# A Uniformly Convergent Orthnormal Expansion for the Bessel Function of the First Kind of Order 0

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**Theorem 1**

*Let $\psi_n(y)$ be defined as*

$$\begin{aligned}\psi_n(y) &= (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, j_{2n}(y)\\ &= (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, \sqrt{\frac{\pi}{2\,y}}\, J_{2n+\frac{1}{2}}(y)\\ &= (-1)^n \sqrt{\frac{(4n+1)\pi}{\pi\,2\,y}}\, J_{2n+\frac{1}{2}}(y)\\ &= (-1)^n \sqrt{\frac{(4n+1)}{2\,y}}\, J_{2n+\frac{1}{2}}(y)\\ &= (-1)^n \sqrt{\frac{2n+\frac{1}{2}}{y}}\, J_{2n+\frac{1}{2}}(y)\end{aligned} \tag{1}$$

*where $J_\nu$ denotes the Bessel function of the first kind and $j_n$ the spherical Bessel function. Then*

$$\begin{aligned}J_0(x) &= \sum_{n=0}^{\infty} \psi_n(x) \int_0^\infty J_0(y)\, \psi_n(y) \mathrm{d}y\\ &= \sum_{n=0}^{\infty} (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, j_{2n}(x) \sqrt{\frac{2\,n+\frac{1}{2}}{\pi}}\, \frac{\Gamma\,(n+\frac{1}{2})^2}{\Gamma\,(n+1)^2}\\ &= \frac{1}{2}\frac{1}{\sqrt{\pi\,x}} \sum_{n=0}^{\infty} (-1)^n\,(4\,n+1)\,\frac{\Gamma\,(n+\frac{1}{2})^2}{\Gamma\,(n+1)^2}\, J_{2n+\frac{1}{2}}(x)\\ &= \frac{1}{\sqrt{4\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(4\,n+1)\,\Gamma\,(n+\frac{1}{2})^2}{\Gamma\,(n+1)^2}\, J_{2n+\frac{1}{2}}(x)\end{aligned} \tag{2}$$

*with uniform convergence $\forall x \in \mathbb{C}$. Moreover, $\{\psi_n\}$ forms an orthonormal system in $L^2([0,\infty))$ satisfying*

$$\int_0^\infty \psi_m(t)\, \psi_n(t) \mathrm{d}t = \delta_{mn} = \begin{cases} 1 & m=n \\ 0 & m \neq n \end{cases} \tag{3}$$

**Proof.**

**Step 1: Orthonormality of $\psi_n(y)$.**

Substituting the definition:

$$\begin{aligned}\langle\psi_m,\psi_n\rangle\ &=(-1)^{m+n}\sqrt{\frac{(4\,m+1)\,(4\,n+1)}{\pi^2}}\,\frac{\pi}{2}\int_0^\infty\frac{J_{2m+\frac{1}{2}}(y)\,J_{2n+\frac{1}{2}}(y)}{y}\mathrm{d}y\\ &=(-1)^{m+n}\sqrt{\frac{(4\,m+1)\,(4\,n+1)}{4\,\pi}}\cdot\frac{\delta_{mn}}{2\,n+\frac{1}{2}}\\ &=\delta_{mn}\end{aligned}\tag{4}$$

The crucial integral follows from Bessel orthogonality:

$$\int_0^\infty\frac{J_\mu(y)\,J_\nu(y)}{y}\mathrm{d}y=\frac{2}{\pi}\,\frac{\delta_{\mu\nu}}{\mu+\nu}\tag{5}$$

**Step 2: Expansion Coefficients.** Using the orthonormal basis:

$$c_n=\int_0^\infty J_0(y)\,\psi_n(y)\mathrm{d}y=(-1)^n\sqrt{\frac{4\,n+1}{2}}\,\frac{\Gamma\,(n+\frac{1}{2})^2}{\Gamma\,(n+1)^2}\tag{6}$$

This emerges from evaluating:

$$\int_0^\infty J_0(y)\frac{J_{2n+\frac{1}{2}}(y)}{\sqrt{y}}\mathrm{d}y=\frac{\Gamma\,(n+\frac{1}{2})^2}{\sqrt{2}\,\Gamma\,(n+1)^2}\tag{7}$$

**Step 3: Series Simplification.** Substitute $c_n$ into the expansion:

$$J_0(x)=\sum_{n=0}^\infty\frac{(-1)^n\,(4\,n+1)}{2\,\sqrt{\pi\,x}}\,\frac{\Gamma\,(n+\frac{1}{2})^2}{\Gamma\,(n+1)^2}\,J_{2n+\frac{1}{2}}(x)\tag{8}$$

TODO: write correct proof, its not due to Wierstrauss, hint, prove that we can always choose an N for a given epsilon such that including that many terms in the expansion results in an error less than epsilon and recognizing that the contribution from the n-th term can be no greater than $\int_0^\infty J_0(y)\,\psi_n(y)\mathrm{d}y$ $\square$