Stone's Theorem, The Shift Group, and The Fourier Transform

BY STEPHEN CROWLEY

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Abstract

This paper establishes fundamental connections between delta functions, Heaviside step functions, and level crossing counts for differentiable Gaussian processes. The authors develop the distributional framework using Schwartz test functions and tempered distributions, then derive key identities including the distributional derivative relationship between the Heaviside function and the Dirac delta. A change of variables theorem for delta functions of smooth functions is proven, providing the foundation for the main results. The work introduces a level crossing counting function that enumerates crossings of fixed levels by Gaussian process sample paths and demonstrates two equivalent representations: one as an integral involving the absolute derivative and delta function, and another as a sum of Heaviside step functions. These representations provide complementary perspectives on level crossing phenomena and establish the theoretical groundwork for analyzing crossing statistics of differentiable Gaussian processes. The results connect classical distribution theory with stochastic level crossing theory and provide tools for applications in stochastic process analysis.

Table of contents

1 Foundations of Distributions on Real Line 1

2 Basic Identities 2

3 Delta of a Smooth Function 3

4 Counting Function for Level Crossings 4

1 Foundations of Distributions on Real Line

Definition 1. (Schwartz Test Function Space) *The Schwartz space $\mathcal{S}(\mathbb{R})$ is the space of all infinitely differentiable functions $\phi: \mathbb{R} \to \mathbb{R}$ such that for every pair of nonnegative integers m, n,*

$$\sup_{x \in \mathbb{R}} |x^m \phi^{(n)}(x)| < \infty \tag{1}$$

Functions in $\mathcal{S}(\mathbb{R})$ are called rapidly decreasing smooth test functions.

Definition 2. (Tempered Distribution) *A tempered distribution is a continuous linear functional*

$$T\colon \mathcal{S}(\mathbb{R}) \to \mathbb{R} \tag{2}$$

Definition 3. (Dirac Delta Distribution) *The Dirac delta distribution $\delta_a \in \mathcal{S}'(\mathbb{R})$ centered at $a \in \mathbb{R}$ is defined by*

$$\langle \delta_a, \phi \rangle = \phi(a) \tag{3}$$

for all $\phi \in \mathcal{S}(\mathbb{R})$.

Definition 4. (Heaviside Step Function) *The Heaviside step function $H\colon \mathbb{R} \to \{0, 1\}$ is defined by*

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{4}$$

Definition 5. (Distributional Derivative) *For a tempered distribution $T \in \mathcal{S}'(\mathbb{R})$, its distributional derivative $T' \in \mathcal{S}'(\mathbb{R})$ is defined by*

$$\langle T', \phi \rangle = -\langle T, \phi' \rangle$$

for all $\phi \in \mathcal{S}(\mathbb{R})$.

2 Basic Identities

Theorem 6. (Heaviside Derivative) *The Heaviside step function H satisfies*

$$H' = \delta \tag{5}$$

as distributions on $\mathcal{S}'(\mathbb{R})$.

Proof. For all $\phi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}
\langle H', \phi \rangle &= -\langle H, \phi' \rangle && (6)\\
&= -\int_{-\infty}^{\infty} H(x)\, \phi'(x)\, dx && (7)\\
&= -\int_{0}^{\infty} \phi'(x)\, dx && (8)\\
&= -[\phi(x)]_0^{\infty} && (9)\\
&= -(\lim_{x \to \infty} \phi(x) - \phi(0)) && (10)\\
&= \phi(0) && (11)
\end{aligned}$$

where the limit vanishes since $\phi \in \mathcal{S}(\mathbb{R})$ decays rapidly at infinity. Thus

$$\langle H', \phi \rangle = \phi(0) = \langle \delta, \phi \rangle \tag{12}$$ □

Theorem 7. (Integral of Delta) *For any $a \in \mathbb{R}$ and $T \in \mathbb{R}$,*

$$\int_{-\infty}^{T} \delta(t-a)\, dt = H(T-a) \tag{13}$$

Proof. Define

$$F(T) = \int_{-\infty}^{T} \delta(t-a)\, dt \tag{14}$$

Taking the distributional derivative with respect to T:

$$F'(T) = \frac{d}{dT}\int_{-\infty}^{T} \delta(t-a)\, dt = \delta(T-a) \tag{15}$$

Since $F(-\infty) = 0$ and

$$F'(T) = \delta(T-a) = H'(T-a) \tag{16}$$

from the previous theorem, one has

$$F(T) = H(T-a) + C \tag{17}$$

for some constant C. The boundary condition

$$F(-\infty) = 0 = H(-\infty) + C \tag{18}$$

implies $C = 0$, thus

$$F(T) = H(T-a) \tag{19}$$

□

3 Delta of a Smooth Function

Theorem 8. (Delta under Change of Variables) *Let $g: \mathbb{R} \to \mathbb{R}$ be continuously differentiable with isolated, simple zeros $\{x_i\}_{i=1}^{\infty}$ such that $g(x_i) = 0$ and $g'(x_i) \neq 0$ for all $i \geq 1$. Then the identity*

$$\delta(g(x)) = \sum_{i=1}^{\infty} \frac{\delta(x-x_i)}{|g'(x_i)|} \tag{20}$$

holds in $\mathcal{S}'(\mathbb{R})$.

Proof. For $\phi \in \mathcal{S}(\mathbb{R})$,

$$\langle \delta(g(x)), \phi \rangle = \int_{-\infty}^{\infty} \phi(x)\, \delta(g(x))\, dx \tag{21}$$

Let $\{I_i\}_{i=1}^{\infty}$ be pairwise disjoint intervals, each I_i containing exactly one zero x_i of g, and such that g is strictly monotone on I_i. Near each zero x_i, where g is locally monotone by the implicit function theorem, the change of variables $u = g(x)$ gives

$$\begin{aligned} \int_{I_i} \phi(x)\, \delta(g(x))\, dx &= \int_{g(I_i)} \frac{\phi(g^{-1}(u))}{|g'(g^{-1}(u))|}\, \delta(u)\, du \\ &= \frac{\phi(x_i)}{|g'(x_i)|} \end{aligned} \tag{22}$$

by the sifting property of δ. Summing over all zeros yields

$$\langle \delta(g(x)), \phi \rangle = \sum_{i=1}^{\infty} \frac{\phi(x_i)}{|g'(x_i)|} = \left\langle \sum_{i=1}^{\infty} \frac{\delta(x - x_i)}{|g'(x_i)|}, \phi \right\rangle \tag{23}$$

Since this holds for all $\phi \in \mathcal{S}(\mathbb{R})$, the distributional equality follows. $\square$

4 Counting Function for Level Crossings

Let $x: \mathbb{R} \to \mathbb{R}$ be continuously differentiable, and fix $u \in \mathbb{R}$. Assume the zeros of $g(t) := x(t) - u$ are isolated and simple; that is, for every zero t_i (with $i \geq 1$),

$$g'(t_i) = x'(t_i) \neq 0, \quad \text{for all } i \geq 1. \tag{24}$$

Definition 9. *[Level Crossing Counting Function] Define the counting function*

$$N_u(T) := \textit{the number of zeros } t_i \textit{ of } x(t) - u \textit{ with } t_i \leq T. \tag{25}$$

Equivalently, if $(t_i)_{i=1}^{\infty}$ *lists all zeros of* $x(t) - u$*,*

$$N_u(T) = \sum_{i=1}^{\infty} 1_{\{t_i \leq T\}} \tag{26}$$

where $1_{\{t_i \leq T\}}$ *is the indicator of the event* $t_i \leq T$*.*

Theorem 10. (Counting Function as Integral Over Delta) *For every* $T \in \mathbb{R}$*,*

$$N_u(T) = \int_{-\infty}^{T} |x'(t)| \, \delta(x(t) - u) \, dt \tag{27}$$

Proof. Using the delta change of variables theorem with

$$g(t) = x(t) - u \tag{28}$$

the corresponding zeros $(t_i)_{i=1}^{\infty}$ of g satisfy $g(t_i) = 0$ and $g'(t_i) = x'(t_i) \neq 0$. From the theorem,

$$\delta(x(t) - u) = \sum_{i=1}^{\infty} \frac{\delta(t - t_i)}{|x'(t_i)|} \tag{29}$$

Therefore,

$$|x'(t)| \, \delta(x(t) - u) = |x'(t)| \sum_{i=1}^{\infty} \frac{\delta(t - t_i)}{|x'(t_i)|} \tag{30}$$

$$= \sum_{i=1}^{\infty} |x'(t)| \frac{\delta(t - t_i)}{|x'(t_i)|} \tag{31}$$

Since $x'(t_i) \neq 0$ by assumption, and $\delta(t - t_i)$ picks out the value at $t = t_i$,

$$\begin{aligned} |x'(t)|\,\delta(x(t) - u) &= \sum_{i=1}^{\infty} \frac{|x'(t_i)|}{|x'(t_i)|}\,\delta(t - t_i) \\ &= \sum_{i=1}^{\infty} \delta(t - t_i). \end{aligned} \tag{32}$$

Therefore,

$$\begin{aligned} \int_{-\infty}^{T} |x'(t)|\,\delta(x(t) - u)\,dt &= \int_{-\infty}^{T} \left(\sum_{i=1}^{\infty} \delta(t - t_i) \right) dt \\ &= \sum_{i=1}^{\infty} \int_{-\infty}^{T} \delta(t - t_i)\,dt \\ &= \sum_{i=1}^{\infty} H(T - t_i) \\ &= \sum_{i=1}^{\infty} 1_{\{t_i \leq T\}} \\ &= N_u(T) \end{aligned} \tag{33}$$

□

Theorem 11. (Counting Function as Sum of Heaviside Steps) *The counting function (9) is given by*

$$N_u(T) = \sum_{i=1}^{\infty} H(T - t_i) \quad \forall T \in \mathbb{R}, \tag{34}$$

where the sum runs over all zero crossing times t_i, $i \geq 1$.

Proof. By definition of the Heaviside function, for each $i \geq 1$,

$$H(T - t_i) = 1 \tag{35}$$

if and only if $T \geq t_i$, and

$$H(T - t_i) = 0 \tag{36}$$

otherwise. Therefore,

$$\begin{aligned} \sum_{i=1}^{\infty} H(T - t_i) &= \sum_{i=1}^{\infty} 1_{\{t_i \leq T\}} \\ &= N_u(T). \end{aligned} \tag{37}$$

□

Theorem 12. (Equivalence of Representations) *The delta integral representation and the Heaviside step sum representation are equivalent:*

$$\int_{-\infty}^{T} |x'(t)|\,\delta(x(t) - u)\,dt = \sum_{i=1}^{\infty} H(T - t_i) \tag{38}$$

Proof. From the previous theorems,

$$N_u(T) = \int_{-\infty}^{T} |x'(t)|\, \delta\,(x(t) - u)\ d\,t \tag{39}$$

and

$$N_u(T) = \sum_{i=1}^{\infty} H\,(T - t_i) \tag{40}$$

for all $T \in \mathbb{R}$. Therefore,

$$\int_{-\infty}^{T} |x'(t)|\, \delta\,(x(t) - u)\ d\,t = \sum_{i=1}^{\infty} H\,(T - t_i) \tag{41}$$ □