Dear Professors Roger and Pol,

Please allow me to introduce myself. I read your book, "Stochastic Finite Elements: The Spectral Approach," back in 2006 and found it fantastic. I revisited it last year while trying to understand a system where I calculated empirically the quantized variance structure function (the variogram) of a function. This was part of testing my code for variance structure calculation in a project called arb4j, a Java interface for the arbitrary precision ball arithmetic library, now part of flintlib.org.

I started this project a few years ago while solving option pricing equations and discovered that floating-point precision errors and the limited precision integration of discretized equations in the time domain were inadequate for practical purposes. Not only was it slow, but it was also very unstable and failed to converge in many instances.

I was working on an implementation of a stochastic process parameter estimator, which involved the empirical calculation of the variance structure function to estimate the roughness exponent (essentially, the fractal dimension). I didn't have any market data, so I discretized a well-known continuous function and was surprised to find the Bessel function of the first kind of order 0 there, manifesting itself as the kernel of the Gaussian process that had generated it, I surmised after the initial disbelief wore off. The variance structure function, directly related simply to the covariance function of a Gaussian process that I had empirically calculated, was stable as the quantization length decreased to 0, and was clearly a scalar multiple of J_0, whose first root is approximately 2.4048.

I set about trying to find the eigenfunctions of

$$\int_0^\infty J_0(x-y)\, f(x)\, \mathrm{d}x \tag{1}$$

and was shocked to find no such results available despite consulting every conceivable reference on the topic of integral transforms. After consulting Yosida's and Riesz-Nagy's functional analysis texts and finding almost all theorems were about situations where the kernel itself was square-integrable and the domain was finite and bounded, neither of which is true here, I read Aronszajn's theorem that if the kernel is positive definite then a unique RKHS for it exists. I naively believed that the RKHS meant a specific sequence of orthogonal polynomials or quasi-polynomial rational functions, and I did not understand at the time that the RKHS is the space of functions capable of being represented by projections onto the integral operator having this kernel as its kernel.

So, I took the existence theorem to mean that one had to exist, of course, and was puzzled as to why no solution had been published or was known. I had an epiphany/hunch and came up with a technique that should solve any (positive definite, of course) stationary covariance kernel (irrational, non-square-integrable is no problem). It goes like this:

1. Identify the orthogonal polynomial sequence associated with the spectral density of the kernel $K = J_0$, which in this case is given by

$$S(\omega) = \int_0^\infty J_0(x)\, e^{ix\omega}\, \mathrm{d}x = \begin{cases} \dfrac{1}{\sqrt{1-\omega^2}} & \omega \in (-1, 1) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

so that we identify the spectral density with the Chebyshev polynomials of the first kind, since their orthogonality measure is, in fact, equal to the spectral density in the case $K(t,s)=J_0(t-s)$. The Chebyshev polynomials' orthogonality relation is

$$\int_{-1}^{1} T_n(\omega)\,T_m(\omega)\,S(\omega)\,\mathrm{d}\omega = \begin{cases} 0 & n\neq m \\ \pi & n=m=0 \\ \frac{\pi}{2} & n=m\neq 0 \end{cases} \tag{3}$$

I had a hunch that if we take the (finite) Fourier transform of the Chebyshev type-I polynomials[1] (which is just the usual infinite Fourier transform with the integration restricted to the range $-1\ldots 1$ since $T_n(x)=0\forall x\notin[-1,1]$)

$$\hat{T}_n(y)=\int_{-1}^{1} T_n(x)\,e^{ixy}\,\mathrm{d}x \tag{4}$$

and orthogonalize them so that our eigenfunctions are their orthogonal complement (via the Gram-Schmidt process)

$$\psi_n(y)=\hat{T}_n^{\perp}(y)=\hat{T}_n(y)-\sum_{m=1}^{n-1}\frac{\langle\hat{T}_n(y),\hat{T}_m^{\perp}(y)\rangle}{\langle\hat{T}_m^{\perp}(y),\hat{T}_n^{\perp}(y)\rangle}\,\hat{T}_n^{\perp}(y) \tag{5}$$

with respect to the unweighted standard Lebesgue inner product measure over 0 to ∞, it would be related to the solution. I was pleasantly surprised to notice that indeed the partial sums of the orthogonalized Fourier transforms of T_n are in fact proportional to the Fourier transforms of the Legendre polynomials over the same interval and are actually eigenfunctions of the given integral covariance operator that the eigenvalues are given by

$$\lambda_n=\int_{-\infty}^{\infty} J_0(x)\,\psi_n(x)\,\mathrm{d}x=\sqrt{\frac{4n+1}{\pi}}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2}=\sqrt{\frac{4n+1}{\pi}}\,(n+1)^2_{-\frac{1}{2}} \tag{6}$$

where $(n+1)^2_{-\frac{1}{2}}$ is the Pochhammer symbol aka rising factorial. There is actually a more involved formula by letting m vary rather than fixed at 0 but my primary aim is to represent sample paths of the J_0 process presently. The eigenfunctions are given by

$$\begin{aligned}\psi_n(y) &=(-1)^n\sqrt{\frac{4n+1}{\pi}}\,j_{2n}(y)\\ &=(-1)^n\sqrt{\frac{4n+1}{\pi}}\sqrt{\frac{\pi}{2y}}\,J_{2n+\frac{1}{2}}(y)\\ &=(-1)^n\sqrt{\frac{4n+1}{2y}}\,J_{2n+\frac{1}{2}}(y)\\ &=\sqrt{\frac{4n+1}{4\pi}}\int_{-1}^{1}P_{2n}(x)\,e^{ixy}\,\mathrm{d}x\end{aligned} \tag{7}$$

where $j_n(x)$ is the spherical Bessel function of the first kind

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x) = \frac{1}{\sqrt{x}} \left(\sin(z) R_{n,\frac{1}{2}}(z) - \cos(z) R_{n,\frac{3}{2}}(z) \right) \tag{8}$$

and where $R_{n,\nu}(z)$ are the (misnamed) Lommel polynomials[2]

$$R_{n,\nu}(z) = \frac{\Gamma(n+\nu)}{\Gamma(\nu)} \left(\frac{z}{2}\right)_2^{-n} F_3\left([-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}]; [\nu, -n, -\nu+1-n]; -z^2 \right) \tag{9}$$

which are actually rational functions of z, not polynomial. It is the modified Lommel polynomials which are actually polynomials defined by

$$h_{n,\nu}(z) = R_{n,\nu}\left(\frac{1}{z}\right) \tag{10}$$

All this leads to the uniformly convergent eigenfunction expansion

$$J_0(x) = \sum_{n=0}^{\infty} \lambda_n \psi_n(x) = \cdots. \tag{11}$$

whose finite version is

$$K_n(x,y) = \sum_{k=0}^{n} \lambda_k \psi_k(x-y) \tag{12}$$

such that

$$\lim_{n\to\infty} K_n(x,y) = J_0(x-y) \tag{13}$$

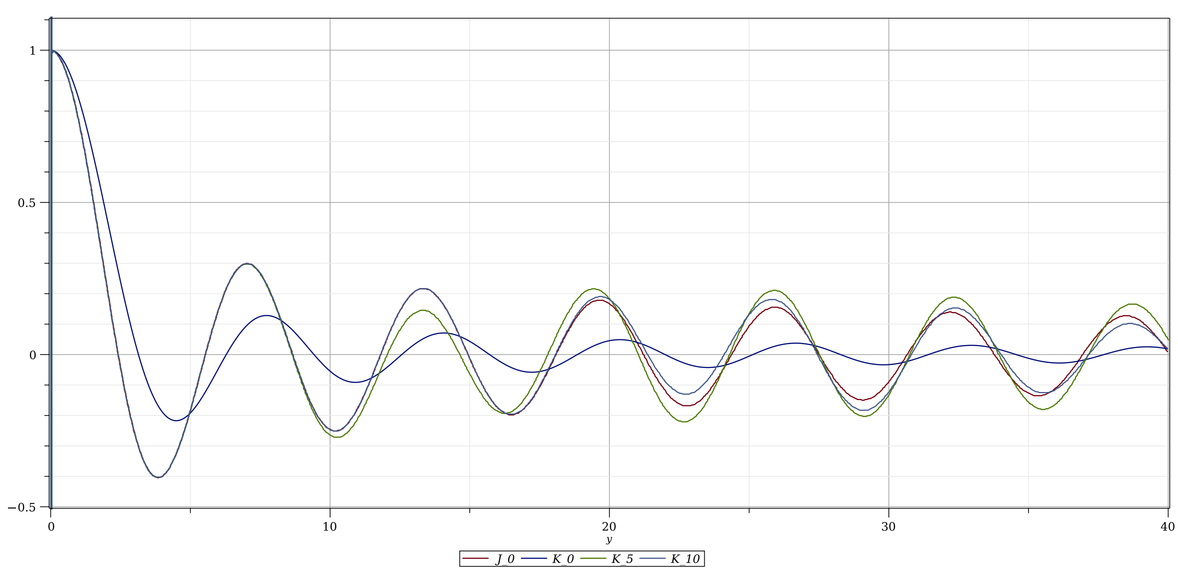

Figure 1. Demonstrating of convergence of $K_n(h)$ to $J_0(h)$ for $n = 0, 5, 10$

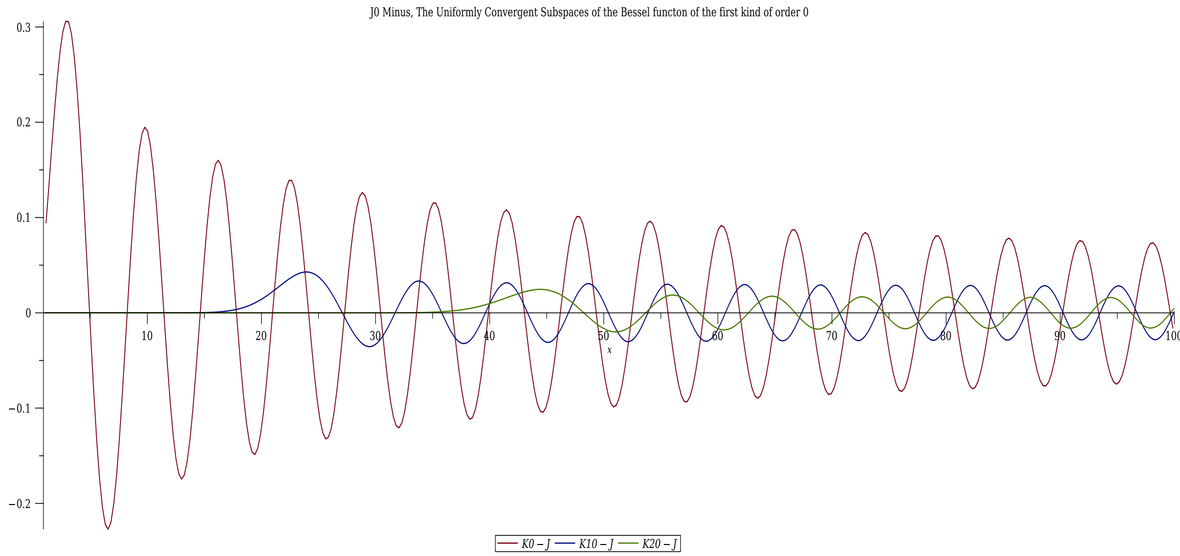

Figure 2. Demonstrating of approximation error between $K_n(h)$ and $J_0(h)$ for $n=0,5,10$

The question is, how to prove it? I am under the impression that this is a noteworthy result and would like to get it published. Hence, I am reaching out to the most esteemed experts on the subject I can think of, given that you wrote the book that helped me understand this method. It's an application of the Galerkin procedure described in the first part of "Stochastic Finite Elements," except instead of just choosing a piecewise or other polynomial basis for the expansion, you apply the orthogonalization procedure to the Fourier transform of the orthogonal polynomials corresponding to the spectral density, and thus the mass and stiffness matrices become the identity, and $c_n = \lambda_n$ becomes an identity.

I do not hold a degree; I'm just a guy who started programming computers when he was about 5 or 6 years old and ran into this interesting solution. There is a bigger picture that this fits into as well that I would love to talk to y'all about. Please do let me know if you could help me out. I think this method of orthogonalization combined with Fourier transform applies to any positive definite stationary kernel, and perhaps it can be extended to harmonizable processes? I have a non-stationary modification of $\int_0^\infty J_0(t-s) f(t) \, dt$ which is particularly interesting, but I figured it was best not to go into that right now until the stationary case is completely squared away.

Sincerely,

Stephen Crowley

Bibliography

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[2] R. Wong K.F. Lee. Asymptotic expansion of the modified lommel polynomials $h_{n,\nu}(x)$ and their zeros. *Proceedings of the American Mathematical Society*, 142(11):3953–3964, 2014.