Let $j_n(x)$ is the spherical Bessel function of the first kind

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x) = \frac{1}{\sqrt{x}} \left(\sin(z) R_{n,\frac{1}{2}}(z) - \cos(z) R_{n,\frac{3}{2}}(z) \right) \tag{1}$$

and $P_n(x)$ be the Legendre polynomials and $R_{n,v}(z)$ be the Lommel (quasi)polynomials[2]

$$R_{n,v}(z) = \frac{\Gamma(n+v)}{\Gamma(v)} \left(\frac{z}{2}\right)_2^{-n} F_3\left([-\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}]; [v, -n, -v+1-n]; -z^2 \right) \tag{2}$$

Theorem 1

The eigenvalues of

$$\int_0^\infty J_0(x-y) f(x)\,\mathrm{d}x \tag{3}$$

are givten by

$$\lambda_n = \int_{-\infty}^\infty J_0(x)\psi_n(x)\,\mathrm{d}x = \sqrt{\frac{4n+1}{\pi}} \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} = \sqrt{\frac{4n+1}{\pi}} (n+1)^2_{-\frac{1}{2}}$$

and its eigenfunctions are

$$\begin{aligned} \psi_n(y) &= (-1)^n \sqrt{\frac{4n+1}{\pi}} j_{2n}(y) \\ &= (-1)^n \sqrt{\frac{4n+1}{\pi}} \sqrt{\frac{\pi}{2y}} J_{2n+\frac{1}{2}}(y) \\ &= (-1)^n \sqrt{\frac{4n+1}{2y}} J_{2n+\frac{1}{2}}(y) \\ &= \sqrt{\frac{4n+1}{4\pi}} \int_{-1}^1 P_{2n}(x) e^{ixy}\,\mathrm{d}x \end{aligned} \tag{4}$$

which satisfy the integral covariance equation

$$\begin{aligned} \int_0^\infty J_0(t-s)\psi_k(t)\mathrm{d}t &= \lambda_k \psi_k(y) \\ &= \sqrt{\frac{4n+1}{\pi}} \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} (-1)^n \sqrt{\frac{4n+1}{\pi}} j_{2n}(y) \\ &= \frac{4n+1}{\pi} \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} j_{2n}(y) \end{aligned} \tag{5}$$

and the eigenfunction expansion given by·

$$J_0(x) = \sum_{n=0}^\infty \lambda_n \psi_n(x) = \frac{4n+1}{\pi} \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} j_{2n}(x) \tag{6}$$

converges uniformly over the entire complex plane except at the irregular removeable singular point at the origin where the limit is however well defined.

Proof.

Let □

$$K_n(x,y)=\sum_{k=0}^{n}\lambda_k\,\psi_k(x-y) \tag{7}$$

then show that

$$\lim_{n\to\infty}K_n(x,y)=J_0(x-y) \tag{8}$$

uniformly by utilizing the results from the uniformly convergent asymptotic series,,.

Remark 2. The solutions in Theorem 1 were derive by identifying the orthogonal polynomial sequence associated with the spectral density of the kernel K, which in the case when $K=J_0$ the spectral density is seen to be equal to

$$S(\omega)=\int_0^\infty J_0(x)\,e^{ix\omega}\,\mathrm{d}x=\begin{cases}\dfrac{1}{\sqrt{1-\omega^2}} & \omega\in(-1,1)\\ 0 & \text{otherwise}\end{cases} \tag{9}$$

so that we identify the spectral density with the Chebyshev polynomials of the first kind, since their orthogonality measure is, in fact, equal to the spectral density in the case $K(t,s)=J_0(t-s)$. The Chebyshev polynomials' orthogonality relation is

$$\int_{-1}^{1}T_n(\omega)\,T_m(\omega)\,S(\omega)\,\mathrm{d}\omega=\begin{cases}0 & n\neq m\\ \pi & n=m=0\\ \dfrac{\pi}{2} & n=m\neq 0\end{cases} \tag{10}$$

2. Orthogonalize the finite Fourier transform of the Chebyshev type-I polynomials[1] (which is just the usual infinite Fourier transform with the integration restricted to the range $-1\ldots 1$ since $T_n(x)=0\forall x\notin[-1,1]$)

$$\hat{T}_n(y)=\int_{-1}^{1}T_n(x)\,e^{ixy}\,\mathrm{d}x \tag{11}$$

by applying the Gram-Schmidt recursions

$$\psi_n(y)=\hat{T}_n^{\perp}(y)=\hat{T}_n(y)-\sum_{m=1}^{n-1}\frac{\langle\hat{T}_n(y),\hat{T}_m^{\perp}(y)\rangle}{\langle\hat{T}_m^{\perp}(y),\hat{T}_n^{\perp}(y)\rangle}\hat{T}_n^{\perp}(y) \tag{12}$$

with respect to the unweighted standard Lebesgue inner product measure over 0 to ∞.

4. Note that the partial sums

$$K_n(x,y)=\sum_{k=0}^{n}\psi_k(x-y) \tag{13}$$

converge pointwisely by applying the appropriate theorem from Fourier analysis.

5. Determine the eigenvalues by projecting them onto the kernel

6. Apply the proof of uniformity of asymptotic series by showing equivalence to…

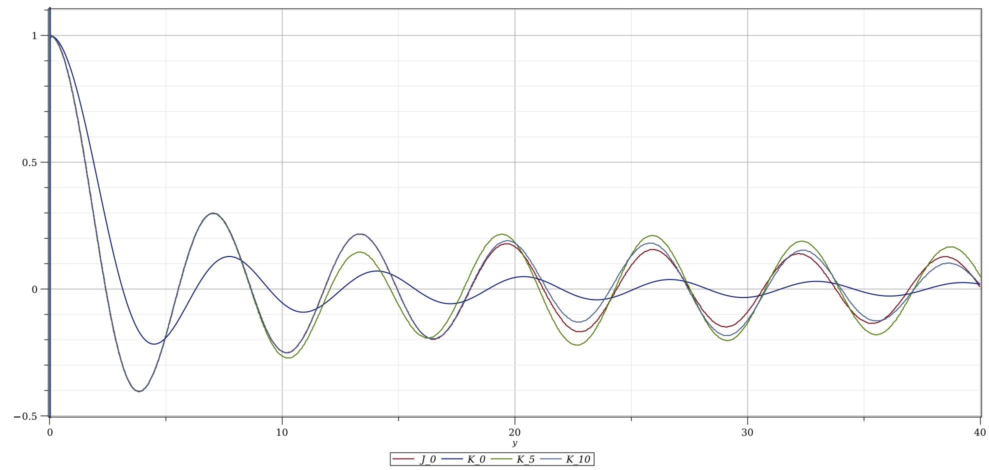

Figure 1. Demonstration of convergence of $K_n(h)$ to $J_0(h)$ for $n = 0, 5, 10$

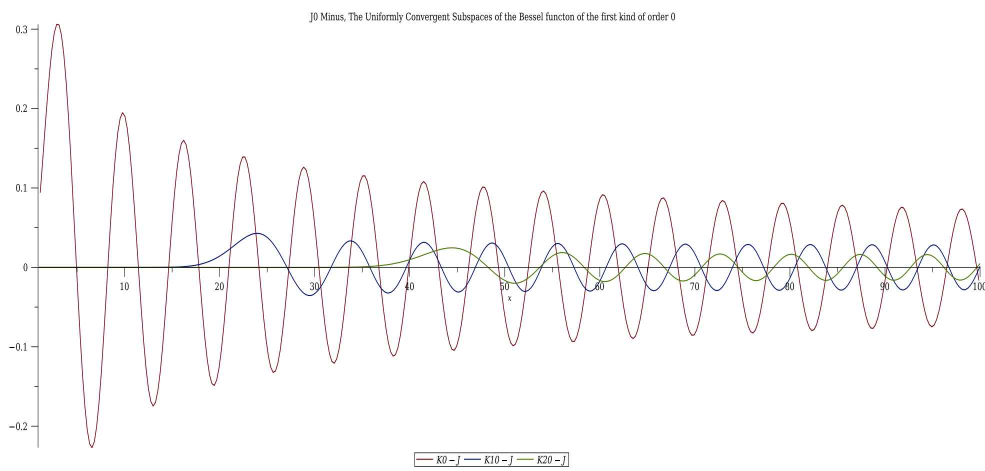

Figure 2. Demonstrating of approximation error between $K_n(h)$ and $J_0(h)$ for $n = 0, 5, 10$

Note 3. This procedure is an application of the Galerkin 'numerical' method in the first part of "Stochastic Finite Elements" except instead of just choosing a piecewise or other polynomial basis for the expansion, you apply the orthogonalization procedure to the Fourier transform of the orthogonal polynomials corresponding to the spectral density, and thus what are termed the the mass and stiffness matrices in the engineering and finite element literature become the identity matrix, and $c_n = \lambda_n$ becomes an identity.

Bibliography

[1] Atul Dixit, Lin Jiu, Victor H Moll, and Christophe Vignat. The finite fourier transform of classical polynomials. *Journal of the Australian Mathematical Society*, 98:145–160, 2015.

[2] R. Wong K.F. Lee. Asymptotic expansion of the modified lommel polynomials $h_{n,\nu}(x)$ and their zeros. *Proceedings of the American Mathematical Society*, 142(11):3953–3964, 2014.