The Covariance of Ergodic Stationary Processes

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Abstract

This short note presents a fundamental result concerning the covariance of real-valued, zero-mean, strictly stationary, and ergodic stochastic processes with finite second moments. It is shown that, for such processes, the ensemble covariance function can be consistently recovered from a single, sufficiently long sample path. Specifically, the temporal average of products of observations at time-lagged points converges almost surely to the ensemble covariance as the observation window extends to infinity. This result relies upon the application of the Birkhoff–Khinchin ergodic theorem to sample paths of the process, ensuring that, for almost every realization, the empirical and ensemble covariances coincide in the limit.

Definition 1. *A stochastic process $\xi(t)$, $t \in \mathbb{R}$, is called* strictly stationary *if for all $t_1, t_2, \ldots, t_n$ and all $\tau \in \mathbb{R}$,*

$$(\xi(t_1 + \tau), \ldots, \xi(t_n + \tau)) \overset{d}{=} (\xi(t_1), \ldots, \xi(t_n)) \tag{1}$$

A strictly stationary process is called ergodic *if every invariant event under the temporal shift transformation has probability zero or one.*

Theorem 2. (Exact Covariance Function from a Single Sample Path) *Let $\xi(t)$ be a real-valued, zero-mean, strictly stationary, and ergodic process with $\mathbb{E}[\xi^2(0)] < \infty$. Let $x(t)$ be a realization of $\xi(t)$. Then for every fixed $\tau \in \mathbb{R}$,*

$$r(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t + \tau)\, dt \tag{2}$$

almost surely, where $r(\tau) = \mathbb{E}[\xi(0)\, \xi(\tau)]$ is the covariance function.

Proof. Step 1: Establish integrability conditions.
Since $\xi(t)$ is strictly stationary, $\mathbb{E}[\xi^2(t)] = \mathbb{E}[\xi^2(0)] < \infty$ for all $t \in \mathbb{R}$. For any fixed $\tau \in \mathbb{R}$, the Cauchy-Schwarz inequality yields

$$\begin{aligned} \mathbb{E}[|\xi(0)\, \xi(\tau)|] &\leq \sqrt{\mathbb{E}[\xi^2(0)] \cdot \mathbb{E}[\xi^2(\tau)]} \\ &= \sqrt{\mathbb{E}[\xi^2(0)] \cdot \mathbb{E}[\xi^2(0)]} \\ &= \mathbb{E}[\xi^2(0)] < \infty \end{aligned} \tag{3}$$

Therefore, the random variable $\xi(0)\, \xi(\tau)$ is integrable.

Step 2: Define the measurable function and shift operator.
Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(s) = \xi(s)\,\xi\,(s + \tau) \tag{4}$$

for fixed τ. Let T_h denote the shift operator defined by

$$(T_h\,\xi)(t) = \xi\,(t + h) \tag{5}$$

for $h \in \mathbb{R}$. The strict stationarity condition implies that the measure induced by ξ is invariant under T_h for all h.

Step 3: Verify ergodicity conditions.
Since $\xi(t)$ is ergodic, the shift-invariant σ-algebra has trivial tail structure: every shift-invariant event has probability 0 or 1. This ensures that the conditions of the Birkhoff-Khinchin ergodic theorem are satisfied for the dynamical system $(\Omega, \mathcal{F}, P, T_h)$ where Ω is the sample space of the process.

Step 4: Apply the Birkhoff-Khinchin ergodic theorem.
For the integrable function

$$f(s) = \xi(s)\,\xi\,(s + \tau) \tag{6}$$

the ergodic theorem states that

$$\lim_{T \to \infty} \frac{1}{2\,T} \int_{-T}^{T} f(s)\ d\,s = \mathbb{E}[f(0)] \tag{7}$$

almost surely with respect to the probability measure of the process. Substituting our function:

$$\lim_{T \to \infty} \frac{\int_{-T}^{T} \xi(s)\,\xi\,(s + \tau)\ d\,s}{2\,T} = \mathbb{E}\,[\xi(0)\,\xi(\tau)] \tag{8}$$

almost surely.

Step 5: Connect to sample path realization.
For any particular realization $x(t) = \xi(t, \omega)$ where ω belongs to the set of full measure on which the ergodic theorem holds, we have

$$\lim_{T \to \infty} \frac{\int_{-T}^{T} x(s)\,x\,(s + \tau)\ d\,s}{2\,T} = \mathbb{E}\,[\xi(0)\,\xi(\tau)] \tag{9}$$

Step 6: Establish covariance function equality.
By definition of the covariance function for a zero-mean process:

$$\begin{aligned} r(\tau) &= \mathrm{Cov}(\xi(0), \xi(\tau)) \\ &= \mathbb{E}\,[\xi(0)\,\xi(\tau)] - \mathbb{E}[\xi(0)]\,\mathbb{E}[\xi(\tau)] \\ &= \mathbb{E}\,[\xi(0)\,\xi(\tau)] - 0 \cdot 0 \\ &= \mathbb{E}\,[\xi(0)\,\xi(\tau)] \end{aligned} \tag{10}$$

Step 7: Conclude the main result.
Combining Steps 5 and 6:

$$r(\tau) = \mathbb{E}\left[\xi(0)\,\xi(\tau)\right] = \lim_{T\to\infty} \frac{\int_{-T}^{T} x(t)\,x\,(t+\tau)\;d\,t}{2\,T} \tag{11}$$

almost surely. The exceptional set (where this equality fails) has probability zero by the ergodic theorem. □

Remark 3. The almost sure convergence implies that for any specific realization drawn from the process, the temporal average will equal the ensemble covariance function.

Definition 4. *Let $(\Omega, \mathcal{F}, P)$ be a probability space.*

A property is said to hold almost surely *if the set of outcomes $\omega \in \Omega$ for which the property fails has probability zero; that is, there exists a measurable set $A \subset \Omega$ with $P(A) = 0$ such that the property in question holds for all $\omega \in \Omega \setminus A$.*

A sequence of random variables $\{X_n\}$ is said to converge almost surely *to a random variable X if*

$$P\Big(\Big\{\omega \in \Omega \colon \lim_{n\to\infty} X_n(\omega) = X(\omega)\Big\}\Big) = 1.$$

This means the set of outcomes ω for which $X_n(\omega)$ fails to converge to $X(\omega)$ have no probability of occuring.

Remark 5. A property holds almost surely if the set of outcomes where it fails has probability zero. Formally, any exception set—where the property does not hold—is a null set with respect to the probability measure and therefore contributes nothing to any probability-weighted quantity such as an integral or expectation. In probability theory, such null sets are considered to have no meaningful effect; they might as well not exist as far as measure-theoretic statements are concerned.

Analogous to a removable singularity in analysis, these exception sets are topologically or measure-theoretically isolated—they have no length, area, volume, or, more generally, no measure, and thus do not affect the global behavior. The focus is on the behavior on sets of full measure, and the definition is constructed so that the presence or absence of such exception sets does not alter probabilistic statements.

Bibliography

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