E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed. (rev. D. R. Heath-Brown), Oxford University Press, 1986, Chapters 9–10.