Theorem: Caputo Fractional Derivative of the Sine Function

Definition 1. *[Caputo Fractional Derivative] For $n-1<\alpha<n$ where $n\in\mathbb{N}$, the Caputo fractional derivative of order α is defined as:*

$$ {}_0^C D_t^\alpha f(t) = \frac{\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau}{\Gamma(n-\alpha)} \tag{1}$$

Definition 2. *[Two-Parameter Mittag-Leffler Function]The Mittag-Leffler function is defined as:*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha,\beta>0 \tag{2}$$

Theorem 3. *For $0<\alpha<1$, the Caputo fractional derivative of $\sin(t)$ is:*

$$ {}_0^C D_t^\alpha \sin(t) = t^{1-\alpha} E_{2,2-\alpha}(-t^2) \tag{3}$$

Proof. Let $f(t)=\sin(t)$. Since $\alpha\in(0,1)$, we have $n=1$ in Definition 1. The first derivative is:

$$f^{(1)}(t) = \cos(t) \tag{4}$$

Substitute into the Caputo definition (1):

$$ {}_0^C D_t^\alpha \sin(t) = \frac{\int_0^t \frac{\cos(\tau)}{(t-\tau)^{\alpha}}\, d\tau}{\Gamma(1-\alpha)} \tag{5}$$

Substitute the Taylor series of $\cos(\tau)$

$$\cos(\tau) = \sum_{k=0}^{\infty} \frac{(-1)^k \tau^{2k}}{(2k)!} \tag{6}$$

then make the substitution

$$u = \frac{\tau}{t} \tag{7}$$

such that

$$
\begin{aligned}
{}_{0}^{C}D_{t}^{\alpha}\sin(t) =& \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\int_0^t \frac{\tau^{2k}}{(t-\tau)^{\alpha}}\,d\tau && (8)\\
=& \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}t^{2k+1-\alpha}\int_0^1 u^{2k}(1-u)^{-\alpha}\,du && (9)\\
=& \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}t^{2k+1-\alpha}B(2k+1,1-\alpha)\\
=& \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}t^{2k+1-\alpha}\frac{\Gamma(2k+1)\Gamma(1-\alpha)}{\Gamma(2k+2-\alpha)}\\
=& \sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1-\alpha}}{\Gamma(2k+2-\alpha)}
\end{aligned}
$$

Factor out $t^{1-\alpha}$:

$$
= t^{1-\alpha}\sum_{k=0}^{\infty}\frac{(-t^2)^k}{\Gamma(2k+2-\alpha)} \tag{10}
$$

Compare with Definition 2:

$$
\sum_{k=0}^{\infty}\frac{(-t^2)^k}{\Gamma(2k+2-\alpha)} = E_{2,2-\alpha}(-t^2) \tag{11}
$$

Thus yielding the result:

$$
{}_{0}^{C}D_{t}^{\alpha}\sin(t) = t^{1-\alpha}E_{2,2-\alpha}(-t^2) \quad \blacksquare \tag{12}
$$
$\square$

For $1<\alpha<2$ $(n=2)$, repeating the process with $f^{(2)}(t) = -\sin(t)$ yields:

$$
{}_{0}^{C}D_{t}^{\alpha}\sin(t) = \cos(t) - \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \sum_{k=1}^{\infty}\frac{(-1)^k t^{2k+2-\alpha}}{\Gamma(2k+3-\alpha)} \tag{13}
$$

which simplifies to:

$$
{}_{0}^{C}D_{t}^{\alpha}\sin(t) = \cos(t) - t^{2-\alpha}E_{2,3-\alpha}(-t^2) \tag{14}
$$

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