Invertibility of Oscillatory Gaussian Processes

by Stephen Crowley Says FUCK ANTHROPIC FUCKING SHITCUNTS

Framework

Let

$$X(t) = \int_{-\infty}^{\infty} A(t, \lambda)\ e^{i\lambda t}\ d\, Z(\lambda), \qquad t \in \mathbb{R},$$

where

- $A: \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a deterministic amplitude function;
- $d\, Z(\lambda)$ is a complex orthogonal increment satisfying

$$\mathbb{E}[d\, Z(\lambda_1)\, \overline{d\, Z(\lambda_2)}] = \delta\, (\lambda_1 - \lambda_2)\ \mu\, (d\, \lambda_1)$$

  in the distributional sense, with μ a finite positive measure.

Fundamental Invertibility Theorem

Theorem 1. *Define*

$$\mathcal{I}[X](\lambda) := \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \overline{A(t, \lambda)}\, e^{-i\lambda t}\ X(t)\ d\, t.$$

Then $\mathcal{I}[X](\lambda) = d\, Z(\lambda)$ for every realisation $X(t)$ if and only if

1. *$A(t, \lambda) \neq 0$ for all $(t, \lambda) \in \mathbb{R}^2$;*
2. ***Kernel orthonormality:***

$$\frac{1}{2\,\pi} \int_{-\infty}^{\infty} \overline{A(t, \lambda_1)}\, A(t, \lambda_2)\ e^{i(\lambda_2 - \lambda_1)t}\ d\, t = \delta\, (\lambda_1 - \lambda_2) \quad (\lambda_1, \lambda_2 \in \mathbb{R}).$$

Moreover, when (1)–(2) hold, $\mathcal{I}$ is the unique linear operator that recovers $d\, Z(\lambda)$ from $X(t)$.

Assume $\mathcal{I}[X] = d\, Z(\lambda)$ for every admissible $X(t)$.

Orthogonality. Fix λ_0 and substitute the representation of $X(t)$:

$$\begin{aligned}\mathcal{I}[X](\lambda_0) &= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \overline{A(t, \lambda_0)} e^{-i\lambda_0 t} \int_{-\infty}^{\infty} A(t, \lambda)\ e^{i\lambda t}\ d\, Z(\lambda)\ d\, t \\ &= \int_{-\infty}^{\infty} \left[\frac{1}{2\,\pi} \int_{-\infty}^{\infty} \overline{A(t, \lambda_0)} A(t, \lambda)\ e^{i(\lambda - \lambda_0)t}\ d\, t \right] d\, Z(\lambda).\end{aligned}$$

Because the left-hand side equals $d\, Z(\lambda_0)$ for *all* increment measures, the bracketed factor must equal $\delta\, (\lambda - \lambda_0)$, yielding (2).

Non-vanishing amplitude. If $A(t_*, \lambda_*) = 0$ on a nontrivial time interval, then observations on that interval contain no information about $d\, Z(\lambda_*)$, contradicting perfect recovery; hence (1) is necessary.

Assume (1)–(2). For any λ_0,

$$\begin{aligned}\mathcal{I}[X](\lambda_0) =& \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{A(t,\lambda_0)}e^{-i\lambda_0 t}\int_{-\infty}^{\infty}A(t,\lambda)\,e^{i\lambda t}\,d\,Z(\lambda)\ d\,t\\ =&\int_{-\infty}^{\infty}\delta\,(\lambda-\lambda_0)\ d\,Z(\lambda)\quad \text{(by (2))}\\ =& d\,Z(\lambda_0).\end{aligned}$$

Thus $\mathcal{I}[X](\lambda) = d\,Z(\lambda)$ for all λ.

Uniqueness

Lemma 2. *If two linear operators L_1, L_2 satisfy $L_i[X](\lambda) = d\,Z(\lambda)$ for every admissible X, then $L_1 = L_2$.*

Proof. Set $L = L_1 - L_2$ and note $L[X] = 0$ for all X. For each fixed λ_0, define the single-component process $X_{\lambda_0}(t) := A(t, \lambda_0)\ e^{i\lambda_0 t}\ d\,Z(\lambda_0)$. By (2) these processes span the same class as $X(t)$, so $L[X_{\lambda_0}] = 0$ for all λ_0. Linearity then implies $L \equiv 0$. □