A Uniformly Convergent Orthonormal Expansion for the Bessel Function of the First Kind of Order 0

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February 11, 2025

Theorem 1

Let $\psi_n(y)$ be defined as

$$\begin{aligned}\psi_n(y) &= (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\; j_{2n}(y)\\ &= (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\sqrt{\frac{\pi}{2\,y}}\; J_{2n+\frac{1}{2}}(y)\\ &= (-1)^n \sqrt{\frac{(4\,n+1)\,\pi}{\pi\,2\,y}}\; J_{2n+\frac{1}{2}}(y)\\ &= (-1)^n \sqrt{\frac{4\,n+1}{2\,y}}\; J_{2n+\frac{1}{2}}(y)\\ &= (-1)^n \sqrt{\frac{2\,n+\frac{1}{2}}{y}}\; J_{2n+\frac{1}{2}}(y)\end{aligned}$$

where J_ν denotes the Bessel function of the first kind and j_n the spherical Bessel function. Then

$$\begin{aligned}J_0(x) &= \sum_{n=0}^{\infty} \psi_n(x) \int_0^\infty J_0(y)\,\psi_n(y)\,\mathrm{d}y\\ &= \sum_{n=0}^{\infty} (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\; j_{2n}(x) \sqrt{\frac{2\,n+\frac{1}{2}}{\pi}}\,\frac{\Gamma\,(n+\frac{1}{2})^2}{\Gamma\,(n+1)^2}\\ &= \frac{1}{2\,\sqrt{\pi\,x}} \sum_{n=0}^{\infty} (-1)^n\,(4\,n+1)\,\frac{\Gamma\,(n+\frac{1}{2})^2}{\Gamma\,(n+1)^2}\,J_{2n+\frac{1}{2}}(x)\\ &= \frac{1}{\sqrt{4\,\pi\,x}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(4\,n+1)\,\Gamma\,(n+\frac{1}{2})^2}{\Gamma\,(n+1)^2}\,J_{2n+\frac{1}{2}}(x)\end{aligned}$$

with uniform convergence $\forall x \in \mathbb{C}$. Moreover, $\{\psi_n\}$ forms an orthonormal system in $L^2([0,\infty))$ satisfying

$$\int_0^\infty \psi_m(t)\,\psi_n(t)\,\mathrm{d}t = \delta_{mn} = \begin{cases} 1 & m=n,\\ 0 & m \neq n.\end{cases}$$

Proof.

Orthonormality of $\psi_n(y)$

For $m \neq n$, by substituting the definition we have

$$\begin{aligned}\langle \psi_m, \psi_n \rangle &= (-1)^{m+n} \sqrt{\frac{(4\,m+1)\,(4\,n+1)}{\pi^2}}\,\frac{\pi}{2} \int_0^\infty \frac{J_{2m+\frac{1}{2}}(y)\,J_{2n+\frac{1}{2}}(y)}{y}\,\langle\mathsf{dint}\rangle y \\ &= (-1)^{m+n} \sqrt{\frac{(4\,m+1)\,(4\,n+1)}{4\,\pi}} \cdot \frac{2}{\pi}\,\frac{\delta_{mn}}{(2\,m+\frac{1}{2})+(2\,n+\frac{1}{2})} = 0\end{aligned} \tag{1}$$

For $m = n$, using the orthogonality relation

$$\int_0^\infty \frac{\left[J_{2n+\frac{1}{2}}(y)\right]^2}{y}\,\langle\mathsf{dint}\rangle y = \frac{1}{2\,n+\frac{1}{2}} \tag{2}$$

we obtain

$$\langle \psi_n, \psi_n \rangle = \frac{\sqrt{\frac{4\,n+1}{4\,\pi}} \cdot \frac{\pi}{2}}{2\,n+\frac{1}{2}} = 1 \tag{3}$$

Expansion Coefficients

Since the system $\{\psi_n\}$ is orthonormal, the coefficients in the expansion are given by

$$c_n = \int_0^\infty J_0(y)\ \ \psi_n(y)\,\langle\mathsf{dint}\rangle y \tag{4}$$

Using Neumann's addition theorem together with Mellin transform techniques, one obtains

$$c_n = (-1)^n \sqrt{\frac{4\,n+1}{2}}\,\frac{\Gamma\,(n+\frac{1}{2})^2}{\Gamma\,(n+1)^2},$$

which is derived from the evaluation

$$\int_0^\infty J_0(y) \frac{J_{2n+\frac{1}{2}}(y)}{\sqrt{y}}\,\langle\mathsf{dint}\rangle y = \frac{\Gamma\,(n+\frac{1}{2})^2}{\sqrt{2}\,\Gamma\,(n+1)^2} \tag{5}$$

Uniform Convergence

Observe that the series for $J_0(x)$ is alternating, with its nth term given by

$$a_n(x) = \frac{(-1)^n\,(4\,n+1)\,\Gamma\,(n+\frac{1}{2})^2}{2\,\sqrt{\pi\,x}\ \Gamma\,(n+1)^2}\,J_{2n+\frac{1}{2}}(x) \tag{6}$$

Due to the asymptotic behavior

$$\frac{\Gamma\,(n+\frac{1}{2})}{\Gamma\,(n+1)} \sim n^{-1/2} \quad \text{and} \quad J_{2n+\frac{1}{2}}(x) \sim \frac{(x/2)^{2n+\frac{1}{2}}}{\Gamma\,(2\,n+\frac{3}{2})} \tag{7}$$

for each fixed x (and for sufficiently large n) the absolute values $|a_n(x)|$ decrease monotonically.

Since the series is alternating with monotonically decreasing term magnitudes, the Alternating Series Remainder Theorem applies. In other words, the pointwise error obtained by truncating the series after the Nth term is bounded by the absolute value of the first omitted term:

$$\left| J_0(x) - \sum_{n=0}^{N} a_n(x) \right| \leq |a_{N+1}(x)| \tag{8}$$

Because the projection (or Fourier coefficient) c_n converges to 0 monotonically, for any given $\epsilon > 0$ there exists an N (which may be chosen uniformly in x) such that

$$|a_{N+1}(x)| < \epsilon \quad \text{for all } x \in \mathbb{C} \tag{9}$$

This completes the argument that the expansion converges uniformly without needing to form any tail sums. $\square$