The Operational Matrix of The Random Wave Process

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Introduction

Let I be defined as the inverse Fourier transform

$$I_{m,n}(y) = \int_{-1}^{1} {}_2F_1\left(-m, m+1; 1; \frac{1}{2} - \frac{x}{2}\right) {}_2F_1\left(-n, n+1; 1; \frac{1}{2} - \frac{x}{2}\right) e^{ixy}\, d\, x \tag{1}$$

where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function, and m, n are non-negative integers.

Hypergeometric Series Expansion

Lemma 1. *For any non-negative integer p and complex numbers b, c with $c \notin \{0, -1, -2, \ldots\}$:*

$${}_2F_1(-p, b; c; z) = \sum_{k=0}^{p} \frac{(-p)_k (b)_k}{(c)_k\, k!} z^k \tag{2}$$

where $(a)_k = a\,(a+1) \cdots (a+k-1)$ is the Pochhammer symbol.

Proof. By definition, ${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k\, k!} z^k$. When $a = -p$ for non-negative integer p:

$$\prod_{i=0}^{k-1} (-p + i) \tag{3}$$

Therefore $(-p)_k = 0$ for all $k > p$ since one of the factors becomes zero. Thus the infinite series terminates at $k = p$. $\square$

Lemma 2. *For the given integral, applying the series expansion:*

$$I_{m,n}(y) = \sum_{k=0}^{m} \sum_{l=0}^{n} \frac{(-m)_k\,(m+1)_k}{k!} \frac{(-n)_l\,(n+1)_l}{l!} \int_{-1}^{1} \left(\frac{1}{2} - \frac{x}{2}\right)^{k+l} e^{ixy}\,dx \tag{4}$$

Proof. Substituting the series expansions for both hypergeometric functions:

$$I_{m,n}(y) = \int_{-1}^{1} \left(\sum_{k=0}^{m} \frac{(-m)_k\,(m+1)_k}{k!} \left(\frac{1}{2} - \frac{x}{2}\right)^k \right) \left(\sum_{l=0}^{n} \frac{(-n)_l\,(n+1)_l}{l!} \left(\frac{1}{2} - \frac{x}{2}\right)^l \right) e^{ixy} dx \tag{5}$$

The series are finite, so we can interchange summation and integration by Fubini's theorem. $\square$

Integral Evaluation

Lemma 3. *For non-negative integer s:*

$$\int_{-1}^{1} \left(\frac{1}{2} - \frac{x}{2}\right)^s e^{ixy}\,dx = \frac{e^{iy}}{2^s} \left[\frac{\Gamma(s+1, -2\,i\,y)}{(-i\,y)^{s+1}} - \frac{\Gamma\,(s+1)}{(-i\,y)^{s+1}} \right] \tag{6}$$

Proof. Make the substitution $u = 1 - x$. Then $dx = -du$ and when $x = -1$, $u = 2$; when $x = 1$, $u = 0$. Thus:

$$\begin{aligned} \int_{-1}^{1} \left(\frac{1}{2} - \frac{x}{2}\right)^s e^{ixy}\,dx &= \frac{1}{2^s} \int_{-1}^{1} (1-x)^s\, e^{ixy}\,dx \\ &= \frac{1}{2^s} \int_{0}^{2} u^s\, e^{iy(1-u)}\,du \\ &= \frac{e^{iy}}{2^s} \int_{0}^{2} u^s\, e^{-iuy}\,du \end{aligned} \tag{7}$$

Let $v = i\,u\,y$. Then $du = \frac{dv}{i\,y}$ and:

$$\begin{aligned} \frac{e^{iy}}{2^s} \int_{0}^{2} u^s\, e^{-iuy}\,du &= \frac{e^{iy}}{2^s} \frac{1}{(i\,y)^{s+1}} \int_{0}^{2iy} v^s\, e^{-v}\,dv \\ &= \frac{e^{iy}}{2^s} \frac{1}{(i\,y)^{s+1}} [\gamma(s+1, 2\,i\,y)] \end{aligned} \tag{8}$$

where $\gamma(a, z) = \Gamma(a) - \Gamma(a, z)$ is the lower incomplete gamma function. $\square$

Double Sum Transformation

Theorem 4. *The double sum can be rewritten as:*

$$\sum_{l=0}^{m+n} \Phi_s(m, n)\, z^l = \sum_{k=0}^{m} \sum_{l=0}^{n} \frac{(-m)_k\,(m+1)_k}{k!} \frac{(-n)_l\,(n+1)_l}{l!} z^{k+l} \tag{9}$$

where

$$\Phi_s(m, n) = {}_3F_2(-m, m+1, -s; 1, 1-s-n; 1) \frac{(-n)_s\,(n+1)_s}{s!} \tag{10}$$

Proof. For fixed $s = k + l$ sum over all k from 0 to $\min(m, s)$ with $l = s - k$ and factor out terms independent of k

$$\begin{aligned} \sum_{k=0}^{\min(m,s)} \frac{(-m)_k\,(m+1)_k}{k!} \frac{(-n)_{s-k}\,(n+1)_{s-k}}{(s-k)!} &= \frac{(-n)_s\,(n+1)_s}{s!} \sum_{k=0}^{\min(m,s)} \frac{(-m)_k\,(m+1)_k\,(-s)_k}{k!(1)_k\,(1-s-n)_k} \\ &= {}_3F_2(-m, m+1, -s; 1, 1-s-n; 1) \end{aligned} \tag{11}$$ $\square$

Final Result

Theorem 5. *The integral evaluates to:*

$$I_{m,n}(y) = e^{iy} \sum_{s=0}^{m+n} \frac{\Phi_s(m, n)}{2^s} \left[\frac{\Gamma(s+1, -2\,i\,y) - \Gamma(s+1)}{(-i\,y)^{s+1}}\right] \tag{12}$$

where $\Phi_s(m, n)$ is as defined above.

Proof. Combining the series expansion from Lemma 2, the integral evaluation from Lemma 3, and the sum transformation from Theorem 1:

$$\begin{aligned} I_{m,n}(y) &= \sum_{k=0}^{m} \sum_{l=0}^{n} \frac{(-m)_k\,(m+1)_k}{k!} \frac{(-n)_l\,(n+1)_l}{l!} \int_{-1}^{1} \left(\frac{1}{2} - \frac{x}{2}\right)^{k+l} e^{ixy}\, dx \\ &= \sum_{s=0}^{m+n} \Phi_s(m, n) \cdot \frac{e^{iy}}{2^s} \left[\frac{\Gamma(s+1, -2\,i\,y) - \Gamma(s+1)}{(-i\,y)^{s+1}}\right] \end{aligned} \tag{13}$$

$\square$