Cramér's Pathwise Inversion Formula for the Orthogonal Random Measure

The Spectral Representation

For a centered stationary Gaussian process $\xi(t)$ on $\mathbb{R}$, there exists an orthogonal random measure $\zeta(\lambda)$ such that

$$\xi(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, d\zeta(\lambda).$$

Recovery of the Spectral Distribution Function

The spectral distribution function $F(\lambda)$ is recovered from the autocovariance $r(t) = \mathbb{E}[\xi(t+\tau)\xi(t)]$ by

$$F(\lambda_2) - F(\lambda_1) = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{-i\lambda_2 t} - e^{-i\lambda_1 t}}{-it} r(t)\, dt.$$

Recovery of the Orthogonal Random Measure Increment

The increment of the orthogonal random measure over the interval $(\lambda_1, \lambda_2]$ is obtained pathwise from the sample trajectory $\xi(t)$ by the quadratic mean limit

$$\zeta(\lambda_2) - \zeta(\lambda_1) = \frac{1}{2\pi} \text{l.i.m.}_{T\to\infty} \int_{-T}^{T} \frac{e^{-i\lambda_2 t} - e^{-i\lambda_1 t}}{-it} \xi(t)\, dt.$$

The Inverse Fourier Transform Identity

The kernel is the inverse Fourier transform of the indicator function of the interval $(\lambda_1, \lambda_2]$:

$$\mathbf{1}_{(\lambda_1,\lambda_2]}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\lambda_2 t} - e^{-i\lambda_1 t}}{-it} e^{it\lambda}\, dt.$$

Real Part of the Orthogonal Random Measure

Setting the lower bound to $\lambda_1 = 0$, the real part $u(\lambda)$ of $\zeta(\lambda)$ is

$$u(\lambda) = \frac{1}{\pi} \text{l.i.m.}_{T\to\infty} \int_{-T}^{T} \frac{\sin(\lambda t)}{t} \xi(t)\, dt.$$

Imaginary Part of the Orthogonal Random Measure

The imaginary part $v(\lambda)$ of $\zeta(\lambda)$ is

$$v(\lambda) = \frac{1}{\pi} \text{l.i.m.}_{T\to\infty} \int_{-T}^{T} \frac{1 - \cos(\lambda t)}{t} \xi(t)\, dt.$$

The Complete Orthogonal Random Measure

The full orthogonal random measure is

$$\zeta(\lambda) = u(\lambda) + iv(\lambda).$$