can this time change be represented as a monotonically increasing function

Yes, the time change in the Lamperti representation can be represented as a monotonically increasing function. The search results provide several key insights into this monotonic property.

Mathematical Structure of the Time Change

The Lamperti time change is defined through an additive functional. For a continuous-state branching process (CSBP), the time change is constructed as[1]:

$$U_i(t) = \int_0^t \mathbf{1}_{s<\tau_i} X_i(s)^2 ds$$

where $X_i(s)$ represents the process value and τ_i is the hitting time to zero. The time change function T_i is then defined as the inverse of this additive functional: $T_i = (U_i|_{[0,\tau_i)})^{-1}$[1].

Monotonicity Properties

The monotonic nature of the time change emerges from several fundamental properties:

Increasing Additive Functional: Since $X_i(s)^2 \geq 0$ for all s, the integral $U_i(t)$ is non-decreasing in t[1]. This ensures that as time progresses, the accumulated "mass" of the process increases monotonically.

Well-Defined Inverse: The fact that $\lim_{t \to \tau_i} U_i(t) = +\infty$ almost surely guarantees that the inverse function T_i exists and is well-defined[1]. Moreover, for all $u \geq 0$, we have $T_i(u) < \tau_i$, and $\lim_{u \to \infty} T_i(u) = \tau_i$[1].

General Formulation

In the broader context of Lamperti transformations, the time change τ is defined as[2]:

$$\tau(t) = \inf \left\{ u \geq 0 : \int_0^u f(r)^\alpha dr > t \right\}$$

where f is a non-increasing càdlàg function and $\alpha \in \mathbb{R}$. This construction inherently produces a monotonically increasing function since it's defined as the inverse of an integral, which is itself monotonic when the integrand is non-negative.

Differentiability and Local Properties

The time change function τ is left and right-differentiable everywhere on its domain, with derivatives given by $\tau'(t_\pm) = f(\tau(t)_\pm)^{-\alpha}$[2]. This local rescaling property means that the time change corresponds to multiplying each infinitesimal time interval by a factor that depends on the current state, while preserving the overall monotonic structure.

The monotonic property is essential for the Lamperti representation to establish a bijective correspondence between continuous-state branching processes and spectrally positive Lévy processes, ensuring that the time transformation preserves the temporal ordering of events.

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1. https://arxiv.org/pdf/2306.02158.pdf
2. https://www.math.univ-paris13.fr/~haas/BivMCSPA