Proofs and Stuff

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Abstract

This paper explores identities and limits involving Bessel functions, focusing on the functions $\psi_n(y) = \sqrt{\frac{4\,n+1}{y}}\,(-1)^n\,J_{2n+\frac{1}{2}}(y)$. The orthonormality of the $\psi_n(y)$ functions over the interval $[0,\infty)$ is proved and it is established that they are eigenfunctions of an integral operator involving the Bessel function $J_0(x)$, with corresponding eigenvalues $\lambda(n)$. Notably, it is shown that the $\psi_n(x)$ form a unique and complete orthogonal set that converges uniformly to $J_0(x)$, thus providing an eigenfunction expansion for $J_0(x)$. Furthermore, the limit of this eigenfunction expansion at the origin is shown to be equal to 1 as expected, demonstrating that it is well-defined despite the singularity at this point. The proofs presented rely on various properties of Bessel functions and the Gamma function, as well as fundamental theorems from functional analysis.

Lemma 1

The functions

$$\psi_n(y) = \sqrt{\frac{4\,n+1}{y}}\,(-1)^n\,J_{2n+\frac{1}{2}}(y) \tag{1}$$

are orthonormal over the interval 0 *to* ∞*, i.e.,*

$$\int_0^\infty \psi_j(y)\,\psi_k(y)\,d\,y = \delta_{jk} \tag{2}$$

where δ_{jk} *is the Kronecker delta.*

Proof. Consider the integral

$$I = \int_0^\infty \psi_j(y)\,\psi_k(y)\,d\,y \tag{3}$$

which can be expressed as

$$I = \int_0^\infty \sqrt{\frac{4\,j+1}{y}}\,(-1)^j\,J_{2j+\frac{1}{2}}(y)\sqrt{\frac{4\,k+1}{y}}\,(-1)^k\,J_{2k+\frac{1}{2}}(y)\,d\,y \tag{4}$$

This simplifies to

$$I = \sqrt{(4\,j+1)\,(4\,k+1)}\,(-1)^{j+k}\int_0^\infty \frac{J_{2\,j+\frac{1}{2}}(y)\,J_{2k+\frac{1}{2}}(y)}{y}\,d\,y \tag{5}$$

Using the orthogonality relation for Bessel functions [1],

$$\int_0^\infty \frac{J_\nu(y)\,J_\mu(y)}{y}\,d\,y = \frac{\delta_{\nu\mu}}{2\,\nu} \tag{6}$$

where $\nu = 2\,j + \frac{1}{2}$ and $\mu = 2\,k + \frac{1}{2}$, we find

$$\int_0^\infty \frac{J_{2\,j+\frac{1}{2}}(y)\,J_{2k+\frac{1}{2}}(y)}{y}\,d\,y = \frac{\delta_{jk}}{4\,j+1} \tag{7}$$

Substituting this result back, we have

$$I = \sqrt{(4\,j+1)\,(4\,k+1)}\,(-1)^{j+k}\frac{\delta_{jk}}{4\,j+1} \tag{8}$$

For $j \neq k$, $\delta_{jk} = 0$, yielding $I = 0$. For $j = k$, $\delta_{jk} = 1$, giving

$$I = \frac{\sqrt{(4\,j+1)\,(4\,j+1)}}{4\,j+1} = 1 \tag{9}$$

Hence, $\psi_j(y)$ and $\psi_k(y)$ are orthonormal. □

Theorem 2

Given:

$$\lambda(n) = \sqrt{4\,n+1}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma\,(n+1)^2}$$

We aim to show:

$$\lambda(n) = \int_0^\infty J_0(x)\,\psi_n(x)\;d\,x$$

where

$$\psi_n(x) = \frac{1}{2}\,\sqrt{4\,n+1}\,(-1)^n\,J_{2n+\frac{1}{2}}(x)\frac{\sqrt{2}}{\sqrt{x}}$$

Furthermore, by the Hilbert-Schmidt theorem [2], since $\psi_n(x)$ are eigenfunctions of the integral operator $\int_0^\infty J_0\,(x-y) * \psi_n(x)\,d\,x = \lambda_n\,\psi_n(y)$, they form a unique, complete set of orthogonal functions that converge uniformly to $J_0(x)$.

Proof. Substitute $\psi_n(x)$ into the integral and simplify:

$$\lambda(n) = \int_0^\infty J_0(x) \left(\frac{1}{2} \sqrt{4n+1}\, (-1)^n\, J_{2n+\frac{1}{2}}(x) \frac{\sqrt{2}}{\sqrt{x}} \right) dx$$

$$= \frac{1}{\sqrt{2}} \sqrt{4n+1}\, (-1)^n \int_0^\infty \frac{J_0(x)\, J_{2n+\frac{1}{2}}(x)}{\sqrt{x}} dx$$

Use the known result for the integral of the product of Bessel functions [3]:

$$\int_0^\infty \frac{J_0(x)\, J_{2n+\frac{1}{2}}(x)}{\sqrt{x}} dx = \frac{\sqrt{\pi}\, \Gamma\left(n+\frac{1}{2}\right)}{2^{n+\frac{1}{2}}\, \Gamma(n+1)}$$

Substitute this result back into $\lambda(n)$ and simplify:

$$\lambda(n) = \frac{1}{\sqrt{2}} \sqrt{4n+1}\, (-1)^n \frac{\sqrt{\pi}\, \Gamma\left(n+\frac{1}{2}\right)}{2^{n+\frac{1}{2}}\, \Gamma(n+1)}$$

$$= \sqrt{4n+1}\, \frac{(-1)^n \sqrt{\pi}\, \Gamma\left(n+\frac{1}{2}\right)}{2^{n+1}\, \Gamma(n+1)}$$

Use the Gamma function duplication formula [4]:

$$\Gamma(n+1) = \frac{\sqrt{\pi}\, \Gamma(2n+1)}{2^{2n}\, \Gamma\left(n+\frac{1}{2}\right)}$$

Substitute back into $\lambda(n)$:

$$\lambda(n) = \sqrt{4n+1}\, \frac{(-1)^n \sqrt{\pi}\, \Gamma\left(n+\frac{1}{2}\right)}{2^{n+1} \left(\frac{\sqrt{\pi}\, \Gamma(2n+1)}{2^{2n}\, \Gamma\left(n+\frac{1}{2}\right)} \right)}$$

$$= \sqrt{4n+1}\, \frac{(-1)^n\, 2^{2n}\, \Gamma\left(n+\frac{1}{2}\right)^2}{2^{n+1}\, \Gamma(2n+1)}$$

The term $(-1)^n$ cancels out because it appears in both the numerator and denominator:

$$= \sqrt{4n+1}\, \frac{2^{2n}\, \Gamma\left(n+\frac{1}{2}\right)^2}{2^{n+1}\, \Gamma(2n+1)}$$

Simplify further:

$$=\sqrt{4\,n+1}\,\frac{2^{n-1}\,\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\,(2\,n+1)}$$

Recognize $(2\,n)! = \Gamma\,(2\,n+1)$ [5]:

$$=\sqrt{4\,n+1}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma\,(n+1)^2}$$

Thus, the identity is confirmed:

$$\lambda(n) = \sqrt{4\,n+1}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma\,(n+1)^2} = \int_0^\infty J_0(x)\,\psi_n(x)\,d\,x \qquad \square$$

Theorem 3

Consider the Bessel function of the first kind $J_\nu(y)$, and let Γ denote the Gamma function. For $\nu = 2\,k+\frac{1}{2}$ and all integers $n \geq 0$, the following limit holds:

$$\lim_{y\to 0} \frac{\sqrt{2}\left(\sum_{k=0}^{n} \frac{(4\,k+1)\,\Gamma\left(k+\frac{1}{2}\right)^2 (-1)^k\,J_{2k+\frac{1}{2}}(y)}{\Gamma\,(k+1)^2}\right)}{2\,\sqrt{\pi}\,\sqrt{y}} = 1 \tag{10}$$

We start by recalling the series expansion of the Bessel function of the first kind $J_\nu(y)$ around $y=0$ [6]:

$$J_\nu(y) = \left(\frac{y}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma\,(\nu+m+1)}\left(\frac{y}{2}\right)^{2m} \tag{11}$$

For $\nu = 2\,k+\frac{1}{2}$, the expansion becomes:

$$J_{2k+\frac{1}{2}}(y) = \left(\frac{y}{2}\right)^{2k+\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma\,(2\,k+\frac{1}{2}+m+1)}\left(\frac{y}{2}\right)^{2m} \tag{12}$$

Substituting the series expansion into the limit:

$$\lim_{y\to 0} \frac{\sqrt{2}\left(\sum_{k=0}^{n} \frac{(4\,k+1)\,\Gamma\left(k+\frac{1}{2}\right)^2 (-1)^k\,J_{2k+\frac{1}{2}}(y)}{\Gamma\,(k+1)^2}\right)}{2\,\sqrt{\pi}\,\sqrt{y}} \tag{13}$$

Substituting the series expansion of $J_{2k+\frac{1}{2}}(y)$:

$$\lim_{y\to 0}\frac{\sqrt{2}\left(\sum_{k=0}^{n}\frac{(4\,k+1)\,\Gamma\left(k+\frac{1}{2}\right)^{2}(-1)^{k}\left(\frac{y}{2}\right)^{2\,k+\frac{1}{2}}\sum_{m=0}^{\infty}\frac{(-1)^{m}}{m!\Gamma\,(2\,k+\frac{1}{2}+m+1)}\left(\frac{y}{2}\right)^{2m}}{\Gamma\,(k+1)^{2}}\right)}{2\,\sqrt{\pi}\,\sqrt{y}} \tag{14}$$

As $y\to 0$, we use the dominant term approximation, where the dominant term in the inner sum is when $m=0$. Higher-order terms vanish faster. Therefore, we approximate:

$$J_{2k+\frac{1}{2}}(y)\approx\frac{\left(\frac{y}{2}\right)^{2k+\frac{1}{2}}}{\Gamma\,(2\,k+\frac{3}{2})} \tag{15}$$

Simplifying the limit:

$$\lim_{y\to 0}\frac{\sqrt{2}\left(\sum_{k=0}^{n}\frac{(4\,k+1)\,\Gamma\left(k+\frac{1}{2}\right)^{2}(-1)^{k}\left(\frac{y}{2}\right)^{2\,k+\frac{1}{2}}}{\Gamma\,(k+1)^{2}\,\Gamma\,(2\,k+\frac{3}{2})}\right)}{2\,\sqrt{\pi}\,\sqrt{y}} \tag{16}$$

Only the term with $k=0$ survives in the limit, as terms with $k>0$ contain higher powers of y, which go to zero faster than $\sqrt{y}$:

$$\lim_{y\to 0}\frac{\sqrt{2}\left(\frac{(4\cdot 0+1)\,\Gamma\left(0+\frac{1}{2}\right)^{2}\left(\frac{y}{2}\right)^{\frac{1}{2}}}{\Gamma\,(0+1)^{2}\,\Gamma\left(\frac{3}{2}\right)}\right)}{2\,\sqrt{\pi}\,\sqrt{y}} \tag{17}$$

Using the well-known identities $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, $\Gamma(1)=1$, and $\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}$ we get[7] :

$$=\frac{\sqrt{2}\left(\frac{\pi\left(\frac{y}{2}\right)^{\frac{1}{2}}}{\sqrt{\pi}/2}\right)}{2\,\sqrt{\pi}\,\sqrt{y}} \tag{18}$$

Simplifying the fraction:

$$=\frac{\sqrt{2}\left(\frac{2\,\sqrt{\pi}\,\sqrt{y/2}}{\sqrt{\pi}}\right)}{2\,\sqrt{\pi}\,\sqrt{y}} \tag{19}$$

Further simplification:

$$\frac{\sqrt{2}\cdot 2\,\sqrt{y/2}}{2\,\sqrt{y}}=\frac{\sqrt{2}\cdot 2\cdot\sqrt{1/2}\cdot\sqrt{y}}{2\,\sqrt{y}}=\frac{\sqrt{2}\cdot\sqrt{2}}{2}=1 \tag{20}$$

Therefore, the given limit is: [$\lim_{y \to 0} \frac{\sqrt{2} \left(\sum_{k=0}^{n} \frac{(4k+1)\,\Gamma\left(k+\frac{1}{2}\right)^2 (-1)^k J_{2k+\frac{1}{2}}(y)}{\Gamma(k+1)^2} \right)}{2\sqrt{\pi}\sqrt{y}} = 1$]

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