Unitary Time Changes of Stationary Processes Yield Oscillatory Processes and a Functional Framework Toward a Hilbert–Pólya Construction

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are square-integrable over compact sets. Applying U_θ to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\; e^{i\lambda\theta(t)}$ and evolutionary spectrum $d\,F_t(\lambda) = \dot{\theta}(t) d\,F(\lambda)$. It is proved that sample paths of any non-degenerate second-order stationary process almost surely lie in $L^2_{\text{loc}}(\mathbb{R})$, making the operator applicable to typical realizations. A zero-localization measure $\mathrm{d}\mu(t) = \delta(Z(t))\,|Z'(t)|\,dt$ induces a Hilbert space $L^2(\mu)$ on the zero set of an oscillatory process Z, and the multiplication operator $(L\,f)(t) = t\,f(t)$ has simple pure point spectrum equal to the zero crossing set of Z. This produces a concrete operator scaffold consistent with a Hilbert–Pólya-type viewpoint.

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TODO: add zero counting function and expected zero counting function!!!!!

1 Function Spaces and Unitary Time Change

1.1 σ-compact sets and local L^2

Definition 1. ***[σ-compact sets]*** *A subset $U \subseteq \mathbb{R}$ is σ-compact if*

$$U = \bigcup_{n=1}^{\infty} K_n \tag{1}$$

with each K_n compact.

Definition 2. ***[Locally square-integrable functions]*** *Define*

$$L^2_{\text{loc}}(\mathbb{R}) := \left\{ f\colon \mathbb{R} \to \mathbb{C}\colon \int_K |f(t)|^2 \, dt < \infty \textit{ for every compact } K \subseteq \mathbb{R} \right\} \tag{2}$$

Remark 3. Every bounded measurable set in $\mathbb{R}$ is compact or contained in a compact set; hence $L^2_{\text{loc}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

1.2 Unitary time-change operator

Definition 4. ***[Unitary time-change]*** *Let the time-scaling function $\theta\colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with $\dot{\theta}(t) > 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero. The function θ maps σ-compact sets to σ-compact sets. Define, for f measurable,*

$$(U_\theta f)(t) = \sqrt{\dot{\theta}(t)}\; f(\theta(t)) \tag{3}$$

Proposition 5. ***[Inverse map]*** *The inverse map is given by*

$$(U_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{4}$$

which is well-defined almost everywhere on every σ-compact set.

Proof. Since $\dot{\theta}(t) = 0$ only on sets of measure zero, and θ^{-1} maps sets of measure zero to sets of measure zero (as absolutely continuous bijective functions preserve measure-zero sets), the denominator $\sqrt{\dot{\theta}(\theta^{-1}(s))}$ is positive almost everywhere. The expression is therefore well-defined almost everywhere on every σ-compact set, which suffices for defining an element of $L^2_{\text{loc}}(\mathbb{R})$. $\square$

Theorem 6. ***[Local unitarity on σ-compact sets]*** *For every σ-compact set $C \subseteq \mathbb{R}$ and $f \in L^2_{\text{loc}}(\mathbb{R})$,*

$$\int_C |(U_\theta f)(t)|^2 \, dt = \int_{\theta(C)} |f(s)|^2 \, ds \tag{5}$$

Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\text{loc}}(\mathbb{R})$.

Proof. Let $f \in L^2_{\text{loc}}(\mathbb{R})$ and let U be any σ-compact set. The local L^2-norm of $U_\theta f$ over C is:

$$\begin{aligned} \int_C |(U_\theta f)(t)|^2 \, dt &= \int_C \left| \sqrt{\dot{\theta}(t)} \, f(\theta(t)) \right|^2 dt \\ &= \int_C \dot{\theta}(t) |f(\theta(t))|^2 \, dt \end{aligned} \tag{6}$$

Since θ is absolutely continuous and strictly increasing, applying the change of variables $s = \theta(t)$ gives

$$ds = \dot{\theta}(t) \, dt \tag{7}$$

almost everywhere. Since θ maps σ-compact sets to σ-compact sets, as t ranges over C, $s = \theta(t)$ ranges over $\theta(C)$, which is σ-compact. Therefore:

$$\int_C \dot{\theta}(t) \, |f(\theta(t))|^2 \, dt = \int_{\theta(C)} |f(s)|^2 \, ds \tag{8}$$

To verify that U_θ^{-1} is indeed the inverse, we compute explicitly. For any $f \in L^2_{\text{loc}}(\mathbb{R})$:

$$\begin{aligned} (U_\theta^{-1} U_\theta f)(s) &= \left(U_\theta^{-1} \sqrt{\dot{\theta}(s)} \, f(\theta(s)) \right)(s) \\ &= \frac{\sqrt{\dot{\theta}(\theta^{-1}(s))}}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \, f(\theta(\theta^{-1}(s))) \\ &= f(s) \end{aligned} \tag{9}$$

since $\theta(\theta^{-1}(s)) = s$. Similarly, for any $g \in L^2_{\mathrm{loc}}(\mathbb{R})$:

$$\begin{aligned}(U_\theta U_\theta^{-1} g)(t) &= \sqrt{\dot{\theta}(t)}\,(U_\theta^{-1} g)(\theta(t)) \\ &= \frac{\sqrt{\dot{\theta}(t)}}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}}\, g(\theta^{-1}(\theta(t))) \\ &= \frac{\sqrt{\dot{\theta}(t)}}{\sqrt{\dot{\theta}(t)}}\, g(t) \\ &= g(t)\end{aligned} \tag{10}$$

since $\theta^{-1}(\theta(t)) = t$. Therefore

$$U_\theta U_\theta^{-1} = U_\theta^{-1} U_\theta = I \tag{11}$$

on $L^2_{\mathrm{loc}}(\mathbb{R})$. □

Theorem 7. ***[Unitarity on $L^2(\mathbb{R})$]*** *$U_\theta: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is unitary:*

$$\int_{\mathbb{R}} |(U_\theta f)(t)|^2\, dt = \int_{\mathbb{R}} |f(s)|^2\, ds \tag{12}$$

and U_θ^{-1} is its inverse.

Proof. For $f \in L^2(\mathbb{R})$, we have:

$$\int_{\mathbb{R}} |(U_\theta f)(t)|^2\, dt = \int_{\mathbb{R}} \dot{\theta}(t) |f(\theta(t))|^2\, dt \tag{13}$$

By the change of variables

$$s = \theta(t) \tag{14}$$

with

$$ds = \dot{\theta}(t)\, dt \tag{15}$$

and since $\theta: \mathbb{R} \to \mathbb{R}$ is bijective:

$$\int_{\mathbb{R}} \dot{\theta}(t)\, |f(\theta(t))|^2\, dt = \int_{\mathbb{R}} |f(s)|^2\, ds \tag{16}$$

The inverse relationship follows from the same computation as in Theorem 6, applied globally. □

2 Oscillatory Processes (Priestley)

Definition 8. ***[Oscillatory process]*** *Let F be a finite nonnegative Borel measure on $\mathbb{R}$. For each $t \in \mathbb{R}$, let $A_t \in L^2(F)$ be the gain function and*

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{17}$$

be the corresponding oscillatory function then an oscillatory process *is a stochastic process which can be represented as*

$$\begin{aligned} Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda)\, \mathrm{d}\,\Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, \mathrm{d}\,\Phi(\lambda) \end{aligned} \tag{18}$$

where Φ is a complex orthogonal random measure with spectral measure F, that is,

$$\mathbb{E}[\Phi\,(d\,\lambda)\,\overline{\Phi\,(d\,\mu)}] = \delta\,(\lambda - \mu)\;\, d\,F(\lambda) \tag{19}$$

and corresponding covariance kernel

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \int_{\mathbb{R}} A_t(\lambda)\,\overline{A_s(\lambda)}\, e^{i\lambda(t-s)}\;\, d\,F(\lambda) \\ &= \int_{\mathbb{R}} \phi_t(\lambda)\overline{\phi_s(\lambda)}\mathrm{d}\,F(\lambda) \end{aligned} \tag{20}$$

Theorem 9. ***[Real-valuedness criterion for oscillatory processes]****Let Z be an oscillatory process with oscillatory function*

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{21}$$

and spectral measure F. Then Z is real-valued if and only if

$$A_t\,(-\lambda) = \overline{A_t(\lambda)} \tag{22}$$

for F-almost every $\lambda \in \mathbb{R}$, equivalently

$$\varphi_t\,(-\lambda) = \overline{\varphi_t(\lambda)} \tag{23}$$

for F-almost every $\lambda \in \mathbb{R}$.

Proof. Assume Z is real-valued, i.e.

$$Z(t) = \overline{Z(t)} \quad \forall t \in \mathbb{R} \tag{24}$$

Writing its oscillatory representation,

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, \mathrm{d}\Phi(\lambda) \tag{25}$$

and taking the complex conjugate gives

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t} \mathrm{d}\overline{\Phi\,(\lambda)} \tag{26}$$

For a real-valued process, the orthogonal random measure Φ must satisfy

$$\mathrm{d}\overline{\Phi\,(\lambda)} = -\mathrm{d}\,\Phi\,(\lambda) \tag{27}$$

which ensures that the spectral representation produces real values. Substituting this identity and using the substitution

$$\mu = -\lambda \tag{28}$$

it is shown that

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t\,(-\mu)} e^{i\mu t}\, \mathrm{d}\,\Phi\,(\mu) \tag{29}$$

Since $Z(t) = \overline{Z(t)}$, comparison of the integrands (which are unique elements of $L^2(F)$) yields

$$A_t(\lambda) = \overline{A_t\,(-\lambda)} \quad \text{for } F\text{-a.e. } \lambda \tag{30}$$

Equivalently, because the oscillatory function (17) is given by

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{31}$$

we have

$$\varphi_t\,(-\lambda) = \overline{\varphi_t(\lambda)} \quad \text{for } F\text{-a.e. } \lambda \tag{32}$$

Conversely, if

$$A_t\,(-\lambda) = \overline{A_t(\lambda)} \tag{33}$$

for F-a.e. λ, then the same substitution shows that

$$\overline{Z(t)} = Z(t) \quad \forall t \in \mathbb{R} \tag{34}$$

so Z is real-valued. This completes the proof. □

Theorem 10. ***[Existence]*** *If F is finite and $(A_t)_{t\in\mathbb{R}}$ is measurable in t with*

$$\int_{\mathbb{R}} |A_t(\lambda)|^2 \, d\,F(\lambda) < \infty \forall t \in \mathbb{R} \tag{35}$$

then there exists a complex orthogonal random measure Φ with spectral measure F such that

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t} \mathrm{d}\Phi\,(\lambda) \tag{36}$$

is well-defined in $L^2(\Omega)$ and has covariance R_Z as in (20) above.

Proof. The proof proceeds by constructing the stochastic integral using the standard extension procedure. First, the integral is defined for simple functions of the form

$$g(\lambda) = \sum_{j=1}^{n} c_j \mathbf{1}_{E_j}(\lambda) \tag{37}$$

where $\{E_j\}$ are disjoint Borel sets with $F(E_j) < \infty$ and $c_j \in \mathbb{C}$:

$$\int_{\mathbb{R}} g(\lambda)\ \Phi\,(d\,\lambda) := \sum_{j=1}^{n} c_j\, \Phi(E_j) \tag{38}$$

For such simple functions, the isometry property holds:

$$\begin{aligned} \mathbb{E}\left[\left|\int_{\mathbb{R}} g(\lambda)\ \Phi\,(d\,\lambda)\right|^2\right] &= \mathbb{E}\left[\left|\sum_{j=1}^{n} c_j\, \Phi(E_j)\right|^2\right] \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \bar{c_k} \mathbb{E}[\Phi(E_j)\overline{\Phi(E_k)}] \\ &= \sum_{j=1}^{n} |c_j|^2\, F(E_j \\ &= \int_{\mathbb{R}} |g(\lambda)|^2\, d\,F(\lambda) \end{aligned} \tag{39}$$

Since simple functions are dense in $L^2(F)$, the integral is extended by continuity to all $g \in L^2(F)$. For each t, since the oscillatory function (17) is defined by

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{40}$$

and $A_t \in L^2(F)$, $\varphi_t \in L^2(F)$ holds. Therefore

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda)\, \mathrm{d}\Phi\,(\lambda) \tag{41}$$

is well-defined in $L^2(\Omega)$. The covariance is computed as:

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\Big[\int_{\mathbb{R}} \varphi_t(\lambda) \mathrm{d}\Phi\,(\lambda) \int_{\mathbb{R}} \overline{\varphi_s(\mu)} \mathrm{d}\overline{\Phi\,(\mu)}\Big] \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\mu)}\mathbb{E}\,[\mathrm{d}\Phi(\lambda)\mathrm{d}\overline{\Phi(\mu)}] \\ &= \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)}\, d\,F(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)} e^{i\lambda(t-s)}\, d\,F(\lambda) \end{aligned} \tag{42}$$

□

3 Unitarily Time-Changed Stationary Processes

3.1 Stationary processes

Definition 11. ***[Cramér representation]*** *A zero-mean stationary process X with spectral measure F admits the sample path representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, \Phi\,(d\,\lambda) \tag{43}$$

which has covariance

$$R_X\,(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)}\, d\,F(\lambda) \tag{44}$$

3.2 Stationary $\rightarrow$ oscillatory via U_θ

Theorem 12. ***[Unitary time change yields oscillatory process]*** *Let X be zero-mean stationary as in Definition 11. For scaling function θ as in Definition 4, define*

$$\begin{aligned} Z(t) &= (U_\theta X)(t) \\ &= \sqrt{\dot{\theta}(t)}\; X(\theta(t)) \end{aligned} \tag{45}$$

Then Z is a realization of an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\; e^{i\lambda\theta(t)} \tag{46}$$

gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\; e^{i\lambda(\theta(t)-t)} \tag{47}$$

and covariance

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\left[\sqrt{\dot{\theta}(t)}\, X(\theta(t))\overline{\sqrt{\dot{\theta}(t)}\, X(\theta(t))}\right] \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\; \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\; R_X\left(\theta(t)-\theta(s)\right) \\ &= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\; \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\; dF(\lambda) \end{aligned} \tag{48}$$

Proof. Applying the unitary time change operator to the spectral representation of $X(t)$:

$$\begin{aligned} Z(t) &= (U_\theta X)(t) \\ &= \sqrt{\dot{\theta}(t)}\; X(\theta(t)) \\ &= \sqrt{\dot{\theta}(t)}\; \int_{\mathbb{R}} e^{i\lambda\theta(t)}\; \mathrm{d}\,\Phi(\lambda) \\ &= \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)}\; e^{i\lambda\theta(t)}\, \mathrm{d}\,\Phi(\lambda) \\ &= \int_{\mathbb{R}} \varphi_t(\lambda)\, \mathrm{d}\,\Phi(\lambda) \end{aligned} \tag{49}$$

where

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \tag{50}$$

To verify this constitutes an oscillatory representation according to Definition 8, $\varphi_t(\lambda)$ has the form $A_t(\lambda)\, e^{i\lambda t}$:

$$\begin{aligned}\varphi_t(\lambda) &= \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \\ &= \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)}\, e^{i\lambda t} \\ &= A_t(\lambda)\, e^{i\lambda t}\end{aligned} \tag{51}$$

where

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)} \tag{52}$$

Since $\dot{\theta}(t) \geq 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of measure zero, $A_t(\lambda)$ is well defined almost everywhere. Moreover, $A_t \in L^2(F)$ for each t since:

$$\begin{aligned}\int_{\mathbb{R}} |A_t(\lambda)|^2\, dF(\lambda) &= \int_{\mathbb{R}} \left|\sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)}\right|^2 dF(\lambda) \\ &= \int_{\mathbb{R}} \dot{\theta}(t) |e^{i\lambda(\theta(t)-t)}|^2\, dF(\lambda) \\ &= \dot{\theta}(t) \int_{\mathbb{R}} dF(\lambda) \\ &= \dot{\theta}(t)\, F(\mathbb{R}) < \infty\end{aligned} \tag{53}$$

where $|e^{i\alpha}| = 1$ for all real α is used. The covariance (48) is computed by substituting the spectral representation and applying Fubuni's theorem to interchange the order of operations.

$$\tag{54}$$ □

Corollary 13. ***[Evolutionary spectrum]*** *The evolutionary spectrum is*

$$\begin{aligned}dF_t(\lambda) &= |A_t(\lambda)|^2\, dF(\lambda) \\ &= \dot{\theta}(t)\, dF(\lambda)\end{aligned} \tag{55}$$

Proof. By definition of the evolutionary spectrum and using the gain function from Theorem 12:

$$\begin{aligned} d\,F_t(\lambda) &= |A_t(\lambda)|^2 \; d\,F(\lambda) \\ &= \left|\sqrt{\dot\theta(t)}\; e^{i\lambda(\theta(t)-t)}\right|^2 \; d\,F(\lambda) \\ &= \dot\theta(t)|e^{i\lambda(\theta(t)-t)}|^2 \; d\,F(\lambda) \\ &= \dot\theta(t) \; d\,F(\lambda) \end{aligned} \tag{56}$$

since

$$|e^{i\alpha}| = 1 \forall a \in \mathbb{R} \tag{57}$$ □

3.3 Covariance operator conjugation

Proposition 14. ***[Operator conjugation]*** *Let*

$$(T_K f)(t) := \int_{\mathbb{R}} K(|t-s|) \; f(s) \; d\,s \tag{58}$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h} \; d\,F(\lambda) \tag{59}$$

Define the transformed kernel

$$K_\theta(s,t) := \sqrt{\dot\theta(t)\,\dot\theta(s)} \;\; K(|\theta(t)-\theta(s)|) \tag{60}$$

and corresponding integral covariance operator

$$(T_{K_\theta} f)(t) := \int_{\mathbb{R}} K_\theta(s,t) \; f(s) \; d\,s \tag{61}$$

Then

$$T_{K_\theta} = U_\theta \; T_K \; U_\theta^{-1} \tag{62}$$

on $L^2_{\mathrm{loc}}(\mathbb{R})$.

Proof. For any $g \in L^2_{\mathrm{loc}}(\mathbb{R})$, compute:

$$\begin{aligned} ((U_\theta\, T_K\, U_\theta^{-1})\, g)(t) &= (U_\theta\, (T_K\, U_\theta^{-1}\, g))(t) \\ &= \sqrt{\dot\theta(t)}\; (T_K\, U_\theta^{-1}\, g)(\theta(t)) \\ &= \sqrt{\dot\theta(t)} \int_{\mathbb{R}} K(|\theta(t)-w|)\, (U_\theta^{-1}\, g)(w) \; d\,w \end{aligned} \tag{63}$$

Substitute $w = \theta(s)$ with $d\,w = \dot{\theta}(s)\ d\,s$:

$$\begin{aligned}
&= \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|) \frac{g(s)}{\sqrt{\dot{\theta}(s)}}\, \dot{\theta}(s)\ d\,s \\
&= \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} K(|\theta(t) - \theta(s)|)\, g(s) \sqrt{\dot{\theta}(s)}\ d\,s \\
&= \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)\dot{\theta}(s)}\, K(|\theta(t) - \theta(s)|)\, g(s)\ d\,s \\
&= \int_{\mathbb{R}} K_\theta(t,s)\, g(s)\ d\,s \\
&= (T_{K_\theta} g)(t)
\end{aligned} \tag{64}$$

where

$$K_\theta(t,s) = \sqrt{\dot{\theta}(t)\dot{\theta}(s)}\, K(|\theta(t) - \theta(s)|) \tag{65}$$

Therefore

$$T_{K_\theta} = U_\theta T_K U_\theta^{-1} \tag{66}$$ □

4 Sample Paths Live in L^2_{loc}

Theorem 15. ***[Sample paths in $L^2_{\text{loc}}(\mathbb{R})$]*** *Let $\{X(t)\}_{t\in\mathbb{R}}$ be a second-order stationary process with*

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \tag{67}$$

then, almost surely, every sample path $t \mapsto X(\omega,t)$ belongs to $L^2_{\text{loc}}(\mathbb{R})$.

Proof. Fix any bounded interval $[a,b]$ and consider the random variable

$$Y_{[a,b]} := \int_a^b X(t)^2\ d\,t \tag{68}$$

By stationarity and Fubini's theorem:

$$\begin{aligned}
\mathbb{E}[Y_{[a,b]}] = \mathbb{E}\left[\int_a^b X(t)^2\ d\,t\right] &= \int_a^b \mathbb{E}[X(t)^2]\ d\,t \\
&= \int_a^b \sigma^2\ d\,t \\
&= \sigma^2 (b-a) < \infty
\end{aligned} \tag{69}$$

By Markov's inequality, for any $M > 0$:

$$P\left(Y_{[a,b]} > M\right) \leq \frac{\mathbb{E}[Y_{[a,b]}]}{M} = \frac{\sigma^2\,(b-a)}{M} \tag{70}$$

Taking $M \to \infty$, the conclusion is

$$P\left(Y_{[a,b]} < \infty\right) = 1 \tag{71}$$

i.e., almost surely the sample path is square-integrable on $[a, b]$. Since $\mathbb{R}$ is the countable union of bounded intervals:

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n] \tag{72}$$

by countable subadditivity of probability:

$$P\left(\bigcap_{n=1}^{\infty} \left\{\int_{-n}^{n} X(t)^2 \, dt < \infty\right\}\right) = 1 \tag{73}$$

Now let K be any compact set. Then K is bounded, so

$$K \subseteq [-N, N] \tag{74}$$

for some N. Therefore:

$$\int_K X(t)^2 \, dt \leq \int_{-N}^{N} X(t)^2 \, dt < \infty \tag{75}$$

almost surely. This holds for every compact set K, so almost surely every sample path lies in $L^2_{\mathrm{loc}}(\mathbb{R})$. □

5 Zero Localization and Hilbert–Pólya Scaffold

5.1 Zero localization measure

Definition 16. ***[Zero localization measure]*** *Let Z be real-valued with $Z \in C^1(\mathbb{R})$ having only simple zeros*

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \tag{76}$$

Define, for Borel $B \subset \mathbb{R}$,

$$\mu(B) = \int_{\mathbb{R}} \boldsymbol{1}_B(t)\ \delta(Z(t))\,|\dot{Z}(t)| \, dt \tag{77}$$

Theorem 17. ***[Atomicity on the zero set]*** *For every* $\phi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \phi(t)\ \delta(Z(t))\,|\dot{Z}(t)|\ dt = \sum_{t_0: Z(t_0)=0} \phi(t_0) \tag{78}$$

hence

$$\mu(t) = \sum_{t_0: Z(t_0)=0} \delta_{t_0}(t) \tag{79}$$

Proof. Since all zeros of Z are simple and $Z \in C^1(\mathbb{R})$, by the inverse function theorem each zero t_0 is isolated. Near each zero t_0, Z is locally monotonic, so the one-dimensional change of variables formula for the Dirac delta can be applied.

Specifically, near t_0 where $Z(t_0) = 0$ and $\dot{Z}(t_0) \neq 0$, locally

$$Z(t) = (t - t_0)\,\dot{Z}(t_0) + O((t - t_0)^2) \tag{80}$$

holds. The distributional identity for the Dirac delta under smooth changes of variables gives:

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|} \tag{81}$$

Therefore:

$$\begin{aligned}
\int_{\mathbb{R}} \phi(t)\ \delta(Z(t))\,|\dot{Z}(t)|\ dt &= \int_{-\infty}^{\infty} \phi(t)\,|\dot{Z}(t)| \sum_{t_0: Z(t_0)=0} \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|}\ dt \\
&= \sum_{t_0: Z(t_0)=0} \int_{\mathbb{R}} \phi(t) \frac{|\dot{Z}(t)|\,\delta\,(t - t_0)}{|\dot{Z}(t_0)|}\ dt \\
&= \sum_{t_0: Z(t_0)=0} \frac{|\dot{Z}(t_0)|}{|\dot{Z}(t_0)|}\,\phi(t_0) \\
&= \sum_{t_0: Z(t_0)=0} \phi(t_0)
\end{aligned} \tag{82}$$

This shows that μ is the discrete measure

$$\mu(t) = \sum_{t_0: Z(t_0)=0} \delta_{t_0}(t) \tag{83}$$

assigning unit mass to each zero. $\square$

5.2 Hilbert space on zeros and multiplication operator

Definition 18. ***[Hilbert space on the zero set]*** *Let* $\mathcal{H} = L^2(\mu)$ *with inner product*

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\,\mathrm{d}\mu(t) \tag{84}$$

Proposition 19. ***[Atomic structure]*** *Let*

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \tag{85}$$

then

$$\mathcal{H} \cong \left\{ f: \{t_0: Z(t_0) = 0\} \to \mathbb{C}: \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \tag{86}$$

with orthonormal basis $\{e_{t_0}\}_{t_0: Z(t_0)=0}$ *where*

$$e_{t_0}(t_1) = \delta_{t_0}(t_1) \tag{87}$$

Proof. By the atomic form of μ, for any $f \in L^2(\mu)$:

$$\|f\|_{\mathcal{H}}^2 = \int |f(t)|^2\,\mathrm{d}\mu(t) \tag{88}$$

$$= \int |f(t)|^2 \sum_{t_0: Z(t_0)=0} \delta_{t_0}(t) \tag{89}$$

$$= \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 \tag{90}$$

This shows the isomorphism with ℓ^2 where the functions e_{t_0} defined by

$$e_{t_0}(t_1) = \delta_{t_0}(t_1) \tag{91}$$

satisfy the relations

$$\begin{aligned} \langle e_{t_0}, e_{t_1}\rangle &= \int e_{t_0}(t)\overline{e_{t_1}(t)}\mathrm{d}\mu(t) \\ &= \sum_{t: Z(t)=0} \delta_{t_0}(t)\,\delta_{t_1}(t) \\ &= \delta_{t_0}(t_1) \\ &= \delta t_1(t_0) \end{aligned} \tag{92}$$

thus forming an orthonormal set. Thus, any $f(t) \in \mathcal{H}$ can be written as

$$f(t) = \sum_{t_0 : Z(t_0) = 0} f(t_0)\, e_{t_0}(t) \tag{93}$$

proving they form a basis. □

Definition 20. ***[Multiplication operator]*** *Define the linear operator*

$$L : \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H} \tag{94}$$

by

$$(L\, f)(t) = t\, f(t) \tag{95}$$

on the support of μ *with domain*

$$\mathcal{D}(L) := \left\{ f \in \mathcal{H} : \int |t\;\, f(t)|^2\;\, \mu\,(d\,t) < \infty \right\} \tag{96}$$

Theorem 21. ***[Self-adjointness and spectrum]*** *L is self-adjoint on $\mathcal{H}$ and has pure point, simple spectrum*

$$\sigma(L) = \overline{\{t \in \mathbb{R} : Z(t) = 0\}} \tag{97}$$

with eigenvalues $\lambda = t_0$ for each zero t_0 and corresponding eigenvectors e_{t_0}.

Proof. First, self-adjointness is verified. For $f, g \in \mathcal{D}(L)$:

$$\begin{aligned} \langle L\, f, g \rangle &= \int (L\, f)(t) \overline{g(t)} \mathrm{d}\mu\,(t) \\ &= \int t\, f(t) \overline{g(t)}\, \mathrm{d}\mu\,(t) \\ &= \int f(t) \overline{t\, g(t)}\, \mathrm{d}\mu\,(t) \\ &= \int f(t) \overline{(L\, g)(t)} \mathrm{d}\mu\,(t) \\ &= \langle f, L\, g \rangle \end{aligned} \tag{98}$$

Thus L is symmetric and acts as

$$(L\, f)(t_0) = t_0\, f(t_0) \tag{99}$$

for each t_0 in the atomic representation where

$$Z(t_0) = 0 \tag{100}$$

This is unitarily equivalent to the diagonal operator on ℓ^2 with diagonal entries

$$\{t_0 \colon Z(t_0) = 0\} \tag{101}$$

Such diagonal operators are self-adjoint. For the spectrum calculation:

$$L\, e_{t_0} = t_0\, e_{t_0} \forall \{t_0 \colon Z(t_0) = 0\} \tag{102}$$

holds, so each t_0 is an eigenvalue of L with eigenvector e_{t_0} and since $\{e_{t_0}\}$ forms an orthonormal basis, L has pure point spectrum. The spectrum of a diagonal operator equals the closure of the set of diagonal entries, hence

$$\sigma(L) = \overline{\{t_0 \colon Z(t_0) = 0\}} \tag{103}$$

The eigenvalues are simple. □

Remark 22. [Operator scaffold] The construction

$$\text{stationary } X \xrightarrow{U_\theta} \text{oscillatory } Z \xrightarrow{\mu = \delta(Z)|\dot{Z}|\, dt} L^2(\mu) \xrightarrow{L: t\cdot} (L, \sigma(L)) \tag{104}$$

produces a concrete self-adjoint operator whose eigenvalues equal the zero set of Z and whose spectrum equals the closure of the zero set, determined by the choice of time-change θ and spectral measure F. This provides an explicit realization consistent with Hilbert–Pólya approaches to encoding arithmetic information in operator spectra.

6 Appendix: Regularity and Simple Zeros

Definition 23. ***[Regularity and simplicity]*** *Assume $Z \in C^1(\mathbb{R})$ and every zero is simple:*

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \tag{105}$$

Lemma 24. ***[Local finiteness and delta decomposition]*** *Under Definition 23, zeros are locally finite and*

$$\delta(Z(t)) = \sum_{t_0 \colon Z(t_0) = 0} \frac{\delta\,(t - t_0)}{|\dot{Z}(t_0)|} \tag{106}$$

whence

$$\mu = \sum_{t_0 \colon Z(t_0) = 0} \delta_{t_0} \tag{107}$$

Proof. Since $Z \in C^1(\mathbb{R})$ and $\dot{Z}(t_0) \neq 0$ at each zero t_0, the inverse function theorem implies that Z is locally invertible near each zero. Specifically, there exists a neighborhood U_{t_0} of t_0 such that $Z|_{U_{t_0}}$ is strictly monotonic and invertible.

This implies zeros are isolated: if $Z(t_0) = 0$ and $\dot{Z}(t_0) \neq 0$, then there exists $\epsilon > 0$ such that $Z(t) \neq 0$ for $0 < |t - t_0| < \epsilon$. Therefore zeros are locally finite (finitely many in any bounded interval).

For the distributional identity, the one-dimensional change of variables formula for the Dirac delta is considered. If $g: I \to \mathbb{R}$ is C^1 on interval I with $\dot{g}(x) \neq 0$ for all $x \in I$, then

$$\delta(g(x)) = \sum_{x_0 : g(x_0) = 0} \frac{\delta(x - x_0)}{|\dot{g}(x_0)|} \tag{108}$$

Applying this locally around each zero t_0 of Z, and since zeros are isolated, the local results can be patched together to obtain the global identity:

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0) = 0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \tag{109}$$

Consequently:

$$\begin{aligned} \mathrm{d}\mu(t) &= \delta(Z(t))|\dot{Z}(t)|\ dt \\ &= \sum_{t_0 : Z(t_0) = 0} \frac{|\dot{Z}(t)|}{|\dot{Z}(t_0)|} \delta(t - t_0)\ dt \\ &= \sum_{t_0 : Z(t_0) = 0} \delta_{t_0}(dt) \end{aligned} \tag{110}$$

where the last equality uses the fact that

$$\frac{|\dot{Z}(t_0)|}{|\dot{Z}(t_0)|} = 1 \tag{111}$$

when evaluating at $t = t_0$. □