Derivation of the Appearance of J_0 in the Random Plane Wave Model

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by Stephen Crowley

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1 Introduction

This document outlines the derivation of the Bessel function J_0 appearing in the expectation of the product of cosine functions in the random plane wave model, where wave vectors are uniformly distributed in all directions with uniformly distributed random phases.

2 Mathematical Setup

Consider a random plane wave model with waves represented as $\cos(\mathbf{k}\cdot\mathbf{x}+\theta)$, where $\mathbf{k}$ is the wave vector, $\mathbf{x}$ is the position vector, and θ is a random phase uniformly distributed between 0 and 2π.

3 Derivation

3.1 Cosine Expansion

Using the cosine product expansion, we have:

$$\cos(a)\cos(b)=\frac{1}{2}[\cos(a-b)+\cos(a+b)] \tag{1}$$

Applying this to our wave model:

$$\cos(\mathbf{k}\cdot\mathbf{x}+\theta)\cos(\mathbf{k}\cdot\mathbf{y}+\theta)=\frac{1}{2}[\cos(\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}))+\cos(\mathbf{k}\cdot(\mathbf{x}+\mathbf{y})+2\theta)]$$

3.2 Expectation Over θ

Averaging over θ and utilizing the uniform distribution of θ, the second term involving cos averaged over 2θ cancels out due to symmetry:

$$\mathbb{E}_\theta[\cos(\mathbf{k}\cdot\mathbf{x}+\theta)\cos(\mathbf{k}\cdot\mathbf{y}+\theta)]=\frac{1}{2}\cos(\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})) \tag{2}$$

3.3 Averaging Over Directions

The integration over all directions of $\mathbf{k}$ isotropically leads to:

$$\int_{\text{sphere}} \cos(\mathbf{k}\cdot\mathbf{r})\; d\,\Omega = 2\,\pi\, J_0(k\,\|\mathbf{r}\|) \tag{3}$$

where $\mathbf{r}=\mathbf{x}-\mathbf{y}$ and $d\,\Omega$ is the differential solid angle element. Normalizing by the total solid angle $4\,\pi$:

$$\frac{1}{4\,\pi}\cdot 2\,\pi\, J_0(k\,\|\mathbf{r}\|) = \frac{1}{2}\, J_0(k\,\|\mathbf{r}\|) \tag{4}$$

3.4 Conclusion

The expectation value of the product of the cosines, after integrating over all directions and averaging out the random phases, results in:

$$\mathbb{E}\left[\cos(\mathbf{k}\cdot\mathbf{x}+\theta)\cos(\mathbf{k}\cdot\mathbf{y}+\theta)\right] = J_0(\|\mathbf{x}-\mathbf{y}\|) \tag{5}$$