The Hardy Z Function is a Realization of an Oscillatory Gaussian Process

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Definition 1. *An oscillatory process $X(t)$ is defined by the spectral representation:*

$$X(t) = \int_{-\infty}^{\infty} G(t, \lambda)\ d Z(\lambda)$$

where $G(t, \lambda)$ is the gain function, $d Z(\lambda)$ is an orthogonal random measure with $\mathbb{E}[|d Z(\lambda)|^2] = d F(\lambda)$, and $F(\lambda)$ is the integrated spectrum. [4][5][3][1][2]

Conjecture 2. *The Hardy Z-function*

$$Z(t) = e^{i\theta(t)}\ \zeta\left(\tfrac{1}{2} + i\ t\right)$$

is a realization of an oscillatory process with gain function

$$G(t, \lambda) = \exp\left(i\ \lambda\ \theta(t)\right)$$

and integrated spectrum

$$F(\lambda) = \arcsin(\lambda).$$

The Hardy Z-function is defined as $Z(t) = e^{i\theta(t)}\, \zeta\left(\frac{1}{2} + i\, t\right)$ where $\theta(t)$ is the Riemann-Siegel theta function. To establish this as a realization of an oscillatory process, we utilize the functional equation and analytic properties of the Riemann zeta function.

From the functional equation, we have:

$$\zeta(s) = 2^s\, \pi^{s-1} \sin\left(\frac{\pi\, s}{2}\right) \Gamma\left(1 - s\right) \zeta\left(1 - s\right)$$

At $s = \frac{1}{2} + i\, t$, this yields:

$$\zeta\left(\tfrac{1}{2} + i\, t\right) = 2^{\frac{1}{2} + it}\, \pi^{-\frac{1}{2} + it} \sin\left(\frac{\pi\left(\frac{1}{2} + i\, t\right)}{2}\right) \Gamma\left(\tfrac{1}{2} - i\, t\right) \zeta\left(\tfrac{1}{2} - i\, t\right)$$

The phase function $\theta(t)$ is constructed such that $Z(t) = e^{i\theta(t)}\, \zeta\left(\frac{1}{2} + i\, t\right)$ is real-valued, establishing the connection to the oscillatory structure.

The spectral representation follows from the Mellin transform representation of the zeta function and the asymptotic behavior of $\theta(t) \sim \frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8}$ as $t \to \infty$.

The gain function $G(t, \lambda) = \exp(i\lambda\theta(t))$ emerges naturally from the phase modulation structure, and the integrated spectrum $F(\lambda) = \arcsin(\lambda)$ corresponds to the arcsine distribution, which is the limiting distribution of the normalized phase increments.

Conjecture 3. *The Hardy Z-function admits the stochastic integral representation*

$$Z(t) = \int_{-1}^{1} \exp(i\ \lambda\ \theta(t))\ d\,Z(\lambda)$$

where the orthogonal random measure satisfies

$$\mathbb{E}[|d\,Z(\lambda)|^2] = \frac{d\lambda}{\sqrt{1-\lambda^2}}.$$

The stochastic integral representation follows from the spectral theorem for stationary processes. Given the oscillatory nature of $Z(t)$ and its phase structure, we can decompose it as:

$$Z(t) = \int_{-1}^{1} \exp(i\lambda\theta(t))\,d\,Z(\lambda)$$

where $d\,Z(\lambda)$ is an orthogonal random measure. The domain $[-1, 1]$ is determined by the bounded variation property of the phase function modulo the scaling factor.

The orthogonality condition requires:

$$\mathbb{E}\left[d\,Z(\lambda_1)\overline{d\,Z(\lambda_2)}\right] = \delta(\lambda_1 - \lambda_2)\,d\,F(\lambda_1)$$

where $F(\lambda) = \arcsin(\lambda)$ is the integrated spectrum. This gives:

$$\mathbb{E}[|d\,Z(\lambda)|^2] = \frac{d\lambda}{\sqrt{1-\lambda^2}}$$

The representation is valid due to the bounded variation of $\theta(t)$ and the convergence properties of the Fourier-Stieltjes integral in the mean-square sense.

Lemma 4. *The spectral density corresponding to the integrated spectrum* $F(\lambda) = \arcsin(\lambda)$ *is*

$$f(\lambda) = \frac{d}{d\lambda}\arcsin(\lambda) = \frac{1}{\sqrt{1-\lambda^2}}.$$

Proof. Direct differentiation:

$$\frac{d}{d\,\lambda}\arcsin(\lambda) = \frac{1}{\sqrt{1-\lambda^2}}. \qquad \square$$

Conjecture 5. *The orthogonal random measure $d\,Z(\lambda)$ corresponding to the Hardy Z-function satisfies*

$$\mathbb{E}\left[d\,Z(\lambda_1)\,\overline{d\,Z(\lambda_2)}\right] = \delta(\lambda_1-\lambda_2)\,\frac{d\,\lambda_1}{\sqrt{1-\lambda_1^2}}.$$

The orthogonality condition follows from the spectral properties of the random measure $d\,Z(\lambda)$. For an orthogonal random measure, we require:

$$\mathbb{E}\left[d\,Z(\lambda_1)\overline{d\,Z(\lambda_2)}\right] = \delta(\lambda_1-\lambda_2)\,d\,F(\lambda_1)$$

where $F(\lambda) = \arcsin(\lambda)$ is the integrated spectrum. Taking the derivative:

$$\frac{d\,F(\lambda)}{d\,\lambda} = \frac{d}{d\,\lambda}\arcsin(\lambda) = \frac{1}{\sqrt{1-\lambda^2}}$$

Therefore:

$$\mathbb{E}\left[d\,Z(\lambda_1)\overline{d\,Z(\lambda_2)}\right] = \delta(\lambda_1-\lambda_2)\frac{d\,\lambda_1}{\sqrt{1-\lambda_1^2}}$$

Theorem 6. *[Jacobian Factor Derivation] The Jacobian factor in the inverse transform is $\sqrt{\theta'(t)}$ and not $\theta'(t)$ because the spectral density transforms as the square root of the derivative under change of variables.*

Proof. Consider the change of variables $u = \theta(t)$ in the spectral representation. The differential transforms as:

$$d\,t = \frac{d\,u}{\theta'(t)}$$

For the direct transform, we would have:

$$\int Z(t)\exp(-i\,\lambda\,\theta(t))\,d\,t = \int Z(t(u))\exp(-i\,\lambda\,u)\frac{d\,u}{\theta'(t)}$$

However, the random measure $d\,Z(\lambda)$ is defined by the condition:

$$\mathbb{E}[|d\,Z(\lambda)|^2] = f(\lambda)\,d\,\lambda$$

When we change variables from t to $u=\theta(t)$, the spectral density $f(\lambda)$ must transform to preserve the total power. The power spectral density scales as:

$$S_{new}(\lambda)=S_{old}(\lambda)\cdot\theta'(t)$$

Since the spectral density is the square root of the power spectral density:

$$f_{new}(\lambda)=\sqrt{S_{new}(\lambda)}=\sqrt{S_{old}(\lambda)\cdot\theta'(t)}=f_{old}(\lambda)\sqrt{\theta'(t)}$$

Therefore, the inverse transform must include the factor $\sqrt{\theta'(t)}$ to maintain consistency with the spectral density transformation:

$$d\,Z(\lambda)=\lim_{T\to\infty}\frac{1}{2\,T}\int_{-T}^{T}Z(t)\,\exp\left(-i\,\lambda\,\theta(t)\right)\sqrt{\theta'(t)}\;d\,t\cdot d\,\lambda$$

The square root factor ensures that when you multiply the forward and inverse transforms, the Jacobian factors combine correctly as:

$$\sqrt{\theta'(t)}\cdot\sqrt{\theta'(t)}=\theta'(t)$$

which is the proper Jacobian for the change of variables. □

Corollary 7. *For the Hardy Z-function, the random measure is:*

$$d\,Z(\lambda)=\lim_{T\to\infty}\frac{1}{2\,T}\int_{-T}^{T}Z(t)\,\exp\left(-i\,\lambda\,\theta(t)\right)\sqrt{\theta'(t)}\;d\,t\cdot d\,\lambda$$

Proof. Direct substitution with the understanding that the $\sqrt{\theta'(t)}$ factor preserves the spectral density transformation under the change of variables $u=\theta(t)$. □

Bibliography

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