Uniform Convergence of an Eigenfunction Expansion for J_0

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Theorem 1

The eigenfunctions and eigenvalues of the stationary integral covariance operator

$$[T\psi_n](x) = \int_0^\infty J_0(x-y)\,\psi_n(x)\mathrm{d}x = \lambda_n \psi_n(x) \tag{1}$$

are given by

$$\psi_n(y) = (-1)^n \sqrt{\frac{4n+1}{\pi}}\, j_{2n}(y) \tag{2}$$

and

$$\begin{aligned}\lambda_n &= \int_{-\infty}^\infty J_0(x)\,\psi_n(x)\,\mathrm{d}x\\ &= \sqrt{\frac{4n+1}{\pi}}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2}\\ &= \sqrt{\frac{4n+1}{\pi}}\,(n+1)_{\frac{1}{2}}^2\end{aligned} \tag{3}$$

Proof. 1. Identifying the orthogonal polynomial sequence associated with the spectral density of the kernel K, which in the case where $K = J_0$ is given by

$$S(\omega) = \int_0^\infty J_0(x)\, e^{ix\omega}\,\mathrm{d}x = \begin{cases}\dfrac{1}{\sqrt{1-\omega^2}} & \omega \in (-1,1)\\ 0 & \text{otherwise}\end{cases} \tag{4}$$

which is equal to the spectral density of the Gaussian process having the kernel $K(t, s) = J_0(t-s)$. Recalling the Chebyshev polynomials' orthogonality relation:

$$\int_{-1}^1 T_n(\omega)\,T_m(\omega)\,S(\omega)\,\mathrm{d}\omega = \begin{cases}0 & n \neq m\\ \pi & n = m = 0\\ \dfrac{\pi}{2} & n = m \neq 0\end{cases} \tag{5}$$

Calculate their (finite) Fourier transforms of the Chebyshev type-I polynomials(which is just the usual infinite Fourier transform with the integration restricted to the range $-1\ldots1$ since $T_n(x)=0\forall x\notin[-1,1]$) or equivalent the spectral density extended to take the value 0 outside $[-1,1]$

$$\begin{aligned}\hat{T}_n(y) &= \int_{-\infty}^{\infty} e^{-ixy}\,T_n(x)d\,y = \int_{-1}^{1} e^{-ixy}\,T_n(x)d\,x\\ &= \int_{-\infty}^{\infty} e^{-ixy}\,{}_2F_1\!\left(\left.\begin{matrix} n, & -n\\ & \frac{1}{2}\end{matrix}\,\right|\frac{1}{2}-\frac{x}{2}\right)d\,x\\ &= \frac{i}{y}\,(e^{-iy}F_n^+(y) - e^{i(\pi n+y)}F_n^-(y))\end{aligned} \tag{6}$$

where

$$F_n^{\pm}(y) = {}_3F_1\!\left(\left.\begin{matrix} 1, & n, & -n\\ & & \frac{1}{2}\end{matrix}\,\right|\frac{\pm i\,y}{2}\right) \tag{7}$$

Then use L^2 norm of $\hat{T}_n(y)$

$$|\hat{T}_n| = \sqrt{\int_{-\infty}^{\infty}\hat{T}_n(y)^2\mathrm{d}y} = \sqrt{\frac{4(-1)^n\pi-(2n^2-1)}{4n^2-1}} \tag{8}$$

to define the normalized Fourier transforms[1] $Y_n(y)$ of $T_n(x)$ by

$$\begin{aligned}Y_n(y) &= \frac{\hat{T}_n(y)}{|\hat{T}_n|}\\ &= \frac{i}{y}\left(\frac{e^{-iy}F_n^+(y) - e^{i(\pi n+y)}F_n^-(y)}{\sqrt{\frac{4(-1)^n\pi-(2n^2-1)}{4n^2-1}}}\right)\end{aligned} \tag{9}$$

then orthogonalize them so that our eigenfunctions are recognized as the the orthogonal complement of the normalized Fourier transformed $Y_n(y)$ of the Type-1 Chebshev polynomials $T_n(x)$ (via the Gram-Schmidt process)

$$\psi_n(y) = Y_n^{\perp}(y) = Y_n(y) - \sum_{m=1}^{n-1}\frac{\langle Y_n(y), Y_m^{\perp}(y)\rangle}{\langle Y_m^{\perp}(y), Y_m^{\perp}(y)\rangle}\,Y_m^{\perp}(y) \tag{10}$$

with respect to the unweighted standard Lebesgue inner product measure over 0 to ∞<

$$\lambda_n = \int_{-\infty}^{\infty} J_0(x)\,\psi_n(x)\,\mathrm{d}x = \sqrt{\frac{4\,n+1}{\pi}}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\,(n+1)^2} = \sqrt{\frac{4\,n+1}{\pi}}\,(n+1)^2_{-\frac{1}{2}} \tag{11}$$

where $(n+1)^2_{-\frac{1}{2}}$ is the Pochhammer symbol aka rising factorial. The eigenfunctions can be equivalently expressed as

$$\begin{aligned}\psi_n(y) &= (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, j_{2n}(y)\\ &= (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\,\sqrt{\frac{\pi}{2\,y}}\, J_{2n+\frac{1}{2}}(y)\\ &= (-1)^n \sqrt{\frac{4\,n+1}{2\,y}}\, J_{2n+\frac{1}{2}}(y)\\ &= \sqrt{\frac{4\,n+1}{4\,\pi}} \int_{-1}^{1} P_{2n}(x)\, e^{ixy}\, \mathrm{d}x\end{aligned} \tag{12}$$

where $P_n(x)$ is the Legendre polynomials, $j_n(x)$ is the spherical Bessel function of the first kind,

$$j_n(x) = \sqrt{\frac{\pi}{2\,x}}\, J_{n+\frac{1}{2}}(x) = \frac{1}{\sqrt{x}}\left(\sin(z)\, R_{n,\frac{1}{2}}(z) - \cos(z)\, R_{n,\frac{3}{2}}(z)\right) \tag{13}$$

and where $R_{n,v}(z)$ are the (misnamed) Lommel polynomials[2]

$$R_{n,v}(z) = \frac{\Gamma(n+v)}{\Gamma(v)} \left(\frac{z}{2}\right)^{-n}_{2} F_3\left([-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}]; [v, -n, -v+1-n]; -z^2\right) \tag{14}$$

which are actually rational functions of z, not polynomial but rather "polynomial in $1/z$". □

Theorem 2

The eigenfunction expansion converges uniformly on $(0, \infty)$ to the covariance kernel function

$$\begin{aligned}J_0(t) &= \sum_{k=0}^{\infty} \lambda_k \psi_k(t)\\ &= \sum_{k=0}^{\infty} \sqrt{\frac{4\,n+1}{\pi}}\, \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, j_{2n}(t)\\ &= \sum_{k=0}^{\infty} \frac{4\,n+1}{\pi}\, \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} (-1)^n\, j_{2n}(t)\end{aligned} \tag{15}$$

Proof. Since T is compact due to its self-adjointness and convergence of the eigenvalues to 0 it converges uniformly since compactness implies uniform convergence of the eigenfunctions. TODO: reproduct relevant theorems from [3, 3. Reproducing Kernel Hilbert Space of a Gaussian Process]Introduction to Gaussian Processes □

Bibliography

[1] Atul Dixit, Lin Jiu, Victor H Moll, and Christophe Vignat. The finite fourier transform of classical polynomials. *Journal of the Australian Mathematical Society*, 98:145–160, 2015.

[2] R. Wong K.F. Lee. Asymptotic expansion of the modified lommel polynomials $h_{n,\nu}(x)$ and their zeros. *Proceedings of the American Mathematical Society*, 142(11):3953–3964, 2014.

[3] Steven P. Lalley. Introduction to gaussian processes. https://galton.uchicago.edu/ lalley/Courses/386/GaussianProcesses.pdf, 2013.