

in light of what we just talked about examine how this paper can be titled more appropriately and how we can make the existing of the unitary bijection more apparent

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\documentclass{article}\usepackage{amsmath,amssymb,amsthm,mathtools}
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Monotonic Time Changes of Stationary Gaussian Processes Yield Oscillatory Processes: Complete Framework

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Abstract

We provide a complete framework for monotonic time changes of stationary Gaussian processes that yield oscillatory processes in the sense of Priestley. The monotonicity is necessary and sufficient for the resulting covariance operator to be self-adjoint and positive definite. We establish bidirectional inversion formulas for reconstructing stationary processes from oscillatory sample paths, recovering random measures, and extracting the underlying white noise. The covariance kernel can be recovered from infinite sample paths using ergodic averaging.

1. Preliminaries

Definition 1. (Stationary Gaussian Process) *Let $Y(u)$ be a mean-zero stationary Gaussian process on \mathbb{R} with covariance kernel*

$$K_0(u - v) = \mathbb{E}[Y(u)Y(v)].$$

The spectral representation is

$$Y(u) = \int_{-\infty}^{\infty} e^{i\lambda u} \sqrt{f_0(\lambda)} dW(\lambda),$$

where $f_0(\lambda)$ is the spectral density and $dW(\lambda)$ is a standard complex white noise random measure:

$$\mathbb{E}[dW(\lambda)\overline{dW(\mu)}] = \delta(\lambda - \mu) d\lambda d\mu.$$

Definition 2. (Time Change) Let $\theta : \mathbb{R} \rightarrow \mathbb{R}$ be strictly increasing, C^2 , with $\dot{\theta}(t) > 0$ for all t . Define the time-changed process

$$X(t) := Y(\theta(t)) \sqrt{\dot{\theta}(t)}.$$

2. Main Theorem and Proof

Theorem 1. (Monotonic Time Change Yields Oscillatory Process) Let $Y(u)$ be a stationary Gaussian process as above, and let θ be strictly increasing, C^2 , with $\dot{\theta}(t) > 0$. Then $X(t)$ is a mean-zero Gaussian process with covariance

$$K(t, s) = K_0(|\theta(t) - \theta(s)|) \sqrt{\dot{\theta}(t) \dot{\theta}(s)},$$

and spectral representation

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda\theta(t)} \sqrt{\dot{\theta}(t)} \sqrt{f_0(\lambda)} dW(\lambda).$$

The process X is oscillatory in the sense of Priestley, and the covariance operator is self-adjoint and positive definite.

Proof. **Step 1: Covariance Structure.**

By construction,

$$\mathbb{E}[X(t)X(s)] = \mathbb{E}[Y(\theta(t))Y(\theta(s))] \sqrt{\dot{\theta}(t) \dot{\theta}(s)} = K_0(|\theta(t) - \theta(s)|) \sqrt{\dot{\theta}(t) \dot{\theta}(s)}.$$

Step 2: Spectral Representation.

The spectral representation for Y gives

$$Y(\theta(t)) = \int_{-\infty}^{\infty} e^{i\lambda\theta(t)} \sqrt{f_0(\lambda)} dW(\lambda).$$

Thus,

$$X(t) = Y(\theta(t)) \sqrt{\dot{\theta}(t)} = \int_{-\infty}^{\infty} e^{i\lambda\theta(t)} \sqrt{\dot{\theta}(t)} \sqrt{f_0(\lambda)} dW(\lambda).$$

Step 3: Oscillatory Class (Priestley).

The process $X(t)$ admits the continuous spectral representation

$$X(t) = \int_{-\infty}^{\infty} A(t, \lambda) d\Phi(\lambda),$$

where

$$A(t, \lambda) = e^{i\lambda\theta(t)} \sqrt{\dot{\theta}(t)},$$

and $d\Phi(\lambda) = \sqrt{f_0(\lambda)} dW(\lambda)$. This matches Priestley's definition of an oscillatory process.

Step 4: Self-Adjointness and Positive Definiteness.

The covariance operator K is positive definite because for any finite collection $\{t_j\}$ and scalars $\{c_j\}$,

$$\sum_{j,k} c_j \overline{c_k} K(t_j, t_k) = \sum_{j,k} c_j \overline{c_k} K_0(|\theta(t_j) - \theta(t_k)|) \sqrt{\dot{\theta}(t_j) \dot{\theta}(t_k)} \geq 0,$$

since K_0 is positive definite and $\dot{\theta}(t) > 0$.

Self-adjointness follows because $K(t, s) = \overline{K(s, t)}$ and the operator is defined by integration against a real, symmetric kernel.

Step 5: Necessity of Monotonicity.

If $\dot{\theta}(t)$ vanishes or changes sign, the scaling factor $\sqrt{\dot{\theta}(t)}$ becomes ill-defined or nonreal, and the kernel can lose positive definiteness. For non-monotonic θ , the deficiency indices n_+ and n_- of the associated operator are unequal, so no self-adjoint extension exists (von Neumann's theorem). \square

3. Bidirectional Inversion Formulas

Theorem 2. (Stationary to Oscillatory Sample Path Inversion) *Let $X(t)$ be an oscillatory process constructed from a stationary process $Y(u)$ via monotonic time change θ . Then the following bidirectional inversion formulas hold:*

(a) Recovering Stationary Process from Oscillatory Sample Path:

Given oscillatory sample path $X(t)$ and time change θ , the stationary process is recovered by:

$$Y(u) = X(\theta^{-1}(u)) \frac{1}{\sqrt{\dot{\theta}(\theta^{-1}(u))}}$$

(b) Recovering Random Measure from Stationary Process:

Given stationary process $Y(u)$ with spectral density $f_0(\lambda)$, the random measure is:

$$d\Phi(\lambda) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T Y(u) \sqrt{f_0(\lambda)} e^{-i\lambda u} du$$

(c) Recovering White Noise from Random Measure:

Given random measure $d\Phi(\lambda) = \sqrt{f_0(\lambda)} dW(\lambda)$, the white noise is:

$$dW(\lambda) = \frac{d\Phi(\lambda)}{\sqrt{f_0(\lambda)}}$$

(d) Direct White Noise Recovery from Oscillatory Sample Path:

Combining the above, the white noise is recovered directly from oscillatory sample path $X(t)$ by:

$$dW(\lambda) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X(\theta^{-1}(u)) \frac{1}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \frac{1}{\sqrt{f_0(\lambda)}} e^{-i\lambda u} du$$

Proof. **Part (a): Stationary Process Recovery.**

By construction of the time-changed process:

$$X(t) = Y(\theta(t)) \sqrt{\dot{\theta}(t)}$$

Setting $u = \theta(t)$, so $t = \theta^{-1}(u)$:

$$X(\theta^{-1}(u)) = Y(\theta(\theta^{-1}(u))) \sqrt{\dot{\theta}(\theta^{-1}(u))} = Y(u) \sqrt{\dot{\theta}(\theta^{-1}(u))}$$

Therefore:

$$Y(u) = X(\theta^{-1}(u)) \frac{1}{\sqrt{\dot{\theta}(\theta^{-1}(u))}}$$

Part (b): Random Measure Recovery.

For a stationary process with spectral representation $Y(u) = \int e^{i\lambda u} \sqrt{f_0(\lambda)} dW(\lambda)$, the Fourier inversion formula gives:

$$\sqrt{f_0(\lambda)} dW(\lambda) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T Y(u) e^{-i\lambda u} du$$

Thus:

$$d\Phi(\lambda) = \sqrt{f_0(\lambda)} dW(\lambda) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T Y(u) e^{-i\lambda u} du$$

Part (c): White Noise Extraction.

By definition, $d\Phi(\lambda) = \sqrt{f_0(\lambda)} dW(\lambda)$, so:

$$dW(\lambda) = \frac{d\Phi(\lambda)}{\sqrt{f_0(\lambda)}}$$

Part (d): Direct Recovery Chain.

Combining parts (a), (b), and (c):

$$dW(\lambda) = \frac{1}{\sqrt{f_0(\lambda)}} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T Y(u) e^{-i\lambda u} du \quad (1)$$

$$= \frac{1}{\sqrt{f_0(\lambda)}} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X(\theta^{-1}(u)) \frac{1}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} e^{-i\lambda u} du \quad (2)$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X(\theta^{-1}(u)) \frac{1}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \frac{1}{\sqrt{f_0(\lambda)}} e^{-i\lambda u} du \quad (3)$$

Convergence Analysis.

The convergence in part (d) is guaranteed because:

1. The inverse scaling $\frac{1}{\sqrt{\dot{\theta}(\theta^{-1}(u))}}$ converts the oscillatory process back to a stationary process
2. The stationary process has well-defined spectral inversion by the classical theory
3. The spectral density normalization $\frac{1}{\sqrt{f_0(\lambda)}}$ ensures proper extraction of white noise

□

4. Kernel Recovery from Infinite Sample Path

Theorem 3. (Kernel Recovery from Infinite Sample Path) *Let $X(t)$ be an oscillatory process on \mathbb{R} with continuous spectral representation*

$$X(t) = \int_{-\infty}^{\infty} A(t, \lambda) d\Phi(\lambda),$$

where $A(t, \lambda)$ is the deterministic amplitude function and $d\Phi(\lambda)$ is the orthogonal random measure with spectral density $f(\lambda)$. If the process is mean-square continuous and ergodic, then the covariance kernel $K(t, s) = \mathbb{E}[X(t)X(s)]$ can be recovered from a single infinitely long sample path by the formula

$$K(t, s) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X(t+u)X(s+u) du$$

almost surely.

Proof. **Step 1: Ergodic Decomposition.**

For an oscillatory process $X(t)$ with continuous spectral representation, the covariance function is

$$K(t, s) = \mathbb{E}[X(t)X(s)] = \int_{-\infty}^{\infty} A(t, \lambda) \overline{A(s, \lambda)} f(\lambda) d\lambda.$$

Step 2: Empirical Average.

Consider the empirical average along a single infinite sample path:

$$\hat{K}_T(t, s) = \frac{1}{2T} \int_{-T}^T X(t+u)X(s+u) du.$$

Step 3: Spectral Substitution.

Substituting the spectral representation:

$$\hat{K}_T(t, s) = \frac{1}{2T} \int_{-T}^T \left(\int_{-\infty}^{\infty} A(t+u, \lambda) d\Phi(\lambda) \right) \left(\int_{-\infty}^{\infty} \overline{A(s+u, \mu)} d\Phi(\mu) \right) du \quad (4)$$

$$= \frac{1}{2T} \int_{-T}^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(t+u, \lambda) \overline{A(s+u, \mu)} d\Phi(\lambda) d\Phi(\mu) du. \quad (5)$$

Step 4: Orthogonality of Random Measure.

Using the orthogonality property $\mathbb{E}[d\Phi(\lambda) \overline{d\Phi(\mu)}] = f(\lambda) \delta(\lambda - \mu) d\lambda d\mu$:

$$\mathbb{E}[\hat{K}_T(t, s)] = \frac{1}{2T} \int_{-T}^T \int_{-\infty}^{\infty} A(t+u, \lambda) \overline{A(s+u, \lambda)} f(\lambda) d\lambda du \quad (6)$$

$$= \int_{-\infty}^{\infty} f(\lambda) \left[\frac{1}{2T} \int_{-T}^T A(t+u, \lambda) \overline{A(s+u, \lambda)} du \right] d\lambda. \quad (7)$$

Step 5: Ergodicity and Convergence.

For oscillatory processes arising from monotonic time changes of stationary processes, the amplitude function $A(t, \lambda) = e^{i\lambda\theta(t)} \sqrt{\dot{\theta}(t)}$ ensures that the process is ergodic. By the ergodic theorem:

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T A(t+u, \lambda) \overline{A(s+u, \lambda)} du = A(t, \lambda) \overline{A(s, \lambda)}$$

almost surely.

Step 6: Dominated Convergence.

Under the assumption of mean-square continuity and finite spectral measure, the dominated convergence theorem applies:

$$\lim_{T \rightarrow \infty} \hat{K}_T(t, s) = \int_{-\infty}^{\infty} A(t, \lambda) \overline{A(s, \lambda)} f(\lambda) d\lambda = K(t, s)$$

almost surely. □

5. Expected Zero Count

If K_0 is twice differentiable and $-\ddot{K}_0(0) > 0$, the expected number of zeros of X in $[0, T]$ is

$$\mathbb{E}[N([0, T])] = \sqrt{-\ddot{K}_0(0)} (\theta(T) - \theta(0)).$$

6. Summary of Inversion Chain

The complete inversion chain for recovering the underlying white noise from an oscillatory sample path is:

$$\text{Oscillatory Process } X(t) \xrightarrow{\text{Inverse Time Change}} \text{Stationary Process } Y(u) \quad (8)$$

$$Y(u) \xrightarrow{\text{Spectral Inversion}} \text{Random Measure } d\Phi(\lambda) \quad (9)$$

$$d\Phi(\lambda) \xrightarrow{\text{Spectral Density Normalization}} \text{White Noise } dW(\lambda) \quad (10)$$

Each step is mathematically rigorous and involves well-defined limiting procedures. The existence of the spectral representation guarantees convergence at each stage.

7. Conclusion

The complete framework establishes a bijective correspondence between stationary Gaussian processes and oscillatory processes via monotonic time changes. All transformations are invertible with explicit formulas, and the underlying white noise can be recovered from oscillatory sample paths through a well-defined chain of operations involving inverse time changes, spectral inversions, and spectral density normalizations.

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Your current paper title, "Monotonic Time Changes of Stationary Gaussian Processes Yield Oscillatory Processes: Complete Framework," accurately reflects the main construction but does not emphasize the precise **existence and role of the unitary bijection** at the heart of your construction. Based on our discussion and standard terminology in ergodic theory and Hilbert space analysis, here is a detailed critique and improvement path:

1. Paper Title: How to Emphasize the Unitary Bijection

A title that foregrounds the algebraic and analytic structure you derive would make the **existence and function of the unitary bijection** much more explicit. Suitable alternatives might be:

- "A Unitary Equivalence Between Stationary and Oscillatory Gaussian Processes via Monotonic Time Change"
- "Unitary Bijections Induced by Monotonic Time Changes for Stationary Gaussian Processes"
- "Monotonic Time Change as Unitary Isomorphism between Stationary and Oscillatory Gaussian Processes"

This focuses directly on the **unitary equivalence** (i.e., existence of a unitary, bijective, structure-preserving map) between the two process classes, rather than only the forward yield or the descriptive framework.

2. Making the Existence of the Unitary Bijection Apparent in the Text

To highlight the unitary bijection explicitly throughout the paper, consider the following modifications and structural improvements:

- **Explicitly Define the Unitary Map as an Operator:**
Introduce a definition, e.g.,

Let $U_\theta: L^2(\mathbb{R}, du) \rightarrow L^2(\mathbb{R}, dt)$ be defined by $(U_\theta f)(t) := f(\theta(t)) \sqrt{\dot{\theta}(t)}$. Then U_θ is a unitary operator between these Hilbert spaces whenever θ is strictly increasing and smooth, with the inverse $(U_\theta^{-1}g)(u) = g(\theta^{-1}(u)) / \sqrt{\dot{\theta}(\theta^{-1}(u))}$.

- **State a Theorem About Unitarity Explicitly:**

Add a theorem:

Theorem (Unitary Correspondence):

The transformation U_θ defined above is a linear bijection and preserves the L^2 inner product:

$$\int_{\mathbb{R}} X(t) \overline{X'(t)} dt = \int_{\mathbb{R}} Y(u) \overline{Y'(u)} du.$$

Thus, U_θ is a unitary isomorphism between the spaces of sample paths of stationary and oscillatory Gaussian processes.

(Include proof using the change of variables $u = \theta(t)$, $du = \dot{\theta}(t)dt$).

- **Clarify Bidirectionality in Operator Terms:**

In the inversion formulas section, clarify that the relationships between X and Y are not just invertible but are explicitly described by the unitary operator and its inverse.

- **Terminology:**

Use precise terms like "**unitary isomorphism**," "**unitary equivalence**," or "**unitary bijection**" throughout when referring to this mapping. In operator-theoretic and ergodic context, a structure-preserving, invertible, linear map on Hilbert space that preserves the inner product is exactly a unitary operator^{[1] [2]}.

- **Abstract and Conclusion:**

Revise sentences so that rather than describing a bijective correspondence, you specify:

"... establishes a unitary isomorphism between stationary and oscillatory Gaussian processes via monotonic time change."

and

"All constructions are given as explicit unitary bijections with concrete inversion formulas."

3. Contextual Support and Standard Terminology

In ergodic theory and Hilbert space operator theory, **unitary isomorphism** (or simply "unitary operator" when the domain and codomain are understood) is precisely the terminology used to denote this structure: a bijective, inner-product-preserving map^{[1] [3] [2]}. By centering this concept, you align your exposition with standard mathematical nomenclature and make the key analytic property (unitarity) front and center.

4. Summary of Actionable Edits

- Change the title to directly mention unitary equivalence or unitary isomorphism.
- Add a clear, formal definition and theorem early in the paper explicitly constructing and proving unitarity of the operator.

- Rephrase "bidirectional invertible formulas," "bijective correspondence," etc., as "**unitary isomorphism**" or "**unitary bijection**."
- Tie this framework to relevant operator-theoretic/ergodic-theoretic language in the abstract and summary.

References to Standard Language:

- A unitary operator is *by definition* a bijective, inner-product-preserving linear transformation between Hilbert spaces^{[1] [3]}.
- In ergodic theory, invertible measure-preserving maps give rise to unitary Koopman operators on function spaces^{[1] [2] [4]}.

Example Modified Title and Abstract (for illustration):

Title:

"Unitary Isomorphisms Induced by Monotonic Time Changes for Stationary Gaussian Processes"

Abstract (first sentence revision):

"We establish an explicit unitary isomorphism between the Hilbert spaces of stationary and oscillatory Gaussian processes, realized via monotonic time changes and their associated Jacobian scaling."

By taking these steps, the central analytic/theoretical structure—the existence and importance of the unitary bijection—becomes immediately clear to skilled readers, connects to standard mathematical language, and positions your work within the broader context of operator algebra and ergodic theory.



1. <https://people.math.harvard.edu/~ctm/papers/home/text/class/notes/ergodic/course.pdf>
2. <https://citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=d35ad2dfd1c7bd9de5a14cd92d0b8975aceef261>
3. <https://math.vanderbilt.edu/peters10/teaching/Spring2011/ErgodicTheoryNotes.pdf>
4. <https://people.tamu.edu/~yvvorobets/MATH614-2014A/Lect4-02web.pdf>