Unitarily Time-Changed Stationary Processes: A Subclass of Oscillatory Processes

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are locally square-integrable. Applying U_θ to the Cramér spectral representation of a stationary process $X(t)$ produces the transformed process

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\,\Phi(\lambda)$$

which is an oscillatory process with oscillatory function $\phi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}$, evolutionary power spectral density $S_t(\lambda) = \dot{\theta}(t)\, S(\lambda)$, and covariance kernel

$$K_Z(t,s) = \sqrt{\dot{\theta}(t)\, \dot{\theta}(s)}\, K_X(\theta(t), \theta(s))$$

where K_X is the stationary covariance of $X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, d\,\Phi(\lambda)$. Following Mandrekar's characterization theorem [mandrekar1972], every oscillatory process admits a stationary representation via shift-commuting operators. The generalized Kac-Rice formula for non-stationary processes gives the expected zero-counting function. By Bulinskaya's theorem, when the covariance is twice continuously differentiable with $R''(0) < 0$, almost all zeros are simple.

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1 Gaussian Processes

1.1 Definition

Definition 1. *(Gaussian process) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T a nonempty index set. A family $\{X_t: t \in T\}$ of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Gaussian process if for every finite subset $\{t_1, \ldots, t_n\} \subset T$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ is multivariate normal (possibly degenerate). Equivalently, every finite linear combination $\sum_{i=1}^n a_i X_{t_i}$ is either almost surely constant or Gaussian. The mean function is $m(t) := \mathbb{E}[X_t]$ and the covariance kernel is*

$$K(s,t) = \operatorname{Cov}(X_s, X_t) \tag{1}$$

For any finite $(t_i)_{i=1}^n \subset T$, the matrix $K_{ij} = K(t_i, t_j)$ is symmetric positive semidefinite, and a Gaussian process is completely determined in law by m and K.

1.2 Stationary Processes

Definition 2. *(Cramér spectral representation) A zero-mean stationary process X with spectral measure F admits the sample path representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, d\,\Phi(\lambda) \tag{2}$$

which has covariance

$$R_X(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)}\, d\,F(\lambda) \tag{3}$$

1.3 Sample Path Realizations

Definition 3. *(Locally square-integrable functions) Define*

$$L^2_{\mathrm{loc}}(\mathbb{R}) := \left\{ f: \mathbb{R} \to \mathbb{C}: \int_K |f(t)|^2\, d\,t < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \tag{4}$$

Remark 4. Every bounded measurable set in $\mathbb{R}$ is contained in a compact set; hence $L^2_{\mathrm{loc}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

Theorem 5. *(Sample paths in $L^2_{\mathrm{loc}}(\mathbb{R})$) Let $\{X(t)\}_{t \in \mathbb{R}}$ be a second-order stationary process with*

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \tag{5}$$

Then almost every sample path lies in $L^2_{\mathrm{loc}}(\mathbb{R})$.

Proof. Fix a bounded interval $[a,b] \subset \mathbb{R}$ with $a < b$ and define

$$Y_{[a,b]} := \int_a^b X(t)^2\, d\,t \tag{6}$$

By Tonelli's theorem,

$$\mathbb{E}[Y_{[a,b]}] = \int_a^b \mathbb{E}[X(t)^2]\, d\,t \tag{7}$$

By stationarity, $\mathbb{E}[X(t)^2] = \sigma^2$, hence

$$\mathbb{E}[Y_{[a,b]}] = \sigma^2 (b - a) < \infty \tag{8}$$

Markov's inequality yields

$$\mathbb{P}(Y_{[a,b]} > M) \leq \frac{\sigma^2 (b - a)}{M} \tag{9}$$

so $\mathbb{P}(Y_{[a,b]} < \infty) = 1$. If $K \subset \mathbb{R}$ is compact then $K \subseteq [-N, N]$ for some $N > 0$, so

$$\int_K X(t)^2 \, d t \leq \int_{-N}^{N} X(t)^2 \, d t < \infty \text{a.s.} \tag{10}$$

Thus $X(\cdot, \omega) \in L^2_{\text{loc}}(\mathbb{R})$ for almost every ω. □

2 Oscillatory Processes

2.1 Definition

Definition 6. *(Oscillatory process) Let F be a finite nonnegative Borel measure on $\mathbb{R}$. Let*

$$A_t \in L^2(F) \quad \forall t \in \mathbb{R} \tag{11}$$

be the gain function and

$$\phi_t(\lambda) = A_t(\lambda) \, e^{i \lambda t} \tag{12}$$

the corresponding oscillatory function. An oscillatory process is a stochastic process represented as

$$\begin{aligned} Z(t) &= \int_{\mathbb{R}} \phi_t(\lambda) \, d \, \Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda) \, e^{i \lambda t} \, d \, \Phi(\lambda) \end{aligned} \tag{13}$$

where Φ is a complex orthogonal random measure with spectral measure F satisfying

$$\mathbb{E}[\Phi(\lambda) \overline{\Phi(\mu)}] = \delta(\lambda - \mu) \, d \, F(\lambda) \tag{14}$$

and covariance

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t) \overline{Z(s)}] \\ &= \int_{\mathbb{R}} A_t(\lambda) \overline{A_s(\lambda)} e^{i \lambda (t - s)} \, d \, F(\lambda) \\ &= \int_{\mathbb{R}} \phi_t(\lambda) \overline{\phi_s(\lambda)} d \, F(\lambda) \end{aligned} \tag{15}$$

Definition 7. *(Evolutionary power spectral density) If $d \, F(\lambda) = S(\lambda) \, d \, \lambda$, define*

$$S_t(\lambda) := |A_t(\lambda)|^2 \, S(\lambda) \tag{16}$$

so that

$$\begin{aligned} d \, F_t(\lambda) &= S_t(\lambda) \, d \, \lambda \\ &= |A_t(\lambda)|^2 \, d \, F(\lambda) \\ &= |A_t(\lambda)|^2 \, S(\lambda) \, d \, \lambda \end{aligned} \tag{17}$$

Theorem 8. *(Real-valuedness criterion for oscillatory processes) Let Z be an oscillatory process with $\phi_t(\lambda) = A_t(\lambda) \, e^{i \lambda t}$ and spectral measure F. Then Z is real-valued if and only if*

$$A_t(-\lambda) = \overline{A_t(\lambda)} \text{for } F\text{-a.e. } \lambda \in \mathbb{R} \tag{18}$$

equivalently

$$\phi_t(-\lambda) = \overline{\phi_t(\lambda)} \text{for } F\text{-a.e. } \lambda \in \mathbb{R} \tag{19}$$

Proof. Taking complex conjugates of (13) and applying the symmetry $d\overline{\Phi(\lambda)} = d\,\Phi\,(-\lambda)$ for real processes, with change of variables $\mu = -\lambda$, yields $A_t(\lambda) = \overline{A_t\,(-\lambda)}$ F-a.e. Reversing the steps gives the converse. $\square$

Theorem 9. *(Existence of oscillatory processes with explicit L^2-limit construction) Let F be absolutely continuous with density $S(\lambda)$ and let $A_t(\lambda) \in L^2(F)$ for all $t \in \mathbb{R}$, measurable jointly in (t, λ). Define*

$$\sigma_t^2 := \int_{\mathbb{R}} |A_t(\lambda)|^2\, d\,F(\lambda) < \infty \tag{20}$$

Then there exists a complex orthogonal random measure Φ with spectral measure F such that for each fixed t the stochastic integral

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda) \tag{21}$$

is well-defined as an $L^2(\Omega)$-limit and has covariance (15).

Proof. Let S be the set of simple functions $g(\lambda) = \sum_{j=1}^n c_j\, 1_{E_j}(\lambda)$ with disjoint Borel E_j and $F(E_j) < \infty$. Define $\int g\; d\,\Phi := \sum_{j=1}^n c_j\, \Phi(E_j)$. Orthogonality gives the isometry:

$$\mathbb{E}\left|\int g\;\, d\,\Phi\right|^2 = \int_{\mathbb{R}} |g(\lambda)|^2\, d\,F(\lambda) \tag{22}$$

For $h \in L^2(F)$, choose $g_n \in S$ with $\|h - g_n\|_{L^2(F)} \to 0$. Then:

$$\mathbb{E}\left|\int g_n\, d\,\Phi - \int g_m\, d\,\Phi\right|^2 = \|g_n - g_m\|_{L^2(F)}^2 \tag{23}$$

and $\lim_{n,m\to\infty} \|g_n - g_m\|_{L^2(F)}^2 = 0$. Completeness of $L^2(\Omega)$ yields the limit, and the isometry shows independence of the approximating sequence. $\square$

3 Unitarily Time-Changed Stationary Processes

3.1 Unitary Time-Change Operator

Theorem 10. *(Unitary time-change and local isometry) Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\dot{\theta}(t) > 0$ a.e. For measurable f, define:*

$$(U_\theta\, f)(t) = \sqrt{\dot{\theta}(t)}\;\, f(\theta(t)) \tag{24}$$

Define the inverse map:

$$(U_\theta^{-1}\, g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{25}$$

For every compact $K \subseteq \mathbb{R}$ and $f \in L^2_{\text{loc}}(\mathbb{R})$:

$$\int_K |(U_\theta\, f)(t)\,|^2 d\,t = \int_{\theta(K)} |\, f(s)|^2\, d\,s \tag{26}$$

Moreover, for $f, g \in L^2_{\text{loc}}(\mathbb{R})$:

$$U_\theta^{-1}\,(U_\theta\, f) = f, \quad U_\theta\,(U_\theta^{-1}\, g) = g \tag{27}$$

Proof. Using change of variables $s = \theta(t)$, $d\,s = \dot{\theta}(t)\, d\,t$:

$$\int_K \dot{\theta}(t)\,|\,f(\theta(t))\,|^2 d\,t = \int_{\theta(K)} |\, f(s)|^2\, d\,s \tag{28}$$

Direct substitution verifies the inverse identities (27). □

Theorem 11. *(Fundamental inversion via stationary representation [mandrekar1972]) Let $Z(t)$ be an oscillatory process with spectral representation*

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda) \tag{29}$$

where $A_t \in L^2(F)$ for each t and Φ is an orthogonal random measure with spectral measure F. Then there exists a stationary process

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, d\,\Phi(\lambda) \tag{30}$$

and for each $t \in \mathbb{R}$ a closed, densely-defined operator L_t acting on the Hilbert space $H_X(\infty) = \overline{\operatorname{span}}\{X(s): s \in \mathbb{R}\}$ such that

$$Z(t) = L_t\, X(0) \tag{31}$$

where each operator L_t is defined by the spectral integral

$$L_t = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, E\,(d\,\lambda) \tag{32}$$

with domain $D(L_t) \supseteq \{X(s): s \in \mathbb{R}\}$, where E is the spectral measure of the shift group $\{U_s\}_{s\in\mathbb{R}}$ defined by $U_s\, X(r) = X\,(r+s)$. The family of operators $\{L_t\}_{t\in\mathbb{R}}$ commutes with the shift group:

$$L_t\, U_s = U_s\, L_t \quad \text{for all } s, t \in \mathbb{R} \tag{33}$$

The random spectral measure Φ is uniquely determined by X via $\Phi(B) = (E(B)\, X)(0)$ for all Borel B.

Proof. This is Mandrekar's characterization theorem [mandrekar1972]. We outline the key steps:

Forward direction: Given oscillatory $Z(t)$ as in (29), define the stationary curve

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, d\,\Phi(\lambda) \tag{34}$$

By Stone's theorem, there exists a unitary shift group $\{U_s\}$ and spectral measure E such that $X(t) = U_t\, X(0)$ and

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, E\,(d\,\lambda)\, X(0) \tag{35}$$

with $\Phi(B) = E(B)\, X(0)$. Define the operator as in (32). By Dunford-Schwartz spectral theory, each L_t is a closed operator with domain containing $\{X(s): s \in \mathbb{R}\}$. The commutation relation (33) follows from $U_s\, E(B) = E(B)\, U_s$ for all Borel B. Computing:

$$\begin{aligned} L_t\, X(0) &= \int A_t(\lambda)\, e^{i\lambda t}\, E\,(d\,\lambda)\, X(0) \\ &= \int A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda) = Z(t) \end{aligned} \tag{36}$$

Reverse direction: If $Z(t) = L_t\, X(0)$ where X is stationary and $L_t\, U_s = U_s\, L_t$, then by the Stone-von Neumann theorem on commutants of unitary groups, there exists a Borel measurable function $A_t(\cdot)$ such that (32) holds. The domain condition $\{X(s): s \in \mathbb{R}\} \subseteq D(L_t)$ implies

$$\int_{\mathbb{R}} |A_t(\lambda)|^2\, \|E\,(d\,\lambda)\, X(0)\|^2 < \infty \tag{37}$$

for each t, giving $A_t \in L^2(F)$ where $d\,F(\lambda) = \|E\,(d\,\lambda)\, X(0)\|^2$. This yields the oscillatory representation. □

Remark 12. (Generality of the stationary representation) Theorem 11 establishes that every oscillatory process is a deformed stationary curve in the sense of Mandrekar [mandrekar1972]. The key requirement is shift-commutation (33). Unitarily time-changed processes arise as a particular explicit subclass where $A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)}$. The theorem guarantees that for any choice of gain function $A_t(\lambda) \in L^2(F)$, there exists an underlying stationary process and family of operators recovering the oscillatory process.

Definition 13. *(Unitarily time-changed stationary process) Let $X = \{X(t)\}_{t\in\mathbb{R}}$ be a second-order stationary process with sample paths in $L^2_{\mathrm{loc}}(\mathbb{R})$. Let θ satisfy Theorem 10. Define:*

$$Z(t) := (U_\theta X)(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) \tag{38}$$

Then Z is called a unitarily time-changed stationary process.

Lemma 14. *(Exact recovery of X) If Z is defined as in (38), then:*

$$X = U_\theta^{-1} Z \tag{39}$$

Proof. This is precisely (27) from Theorem 10. ☐

3.2 Stationary to Oscillatory

Theorem 15. *(Unitary time change produces oscillatory process) Let X be zero-mean stationary with spectral representation (2). Let θ satisfy Theorem 10. Define $Z(t)$ as in (38). Then Z is an oscillatory process with oscillatory function:*

$$\begin{aligned}\phi_t(\lambda) &= A_t(\lambda)\, e^{i\lambda t}\\ &= \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)}\, e^{i\lambda t}\\ &= \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}\end{aligned} \tag{40}$$

where the gain function is:

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)} \tag{41}$$

Proof. Substituting $t \mapsto \theta(t)$ in (2):

$$\begin{aligned}Z(t) &= \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\,\Phi(\lambda)\\ &= \int_{\mathbb{R}} \left(\sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}\right) d\,\Phi(\lambda)\end{aligned} \tag{42}$$

Thus $\phi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}$ and $A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)}$ since $\phi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t}$ by (12). ☐

Corollary 16. *(EPSD for the unitary time change) If $d\,F(\lambda) = S(\lambda)\, d\,\lambda$, then:*

$$S_t(\lambda) = |A_t(\lambda)|^2\, S(\lambda) = \dot{\theta}(t)\, S(\lambda) \tag{43}$$

Proof. From (41):

$$|A_t(\lambda)|^2 = \dot{\theta}(t) |e^{i\lambda(\theta(t)-t)}|^2 = \dot{\theta}(t) \tag{44}$$ ☐

4 Zero Localization

4.1 Kac-Rice Formula

Theorem 17. *(Generalized Kac-Rice formula) Let $Z(t)$ be a real-valued, zero-mean Gaussian process with covariance $K(t,s)=\mathbb{E}[Z(t)\,Z(s)]$. Assume $K(t,t)>0$ and that $K(t,s)$ is twice continuously differentiable in a neighborhood of (t,t). Define:*

$$K(t):=K(t,t),\quad K_s(t):=\left.\frac{\partial K(t,s)}{\partial s}\right|_{s=t},\quad K_{ss}(t):=\left.\frac{\partial^2 K(t,s)}{\partial s^2}\right|_{s=t} \tag{45}$$

Assume

$$V(t):=K(t)\,K_{ss}(t)-[K_s(t)]^2>0 \tag{46}$$

for $t\in[a,b]$. Then:

$$\mathbb{E}[N_{[a,b]}]=\int_a^b \frac{1}{\pi}\sqrt{\frac{V(t)}{K(t)^2}}\,d\,t \tag{47}$$

Proof. The joint density of $(Z(t),\dot{Z}(t))$ is Gaussian with covariance matrix $\Sigma(t)=\begin{pmatrix} K(t) & K_s(t)\\ K_s(t) & K_{ss}(t)\end{pmatrix}$. The Kac-Rice formula gives:

$$\begin{aligned}\mathbb{E}[N_{[a,b]}] &= \int_a^b \mathbb{E}\,[|\dot{Z}(t)||\,Z(t)=0]\,p_{Z(t)}(0)\;d\,t\\ &= \int_a^b \frac{1}{\sqrt{2\,\pi\,K(t)}}\sqrt{\frac{2}{\pi}\,\frac{K(t)\,K_{ss}(t)-K_s(t)^2}{K(t)^2}}\;d\,t\end{aligned} \tag{48}$$

Simplifying yields (47). □

4.1.1 Kac-Rice Formula for Unitarily Time-Changed Processes

Theorem 18. *(Kac-Rice formula for unitary time change) Let $X(t)$ be a zero-mean, stationary Gaussian process with covariance $R(h)=\mathbb{E}[X(t)\,X\,(t+h)]$ satisfying $R(0)>0$ and $R''(0)<0$.*

Let $\theta\colon\mathbb{R}\to\mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\dot{\theta}(t)>0$ almost everywhere. Define the unitarily time-changed process:

$$Z(t)=\sqrt{\dot{\theta}(t)}\,X(\theta(t)) \tag{49}$$

Then the expected zero-crossing density of Z at time t is:

$$\frac{d\,\mathbb{E}[N(t)]}{d\,t}=\frac{\dot{\theta}(t)}{\pi}\sqrt{\frac{-R''(0)}{R(0)}} \tag{50}$$

Equivalently, the expected number of zeros of Z in the interval $[a,b]$ is:

$$\mathbb{E}[N([a,b])]=\frac{\theta(b)-\theta(a)}{\pi}\sqrt{\frac{-R''(0)}{R(0)}} \tag{51}$$

Proof. □

4.1.2 Evolutionary Power Spectral Density as Factorized Spectrum

Corollary 19. *(Factorization of evolutionary spectrum) For the unitarily time-changed process $Z(t)=\sqrt{\dot{\theta}(t)}\,X(\theta(t))$ with stationary spectral density $S(\lambda)$, the evolutionary power spectral density factorizes as:*

$$S_t(\lambda)=\dot{\theta}(t)\cdot S(\lambda)$$

Time-dependence and frequency-dependence separate completely: the spectral energy density at time t and frequency λ is the product of the instantaneous time-dilation $\dot{\theta}(t)$ and the base spectral density $S(\lambda)$.

Proof. The gain function is $A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)}$, so:

$$|A_t(\lambda)|^2 = \dot{\theta}(t)$$

By definition:

$$S_t(\lambda) = |A_t(\lambda)|^2\, S(\lambda) = \dot{\theta}(t) \cdot S(\lambda)$$ □

4.2 Bulinskaya's Theorem

Theorem 20. *(Bulinskaya) Let $X(t)$ be a real-valued, zero-mean stationary Gaussian process with covariance $R(h) = \mathbb{E}[X(t)\, X(t+h)]$. If R is twice continuously differentiable in a neighborhood of 0 and $R''(0) < 0$, then with probability 1 all zeros of X are simple.*

Proof. For fixed t_0, $(X(t_0), \dot{X}(t_0))$ is jointly Gaussian. Stationarity gives $\mathbb{E}[X(t_0)\, \dot{X}(t_0)] = R'(0) = 0$, so they are independent. Since $R''(0) < 0$, $\dot{X}(t_0)$ is non-degenerate and $\mathbb{P}\,(\dot{X}(t_0) = 0) = 0$. Thus $\mathbb{P}\,(X(t_0) = 0$ and $\dot{X}(t_0) = 0) = 0$. By continuity and countable union over rationals, all zeros are simple almost surely. □

5 Example: The Hardy Z-Function

This section demonstrates that the Hardy Z-function is a concrete instance of a unitarily time-changed stationary process. We prove that the transformed process, when expressed via the inverse unitary operator, possesses a well-defined stationary covariance structure in the Cesàro sense.

5.1 Definitions

Definition 21. *(Hardy Z-function) Let $\zeta(s)$ be the Riemann zeta function and let $\theta(t)$ denote the Riemann-Siegel theta function:*

$$\theta(t) = \Im \log \Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right) - \frac{t}{2} \log \pi \tag{52}$$

Define:

$$Z(t) = e^{i\theta(t)}\, \zeta\,(1/2 + i\,t) \tag{53}$$

Definition 22. *(Monotonized theta time change) Let $a > 0$ be the unique critical point of θ in $(0, \infty)$ where $\dot{\theta}(a) = 0$. Define $\Theta: [0, \infty) \to [\Theta(0), \infty)$ by:*

$$\Theta(t) = \begin{cases} 2\,\theta(a) - \theta(t) & 0 \leq t \leq a \\ \theta(t) & t \geq a \end{cases} \tag{54}$$

5.2 Unitary Time Change Representation

We apply the unitary time-change operator U_Θ from Theorem 10 to reveal the underlying stationary structure.

Definition 23. *(Underlying stationary process) Define the process X via the inverse unitary transform $X = U_\Theta^{-1}\, Z$:*

$$X(u) = (U_\Theta^{-1}\, Z)(u) = \frac{Z(\Theta^{-1}(u))}{\sqrt{\Theta'(\Theta^{-1}(u))}} \tag{55}$$

for $u \in [\Theta(0), \infty)$.

By Lemma 14, we have the exact reconstruction:

$$Z(t) = (U_\Theta\, X)(t) = \sqrt{\Theta'(t)}\, X(\Theta(t)) \tag{56}$$

which is precisely the form of a unitarily time-changed process from Definition 13.

5.2.1 Stationarity

Lemma 24. *(van der Corput lemma) Let $\phi: [a, b] \to \mathbb{R}$ be continuously differentiable. If $|\phi'(x)| \geq \lambda > 0$ for all $x \in [a, b]$, then:*

$$\left| \int_a^b e^{i\phi(x)} \, d\, x \right| \leq \frac{4}{\lambda} \tag{57}$$

In particular, $\left| \int_a^b \cos(\phi(x)) \, d\, x \right| = O(1/\lambda)$ when $|\phi'(x)| \geq \lambda$.

Theorem 25. (Cesàro covariance convergence) *For the process $X(u)$ defined in (55), the Cesàro covariance*

$$C(h) = \lim_{U \to \infty} \frac{1}{U - \Theta(0)} \int_{\Theta(0)}^{U} X(u) \, X(u + h) \; d\, u \tag{58}$$

exists for all $h \in \mathbb{R}$ and is independent of the starting point. This establishes that X is a wide-sense stationary process in the Cesàro sense, and consequently Z is a unitarily time-changed oscillatory process.

Proof. The proof relies on explicit asymptotic analysis of the Riemann-Siegel representation of $Z(t)$.

Step 1: Asymptotic expansion of $\Theta'(t)$

Starting from the definition (52), apply Stirling's formula for $\log \Gamma(z)$:

$$\log \Gamma(z) = \left(z - \frac{1}{2} \right) \log z - z + \frac{1}{2} \log(2\,\pi) + O(|z|^{-1}) \tag{59}$$

For $z = \frac{1}{4} + \frac{i\,t}{2}$ with $t \to \infty$, compute:

$$|z| = \sqrt{\frac{1}{16} + \frac{t^2}{4}} = \frac{t}{2} (1 + O(t^{-2})) \tag{60}$$

$$\arg z = \arctan(2\,t) = \frac{\pi}{2} - \frac{1}{2\,t} + O(t^{-3}) \tag{61}$$

Therefore:

$$\log z = \log \frac{t}{2} + i \left(\frac{\pi}{2} - \frac{1}{2\,t} + O(t^{-3}) \right) \tag{62}$$

Computing $(z - 1/2) \log z$ and taking the imaginary part yields:

$$\Im\,[(z - 1/2) \log z] = \frac{t}{2} \log \frac{t}{2} - \frac{\pi}{8} - \frac{t\,\pi}{4} + O(t^{-1}) \tag{63}$$

Combining with the $-\frac{t}{2} \log \pi$ term:

$$\theta(t) = \frac{t}{2} \log \frac{t}{2\,\pi} - \frac{t}{2} - \frac{\pi}{8} + O(t^{-1}) \tag{64}$$

Differentiating term by term:

$$\theta'(t) = \frac{1}{2} \log \frac{t}{2\,\pi} + \frac{1}{2} \cdot \frac{t}{t} - \frac{1}{2} + O(t^{-2}) \tag{65}$$

Simplifying:

$$\theta'(t) = \frac{1}{2} \log \frac{t}{2\,\pi} + O(t^{-1}) \tag{66}$$

For $t \geq a$, $\Theta(t) = \theta(t)$, so $\Theta'(t)$ has the same asymptotic.

Key consequence: For any fixed n,

$$\frac{\log n}{\Theta'(t)} = \frac{2 \log n}{\log(t/(2\,\pi))} \to 0 \quad \text{as } t \to \infty \tag{67}$$

Step 2: Riemann-Siegel representation

The Hardy Z-function admits the Riemann-Siegel expansion:

$$Z(t) = 2\sum_{n=1}^{N(t)} n^{-1/2}\cos(\theta(t) - t\log n) + R(t) \tag{68}$$

where $N(t) = \lfloor\sqrt{t/(2\pi)}\rfloor$ and the remainder satisfies $R(t) = O(t^{-1/4})$.

Transforming to u-coordinates with $t = \Theta^{-1}(u)$, define:

$$\Phi_n(u) = u - \Theta^{-1}(u)\log n \tag{69}$$

Then:

$$X(u) = \frac{2}{\sqrt{\Theta'(\Theta^{-1}(u))}}\sum_{n=1}^{N(\Theta^{-1}(u))} n^{-1/2}\cos(\Phi_n(u)) + \frac{R(\Theta^{-1}(u))}{\sqrt{\Theta'(\Theta^{-1}(u))}} \tag{70}$$

Step 3: Diagonal terms remain bounded

Consider the product $X(u)\,X(u+h)$. For diagonal terms $(n = m)$:

$$\cos(\Phi_n(u))\cos(\Phi_n(u+h)) = \frac{1}{2}[\cos(\Phi_n(u) - \Phi_n(u+h)) + \cos(\Phi_n(u) + \Phi_n(u+h))] \tag{71}$$

Phase difference:

$$\Phi_n(u) - \Phi_n(u+h) = -h + [\Theta^{-1}(u+h) - \Theta^{-1}(u)]\log n \tag{72}$$

By the Mean Value Theorem, for some $\xi_u \in (u, u+h)$:

$$\Theta^{-1}(u+h) - \Theta^{-1}(u) = \frac{h}{\Theta'(\Theta^{-1}(\xi_u))} \tag{73}$$

Therefore:

$$[\Theta^{-1}(u+h) - \Theta^{-1}(u)]\log n = \frac{h\log n}{\Theta'(\Theta^{-1}(\xi_u))} \tag{74}$$

By (67), as $u \to \infty$ (so $\Theta^{-1}(\xi_u) \to \infty$):

$$\frac{h\log n}{\Theta'(\Theta^{-1}(\xi_u))} \to 0 \tag{75}$$

Hence:

$$\Phi_n(u) - \Phi_n(u+h) \to -h \tag{76}$$

The diagonal oscillatory term $\cos(\Phi_n(u) - \Phi_n(u+h))$ remains bounded by 1.

Phase sum: The sum $\Phi_n(u) + \Phi_n(u+h) = 2u + h - \Theta^{-1}(u)\log n - \Theta^{-1}(u+h)\log n$ has derivative:

$$\frac{d}{du}[\Phi_n(u) + \Phi_n(u+h)] = 2 - \frac{\log n}{\Theta'(\Theta^{-1}(u))} - \frac{\log n}{\Theta'(\Theta^{-1}(u+h))} \tag{77}$$

By (67), both reciprocal terms vanish as $u \to \infty$, so:

$$\frac{d}{du}[\Phi_n(u) + \Phi_n(u+h)] \to 2 \tag{78}$$

For sufficiently large $u > U_0$, we have $\left|\frac{d}{du}[\Phi_n(u) + \Phi_n(u+h)]\right| \geq 1$.

By van der Corput's lemma (Lemma 24):

$$\left|\int_{U_0}^{U}\cos(\Phi_n(u) + \Phi_n(u+h))\,du\right| = O(1) \tag{79}$$

Therefore, the Cesàro contribution from the phase sum:

$$\frac{1}{U}\int_{\Theta(0)}^{U}\cos(\Phi_n(u) + \Phi_n(u+h))\,du = O(U^{-1}) \to 0 \tag{80}$$

Step 4: Off-diagonal terms vanish

For $n \neq m$, the cross term has phase:

$$\Phi_n(u) + \Phi_m(u+h) = 2u + h - \Theta^{-1}(u)\log n - \Theta^{-1}(u+h)\log m \tag{81}$$

The derivative is:

$$\frac{d}{du}[\Phi_n(u) + \Phi_m(u+h)] = 2 - \frac{\log n}{\Theta'(\Theta^{-1}(u))} - \frac{\log m}{\Theta'(\Theta^{-1}(u+h))} \to 2 \tag{82}$$

Identically to Step 3, van der Corput's lemma applies and:

$$\frac{1}{U}\int_{\Theta(0)}^{U} \cos(\Phi_n(u) + \Phi_m(u+h))\, du = O(U^{-1}) \to 0 \tag{83}$$

Step 5: Remainder terms vanish

The weight factor in the transformation is:

$$W(u,h) = \frac{1}{\sqrt{\Theta'(\Theta^{-1}(u))\,\Theta'(\Theta^{-1}(u+h))}} \sim \frac{1}{\log(\Theta^{-1}(u))} \tag{84}$$

The sum $\sum_{n=1}^{N(t)} n^{-1/2}\cos(\Phi_n(u))$ is bounded by $O(\sqrt{N(t)}) = O(t^{1/4})$ where $t = \Theta^{-1}(u)$.

Cross terms involving the remainder $R(\Theta^{-1}(u+h)) = O(t^{-1/4})$ give:

$$W(u,h)\cdot O(t^{1/4})\cdot O(t^{-1/4}) = O((\log t)^{-1}) \tag{85}$$

Changing to t-coordinates with $u = \Theta(t)$ and $du = \Theta'(t)\,dt$:

$$\frac{1}{U}\int_{t_0}^{t_1} \frac{\Theta'(t)}{\log t}\, dt \sim \frac{1}{U}\int_{t_0}^{t_1} \frac{\frac{1}{2}\log t}{\log t}\, dt = \frac{t_1 - t_0}{2U} \tag{86}$$

Since $U = \Theta(t_1) - \Theta(t_0) \sim \frac{t_1}{2}\log t_1$ for large t_1:

$$\frac{t_1 - t_0}{2U} \sim \frac{t_1}{t_1 \log t_1} = (\log t_1)^{-1} \to 0 \tag{87}$$

Similarly, $R(\Theta^{-1}(u))\, R(\Theta^{-1}(u+h)) = O(t^{-1/2})$ gives Cesàro average $O(t^{-1/2}) \to 0$.

Step 6: Independence of starting point

For any bounded integrable function f and starting points $u_0, \tilde{u}_0 \geq \Theta(0)$:

$$\left|\frac{1}{U}\int_{u_0}^{u_0+U} f\, du - \frac{1}{U}\int_{\tilde{u}_0}^{\tilde{u}_0+U} f\, du\right| \leq \frac{2\,|\tilde{u}_0 - u_0|\sup|f|}{U} \to 0 \tag{88}$$

Conclusion

Combining Steps 3–6, the Cesàro covariance limit

$$C(h) = \lim_{U\to\infty} \frac{1}{U - \Theta(0)}\int_{\Theta(0)}^{U} X(u)\, X(u+h)\, du \tag{89}$$

exists and is independent of the starting point $\Theta(0)$. This establishes that $X(u)$ is wide-sense stationary in the Cesàro sense. □

Corollary 26. *The Hardy Z-function is a unitarily time-changed stationary process $Z = U_\Theta X$, where X is the Cesàro-stationary process characterized by Theorem 25. Therefore, Z is an oscillatory process with evolutionary power spectral density*

$$S_t(\lambda) = \Theta'(t)\, S_X(\lambda) \tag{90}$$

where $S_X(\lambda)$ is the spectral density of X.

Remark 27. The convergence of the Cesàro covariance rigorously establishes that the Hardy Z-function, when viewed through theta-time coordinates, admits a well-defined stationary structure. The explicit form of $C(h)$ encodes the deep spectral properties of the Riemann zeta function and requires detailed harmonic analysis of the Riemann-Siegel coefficients.

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