# The Lemniscate Constants

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**The lemniscate constants, and indeed some of the methods used for actually computing them, have played an enormous part in the development of mathematics. An account is given here of some of the methods used—most of the derivations can be made by elementary methods. This material can be used for teaching purposes, and there is much relevant and interesting historical material. The acceleration methods developed for the purpose of evaluating these constants are useful in other problems.**

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## 1. Introduction

The lemniscate constants are, in standard notation,

$$A = \int_0^1 \frac{dt}{\sqrt{1 - t^4}} = \frac{1}{\sqrt{2}} K\left(\frac{1}{2}\right) = \left(\frac{1}{4}\right)(2\pi)^{-\frac{1}{2}}\left(\Gamma\left(\frac{1}{4}\right)\right)^2,$$

$$B = \int_0^1 \frac{t^2 dt}{\sqrt{1 - t^4}} = (2\pi)^{-\frac{1}{2}}\left(\Gamma\left(\frac{3}{4}\right)\right)^2.$$

THEOREM 1. *The length of a quadrant of the lemniscate of Bernoulli, $r^2 = \cos 2\theta$, is $2A$.*

PROOF. Trivial.

THEOREM 2. $AB = \frac{1}{4}\pi$.

An elementary proof, based essentially on Wallis' formula, is due to Euler [10]. This result was also established by Landen in 1755. It is also evident from the Γ-function representations given above.

THEOREM 3. *$A$ is a transcendental number.*

THEOREM 4. *$B$ is a transcendental number.*

These theorems were proved by Theodore Schneider in 1937 and 1941 respectively. (See Siegel [35, 36].)

## 2. Computation by Integration of Power Series

The obvious approach to the computation of $A$ is to expand the integrand by the binomial theorem and to integrate term by term.

THEOREM 5.

$$A \equiv \sum a_n \equiv 1 + \sum_1^\infty \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{4n+1}.$$

The representation of $A$ just given is of no practical use for computation. In fact, by the use of Wallis' formula, or Stirling's approximation for $n!$, we find that $a_n \simeq (4\pi^{\frac{1}{2}}n^{\frac{3}{2}})^{-1}$ which implies that the remainder after $n$ terms is $O(n^{-\frac{1}{2}})$.

We can obtain an alternating series for $A$ by writing the integrand as $(1 - t^2)^{-\frac{1}{2}}(1 + t^2)^{-\frac{1}{2}}$ and expanding the second factor by the binomial theorem and then integrating term by term using the fact that

$$\int_0^1 t^{2n}(1 - t^2)^{-\frac{1}{2}}\, dt = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2}. \tag{2.1}$$

THEOREM 6.

$$A \equiv \sum b_n \equiv \frac{\pi}{2}\left\{1 + \sum_1^\infty (-1)^n \left[\frac{(2n-1)!!}{(2n)!!}\right]^2\right\}.$$

The representation just given is of little practical use in computation since $|b_n| \simeq (2n)^{-1}$.

In view of the slow convergence of the series in Theorems 5 and 6, the question of acceleration arises. This question has been studied for a long time: among those who have contributed recently are J.R. Airey [1], W.G. Bickley and J.C.P. Miller [4], K. Nickel and D. Shanks [33], H.C. Thacher, Jr. [39], and P. Wynn [50], who has provided ALGOL algorithms in many cases. We discuss here only the Euler transform and the results of Stirling and A.A. Markoff.

### Euler

It is well known that the Euler transformation is quite effective in accelerating the convergence of series such as $\sum b_n$. It is known (Knopp [24]) that if the sequence $(-1)^n b_n$ is completely monotone and

$$(b_{n+1}/b_n) \sim 1,$$

then the Euler transform will be convergent like a geometric series with common ratio 1/2. This makes computation feasible, and things can be improved by an appropriate delay in starting the transform. These matters are discussed by Dahlquist, et al. [7] and Todd [40].

The fact that the sequence $\{b_n\}$ is completely monotone can be established as follows. Since (Knopp [24]) the term-by-term product of two completely monotone sequences is completely monotone, it is enough to show that $\{b_n'\}$ is completely monotone where

$$b_n' \equiv \frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots (2n)} \equiv \frac{(2n-1)!!}{(2n)!!}.$$

Putting $t^2 = T$ in (2.1), we find

$$b_n' = (1/\pi)\int_0^1 T^n[T(1-T)]^{-\frac{1}{2}}\,dT,$$

exhibiting $b_n'$ as the $n$th moment of the distribution $[T(1-T)]^{-\frac{1}{2}}$; it follows (Widder [47, p. 108]) that $\{b_n'\}$ is completely monotone.

### Stirling

The difficulty in using a power series directly was clear to Stirling, and he devised an acceleration method in 1730 which led to the value

$A = 1.31102\,87771\,46059\,87$

(which is correct to 15D) by dealing with a head of nine terms of the series of Theorem 6 and transforming the next nine terms. We shall explain this briefly.

Stirling discusses the general Beta-function

$$B(\nu, \mu) = \int_0^1 t^{\nu-1}(1-t)^{\mu-1}\,dt$$

which reduces to $4A$ when $\mu = \frac{1}{2}$, $\nu = \frac{1}{4}$. He expands the factor $(1-t)^{\mu-1}$ as a binomial series, then integrates term by term, and separates the resulting series into a head and tail

$S = [u(1) + \cdots + u(n-1)] + [u(n) + \cdots]$

which he writes as $S = S(n-1) + C(n)u(n)$ where $C(n)$ is a "converging factor" which indicates the ap~~propriate modification of the nth term~~ which gives a correct truncation.

Stirling proposes to find an *approximation* to $C(n)$ in the form $C(n) \doteqdot (n+p)/q$ where $p,q$ are independent of $n$ so that, accurately,

$S = S(n-1) + \{(n+p)u(n)/q\} + \mathrm{R}(n+1)$.

If we write $\sigma(n+1) = \mathrm{R}(n+1) - \mathrm{R}(n+2)$, then

$$\mathrm{R}(n+1) = \sigma(n+1) + \sigma(n+2) + \cdots \qquad (2.2)$$

and we want $\sigma$ to be small. We can express $\sigma/u$ as a ratio of two quadratics in $n$, and Stirling chooses $p,q$ to make the $n^2,n$ terms in the numerator vanish,

$q = \mu$, $p = (\nu - 1)/(1+\mu)$, so that the first improvement is to give

$$S \doteqdot S(n-1) + \frac{n(\mu+1) - 1(1-\nu)}{\mu(\mu+1)}\,u(n).$$

Stirling then repeats this process on the series (2.2)—the $\sigma$'s satisfy a recurrence relation similar to that for the $u$'s. He then finds the next improvement in the form

$$S \doteqdot S(n-1) + \frac{n(\mu+1) - 1(1-\nu)}{\mu(\mu+1)}\,u(n) + \frac{(n+2)(\mu+3) - 2(2-\nu)}{(\mu+2)(\mu+3)}\,\sigma(n+1).$$

This process can be repeated indefinitely and the convergence of the various series, e.g. (2.2), is clear.

Stirling also found $B$ in the same way:

$B = .59907\,01173\,67796\,11$.

### A.A. Markoff

Stirling's device is essentially a transformation of ordinary hypergeometric series $F(\alpha,\beta,\gamma,x)$ applied to the series $\Sigma a_n$. Markoff [27] works on the alternating series $\Sigma b_n$ and apparently uses transformations of generalized hypergeometric series. His virtuosity in these matters was well known. Hermite wrote in 1889 (see Ogigova [30]), "Par quelle voie vous êtes parvenu à une telle transformation, je ne puis même de loin l'entrevoir, et il me faut vous laisser votre secret."

We discuss here one of the two methods given by Markoff. He begins with the remark that if

$$F(a,b) = C\left[2a+b+\alpha-1 + \frac{(2a+b+\alpha+1)\varphi(a)}{\varphi(-a-b-\alpha)} + \frac{(2a+b+\alpha+3)\varphi(a)\varphi(a+1)}{\varphi(-a-b-\alpha)\varphi(-a-b-\alpha-1)} + \cdots\right]$$

when $\varphi(t) = t^3 + \alpha t^2 + \beta t + \gamma$, with arbitrary $C$, $a$, $b$, $\alpha$, $\beta$, $\gamma$, then $F(a,b) = F(b,a)$. In this he first puts

$$b = \tfrac{1}{2}(\delta+1),\ \alpha = \tfrac{1}{2}(\delta-1),$$

$$\varphi(t) = (t + \tfrac{1}{2}(\delta-1))t^2, \text{ and } C = D/(2a+\delta-1).$$

This gives

$$F(a,b) = D\left[1 - \left(\frac{a}{a+\delta}\right)^2 + \left(\frac{a}{a+\delta}\cdot\frac{a+1}{a+\delta+1}\right)^2 - \left(\frac{a}{a+\delta}\cdot\frac{a+1}{a+\delta+1}\cdot\frac{a+2}{a+\delta+2}\right)^2 + \cdots\right];$$

$$F(b,a) = (2a+3\delta-1)E_0 - (2a+3\delta+3)E_1 + (2a+3\delta+7)E_2 - (2a+3\delta+11)E_3 + \cdots,$$

where $E_0 = D/(4a + 2\delta - 2)$,

$$E_1 = \frac{\delta(\delta+1)^2}{2(2a+\delta+1)(a+\delta)^2} E_0,$$

and, generally,

$$E_{n+1} = \frac{(n+\delta)(2n+\delta+1)^2}{2(2a+2n+\delta+1)(a+n+\delta)^2} E_n.$$

He then puts $a = 23/2$, $\delta = 1/2$,

$$D = \left\{\frac{1\cdot3\cdot5\cdot7\cdots21}{2\cdot4\cdot6\cdot8\cdots22}\right\}^2$$

$$= 0.02828\ 72353\ 30935\ 80044\ 80600\ldots$$

and sums the first 13 terms of the $E$-series to 0.0147694731 6168476217 682. . . . Subtracting this from the head

$$1 - \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1\cdot3\cdot5\cdots19}{2\cdot4\cdot6\cdots20}\right)^2$$

$$= \sum_{n=0}^{11}(-1)^n\left(\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}\right)^2$$

$$= 0.8493963148\ 3575794845\ 819\ldots$$

he finds

$$\frac{2}{\pi}A = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots(2n)}\right)^2$$

$$= 0.8346268416\ 7407318628\ 1$$

correct to 21 places.

The only proofs of the basic relation $F(a, b) = F(b, a)$ known to me are one given by H.M. Srivastava—it uses a transformation of a ${}_6F_5$ due to F.J.W. Whipple (cf *SIAM Rev.* (1974), p. 260)—and one due to R. A. Askey—this uses a transformation of a ${}_9F_8$ due to W. N. Bailey.

## 3. The Arithmetic-Geometric Mean Calculations of Gauss and the Carlson Sequences

As a teenager, Gauss experimented numerically with the algorithm

$a_0 = a$, $b_0 = b$, $a \geq b > 0$,

$a_{n+1} = \frac{1}{2}(a_n + b_n)$, $b_{n+1} = (a_nb_n)^{\frac{1}{2}}$, $n = 0, 1, 2, \ldots$.

It is convenient to write $c_n = (a_n^2 - b_n^2)^{\frac{1}{2}}$. The sequences $\{a_n\}$, $\{b_n\}$ converge monotonically and quadratically to a limit which we denote by $M(a, b)$. It was clear that Gauss was interested in finding the form of the function $M(a, b)$. Among his computations is the following for $M(2^{\frac{1}{2}}, 1)$:

| $n$ | $a_n$ | $b_n$ |
|---|---|---|
| 0 | 1.41421 35623 73095 04880 2 | 1.00000 00000 00000 00000 0 |
| 1 | 1.20710 67811 86547 52440 1 | 1.18920 71150 02721 06671 7 |
| 2 | 1.19815 69480 94634 29555 9 | 1.19812 35214 03120 12260 7 |
| 3 | 1.19814 02347 93877 20908 3 | 1.19814 02346 77307 20579 8 |
| 4 | 1.19814 02347 35592 20744 1 | 1.19814 02347 35592 20743 9 |

In his Notebook [19, p. 542], on May 30, 1799, Gauss observed that $\pi/(2M(2^{\frac{1}{2}}, 1))$ coincided to 11D with $A$, and wrote, "Terminum medium arithmetico-geometricum inter 1 et $\sqrt{2}$ esse $= \pi/\omega$ usque ad figuram undeciman comprobavimus, qua re demonstrata prorsus novus campus in analysis certo aperietur."

Much of the work of Gauss in this area was never published by him; it was edited from his papers after his death, for his Collected Works, by various mathematicians including Schering, Klein, Fricke, Schlesinger. See Geppert [21]. It was not until several months later, on December 23, 1799, that Gauss established the relation between $M$ and an elliptic integral.

THEOREM 7. *If* $0 \leq k \leq 1$, *then*

$$\frac{\pi}{2M(1,(1-k^2)^{\frac{1}{2}})} = \int_0^{\pi/2}\frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{1}{2}}} = \int_0^1\frac{dt}{((1-t^2)(1-k^2t^2))^{\frac{1}{2}}} = K(k^2).$$

In his Notebook [19, p. 544] he wrote, "Medium Arithmetico-Geometricum ipsum est quantitas integralis. Dem[onstratum]."

There seems to be no easy proof of this theorem. All available use the following idea. In the usual notation,

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(a_n^2\cos^2\theta + b_n^2\sin^2\theta)^{\frac{1}{2}}} = \int_0^{\frac{1}{2}\pi}\frac{d\varphi}{(a_{n+1}^2\cos^2\varphi + b_{n+1}^2\sin^2\varphi)^{\frac{1}{2}}} \tag{3.1}$$

and so, passing to the limit,

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(a_0^2\cos^2\theta + b_0^2\sin^2\theta)^{\frac{1}{2}}} = \int_0^{\frac{1}{2}\pi}\frac{d\varphi}{M(a_0, b_0)},$$

which is the result required. All depends on the proof of (3.1), which can be done by the transformation

$$\sin\theta = \frac{2a_n}{(a_n + b_n)\operatorname{cosec}\varphi + (a_n - b_n)\sin\varphi}.$$

For arecent exposition of this material see Fuchs [51].

THEOREM 8. *If* $x_0 = 1$, $y_0 = 0$, *and if for* $n \geq 0$, $x_{n+1} = \frac{1}{2}(x_n + y_n)$, $y_{n+1} = (\frac{1}{2}x_n(x_n + y_n))^{\frac{1}{2}}$, *then* $\lim x_n = \lim y_n = A^{-2}$. *Further, convergence is geometric with multiplier* $\frac{1}{4}$.

PROOF. It is easy to check using the transformation $x = (1 + T)^{-\frac{1}{4}}$ that

$$A = \int_0^1(1 - x^4)^{-\frac{1}{2}}\,dx = \frac{1}{4}\int_0^\infty(1 + T)^{-\frac{3}{4}}T^{-\frac{1}{2}}\,dT. \tag{3.2}$$

Consider the transformation of the integral

$$I(x^2, y^2) = \int_0^\infty(t + x^2)^{-\frac{3}{4}}(t + y^2)^{-\frac{1}{2}}\,dt$$

by changing the variable from $t$ to $s$ where $t = s(s + xy)/[s + (1/2)(x + y)^2]$. A little algebra gives

$$I(x_0^2, y_0^2) = I(x_1^2, y_1^2). \tag{3.3}$$

The existence of the common limit of $l(x_0, y_0)$ of the sequences $\{x_n\}$, $\{y_n\}$ and the rate of convergence is easy to check. Repeated use of (3.3) gives

$$I(x_0^2, y_0^2) = \int_0^\infty(t + l^2)^{-\frac{5}{4}}\,dt = 4[l(x_0, y_0)]^{-\frac{1}{2}}.$$

From (3.2), taking $x_0 = 1$, $y_0 = 0$, we find $A = [l(1,0)]^{-\frac{1}{2}}$ as required.

Theorems 7 and 8 and many related results were obtained by a uniform method by B.C. Carlson [6].

## 4. Arcsl Relations

Since

$$\int_0^x \frac{dt}{(1-t^2)^{\frac{1}{2}}} = \arcsin x, \ \arcsin 1 = \tfrac{1}{2}\pi,$$

it is natural to introduce lemniscate functions defined in the first place by

$$\int_0^x \frac{dt}{(1-t^4)^{\frac{1}{2}}} = \operatorname{arcsl} x, \ \operatorname{arcsl} 1 = A,$$

$\operatorname{arccl} x + \operatorname{arcsl} x = A$.

There is no definitive account of the lemniscate functions. A Cambridge Tract announced by G.B. Matthews never appeared; there is some account of the theory in various books such as Markushevich [28], C.L. Siegel [35]. The explicit expressions in terms of the Jacobian functions with modulus $k^2 = \frac{1}{2}$ are

$\operatorname{sl} x = 2^{-\frac{1}{2}} \operatorname{sn}(2^{\frac{1}{2}} x)/\operatorname{dn}(2^{\frac{1}{2}} x)$,
$\operatorname{cl} x = \operatorname{cn}(2^{\frac{1}{2}} x)$.

It is clear that if we could obtain a relation of the form

$$A = \sum_{i=1}^{n} a_i \operatorname{arcsl} x_i, \tag{4.1}$$

where the $x_i$ satisfy, e.g. $|x_i| \le \frac{1}{2}$, then we would have a feasible method for evaluating $A$ by

$\operatorname{arcsl} x = x + \frac{1}{10} x^5 + \frac{1}{24} x^9 + \cdots$

since this will mean we are working with series with effective ratio $2^{-4} = .0625$. This is similar to the classical ways of evaluating $\pi$, e.g. using

$$\tfrac{1}{4}\pi = 4 \operatorname{arccot} 5 - \operatorname{arccot} 239. \tag{4.2}$$

Gauss gave the following result:

THEOREM 9. $A = \mathit{arcsl}\, \frac{7}{23} + 2\, \mathit{arcsl}\, \frac{1}{2}$.

PROOF. This result can be written as

$$2\int_0^{\frac{1}{2}} \frac{dt}{(1-t^4)^{\frac{1}{2}}} = \int_{\frac{7}{23}}^1 \frac{dT}{(1-T^4)^{\frac{1}{2}}},$$

which can be established by using the transformation $T = (1 - 2t^2 - t^4)/(1 + 2t^2 - t^4)$.

The effective ratio of terms in the series is $2^{-4} = .0625$.

A somewhat similar result was given much earlier (1714) by Fagnano [13], who discussed the bisection of the arc of the lemniscate.

THEOREM 10. $A = 2\, \mathit{arcsl}(2^{\frac{1}{2}} - 1)^{\frac{1}{2}}$.

PROOF. This result is equivalent to

$$\int_0^1 \frac{dt}{(1-t^4)^{\frac{1}{2}}} = 2\int_0^{(2^{\frac{1}{2}}-1)^{\frac{1}{2}}} \frac{d\tau}{(1-\tau^4)^{\frac{1}{2}}},$$

and can be established by the transformation $t = 2\,\tau(1-\tau^4)^{\frac{1}{2}}/(1+\tau^4)$.

The trigonometric analog of this theorem is: $\frac{1}{2}\pi = 2 \arcsin 2^{-\frac{1}{2}}$.

The effective ratio of the terms in the series is $(2^{\frac{1}{2}} - 1)^2 \doteqdot .1716$, so a computation of $A$ via Theorem 10 is somewhat less convenient than that via Theorem 9. However, Fagnano showed that it is possible to divide the arc of the lemniscate into three or five parts, again at the expense of solving quadratics only; these results also lead to convenient methods of computation of $A$.

THEOREM 11. $A = 3\, \mathit{arcsl}\{\frac{1}{2}[1 + \sqrt{3} - \sqrt{(2\sqrt{3})}]\}$.

This can be established using the transformation employed in Theorem 9. The trigonometric analog of this theorem is: $\frac{1}{2}\pi = 3 \arcsin 3^{-\frac{1}{2}}$. The effective ratio of terms in the series is $(.43542)^4 = .03594$.

THEOREM 12.

$$A = 5\, \mathit{arcsl}\left[\frac{1 - \{(6\sqrt{5}-13) + (340 - 152\sqrt{5})^{\frac{1}{2}}\}^{\frac{1}{2}}}{1 + \{(6\sqrt{5}-13) + (340 - 152\sqrt{5})^{\frac{1}{2}}\}^{\frac{1}{2}}}\right]^{\frac{1}{2}}.$$

PROOF. Use the fact established by Gauss:

$$\operatorname{sl} 5\varphi = \frac{s(5 - 2s^4 + s^8)(1 - 12s^4 - 26s^8 + 52s^{12} + s^{16})}{(1 - 2s^4 + 5s^8)(1 + 52s^4 - 26s^8 + 12s^{12} + s^{16})},$$

where $s = \operatorname{sl} \varphi$, and ideas of Fagnano and Watson.

The trigonometric analog of this theorem is: $\frac{1}{2}\pi = 5 \arcsin (\sqrt{5} - 1)/4$. The effective ratio of terms in the series is $\operatorname{sl}^4 \frac{1}{5} A \doteqdot .0047$.

Using the lemniscate trigonometry it is natural to investigate relations of the form (4.1) in the same way as Lehmer [26] has evaluated arctan relations typified by (4.2) as an efficient means of calculating $\frac{1}{4}\pi$. See also Størmer [38], Todd [41, 42]. For instance, the arcsl relation $A = 2 \operatorname{arcsl} \frac{1}{3} + \operatorname{arcsl} (31/49)$ is less efficient than that given in Theorem 9.

## 5. Quadrature Formulas

Another obvious approach to the evaluation of $A$ is to use appropriate quadrature formulas.

We mention here briefly the Gauss-Chebyshev quadrature [40]:

$$A = \frac{1}{2}\int_{-1}^{1} \frac{1}{(1+t^2)^{\frac{1}{2}}} \cdot \frac{dt}{(1-t^2)^{\frac{1}{2}}}$$
$$\doteqdot Q_n = .5 \sum_{r=1}^{n} \frac{(\pi/n)}{(1 + \cos^2((2n-1)\pi/2n))^{\frac{1}{2}}}.$$

It is awkward to estimate the error here, but there are some possibilities (see e.g. G. Freud [17]).

In view of the singularity at $t = 1$, Romberg methods are not directly applicable. Appropriate modifications have been developed by L. Fox [15, 16] and J.A. Shanks [34].

## 6. Theta Series

This seems to be the most powerful method. Formulas involving $\vartheta$-functions have proved effective in many computational problems, e.g. that of Ewald [12] on crystal structure, and especially in calculations involving elliptic functions and integrals.

The idea is that the Jacobian elliptic functions, sn, cn, dn, and the complete elliptic integrals, $K$, $K'$, for a given modulus $k^2$ can be expressed in terms of $\vartheta$-functions with $q$ determined by the equation
$q = \exp(-\pi K'/K)$.

In the lemniscate case, $k^2 = \frac{1}{2} = k'^2$ and so $K = K'$ (the period parallelogram is a square), which implies that $q = e^{-\pi} = .04321\ldots$.

We shall give an outline of a direct proof of our representations rather than relying on the general theory of elliptic functions.

Enlightened experimentation (cf. von Dávid [8]) suggests that we consider the series
$\alpha(x) = 1 + 2x + 2x^4 + 2x^9 + \cdots,$
$\beta(x) = 1 - 2x + 2x^4 - 2x^9 + \cdots.$
The following two results are needed.

THEOREM 13. *If* $|x| < 1$, *then*
$[\alpha(x)]^2 + [\beta(x)]^2 = 2[\alpha(x^2)]^2$, $\alpha(x)\beta(x) = [\beta(x^2)]^2$.

THEOREM 14. *For any a,b such that* $0 \le b \le a$ *there is an x,* $|x| < 1$, *such that* $\alpha(x)/\beta(x) = (a/b)^{\frac{1}{2}}$. *Indeed* $x = exp(-\pi M(a_0, b_0)/M(a_0, c_0))$. *In particular, for* $a = \sqrt{2}$, $b = 1$, *we have* $x = e^{-\pi}$.

Theorem 13 is essentially combinatorial in nature and expresses some of the basic relations among the $\vartheta$-functions. For a proof see e.g. van der Pol [31].

Theorem 14 depends essentially on Theorem 15.

THEOREM 15. *This* $M(1,x)\log x^{-1} \to \frac{1}{2}\pi$, *as* $x \to 0$.

Barna [2] has given a neat proof of this.

The point of these results is that they permit a parameterization of the Gaussian algorithm. Specifically, given $a_0 \ge b_0$, Theorem 14 implies the existence of an $x$, $|x| < 1$, such that
$a_0 = M(a_0, b_0)(\alpha(x))^2$, $b_0 = M(a_0, b_0)(\beta(x))^2$, and then repeated application of Theorem 13 gives

$$a_n = M(a_0, b_0)(\alpha(x^{2^n}))^2, \quad b_n = M(a_0, b_0)(\beta(x^{2^n}))^2. \tag{6.1}$$

It will now be more convenient to use the standard $\vartheta$-function notation in which $\alpha(q) = \vartheta_3(q)$, $\beta(q) = \vartheta_4(q)$, $\vartheta_3(q) = 2q^{\frac{1}{4}}[1 + q^2 + q^6 + q^{12} + \cdots]$.

From (6.1) we have, in the lemniscate case,

$$A = (\pi/2a_n)\{\vartheta_3(\exp(-2^n\pi))\}^2. \qquad (6.2)_n$$

$$A = (\pi/2b_n)\{\vartheta_4(\exp(-2^n\pi))\}^2. \qquad (6.3)_n$$

$$A = (\pi/2c_n)\{\vartheta_2(\exp(-2^n\pi))\}^2. \qquad (6.4)_n$$

In (6.2) and (6.3) the factor in braces is a correction factor, which brings the $a_n$,$b_n$ to their limit $\pi/(2A)$. Since the argument in the $\vartheta$ series can be made arbitrarily small by choice of $n$, we can optimize the computation of $A$ by balancing the work done in computing $a_n$,$b_n$ with that done in estimating the sums of the series. Since $\vartheta_3$,$\vartheta_4$ differ only by the signs of alternate terms, use of (6.2) *and* (6.3) gives a convenient check. The formula (6.4) does not appear to be so convenient as the right-hand side approaches the form 0/0.

THEOREM 16. $A = \frac{1}{2}\pi\{\vartheta_4(e^{-\pi})\}^2 = \frac{1}{2}\pi\{\vartheta_2(e^{-\pi})\}^2$.

These were given by Gauss [18]; they come from $(6.3)_0$ and the fact that $\vartheta_2 = \vartheta_4$ in the lemniscate case.

THEOREM 17. $A = \frac{1}{4}\pi 2^{\frac{1}{2}}\{\vartheta_3(e^{-\pi})\}^2$.

This was used by Wrench [48] in his fundamental calculations to 164 + D. It comes from $(6.2)_0$ and the fact that $\vartheta_3 = 2^{\frac{1}{4}}\vartheta_4$ in the lemniscate case.

If we take $(6.4)_1$ where
$a_1 = (1 + 2^{\frac{1}{2}})/2$, $b_1 = 2^{\frac{1}{4}}$, $c_1 = (2^{\frac{1}{2}} - 1)/2$, we get
$A = (\pi/(2^{\frac{1}{2}} - 1))\{\vartheta_2(e^{-2\pi})\}^2$. If we take $(6.4)_2$ we get

$$A = (2\pi/(2^{\frac{1}{4}} - 1)^2)\{\vartheta_2(e^{-4\pi})\}^2. \tag{6.6}$$

This result appeared essentially in a Cambridge examination in 1881 (Whittaker and Watson [46]).

## 7. Three Additional Representations

THEOREM 18.

$$A = 2^{\frac{1}{4}}\pi \exp(-\pi/3)\left\{\sum_{-\infty}^{\infty} (-1)^n \exp(-2\pi(3n^2 + n))\right\}^2.$$

This was given by Lehmer [25] without proof and used by him for 50D calculations. It can be proved using the product representations of the $\vartheta$-functions such as $\vartheta_4(q) = \Pi(1 - q^{2n+1})^2(1 - q^{2n})$ and the fact that $\sum(-1)^n x^{(3n^2+n)/2} = \Pi(1 - x^n)$. Proof of these results are given e.g. in Hardy and Wright [22].

THEOREM 19.

$$A^4 = \frac{11}{48} - \frac{5\pi}{8} + \frac{15}{8}\sum_{1}^{\infty} \mathit{cosech}^4\, n\pi.$$

This result appeared in Muckenhoupt [29].

THEOREM 20.

$$(1 - A^2\pi^{-2})^2 = 1 - \pi^{-1} + 6\sum_{0}^{\infty} \mathit{cosech}^4\,(2n + 1)\pi.$$

This result has been derived by Carlson (unpublished) from results of Kiyek and Schmidt [23].

## 8. Historical Notes

*G.C. di Fagnano* (1682–1766) was called for advice, in 1743, by Pope Benedict XIV, when it was discovered that St. Peter's was threatened with collapse. After restoring the foundations Fagnano was honored by being created a marquis and by having his Collected Works published. In his portrait, in his birthplace Senigallia, Fagnano has a diagram of a lemniscate in his hand; a lemniscate also appears on the title page of his book [13] *Produzioni Matematiche* and beneath it the words "Multifariam Divisa Atque Dimensa. Deo Veritatis Gloria." This book reached Euler on December 23, 1751, exactly 48 years before Gauss' proof of Theorem 7.

The part played by Fagnano in the founding of the theory of elliptic functions is now fully recognized. (See e.g. C.L. Siegel [36].) For further information see G.N. Watson [45].

*J. Stirling* (1692–1770) held a Snell Exhibition at Oxford. This award was restricted to students from Glasgow, but there is no record that he ever attended that university; it has been pointed out that a kinsman was rector of the university at the time.

Stirling was arrested in Oxford in 1715 for cursing the king; he was tried and found "not guilty." He lost his scholarships because he refused to sign an oath of allegiance. He was offered a chair in Italy but apparently rejected it on account of religious difficulties. He spent the last half of his life as manager of a mine in Scotland.

The English translation of his book [37], which is really about finite differences, seems very rare; a copy is in the Bodleian Library (Rignaud d. 158). For further historical information see C. Tweedie [44].

*A.A. Markoff* (1856–1922) with his brother W.A. Markoff (1871–1897) began work in the constructive theory of functions, established by P.L. Chebyshev. His book [27] on finite differences is one of the classics; he later made contributions to the theory of probability, introducing what are now called Markoff processes in 1907. A.A. Markoff, who was born in 1903, and is well known for his work in logic and in particular for his book on *The Theory of Algorithms*, is a son.

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