Fourier Transform of the Jacobi Weight Function

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Theorem 1

For $\alpha, \beta > -1$, the Fourier transform of the Jacobi weight function

$$w(x) = (1-x)^{\alpha} (1+x)^{\beta} \quad \text{on } [-1,1]$$

is given by

$$\hat{w}(t) = 2^{\alpha+\beta+1} \Gamma(\alpha+1) \Gamma(\beta+1) \frac{J_{\alpha+\beta+1}(2t)}{(2t)^{\alpha+\beta+1}} e^{it}$$

where J_ν denotes the Bessel function of the first kind of order ν.

Proof. 1. Initial Setup and Conditions:
The conditions $\alpha, \beta > -1$ ensure:

- The weight function is integrable on $[-1, 1]$
- The Beta function $B(\alpha+1, \beta+1)$ is well-defined
- The resulting Bessel function expression converges

We need to compute the Fourier transform:

$$\hat{w}(t) = \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} e^{-ixt} \, dx \tag{1}$$

2. Change of Variables:
Let

$$u = \frac{1+x}{2} \tag{2}$$

then:

$$\begin{aligned} x &= 2u - 1 \\ dx &= 2\, du \\ \text{when } x &= -1, u = 0 \\ \text{when } x &= 1, u = 1 \end{aligned} \tag{3}$$

The integral becomes:

$$\hat{w}(t)=2^{1+\alpha+\beta}\int_0^1(1-u)^{\alpha}\,u^{\beta}\,e^{-i(2u-1)t}\,d\,u \tag{4}$$

3. Exponential Splitting:

$$e^{-i(2u-1)t}=e^{-i2ut}\,e^{it} \tag{5}$$

4. Connection to Hypergeometric Functions:
The integral now takes the form:

$$2^{1+\alpha+\beta}\,e^{it}\int_0^1(1-u)^{\alpha}\,u^{\beta}\,e^{-i2ut}\,d\,u \tag{6}$$

This integral relates to the generalized hypergeometric function ${}_1F_1$ through:

$$\int_0^1 u^{\beta}\,(1-u)^{\alpha}\,e^{-i2ut}\,d\,u=B(\alpha+1,\beta+1)_1\,F_1(\beta+1;\alpha+\beta+2;-2\,i\,t) \tag{7}$$

5. Transformation to Bessel Functions:
The hypergeometric function transforms to Bessel form through three key steps:

First, applying the Kummer transformation:

$${}_1F_1(a;b;z)=e^z_1\,F_1(b-a;b;-z) \tag{8}$$

Second, using the limiting relation between confluent hypergeometric and Bessel functions:

$$J_{\nu}(z)=\frac{(z/2)^{\nu}}{\Gamma\,(\nu+1)}{}_0\,F_1(;\nu+1;-z^2/4) \tag{9}$$

Finally, through Hankel's contour integral representation:

$$J_{\nu}(z)=\frac{z^{\nu}}{2^{\nu}\,\Gamma\,(\nu+1)}{}_0\,F_1\!\left(;\nu+1;-\frac{z^2}{4}\right) \tag{10}$$

These transformations yield:

$$\int_0^1(1-u)^{\alpha}\,u^{\beta}\,e^{-i2ut}\,d\,u=B(\alpha+1,\beta+1)\,\frac{J_{\alpha+\beta+1}\,(2\,t)}{(2\,t)^{\alpha+\beta+1}} \tag{11}$$

6. Final Result:
Combining all terms:

$$\hat{w}(t)=2^{1+\alpha+\beta}\,B(\alpha+1,\beta+1)\,\frac{J_{\alpha+\beta+1}\,(2\,t)}{(2\,t)^{\alpha+\beta+1}}\,e^{it} \tag{12}$$

Using the Beta function relation $B(a,b)=\frac{\Gamma(a)\,\Gamma(b)}{\Gamma\,(a+b)}$, we obtain our final result:

$$\hat{w}(t)=2^{\alpha+\beta+1}\,\Gamma\,(\alpha+1)\,\Gamma\,(\beta+1)\,\frac{J_{\alpha+\beta+1}\,(2\,t)}{(2\,t)^{\alpha+\beta+1}}\,e^{it} \tag{13}$$

The e^{it} term carries the essential phase information of the Fourier transform, completing the proof. $\square$