A Bijective Modification of the Riemann-Siegel Theta Function

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Abstract

A monotonically increasing version $\vartheta^+(t)$ of the Riemann–Siegel theta function $\vartheta(t)$ is constructed by modifying through reflection about its unique nonzero critical point. ·

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1 The Riemann–Siegel Theta Function

Definition 1. *[Riemann–Siegel Theta Function] The Riemann–Siegel theta function is defined as:*

$$\vartheta(t) = \arg\Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right) - \frac{t}{2}\log\pi \tag{1}$$

where Γ is the gamma function and arg denotes the principal argument, taken continuously along the path.

Definition 2. *[Digamma and Trigamma Functions] The digamma function $\psi^{(0)}(z)$ and trigamma function $\psi^{(1)}(z)$ are defined by:*

$$\psi^{(0)}(z) = \frac{d}{d\,z}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad [DLMF5.2.1] \tag{2}$$

$$\psi^{(1)}(z) = \frac{d}{d\,z}\psi^{(0)}(z) = \sum_{n=0}^{\infty}\frac{1}{(n+z)^2} \quad [DLMF5.4.2] \tag{3}$$

for $\Re(z) > 0$.

Proposition 3. *[Derivative Properties] The derivative of the Riemann–Siegel theta function is:*

$$\vartheta'(t) = \frac{1}{2}\,\Im\left[\psi^{(0)}\left(\frac{1}{4} + \frac{i\,t}{2}\right)\right] - \frac{\log \pi}{2} \tag{4}$$

Proof. Let $w(t) = \frac{1}{4} + \frac{i\,t}{2}$. Along the curve $t \mapsto w(t)$ the principal argument of Γ can be chosen continuously, so

$$\frac{d}{d\,t}\arg\Gamma(w(t)) = \Im\left(\frac{d}{d\,t}\log\Gamma(w(t))\right) = \Im(\psi^{(0)}(w(t))\ w'(t)) \tag{5}$$

Since $w'(t) = i/2$, this derivative equals $\frac{1}{2}\,\Im\,\psi^{(0)}(w(t))$. Differentiating $-\frac{t}{2}\log\pi$ gives $-\frac{\log\pi}{2}$. $\square$

Theorem 4. *[Limit at the Origin]*

$$\lim_{t \to 0^+} \Im\left[\psi^{(0)}\left(\frac{1}{4} + \frac{i\,t}{2}\right)\right] = 0 \tag{6}$$

Proof. Using the integral representation [AbramowitzStegun6.3.1] and dominated convergence (or by analyticity and Taylor expansion in t), the imaginary part vanishes as $t \to 0^+$. $\square$

Theorem 5. *[Monotonicity of the Digamma Imaginary Part] For fixed $\sigma > 0$, the function $t \mapsto \Im\,[\psi^{(0)}\,(\sigma + i\,t)]$ is strictly increasing for $t > 0$.*

Proof. Differentiating with respect to t gives

$$\frac{\partial}{\partial t}\Im\,[\psi^{(0)}\,(\sigma + i\,t)] = \Re\left[\frac{\partial}{\partial t}\,\psi^{(0)}\,(\sigma + i\,t)\right] = \Re[i\ \psi^{(1)}\,(\sigma + i\,t)] = -\,\Im[\psi^{(1)}\,(\sigma + i\,t)] \tag{7}$$

Using the absolutely convergent series [DLMF5.4.2]

$$\psi^{(1)}(z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2}, \quad \Re\,z > 0 \tag{8}$$

and setting $z = \sigma + i\,t$, we have

$$\frac{1}{(n + \sigma + i\,t)^2} = \frac{(n+\sigma)^2 - t^2 - 2\,i\,(n+\sigma)\,t}{((n+\sigma)^2 + t^2)^2} \tag{9}$$

Thus

$$\Im[\psi^{(1)}(\sigma+i\,t)]=\sum_{n=0}^{\infty}\frac{-2\,(n+\sigma)\,t}{((n+\sigma)^2+t^2)^2}<0 \tag{10}$$

for $\sigma>0$, $t>0$. Hence $\frac{\partial}{\partial t}\Im\,[\psi^{(0)}\,(\sigma+i\,t)]>0$. □

Theorem 6. *[Asymptotic Limit]*

$$\lim_{t\to\infty}\Im\left[\psi^{(0)}\left(\frac{1}{4}+\frac{i\,t}{2}\right)\right]=\frac{\pi}{2} \tag{11}$$

Proof. From the asymptotic expansion [DLMF5.11.1], $\psi^{(0)}(z)=\log z-\frac{1}{2\,z}+O(|z|^{-2})$ as $|z|\to\infty$ with $|\arg z|<\pi$. Writing $z=\frac{1}{4}+\frac{i\,t}{2}$, we have $\Im\log z=\arg z\to\frac{\pi}{2}$, and $\Im\,(-1/(2\,z))=O(1/t)$, so the limit is $\pi/2$. □

Theorem 7. *[Unique Critical Point] There exists a unique $a>0$ such that $\vartheta'(a)=0$, equivalently*

$$\Im\left[\psi^{(0)}\left(\frac{1}{4}+\frac{i\,a}{2}\right)\right]=\log\pi \tag{12}$$

Moreover:

- $\vartheta'(t)<0$ *for* $t\in(0,a)$
- $\vartheta'(t)=0$ *at* $t=a$
- $\vartheta'(t)>0$ *for* $t>a$

Proof. By Theorems 4, 5, and 6, the function $t\mapsto\Im\,[\psi^{(0)}\,(1/4+i\,t/2)]$ is continuous, strictly increasing from 0 to $\pi/2$, and therefore attains the value $\log\pi$ at a unique $a>0$. The sign changes for $\vartheta'(t)$ follow from Proposition 3. □

2 Monotonization Construction

Definition 8. *[Monotonized Theta Function] Define the monotonized Riemann–Siegel theta function*

$$\vartheta^{+}(t)=\begin{cases}2\,\vartheta(a)-\vartheta(t) & t\in[0,a]\\ \vartheta(t) & t>a\end{cases} \tag{13}$$

where a is the unique critical point from Theorem 7.

Theorem 9. *[Monotonicity of ϑ^+] The function $\vartheta^+(t)$ is nondecreasing on $[0,\infty)$ and strictly increasing on $[0,\infty)\setminus\{a\}$:*

$$\frac{d}{dt}\vartheta^+(t)=\begin{cases}-\vartheta'(t)>0, & t\in(0,a)\\ 0, & t=a\\ \vartheta'(t)>0 & t>a\end{cases} \tag{14}$$

Proof. Immediate from the definition and the sign of $\vartheta'(t)$. □

Proposition 10. *[Continuity and Differentiability] The function $\vartheta^+(t)$ is continuous and differentiable everywhere, including at $t=a$.*

Proof. Continuity at a:

$$\lim_{t\to a^-}\vartheta^+(t)=2\,\vartheta(a)-\vartheta(a)=\vartheta(a)=\lim_{t\to a^+}\vartheta^+(t)=\vartheta^+(a) \tag{15}$$

Differentiability at a:

$$\lim_{t\to a^-}\frac{d}{dt}\vartheta^+(t)=-\vartheta'(a)=0=\lim_{t\to a^+}\frac{d}{dt}\vartheta^+(t)$$ □

3 Phase Information Preservation

Definition 11. *[Phase Representation] On the critical line,*

$$\zeta\left(\frac{1}{2}+i\,t\right)=e^{-i\vartheta(t)}\;Z(t) \tag{16}$$

where $Z(t)$ is real-valued (the Hardy Z-function).

Theorem 12. *[Phase Preservation] Define*

$$\tilde{Z}(t)=e^{i\vartheta^+(t)}\;\zeta\left(\frac{1}{2}+i\,t\right) \tag{17}$$

Then

$$\tilde{Z}(t)=\begin{cases}e^{2i\vartheta(a)}\;Z(t), & t\in[0,a],\\ Z(t), & t>a.\end{cases} \tag{18}$$

Proof. For $t > a$, $\vartheta^+(t) = \vartheta(t)$, so $\tilde{Z}(t) = Z(t)$. For $t \in [0, a]$,

$$\tilde{Z}(t) = e^{i(2\vartheta(a) - \vartheta(t))}\,\zeta\left(\tfrac{1}{2} + i\,t\right) = e^{2i\vartheta(a)}\ e^{-i\vartheta(t)}\,\zeta\left(\tfrac{1}{2} + i\,t\right) = e^{2i\vartheta(a)}\ Z(t)$$ □

Corollary 13. *[Zero Preservation] The zeros of* $\zeta\left(\frac{1}{2} + i\,t\right)$ *correspond exactly to the zeros of both* $Z(t)$ *and* $\tilde{Z}(t)$ *for* $t > 0$.

Proof. Multiplication by nonzero phase factors preserves zeros. □

Proposition 14. *[Bijectivity] The function* $\vartheta^+(t) \colon [0, \infty) \to [\vartheta^+(0), \infty)$ *is bijective.*

Proof. Injectivity: By Theorem 9, ϑ^+ is strictly increasing except at $t = a$ where the derivative is 0 but the function increases through a; hence injective. Surjectivity: ϑ^+ is continuous and increases without bound as $t \to \infty$ (matching ϑ asymptotically), so by the Intermediate Value Theorem its range is $[\vartheta^+(0), \infty)$. □

Theorem 15. *[Modulating Function Criteria] The function* $\vartheta^+(t)$ *satisfies:*

1. *Piecewise* C^1 *with piecewise continuous first derivative and matching at* $t = a$,
2. *Monotonically nondecreasing with* $\frac{d}{d\,t}\vartheta^+(t) \geq 0$ *and equality only at* $t = a$,
3. *Bijective with* $\lim_{t \to \infty} \vartheta^+(t) = \infty$.

Proof. Immediate from Propositions 10, 14 and Theorem 9. □

4 Conclusion

The monotonized Riemann–Siegel theta function $\vartheta^+(t)$ constructed through geometric reflection about its unique critical point provides a bijective, monotonically increasing transformation that preserves all essential phase information of the original theta function. This construction maintains exact correspondence with zeros of the Riemann zeta function while enabling applications requiring monotonic phase functions.

Bibliography

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