The Measurability of a Stochastic Process of Second Order and its Linear Space*

Stamatis Cambanis
Department of Statistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 832
June, 1972

1 The Measurability of a Stochastic Process of Second Order

Let T be a separable metric space and $\mathcal{B}(T)$ the σ-algebra of Borel sets of T, and let X_t, $t \in T$, be a real stochastic process on the probability space $(\Omega, \mathcal{F}, P)$. X_t, $t \in T$, is called measurable if the map $(t, \omega) \to X_t(\omega)$ is $\mathcal{B}(T) \times \mathcal{F}$-measurable. A process Y_t, $t \in T$, on $(\Omega, \mathcal{F}, P)$ is called a modification of X_t, $t \in T$, if $P\{X_t = Y_t\} = 1$ for all t in T. X_t, $t \in T$, is of second order if $E(X_t^2) < +\infty$ for all t in T, and then its autocorrelation R is defined by

$$R(t, s) = E(X_t X_s) \quad \text{for all } t, s \in T. \tag{1}$$

It is clear from Fubini's theorem that if a second order process X_t, $t \in T$, has a measurable modification then R is $\mathcal{B}(T) \times \mathcal{B}(T)$-measurable. That the measurability of R is not sufficient for the existence of a measurable modification of X_t, $t \in T$, is demonstrated in Remark 2. It is thus of interest to find a condition which along with the measurability of R would imply the existence of a measurable modification of X_t, $t \in T$. This question is answered in Theorem 1, where in fact necessary and sufficient conditions are given for a second order process to have a measurable modification. A remarkable consequence of these conditions is that the existence of a measurable modification of a second order process is a second order property.

The proof of Theorem 1 is based on the necessary and sufficient conditions for a process (not necessarily of second order) to have a measurable modification given in [5], which are expressed as follows (here the terminology of [6] is followed). Let M be the space of all real random variables on $(\Omega, \mathcal{F}, P)$ with the topology of convergence in probability, where random variables that are equal a.e. $[P]$ are considered identical. If ξ is a real random variable, its class in M is denoted by $[\xi]$. Then X_t, $t \in T$, has a measurable modification if and only if the

*This research was supported by the Air Force Office of Scientific Research under Grant AFOSR-68-1415.

map from T to M taking t to $[X_t]$ is measurable and has separable range [5, 6]. Moreover, the measurable modification can be taken to be separable and also progressively measurable, the latter if T is an interval and a nondecreasing family $\mathcal{F}_t$, $t \in T$, of sub-σ-algebras of $\mathcal{F}$ is given.

For a second order process X_t, $t \in T$, we denote by $H(X)$ the closure in $L_2(\Omega, \mathcal{F}, P)$ of the linear space of the random variables $\{X_t, t \in T\}$ and we call it the linear space of the process. We also denote by $R(K)$ the reproducing kernel Hilbert space of a real, symmetric, nonnegative definite function K on $T \times T$. It is well known that $R(R)$ consists of all functions f on T of the form $f(t) = E(\xi X_t)$, $t \in T$, for some $\xi \in H(X)$, and that the map $\xi \to E(\xi X_t)$ defines an inner product preserving isomorphism between $H(X)$ and $R(R)$ [16].

Theorem 1. *Let X_t, $t \in T$, be a real, second order process with autocorrelation R. The following are equivalent:*

(i) X_t, $t \in T$, has a measurable modification.

(ii) R is $\mathcal{B}(T) \times \mathcal{B}(T)$-measurable and $H(X)$ (or $R(R)$) is separable.

Proof. (a) We first show that ((ii)) implies ((i)). It suffices to verify the conditions of [5, 6]; the construction of a measurable modification is the same as in [5] or in [6].

Since convergence in $L_2(\Omega, \mathcal{F}, P)$ implies convergence in probability, the separability of $H(X)$ as a subset of $L_2(\Omega, \mathcal{F}, P)$ implies its separability as a subset of M. Thus its subset $\{[X_t], t \in T\}$ is separable in M. To complete the proof it suffices to show that the map $X : T \to M$ defined by $X(t) = [X_t]$ is measurable. The metric ρ on M defined by $\rho(\xi, \eta) = E\left[\frac{|\xi - \eta|}{1 + |\xi - \eta|}\right]$, $\xi, \eta \in M$ metrizes the topology of convergence in probability. Thus for the measurability of X it suffices to show that $X^{-1}(B) \in \mathcal{B}(T)$ for every set B in M of the form $B = \{Y \in M : \rho(Y, Y_0) \le r\}$, where $Y_0 \in M$ and $r > 0$. Since $X^{-1}(B) = \{t \in T : \rho([X_t], Y_0) < r\}$, it suffices to prove that the real function $\rho([X_t], Y_0)$ on T is $\mathcal{B}(T)$-measurable for all $Y_0 \in M$.

Let $\{\xi_n\}_{n=1}^{\infty}$ be a complete orthonormal sequence in $H(X)$ (which exists because $H(X)$ is separable). Then for all $t \in T$ we have

$$X_t = \sum_{n=1}^{\infty} a_n(t) \xi_n \quad \text{in } L_2(\Omega, \mathcal{F}, P), \tag{2}$$

where

$$a_n(t) = E(\xi_n X_t). \tag{3}$$

Thus $a_n \in R(R)$, and in fact $\{a_n\}_{n=1}^{\infty}$ is a complete orthonormal sequence in $R(R)$. If for every $t \in T$ we let

$$X_t^{(N)} = \sum_{n=1}^{N} a_n(t) \xi_n, \tag{4}$$

then $X_t^{(N)}$ converges to X_t in $L_2(\Omega, \mathcal{F}, P)$ and thus in probability. Let $Y_0 \in M$ be fixed and $Y_t = X_t - Y_0$ and $Y_t^{(N)} = X_t^{(N)} - Y_0$ for all $t \in T$. Then $Y_t^{(N)}$ converges to Y_t in probability, i.e., $\rho([Y_t^{(N)}],[Y_t]) \to 0$ as $N \to \infty$. Dropping the index t for simplicity we have

$$\rho([X^{(N)}],[X]) \leq \frac{E\left[\frac{|X^{(N)}-X|}{1+|X^{(N)}-X|}\right]}{E\left[\frac{|X|}{1+|X|}\right]}. \tag{5}$$

Thus

$$\rho([X_t],Y_0) = \lim_{N\to\infty} E\left[\frac{|X_t^{(N)}-Y_0|}{1+|X_t^{(N)}-Y_0|}\right] \quad \text{for all } t \in T. \tag{6}$$

Note that every function in $R(R)$ is either a finite linear combination of the functions $\{R(t,\cdot), t \in T\}$ or a pointwise limit on T of such functions. Hence, since R is $\mathcal{B}(T)\times\mathcal{B}(T)$-measurable, $R(t,\cdot)$ is $\mathcal{B}(T)$-measurable for all $t \in T$, and it follows that every f in $R(R)$ is $\mathcal{B}(T)$-measurable. Consequently $Y_t^{(N)} = \sum_{n=1}^{N} a_n(t)\xi_n - Y_0$ is $\mathcal{B}(T)\times\mathcal{F}$-measurable. By Fubini's theorem $E\left[\frac{|X_t^{(N)}-Y_0|}{1+|X_t^{(N)}-Y_0|}\right]$ is $\mathcal{B}(T)$-measurable, and thus so is $\rho([X_t],Y_0)$, which completes the proof.

(b) We now show that ((i)) implies ((ii)). The measurability of R follows from Fubini's theorem and ((i)). In order to prove the separability of $H(X)$ we first assume that R is uniformly bounded on T:

$$R(t,t) \leq C < +\infty \quad \text{for all } t \in T. \tag{7}$$

We will show that this implies the uniform integrability of the family of random variables $\{X_t, t\in T\}$. Indeed we have for all $a>0$,

$$\int_{\{|X_t|>a\}} |X_t|dP \leq \left[P\{|X_t|>a\}\cdot E(X_t^2)\right]^{1/2} \leq \left[\frac{E(X_t^2)}{a^2}\cdot E(X_t^2)\right]^{1/2} = \frac{R(t,t)}{a}.$$

Thus

$$\lim_{a\to\infty}\sup_{t\in T}\int_{\{|X_t|>a\}}|X_t|dP = 0, \tag{8}$$

and $\{X_t, t\in T\}$ is uniformly integrable.

Now ((i)) implies that $\{[X_t], t \in T\}$ is separable in M. Thus there exists a countable subset M' of $\{[X_t], t\in T\}$ such that for every $t \in T$, $[X_t]$ is the limit in probability of a sequence in M', and hence also in $L_2(\Omega,\mathcal{F},P)$, since M' is uniformly integrable [14]. It follows that $H(X)$ equals the $L_2(\Omega,\mathcal{F},P)$ closure of the linear span of M' and, since M' is countable, $H(X)$ is separable.

We now consider the general case and define for $N = 1,2,\ldots$

$$T_N = \{t \in T : R(t,t) \leq N\}. \tag{9}$$

Since R is measurable, $T_N \in \mathcal{B}(T)$ and by ((i)) $\{X_t, t\in T_N\}$ has a measurable modification. It follows by what has been proven that the $L_2(\Omega,\mathcal{F},P)$ closure of the linear span of the random variables $\{X_t, t\in T_N\}$, $H_N(X)$, is separable. Since X_t is of second order, R is finite valued and thus $\bigcup_{N=1}^{\infty} T_N = T$. It follows that $H(X)$ is the $L_2(\Omega,\mathcal{F},P)$ closure of $\bigcup_{N=1}^{\infty} H_N(X)$ and thus $H(X)$ is separable.

Thus a $\mathcal{B}(T)\times\mathcal{B}(T)$-measurable, symmetric, nonnegative definite, real function R on $T\times T$ is the autocorrelation of a measurable process if and only if $R(R)$ is separable. $\square$

Remark 1. *The mean m and the covariance C of a real second order process X_t, $t \in T$, are defined by $m(t) = E(X_t)$ and $C(t,s) = E([X_t - m(t)][X_s - m(s)])$ for all $t, s \in T$. Then $R(t,s) = m(t)m(s) + C(t,s)$. In connection with ((ii)) of Theorem 1 it should be noted that R is $\mathcal{B}(T) \times \mathcal{B}(T)$-measurable if and only if m is $\mathcal{B}(T)$-measurable and C is $\mathcal{B}(T) \times \mathcal{B}(T)$-measurable.*

The "if" part is obvious. The "only if" part is shown as follows. We have $m(t) = E(X_t I_\Omega)$ for all $t \in T$, where I_Ω is the indicator function. Denote by ξ the projection of $I_\Omega \in L_2(\Omega, \mathcal{F}, P)$ onto the subspace $H(X)$. Then $m(t) = E(X_t \xi)$ for all $t \in T$ and $\xi \in H(X)$, and thus $m \in R(R)$. Since R is $\mathcal{B}(T) \times \mathcal{B}(T)$-measurable, m is $\mathcal{B}(T)$-measurable (see part (a) of the proof of Theorem 1) and $C(t,s) = R(t,s) - m(t)m(s)$ is $\mathcal{B}(T) \times \mathcal{B}(T)$-measurable.

Remark 2. *Let $T = [0,1]$ and $R(t,s) = 1$ for $t = s$ in T and $R(t,s) = 0$ for $t \neq s$ in T. Since R is symmetric and nonnegative definite, there exists a probability space $(\Omega, \mathcal{F}, P)$ and a real process X_t, $t \in T$, on it with autocorrelation R. R is clearly $\mathcal{B}(T) \times \mathcal{B}(T)$-measurable, but since the values of X_t are orthogonal in $L_2(\Omega, \mathcal{F}, P)$, $E(X_t X_s) = 0$ for $t \neq s$ in T, $H(X)$ is not separable and by Theorem 1, X_t, $t \in T$, does not have a measurable modification. This can be also shown without using Theorem 1. Indeed, assume that X_t, $t \in T$, has a measurable modification Y_t, $t \in T$. Then*

$$\int_T R(t,t)dt = 1 < \infty \tag{10}$$

implies that $\int_T Y_t^2 dt < +\infty$ a.e. $[P]$. If $\{\Phi_n\}_{n=1}^\infty$ is a complete orthonormal set in $L_2(T, \mathcal{B}(T), Leb)$, then

$$Y_t = \sum_{n=1}^\infty \Phi_n(t) \int_T Y_s \Phi_n(s) ds \quad a.e.\ [P], \tag{11}$$

where the convergence is in $L_2(T)$ a.e. $[P]$, and thus

$$E\left[\int_T Y_t^2 dt\right] = \sum_{n=1}^\infty \int_T \int_T R(t,s)\Phi_n(t)\Phi_n(s) dt ds = 0 \quad a.e.\ [P],$$

which contradicts $E\left[\int_T Y_t^2 dt\right] = 1$. It follows that X_t, $t \in T$, does not have a measurable modification.

Remark 3. *For Gaussian processes it can be easily shown that ((ii)) implies ((i)) without relying on the results of [5]; this is done in [15].*

Corollary 1. *Let R be a symmetric, nonnegative definite, real function on $T \times T$. If $R(R)$ is separable the following are equivalent:*

(i) $R(t, \cdot)$ is $\mathcal{B}(T)$-measurable for all $t \in T$.

(ii) R is $\mathcal{B}(T) \times \mathcal{B}(T)$-measurable.

Proof. It suffices to show that ((i)) implies ((ii)). Since R is symmetric, nonnegative definite and real, there exists a probability space $(\Omega, \mathcal{F}, P)$ and a real process X_t, $t \in T$, on it with autocorrelation R. It is clear from part (a) of the proof of Theorem 1 that the separability of $R(R)$ and ((i)) imply the existence of a measurable modification of X_t, $t \in T$, and thus ((ii)). This result can be shown in a simpler way without using an associated process. Indeed, if $\{f_n\}_{n=1}^{\infty}$ is a complete orthonormal set in $R(R)$, then it is easily seen that $R(t,s) = \sum_{n=1}^{\infty} f_n(t)f_n(s)$ for all $t, s \in T$. Now ((i)) implies as in part (a) of the proof of Theorem 1 that every f_n is $\mathcal{B}(T)$-measurable and thus ((ii)) holds. □

Corollary 2. *A second order process X_t, $t \in T$, which satisfies any of the following conditions has a measurable modification (in ((iii)) also progressively measurable):*

(i) X_t, $t \in T$, is weakly continuous on T.

(ii) T is an arbitrary interval and X_t, $t \in T$, has orthogonal increments.

(iii) T is an arbitrary interval and X_t, $t \in T$, is a martingale.

Proof. ((i)): Since T is separable and X_t weakly continuous on T, $H(X)$ is separable [16]. By the weak continuity of X_t, $R(t, \cdot)$ is continuous, hence $\mathcal{B}(T)$-measurable, for all $t \in T$. The conclusion follows from Corollary 1 and Theorem 1.

((ii)): It is known that $H(X)$ is separable [8]. Also, X_t has left and right $L_2(\Omega, \mathcal{F}, P)$ limits on T and that except on a countable subset of T, $X_{t-} = X_t = X_{t+}$. This implies the measurability of R and the result follows from Theorem 1.

((iii)): Define the function F by $F(t) = E(X_t^2)$ for all $t \in T$. By the martingale property, with respect to the nondecreasing family $\mathcal{F}_t$, $t \in T$, of sub-σ-algebras of $\mathcal{F}$, we have for all $s < t$ in T,

$$E(X_t X_s) = E[E(X_t X_s \mid \mathcal{F}_s)] = E[X_s E(X_t \mid \mathcal{F}_s)] = E(X_s^2)$$

and thus

$$E(\{X_t - X_s\}^2) = F(t) - F(s).$$

It follows from this relationship, as in [8] and in ((ii)), that $H(X)$ is separable and R is $\mathcal{B}(T) \times \mathcal{B}(T)$-measurable. □

Remark 4. *Let X_t, $t \in T$, T an arbitrary interval, be a real separable process of second order with autocorrelation R. If X_t is mean square differentiable on T and $\frac{\partial R(t,s)}{\partial t}$ and $\frac{\partial R(t,s)}{\partial s}$ are locally Lebesgue integrable in T and in t, s respectively, then with probability one the paths of X_t, $t \in T$, are absolutely continuous on every compact subinterval of T. This is shown in [10] with the additional assumption that the mean square derivative $\dot{X}_t$ of X_t has a measurable modification, which is always satisfied because of Theorem 1. Indeed, since X_t is mean square differentiable on T, it is mean square continuous on T. Thus $H(X)$ is separable and the continuity of R implies the measurability of R. Since $\dot{X}_t$ is the autocorrelation of X_t and since $H(\dot{X}) \subset H(X)$, the conclusion follows from Theorem 1.*

Corollary 3. *If a real process X_t, $t \in T$, has a measurable modification, then $\mathcal{F}(X)$ coincides mod 0 with a separable σ-algebra.*

Proof. Since X_t, $t \in T$, has a measurable modification, $\{[X_t], t \in T\}$ is a separable subset of M. Thus there exists a countable subset $M' = \{[X_t], t \in S\}$ of $\{[X_t], t \in T\}$ (S is a countable subset of T) such that for every $t \in T$, $[X_t]$ is the limit in probability of a sequence from M', and thus X_t is the a.e. $[P]$ limit of a sequence from $\{X_t, t \in S\}$. If $\mathcal{F}'$ is the sub-σ-algebra of $\mathcal{F}$ generated by the random variables $\{X_t, t \in S\}$, then $\mathcal{F}' \subset \mathcal{F}(X)$, $\mathcal{F}'$ is separable and $\mathcal{F}(X)$ coincides with $\mathcal{F}'$ mod 0. □

2 On the Separability of the Linear Space of a Second Order Process

The linear space $H(X)$ of a second order process X_t, $t \in T$, plays an important role in the structure of such processes and in a variety of problems in statistical inference. If $H(X)$ is separable then X_t admits series representations and also integral representations (Theorem 2) that can be effectively used in problems such as linear mean square estimation. Also the separability of $H(X)$ is the only condition needed for a second order process to have the Hida-Cramér representation [11]. It is thus of interest that a measurable second order process has a separable linear space. $H(X)$ is known to be separable when the process X_t, $t \in T$, is weakly continuous [16], has orthogonal increments [8], or is a martingale (Corollary 2((iii))). In Theorem 2 necessary and sufficient conditions are given for $H(X)$ to be separable in terms of integral representations of X_t.

Before stating the theorem we mention a few basic facts about random measures, that can be found for instance in [7, 8]. Let $(V, \mathcal{V})$ be a measurable space. A random measure Z on $(V, \mathcal{V})$ is a countably additive map from $\mathcal{V}$ to $L_2(\Omega, \mathcal{F}, P)$; i.e., whenever A is the disjoint union of the sets $A_k \in \mathcal{V}$, $Z(A) = \sum_{k=1}^{\infty} Z(A_k)$ in $L_2(\Omega, \mathcal{F}, P)$. (Here we consider the case where Z is defined on the entire σ-algebra $\mathcal{V}$). To each random measure Z on V there corresponds a finite signed measure μ on $V \times V$ by $\mu(A \times B) = E[Z(A)Z(B)]$, $A, B \in \mathcal{V}$. μ is symmetric and nonnegative definite on the measurable rectangles of $V \times V$. A random measure Z is called orthogonal if $\mu(A \times B) = 0$ whenever A and B are disjoint, and to each orthogonal random measure there corresponds a finite nonnegative measure ν on V by $\nu(A) = E[Z^2(A)]$, $A \in \mathcal{V}$. Let $H(Z)$ be the closure in $L_2(\Omega, \mathcal{F}, P)$ of the linear span of $\{Z(A), A \in \mathcal{V}\}$, and let $\Lambda_2(\mu)$ be the Hilbert space of real, $\mathcal{V}$-measurable functions on V with inner product $\langle f, g\rangle_{\Lambda_2(\mu)} = \int_V \int_V f(u)g(v)d\mu(u, v)$ (of course $\Lambda_2(\mu)$ consists of equivalence classes of functions, two functions f and g considered identical if $\langle f-g, f-g\rangle_{\Lambda_2(\mu)} = 0$). There is an inner product preserving isomorphism between $\Lambda_2(\mu)$ and $H(Z)$, denoted by $\leftrightarrow$, such that $I_A \leftrightarrow Z(A)$, $A \in \mathcal{V}$, and integration of functions $f \in \Lambda_2(\mu)$ with respect to Z is defined by $\int_V f(u)dZ(u) \leftrightarrow f$. If Z is orthogonal, there is an inner product preserving isomorphism between $L_2(\nu) = L_2(V, \mathcal{V}, \nu)$ and $H(Z)$, denoted again by $\leftrightarrow$, such that $I_A \leftrightarrow Z(A)$, $A \in \mathcal{V}$, and integration of functions in $L_2(\nu)$ with respect to Z is defined by $\int_V f(u)dZ(u) \leftrightarrow f$.

Theorem 2. *Let X_t, $t \in T$, be a second order process.*

(i) If $H(X)$ is separable then for every finite measure space $(V, \mathcal{V}, \nu)$ such that $L_2(\nu) = L_2(V, \mathcal{V}, \nu)$ is separable and infinite dimensional, X_t has a representation

$$X_t = \int_V f(t, u)dZ(u) \quad \text{for all } t \in T, \tag{12}$$

where Z is an orthogonal measure on V with corresponding measure ν and $f(t, \cdot) \in L_2(\nu)$ for all $t \in T$. Conversely, if X_t has such a representation, $H(X)$ is separable.

(ii) If $H(X)$ is separable, then for every measurable space $(V, \mathcal{V})$ and every finite signed measure μ on $V \times V$ which is symmetric and nonnegative definite on the measurable

rectangles of $V \times V$, and such that $\Lambda_2(\mu)$ is separable and infinite dimensional, X_t has a representation

$$X_t = \int_V f(t,u)dZ(u) \quad \text{for all } t \in T, \tag{13}$$

where Z is a random measure on V with corresponding measure μ and $f(t,\cdot) \in \Lambda_2(\mu)$ for all $t \in T$. Conversely, if X_t has such a representation, $H(X)$ is separable.

Proof. ((ii)) being a particular case of ((i)), we will prove only ((ii)). We start with the second claim. If X_t has such a representation then $X_t \in H(Z)$ for all $t \in T$, hence $H(X) \subset H(Z)$ and the conclusion follows from the isomorphism between $H(Z)$ and $\Lambda_2(\mu)$ and the separability of the latter.

We now prove the first claim. Assume that $H(X)$ is separable and let $\{\xi_n\}_{n=1}^{\infty}$ be a complete orthonormal set. Then for all $t \in T$,

$$X_t = \sum_{n=1}^{\infty} a_n(t)\xi_n \quad \text{in } L_2(\Omega, \mathcal{F}, P), \tag{14}$$

where $a_n(t) = E(X_t\xi_n)$. Let $\{f_n\}_{n=1}^{\infty}$ be a complete orthonormal set in $\Lambda_2(\mu)$. Since μ is finite, $I_A \in \Lambda_2(\mu)$ for all $A \in \mathcal{V}$. Then

$$I_A = \sum_{n=1}^{\infty} \langle I_A, f_n\rangle f_n \quad \text{in } \Lambda_2(\mu), \tag{15}$$

where $\langle I_A, f_n\rangle = \int_V \int_V f_n(v)d\mu(u,v)$. Throughout the proof we will write $\langle\cdot,\cdot\rangle$ for $\langle\cdot,\cdot\rangle_{\Lambda_2(\mu)}$. Thus for all n, λ_n is a finite signed measure on $(V, \mathcal{V})$. We also have

$$\langle I_A, I_A\rangle = \sum_{n=1}^{\infty} \lambda_n^2(A) = \mu(A \times A) < +\infty. \tag{16}$$

Hence

$$Z(A) = \sum_{n=1}^{\infty} \lambda_n(A)\xi_n \tag{17}$$

defines a function from $\mathcal{V}$ to $L_2(\Omega, \mathcal{F}, P)$ (the convergence being in $L_2(\Omega, \mathcal{F}, P)$). We will show that Z is a random measure with corresponding measure μ. The latter is clear since for all $A, B \in \mathcal{V}$ we have

$$E[Z(A)Z(B)] = \sum_{n=1}^{\infty} \lambda_n(A)\lambda_n(B) = \langle I_A, I_B\rangle = \mu(A \times B). \tag{18}$$

For the countable additivity of Z let $A = \bigcup_{k=1}^{\infty} A_k$ where $\{A_k\}_{k=1}^{\infty}$ is a disjoint sequence of sets in $\mathcal{V}$. Then

$$E\left[\left(Z(A) - \sum_{k=1}^{K} Z(A_k)\right)^2\right] = \mu\left(\left(\bigcup_{k=K+1}^{\infty} A_k\right) \times \left(\bigcup_{k=K+1}^{\infty} A_k\right)\right) \to 0 \quad \text{as } K \to \infty. \tag{19}$$

Thus $Z(A) = \sum_{k=1}^{\infty} Z(A_k)$.

We now show that for every $g \in \Lambda_2(\mu)$,

$$\int_V g(u)dZ(u) = \sum_{n=1}^{\infty} \langle g, f_n \rangle \xi_n \quad \text{in } L_2(\Omega, \mathcal{F}, P). \tag{20}$$

This is true for indicator functions by definition of Z, and therefore also for simple functions. Since $H(Z)$ is defined as the $L_2(\Omega, \mathcal{F}, P)$ closure of the linear space of $\{Z(A), A \in \mathcal{V}\}$, it follows by the isomorphism between $\Lambda_2(\mu)$ and $H(Z)$ that the linear span of $\{I_A, A \in \mathcal{V}\}$ is dense in $\Lambda_2(\mu)$. Thus every $g \in \Lambda_2(\mu)$ is the limit of a sequence of simple functions. The result follows from

$$E\left[\left(\int_V g dZ - \sum_{n=1}^{\infty} \langle g, f_n \rangle \xi_n\right)^2\right] = \langle g - g_k, g - g_k \rangle \to 0, \tag{21}$$

which implies that $H(Z) = H(X)$.

In particular we have

$$\int_V f(t,u)dZ(u) = \sum_{n=1}^{\infty} a_n(t)\xi_n. \tag{22}$$

Now since $f(t) \in \Lambda_2(\mu)$ for all $t \in T$ and $f(t) \leftrightarrow X_t$, we can define $f(t, \cdot) \in \Lambda_2(\mu)$ for all $t \in T$ by

$$f(t,u) = \sum_{n=1}^{\infty} a_n(t) f_n(u), \tag{23}$$

where the convergence is in $\Lambda_2(\mu)$. It follows from the property of the integral just proven that for all $t \in T$ we have the following equality in $L_2(\Omega, \mathcal{F}, P)$,

$$\int_V f(t,u)dZ(u) = \sum_{n=1}^{\infty} a_n(t)\xi_n = X_t, \tag{24}$$

which concludes the proof. □

Remark 5. *We assume throughout this remark that $H(X)$ is separable. Then it is clear that the first claim in ((i)) and ((ii)) is valid provided the dimensionality of $L_2(\nu)$ and $\Lambda_2(\mu)$ is no less than the dimensionality of the integers. Also, one can take $(V, \mathcal{V}) = (T, \mathcal{B}(T))$ or as V any interval and $\mathcal{V}$ its Borel sets; in the latter case ν may be taken the Lebesgue measure or one absolutely continuous to it, and μ may be taken absolutely continuous to the Lebesgue measure on $V \times V$. If a series (respectively, integral) representation of X_t is known then one can obtain integral (respectively, series) representations of X_t as indicated in the proof of Theorem 2. These representations will be explicitly obtained if one can find complete orthonormal sets in the spaces $L_2(\nu)$ and $\Lambda_2(\mu)$. If V is an interval and $\mathcal{V}$ its Borel sets, complete orthonormal sets in $L_2(\nu)$ are given in [2] (see also [13]), and complete sets in $\Lambda_2(\mu)$ are given in [3] (In [3] the case where V is the entire real line is treated and the case where V is an interval can be treated similarly). If neither an integral nor a series representation of X_t is available, the problem arises how to obtain explicitly such a representation (in terms of the process X_t, $t \in T$, and its autocorrelation R). This problem is solved in [4] for weakly continuous processes X_t, $t \in T$, and T an arbitrary interval.*

Remark 6. *Theorem 2 may also be stated in terms of integral representation of the autocorrelation R, which for ((i)) and ((ii)) are respectively*

$$R(t,s) = \int_V f(t,u) f(s,u) d\nu(u) \quad \textit{for all } t, s \in T, \tag{25}$$

and

$$R(t,s) = \int_V \int_V f(t,u) f(s,v) d\mu(u,v) \quad \textit{for all } t, s \in T. \tag{26}$$

Remark 7. *In [12] a second order process X_t, $t \in \mathbb{R} = (-\infty, +\infty)$, is called oscillatory if it has a representation*

$$X_t = \int_{-\infty}^{\infty} e^{itu} a_t(u) dZ(u) \quad \textit{for all } t \in \mathbb{R}, \tag{27}$$

where Z is an orthogonal random measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with corresponding measure ν and $a_t(\cdot) \in L_2(\nu)$ for all $t \in T$ (this is a generalization of a concept introduced by Priestley). If X_t, $t \in \mathbb{R}$, is oscillatory then $H(X)$ is separable, since $L_2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$ is separable. Conversely, if $H(X)$ is separable it follows by Theorem 2((i)) that for any finite measure ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we have $X_t = \int_{-\infty}^{\infty} f(t,u) dZ(u)$ for all $t \in T$, where Z is an orthogonal random measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with corresponding measure ν and $f(t, \cdot) \in L_2(\nu)$ for all $t \in T$. If we define $a_t(u) = e^{-itu} f(t,u)$, it becomes clear that X_t, $t \in \mathbb{R}$, is oscillatory. Thus a second order process is oscillatory if and only if its linear space is separable.

Remark 8. *Some simple sufficient conditions for $H(X)$ to be separable are as follows. If X_t, $t \in T$, is a linear operation on a second order process Y_s, $s \in S$, with separable linear space, then $H(X) \subset H(Y)$ and the separability of $H(X)$ follows from that of $H(Y)$. Also, because of the isomorphism between $H(X)$ and $R(R)$, $H(X)$ is separable if there is a symmetric, nonnegative definite function K on $T \times T$ such that $R(R) \subset R(K)$ and $R(K)$ is separable. A sufficient condition for $R(R) \subset R(K)$ is that $K - R$ be nonnegative definite [1].*

References

[1] Aronszajn, N.: Theory of reproducing kernels. Trans. Amer. Math. Soc. **68**, 337–404 (1950).

[2] Cambanis, S.: Bases in L_2 spaces with applications to stochastic processes with orthogonal increments. Proc. Amer. Math. Soc. **29**, 284–290 (1971).

[3] Cambanis, S., Liu, B.: On harmonizable stochastic processes. Information and Control **17**, 183–202 (1970).

[4] Cambanis, S., Masry, E.: On the representation of weakly continuous processes. Information Sci. **3**, 277–290 (1971).

[5] Chung, K.L., Doob, J.L.: Fields, optionality and measurability. Amer. J. Math. **87**, 397–424 (1965).

[6] Cohn, D.L.: Measurable choice of limit points and the existence of separable and measurable processes. Z. Wahrscheinlichkeitstheorie verw. Geb. **22**, 161–165 (1972).

[7] Cramer, H.: A contribution to the theory of stochastic processes. Proc. Second Berkeley Symp. Math. Statist. Prob., 329–339. Berkeley: Univ. California Press 1951.

[8] Cramer, H., Leadbetter, M.R.: Stationary and Related Stochastic Processes. New York: Wiley 1967.

[9] Doob, J.L.: Stochastic Processes. New York: Wiley 1953.

[10] Gikhman, I.I., Skorokhod, A.V.: Introduction to the Theory of Random Process. Philadelphia: Saunders 1969.

[11] Kallianpur, G., Mandrekar, V.: Multiplicity and representation theory of purely non-deterministic stochastic processes. Theor. Probability Appl. **10**, 553–581 (1965).

[12] Mandrekar, V.: A characterization of oscillatory processes and their prediction. Proc. Amer. Math. Soc. **32**, 280–284 (1972).

[13] Masry, E., Liu, B., Steiglitz, K.: Series expansion of wide-sense stationary random processes. IEEE Trans. Information Theory **14**, 792–796 (1968).

[14] Neveu, J.: Mathematical Foundations of the Calculus of Probability. San Francisco: Holden-Day 1965.

[15] Neveu, J.: Processus Aléatoires Gaussiens. Montréal: Presses Univ. Montréal 1968.

[16] Parzen, E.: Time Series Analysis Papers. San Francisco: Holden-Day 1967.

[17] Rasanov, Yu.A.: Stationary Random Processes. San Francisco: Holden-Day 1967.