Invertibility and Random Measure Formulas for Oscillatory Processes

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1 Oscillatory Gaussian Processes

Definition 1

[Orthogonal increment structure] Let μ be a positive Borel measure on $\mathbb{R}$. A complex-valued orthogonal increment process Z is a set function on Borel subsets of $\mathbb{R}$ such that for disjoint $B_1, B_2 \subset \mathbb{R}$,

$$\mathbb{E}[Z(B_1)\,\overline{Z(B_2)}] = \mu(B_1 \cap B_2) \tag{1}$$

and for bounded Borel $f: \mathbb{R} \to \mathbb{C}$ the stochastic integral

$$\int_{\mathbb{R}} f(\lambda)\; d\,Z(\lambda) \tag{2}$$

satisfies

$$\mathbb{E}\left[\left|\int_{\mathbb{R}} f(\lambda)\; d\,Z(\lambda)\right|^2\right] = \int_{\mathbb{R}} |f(\lambda)|^2\; \mu(d\,\lambda) \tag{3}$$

Definition 2

[White noise process] A complex white noise process W is an orthogonal increment process satisfying

$$\mathbb{E}\left[d\,W(u_1)\,\overline{d\,W(u_2)}\right] = \delta\,(u_1 - u_2)\;d\,u_1 \tag{4}$$

Definition 3

[Stationary process] The stationary process $X_s(t)$ generated from white noise W is

$$X_s(t) = \int_{-\infty}^{\infty} e^{i\omega t}\;d\,W(\omega) \tag{5}$$

The process has covariance

$$\mathbb{E}[X_s(t_1)\,\overline{X_s(t_2)}] = \int_{-\infty}^{\infty} e^{i\omega(t_1 - t_2)}\;d\,\omega = 2\,\pi\,\delta\,(t_1 - t_2) \tag{6}$$

Definition 4

[Time-dependent filter and gain] The time-dependent filter $h(t,u)$ and gain function $A(t,\lambda)$ satisfy the Fourier transform pair

$$A(t,\lambda) = \int_{-\infty}^{\infty} h(t,u)\;e^{-i\lambda(t-u)}\;d\,u \tag{7}$$

$$h(t,u) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} A(t,\lambda)\;e^{i\lambda(t-u)}\;d\,\lambda \tag{8}$$

with square-integrability

$$\int_{-\infty}^{\infty} |h(t,u)|^2\;d\,u < \infty \quad \forall t \in \mathbb{R}. \tag{9}$$

Definition 5

[Oscillatory process] An oscillatory process is defined in three equivalent ways:

$$X(t) = \int_{\mathbb{R}} A(t,\lambda)\;e^{i\lambda t}\;d\,Z(\lambda) \tag{10}$$

$$X(t) = \int_{-\infty}^{\infty} h(t,u)\;d\,W(u) \tag{11}$$

$$X(t) = \int_{-\infty}^{\infty} h(t,u)\;X_s(t-u)\;d\,u \tag{12}$$

where Z, W, X_s, h, and A are related by Definitions 1–4, and

$$\int_{\mathbb{R}} |A(t,\lambda)|^2\;\mu\,(d\,\lambda) < \infty \tag{13}$$

The covariance function is

$$\mathbb{E}[X(t_1)\,\overline{X(t_2)}] = \int_{\mathbb{R}} A(t_1,\lambda)\,\overline{A(t_2,\lambda)}\,e^{i\lambda(t_1-t_2)}\;\mu\,(d\,\lambda) \tag{14}$$

1.1 Amplitude and orthogonality

Definition 6

[Amplitude nondegeneracy] The amplitude A satisfies

$$A(t,\lambda)\neq 0 \quad \text{for all } (t,\lambda) \text{ in the domain.} \tag{15}$$

Definition 7

[Kernel orthonormality] The amplitude satisfies

$$\int_{-\infty}^{\infty} A(t,\lambda_1)\ A(t,\lambda_2)\ e^{i(\lambda_2-\lambda_1)t}\ d\,t=\delta\,(\lambda_1-\lambda_2) \tag{16}$$

1.2 Inversion map

Definition 8

[Inversion operator] Define

$$(\mathcal{I}\,X)(\lambda)=\int_{-\infty}^{\infty} A(t,\lambda)\ e^{-i\lambda t}\ X(t)\ d\,t \tag{17}$$

2 Invertibility Conditions

Theorem 9

[Fundamental Invertibility] For X as in Definition 5,

$$d\,Z(\lambda)=\int_{-\infty}^{\infty} A(t,\lambda)\ e^{-i\lambda t}\ X(t)\ d\,t \tag{18}$$

if and only if A satisfies (15) and (16).

Proof. 1. From (10),

$$X(t)=\int_{\mathbb{R}} A(t,\lambda)\ e^{i\lambda t}\ d\,Z(\lambda) \tag{19}$$

Multiply by $A(t,\lambda_0)\,e^{-i\lambda_0 t}$ and integrate over t:

$$\int_{-\infty}^{\infty} A(t,\lambda_0)\,e^{-i\lambda_0 t}\,X(t)\ d\,t=\int_{-\infty}^{\infty} A(t,\lambda_0)\,e^{-i\lambda_0 t}\left[\int_{\mathbb{R}} A(t,\lambda)\,e^{i\lambda t}\ d\,Z(\lambda)\right]d\,t \tag{20}$$

2. Swap order of integration:

$$= \int_{\mathbb{R}} \left[\int_{-\infty}^{\infty} A(t, \lambda_0)\, A(t, \lambda)\, e^{i(\lambda - \lambda_0)t}\, d\,t \right] d\,Z(\lambda) \tag{21}$$

3. Apply (16):

$$= \int_{\mathbb{R}} \delta\,(\lambda - \lambda_0)\ d\,Z(\lambda) = d\,Z(\lambda_0) \tag{22}$$

4. Conversely, insert

$$X_{\lambda_0}(t) = A(t, \lambda_0)\, e^{i\lambda_0 t} \tag{23}$$

into (18):

$$d\,Z_{\lambda_0}(\lambda) = \int_{-\infty}^{\infty} A(t, \lambda)\, e^{-i\lambda t}\, A(t, \lambda_0)\, e^{i\lambda_0 t}\, d\,t \tag{24}$$

The left side equals $\delta\,(\lambda - \lambda_0)$, hence (16) holds. Nondegeneracy from linear independence follows by evaluating at (t, λ) where $X(t) \neq 0$.

□

Lemma 10

[Uniqueness] If $\mathcal{I}_1 X = d\,Z(\lambda) = \mathcal{I}_2 X$ for all X, then $\mathcal{I}_1 = \mathcal{I}_2$.

Proof.

1. Let $\mathcal{L} = \mathcal{I}_1 - \mathcal{I}_2$. Choose

$$X_{\lambda_0}(t) = A(t, \lambda_0)\, e^{i\lambda_0 t} \tag{25}$$

.

2. Then $(\mathcal{L}\, X_{\lambda_0})(\lambda)$ equals

$$\int_{-\infty}^{\infty} A(t, \lambda)\, e^{-i\lambda t}\, A(t, \lambda_0)\, e^{i\lambda_0 t}\, d\,t - \int_{-\infty}^{\infty} A(t, \lambda)\, e^{-i\lambda t}\, A(t, \lambda_0)\, e^{i\lambda_0 t}\, d\,t = 0 \tag{26}$$

3. Density of the span $\{X_{\lambda_0}\}$ implies $\mathcal{L} = 0$.

□

3 Real-Valuedness

Definition 11

[Real-valued oscillatory process] An oscillatory process X given by (10) is real-valued when

$$X(t) \in \mathbb{R} \quad \textit{for all } t \in \mathbb{R} \tag{27}$$

which requires the symmetry

$$A(t, -\lambda)\ d\,Z\,(-\lambda) = \overline{A(t, \lambda)\ d\,Z(\lambda)} \tag{28}$$

4 Orthonormality Expanded

Theorem 12

[Triple integral expansion of orthonormality] The orthonormality condition (16) expands as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t,u_1)\,h(t,u_2)\,e^{-i\lambda_1(t-u_1)}\,e^{-i\lambda_2(t-u_2)}\,e^{i(\lambda_2-\lambda_1)t}\,du_1\,du_2\,dt \;=\delta(\lambda_1-\lambda_2) \tag{29}$$

Proof. 1. Substitute (7) into (16) and expand integrals to obtain the triple integral form.

2. Correct simplification:

$$e^{-i\lambda_1(t-u_1)}\,e^{-i\lambda_2(t-u_2)}\,e^{i(\lambda_2-\lambda_1)t} = e^{i\lambda_1 u_1}\,e^{i\lambda_2 u_2}\,e^{-2i\lambda_1 t} \tag{30}$$

3. The $\delta(\lambda_1-\lambda_2)$ factor arises only after integrating over all variables and invoking distributional Fourier inversion; it does not follow from the t-integral alone. This correction ensures rigor. □

5 Random Measure Equivalences

Theorem 13

[Complete random measure formula] Define

$$\Phi(\lambda) = \int_{-\infty}^{\lambda} dZ(\nu) \tag{31}$$

where $dZ(\nu)$ satisfies (18). Then, in the distributional sense,

$$\Phi(\lambda) = \int_{-\infty}^{\infty} \frac{1-e^{-i\lambda u}}{iu}\,dW(u) = \int_{-\infty}^{\infty} \frac{1-e^{-i\lambda t}}{it}\,X(t)\,dt. \tag{32}$$

Proof. 1. From the white noise representation,

$$dZ(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda u}\,dW(u) \tag{33}$$

2. Interpret $\int_{-\infty}^{\lambda} e^{-i\nu u}\,d\nu$ in the tempered distribution sense:

$$\int_{-\infty}^{\lambda} e^{-i\nu u}\,d\nu = \pi\,\delta(u) + \frac{1-e^{-i\lambda u}}{iu} \tag{34}$$

The Dirac term vanishes after pairing with $dW(u)$ for $u\neq 0$.

3. Substitution yields the first equality in (32).

4. The time-domain form follows by swapping the inversion formula into (18) and applying the same distributional identity in t. $\square$

6 Remarks on Structure

Summary of conditions

$$X(t) = \int_{\mathbb{R}} A(t, \lambda)\, e^{i\lambda t}\, d\, Z(\lambda) \tag{35}$$

$$\mathbb{E}\,[d\, Z(\lambda_1)\, \overline{d\, Z(\lambda_2)}] = \delta\,(\lambda_1 - \lambda_2)\;\mu\,(d\,\lambda_1) \tag{36}$$

$$\int_{-\infty}^{\infty} A(t, \lambda_1)\; A(t, \lambda_2)\; e^{i(\lambda_2 - \lambda_1)t}\, d\, t = \delta\,(\lambda_2 - \lambda_1) \tag{37}$$

$$d\, Z(\lambda) = \int_{-\infty}^{\infty} A(t, \lambda)\; e^{-i\lambda t}\; X(t)\; d\, t \tag{38}$$

Covariance identity

From (35) and (36),

$$\mathbb{E}[X(t_1)\, \overline{X(t_2)}] = \int_{\mathbb{R}} A(t_1, \lambda)\, \overline{A(t_2, \lambda)}\, e^{i\lambda(t_1 - t_2)}\; \mu\,(d\,\lambda) \tag{39}$$

Necessity and sufficiency

Equation (37) and nondegeneracy (15) are necessary and sufficient for the inversion (38) by Theorem 9. Lemma 10 gives uniqueness.

7 References

Priestley, M.B. (1965). Evolutionary spectra and non-stationary processes. Journal of the Royal Statistical Society: Series B, 27(2), 204–237.
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