A Pair of Orthogonal Polynomial Sequences on $[0,\infty]$ That Uniformly Converge to The Bessel functions of the First Kind of Orders 0 and 1

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Abstract

The even-indexed orthonormalized Fourier transforms of the Chebyshev polynomials of the first kind form a basis in a reproducing-kernel Hilbert space for the Bessel function of the first kind J_0 and likewise for the odd-indexed functions which form a basis that reproduces $\dot{J}_0 = -J_1$. Suprisingly, such a basis for these functions was not known to exist before this.

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1 The Type-I Chebyshev Polynomials $T_n(x)$

Let T_n be the Chebyshev polynomials of the first kind, also said to be of Type-I, defined by

$$\begin{aligned} T_n(x) &= {}_2F_1\left(\begin{matrix} n, & -n \\ & \frac{1}{2} \end{matrix} \middle| \frac{1}{2} - \frac{x}{2} \right) \\ &= \int_{-\infty}^{\infty} e^{ixy} \hat{T}_n(y) d y \\ &= \int_{-\infty}^{\infty} e^{ixy} \frac{i}{y} (\mathrm{e}^{-\mathrm{i}y} F_n^{+}(y) - \mathrm{e}^{\mathrm{i}y} (-1)^n F_n^{-}(y)) d y \\ &= \int_{-\infty}^{\infty} e^{ixy} \int_{-\infty}^{\infty} e^{-iyz} T_n(z) d z d y \end{aligned} \tag{1}$$

where ${}_2F_1$ is the (Gauss) hypergeometric function. [?, (13.140)]

1.1 The Fourier Transforms $\hat{T}_n(y)$ of $T_n(x)$

The functions $\hat{T}_n(y)$ are Fourier transforms of $T_n(x)$ defined by

$$\begin{aligned}\hat{T}_n(y) &= \int_{-\infty}^{\infty} e^{-ixy} T_n(x) d\, y = \int_{-1}^{1} e^{-ixy} T_n(x) d\, x \\ &= \int_{-\infty}^{\infty} e^{-ixy}\, {}_2F_1\left(\begin{matrix} n, & -n \\ & \frac{1}{2} \end{matrix} \middle| \frac{1}{2} - \frac{x}{2}\right) d\, x \\ &= \frac{i}{y}\left(e^{-iy} F_n^{+}(y) - e^{i(\pi n + y)} F_n^{-}(y)\right)\end{aligned} \tag{2}$$

where

$$F_n^{\pm}(y) = {}_3F_1\left(\begin{matrix} 1, & n, & -n \\ & & \frac{1}{2} \end{matrix} \middle| \frac{\pm i y}{2}\right) \tag{3}$$

The L^2 norm of $\hat{T}_n(y)$ is

$$\|\hat{T}_n\| = \sqrt{\int_{-\infty}^{\infty} \hat{T}_n(y)^2 \mathrm{d}y} = \sqrt{\frac{4(-1)^n \pi - (2n^2 - 1)}{4n^2 - 1}} \tag{4}$$

then define the normalized Fourier transforms $Y_n(y)$ of $T_n(x)$ by

$$\begin{aligned}Y_n(y) &= \frac{\hat{T}_n(y)}{\|\hat{T}_n\|} \\ &= \frac{i}{y}\left(\frac{e^{-iy} F_n^{+}(y) - e^{i(\pi n + y)} F_n^{-}(y)}{\sqrt{\frac{4(-1)^n \pi - (2n^2 - 1)}{4n^2 - 1}}}\right)\end{aligned} \tag{5}$$

1.2 Orthogonalizing $Y_n(y)$ Via The Gram-Schmidt Process

Apply the Gram-Schmidt process to the normalized Fourier transforms of the Type I Chebyshev polynomials $Y_n(y)$ to get $\psi_n(y)$

$$Y_n^{\perp}(y) = Y_n(y) - \sum_{m=1}^{n-1} \frac{\langle Y_n(y), Y_m^{\perp}(y)\rangle}{\langle Y_m^{\perp}(y), Y_m^{\perp}(y)\rangle} Y_m^{\perp}(y) \tag{6}$$

Let

$$A_{k,n} = -(-1)^{n + \binom{k}{2}} (k - 2n + 1)!\, 2^{2n - 1 - k} \binom{k+1}{k - 2n + 1} \binom{2k + 2 - 2n}{k + 1} \tag{7}$$

and

$$B_{k,n} = \frac{(-1)^{n + \binom{k}{2}}\, 2^{k - 2n} (k - n)! \binom{\frac{1}{2} - n + k}{k - 2n}}{n!} \tag{8}$$

then defined the associated functions

$$\Psi_n^{\sin}(y) = \frac{\sin(y)\sqrt{2n-1}}{x^n\sqrt{\pi}} \sum_{k=0}^{n-2} x^{2k} A_{k,n-2} \tag{9}$$

and

$$\Psi_n^{\cos}(y) = \frac{\cos(y)\sqrt{2n-1}}{x^n\sqrt{\pi}} \sum_{k=0}^{n-2} x^{2k+1} B_{k,n-2} \tag{10}$$

then $Y_n^{\perp}(y)$ can be expressed as

$$Y_n^{\perp}(y) = \Psi_n^{\sin}(y) + \Psi_n^{\cos}(y) \tag{11}$$

so that

$$\psi_n(y) = \lambda_h Y_n^{\perp}(y) \tag{12}$$

$$\begin{bmatrix}
-3 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
15 & -6 & 0 & 0 & 0 & 0 & 0 & \ldots \\
105 & -45 & 1 & 0 & 0 & 0 & 0 & \ldots \\
-945 & 420 & -15 & 0 & 0 & 0 & 0 & \ldots \\
-10395 & 4725 & -210 & 1 & 0 & 0 & 0 & \ldots \\
135135 & -62370 & 3150 & -28 & 0 & 0 & 0 & \ldots \\
2027025 & -945945 & 51975 & -630 & 1 & 0 & 0 & \ldots \\
-34459425 & 16216200 & -945945 & 13860 & -45 & 0 & 0 & \ldots \\
-654729075 & 310134825 & -18918900 & 315315 & -1485 & 1 & 0 & \ldots \\
13749310575 & -6547290750 & 413513100 & -7567560 & 45045 & -66 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{bmatrix}$$

Table 1. The first 10 row-vectors of $A_{k,n}$ matrix

2 An Integral Covariance Operator

The eigenvalues λ_k of the integral covariance operator

$$Tf(x) = \int_0^\infty J_0(x-y) f(x)\mathrm{d}x \tag{13}$$

where J_0 is the Bessel function of the first kind of order 0 are given by

$$\begin{aligned} \lambda_k &= \sqrt{\frac{4k+1}{\pi}} (k+1)^2_{-\frac{1}{2}} \\ &= \sqrt{\frac{4k+1}{\pi}} \frac{\Gamma\left(k+\frac{1}{2}\right)^2}{\Gamma(k+1)^2} \end{aligned} \tag{14}$$

which, together with the eigenfunctions

$$\psi_k(x) = e^{i\pi k}\sqrt{\frac{4k+1}{\pi}}\, j_{2k}(y) \tag{15}$$

solve the characteristic(eigenfunction) equation

$$\psi_k(y) = \lambda_k \int_0^\infty J_0(x-y)\,\psi_k(x)\mathrm{d}x \tag{16}$$

so that

$$J_0(x) = \sum_{k=0}^{\infty} \lambda_k \psi_k(y)$$