

Oscillatory Process Inversion

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Theorem 1. *[Inversion of Oscillatory Processes] Given*

$$Z(s) = \int_{-\infty}^{\infty} \phi_s(\lambda) d\Phi(\lambda) \quad (1)$$

where $\phi_s(\lambda)$ is measurable, invertible ($\phi_s(\lambda) \neq 0$ for all s, λ), and satisfies the **quadratic integrability condition**:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\phi_t(\lambda)|^2 dt d\lambda < \infty \quad (2)$$

and $\Phi(\lambda)$ is a complex orthogonal random measure with

$$E(|d\Phi(\lambda)|^2) = d\mu(\lambda) \quad (3)$$

Define the stationary process:

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda t} d\Phi(\lambda) \quad (4)$$

Define the inverse kernel:

$$b(t, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\lambda t}}{\phi_s(\lambda)} e^{-i\lambda s} d\lambda \quad (5)$$

Invertibility condition: Require

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{1}{\phi_t(\lambda)} \right|^2 dt d\lambda < \infty \quad (6)$$

Then:

$$X(t) = \int_{-\infty}^{\infty} b(t, s) Z(s) ds \quad (7)$$

Proof. • Expand the convolution:

$$\begin{aligned} \int_{-\infty}^{\infty} b(t, s) Z(s) ds &= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\lambda t}}{\phi_s(\lambda)} e^{-i\lambda s} d\lambda \right] \\ &\quad \times \left[\int_{-\infty}^{\infty} \phi_s(\omega) d\Phi(\omega) \right] ds \end{aligned} \quad (8)$$

- Exchange order of integration by Fubini's theorem (justified by the quadratic integrability conditions (6)):

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{i\lambda t} \frac{\phi_s(\omega)}{\phi_s(\lambda)} e^{-i\lambda s} ds \right] d\lambda d\Phi(\omega) \quad (9)$$

- Evaluate the inner integral over s :

$$\int_{-\infty}^{\infty} \frac{\phi_s(\omega)}{\phi_s(\lambda)} e^{-i\lambda s} ds \quad (10)$$

For $\phi_s(\lambda) = A_s(\lambda) e^{i\lambda s}$:

$$\frac{\phi_s(\omega)}{\phi_s(\lambda)} e^{-i\lambda s} = \frac{A_s(\omega)}{A_s(\lambda)} e^{i(\omega-\lambda)s} \quad (11)$$

The integral becomes:

$$\int_{-\infty}^{\infty} \frac{A_s(\omega)}{A_s(\lambda)} e^{i(\omega-\lambda)s} ds = 2\pi \delta(\omega - \lambda) \quad (12)$$

where this holds as a distributional identity under the stated integrability and invertibility conditions.

- Substitute (12) back:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\lambda t} \cdot 2\pi \delta(\omega - \lambda) d\lambda d\Phi(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} d\Phi(\omega) = X(t) \quad (13)$$

□

Summary

Under the conditions:

1. $\iint |\phi_t(\lambda)|^2 dt d\lambda < \infty$
2. $\phi_s(\lambda) \neq 0$ everywhere
3. $\iint |1/\phi_t(\lambda)|^2 dt d\lambda < \infty$

The stationary process $X(t)$ is exactly inverted from the oscillatory process $Z(t)$ by equation (7) with kernel (5).