# The Eigenfunctions of Stationary and Oscillatory Processes on the Real Line

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**Definition 1. (Stationary Process)** *A stochastic process* $\{X(t), t \in \mathbb{R}\}$ *is stationary when* $R(s,t) = R(t-s)$ *for all* $s, t \in \mathbb{R}$.

**Definition 2. (Oscillatory Process (Priestley))** *A process* $\{X(t), t \in \mathbb{R}\}$ *admits the evolutionary spectral representation*

$$X(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, dZ(\omega)$$

*where* $A(t,\omega)$ *is a gain function and* $Z(\omega)$ *is an orthogonal increment process.*

**Theorem 3. (Eigenfunction Property for Stationary Processes)** *Let* $R(\tau)$ *be a stationary covariance function. Define*

$$(Kf)(t) = \int_{-\infty}^{\infty} R(t-s)\, f(s)\, ds.$$

*For any* $\omega \in \mathbb{R}$,

$$K e^{i\omega t} = S(\omega)\, e^{i\omega t}, \qquad S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau.$$

**Proof.**

$$\begin{aligned}(Kf)(t) &= \int_{-\infty}^{\infty} R(t-s)\, e^{i\omega s}\, ds \\ &= \int_{-\infty}^{\infty} R(\tau)\, e^{i\omega(t-\tau)}\, d\tau \\ &= e^{i\omega t} \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau = S(\omega)\, e^{i\omega t}.\end{aligned}$$

□

**Theorem 4. (Eigenfunction Property for Oscillatory Processes)** *Let*

$$C(s,t) = \int_{-\infty}^{\infty} A(s,\omega)\, A^*(t,\omega)\, dF(\omega)$$

*and*

$$(Kf)(t) = \int_{-\infty}^{\infty} C(t,s)\, f(s)\, ds.$$

*Then for any* $\omega$*,*

$$\phi(t,\omega) = A(t,\omega)\, e^{i\omega t}$$

*is an eigenfunction of* $K$ *with eigenvalue* $dF(\omega)$*.*

**Proof.**

$$\begin{aligned}
K\phi(\cdot,\omega)(t) =& \int_{-\infty}^{\infty} C(t,s)\, \phi(s,\omega)\, ds \\
=& \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} A(t,\lambda)\, A^*(s,\lambda)\, dF(\lambda) \right) A(s,\omega)\, e^{i\omega s}\, ds \\
=& \int_{-\infty}^{\infty} A(t,\lambda) \left[ \int_{-\infty}^{\infty} A^*(s,\lambda)\, A(s,\omega)\, e^{i\omega s}\, ds \right] dF(\lambda) \\
=& \int_{-\infty}^{\infty} A(t,\lambda)\, \delta(\lambda-\omega)\, dF(\lambda) \\
=& A(t,\omega)\, dF(\omega) = \phi(t,\omega)\, dF(\omega).
\end{aligned}$$

□

**Theorem 5. (Reality and Conjugate Symmetry)** $X(t)$ *is real-valued if and only if* $A(t,\omega) = A^*(t,-\omega)$ *and* $dZ(-\omega) = dZ^*(\omega)$ *for all* $t,\omega$*. Also,* $\phi^*(t,\omega) = \phi(t,-\omega)$*.*

**Proof.** By definition,

$$X(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, dZ(\omega)$$

and

$$X^*(t) = \left( \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, dZ(\omega) \right)^* = \int_{-\infty}^{\infty} A^*(t,\omega)\, e^{-i\omega t}\, dZ^*(\omega).$$

Substitute $\nu = -\omega$ so $\omega = -\nu$ and $d\,\omega = -d\,\nu$:

$$X^*(t) = \int_{+\infty}^{-\infty} A^*(t, -\nu)\, e^{i\nu t}\, d\, Z^*\,(-\nu)\,(-d\,\nu) = \int_{-\infty}^{\infty} A^*(t, -\nu)\, e^{i\nu t}\, d\, Z^*\,(-\nu)\, d\,\nu$$

Rename dummy variable $\nu \mapsto \omega$:

$$X^*(t) = \int_{-\infty}^{\infty} A^*(t, -\omega)\, e^{i\omega t}\, d\, Z^*\,(-\omega)$$

For real-valued $X(t)$, require $X^*(t) = X(t)$ for all $t$. Uniqueness of the spectral representation forces

$$A(t, \omega) = A^*(t, -\omega), \qquad d\, Z\,(-\omega) = d\, Z^*(\omega).$$

For the eigenfunction conjugate pairs:

$$\phi^*(t, \omega) = [A(t, \omega)\, e^{i\omega t}]^* = A^*(t, \omega)\, e^{-i\omega t} = A(t, -\omega)\, e^{-i\omega t} = A(t, -\omega)\, e^{i(-\omega)t} = \phi(t, -\omega). \quad \square$$

**Theorem 6. (Filter Kernel: Dual Fourier Formula)**

$$h(t, u) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} A(t, \omega)\, e^{i\omega(t-u)}\, d\,\omega = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \phi(t, \omega)\, e^{-i\omega u}\, d\,\omega$$

**Proof.**

$$\begin{aligned} \int_{-\infty}^{\infty} \phi(t, \omega)\, e^{-i\omega u}\, d\,\omega &= \int_{-\infty}^{\infty} [A(t, \omega)\, e^{i\omega t}]\, e^{-i\omega u}\, d\,\omega \\ &= \int_{-\infty}^{\infty} A(t, \omega)\, e^{i\omega(t-u)}\, d\,\omega \end{aligned}$$

$\square$

**Theorem 7. (Inverse Relations)**

$$A(t, \omega) = \int_{-\infty}^{\infty} h(t, u)\, e^{-i\omega(t-u)}\, d\,u, \qquad \phi(t, \omega) = \int_{-\infty}^{\infty} h(t, u)\, e^{-i\omega u}\, d\,u$$

**Proof.** Start from

$$h(t, u) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} A(t, \lambda)\, e^{i\lambda(t-u)}\, d\,\lambda,$$

then multiply both sides by $e^{-i\omega(t-u)}$ and integrate over $u$:

$$\int_{-\infty}^{\infty} h(t,u)\, e^{-i\omega(t-u)}\, du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} A(t,\lambda)\, e^{i\lambda(t-u)}\, d\lambda \right) e^{-i\omega(t-u)}\, du$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} A(t,\lambda) \left( \int_{-\infty}^{\infty} e^{i(\lambda-\omega)(t-u)}\, du \right) d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} A(t,\lambda) \cdot 2\pi\, \delta(\lambda-\omega)\, d\lambda = A(t,\omega)$$

The result for $\phi(t,\omega)$ follows similarly using the inverse Fourier transform in $u$. □

**Theorem 8. (Filter Representation of Nonstationary Process)** *If* $X(u) = \int_{-\infty}^{\infty} e^{i\omega u}\, dZ(\omega)$, *then*

$$Y(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, dZ(\omega) = \int_{-\infty}^{\infty} h(t,u)\, X(u)\, du$$

*where* $h(t,u)$ *is as above.*

**Proof.** Substitute the spectral representation for $X(u)$:

$$\int_{-\infty}^{\infty} h(t,u)\, X(u)\, du = \int_{-\infty}^{\infty} h(t,u) \left( \int_{-\infty}^{\infty} e^{i\omega u}\, dZ(\omega) \right) du = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} h(t,u)\, e^{i\omega u}\, du \right) dZ(\omega)$$

Now substitute the kernel expression:

$$\int_{-\infty}^{\infty} h(t,u)\, e^{i\omega u}\, du = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(t,\lambda) \left( \int_{-\infty}^{\infty} e^{i\lambda(t-u)}\, e^{i\omega u}\, du \right) d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} A(t,\lambda)\, e^{i\lambda t} \left( \int_{-\infty}^{\infty} e^{i(\omega-\lambda)u}\, du \right) d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} A(t,\lambda)\, e^{i\lambda t}\, 2\pi\, \delta(\omega-\lambda)\, d\lambda = \int_{-\infty}^{\infty} A(t,\lambda)\, e^{i\lambda t}\, \delta(\omega-\lambda)\, d\lambda = A(t,\omega)\, e^{i\omega t}$$

Therefore,

$$Y(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, dZ(\omega)$$ □