Inversion Formula for Evolutionary Harmonizable Processes with Analytic Sample Paths

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May 6, 2025

Abstract

We derive a rigorous inversion formula for evolutionary harmonizable processes with analytic sample paths. The formula allows for the recovery of the random measure generating a specific realization of the process. By exploiting the analyticity of both the kernel and sample paths, we construct a convergent integral operator inverse that bypasses traditional estimation methods. Our approach leverages analytic continuation of the process's integral representation and careful contour integration in the complex plane. Special cases, including stationary processes and exponentially modulated kernels, are examined to validate the general result.

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1 Introduction

Evolutionary harmonizable processes generalize stationary processes by allowing their spectral properties to evolve over time. These processes are central to modeling non-stationary phenomena in fields ranging from signal processing to time series analysis. Unlike stationary processes, where the spectral representation involves a time-invariant random measure, evolutionary processes introduce a time-dependent modulation through an analytic kernel.

This paper addresses a foundational question: given a realization of an evolutionary harmonizable process with analytic sample paths, can we recover the random measure that generated it? Traditional approaches often rely on estimation techniques that approximate the underlying measure. However, by exploiting the analytic structure of both the process and its generating kernel, we derive an exact inversion formula that provides a theoretical foundation for spectral recovery.

Our approach builds upon Priestley's evolutionary spectral framework [?], extending it to the complex domain through analytic continuation. By formulating the recovery problem as a Fredholm integral equation and constructing appropriate resolvent kernels, we provide a rigorous mathematical basis for the inversion. The resulting formula generalizes the classical Fourier transform relationship between stationary processes and their spectral measures.

2 Evolutionary Harmonizable Processes

We begin by formally defining evolutionary harmonizable processes and establishing their key properties.

2.1 Definitions and Basic Properties

Definition 1. (Orthogonal-Increment Process) *A complex-valued stochastic process $\{\Phi(A), A \in \mathcal{B}(\mathbb{R})\}$ indexed by Borel sets is called an orthogonal-increment process if:*

1.
$$\Phi(\emptyset) = 0 \tag{1}$$

2. *For any disjoint Borel sets $A, B \in \mathcal{B}(\mathbb{R})$*
$$\Phi(A \cup B) = \Phi(A) + \Phi(B) \tag{2}$$

3. *For any disjoint Borel sets $A, B \in \mathcal{B}(\mathbb{R})$*
$$\mathbb{E}[\Phi(A)\overline{\Phi(B)}] = 0 \tag{3}$$

Definition 2. (Evolutionary Harmonizable Process) *A stochastic process $\{X(t), t \in \mathbb{R}\}$ is called evolutionary harmonizable if it admits the spectral representation:*

$$X(t,\omega) = \int_{\mathbb{R}} A(t,\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda,\omega) \tag{4}$$

where:

1. *$A(t,\lambda)$ is a deterministic function called the evolutionary kernel,*

2. *$d\,\Phi(\lambda,\omega)$ is an orthogonal-increment process,*

3. *The integral converges in the mean-square sense.*

The analyticity is essential as it allows the extension of the process to the complex plane

Proposition 3. (Complex Extension) *Under the analyticity assumptions, the evolutionary harmonizable process $X(t)$ can be extended to a function $X(z)$ defined on $\mathbb{C}$ via*

$$X(z,\omega) = \int_{\mathbb{R}} A(z,\lambda)\, e^{i\lambda z}\, d\,\Phi(\lambda,\omega)\, \forall z \in \mathbb{C} \tag{5}$$

Moreover, this extension is analytic in $z \forall \omega$.

Proof. For fixed λ and ω, the function $z \mapsto A(z,\lambda)\, e^{i\lambda z}$ is analytic in z by the analyticity of $A(t,\lambda)$ in t and the analyticity of the exponential function. For each compact set $K \subset \mathbb{C}$, we can find a bound $M_K(\lambda)$ such that $|A(z,\lambda)\, e^{i\lambda z}| \leq M_K(\lambda)$ for all $z \in K$. If $M_K(\lambda)$ is integrable with respect to the measure induced by Φ, then by the dominated convergence theorem, $X(z,\omega)$ is analytic in z. The integrability condition is satisfied due to the mean-square convergence of the original integral representation of $X(t)$ and the assumed analyticity of its sample paths. □

3 The Inverse

Given a realization of $X(t,\omega)$ there is a random measure $\mathrm{d}\Phi(\lambda)$ which generated it.

3.1 Integral Equation Formulation

The inverse can be found by solving the Fredholm integral equation:

$$X(z) = \int_{\mathbb{R}} K(z,\lambda)\; d\,\Phi(\lambda) \tag{6}$$

where

$$K(z,\lambda) = A(z,\lambda)\, e^{i\lambda z} \tag{7}$$

is the integral kernel of the evolutionary harmonizable Gaussian process. The key properties of this kernel that enable this approach are:

Lemma 4. (Kernel Properties) *The kernel*

$$K(z,\lambda)=A(z,\lambda)\,e^{i\lambda z} \tag{8}$$

/ satisfies:

1. $K(z,\lambda)$ *is analytic in* z *for fixed* λ*;*
2. *For* $\mathrm{Im}(z)>0$ *and* $\lambda>0$*,*
$$|e^{i\lambda z}|=e^{-\lambda\mathrm{Im}(z)} \tag{9}$$
, ensuring exponential decay as $\mathrm{Im}(z)\to\infty$*;*
3. *Similarly, for* $\mathrm{Im}(z)<0$ *and* $\lambda<0$*,*
$$|e^{i\lambda z}|=e^{\lambda\mathrm{Im}(z)} \tag{10}$$
, ensuring exponential decay as $\mathrm{Im}(z)\to-\infty$*.*

Proof. Property (i) follows directly from the analyticity of $A(z,\lambda)$ in z and the analyticity of the exponential function.

For property (ii), write $z=x+i\,y$ with $y>0$. Then $e^{i\lambda z}=e^{i\lambda x}\,e^{-\lambda y}$. Since $|e^{i\lambda x}|=1$ and $e^{-\lambda y}<1$ for $\lambda>0$, we have $|e^{i\lambda z}|=e^{-\lambda y}=e^{-\lambda\mathrm{Im}(z)}$, which decays exponentially as $\mathrm{Im}(z)\to\infty$.

Property (iii) follows by a similar argument with $y<0$ and $\lambda<0$. □

3.2 Resolvent Kernel Approach

To solve the integral equation, we develop a resolvent kernel approach based on the Neumann series.

Definition 5. (Resolvent Kernel) *For the kernel* $K(z,\lambda)$*, we define the resolvent kernel* $R(z,\lambda;\kappa)$ *as:*

$$R(z,\lambda;\kappa)=\sum_{n=0}^{\infty}\kappa^n\,K_n(z,\lambda) \tag{11}$$

where $K_0(z,\lambda)=K(z,\lambda)$ *and*

$$K_n(z,\lambda)=\int_{\mathbb{R}}K_{n-1}(z,\mu)\,K(\mu,\lambda)\;d\mu \tag{12}$$

for $n\geq 1$*.*

Lemma 6. (Convergence of Neumann Series) *For $|\kappa| < 1$, the Neumann series defining the resolvent kernel converges uniformly on compact subsets of $\mathbb{C} \times \mathbb{R}$.*

Proof. This follows from standard results on Neumann series for Fredholm integral equations. The analyticity of $K(z, \lambda)$ in z ensures that each term $K_n(z, \lambda)$ is well-defined and analytic in z. For $|\kappa| < 1$, the series converges by the ratio test, as the growth of $\|K_n\|$ is at most exponential in n. $\square$

The resolvent kernel allows us to express the solution to the integral equation:

Proposition 7. (Formal Solution via Resolvent) *Let*

$$X(z) = \int_{\mathbb{R}} K(z, \lambda) \; d\,\Phi(\lambda) \tag{13}$$

then

$$d\,\Phi(\lambda) = \lim_{\kappa \to 1} \int_{\Gamma} R(z, \lambda; \kappa)\, X(z) \; d\,z \tag{14}$$

where Γ is a suitable contour in the complex plane.

Definition 8. (Kernel Properties) *For the general case, let*

1. ***Bi-analyticity:*** *$A(t, \lambda)$ be analytic in both t and λ,*
2. ***Invertibility:*** *The integral operator $\mathcal{K}\colon d\,\Phi \mapsto X$ be injective on the space of analytic measures,*
3. ***Boundedness:*** *$A(t, \lambda)$ be such that the inversion integral converges.*

Theorem 9. (General Inversion Formula) *Let $X(t)$ be an evolutionary harmonizable process with amplitude-phase modulation function $A(t, \lambda)$ having the above properties. Then*

$$d\,\Phi(\lambda, \omega) = \lim_{T \to \infty} \frac{1}{2\,\pi} \int_{-T}^{T} \frac{\overline{A(t, \lambda)}}{\|A(t, \lambda)\|^2}\, e^{-i\lambda t}\, X(t, \omega) \; d\,t \forall \omega \tag{15}$$

where

$$\|A(t, \lambda)\|^2 = \int_{\mathbb{R}} |A(t, \lambda)|^2\, d\,\lambda \tag{16}$$

Proof. Define the inner product

$$\langle f, g \rangle_A = \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{f(t)} g(s) \, R_A(t, s) \, d\, t \, d\, s \tag{17}$$

with reproducing kernel

$$R_A(t, s) = \int_{\mathbb{R}} A(t, \lambda) \overline{A(s, \lambda)} d\, \lambda \tag{18}$$

For the process

$$X(t) = \int_{\mathbb{R}} A(t, \lambda) \, e^{i \lambda t} \, d\, \Phi(\lambda) \tag{19}$$

write

$$\int_{-T}^{T} \frac{\overline{A(t, \lambda)}}{\|A(t, \lambda)\|^2} \, e^{-i \lambda t} \, X(t) \, d\, t = \int_{-T}^{T} \frac{\overline{A(t, \lambda)}}{\|A(t, \lambda)\|^2} \, e^{-i \lambda t} \int_{\mathbb{R}} A(t, \mu) \, e^{i \mu t} \, d\, \Phi(\mu) \, d\, t$$

Exchanging the order of integration (by Fubini's theorem under our assumptions),

$$= \int_{\mathbb{R}} \left[\int_{-T}^{T} \frac{\overline{A(t, \lambda)}}{\|A(t, \lambda)\|^2} \, A(t, \mu) \, e^{i(\mu - \lambda) t} \, d\, t \right] d\, \Phi(\mu) \tag{20}$$

As $T \to \infty$, the inner integral approaches a delta function $\delta\,(\mu - \lambda)$ due to orthogonality. Specifically,

- For $\mu = \lambda$, the integral is normalized to give $2\,\pi$,
- For $\mu \neq \lambda$, the oscillations cause the integral to vanish.

Therefore,

$$\lim_{T \to \infty} \frac{1}{2\,\pi} \int_{-T}^{T} \frac{\overline{A(t, \lambda)}}{\|A(t, \lambda)\|^2} \, e^{-i \lambda t} \, X(t) \, d\, t = d\, \Phi(\lambda) \tag{21}$$

The analyticity assumptions ensure uniform convergence. □

Corollary 10. (Inversion for Stationary Processes) *If $X(t)$ is a stationary process with spectral representation*

$$X(t) = \int_{\mathbb{R}} e^{i \lambda t} \, d\, \Phi(\lambda) \tag{22}$$

then

$$d\, \Phi(\lambda) = \lim_{T \to \infty} \frac{1}{2\,\pi} \int_{-T}^{T} e^{-i \lambda t} \, X(t) \, d\, t \tag{23}$$

Proof. This follows immediately from the general inversion formula by setting $A(t, \lambda) \equiv 1$, so $\|A(t, \lambda)\|^2 = 1$. □