The Eigenfunctions of $\int_0^\infty J_0(|x-y|)\, f(x) \mathrm{d}x$ and a Technique For Deriving The Eigenfunctions of Stationary Gaussian Process Integral Covariance Operators

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Abstract

The null space of Gaussian process kernel inner product operators are shown to be the Fourier transforms of the polynomials orthogonal with respect to the spectral densities of the processes and it is furthermore shown that the orthogonal complements of the null spaces as given by the Gram-Schmidt recursions enumerate the products $g_k(t) = \sqrt{\frac{c_p}{c_q}} \frac{\prod_{i=1}^{n_k} (t - \alpha_{k,i})}{\prod_{j=1}^{m_k} (t - \beta_{k,j})} = f_k(t)\, f_k(s)$ of the eigenfunctions f_k of the corresponding integral covariance operators.

Let $C(x)$ be the covariance function of a stationary Gaussian process on $[0, \infty)$. Define the integral covariance operator T by:

$$(T f)(x) = \int_0^\infty C\,(x - y)\, f(y)\, \mathrm{d}\, y \tag{1}$$

Let $S(\omega)$ be the spectral density related to $C(x)$ by the Wiener-Khinchin theorem:

$$C(x) = \frac{1}{\pi} \int_{-\infty}^\infty e^{i\omega x}\, S(\omega)\, \mathrm{d}\, \omega \tag{2}$$

$$S(\omega) = \int_0^\infty C(x)\, e^{-i x\omega}\, \mathrm{d}\, x \tag{3}$$

Consider polynomials $\{p_n(\omega)\}$ orthogonal with respect to $S(\omega)$:

$$\int_{-\infty}^\infty p_n(\omega)\, p_m(\omega)\, S(\omega)\, \mathrm{d}\, \omega = \delta_{nm} \tag{4}$$

Define $r_n(x)$ as the inverse Fourier transforms of $p_n(\omega)$:

$$r_n(x) = \int_{-\infty}^\infty p_n(\omega)\, e^{i\omega x}\, \mathrm{d}\, \omega \tag{5}$$

Lemma 1. *The functions $r_n(x)$ form the null space of the kernel inner product:*

$$\int_0^\infty C(x)\, r_n(x)\, \mathrm{d}\, x = 0 \tag{6}$$

Proof. Proof: Substitute the definitions of $C(x)$ and $r_n(x)$, and apply Fubini's theorem:

$$\int_0^\infty C(x)\, r_n(x)\, \mathrm{d}\, x = \int_0^\infty \frac{1}{\pi} \int_{-\infty}^\infty e^{i\omega x}\, S(\omega)\, \mathrm{d}\,\omega \int_{-\infty}^\infty p_n(\omega')\, e^{i\omega' x}\, \mathrm{d}\,\omega'\, \mathrm{d}\, x \tag{7}$$

By Fubini's theorem, we can swap the integrals:

$$= \frac{1}{\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty p_n(\omega')\, S(\omega) \int_0^\infty e^{i(\omega+\omega')x}\, \mathrm{d}\, x\, \mathrm{d}\,\omega'\, \mathrm{d}\,\omega \tag{8}$$

The integral over x yields the Dirac delta function $\delta\,(\omega - \omega')$:

$$= \frac{1}{\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty p_n(\omega')\, S(\omega)\, \pi\, \delta\,(\omega - \omega')\, \mathrm{d}\,\omega'\, \mathrm{d}\,\omega \tag{9}$$

Now, integrate over ω' using the delta function:

$$= \int_{-\infty}^\infty p_n(\omega)\, S(\omega)\, \mathrm{d}\,\omega \tag{10}$$

By the orthogonality of $p_n(\omega)$ with respect to $S(\omega)$, we conclude:

$$\int_{-\infty}^\infty p_n(\omega)\, S(\omega)\, \mathrm{d}\,\omega = 0 \tag{11}$$

Thus, $\int_0^\infty C(x)\, r_n(x)\, \mathrm{d}\, x = 0$, which completes the proof. □

1 Eigenfunctions from Orthogonalized Null Space

By orthogonalizing the null space $\{r_n(x)\}$, we obtain the eigenfunctions $\{\psi_n(x)\}$ of the covariance operator T. The orthogonalization process gives:

$$r_n^\perp(y) = \psi_n(x) = \sum_{k=0}^{n} a_{nk}\, r_k(x) = r_n(y) - \sum_{m=0}^{n-1} \frac{\langle r_n(y), r_m^\perp(y)\rangle}{\langle r_m^\perp(y), r_m^\perp(y)\rangle}\, r_m^\perp(y)$$

where the coefficients a_{nk} are given by:

$$a_{nk} = \begin{cases} 1 & \text{if } k = n \\ -\sum_{j=k}^{n-1} a_{nj} \langle r_n, \psi_j \rangle & \text{if } k < n \\ 0 & \text{if } k > n \end{cases} \tag{12}$$

Theorem 2. *For any rational function $f(t-s) = \frac{P(t-s)}{Q(t-s)}$, where P and Q are polynomials, there exist rational functions $g(t)$ and $g(s)$ such that $f(t-s) = g(t)\, g(s)$.*

Proof. Let

$$P(t-s) = c_p \prod_{i=1}^{n} (t - s - \alpha_i) \tag{13}$$

and

$$Q(t-s) = c_q \prod_{j=1}^{m} (t - s - \beta_j) \tag{14}$$

be the complete factorizations over $\mathbb{C}$. Define:

$$g(t) = \sqrt{\frac{c_p}{c_q}} \frac{\prod_{i=1}^{n} (t - \alpha_i)}{\prod_{j=1}^{m} (t - \beta_j)}$$

Then:

$$\begin{aligned} g(t)\, g(s) &= \frac{c_p}{c_q} \frac{\prod_{i=1}^{n} (t - \alpha_i)}{\prod_{j=1}^{m} (t - \beta_j)} \cdot \frac{\prod_{i=1}^{n} (s - \alpha_i)}{\prod_{j=1}^{m} (s - \beta_j)} \\ &= \frac{c_p}{c_q} \frac{\prod_{i=1}^{n} (t - \alpha_i)(s - \alpha_i)}{\prod_{j=1}^{m} (t - \beta_j)(s - \beta_j)} \\ &= \frac{c_p \prod_{i=1}^{n} ((t-s) - \alpha_i)}{c_q \prod_{j=1}^{m} ((t-s) - \beta_j)} \\ &= f(t-s) \end{aligned}$$

For complex roots, we pair each α_i or β_j with its complex conjugate in the factorization of $g(t)$. This ensures that the product $(t - \alpha_i)(t - \bar{\alpha_i})$ results in a quadratic polynomial with real coefficients, making $g(t)$ a real-valued function. □

Theorem 3. *Let $\{\psi_n(x)\}$ be the orthogonal complement of $\{r_n(x)\}$. Then $\psi_n(x)$ are eigenfunctions of T, with eigenvalues:*

$$\lambda_n = \int_0^\infty C(z)\,\psi_n(z)\,\mathrm{d}\,z \tag{15}$$

Proof. This is not quite right, they have to be factorized as in Theorem 2. I think the infinite-dimensional version of this is the Hadamard product factorization? □

Definition 4. *Let $j_n(x)$ is the spherical Bessel function of the first kind,*

$$\begin{aligned} j_n(z) &= \sqrt{\frac{\pi}{2\,z}}\,J_{n+\frac{1}{2}}(x) \\ &= \frac{1}{\sqrt{z}}\left(\sin(z)\,R_{n,\frac{1}{2}}(z) - \cos(z)\,R_{n-1,\frac{3}{2}}(z)\right) \end{aligned} \tag{16}$$

where $R_{n,v}(z)$ are the (misnamed) Lommel polynomials [3]

$$R_{n,v}(z) = \frac{\Gamma(n+v)}{\Gamma(v)}\left(\frac{2}{z}\right)^n {}_2F_3\left(\left[-\frac{n}{2},\frac{1}{2}-\frac{n}{2}\right];[v,-n,1-v-n];-z^2\right) \tag{17}$$

where ${}_2F_3$ is a generalized hypergeometric function. The "Lommel polynomials" are actually rational functions of z, not polynomial; but rather "polynomial in $\frac{1}{z}$".

Conjecture 5. *The series*

$$\begin{aligned} J_0(t) &= \sum_{k=0}^\infty \lambda_k \psi_k(t) \\ &= \sum_{k=0}^\infty \sqrt{\frac{4\,n+1}{\pi}}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2}(-1)^n\sqrt{\frac{4\,n+1}{\pi}}\,j_{2n}(t) \\ &= \sum_{k=0}^\infty \frac{4\,n+1}{\pi}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2}(-1)^n\,j_{2n}(t) \end{aligned} \tag{18}$$

converges uniformly for all complex t except the origin where it has a regular singular point where $\lim_{t\to 0} J_0(t) = 1$.

Conjecture 6. *The eigenfunctions of the stationary integral covariance operator*

$$[T\psi_n](x) = \int_0^\infty J_0\,(x-y)\;\psi_n(x)\mathrm{d}x = \lambda_n \psi_n(x) \tag{19}$$

are given by

$$\psi_n(y) = (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\; j_{2n}(y) \tag{20}$$

and the eigenvalues are given by

$$\begin{aligned}\lambda_n &= \int_{-\infty}^{\infty} J_0(x)\;\psi_n(x)\,\mathrm{d}x\\ &= \sqrt{\frac{4\,n+1}{\pi}}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\,(n+1)^2}\\ &= \sqrt{\frac{4\,n+1}{\pi}}\,(n+1)^2_{-\frac{1}{2}}\end{aligned} \tag{21}$$

where $(n+1)^2_{-\frac{1}{2}}$ *is the Pochhammer symbol(ascending/rising factorial).*

Note 7. *TODO: change* $J_0(x)$ *to* $J_0(|x|)$ *and* $\psi_n(y)$ *to* $\psi_n(|y|)$*, by doing so the range of integration can be extended from* $(0,\infty)$ *to* $(-\infty,\infty)$ *without the integral vanishing due to the fact that the odd-idexed Bessel functions are odd and therefore integration over the full interval vanishes unless the absolute value is used. The absolute value is the natural way to utilize the functions sice they are symmetric or anti-symmetric depending upon the parity of the index and thus no information is lost in making this change and indeed doing so will make all the integer-indexed Type-1 Bessel functions representable via this expansion rather than only the even indexed ones as it is now. The primary aim is to conclusively prove that the proposed functions are the eigenfunctions and eigenvalues of* $\int_0^\infty J_0\,(x-y)\;\psi_n(x)\mathrm{d}x$ *but solving the operator for each* $\int_0^\infty J_m(x-y)\;\psi_n(x)\mathrm{d}x$ *seems inevitable since the same cofactor appears in*

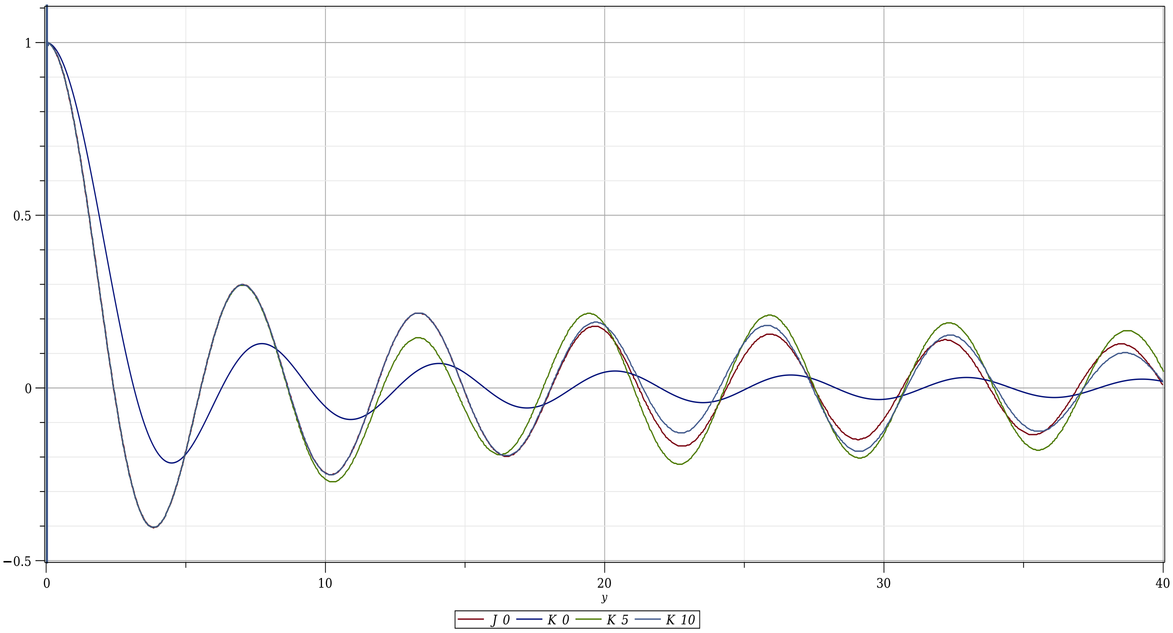

Figure 1. J_0 compared to the finite-rank approximations for rank $0, 5$, and 10. The figure shows the excellent convergence properties of the proposed eigenfunction expansion $J_0(x) = \lim_{n\to\infty}\sum_{k=0}^{n}\lambda_k\psi_k(x)$.

Remark 8. There is no doubt that the expansion $J_0(x) = \lim_{n\to\infty}\sum_{k=0}^{n}\lambda_k\psi_k(x)$ is correct, but only finding the eigenfunctions of the finite-rank operators and confirming via Rellich's theorem that the finite-rank subspace operators have eigenfunctions which converge to the full-rank operator eigenfunctions $\psi_n(x)$ whose eigenvalues are $\lambda_n = \int_{-\infty}^{\infty} J_0(|x|)\psi_n(x)\mathrm{d}x$

Theorem 9. *The operator defined by Equation (19) is compact relative to the canonical metric induced by the covariance kernel $J_0(|x-y|)$.*

Proof. Due to the spectral theorem for compact operators, if an operator is self-adjoint and has eigenvalues converging to 0, then it is necessarily compact. It is easy to see that

$$\lim_{n\to\infty}\lambda_n = 0 \tag{22}$$

and due to the symmetry of $J_0(|t|)$ it is self-adjoint and therefore the operator T defined in Equation (19) is compact. [1] $\square$

Definition 10. *The spectral density of a stationary process is the Fourier tranform of the covariance kernel due to Wiener-Khinchine theorem.*

Definition 11. *Let $S_n(x)$ be the orthogonal polynomials whose orthogonality measure is equal to the spectral density of the process. These polynomials shall be called the spectral polynomials corresponding to the process.*

Example 12. Let the kernel function be given by $K(t,s)=J_0(t-s)$ then identify the orthogonal polynomial sequence associated with the spectral density

$$S(\omega)=\int_{-\infty}^{\infty} J_0(x)\, e^{ix\omega}\, \mathrm{d}x=\begin{cases} \dfrac{2}{\sqrt{1-\omega^2}} & \omega\in(-1,1) \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

as being twice the orthogonality measure of the Type-I Chebyshev polynomials $T_n(x)$ so that the orthogonal polynomial sequence is identified as

$$S_n(x)=\sqrt{2}T_n(x) \tag{24}$$

so that

$$\int_{-1}^{1} S_n(\omega)\, S_m(\omega)\, S(\omega)\, \mathrm{d}\omega=\begin{cases} 0 & n\neq m \\ 2\pi & n=m=0 \\ \pi & n=m\neq 0 \end{cases} \tag{25}$$

Remark 13. If the spectral density does not equal the orthogonality measure of a known set of orthogonal polynomials then such a set can always be generated by applying the Gram-Schmidt process to the monomials so that they are transformed into a set that is orthogonal with respect any given spectral density (of a stationary process).

Definition 14. *The sequence $\hat{S}_n(y)$ of Fourier transforms of the spectral polynomials $S_n(x)$ is given by*

$$\hat{S}_n(y)=\int_{-1}^{1} S_n(x)e^{ixy}\mathrm{d}x \tag{26}$$

Example 15. The Fourier transforms of the Chebyshev polynomials are just the usual infinite Fourier transforms with the integration restricted to the range $-1\ldots 1$ since $T_n(x)=0\forall x\notin[-1,1]$). Equivalently, the spectral density function can be extended to take the value 0 outside the interval $[-1,1]$. The derivation of

$$\begin{aligned} \hat{T}_n(y) &= \int_{-\infty}^{\infty} e^{-ixy}\, T_n(x) d\, y=\int_{-1}^{1} e^{-ixy}\, T_n(x) d\, x \\ &= \int_{-\infty}^{\infty} e^{-ixy}\, {}_2F_1\left(\begin{matrix} n, & -n \\ & \frac{1}{2} \end{matrix} \middle| \frac{1}{2}-\frac{x}{2} \right) d\, x \\ &= \frac{i}{y}\, (e^{-iy}F_n^{+}(y)-e^{i(\pi n+y)}F_n^{-}(y)) \end{aligned} \tag{27}$$

where

$$F_n^{\pm}(y)={}_3F_1\left(\begin{matrix} 1, & n, & -n \\ & & \frac{1}{2} \end{matrix} \middle| \frac{\pm iy}{2} \right) \tag{28}$$

can be found in [2].

Definition 16. *Let $Y_n(y)$ be the normalized spectral polynomials $S_n(x)$*

Example 17. When $K = J_0$ the spectral polynomials are given by

$$S_n(x) = \sqrt{2} T_n(x) \tag{29}$$

so that

$$\begin{aligned} Y_n(y) &= \frac{\hat{T}_n(y)}{|\hat{T}_n|} \\ &= \frac{i}{y}\left(\frac{e^{-iy} F_n^+(y) - e^{i(\pi n + y)} F_n^-(y)}{\sqrt{\frac{4(-1)^n \pi - (2n^2 - 1)}{4n^2 - 1}}} \right) \end{aligned} \tag{30}$$

where the L^2 norm of $\hat{T}_n(y)$ is given by

$$\begin{aligned} |\hat{T}_n| &= \sqrt{\int_{-\infty}^{\infty} \hat{T}_n(y)^2 \mathrm{d}y} \\ &= \sqrt{\frac{4(-1)^n \pi - (2n^2 - 1)}{4n^2 - 1}} \end{aligned} \tag{31}$$

Conjecture 18. *The eigenfunctions of the integral covariance operator (19) are given by the orthogonal complement of the normalized Fourier transforms $Y_n(y)$ of the spectral polynomials (via the Gram-Schmidt process)*

$$\psi_n(y) = Y_n^{\perp}(y) = Y_n(y) - \sum_{m=1}^{n-1} \frac{\langle Y_n(y), Y_m^{\perp}(y) \rangle}{\langle Y_m^{\perp}(y), Y_m^{\perp}(y) \rangle} Y_m^{\perp}(y) \tag{32}$$

can be equivalently expressed as

$$\begin{aligned} \psi_n(y) &= (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\, j_{2n}(y) \\ &= (-1)^n \sqrt{\frac{4\,n+1}{\pi}} \sqrt{\frac{\pi}{2\,y}}\, J_{2n+\frac{1}{2}}(y) \\ &= (-1)^n \sqrt{\frac{4\,n+1}{2\,y}}\, J_{2n+\frac{1}{2}}(y) \\ &= \sqrt{\frac{4\,n+1}{4\,\pi}} \int_{-1}^{1} P_{2n}(x)\, e^{ixy}\, \mathrm{d}x \end{aligned} \tag{33}$$

Remark 19. Since T is compact due to its self-adjointness and convergence of the eigenvalues to 0 it converges uniformly since compactness implies uniform convergence of the eigenfunctions. TODO: cite/theorems from [4, 3. Reproducing Kernel Hilbert Space of a Gaussian Process]

Theorem 20. *The Bessel function identity*

$$J_0(x-y) = \sum_{k=-\infty}^{\infty} J_k(x)\, J_k(y) \tag{34}$$

can be used to expression Equation (19) as a series

$$\begin{aligned} [T f](y) &= \int_0^{\infty} J_0(x-y)\, f(x)\, d x \\ &= \int_0^{\infty} \sum_{k=-\infty}^{\infty} J_k(x)\, J_k(y)\, f(x)\, d x \\ &= \sum_{k=-\infty}^{\infty} J_k(y) \int_0^{\infty} J_k(x)\, f(x)\, d x \end{aligned}$$

by applying Fubini's theorem to exchange the sum with the integral when f is absolutely integrable.

Proof. TODO: demonstrate the identity is well-known and the interchange is justified when $f \in L^1_{0,\infty}$, e.g., f is Lebesgue absolutely integrable over $[0,\infty]$ □

Example 21. Simplifying The Convolution

Apply the addition theorem

$$J_0(x-y) = \sum_{k=-\infty}^{\infty} J_k(x)\, J_k(y) \tag{35}$$

to the integral covariance operator from Conjecture 20

$$\begin{aligned} [T\psi_n](x) &= \int_0^{\infty} J_0(x-y)\, \psi_n(y)\, d y \\ &= \int_0^{\infty} \sum_{k=-\infty}^{\infty} J_k(x)\, J_k(y)\, \psi_n(y)\, d y \\ &= \sum_{k=-\infty}^{\infty} J_k(x) \int_0^{\infty} J_k(y)\, \psi_n(y)\, d y \\ &= \sum_{k=-\infty}^{\infty} J_k(x) \int_0^{\infty} J_k(y)\, (-1)^n \sqrt{\frac{4n+1}{\pi}}\, j_{2n}(y)\, d y \end{aligned} \tag{36}$$

Where $\psi_n(y)$ is:

$$\psi_n(y) = (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\; j_{2n}(y) = (-1)^n \sqrt{\frac{4\,n+1}{\pi}}\; \sqrt{\frac{\pi}{2\,y}}\; J_{2n+\frac{1}{2}}(y) \tag{37}$$

Substituting

$$\begin{aligned}\int_0^\infty J_k(y)\,\psi_n(y)\; d\,y \;&= \int_0^\infty J_k(y)\,(-1)^n \sqrt{\frac{4\,n+1}{\pi}}\; j_{2n}(y)\; d\,y \\ &= \frac{\sqrt{4\,n+1}\,(-1)^n \sqrt{\pi}\,\Gamma\left(\frac{k}{2}+n+\frac{1}{2}\right)}{2\,\Gamma\left(-n+\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{k}{2}+n+1\right)\Gamma\left(n+1-\frac{k}{2}\right)}\end{aligned} \tag{38}$$

Now, putting it all back into the expansion for $[T\psi_n](x)$:

$$[T\psi_n](x) = \sum_{k=-\infty}^{\infty} J_k(x) \frac{\sqrt{4\,n+1}\,(-1)^n \sqrt{\pi}\,\Gamma\left(\frac{k}{2}+n+\frac{1}{2}\right)}{2\,\Gamma\left(-n+\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{k}{2}+n+1\right)\Gamma\left(n+1-\frac{k}{2}\right)} \tag{39}$$

Remark 22. I checked a few points of Equation 39 and found it to only be correct for $x = 0$; I think the calculations need to be redone with the kernel $J_0(|x|)$ instead of $J_0(x)$ because as is, the reproduction of the odd-indexed Bessel functions does not hold, only for the even functions J_{2n} is it true that

$$J_{2n}(x) = \sum_{k=0}^{\infty} \psi_k(x) \int_0^\infty J_{2n}(y)\psi_k(y)\mathrm{d}k \tag{40}$$

which is the same as Equation 38 after making the change-of-variables $2n \to n$. Now that I think about that as I write it, I realize that's probably why Equation 39 doesn't quite work as written…hmm.

Theorem 23.

$$\sum_{k=0}^{\infty} \psi_k(x)^2 = \frac{1}{\pi} \tag{41}$$

Theorem 24.

$$\int_0^\infty \psi_n(x)^2 \mathrm{d}x = 1 = K(s,s) = K(s-s) = K(0) \tag{42}$$

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