Stone's Theorem, The Shift Group, and The Fourier Transform

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Definitions

Shift Group: For $f \in L^2(\mathbb{R})$, define the family of unitary operators $(S_t)_{t\in\mathbb{R}}$ by

$$(S_t f)(x) = f(x+t).$$

Generator of Shift Group: Define $A = \frac{d}{dx}$ on the domain

$$D(A) = \{f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R})\},$$

where f' is in the distributional sense.

Momentum Operator: Define $P = -i\,A = -i\,\frac{d}{dx}$ on the same domain $D(P) = D(A)$.

Fourier Transform:

$$\mathcal{F}[f](\omega) = \hat{f}(\omega) := \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx.$$

Theorems and Proofs

Theorem 1. *[Stone's Theorem Applied to Shift Group] The strongly continuous unitary group (S_t) on $L^2(\mathbb{R})$ has a densely defined skew-adjoint generator $A = \frac{d}{dx}$ such that $S_t = e^{tA}$. The generator satisfies*

$$A f = \lim_{h\to 0} \frac{S_h f - f}{h}$$

in the L^2 topology on the domain $D(A)$.

Proof. Let $f \in D(A)$. Then

$$\frac{S_h f(x) - f(x)}{h} = \frac{f(x+h) - f(x)}{h} \to f'(x)$$

as $h \to 0$ in L^2 norm. Thus, the infinitesimal generator of S_t is $A = \frac{d}{dx}$.

To verify A is skew-adjoint, for $f, g \in D(A)$:

$$\begin{aligned} \langle Af, g \rangle &= \int_{-\infty}^{\infty} f'(x) \overline{g(x)} dx && (1) \\ &= f(x) \overline{g(x)} |_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) \overline{g'(x)} dx && (2) \\ &= 0 - \int_{-\infty}^{\infty} f(x) \overline{g'(x)} dx && (3) \\ &= -\langle f, Ag \rangle && (4) \end{aligned}$$

Therefore $A^* = -A$, confirming A is skew-adjoint. □

Theorem 2. *[Relation Between Generators] The shift group is generated by both the skew-adjoint operator $A = \frac{d}{dx}$ and the self-adjoint momentum operator $P = -iA$:*

$$S_t = e^{tA} = e^{-itP}$$

Proof. Since $P = -iA$, we have $-itP = -it(-iA) = -i^2 tA = tA$. Therefore:

$$e^{-itP} = e^{tA}$$

For $f \in D(A)$, using the Taylor expansion:

$$e^{tA} f(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n f(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f^{(n)}(x) = f(x+t) = S_t f(x)$$ □

Theorem 3. *[Complex Exponentials Are Eigenfunctions] For any $\omega \in \mathbb{R}$:*

1. *$A e^{i\omega x} = i\omega e^{i\omega x}$ (eigenvalue $i\omega$ for skew-adjoint A)*
2. *$P e^{i\omega x} = \omega e^{i\omega x}$ (eigenvalue ω for self-adjoint P)*

Proof. Direct calculations:

1. $A e^{i\omega x} = \frac{d}{dx} e^{i\omega x} = i\omega e^{i\omega x}$

2. $P e^{i\omega x} = -i\,\frac{d}{d\,x}\,e^{i\omega x} = -i\,(i\,\omega)\,e^{i\omega x} = \omega\,e^{i\omega x}$ □

Theorem 4. *[Fourier Transform of Complex Exponential] Let $\omega_0 \in \mathbb{R}$. In the distributional sense,*

$$\mathcal{F}[e^{i\omega_0 x}](\omega) = 2\,\pi\,\delta\,(\omega - \omega_0)$$

where δ is the Dirac delta distribution.

Proof. We prove this by showing that for any test function $\phi \in \mathcal{S}(\mathbb{R})$ (Schwartz space):

$$\langle \mathcal{F}[e^{i\omega_0 x}], \phi \rangle = 2\,\pi\,\phi(\omega_0)$$

By definition of the Fourier transform of distributions:

$$\begin{aligned} \langle \mathcal{F}[e^{i\omega_0 x}], \phi \rangle &= \langle e^{i\omega_0 x}, \mathcal{F}[\phi] \rangle && (5) \\ &= \int_{-\infty}^{\infty} e^{i\omega_0 x}\,\hat{\phi}(x)\,d\,x && (6) \\ &= \int_{-\infty}^{\infty} e^{i\omega_0 x} \int_{-\infty}^{\infty} \phi(\omega)\,e^{-i\omega x}\,d\,\omega\,d\,x && (7) \end{aligned}$$

By Fubini's theorem (valid for $\phi \in \mathcal{S}(\mathbb{R})$):

$$\begin{aligned} &= \int_{-\infty}^{\infty} \phi(\omega) \int_{-\infty}^{\infty} e^{i(\omega_0 - \omega)x}\,d\,x\,d\,\omega && (8) \\ &= \int_{-\infty}^{\infty} \phi(\omega) \cdot 2\,\pi\,\delta\,(\omega_0 - \omega)\,d\,\omega && (9) \\ &= 2\,\pi\,\phi(\omega_0) && (10) \\ &= \langle 2\,\pi\,\delta\,(\omega - \omega_0), \phi \rangle && (11) \end{aligned}$$

Therefore, $\mathcal{F}[e^{i\omega_0 x}] = 2\,\pi\,\delta\,(\omega - \omega_0)$. □

Theorem 5. *[Inverse Fourier Transform of Dirac Delta] In the distributional sense,*

$$\mathcal{F}^{-1}\,[\delta\,(\omega - \omega_0)](x) = \frac{1}{2\,\pi}\,e^{i\omega_0 x}$$

Proof. From the previous theorem, we have $\mathcal{F}[e^{i\omega_0 x}] = 2\,\pi\,\delta\,(\omega - \omega_0)$. Applying $\mathcal{F}^{-1}$ to both sides:

$$e^{i\omega_0 x} = \mathcal{F}^{-1}\,[2\,\pi\,\delta\,(\omega - \omega_0)]$$

Therefore:

$$\mathcal{F}^{-1}\left[\delta\left(\omega-\omega_0\right)\right]=\frac{1}{2\,\pi}\,e^{i\omega_0 x} \qquad \square$$

Theorem 6. *[Spectral Decomposition via Fourier Transform] Under the Fourier transform $\mathcal{F}$:*

1. *The self-adjoint momentum operator becomes multiplication by ω: $\mathcal{F}\,[P\,f](\omega)=\omega\,\hat{f}(\omega)$*

2. *The shift group becomes multiplication by a phase: $\mathcal{F}\,[S_t\,f](\omega)=e^{i\omega t}\,\hat{f}(\omega)$*

Proof. For part 1, if $f\in D(P)$:

$$\begin{aligned}\mathcal{F}\,[P\,f](\omega)&=\int_{-\infty}^{\infty}(-i\,f'(x))\,e^{-i\omega x}\;d\,x &(12)\\ &=-i\int_{-\infty}^{\infty}f'(x)\,e^{-i\omega x}\;d\,x &(13)\end{aligned}$$

Integration by parts (boundary terms vanish):

$$=-i\,[0+i\,\omega\,\hat{f}(\omega)]=\omega\,\hat{f}(\omega) \tag{14}$$

For part 2:

$$\mathcal{F}\,[S_t\,f](\omega)=\int_{-\infty}^{\infty}f\,(x+t)\,e^{-i\omega x}\,d\,x \tag{15}$$

Let $u=x+t$, so $x=u-t$, $d\,x=d\,u$:

$$\begin{aligned}&=\int_{-\infty}^{\infty}f(u)\,e^{-i\omega(u-t)}\,d\,u &(16)\\ &=e^{i\omega t}\int_{-\infty}^{\infty}f(u)\,e^{-i\omega u}\,d\,u=e^{i\omega t}\,\hat{f}(\omega) &(17)\end{aligned}$$

$\square$

Theorem 7. *[Eigenfunction Property of Shift Group] Complex exponentials are eigenfunctions of the shift group:*

$$S_t\,e^{i\omega x}=e^{i\omega t}\,e^{i\omega x}$$

with eigenvalue $e^{i\omega t}$.

Proof. $$S_t\,e^{i\omega x}=e^{i\omega(x+t)}=e^{i\omega x}\,e^{i\omega t}=e^{i\omega t}\,e^{i\omega x} \qquad \square$$

Theorem 8. *[Spectral Representation of Identity] The identity operator can be represented using the Dirac delta:*

$$I = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} |e^{i\omega\cdot}\rangle\langle e^{i\omega\cdot}|\,d\,\omega$$

where in distributional form:

$$\delta\,(x-y) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} e^{i\omega\,(x-y)}\,d\,\omega$$

Proof. For any test function f:

$$\frac{1}{2\,\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)\,e^{i\omega\,(x-y)}\,d\,y\,d\,\omega = \int_{-\infty}^{\infty} f(y)\left[\frac{1}{2\,\pi}\int_{-\infty}^{\infty} e^{i\omega\,(x-y)}\,d\,\omega\right]d\,y \tag{18}$$

$$= \int_{-\infty}^{\infty} f(y)\,\delta\,(x-y)\,d\,y \tag{19}$$

$$= f(x) \tag{20}$$

This shows the spectral representation of the identity using the continuous spectrum of the momentum operator. □

Corollary 9. *[Consistency Check] The eigenvalue relationships are consistent:*

$$S_t\,e^{i\omega x} = e^{t\,A}\,e^{i\omega x} = e^{t\,(i\omega)}\,e^{i\omega x} = e^{i\omega t}\,e^{i\omega x}$$

since A has eigenvalue $i\,\omega$ on $e^{i\omega x}$.

Conclusion

Stone's theorem ensures that the shift group (S_t) has a skew-adjoint generator $A = \frac{d}{d\,x}$, whose eigenfunctions are the complex exponentials $e^{i\omega x}$ with purely imaginary eigenvalues $i\,\omega$. The related **self-adjoint momentum operator** $P = -i\,A$ has the same eigenfunctions but with real eigenvalues ω.

The Dirac delta function emerges naturally as the Fourier transform of complex exponentials, providing the spectral measure for the continuous spectrum of the momentum operator. This gives us the fundamental relationships:

- Complex exponentials $e^{i\omega_0 x} \leftrightarrow 2\,\pi\,\delta\,(\omega-\omega_0)$ under Fourier transform
- The identity operator has the spectral representation $I = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} |e^{i\omega\cdot}\rangle\langle e^{i\omega\cdot}|\,d\,\omega$

- The delta function provides the orthogonality relation for the continuous eigenfunction basis

The Fourier transform provides the spectral decomposition that diagonalizes both operators, with the Dirac delta serving as the key distributional tool that makes the continuous spectrum rigorous. This mathematical structure underlies all of Fourier analysis and quantum mechanics, where complex exponentials are the fundamental building blocks precisely because they diagonalize translation-invariant systems.