Unitary Time Changes of Stationary Processes Yield Oscillatory Processes

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Definition 1. *[Unitary time change operator on $L^2(\mathbb{R})$] Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous with $\theta'(t) \neq 0$ almost everywhere. The unitary time change operator U_θ on $L^2(\mathbb{R})$ is defined by*

$$(U_\theta f)(t) := \sqrt{|\theta'(t)|}\ f(\theta(t)) \qquad \textit{for } f \in L^2(\mathbb{R}) \tag{1}$$

Theorem 2. *[Unitarity of U_θ] The operator U_θ defined above is unitary on $L^2(\mathbb{R})$.*

Proof. Absolute continuity with $\theta'(t) \neq 0$ a.e. implies the change-of-variables formula

$$\int_{\mathbb{R}} |(U_\theta f)(t)|^2 \, d t = \int_{\mathbb{R}} |\theta'(t)| \, |f(\theta(t))|^2 \, d t = \int_{\mathbb{R}} |f(u)|^2 \, d u \tag{2}$$

so U_θ is isometric. Surjectivity follows from the same change-of-variables applied to $U_{\theta^{-1}}$, which exists almost everywhere under these hypotheses. Hence U_θ is unitary. □

Definition 3. *[Oscillatory processes in the sense of Priestley] An oscillatory process Z is specified by a measurable gain function $A_t(\lambda)$ and has oscillatory function*

$$\varphi_t(\lambda) := A_t(\lambda)\ e^{i\lambda t} \tag{3}$$

The process Z has spectral representation

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda)\ \Phi(d\lambda) = \int_{\mathbb{R}} A_t(\lambda)\ e^{i\lambda t}\ \Phi(d\lambda) \tag{4}$$

where Φ is a complex orthogonal random measure on $\mathbb{R}$ with spectral measure F satisfying

$$E[\Phi(d\lambda)\, \overline{\Phi(d\mu)}] = 1_{\{\lambda=\mu\}}\ d F(\lambda) \tag{5}$$

The covariance kernel of Z is

$$R_Z(t,s) := E[Z(t)\overline{Z(s)}] = \int_{\mathbb{R}} A_t(\lambda)\, \overline{A_s(\lambda)}\, e^{i\lambda(t-s)}\ d F(\lambda) \tag{6}$$

Remark 4. [Real-valuedness condition] The oscillatory process Z is real-valued if and only if the gain satisfies conjugate symmetry:

$$A_t(-\lambda) = \overline{A_t(\lambda)} \quad \text{for } F\text{-almost every } \lambda, \text{for each fixed } t \tag{7}$$

Theorem 5. *[Unitary time change of stationary process yields oscillatory process] Let X be a zero-mean stationary Gaussian process with Cramér spectral representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\ \Phi(d\lambda) \tag{8}$$

where Φ is the same complex orthogonal random measure with spectral measure F as in the oscillatory definition. Let U_θ be a unitary time change operator as defined above. Then the transformed process

$$Z(t) := (U_\theta X)(t) = \sqrt{|\theta'(t)|}\ X(\theta(t)) \tag{9}$$

is an oscillatory process in the sense of Priestley with oscillatory function

$$\varphi_t(\lambda) = \sqrt{|\theta'(t)|}\ e^{i\lambda\theta(t)} \tag{10}$$

Proof. Starting from the stationary representation, we compute

$$\begin{aligned} Z(t) &= \sqrt{|\theta'(t)|}\ X(\theta(t)) && (11)\\ &= \sqrt{|\theta'(t)|} \int_{\mathbb{R}} e^{i\lambda\theta(t)}\ \Phi(d\lambda) && (12)\\ &= \int_{\mathbb{R}} \sqrt{|\theta'(t)|}\ e^{i\lambda\theta(t)}\ \Phi(d\lambda) && (13) \end{aligned}$$

Defining

$$\varphi_t(\lambda) := \sqrt{|\theta'(t)|}\ e^{i\lambda\theta(t)} \tag{14}$$

we have

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda)\ \Phi(d\lambda)$$

which is precisely the oscillatory form. The covariance kernel becomes

$$R_Z(t,s) = \int_{\mathbb{R}} \varphi_t(\lambda)\ \overline{\varphi_s(\lambda)}\, dF(\lambda) = \int_{\mathbb{R}} \sqrt{|\theta'(t)||\theta'(s)|}\ e^{i\lambda(\theta(t)-\theta(s))}\, dF(\lambda) \qquad \square$$

Theorem 6. *[Explicit gain function for unitary time change] In the setting of the previous theorem, the gain function for the oscillatory process*

$$Z(t) = (U_\theta X)(t) \tag{15}$$

is given by

$$A_t(\lambda) = \sqrt{|\theta'(t)|}\; e^{i\lambda(\theta(t)-t)} \tag{16}$$

The oscillatory function is

$$\varphi_t(\lambda) = A_t(\lambda)\; e^{i\lambda t} = \sqrt{|\theta'(t)|}\; e^{i\lambda\theta(t)} \tag{17}$$

and the covariance kernel takes the form

$$R_Z(t,s) = \int_{\mathbb{R}} A_t(\lambda)\, \overline{A_s(\lambda)}\, e^{i\lambda(t-s)}\; d\,F(\lambda) \tag{18}$$

Proof. From the previous theorem, we have

$$\varphi_t(\lambda) = \sqrt{|\theta'(t)|}\; e^{i\lambda\theta(t)} \tag{19}$$

Since the oscillatory function must satisfy

$$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{20}$$

one solves for the gain:

$$A_t(\lambda) = \frac{\varphi_t(\lambda)}{e^{i\lambda t}} = \frac{\sqrt{|\theta'(t)|}\; e^{i\lambda\theta(t)}}{e^{i\lambda t}} = \sqrt{|\theta'(t)|}\; e^{i\lambda(\theta(t)-t)}$$

and substitutes back into the covariance formula:

$$\begin{aligned} R_Z(t,s) &= \int_{\mathbb{R}} \varphi_t(\lambda)\;\, \bar{\varphi}_s(\lambda)\}\, d\,F(\lambda) & (21)\\ &= \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, \overline{A_s(\lambda)\, e^{i\lambda s}}\, d\,F(\lambda) & (22)\\ &= \int_{\mathbb{R}} A_t(\lambda)\, \overline{A_s(\lambda)}\, e^{i\lambda(t-s)}\; d\,F(\lambda) & (23)\end{aligned}$$

□

Theorem 7. *[Unitary time change on $L^2(\mathbb{R})$] Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous with $\theta'(t) \neq 0$ almost everywhere. Define the operator*

$$(U_\theta f)(t) := \sqrt{|\theta'(t)|}\;\, f(\theta(t)) \qquad \textit{for } f \in L^2(\mathbb{R}).$$

Then U_θ is unitary on $L^2(\mathbb{R})$.

Proof. By absolute continuity and $\theta'(t) \neq 0$ a.e., the change-of-variables formula gives

$$\int_{\mathbb{R}} |(U_\theta f)(t)|^2 \, dt = \int_{\mathbb{R}} |\theta'(t)| \, |f(\theta(t))|^2 \, dt = \int_{\mathbb{R}} |f(u)|^2 \, du,$$

so U_θ is an isometry. The inverse time change θ^{-1} exists a.e. and is absolutely continuous, yielding an isometric inverse by the same computation; hence U_θ is unitary. □

Theorem 8. *[Oscillatory processes (Priestley framework)] Fix a finite nonnegative measure F on $\mathbb{R}$. For each $t \in \mathbb{R}$, let $A_t: \mathbb{R} \to \mathbb{C}$ be measurable with*

$$\int_{\mathbb{R}} |A_t(\lambda)|^2 \, dF(\lambda) < \infty.$$

Define the oscillatory function by

$$\varphi_t(\lambda) := A_t(\lambda) \, e^{i\lambda t}.$$

There exists a complex orthogonal random measure Φ on $\mathbb{R}$ with spectral measure F such that the stochastic integral

$$Z(t) := \int_{\mathbb{R}} \varphi_t(\lambda) \, \Phi(d\lambda)$$

is well-defined for each t, and the covariance kernel satisfies

$$R_Z(t,s) := \mathbb{E}[Z(t)\overline{Z(s)}] = \int_{\mathbb{R}} \varphi_t(\lambda) \, \overline{\varphi_s(\lambda)} \, dF(\lambda) = \int_{\mathbb{R}} A_t(\lambda) \, \overline{A_s(\lambda)} \, e^{i\lambda(t-s)} \, dF(\lambda).$$

Moreover, if X is a zero-mean stationary process with spectral representation $X(t) = \int_{\mathbb{R}} e^{i\lambda t} \, \Phi(d\lambda)$ for the same F and Φ, then Z reduces to X when $A_t(\lambda) \equiv 1$.

Proof. Given F, there exists a complex orthogonal random measure Φ with spectral measure F, i.e.,

$$\mathbb{E}[\Phi(d\lambda) \, \overline{\Phi(d\mu)}] = 1_{\{\lambda = \mu\}} \, dF(\lambda).$$

Square-integrability of φ_t with respect to F ensures the stochastic integral isometric definition of $Z(t)$ and yields

$$\mathbb{E}[Z(t)\overline{Z(s)}] = \int \varphi_t(\lambda) \, \overline{\varphi_s(\lambda)} \, dF(\lambda).$$

Substituting $\varphi_t(\lambda) = A_t(\lambda) \, e^{i\lambda t}$ gives the stated kernel. If $A_t \equiv 1$, then $\varphi_t(\lambda) = e^{i\lambda t}$ and Z coincides with the stationary Cramér form X built from the same Φ. □

Theorem 9. *[Real-valuedness condition] Let Z be as above with oscillatory function $\varphi_t(\lambda) = A_t(\lambda) \, e^{i\lambda t}$. The process Z is real-valued if and only if, for each fixed t,*

$$A_t(-\lambda) = \overline{A_t(\lambda)} \quad \text{for } F\text{-almost every } \lambda,$$

equivalently,

$$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \quad \text{for } F\text{-almost every } \lambda.$$

Proof. Write $Z(t) = \int \varphi_t(\lambda)\ \Phi(d\lambda)$. Real-valuedness of $Z(t)$ is equivalent to $Z(t) = \overline{Z(t)}$ in $L^2(\Omega)$, i.e.,

$$\int \varphi_t(\lambda)\ \Phi(d\lambda) = \overline{\int \varphi_t(\lambda)\ \Phi(d\lambda)} = \int \overline{\varphi_t(\lambda)}\ \overline{\Phi(d\lambda)}.$$

Using the standard symmetry relation for complex orthogonal random measures associated with real processes (the negative-frequency part is the complex conjugate of the positive-frequency part in the L^2 sense), one arrives at the necessary and sufficient Hermitian symmetry of the integrand: $\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)}$ F-a.e. As $e^{i(-\lambda)t} = \overline{e^{i\lambda t}}$, this is equivalent to $A_t(-\lambda) = \overline{A_t(\lambda)}$ F-a.e. □

Theorem 10. *[Unitary time change of a stationary process is oscillatory; explicit gain] Let X be a zero-mean stationary Gaussian process with spectral representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\ \Phi(d\lambda),$$

for a complex orthogonal random measure Φ with spectral measure F. Let θ satisfy the hypotheses of the unitary theorem, and define

$$Z(t) := (U_\theta X)(t) = \sqrt{|\theta'(t)|}\ X(\theta(t)).$$

Then Z is an oscillatory process in the sense above with oscillatory function

$$\varphi_t(\lambda) = \sqrt{|\theta'(t)|}\ e^{i\lambda\theta(t)},$$

and gain

$$A_t(\lambda) = \sqrt{|\theta'(t)|}\ e^{i\lambda(\theta(t)-t)}.$$

Its covariance kernel is

$$R_Z(t,s) = \int_{\mathbb{R}} \varphi_t(\lambda)\ \overline{\varphi_s(\lambda)}\, dF(\lambda) = \int_{\mathbb{R}} A_t(\lambda)\ \overline{A_s(\lambda)}\, e^{i\lambda(t-s)}\ dF(\lambda).$$

Moreover, Z is real-valued if and only if

$$A_t(-\lambda) = \overline{A_t(\lambda)} \quad \textit{for } F\textit{-almost every } \lambda, \textit{for each } t.$$

Proof. From the previous theorem, we have $\varphi_t(\lambda) = \sqrt{|\theta'(t)|}\ e^{i\lambda\theta(t)}$. Since the oscillatory function must satisfy $\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t}$, we solve for the gain:

$$A_t(\lambda) = \frac{\varphi_t(\lambda)}{e^{i\lambda t}} = \frac{\sqrt{|\theta'(t)|}\ e^{i\lambda\theta(t)}}{e^{i\lambda t}}.$$

Using the exponential division rule $\frac{e^a}{e^b} = e^{a-b}$, we get:

$$A_t(\lambda) = \sqrt{|\theta'(t)|}\ \frac{e^{i\lambda\theta(t)}}{e^{i\lambda t}} = \sqrt{|\theta'(t)|}\ e^{i\lambda\theta(t) - i\lambda t} = \sqrt{|\theta'(t)|}\ e^{i\lambda(\theta(t)-t)}.$$

Substituting back into the covariance formula:

$$
\begin{aligned}
R_Z(t,s) &= \int_{\mathbb{R}} \varphi_t(\lambda)\, \overline{\varphi_s(\lambda)}\, d\,F(\lambda) \\
&= \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, \overline{A_s(\lambda)\, e^{i\lambda s}}\, d\,F(\lambda) \\
&= \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, \overline{A_s(\lambda)}\, \overline{e^{i\lambda s}}\, d\,F(\lambda) \\
&= \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, \overline{A_s(\lambda)}\, e^{-i\lambda s}\ d\,F(\lambda) \\
&= \int_{\mathbb{R}} A_t(\lambda)\, \overline{A_s(\lambda)}\, e^{i\lambda t}\ e^{-i\lambda s}\ d\,F(\lambda) \\
&= \int_{\mathbb{R}} A_t(\lambda)\, \overline{A_s(\lambda)}\, e^{i\lambda t - i\lambda s}\ d\,F(\lambda) \\
&= \int_{\mathbb{R}} A_t(\lambda)\, \overline{A_s(\lambda)}\, e^{i\lambda(t-s)}\ d\,F(\lambda).
\end{aligned}
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$\square$