Schur Factorization of Square Matrices

The Schur factorization is a fundamental result in linear algebra that states that any square matrix $A \in \mathbb{C}^{n \times n}$ can be factorized into a unitary matrix Q and an upper triangular matrix U such that:

$$A = Q U Q^* \tag{1}$$

Statement of the Schur Factorization

For any complex square matrix $A \in \mathbb{C}^{n \times n}$, there exists a unitary matrix Q and an upper triangular matrix U such that $A = Q U Q^*$, where:

- Q is a unitary matrix ($Q^* Q = I$).
- U is an upper triangular matrix with the eigenvalues of A on its diagonal.

Proof of Existence

The proof of the Schur factorization can be approached using the concept of unitary similarity transformations and relies on induction, properties of eigenvalues, and specific unitary transformations like Householder transformations.

1. Base Case ($n = 1$)

For a 1×1 matrix $A = [a]$, the Schur factorization is trivially $A = Q U Q^*$ where $Q = 1$ and $U = A$. Thus, the base case holds.

2. Inductive Step

Assume that any $(n-1) \times (n-1)$ matrix has a Schur factorization. We now prove that an $n \times n$ matrix A also has a Schur factorization.

3. Finding an Eigenvalue and Eigenvector

By the fundamental theorem of algebra, the complex matrix A has at least one eigenvalue, say λ, with a corresponding unit eigenvector $x \in \mathbb{C}^n$.

4. Constructing the Householder Transformation

To simplify the process, we use a Householder transformation to construct a unitary matrix Q_1 that maps x to the first basis vector e_1. The Householder transformation is defined as:

$$Q_1 = I - 2\frac{v\,v^*}{v^*\,v} \tag{2}$$

where $v = x - \alpha\, e_1$ and $\alpha = \|x\|$ or $-\|x\|$, with the sign chosen to avoid cancellation errors. This choice ensures that $v \neq 0$ and helps maintain numerical stability.

As a result, Q_1 is unitary, and $Q_1^*\, x = \alpha\, e_1$.

5. Transforming A

Compute $A_1 = Q_1^*\, A\, Q_1$.

The first column of A_1 is given by:

$$A_1\, e_1 = Q_1^*\, A\, Q_1\, e_1 = Q_1^*\, A\, x = Q_1^*\, (\lambda\, x) = \lambda\, Q_1^*\, x = \lambda\, e_1 \tag{3}$$

Since $A_1\, e_1 = \lambda\, e_1$, the first column of A_1 is λ in the first position and zeros elsewhere (below the first position), confirming that the entries below the first element are zeros.

6. Partitioning A_1

Partition A_1 as:

$$A_1 = \begin{bmatrix} \lambda & y^* \\ 0 & B \end{bmatrix} \tag{4}$$

where $y \in \mathbb{C}^{n-1}$ and $B \in \mathbb{C}^{(n-1)\times(n-1)}$.

7. Inductive Application

By the inductive hypothesis, the $(n-1)\times(n-1)$ matrix B has a Schur factorization. Thus, there exists a unitary matrix $Q_2 \in \mathbb{C}^{(n-1)\times(n-1)}$ such that:

$$B = Q_2\, U'\, Q_2^* \tag{5}$$

where U' is upper triangular.

8. Constructing the Final Unitary Matrix

Define the unitary matrix:

$$Q = Q_1 \begin{bmatrix} 1 & 0 \\ 0 & Q_2 \end{bmatrix} \tag{6}$$

9. Forming the Upper Triangular Matrix U

Compute:

$$Q^* A Q = \begin{bmatrix} 1 & 0 \\ 0 & Q_2^* \end{bmatrix} Q_1^* A Q_1 \begin{bmatrix} 1 & 0 \\ 0 & Q_2 \end{bmatrix} = \begin{bmatrix} \lambda & y^* Q_2 \\ 0 & U' \end{bmatrix} = U \tag{7}$$

Since U' is upper triangular and $y^* Q_2$ is a row vector, the resulting matrix U is upper triangular. The presence of $y^* Q_2$ in the first row doesn't introduce any non-zero elements below the main diagonal, ensuring U remains upper triangular.

10. Diagonal Elements as Eigenvalues

The first diagonal element of U is λ, which is an eigenvalue of A by construction. By the inductive hypothesis, the diagonal elements of U' are the eigenvalues of B, which are also eigenvalues of A. Thus, all diagonal elements of U are eigenvalues of A.

11. Conclusion

We have shown that for any $n \times n$ complex matrix A, there exists a unitary matrix Q and an upper triangular matrix U such that $A = Q U Q^*$. This completes the proof.