Rigorous Derivation of Caputo Fractional Derivative for $\sin(t)$

Definition 1. *[Caputo Fractional Derivative] For any $\alpha > 0$, the Caputo derivative of order α is:*

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^t \frac{f^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha - \lceil\alpha\rceil + 1}} \, d\tau \tag{1} $$

where $\lceil\alpha\rceil$ denotes the ceiling function.

Definition 2. *[Mittag-Leffler Function] The two-parameter Mittag-Leffler function:*

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{2} $$

Lemma 3. *[Higher Derivatives of Sine] For $m \in \mathbb{N}$:*

$$ \sin^{(m)}(t) = \sin\left(t + \frac{m\pi}{2}\right) \tag{3} $$

Lemma 4. *[Beta-Gamma Relationship] For $a, b > 0$:*

$$ \int_0^1 u^{a-1} (1-u)^{b-1} \, du = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} \tag{4} $$

Theorem 5. *[Caputo Derivative of Sine] For any $\alpha > 0$:*

$$ {}_0^C D_t^\alpha \sin(t) = t^{\lceil\alpha\rceil - \alpha} E_{2, \lceil\alpha\rceil - \alpha + 1}(-t^2) \tag{5} $$

Proof. 1. **Start with Caputo definition**:

$$ {}_0^C D_t^\alpha \sin(t) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^t \frac{\sin^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha - \lceil\alpha\rceil + 1}} \, d\tau \tag{6} $$

2. **Express higher derivative**:

$$\sin^{(\lceil\alpha\rceil)}(\tau) = \sin\left(\tau + \frac{\lceil\alpha\rceil\,\pi}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k\,\tau^{2k+p}}{(2\,k+p)!}, \quad p = \lceil\alpha\rceil \bmod 2 \tag{7}$$

3. **Substitute series into integral**:

$${}_0^C D_t^\alpha \sin(t) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\,k+p)!} \int_0^t \frac{\tau^{2k+p}}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}}\,d\,\tau \tag{8}$$

4. **Change variables $\tau = t\,u$**:

$$\int_0^t \frac{\tau^{2k+p}}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}}\,d\,\tau = t^{2k+p+\lceil\alpha\rceil-\alpha}\,\frac{\Gamma(2\,k+p+1)\,\Gamma(\lceil\alpha\rceil-\alpha)}{\Gamma(2\,k+p+\lceil\alpha\rceil-\alpha+1)} \tag{9}$$

5. **Simplify expression**:

$${}_0^C D_t^\alpha \sin(t) = t^{\lceil\alpha\rceil-\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k\,t^{2k}}{\Gamma(2\,k+\lceil\alpha\rceil-\alpha+p+1)} \tag{10}$$

6. **Unify using $p = \lceil\alpha\rceil \bmod 2$**:

$$\lceil\alpha\rceil - \alpha + p + 1 = \lceil\alpha\rceil - \alpha + 1 + (\lceil\alpha\rceil \bmod 2) \tag{11}$$

$$= \lceil\alpha\rceil - \alpha + 1 + \lceil\alpha\rceil - 2\,\lfloor\lceil\alpha\rceil/2\rfloor \tag{12}$$

$$= 2\lceil\alpha\rceil - \alpha + 1 - 2\,\lfloor\lceil\alpha\rceil/2\rfloor \tag{13}$$

7. **Recognize Mittag-Leffler pattern**:

$$\sum_{k=0}^{\infty} \frac{(-t^2)^k}{\Gamma(2\,k+\lceil\alpha\rceil-\alpha+1)} = E_{2,\lceil\alpha\rceil-\alpha+1}(-t^2) \tag{14}$$

8. **Final result**:

$${}_0^C D_t^\alpha \sin(t) = t^{\lceil\alpha\rceil-\alpha}\,E_{2,\lceil\alpha\rceil-\alpha+1}(-t^2) \quad \blacksquare \tag{15}$$

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