On Compact Operators

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Abstract

In this basic note, we consider some basic properties of compact operators. We also consider the spectrum of compact operators on Hilbert spaces. A basic numerical example involving a compact integral operator is provided for further illustration.

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1 Introduction

The goal of this brief note is to collect some of the basic properties of compact operators on normed linear spaces. The results discussed here are all standard and be found in standard references such as [4, 9, 8, 7, 2, 5] to name a few. The point of this note is to provide an accessible introduction to some basic properties of compact operators. After stating some preliminaries and basic definitions in Section 2, we follow up by discussing some examples of compact operators in Section 3. Then, in Section 4, we discuss the range space of a compact operator, where we will see that the range of a compact operator is "almost finite-dimensional". In Section 5, we discuss a basic result on approximation of compact operators by finite-dimensional operators. Section 6 recalls some basic facts regarding the spectrum of linear operators on Banach spaces. Sections 7–9 are concerned with spectral properties of compact operators on Hilbert spaces and Fredholm's theorem of alternative. Finally a numerical example is presented in Section 11.

2 Preliminaries

Let us begin by recalling the notion of precompact and relatively compact sets.

Definition 2.1. (Relatively Compact). Let X be a metric space; $A \subset X$ is relatively compact in X, if A is compact in X

Definition 2.2. (Precompact). Let X be a metric space; $A \subset X$ is precompact (also called totally bounded) if for every $\epsilon > 0$, there exist finitely many points $x_1, \ldots, x_n$ in A such that $\bigcup_{i=1}^{n} B(x_i, \epsilon)$ covers A.

The following Theorem shows that when we are working in a complete metric space, precompactness and compactness are equivalent.

Theorem 1. *Let X be a metric space. If $A \subset X$ is relatively compact then it is precompact. Moreover, if X is complete then the converse holds also.*

Then, we define a compact operator as below

Finite-dimensional operators

Let $T: X \to Y$ be a continuous linear mapping between normed linear spaces. If the range space $Ran(T)$ is of finite dimension, $\dim(Ran(T)) < \infty$, we call T a *finite-dimensional* operator. It is straightforward to see that finite-dimensional operators are compact. This is seen by noting that for a bounded set $E \subseteq X$, $T(E)$ is closed and bounded in the finite-dimensional subspace $Ran(T) \subset Y$. Therefore, Heine-Borel Theorem applies, and $T(E)$ is compact in $Ran(T) \subset Y$.

Hilbert-Schmidt operators

Let $D \subseteq \mathbb{R}^n$ be a bounded domain. We call a function $k: D \times D \to \mathbb{R}$ a *Hilbert-Schmidt kernel* if

$$\int_D \int_D |k(x, y)|^2 \; d y \; d x < \infty$$

that is, $k \in L^2 (D \times D)$ (note that one special case is when k is a continuous function on $D \times D$). Define the integral operator K on $L^2(D)$, $K u$ for a $u \in L^2(D)$, by

$$[K u](x) = \int_D k(x, y) \, u(y) \; d y$$

Clearly, K is linear; moreover, it is simple to show that $K: L^2(D) \to L^2(D)$.

Let $u \in L^2(D)$, then

$$\int_D |[K u](x)|^2 \; d x = \int_D \left| \int_D k(x, y) \, u(y) \; d y \right|^2 \; d x \leq \|k\|_{L^2(D \times D)} \|u\|^2_{L^2(D)},$$

so that $K u \in L^2(D)$. The mapping K is what we call a *Hilbert-Schmidt* operator.

Lemma 3.1. Let D be a bounded domain in $\mathbb{R}^n$ and let K be a Hilbert-Schmidt kernel. Then, the integral operator $K: L^2(D) \to L^2(D)$ given by $[K u](x) = \int_D k(x, y) \, u(y) \; d y$ is a compact operator.

Remark 3.2. One can think of Hilbert-Schmidt operators as generalizations of the idea of matrices to infinite-dimensional spaces. Note that if A is an $n \times n$ matrix (a linear mapping on $\mathbb{R}^n$), then, the action $A x$ of A on a vector $x \in \mathbb{R}^n$ is given by

$$[A x]_i = \sum_{j=1}^{n} A_{i j} x_j$$

Now note that,

$$[K u](x) = \int_D k(x, y) \, u(y) \; d y$$

is an analog of (3.2) with the summation replaced with an integral.

Range of a compact operator

We saw in the previous section that a continuous linear map with a finite-dimensional range is compact. While the converse is not true in general, we can say something to the effect that the range of a compact operator is "almost finite-dimensional".

More precisely, the range of compact operators can be approximated to a finite-dimensional subspace within a prescribed ϵ-distance, as described in the following result. The proof given below follows that of [8] closely.

Theorem 4.1

Let $T: X \to Y$ be a compact linear transformation between Banach spaces X and Y. Then, given any $\epsilon > 0$, there exists a finite-dimensional subspace M in $Ran(T)$ such that, for any $x \in X$,

$$\inf_{m \in M} \|T(x) - m\| \leq \epsilon \|x\|$$

Proof

Let $\epsilon > 0$ be fixed but arbitrary. Let B_X denote the closed unit ball of X. Note that $T(B_X)$ is precompact, and thus can be covered with a finite cover, $\bigcup_{i=1}^{N} B(y_i, \epsilon)$, with $y_i \in T(B_X) \cap Ran(T)$. Now let M be the span of $y_1, \ldots, y_N$, and note that $M \subseteq Ran(T)$ and $\text{dist}(T(z), M) \leq \epsilon$ for any $z \in B_X$. Therefore, for any $x \in X$,

$$\inf_{m \in M} \|T(x / \|x\|) - m\| \leq \epsilon$$

And thus,

$$\inf_{m \in M} \|T(x) - m\| \leq \epsilon \|x\|, \quad m' = m / \|x\|, m \in M$$

5 Approximation by finite-dimensional operators

We have already noted that finite-dimensional operators on normed linear spaces are compact. Moreover, we know by Theorem 2.6 that the limit (in the operator norm) of a sequence of finite-dimensional operators is a compact operator. Moreover, we have seen that the range of compact operators can be approximated. A natural follow up is the following question: Let $T: X \to Y$ be a compact operator between normed linear spaces X and Y, is it then true that T is a limit (in operator norm) of a sequence of finite-dimensional operators? The answer to this question is negative for Banach spaces in general (see [3]). However, the result holds in the case Y is a Hilbert space, as given by the following known result:

Theorem 5.1

Let $T: X \to Y$ be a compact operator, where X is a Banach space, and Y is a Hilbert space. Then, T is the limit (in operator norm) of a sequence of finite-dimensional operators.

Proof

Let B_X denote the closed unit ball of X. Since T is compact, we know $T(B_X)$ is precompact; thus for any $n \geq 1$ there exists $y_1, \ldots, y_N$ in $T(B_X)$ such that $T(B_X) \subseteq \bigcup_{i=1}^{N} B(y_i, 1/n)$. Let $M_n = \text{span}\{y_1, \ldots, y_N\}$ and let Π_n be the orthogonal projection of Y onto M_n. First note that for any $y \in Y$, we have

$$\|\Pi_n(y) - y\| \leq \|y - y\|, \quad i = 1, \ldots, N. \quad (5.1)$$

Next define the mapping T_n by

$$T_n = \Pi_n \circ T$$

We know, by construction, for any $x \in B_X$, $\|T(x) - y\| \leq 1/n$ for some $i \in \{1, \ldots, N\}$. Moreover, by (5.1)

$$\|T_n x - y\| = \|\Pi_n(T(x)) - y\| \leq \|T(x) - y\| \leq 1/n$$

Therefore, for any $x \in X$, with $\|x\| \leq 1$

$$\|T - T_n\| \leq \frac{2}{n} \quad \text{as} \quad n \to \infty.$$

6 Spectrum of linear operators on a Banach space

Recall that for a linear operator A on a finite dimensional linear space, we define its spectrum $\sigma(A)$ as the set of its eigenvalues. On the other hand, for a linear operator T on an infinite dimensional (complex) normed linear space the spectrum $\sigma(T)$ of T is defined by,

$$\sigma(T) = \{\lambda \in \mathbb{C}: T - \lambda I \text{ is not invertible in } B[X]\}$$

and $\sigma(T)$ is the disjoint union of the point spectrum $\sigma_p(T)$ (set of eigenvalues), continuous spectrum $\sigma_c(T)$, and residual spectrum $\sigma_r(T)$. Let us recall that the continuous spectrum is given b

$$\sigma_c(T) = \{\lambda \in \mathbb{C}: \mathrm{Ker}(T - \lambda I) = \{0\}, \mathrm{Ran}(T - \lambda I) \neq X, \mathrm{Ran}(T - \lambda I) \text{ is dense in } X\}$$

and residual spectrum of T is given by

$$\sigma_r(T) = \{\lambda \in \mathbb{C}: \mathrm{Ker}(T - \lambda I) = \{0\}, \mathrm{Ran}(T - \lambda I) \neq X\}$$

7 Some spectral properties of compact operators on Hilbert spaces

As we saw in Remark 2.8, a compact operator T on an infinite dimensional normed linear space X cannot be invertible in $B[X]$. Therefore, we always have $0 \in \sigma(T)$. However, in general, not much can be said on whether $\lambda = 0$ is in point spectrum (i.e. an eigenvalue) or the other parts of the spectrum. However, we mention the following special case:

Lemma 7.1

Let H be a complex Hilbert space, and let $T \in B[H]$ be a one-to-one compact self-adjoint operator. Then, $0 \in \sigma_c(T)$.

Proof

Since zero is not in the point spectrum, it must be in $\sigma_c(T)$ or in $\sigma_r(T)$. We show $0 \in \sigma_c(T)$ by showing that the range of T is dense in H. We first claim that $\mathrm{Ran}(T)^{\perp} = \{0\}$. To show this we proceed as follows. Let $z \in \mathrm{Ran}(T)^{\perp}$, and note that for every $y \in H$

$$0 = \langle T z, y \rangle = \langle z, T^* y \rangle$$

Hence, $Tz = 0$ which, since T is one-to-one, implies that $z = 0$. This shows that $\mathrm{Ran}(T)^{\perp} = \{0\}$. Thus, we have,

$$\mathrm{Ran}(T) = (\mathrm{Ran}(T)^{\perp})^{\perp} = \{0\}^{\perp} = H.$$

The next result sheds further light on the spectrum of a compact operator on a complex Hilbert space.

Lemma 7.2

Let T be a compact operator on a complex Hilbert space H. Suppose λ is a non-zero complex number. Then,

1. $\mathrm{Ker}(T - \lambda I)$ is finite dimensional
2. $\mathrm{Ran}(T - \lambda I)$ is closed
3. $T - \lambda I$ is invertible if and only if $\mathrm{Ran}(T - \lambda I) = H$

8 Fredholm's Theorem of Alternative

The following result, known as Fredholm's theorem of alternative, has a straightforward interpretation for compact operators on a Hilbert space H. Assume $T \in B[H]$ and $\lambda \neq 0$.

Theorem 2. *The generalized inverse of $T - \lambda I$, denoted S, exists if and only if $Ker(T - \lambda I) = \{0\}$ and $Ran(T - \lambda I) = H$. That is, $T - \lambda I$ is invertible if and only if the equation $(T - \lambda I) x = y$ has a solution for every $y \in H$ and the solution is unique. Moreover, if these conditions are met, $S = (T - \lambda I)^{-1}$ and we have the following decomposition:*

$$H = Ker(T - \lambda I) \oplus Ran(T - \lambda I)$$

This result follows from the spectral properties of T where $\sigma(T) \neq \{0\}$ if T is a compact operator. It implies that the only possible accumulation point of the spectrum of a compact operator is zero, and thus, any non-zero λ must satisfy the conditions of the theorem or $T - \lambda I$ is not invertible.

9 Spectral Theorem for Compact Self-Adjoint Operators

It is a notable result that for compact self-adjoint operators on a complex Hilbert space H, the spectrum consists almost entirely of eigenvalues with 0 as a possible accumulation point. The spectral theorem for compact self-adjoint operators provides a deeper understanding of the structure of these operators:

Theorem 3. *Let T be a compact self-adjoint operator on a Hilbert space H. Then the spectrum $\sigma(T)$ consists entirely of eigenvalues, except possibly for $\lambda = 0$. Moreover, each eigenvalue $\lambda \neq 0$ has finite multiplicity and the corresponding eigenvectors form an orthonormal basis for H.*

Proof

For any compact self-adjoint operator T, we can diagonalize T by an orthonormal basis consisting of eigenvectors corresponding to the non-zero eigenvalues. Each of these eigenvalues has a finite multiplicity, which means there are only finitely many linearly independent eigenvectors corresponding to each eigenvalue.

If $\lambda \neq 0$ is an eigenvalue of T, then there exists an eigenvector $v \neq 0$ such that $Tv = \lambda v$. Since T is self-adjoint, all eigenvalues are real, and the eigenvectors corresponding to distinct eigenvalues are orthogonal. By the property of compact operators, any limit point of the set of eigenvalues can only be 0, and thus, the only possible accumulation point of $\sigma(T)$ is 0. This completes the diagonalization of T with respect to these eigenvalues and corresponding eigenvectors, providing the spectral decomposition of T into its eigencomponents.

Proof of Theorem

The proof of this result is standard. See e.g., [8] for a proof. Here we just give the proof for the first statement of the theorem. Let $M = \mathrm{Ker}(T - \lambda I)$. Note that since T is continuous M is closed. Also, note that $T|_M = \lambda I$. We show M is finite-dimensional by showing that any bounded sequence in M has a convergent subsequence. Take a bounded sequence $\{x_n\}$ in M. Then, there exists a subsequence $\{T_{n_k}\}$ that converges. However, $T_{n_k} = \lambda x_{n_k}$, thus (also using $\lambda \neq 0$), it follows that $\{x_{n_k}\}$ has a convergent subsequence.

Note that by the above theorem it follows immediately that a compact operator on a complex Hilbert space has empty non-zero continuous spectrum. In the next section, we will refine this result further by showing that in fact the same holds for the residual spectrum of $\sigma_r(T)$ then it must be an eigenvalue of T. That is, if a non-zero $\lambda \in \sigma(T)$ then operator $T - \lambda I$ cannot have a dense range.

10 Singular value decomposition and some basic results

Here we briefly recall the notion of the singular value decomposition (SVD) of compact linear transformations between Hilbert spaces.

First we recall the following definition [6]:

Definition 10.1. Let H and K be Hilbert spaces, and let $A: H \to K$ be a compact linear transformation, and $A^*: K \to H$ be its adjoint. The nonnegative square roots of the eigenvalues of the (positive self-adjoint) compact operator $A^* A: H \to H$ are called the singular values of A.

The following result describes the singular value decomposition of compact linear transformations [6]: **Theorem 10.2.** Let H and K be real Hilbert spaces, and let $A: H \to K$ be a (nonzero) compact operator. Let $\{\sigma_n\}$ be the sequence of nonzero singular values of A repeated according to their multiplicity and in descending order. Then, there exist orthonormal sequences $\{u_n\}$ and $\{v_n\}$ in H and K, respectively, such that

$$A u_n = \sigma_n v_n, \quad A^* v_n = \sigma_n u_n \forall n \geq 1$$

For each $x \in H$, we have

$$x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n + Q x$$

where Q is the orthogonal projection operator onto the kernel of A. Moreover,

$$A x = \sum_{n=1}^{\infty} \sigma_n \langle x, u_n \rangle u_n, \quad x \in H$$

We can use the SVD of a compact operator $A: H \to K$ in a similar way as we do with matrices. Let us for example note, the following result regarding the operator norm of A. Recall that for $A \in B[H, K]$, the operator norm is given by

$$\|A\| = \sup_{\|x\|=1} \|A x\|_K$$

Theorem 10.3. Let $A: H \to K$ be compact, and let $\{\sigma_n\}$ be the sequence of its nonzero singular values ordered according to

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots$$

Then, $\|A\| = \sigma_1$.

Proof. Let $x \in H$ be arbitrary and consider its decomposition (10.1). We note that

$$\|x\|^2 = \sum_{n=1}^{\infty} \langle x, u_n \rangle^2 + \|Q x\|^2$$

Moreover, we have

$$A x = \sum_{n=1}^{\infty} \sigma_n \langle x, u_n \rangle u_n$$

and thus

$$\|A x\|^2 = \sum_{n=1}^{\infty} \sigma_n^2 \langle x, u_n \rangle^2 \leq \sigma_1^2 \sum_{n=1}^{\infty} \langle x, u_n \rangle^2 \leq \sigma_1^2 \|x\|^2$$

Thus, we have that $\|A\,x\| \leq \sigma_1\|x\|$, for every $x \in H$. Moreover, $\|A\| = \sigma_1$ follows since $\|A\| = \sup_{\|x\|=1} \|A\,x\|$.

The following result is an analogue of best rank-k approximation result for matrices. The proof given below follows in similar lines as that given in [11].

Theorem 10.4. Let $A\colon H \to K$ be compact. Consider SVD of A:

$$A\,x = \sum_{n=1}^{\infty} \sigma_n \langle x, u_n\rangle\, u_n \forall x \in H$$

as in Theorem 10.2. We have,

$$\sigma_k = \min_{\|A-F\|:\mathrm{rank}(F)\leq k-1} \forall k = 1, 2, \ldots$$

Proof. The result is immediate for $k=1$. Let $k>1$, and let F in $B[H,K]$ have rank at most $k-1$, and consider $\{v_1, \ldots, v_{k-1}\}$. Since rank(F) $\leq k-1$, the set $\{F v_1, \ldots, F v_k\}$ must be linearly dependent. That is there exists, $a_1, \ldots, a_k$, not all zeros, with $0 = \sum_{n=1}^{k} a_n\, F v_n$. Let $z = \sum_{n=1}^{k} a_n\, v_n$, and let $x = z/\|z\|$ so that $\|x\| = 1$ and $F x = 0$. Now,

$$\|A-F\|^2 \geq \|(A-F)\,x\|^2 = \|A\,x\|^2 = \sum_{n=k}^{\infty} \sigma_n^2 \langle x, u_n\rangle^2 \geq \sigma_k^2 \|x\|^2 = \sigma_k^2$$

Therefore, $\|A-F\| \geq \sigma_k$. Next, define

$$F = \sum_{n=1}^{k-1} \sigma_n \langle \cdot, u_n\rangle\, u_n$$

Clearly, F has rank $k-1$, and we have $A - F = \sum_{n=k}^{\infty} \sigma_n \langle \cdot, u_n\rangle\, u_n$, from which we have $\|A-F\| = \sigma_k$.

The following singular value inequality is useful in applications, and is analogue of results for matrices. The proof here is adapted from that in [1, page 188].

Theorem 10.5. Let $A\colon H \to K$ be compact and assume $B \in B[H]$ Then,

$$\sigma_k\,(A\,B) \leq \sigma_k(A)\|B\|, \quad k \geq 1$$

Proof. First note that by the assumptions of the theorem $A\,B$ is compact in $B[H,K]$. Consider,

$$A = \sum_{n=1}^{\infty} \sigma_n \langle \cdot, u_n\rangle\, u_n$$

$$F = \sum_{n=1}^{k-1} \sigma_n \langle \cdot, u_n\rangle\, u_n$$

Note that rank$(F B) \leq k-1$, therefore

$$\sigma_k\,(A\,B) \leq \|A\,B - F B\| \leq \|B\|\, \|A-F\| = \sigma_k(A)\|B\|$$

11 A numerical illustration

Here we study the convolution operator, $F: L^2([0,1]) \to L^2([0,1])$ given by

$$(F u)(x) = \int_0^1 k\,(x-y)\,u(y)\;d\,y$$

$$k(x) = \frac{e^{-\frac{x^2}{2\gamma^2}}}{\sqrt{2\,\pi\,\gamma^2}}$$

$$\gamma = 0.03$$

Notice that this is a compact operator (cf. e.g., Lemma 3.1). To study the spectral properties of F, we compute the eigenvalues and eigenvectors of the operator numerically by first discretizing the integral operator via quadrature and then computing the spectrum of the discretized operator. Using an n-point composite mid-point rule we obtain the discretized operator K:

$$K_{ij} = h\,C\,e^{-\frac{((i-j)h)^2}{2\gamma^2}}$$

where $h = \frac{1}{n}$ with $i, j \in \{1, \ldots, n\}$.

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