THE FINITE FOURIER TRANSFORM OF CLASSICAL POLYNOMIALS

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Abstract. The finite Fourier transform of a family of orthogonal polynomials $A_n(x)$, is the usual transform of the polynomial extended by 0 outside their natural domain. Explicit expressions are given for the Legendre, Jacobi, Gegenbauer and Chebyshev families.

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1. Introduction

Compendia of formulas, such as the classical *Table of Integrals, Series and Products* by I. S. Gradshteyn and I. M. Ryzhik [gradshteyn-2007a] and the recent *NIST Handbook of Mathematical Functions* [olver-2010a] do not contain a systematic collection of Fourier transforms of orthogonal polynomials.

Special cases do appear. For instance, [olver-2010a, formula 18.17.19] contains the identity

$$\int_{-1}^{1} P_n(x)\, e^{\imath \lambda x}\, d\, x = \imath^n \sqrt{\frac{2\,\pi}{\lambda}}\, J_{n+\frac{1}{2}}(\lambda), \tag{1}$$

for the *finite Fourier transform* of the Legendre polynomial P_n. Here J_α is the Bessel function defined by

$$J_\alpha(\lambda) = \sum_{k=0}^{\infty} \frac{(-1)^k\, (\lambda/2)^{2k+\alpha}}{k!\Gamma\,(k+\alpha+1)}. \tag{2}$$

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A second example is [erderly-1954a, formula 3.3(7), page 123]

$$\int_{-1}^{1} P_\nu(x)\, e^{\imath\lambda x}\, d\,x = \frac{2\,\pi \sin \pi\,\nu}{\nu\,(\nu+1)}\, e_2^{-\imath\lambda} F_2\left(\begin{matrix} 1, 1 \\ -\nu, 2+\nu \end{matrix} \middle| \, 2\,\imath\lambda \right). \tag{3}$$

The more natural situation, where the corresponding kernel appears in the integrand, is included in the tables. For instance, for the Jacobi polynomial, [olver-2010a, 18.17.16] gives

$$\int_{-1}^{1} (1 - x)^{\alpha}\ (1 + x)^{\beta}\ P_n^{(\alpha,\beta)}(x)\ e^{\imath\lambda x}\ d\ x = X_n(\lambda;\ \alpha, \beta)_1 F_1\left(\begin{matrix} n+\alpha+1 \\ 2\,n+\alpha+\beta+2 \end{matrix} \middle| -2\,\imath\lambda \right), \tag{4}$$

with

$$X_n(\lambda;\alpha,\beta) = \frac{(\imath\lambda)^n\, e^{\imath\lambda}}{n!}\, 2^{n+\alpha+\beta+1} \times B(n+\alpha+1, n+\beta+1) \tag{5}$$

The work presented here was stimulated by results of A. Fokas et al. [fokas-2014a]. A second motivation was the fact that the authors were unable to find the finite Fourier transform of classical orthogonal polynomials readily available in the literature. These results were also developed in [fokas-2014a] and some of them appear in [greenen-2008a]. The authors wish to thank A. Fokas and T. Koorwinder for correspondence on the questions discussed here.

The goal of this project is to produce closed-form evaluations of definite integrals of the form

$$\hat{P}(\lambda) := \int_a^b e^{\imath\lambda x}\, P(x)\ d\,x \tag{6}$$

for a variety of polynomials P, orthogonal on the interval $[a,b]$. The function $\hat{P}(\lambda)$ is called the *finite Fourier transform* of the polynomial P. The case considered here includes the Legendre polynomial $P_n(x)$, the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, from which the Gegenbauer polynomials $C_n^{(\nu)}(x)$ and both types of Chebyshev polynomials $T_n(x)$ and $U_n(x)$ are derived.

Naturally, depending on the representation given of the polynomial P, it is possible to obtain a variety of expressions for $\hat{P}$. For instance, if an expression for the coefficients of P is available, the identity in Lemma 1 and a simple scaling give directly a double-sum representation for $\hat{P}(\lambda)$.

It is convenient to introduce the notation

$$E_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!} \tag{7}$$

for the partial sums of the exponential function. Many of the results may be expressed in terms of E_n. The result is elementary and it appears in [gradshteyn-2007a, formula 2.323].

Lemma 1. *Let $k \geq 0$ be an integer and λ an indeterminate. Then,*

$$\int_{-1}^{1} x^k e^{\imath\lambda x}\, d\,x = \frac{(-1)^k\, k!}{(\imath\lambda)^{k+1}}\, [e^{\imath\lambda}\, E_k\,(-\imath\lambda) - e^{-\imath\lambda}\, E_k(\imath\lambda)], \tag{8}$$

and

$$\int_{0}^{1} x^k e^{\imath\lambda x}\, d\,x = \frac{(-1)^k\, k!}{(\imath\lambda)^{k+1}}\, [e^{\imath\lambda}\, E_k\,(-\imath\lambda) - 1]. \tag{9}$$

Proof. Integrate by parts. □

Note 2. *The notation is standard. The symbol $(a)_n$ denotes the shifted factorial, defined by $(a)_n = a\,(a+1)\cdots(a+n-1)$ and $(a)_0 = 1$. The elementary properties*

$$\begin{aligned}
(1)_n &= n! && (10)\\
(a)_n &= \frac{\Gamma\,(a+n)}{\Gamma(a)} && (11)\\
(a+\tfrac{1}{2})_n &= \frac{(2\,a)_{2n}}{2^{2n}(a)_n} && (12)\\
(-n)_k &= \frac{(-1)^k\, n!}{(n-k)!} \qquad \textit{for } n,\ k \in \mathbb{N} && (13)\\
(n+1)_k &= \frac{(n+k)!}{n!} \qquad \textit{for } n,\ k \in \mathbb{N}, && (14)\\
(-a)_n &= (-1)^n\,(a-n+1)_n, && (15)
\end{aligned}$$

are used throughout.

2. Legendre polynomials

This section contains a variety of formulas for the finite Fourier transform of the Legendre polynomials $P_n(x)$. These are orthogonal polynomials on the interval $[-1,1]$, with weight $w(x) \equiv 1$. The next theorem gives all the results.

Theorem 3. *The finite Fourier transform of the Legendre polynomial $P_n(x)$ is given by one of the four equivalent forms:*

$$\begin{aligned}
\widehat{P}_n(\lambda) &= 2^n \sum_{k=0}^{n} \binom{n}{k} \binom{\frac{1}{2}(n+k-1)}{n} \frac{(-1)^k\, k!}{(\imath\lambda)^{k+1}}\, [e^{\imath\lambda}\, E_k\,(-\imath\lambda) - e^{-\imath\lambda}\, E_k(\imath\lambda)]\\
&= \imath^n \sqrt{\frac{2\,\pi}{\lambda}}\, J_{n+1/2}(\lambda)\\
&= 2 \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!\, k!} \frac{[e^{-\imath\lambda}\, E_k\,(2\,\imath\lambda) - e^{\imath\lambda}]}{(-2\,\imath\,\lambda)^{k+1}}\\
&= 2 \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!\, k!} \frac{[(-1)^{n+k}\, e^{-\imath\lambda} - e^{\imath\lambda}]}{(-2\,\imath\lambda)^{k+1}}.
\end{aligned}$$

Proof. The first formula follows from the explicit representation

$$P_n(x) = 2^n \sum_{k=0}^{n} \binom{n}{k} \binom{\frac{1}{2}(n+k-1)}{n} x^k \tag{16}$$

given in ?? and Lemma 1. The second expression for $\hat{P}_n(\lambda)$ comes from their Rodrigues formula

$$P_n(x) = \frac{1}{2^n\, n!}\left(\frac{d}{d\,x}\right)^n (x^2-1)^n, \tag{17}$$

(see [gradshteyn-2007a, Formula 8.910.2]) and it appears as entry 7.242.5 in [gradshteyn-2007a]. Then

$$\hat{P}_n(\lambda) = \frac{1}{2^n\, n!}\int_{-1}^{1} e^{\imath\lambda x}\left(\frac{d}{d\,x}\right)^n (x^2-1)^n \; d\,x \tag{18}$$

and integrating by parts n-times yields

$$\hat{P}_n(\lambda) = \frac{(-\imath\,\lambda)^n}{2^n\, n!}\int_{-1}^{1}(x^2-1)^n\, e^{\imath\lambda x}\; d\,x. \tag{19}$$

Entry 3.387.2 of [gradshteyn-2007a] states that

$$\int_{-1}^{1}(1-x^2)^{\nu-1}\, e^{\imath\mu x}\; d\,x = \sqrt{\pi}\left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}}\Gamma(\nu)\, J_{\nu-\frac{1}{2}}(\mu). \tag{20}$$

The result is obtained by choosing $\mu=\lambda$ and $\nu=n+1$.

The third form of the finite Fourier transform of the Legendre polynomials is obtained from their hypergeometric representation

$$P_n(x) = {}_2F_1\left(\begin{matrix}-n & n+1\\ & 1\end{matrix}\,\middle|\,\frac{1-x}{2}\right) = \sum_{k=0}^{n}\frac{(-n)_k\,(n+1)_k}{(1)_k\; k!}\left(\frac{1-x}{2}\right)^k, \tag{21}$$

that gives

$$\hat{P}_n(\lambda) = \sum_{k=0}^{n}\frac{(-n)_k\,(n+1)_k}{k!^2}\int_{-1}^{1} e^{\imath\lambda x}\left(\frac{1-x}{2}\right)^k\; d\,x. \tag{22}$$

A change of variables and the formulas (14) and (15) give

$$\hat{P}_n(\lambda) = 2\, e^{\imath\lambda}\sum_{k=0}^{n}\frac{(-1)^k\,(n+k)!}{(n-k)!k!^2}\int_0^1 t^k\, e^{-2\imath\lambda t}\; d\,t. \tag{23}$$

Lemma 1 now gives the stated result.

To produce the last form for $\hat{P}_n(\lambda)$, let $t=2\,\imath\,\lambda$ in the third expression for this transform. Then, after multiplication by t^n and some simplification, the claim is equivalent to the polynomial identity

$$\sum_{k=0}^{n}\frac{(2\,n-k)!}{k!(n-k)!}(-1)^k\, t^k\sum_{j=0}^{n-k}\frac{t^j}{j!} = \sum_{k=0}^{n}\frac{(2\,n-k)!}{k!(n-k)!}\, t^k. \tag{24}$$

To simplify the sum, let $\nu=k+j$ on the left-hand side to show that the desired identity is equivalent to

$$\sum_{\nu=0}^{n}\left[\sum_{k=0}^{\nu}\frac{(-1)^k\,(2\,n-k)!}{k!\,(n-k)!\,(\nu-k)!}\right]t^{\nu} = \sum_{k=0}^{n}\frac{(2\,n-k)!}{k!\,(n-k)!}\, t^k. \tag{25}$$

Matching coefficients, the result follows from

$$\sum_{j=0}^{k} \frac{(-1)^j\,(2\,n-j)!}{j!\,(n-j)!\,(k-j)!} = \frac{(2\,n-k)!}{k!\,(n-k)!} \tag{26}$$

for every $0 \leq k \leq n$. This is equivalent to the binomial identity given in Lemma 4 below. The proof is complete. $\square$

Lemma 4. *For $n \in \mathbb{N}$ and $0 \leq k \leq n$*

$$\sum_{j=0}^{k} (-1)^j \binom{n}{j} \binom{2\,n-j}{2\,n-k} = \binom{n}{k}. \tag{27}$$

Proof. The proof uses $\binom{r}{k} = (-1)^k \binom{k-r-1}{k}$ to write

$$\binom{2\,n-j}{2\,n-k} = \binom{2\,n-j}{k-j} = (-1)^{k-j} \binom{k-2\,n-1}{k-j} \tag{28}$$

and then (27) is converted into Vandermonde identity

$$\sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} = \binom{a+b}{n}. \tag{29}$$ $\square$

3. Jacobi polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{\alpha+n}{k} \binom{\beta+n}{n-k} (x-1)^{n-k}\,(x+1)^k \tag{30}$$

are orthogonal on $[-1,1]$ with respect to the weight

$$w(x) = (1-x)^{\alpha}\,(1+x)^{\beta}. \tag{31}$$

This section contains expressions for their finite Fourier transform. The hypergeometric representation

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, & n+\alpha+\beta+1 \\ & \alpha+1 \end{matrix} \,\middle|\, \frac{1-x}{2} \right), \tag{32}$$

is used in the calculations.

Theorem 5. *The finite Fourier transform of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ is given by*

$$\begin{aligned} \widehat{P_n^{(\alpha,\beta)}}(\lambda) &= 2\,e^{\imath\lambda}\,(\alpha+1)_n \sum_{k=0}^{n} \frac{(n+\alpha+\beta+1)_k}{(n-k)!(\alpha+1)_k} \left[\frac{e^{-2\imath\lambda}\,E_k\,(2\,\imath\lambda) - 1}{(-2\,\imath\lambda)^{k+1}} \right] \\ &= 2 \sum_{k=0}^{n} \frac{(n+\alpha+\beta+1)_k}{(-2\,\imath\lambda)^{k+1}\,(n-k)!} \\ &\quad \times [(-1)^{n-k}\,e^{-\imath\lambda}\,(\beta+k+1)_{n-k} - e^{\imath\lambda}\,(\alpha+k+1)_{n-k}], \end{aligned}$$

for $\lambda \neq 0$. *For* $\lambda = 0$,

$$\widehat{P_n^{(\alpha,\beta)}}(0) = \frac{(n+\alpha+\beta+1)}{2}\left[\binom{\alpha+n}{n-1} - (-1)^{n-1}\binom{\beta+n}{n-1}\right]. \tag{33}$$

Proof. The first statement comes from the hypergeometric form

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, & n+\alpha+\beta+1\\ & \alpha+1\end{matrix}\,\middle|\,\frac{1-x}{2}\right) \\ &= \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k\,(n+\alpha+\beta+1)_k}{(\alpha+1)_k\,k!2^k}\,(1-x)^k \end{aligned} \tag{34}$$

and use Lemma 1 to produce

$$\int_{-1}^{1}(1-x)^k\,e^{\imath\lambda x}\,dx = -e^{\imath\lambda}\,\frac{k!}{(\imath\lambda)^{k+1}}\,[e^{-2\imath\lambda}\,E_k\,(2\,\imath\lambda)-1] \tag{35}$$

and then $(-n)_k = (-1)^k\,n!/(n-k)!$ to simplify the result.

Now use identity (the case $m=1$ of [gradshteyn-2007a, 8.961.4]:

$$\frac{d}{dx}\,P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2}\,P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{36}$$

and integrate by parts to obtain

$$\widehat{P_n^{(\alpha,\beta)}}(\lambda) = \frac{e^{\imath\lambda x}}{\imath\lambda}\,P_n^{(\alpha,\beta)}(x)|_{-1}^{1} - \frac{(n+\alpha+\beta+1)}{2\,\imath\lambda}\widehat{P_{n-1}^{(\alpha+1,\beta+1)}}(\lambda).$$

Introduce the notation for the boundary term

$$a_n^{(\alpha,\beta)} = \frac{e^{\imath\lambda x}}{\imath\lambda}\,P_n^{(\alpha,\beta)}(x)|_{-1}^{1}. \tag{37}$$

to write the previous computation as the recurrence

$$\widehat{P_n^{(\alpha,\beta)}}(\lambda) = a_n^{(\alpha,\beta)}(\lambda) - \frac{(n+\alpha+\beta+1)}{2\,\imath\lambda}\widehat{P_{n-1}^{(\alpha+1,\beta+1)}}(\lambda). \tag{38}$$

Iteration yields

$$\begin{aligned} \widehat{P_n^{(\alpha,\beta)}}(\lambda) &= \sum_{k=1}^{n}\,(-1)^{n-k}\frac{(n+\alpha+\beta+1)_{n-k}}{(2\,\imath\lambda)^{n-k}}\,a_k^{(\alpha+n-k,\beta+n-k)}(\lambda) \\ &+\ (-1)^n\frac{(n+\alpha+\beta+1)_n}{(2\,\imath\lambda)^n}\widehat{P_0^{(\alpha+n,\beta+n)}}(\lambda). \end{aligned}$$

Evaluate the last term is evaluated as $a_0^{(\alpha,\beta)}(\lambda)$ and use

$$P_n^{(\alpha,\beta)}(1) = \binom{\alpha+n}{n}\text{and } P_n^{(\alpha,\beta)}\,(-1) = (-1)^n\binom{\beta+n}{n} \tag{39}$$

from (30) to obtain

$$a_n^{(\alpha,\beta)} = \frac{1}{\imath\lambda}\left[e^{\imath\lambda}\binom{\alpha+n}{n} - (-1)^n\,e^{-\imath\lambda}\binom{\beta+n}{n}\right]. \tag{40}$$

Some algebraic simplification now gives the stated result. The value for $\lambda=0$ comes directly from (36). ☐

The next statement represents a hypergeometric rewrite of the last formula in Theorem 5.

Theorem 6. *The finite Fourier transform of the Jacobi polynomial is given by*

$$\begin{aligned}\widehat{P_n^{(\alpha,\beta)}}(\lambda) &= \frac{(\beta+1)_n}{\imath\lambda n!}(-1)^{n+1}e^{-\imath\lambda}{}_3F_1\left(\begin{matrix}n+\alpha+\beta+1,-n,1\\ \beta+1\end{matrix}\,\middle|\,\frac{-1}{2\imath\lambda}\right)+\\ &\quad+\frac{(\alpha+1)_n}{\imath\lambda n!}e^{\imath\lambda}{}_3F_1\left(\begin{matrix}n+\alpha+\beta+1,-n,1\\ \alpha+1\end{matrix}\,\middle|\,\frac{1}{2\imath\lambda}\right).\end{aligned}$$

Proof. The first term in the expression the last formula of Theorem 5 is simplified using (14) and $(\beta+k+1)_{n-k}=\frac{(\beta+1)_n}{(\beta+1)_k}$ to obtain

$$\frac{(-1)^{n-k}(n+\alpha+\beta+1)_k(\beta+k+1)_k}{(-2\imath\lambda)^{k+1}(n-k)!}= \frac{(-1)^{n+1}(\beta+1)_n}{2\imath\lambda}\frac{(n+\alpha+\beta+1)_k(-n)_k(1)_k}{(\beta+1)_k}\frac{t^k}{k!}$$

with $t=-1/2\imath\lambda$. Summing from $k=0$ to n gives the first term in the answer. A similar argument simplifies the second term in Theorem 5. ☐

Note 7. *Define*

$$A_n^{(a,b)}(t)=\frac{(a+1)_n}{n!}{}_3F_1\left(\begin{matrix}n+a+b+1,-n,1\\ a+1\end{matrix}\,\middle|\,\frac{1}{t}\right). \tag{41}$$

then the finite Fourier transform of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is given by

$$\widehat{P_n^{(\alpha,\beta)}}(\lambda)=\frac{1}{\imath\lambda}\left[(-1)^{n+1}e^{-\imath\lambda}A_n^{(\beta,\alpha)}(-2\imath\lambda)+e^{\imath\lambda}A_n^{(\alpha,\beta)}(2\imath\lambda)\right]. \tag{42}$$

4. A collection of special examples

This section presents a collection of special cases of the Jacobi polynomials and their respective finite Fourier transforms.

4.1. **Legendre polynomials.**

These polynomials were discussed in Section 3 and correspond to the special case $\alpha=\beta=0$; that is,

$$P_n(x)=P_n^{(0,0)}(x). \tag{43}$$

The first formula in Theorem 5 reproduces the third formula in Theorem 3. Similarly, the second formula in Theorem 5 gives the last expression for the finite Fourier transform of Legendre polynomials in Theorem 3.

4.2. **Gegenbauer polynomials.**

These polynomials are also special cases of $P_n^{(\alpha,\beta)}(x)$:

$$C_n^{(\nu)}(x) = \frac{(2\,\nu)_n}{(\nu+1/2)_n} P_n^{(\nu-1/2,\nu-1/2)}(x). \tag{44}$$

Theorem 8. *The finite Fourier transform of the Gegenbauer polynomial $C_n^{(\nu)}(x)$ is given by*

$$\begin{aligned}\widehat{C_n^{(\nu)}}(\lambda) &= 2\,(2\,\nu)_n\, e^{\imath\lambda} \sum_{k=0}^{n} 2^{2k} \frac{(n+2\,\nu)_k(\nu)_k}{(n-k)!(2\,\nu)_{2k}} \left[\frac{e^{-2\imath\lambda}\, E_k\,(2\,\imath\lambda)-1}{(-2\,\imath\lambda)^{k+1}} \right] \\ &= \frac{2\,(2\,\nu)_n(\nu)_n}{(2\,\nu)_{2n}} \sum_{k=0}^{n} 2^{2k} \frac{(n+2\,\nu)_k\,(2\,\nu+2\,k)_{2n-2k}}{(n-k)!(\nu+k)_{n-k}} \left[\frac{(-1)^{n-k}\, e^{-\imath\lambda} - e^{\imath\lambda}}{(-2\,\imath\lambda)^{k+1}} \right]\end{aligned}$$

and also

$$\begin{aligned}\widehat{C_n^{(\nu)}}(\lambda) = \frac{(2\,\nu)_n}{\imath\lambda n!} \times \Bigg[(-1)^{n+1}\, e^{-\imath\lambda}_3F_1 \left(\left. \begin{matrix} n+2\,\nu, -n, 1 \\ \nu+\frac{1}{2} \end{matrix} \right| -\frac{1}{2\,\imath\lambda} \right) + \\ e^{\imath\lambda}_3F_1 \left(\left. \begin{matrix} n+2\,\nu, -n, 1 \\ \nu+\frac{1}{2} \end{matrix} \right| \frac{1}{2\,\imath\lambda} \right) \Bigg].\end{aligned}$$

4.3. **Chebyshev polynomials.**

The Chebyshev polynomial are related to Gegenbauer polynomials by

$$U_n(x) = C_n^{(1)}(x) \text{ and } T_n(x) = \lim_{\nu\to 0} \frac{n\,C_n^{(\nu)}(x)}{2\,\nu}, \text{for } n \geq 1. \tag{45}$$

These formulas are now used to evaluate the finite Fourier transform of Chebyshev polynomials.

Theorem 9. *The finite Fourier transform of the Chebyshev polynomial is given by*

$$\begin{aligned}\widehat{U_n}(\lambda) &= e^{\imath\lambda} \sum_{k=0}^{n} 2^{2k+1}\, k! \binom{n+k+1}{n-k} \left[\frac{e^{-2\imath\lambda}\, E_k\,(2\,\imath\lambda)-1}{(-2\,\imath\lambda)^{k+1}} \right] \\ &= \sum_{k=0}^{n} \frac{2^{2k+1}\,(n+k+1)!\,k!}{(2\,k+1)!\,(n-k)!} \frac{[(-1)^{n-k}\, e^{-\imath\lambda} - e^{\imath\lambda}]}{(-2\,\imath\lambda)^{k+1}}\end{aligned}$$

and

$$\hat{T}_n(\lambda) = \sum_{k=0}^{n} (-1)^{k+1} \frac{n\,2^k\,(n+k)!k!}{(n-k)!(2\,k)!(n+k)} \frac{[(-1)^{n-k}\, e^{-\imath\lambda} - e^{\imath\lambda}]}{(\imath\lambda)^{k+1}}$$

5. Biorthogonality for the Jacobi polynomials

The sequence of functions $\{\frac{1}{\sqrt{2}} e^{\pi\imath jx}\colon j \in \mathbb{Z}\}$ forms an orthonormal family on the Hilbert space $L^2[-1,1]$. Therefore, every continuous function f defined on $[-1,1]$ may be expanded in the form

$$f(x) = \frac{1}{\sqrt{2}} \sum_{j=-\infty}^{\infty} a_j(f)\, e^{\pi\imath jx}, \tag{46}$$

indent where the Fourier coefficients are given by

$$a_j(f) = \frac{1}{\sqrt{2}} \int_{-1}^{1} f(x)\, e^{-\pi \imath j x}\, dx. \tag{47}$$

Parseval's identity [hardy-1950a, Theorem 14] states that

$$\int_{-1}^{1} f(x) \overline{g(x)}\, dx = \sum_{j=-\infty}^{\infty} a_j(f) \overline{a_j(g)}. \tag{48}$$

This identity is now made explicit for the case

$$f(x) = P_n^{(\alpha,\beta)}(x) \text{ and } g(x) = Q_n^{(\alpha,\beta)}(x) := (1-x)^{\alpha}\, (1+x)^{\beta}\, P_n^{(\alpha,\beta)}(x). \tag{49}$$

The Fourier coefficients $a_j(Q_m^{(\alpha,\beta)}(x))$ are given in (4) and $a_j(P_n^{(\alpha,\beta)}(x))$ have been evaluated in Theorem 5. Parseval's identity and the orthogonality of Jacobi polynomials give

$$\sum_{j=-\infty}^{\infty} a_j(P_n^{(\alpha,\beta)}(x)) \overline{a_j(Q_m^{(\alpha,\beta)}(x))} = \frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\, n!\,\Gamma(n+\alpha+\beta+1)} \delta_{n,m},$$

where $\delta_{n,m}$ is Kronecker's delta (1 if $n=m$ and 0 if $n \neq m$). Only the case $n \neq m$ leads to an interesting relation. A direct calculation shows that $a_0(Q_m^{(\alpha,\beta)}(x) = 0$, so that Parseval's identity is written as

$$\sum_{j \in \mathbb{Z}, j \neq 0} a_j(P_n^{(\alpha,\beta)}(x)) \overline{a_j(Q_m^{(\alpha,\beta)}(x))} = 0, \quad \text{for } n \neq m.$$

To simplify the previous relation, replace $\lambda = -\pi j$ in (4) and use Kummer's identity

$${}_1F_1\left(\left. \begin{matrix} u \\ u+v \end{matrix} \right| z \right) = e^z_1 F_1\left(\left. \begin{matrix} v \\ u+v \end{matrix} \right| -z \right) \tag{50}$$

to obtain

$$\overline{a_j(Q_m^{(\alpha,\beta)})} = \frac{(-1)^j j^m}{m!} 2^{m+\alpha+\beta+1/2} B(m+\alpha+1, m+\beta+1) \\ {}_1F_1\left(\left. \begin{matrix} m+\beta+1 \\ 2m+\alpha+\beta+2 \end{matrix} \right| 2\pi \imath j \right).$$

Similiarly, (5) with $\lambda = -\pi j$ gives

$$a_j(P_n^{(\alpha,\beta)}) = \frac{(-1)^j}{2\pi \imath j n!} \\ \left[(-1)^n (\beta+1)_n\, {}_3F_1\left(\left. \begin{matrix} n+\alpha+\beta+1, -n, 1 \\ \beta+1 \end{matrix} \right| \frac{1}{2\pi \imath j} \right) - \right. \\ \left. (\alpha+1)_n\, {}_3F_1\left(\left. \begin{matrix} n+\alpha+\beta+1, -n, 1 \\ \alpha+1 \end{matrix} \right| -\frac{1}{2\pi \imath j} \right) \right].$$

Parseval's identity now produces the next result.

Theorem 10. *Define*

$$W_{n,m}^{(\alpha,\beta)}(t;\ j) = (\alpha+1)_n\, j_3^{m-1} F_1\left(\begin{matrix} n+\alpha+\beta+1, -n, 1 \\ \alpha+1 \end{matrix} \middle| \frac{1}{t}\right)_1 F_1\left(\begin{matrix} m+\alpha+1 \\ 2m+\alpha+\beta+2 \end{matrix} \middle| t\right).$$

Then

$$(-1)^n \sum_{j\in\mathbb{Z},\, j\neq 0} W_{n,m}^{(\beta,\alpha)}(2\pi\imath j; j) = (-1)^{m-1} \sum_{j\in\mathbb{Z},\, j\neq 0} W_{n,m}^{(\alpha,\beta)}(2\pi\imath j; j). \tag{51}$$

In particular, if n and m have opposite parity, then

$$\sum_{j\in\mathbb{Z},\, j\neq 0} W_{n,m}^{(\beta,\alpha)}(2\pi\imath j; j) = \sum_{j\in\mathbb{Z},\, j\neq 0} W_{n,m}^{(\alpha,\beta)}(2\pi\imath j; j). \tag{52}$$

6. An operator point of view

To obtain the finite Fourier transform of a polynomial start with

$$\int_{-1}^{1} x^k e^{\imath\lambda x}\, dx = (-\imath D)^k (2\text{sinc } \lambda) \tag{53}$$

where the *sinc* function is

$$\text{sinc } \lambda = \frac{\sin\lambda}{\lambda} \tag{54}$$

and $D = \frac{d}{d\lambda}$. The action is extended by linearity to obtain

$$\hat{P}(\lambda) = P(-\imath D)(2\text{sinc } \lambda). \tag{55}$$

For instance, for the Chebyshev polynomial

$$U_n(x) = \sum_{k=0}^{n} (-2)^k \binom{n+k+1}{n-k} (1-x)^k \tag{56}$$

leads to

$$\begin{aligned} \widehat{U_n}(\lambda) &= \sum_{k=0}^{n} (-2)^k \binom{n+k+1}{n-k} (1+\imath D)^k (2\text{sinc } \lambda) \\ &= U_n(-\imath D)(2\text{sinc } \lambda). \end{aligned} \tag{57}$$

It is elementary to check that

$$\left(\frac{d}{d\lambda}\right)^n \text{sinc } \lambda = A_n(\lambda)\sin\lambda + B_n(\lambda)\cos\lambda \tag{58}$$

where A_n, B_n are polynomials in $1/\lambda$ that satisfy the recurrences

$$\begin{aligned} A_{n+1}(\lambda) &= A_n'(\lambda) - B_n(\lambda) \\ B_{n+1}(\lambda) &= A_n(\lambda) + B_n'(\lambda), \end{aligned}$$

with initial values $A_0(\lambda) = 1/\lambda$ and $B_0(\lambda) = 0$. An explicit expression for these polynomials can be obtain from

$$\left(\frac{d}{d\lambda} \right)^n \text{sinc } \lambda = \sum_{j=0}^{n} \frac{n!}{(n-j)!} \frac{\sin\left(\lambda + (n+j)\frac{\pi}{2}\right)}{\lambda^{j+1}}. \tag{59}$$

Details of this approach to finite Fourier transform of orthogonal polynomials will be given elsewhere.

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