Locally Convex Topological Vector Spaces

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Definition 1. *[Seminorm] Let X be a vector space over $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$). A seminorm is a function $p: X \to [0, \infty)$ satisfying:*

1. *$p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$ (subadditivity)*
2. *$p(\alpha x) = |\alpha|\, p(x)$ for all $x \in X$, $\alpha \in \mathbb{K}$ (homogeneity)*

Definition 2. *[Locally Convex Topological Vector Space] A locally convex topological vector space (LCTVS) is a pair (X, P) where:*

- *X is a vector space over $\mathbb{K}$*
- *$P = \{p_\alpha: \alpha \in A\}$ is a family of seminorms where A is an arbitrary index set*
- *For any $x \neq 0$, there exists $\alpha \in A$ such that $p_\alpha(x) \neq 0$ (separation condition)*

The topology τ on X is generated by the base of neighborhoods of 0 of the form:

$$V(\alpha_1, \ldots, \alpha_n; \varepsilon) = \{x \in X: p_{\alpha_i}(x) < \varepsilon \text{ for } i = 1, \ldots, n\}$$

where $\varepsilon > 0$, $n \in \mathbb{N}$, and $\alpha_1, \ldots, \alpha_n \in A$

Example 3. [Continuous Functions] Let $C[a, b]$ be the space of continuous functions on $[a, b]$. We define three families of seminorms:

1. $p_1(f) = \max\{|f(x)|: x \in [a, b]\}$
2. $p_2(f) = \int_a^b |f(x)|\, d x$
3. $p_t(f) = |f(t)|\ \forall t \in [a, b]$

with seminorms:

For p_1:

- Subadditivity:

$$\begin{aligned} p_1(f + g) &= \max|f + g| \leq \max(|f| + |g|) \\ &\leq \max|f| + \max|g| = p_1(f) + p_1(g) \end{aligned} \tag{1}$$

- Homogeneity:

$$p_1(\alpha f) = \max|\alpha f| = |\alpha| \max|f| = |\alpha|\, p_1(f) \tag{2}$$

For p_2:

1. Subadditivity:

$$\begin{aligned} p_2(f+g) &= \int |f+g| \leq \int (|f|+|g|) \\ &= \int |f| + \int |g| = p_2(f) + p_2(g) \end{aligned} \tag{3}$$

2. Homogeneity:

$$p_2(\alpha f) = \int |\alpha f| = |\alpha| \int |f| = |\alpha|\, p_2(f) \tag{4}$$

For p_t:

- Subadditivity: $p_t(f+g) = |f(t)+g(t)| \leq |f(t)| + |g(t)| = p_t(f) + p_t(g)$

- Homogeneity: $p_t(\alpha f) = |\alpha f(t)| = |\alpha||f(t)| = |\alpha|\, p_t(f)$

The separation condition is satisfied: if $f \neq 0$, then $p_1(f) > 0$

Example 4. [Smooth Functions] Let $C^\infty(\mathbb{R})$ be the space of smooth functions. Define:

$$p_{k,n}(f) = \sup\{|f^{(n)}(x)| : x \in [-k,k]\} \tag{5}$$

for $k, n \in \mathbb{N}$.

Proposition 5. *These are seminorms:*

1. *Subadditivity:*

$$\begin{aligned} p_{k,n}(f+g) &= \sup|f^{(n)} + g^{(n)}| \\ &\leq \sup|f^{(n)}| + \sup|g^{(n)}| \\ &= p_{k,n}(f) + p_{k,n}(g) \end{aligned} \tag{6}$$

2. *Homogeneity:*

$$p_{k,n}(\alpha f) = \sup |\alpha f^{(n)}| = |\alpha| \sup |f^{(n)}| = |\alpha| p_{k,n}(f) \tag{7}$$

The separation condition is satisfied: if $f \neq 0$, some derivative must be nonzero at some point, so some $p_{k,n}(f) > 0$.

Example 6. [Sequence Space] Let $\mathbb{C}^{\mathbb{N}}$ be the space of complex sequences. Define:

1. $p_k(x) = |x_k|$ for each $k \in \mathbb{N}$
2. $q_n(x) = \max \{|x_k|: 1 \leq k \leq n\}$

These are seminorms:

For p_k:

- Subadditivity:

$$p_k(x+y) = |x_k + y_k| \leq |x_k| + |y_k| = p_k(x) + p_k(y) \tag{8}$$

- Homogeneity:

$$q_n(x+y) \quad = \max |x_k + y_k| \leq \max (|x_k| + |y_k|) \tag{9}$$

For q_n:

- Subadditivity:

$$\begin{aligned} q_n(x+y) &= \max |x_k + y_k| \leq \max (|x_k| + |y_k|) \\ &\leq \max |x_k| + \max |y_k| = q_n(x) + q_n(y) \end{aligned} \tag{10}$$

- Homogeneity:

$$q_n(\alpha x) = \max |\alpha x_k| = |\alpha| \max |x_k| = |\alpha| q_n(x) \tag{11}$$

The separation condition is satisfied: if $x \neq 0$, then some $x_k \neq 0$, so $p_k(x) > 0$

In each example, the topology generated by these seminorms makes the space into a LCTVS(locally convex topological vector space) because:

1. Addition is continuous (which follows from subadditivity)

2. Scalar multiplication is continuous (which follows from homogeneity)

3. The separation condition ensures the topology is Hausdorff

Axiom 7. *The separation condition ensures our LCTVS is a Hausdorff space, also called a T_2 space or a separated space - these are all synonymous terms. A topological space is Hausdorff/T_2/separated if:*

$$\forall x, y \in X, x \neq y \Longrightarrow \exists \text{open sets } U, V \text{with } x \in U, y \in V, U \cap V = \emptyset \tag{12}$$

In an LCTVS, this follows from the separation condition:

- *Given $x \neq y$, let $z = x - y \neq 0$*

- *By separation condition, $\exists \alpha$ with $p_\alpha(z) > 0$*

- *Let*

$$U = \{u : p_\alpha(u - x) < \varepsilon\} \tag{13}$$

and

$$V = \{v : p_\alpha(v - y) < \varepsilon\} \tag{14}$$

where

$$\varepsilon = \frac{p_\alpha(z)}{2} \tag{15}$$

- *Then these open sets separate $x \in U$ and $y \in V$ such that $U \cap V = \emptyset$*

Note: While T_2, Hausdorff, and separated are synonymous, they are stronger than T_1 (which only requires points to be closed) and weaker than T_3 (regular) spaces.

Proposition 8. (The Separation Condition) *The condition "for any $x \neq 0$, there exists $\alpha \in A$ such that $p_\alpha(x) \neq 0$" is called the separation condition because:*

1. *It separates points from zero: Any nonzero point can be separated from zero by at least one seminorm*

2. *It ensures the topology is Hausdorff: Any two distinct points can be separated by disjoint neighborhoods*

To see (2): Let $x \neq y$ be distinct points. Then $z = x - y \neq 0$. By the separation condition, there exists α with $p_\alpha(z) > 0$. Let $\varepsilon = p_\alpha(z)/2$. Then:

- $U = \{u \colon p_\alpha(u - x) < \varepsilon\}$ *is a neighborhood of x*

- $V = \{v \colon p_\alpha(v - y) < \varepsilon\}$ *is a neighborhood of y*

Moreover, $U \cap V = \emptyset$, because if $w \in U \cap V$:

$$p_\alpha(z) = p_\alpha(x - y) \leq p_\alpha(w - y) + p_\alpha(x - w) < \varepsilon + \varepsilon = p_\alpha(z)$$

which is a contradiction.