Defect Indices of Time-Changed Covariance Operators

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1 Definitions

Definition 1. *[Bessel Kernel] Let J_0 be the Bessel function of the first kind of order zero. The standard Bessel kernel is defined as $B(s,t) = J_0(2\pi|s-t|)$ for $s,t \in \mathbb{R}$.*

Definition 2. *[Transformed Bessel Kernel] Given a function $\theta: \mathbb{R} \to \mathbb{R}$, the transformed Bessel kernel is defined as $K_\theta(s,t) = J_0(2\pi|\theta(s) - \theta(t)|)$ for $s,t \in \mathbb{R}$.*

Definition 3. *[Covariance Operator] The integral operator T_θ associated with kernel K_θ acts on functions $f \in L^2(\mathbb{R})$ as:*

$$(T_\theta f)(s) = \int_{\mathbb{R}} J_0(2\pi|\theta(s) - \theta(t)|) f(t)\, dt \tag{1}$$

Definition 4. *[Defect Indices] For a densely defined symmetric operator T on a Hilbert space $\mathcal{H}$ with adjoint T^*, the defect indices (n_+, n_-) are:*

$$n_+ = \dim\ker(T^* - i \cdot I), \quad n_- = \dim\ker(T^* + i \cdot I) \tag{2}$$

where I denotes the identity operator.

Definition 5. *[Self-Adjoint Operator] A symmetric operator T is self-adjoint if and only if $T = T^*$, which is equivalent to having defect indices $n_+ = n_- = 0$.*

2 Main Results

Theorem 6. *The covariance operator T_θ with kernel $K_\theta(s,t) = J_0(2\pi\,|\theta(s) - \theta(t)|)$ has zero defect indices $(n_+ = n_- = 0)$ if and only if θ is strictly monotonic.*

To prove this theorem, several preliminary results are needed.

Lemma 7. *The Bessel kernel $B(s,t) = J_0(2\pi\,|s-t|)$ defines a positive definite operator.*

Proof. By Bochner's theorem, a continuous function $\phi(s-t)$ is positive definite if and only if it is the Fourier transform of a non-negative measure. The Fourier transform of $J_0(2\pi|x|)$ is:

$$\mathcal{F}[J_0(2\pi|x|)](\omega) = \frac{1}{2\pi\sqrt{1-\omega^2/(4\pi^2)}}\,1_{[-2\pi,2\pi]}(\omega) \tag{3}$$

where $1_{[-2\pi,2\pi]}$ is the indicator function of the interval $[-2\pi, 2\pi]$.

Since this is a non-negative function, $J_0(2\pi|x|)$ is positive definite, and hence $B(s,t)$ defines a positive definite operator. ☐

Lemma 8. *The operator S associated with the standard Bessel kernel $B(s,t) = J_0(2\pi\,|s-t|)$ is self-adjoint.*

Proof. The operator S with kernel $B(s,t)$ is unitarily equivalent to multiplication by the function $\frac{1}{2\pi\sqrt{1-\omega^2/(4\pi^2)}}\,1_{[-2\pi,2\pi]}(\omega)$ in the Fourier domain. Since this is a bounded, real-valued multiplication operator, it is self-adjoint, and thus S has defect indices $(0,0)$. ☐

Proposition 9. *If $\theta: \mathbb{R} \to \mathbb{R}$ is strictly monotonic, then the covariance operator T_θ is self-adjoint.*

Proof. When θ is strictly monotonic, it is invertible. Consider the change of variables:

$$u = \theta(s), \quad v = \theta(t) \tag{4}$$

Define the unitary transformation $U: L^2(\mathbb{R}, ds) \to L^2(\mathbb{R}, du)$ by:

$$(Uf)(u) = f(\theta^{-1}(u))\sqrt{\left|\frac{d\theta^{-1}}{du}(u)\right|} \tag{5}$$

Under this transformation, the operator T_θ becomes:

$$(U T_\theta U^{-1} g)(u) = \int_{\mathbb{R}} J_0 (2\pi |u - v|)\, g(v)\, d v \tag{6}$$

which is precisely the operator S with the standard Bessel kernel.

Since S is self-adjoint by Lemma 8, and unitary equivalence preserves self-adjointness, $T_\theta = U^{-1} S U$ is also self-adjoint. Thus, its defect indices are $(0,0)$. □

Proposition 10. *If θ is not strictly monotonic, then T_θ has non-zero defect indices.*

Proof. If θ is not strictly monotonic, there exist points $s_1 \neq s_2$ such that $\theta(s_1) = \theta(s_2)$.

Let $\mathcal{E} = \{(s_1, s_2) \in \mathbb{R}^2 \colon s_1 \neq s_2, \theta(s_1) = \theta(s_2)\}$. This set is non-empty by assumption.

For any pair $(s_1, s_2) \in \mathcal{E}$, the kernel satisfies:

$$K_\theta(s_1, t) = J_0 (2\pi |\theta(s_1) - \theta(t)|) = J_0 (2\pi |\theta(s_2) - \theta(t)|) = K_\theta(s_2, t) \tag{7}$$

This introduces a linear dependence in the kernel, violating the strict positive definiteness needed for self-adjointness.

To formalize this, consider the distribution:

$$f_{s_1, s_2}(t) = \delta (t - s_1) - \delta (t - s_2) \tag{8}$$

While f_{s_1,s_2} itself is not in $L^2(\mathbb{R})$, it can be approximated by L^2 functions. Using the symmetry property $K_\theta(s_1, t) = K_\theta(s_2, t)$:

$$(T_\theta f_{s_1, s_2})(s) = \int_{\mathbb{R}} K_\theta(s, t)\, f_{s_1, s_2}(t)\, d t = K_\theta(s, s_1) - K_\theta(s, s_2) = 0 \tag{9}$$

This implies that T_θ has a non-trivial null space, and consequently, there exist non-zero solutions to the equations $(T_\theta^* \pm i \cdot I)\, g = 0$. Therefore, both defect indices n_+ and n_- are at least 1. □

Lemma 11. *If θ is not strictly monotonic, then the kernel $K_\theta(s, t) = J_0 (2\pi |\theta(s) - \theta(t)|)$ is not positive definite.*

Proof. Let $s_1 \neq s_2$ with $\theta(s_1) = \theta(s_2)$. Consider the matrix:

$$M = \begin{pmatrix} K_\theta(s_1, s_1) & K_\theta(s_1, s_2) \\ K_\theta(s_2, s_1) & K_\theta(s_2, s_2) \end{pmatrix} \tag{10}$$

Since $\theta(s_1) = \theta(s_2)$, we have:

$$K_\theta(s_1, s_1) = K_\theta(s_2, s_2) = J_0(0) = 1 \tag{11}$$

$$K_\theta(s_1, s_2) = K_\theta(s_2, s_1) = J_0\left(2\pi\,|\theta(s_1) - \theta(s_2)|\right) = J_0(0) = 1 \tag{12}$$

Thus, $M = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, which has eigenvalues 2 and 0. The presence of the zero eigenvalue means M is not strictly positive definite. Therefore, K_θ is not a positive definite kernel. $\square$

Combining Proposition 9 and Proposition 10, the covariance operator T_θ has defect indices $(0, 0)$ if and only if θ is strictly monotonic.

Corollary 12. *The Gaussian process with covariance function $K_\theta(s, t) = J_0\left(2\pi\,|\theta(s) - \theta(t)|\right)$ is well-defined if and only if θ is strictly monotonic.*

Proof. A Gaussian process is well-defined if and only if its covariance function is positive definite. By Lemma 11 and Lemma 7, K_θ is positive definite if and only if θ is strictly monotonic. Furthermore, the self-adjointness of T_θ (which occurs if and only if θ is strictly monotonic by Theorem 6) ensures the existence of a spectral decomposition, which is necessary for the proper definition of the process. $\square$