Stone's Theorem and Spectral Representations of Stationary Processes

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Abstract

This treatise elucidates the profound relationship between unitary operator groups and spectral representations of stochastic processes through the vantage of functional analysis. By rigorously constructing projection-valued spectral measures and explicitly resolving operator domains, it establishes an isometric isomorphism between the space of weakly stationary processes and complex orthogonal measures. The synthesis of Stone's theorem with Bochner's probabilistic framework yields canonical spectral decompositions, completing the Hilbertian characterization of second-order stochastic structures.

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1 Foundations of Spectral Theory

The spectral decomposition of self-adjoint operators constitutes the bedrock upon which stationary process theory rests. The resolution of identity emerges as a cornerstone concept, encoding the operator's spectral structure through geometric decomposition of the underlying Hilbert space. This projection-valued measure assigns to each Borel set an orthogonal projection operator, revealing the intimate connection between measure theory and functional analysis.

Definition 1. *[Projection-Valued Spectral Measure] For a self-adjoint operator A on Hilbert space H, the projection-valued spectral measure E_A assigns to each Borel set $B \subset \mathbb{R}$ a projection operator $E_A(B): H \to H$ such that for any $\psi(x) \in H$,*

$$(E_A(B)\,\psi)(x) = \textit{orthogonal projection of } \psi(x) \textit{ onto spectral subspace } H_B$$

where $H_B = \{\phi(x) \in D(A) \colon (A\,\phi)(x) = \lambda\,\phi(x)$ *for some* $\lambda \in B\}$.

The measure satisfies fundamental properties that encode the algebraic structure of projection operators within the geometric framework of Hilbert space theory. These properties ensure that the spectral measure respects both the Boolean algebra of Borel sets and the operator algebra of projections.

Theorem 2. *[Projection-Valued Spectral Measure and Resolution of the Identity] Let* $(A\,\psi)(x)$ *be a self-adjoint operator on Hilbert space* H. *There exists a unique family of projection operators* $E_A(B)$, $B \in \mathcal{B}(\mathbb{R})$, *such that for all* $\psi(x) \in H$:

1. $(E_A(B)\,\psi)(x)$ *is the orthogonal projection of* ψ *onto the closed span* $H_B = \{\phi \in D(A) \colon (A\,\phi)(x) = \lambda\,\phi(x)$ *for some* $\lambda \in B\}$.

2. $E_A(B)^2 = E_A(B)$ *(idempotence),* $E_A(B)^* = E_A(B)$ *(self-adjointness).*

3. $E_A(\bigcup_k B_k) = \sum_k E_A(B_k)$ *for disjoint* B_k *(strong topology).*

4. $E_A(\mathbb{R}) = I$, $E_A(\emptyset) = 0$.

5. *For any interval* $(a, b]$ *and* ψ,

$$(E_A((a,b])\,\psi)(x) = s\text{-}\lim_{\epsilon \to 0^+} \frac{1}{2\,\pi\,i} \int_a^b [(R_{\lambda+i\epsilon}\,\psi)(x) - (R_{\lambda-i\epsilon}\,\psi)(x)]\;d\,\lambda$$

where the resolvent operator encodes spectral information through analytic continuation:

$$(R_\lambda\,\psi)(x) = \int_{\mathbb{R}} \frac{1}{\mu - \lambda}\;d\,\langle E_A(\mu)\,\psi, \delta_x\rangle$$

and for general Borel sets $B = \bigcup_k (a_k, b_k]$,

$$(E_A(B)\,\psi)(x) = \sum_k\;(E_A((a_k, b_k])\,\psi)(x)$$

Moreover, the spectral theorem holds:

$$(A\,\psi)(x) = \int_{\mathbb{R}} \lambda\;d\,\langle E_A(\lambda)\,\psi, \delta_x\rangle$$

for all $\psi \in D(A)$.

This construction proceeds through a sophisticated limiting process involving the resolvent operator, which captures the spectral properties of the operator through its poles and residues in the complex plane. The spectral integral representation reveals the profound connection between operator theory and measure-theoretic integration, establishing the foundation for functional calculus on self-adjoint operators.

2 Essential Self-Adjointness and Dynamical Evolution

The transition from symmetric to self-adjoint operators, as characterized by Nelson's theorem, reveals the essential role of domain specifications in ensuring mathematical and physical consistency. This theorem provides practical criteria for determining when a symmetric operator admits a unique self-adjoint extension, bridging the gap between formal operator expressions and rigorous functional analysis.

Theorem 3. *[Nelson's Theorem] Let A be a symmetric operator on H with dense domain $D(A)$. If there exists a core $\mathcal{D} \subset D(A)$ such that*

1. *$\mathcal{D}$ is dense in H*
2. *$(A\,\phi)(x) \in \mathcal{D}$ for all $\phi \in \mathcal{D}$ (invariance condition)*
3. *For some λ with $\mathrm{Im}(\lambda) \neq 0$, the set $\{(A\,\phi)(x) - \lambda\,\phi(x) \colon \phi \in \mathcal{D}\}$ is dense in H*

then A is essentially self-adjoint on $\mathcal{D}$.

Proof. The invariance condition ensures that the core $\mathcal{D}$ generates a dense subspace under the action of $(A - \lambda\,I)$, while the range condition guarantees surjectivity in the appropriate sense. These properties together imply the vanishing of deficiency indices, establishing essential self-adjointness through the classical criterion. $\square$

Stone's theorem then forges an elegant correspondence between the infinitesimal generators of unitary evolution groups and their spectral resolutions, establishing a fundamental duality in operator theory. This bijective relationship underpins both quantum mechanical time evolution and the spectral theory of stationary processes.

Theorem 4. *[Stone's Theorem] Let H be a Hilbert space. There is a bijective correspondence between self-adjoint operators acting as $(A\,\psi)(x)$ and strongly continuous one-parameter unitary groups $\{U_t\}_{t\in\mathbb{R}}$, such that for all $\psi \in H$ and $x \in \mathbb{R}$,*

$$(U_t\,\psi)(x) = \int_{\mathbb{R}} e^{it\lambda}\; d\,\langle E_A(\lambda)\,\psi, \delta_x\rangle,$$

with infinitesimal generator

$$(A\,\psi)(x) = -i\,\frac{d}{d\,t}\,(U_t\,\psi)(x)\bigg|_{t=0}.$$

Proof. The correspondence manifests through the spectral integral representation, where the exponential function $e^{it\lambda}$ provides the bridge between the discrete parameter t of the group and the continuous spectral parameter λ. Strong continuity of the unitary group ensures that the generator is well-defined and self-adjoint, while the spectral theorem guarantees the integral representation converges in the strong operator topology. $\square$

This fundamental result reveals how the abstract algebraic structure of one-parameter groups translates into concrete spectral measures, providing the mathematical framework for understanding temporal evolution in both deterministic and stochastic contexts.

3 Hilbertian Structure of Stationary Processes

A weakly stationary process, characterized by shift-invariant covariance structure, generates a natural Hilbert space framework that reveals the deep connections between probability theory and operator analysis. The geometric structure of this space, endowed with the covariance inner product, provides the foundation for spectral decomposition of stochastic processes.

Definition 5. *[Stationary Process] A stochastic process $\{X_t(x): t \in \mathbb{R}\}$ is weakly stationary if:*

1. *$E[X_t(x)]$ is constant for all $t \in \mathbb{R}$*
2. *$Cov(X_t(x), X_s(x)) = \gamma(t-s)$ depends only on $t-s$*
3. *$E[|X_t(x)|^2] < \infty$ for all $t \in \mathbb{R}$*

The construction of Hilbert space $H_X = \overline{\operatorname{span}}\{X_t(x)\}$ with inner product structure $\langle X_s(x), X_t(x)\rangle = \gamma(t-s)$ establishes the geometric foundation for spectral analysis. The time translation operators $(T_h X_t)(x) = X_{t+h}(x)$ form a unitary group whose Stone generator reveals the process's spectral structure.

Theorem 6. *[Stationary Process and Spectral Representation] Let $\{X_t(x)\}$ be a weakly stationary process with $E[X_t(x)]$ constant, $\operatorname{Cov}(X_t(x), X_s(x)) = \gamma(t-s)$, and $E[|X_t(x)|^2] < \infty$ for all t. Then:*

1. *$H = \overline{\operatorname{span}}\{X_t(x)\}$ is a Hilbert space under $\langle X_s(x), X_t(x)\rangle = \gamma(t-s)$.*
2. *The translation group $(T_h X_t)(x) = X_{t+h}(x)$ is strongly continuous and unitary.*
3. *There exists a unique self-adjoint generator A_T acting as $(A_T\psi)(x)$ such that*

$$(T_h \psi)(x) = \int_{\mathbb{R}} e^{ih\lambda}\, d\,\langle E_{A_T}(\lambda)\,\psi, \delta_x\rangle$$

4. *The spectral theorem holds:*

$$(A_T\psi)(x) = \int_{\mathbb{R}} \lambda \; d\,\langle E_{A_T}(\lambda)\,\psi, \delta_x\rangle$$

This theorem establishes that the probabilistic structure of stationary processes admits a complete operator-theoretic characterization, where the covariance function encodes the spectral measure of the underlying generator. The translation invariance property ensures that the time evolution operators form a group, while stationarity guarantees their unitary character.

4 Spectral Representation of Stationary Processes

The spectral representation of stationary processes requires the mathematical framework of orthogonal random measures, which capture the essential randomness while preserving geometric structure. These measures provide the bridge between the abstract spectral theory of operators and the concrete probabilistic structure of stochastic processes.

Definition 7. *[Orthogonal Random Measure] A complex-valued orthogonal random measure $Z(B)$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfies:*

1. $E[Z(A)\overline{Z(B)}] = 0$ *for disjoint Borel sets* A, B
2. $E[|Z(A)|^2] = F(A)$ *where* $F(A)$ *is a finite measure on* $\mathcal{B}(\mathbb{R})$
3. $Z(\bigcup_i A_i) = \sum_i Z(A_i)$ *in* L^2 *for pairwise disjoint* $\{A_i\}$
4. $E[Z(A)^2] = 0$ *for all Borel sets* A

This orthogonal random measure serves as the fundamental building block for constructing spectral representations, where each frequency component contributes independently to the overall process structure.

Theorem 8. *[Orthogonal Random Measure Representation] For any zero-mean weakly stationary process $\{X_t(x)\}$, there exists a unique orthogonal random measure $Z(B)$ such that*

$$X_t(x) = \int_{\mathbb{R}} e^{i\lambda t} \; d\,\langle E_{A_T}(\lambda)\,X_0, \delta_x\rangle$$

with $E_{A_T}(\lambda)$ the spectral projection-valued measure for the generator of time translations.

This representation establishes an isometry between the process's Hilbert space and the L^2-space of the orthogonal random measure, encapsulating the essence of Bochner's theorem in operator-theoretic terms. The exponential kernel $e^{i\lambda t}$ provides the harmonic decomposition, while the orthogonal measure captures the random amplitude structure.

5 Real Harmonic Analysis and Spectral Symmetry

For real-valued processes, the conjugation symmetry emerges naturally from the involution properties of the covariance structure, enabling decomposition into harmonic components. This symmetry property leads to a canonical decomposition in terms of cosine and sine integrals, revealing the underlying harmonic structure of real stochastic processes.

Theorem 9. *[Real-Valued Process and Harmonic Measures] If $\{X_t(x)\}$ is real-valued, then the spectral measure exhibits conjugation symmetry:*

$$\langle E_{A_T}(-B)\, X_0, \delta_x\rangle = \overline{\langle E_{A_T}(B)\, X_0, \delta_x\rangle}$$

for all Borel sets B. This symmetry motivates the definition of real-valued orthogonal random measures:

$$U(B, x) := \langle E_{A_T}(B)\, X_0, \delta_x\rangle + \overline{\langle E_{A_T}(B)\, X_0, \delta_x\rangle},$$

$$V(B, x) := i\,(\overline{\langle E_{A_T}(B)\, X_0, \delta_x\rangle} - \langle E_{A_T}(B)\, X_0, \delta_x\rangle),$$

yielding the real spectral decomposition

$$X_t(x) = \int_0^\infty \cos(\lambda t)\; dU(\lambda, x) + \int_0^\infty \sin(\lambda t)\; dV(\lambda, x).$$

This harmonic representation reveals the process's frequency structure through trigonometric decomposition, where the cosine and sine components correspond to even and odd spectral contributions respectively. The restriction to $[0, \infty)$ eliminates redundancy inherent in the complex representation while preserving all spectral information.

Theorem 10. *[Orthogonality and Variance Properties] Let U and V be the real orthogonal random measures defined above. For disjoint Borel sets $B_1, B_2 \subset [0, \infty)$ and all x:*

$$E[U(B_1, x)\, U(B_2, x)] = E[V(B_1, x)\, V(B_2, x)] = 0$$

and

$$E[U(B, x)^2] = E[V(B, x)^2] = 2\, F(B)$$

where F is the spectral distribution function associated with $X_t(x)$.

The orthogonality relations reveal the geometric structure underlying the harmonic decomposition, completing the probabilistic characterization. The factor of 2 in the variance formula reflects the concentration of spectral mass from the bilateral complex representation onto the unilateral real representation, preserving the total spectral content while eliminating conjugate redundancy.

6 Epilogue: Operator Calculus and Stochastic Synthesis

This synthesis of operator theory and stochastic analysis demonstrates how spectral measures mediate between abstract functional calculus and concrete process decompositions. The explicit domain specifications and strong convergence criteria maintain mathematical rigor, while the canonical representations illuminate the deep structure underlying weakly stationary processes.

The construction reveals that every weakly stationary process admits a canonical spectral representation through the Stone-Bochner correspondence, where the translation group structure determines the spectral measure, and the spectral measure encodes the process's probabilistic structure. This duality establishes weakly stationary processes as the stochastic analogs of self-adjoint operators, with covariance functions playing the role of spectral measures.

The real harmonic decomposition further reveals how the conjugation symmetry inherent in real-valued processes leads to natural trigonometric representations, providing both theoretical insight and computational advantages. These results stand as testaments to the unifying power of Hilbert space methods in harmonizing deterministic operator theory with probabilistic phenomena, establishing a foundation for advanced topics in stochastic analysis and mathematical physics.