Orthonormal Galerkin Method for Stationary Integral Covariance Operator Eigenfunction Expansions

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1 Given

1. $K(s,t) = K(t-s)$
2. $K(t-s) = \sum_{n=0}^{\infty} \psi_n(t-s)$ (uniformly convergent)
3. Eigenvalue equation: $\int_{-\infty}^{\infty} K(t-s)\, \phi_k(t)\, d t = \lambda_k\, \phi_k(s)$
4. Eigenfunction expansion: $\phi_k(t) = \sum_{n=0}^{\infty} c_{n,k}\, \psi_n(t)$
5. The basis functions $\{\psi_n\}$ are orthonormal, i.e., $\int_{-\infty}^{\infty} \psi_m(s)\, \psi_n(s)\, d s = \delta_{mn}$

2 Objective

The goal is to solve for the coefficient matrix $c_{n,k}$ of the eigenfunctions

$$T\phi_k(s) = \lambda_k\, \phi_k(s) \tag{1}$$

of the integral covariance operator

$$T f(s) = \int_{-\infty}^{\infty} K(t-s)\, f(t)\, d t \tag{2}$$

3 Proof

1. The eigenfunction expansion is substituted into the eigenvalue equation:

$$\int_{-\infty}^{\infty} K\,(t-s) \sum_{n=0}^{\infty} c_{n,k}\,\psi_n(t)\ d\,t = \lambda_k \sum_{n=0}^{\infty} c_{n,k}\,\psi_n(s) \tag{3}$$

2. Using the uniform expansion of K:

$$\int_{-\infty}^{\infty} \sum_{j=0}^{\infty} \psi_j\,(t-s) \sum_{n=0}^{\infty} c_{n,k}\,\psi_n(t)\ d\,t = \lambda_k \sum_{n=0}^{\infty} c_{n,k}\,\psi_n(s) \tag{4}$$

3. Applying Fubini's theorem (justified by uniform convergence):

$$\sum_{j=0}^{\infty} \sum_{n=0}^{\infty} c_{n,k} \int_{-\infty}^{\infty} \psi_j\,(t-s)\,\psi_n(t)\ d\,t = \lambda_k \sum_{n=0}^{\infty} c_{n,k}\,\psi_n(s) \tag{5}$$

4. Let $G_{j,n}(s) = \int_{-\infty}^{\infty} \psi_j\,(t-s)\,\psi_n(t)\ d\,t$:

$$\sum_{n=0}^{\infty} c_{n,k} \sum_{j=0}^{\infty} G_{j,n}(s) = \lambda_k \sum_{n=0}^{\infty} c_{n,k}\,\psi_n(s) \tag{6}$$

5. Projecting onto the basis $\{\psi_m(s)\}$ by multiplying both sides by $\psi_m(s)$ and integrating over s:

$$\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} c_{n,k} \sum_{j=0}^{\infty} G_{j,n}(s)\,\psi_m(s)\ d\,s = \lambda_k \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} c_{n,k}\,\psi_n(s)\,\psi_m(s)\ d\,s \tag{7}$$

6. Interchanging summation and integration:

$$\sum_{n=0}^{\infty} c_{n,k} \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} G_{j,n}(s)\,\psi_m(s)\ d\,s = \lambda_k \sum_{n=0}^{\infty} c_{n,k} \int_{-\infty}^{\infty} \psi_n(s)\,\psi_m(s)\ d\,s \tag{8}$$

7. The right-hand side simplifies to $\lambda_k\,c_{m,k}$ by orthonormality of $\{\psi_n\}$. Define:

$$b_{m,n} = \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} G_{j,n}(s)\,\psi_m(s)\ d\,s \tag{9}$$

8. The equation becomes:

$$\sum_{n=0}^{\infty} b_{m,n}\, c_{n,k} = \lambda_k\, c_{m,k} \tag{10}$$

9. This reduces to a standard eigenvalue problem:

$$B\,\vec{c}_k = \lambda_k\,\vec{c}_k \tag{11}$$

where $B = (b_{m,n})$ and $\vec{c}_k = (c_{0,k}, c_{1,k}, \ldots)^T$

4 Verification that Solutions are Eigenfunctions

A verification that the solutions obtained are indeed eigenfunctions of the original integral equation follows:

1. Let λ_k and $\vec{c}_k = (c_{0,k}, c_{1,k}, \ldots)^T$ be the eigenvalues and eigenvectors of the matrix equation:

$$B\,\vec{c}_k = \lambda_k\,\vec{c}_k \tag{12}$$

where $B = (b_{m,n})$ as derived above.

2. The functions $\phi_k(t)$ are constructed as:

$$\phi_k(t) = \sum_{n=0}^{\infty} c_{n,k}\,\psi_n(t) \tag{13}$$

3. Substituting into the original integral equation:

$$\int_{-\infty}^{\infty} K\,(t-s)\,\phi_k(t)\,d\,t = \int_{-\infty}^{\infty} K\,(t-s)\left[\sum_{n=0}^{\infty} c_{n,k}\,\psi_n(t)\right] d\,t \tag{14}$$

4. Using the expansion of $K\,(t-s)$ and interchanging summations:

$$= \int_{-\infty}^{\infty}\left[\sum_{j=0}^{\infty} \psi_j\,(t-s)\right]\left[\sum_{n=0}^{\infty} c_{n,k}\,\psi_n(t)\right] d\,t \tag{15}$$

$$= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} c_{n,k} \int_{-\infty}^{\infty} \psi_j\,(t-s)\,\psi_n(t)\,d\,t \tag{16}$$

5. Recalling the definitions:

$$\begin{aligned} G_{j,n}(s) &= \int_{-\infty}^{\infty} \psi_j\,(t-s)\,\psi_n(t)\;d\,t \\ b_{m,n} &= \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} G_{j,n}(s)\,\psi_m(s)\;d\,s \end{aligned} \tag{17}$$

6. The left-hand side of the integral equation can be rewritten:

$$\begin{aligned} T\phi_k(s) &= \int_{-\infty}^{\infty} K\,(t-s)\,\phi_k(t)\;d\,t \\ &= \int_{-\infty}^{\infty} K\,(t-s) \left[\sum_{n=0}^{\infty} c_{n,k}\,\psi_n(t) \right] d\,t \\ &= \sum_{n=0}^{\infty} c_{n,k} \int_{-\infty}^{\infty} K\,(t-s)\,\psi_n(t)\;d\,t \\ &= \sum_{n=0}^{\infty} c_{n,k} \left[\sum_{j=0}^{\infty} \int_{-\infty}^{\infty} \psi_j\,(t-s)\,\psi_n(t)\;d\,t \right] \\ &= \sum_{n=0}^{\infty} c_{n,k} \left[\sum_{j=0}^{\infty} G_{j,n}(s) \right] \end{aligned} \tag{18}$$

7. Projecting onto $\psi_m(s)$ by multiplying by $\psi_m(s)$ and integrating over s:

$$\begin{aligned} \int_{-\infty}^{\infty} \psi_m(s)\,T\phi_k(s)\;d\,s &= \int_{-\infty}^{\infty} \psi_m(s) \left[\sum_{n=0}^{\infty} c_{n,k} \left[\sum_{j=0}^{\infty} G_{j,n}(s) \right] \right] d\,s \\ &= \sum_{n=0}^{\infty} c_{n,k} \left[\sum_{j=0}^{\infty} \int_{-\infty}^{\infty} G_{j,n}(s)\,\psi_m(s)\;d\,s \right] \\ &= \sum_{n=0}^{\infty} c_{n,k}\,b_{m,n} \\ &= (B\,\vec{c}_k)_m \\ &= \lambda_k (\vec{c}_k)_m \\ &= \lambda_k\,c_{m,k} \end{aligned} \tag{19}$$

8. Since this holds for all m, and $\{\psi_m\}$ is a complete orthonormal basis, the conclusion follows:

$$\int_{-\infty}^{\infty} K\,(t-s)\,\phi_k(t)\,d\,t = \lambda_k\,\phi_k(s) \tag{20}$$

Therefore, the $\phi_k(s)$ constructed from the eigenvectors of B are indeed eigenfunctions of the original integral equation with eigenvalues λ_k.