# Spectral Support and Bandlimited Gaussian Processes

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# 1 Fundamental Definitions

**Definition 1.** *[Heaviside Step Function] The Heaviside step function* $H\colon\mathbb{R}\to\{0,1\}$ *is defined as*

$$H(x)=\begin{cases}1 & \text{if } x\geq 0\\ 0 & \text{if } x<0\end{cases} \tag{1}$$

**Definition 2.** *[Rectangular Function] The rectangular function* $\operatorname{rect}_{[a,b]}\colon\mathbb{R}\to\{0,1\}$ *for* $a<b$ *is defined as*

$$\operatorname{rect}_{[a,b]}(\omega)=H(\omega-a)-H(\omega-b) \tag{2}$$

*which equals 1 for* $\omega\in[a,b]$ *and 0 otherwise.*

**Definition 3.** *[Spectral Density] Let* $\{X_t\}_{t\in\mathbb{R}}$ *be a zero-mean, stationary Gaussian process with covariance function* $K(\tau)=\mathbb{E}[X_t X_{t+\tau}]$*. The spectral density* $S\colon\mathbb{R}\to[0,\infty)$ *is the Fourier transform of the covariance function:*

$$S(\omega)=\int_{-\infty}^{\infty}K(\tau)\,e^{-i\omega\tau}\,d\tau \tag{3}$$

*provided this integral exists.*

**Definition 4.** *[Spectral Support] The spectral support of a Gaussian process with spectral density* $S(\omega)$ *is the set*

$$\operatorname{supp}(S)=\overline{\{\omega\in\mathbb{R}\colon S(\omega)>0\}} \tag{4}$$

*where* $\bar{A}$ *denotes the closure of set* $A$*.*

**Definition 5.** *[Bandlimited Process] A stationary Gaussian process is called bandlimited if its spectral support is bounded, i.e., if there exist constants* $a, b \in \mathbb{R}$ *with* $a < b$ *such that*

$$\mathrm{supp}(S) \subseteq [a, b] \tag{5}$$

# 2 Main Results

**Theorem 6.** *[Sinc Kernel Spectral Density] Consider the covariance function*

$$K(\tau) = \frac{\sin(2\pi\tau)}{2\pi\tau} \tag{6}$$

*with the convention that* $K(0) = 1$*. The corresponding spectral density is*

$$S(\omega) = \frac{1}{2}\mathrm{rect}_{[-1,1]}(\omega) = \frac{1}{2}[H(\omega+1) - H(\omega-1)] \tag{7}$$

**Proof.** The Fourier transform of $K(\tau) = \frac{\sin(2\pi\tau)}{2\pi\tau}$ is computed as follows. Using the identity $\sin(2\pi\tau) = \frac{e^{i2\pi\tau} - e^{-i2\pi\tau}}{2i}$, one has

$$S(\omega) = \int_{-\infty}^{\infty} \frac{\sin(2\pi\tau)}{2\pi\tau} e^{-i\omega\tau} d\tau \tag{8}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i2\pi\tau} - e^{-i2\pi\tau}}{2i\tau} e^{-i\omega\tau} d\tau \tag{9}$$

$$= \frac{1}{4\pi i}\left[\int_{-\infty}^{\infty} \frac{e^{-i(\omega-2\pi)\tau}}{\tau} d\tau - \int_{-\infty}^{\infty} \frac{e^{-i(\omega+2\pi)\tau}}{\tau} d\tau\right] \tag{10}$$

By the well-known result that the Fourier transform of $\frac{\sin(a\tau)}{\pi\tau}$ is $\mathrm{rect}_{[-a,a]}(\omega)$, it follows that

$$S(\omega) = \frac{1}{2}\mathrm{rect}_{[-1,1]}(\omega) = \frac{1}{2}[H(\omega+1) - H(\omega-1)] \tag{11}$$ □

**Proposition 7.** *[General Bandlimited Spectral Density] A Gaussian process is bandlimited with spectral support* $[a, b]$ *if and only if its spectral density can be written as*

$$S(\omega) = f(\omega) \cdot \mathrm{rect}_{[a,b]}(\omega) = f(\omega) \cdot [H(\omega-a) - H(\omega-b)] \tag{12}$$

*for some non-negative function* $f\colon [a, b] \to [0, \infty)$*.*

**Proof.** ($\Rightarrow$) If the process is bandlimited with spectral support $[a, b]$, then $S(\omega) = 0$ for $\omega \notin [a, b]$. Define $f(\omega) = S(\omega)$ for $\omega \in [a, b]$ and extend arbitrarily to $\mathbb{R}$. Then $S(\omega) = f(\omega) \cdot \mathrm{rect}_{[a, b]}(\omega)$.

($\Leftarrow$) If $S(\omega) = f(\omega) \cdot \mathrm{rect}_{[a, b]}(\omega)$, then $S(\omega) = 0$ for $\omega \notin [a, b]$, implying $\mathrm{supp}(S) \subseteq [a, b]$. $\square$

**Example 8.** [Band-pass Process] Consider a bandlimited process with spectral support $[-\Omega, -\omega_0] \cup [\omega_0, \Omega]$ where $0 < \omega_0 < \Omega$. The spectral density can be expressed as

$$\begin{aligned} S(\omega) &= f(\omega) \cdot [\mathrm{rect}_{[-\Omega, -\omega_0]}(\omega) + \mathrm{rect}_{[\omega_0, \Omega]}(\omega)] && (13) \\ &= f(\omega) \cdot [H(\omega + \Omega) - H(\omega + \omega_0) + H(\omega - \omega_0) - H(\omega - \Omega)] && (14) \end{aligned}$$

for some appropriate function $f$.

**Theorem 9.** *[Wiener-Khintchine Relation for Bandlimited Processes] If a Gaussian process has spectral density $S(\omega) = g(\omega) \cdot \mathrm{rect}_{[a, b]}(\omega)$, then its covariance function is given by*

$$K(\tau) = \frac{1}{2\pi} \int_a^b g(\omega)\, e^{i\omega\tau}\, d\omega \tag{15}$$

**Proof.** By the inverse Fourier transform relation,

$$\begin{aligned} K(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{i\omega\tau}\, d\omega && (16) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) \cdot \mathrm{rect}_{[a, b]}(\omega)\, e^{i\omega\tau}\, d\omega && (17) \\ &= \frac{1}{2\pi} \int_a^b g(\omega)\, e^{i\omega\tau}\, d\omega && (18) \end{aligned}$$

since $\mathrm{rect}_{[a, b]}(\omega) = 0$ outside $[a, b]$. $\square$

# 3 Conclusion

The spectral support serves as the fundamental concept for characterizing bandlimited Gaussian processes. The Heaviside step function provides a natural mathematical framework for expressing the boundaries of spectral support, enabling precise characterization of the frequency domain properties of such processes.