The Eigenfunctions of $\int_0^\infty J_0(|x-y|)\, f(x) \mathrm{d}x$ and a Technique For Deriving The Eigenfunctions of Stationary Gaussian Process Integral Covariance Operators

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Abstract

The null space of Gaussian process kernel inner product operators are shown to be the Fourier transforms of the polynomials orthogonal with respect to the spectral densities of the processes and it is furthermore shown that the orthogonal complements of the null spaces as given by the Gram-Schmidt recursions enumerate the products $g_k(t) = \sqrt{\frac{c_p}{c_q}} \frac{\prod_{i=1}^{n_k} (t - \alpha_{k,i})}{\prod_{j=1}^{m_k} (t - \beta_{k,j})} = f_k(t)\, f_k(s)$ of the eigenfunctions f_k of the corresponding integral covariance operators.

Let $C(x)$ be the covariance function of a stationary Gaussian process on $[0, \infty)$. Define the integral covariance operator T by:

$$(T f)(x) = \int_0^\infty C\,(x-y)\, f(y)\ \mathrm{d}\, y \tag{1}$$

Let $S(\omega)$ be the spectral density related to $C(x)$ by the Wiener-Khinchin theorem:

$$C(x) = \frac{1}{\pi} \int_{-\infty}^\infty e^{i\omega x}\, S(\omega)\ \mathrm{d}\,\omega \tag{2}$$

$$S(\omega) = \int_0^\infty C(x)\, e^{-ix\omega}\, \mathrm{d}\, x \tag{3}$$

Consider polynomials $\{p_n(\omega)\}$ orthogonal with respect to $S(\omega)$:

$$\int_{-\infty}^\infty p_n(\omega)\, p_m(\omega)\, S(\omega)\ \mathrm{d}\,\omega = \delta_{nm} \tag{4}$$

Define $r_n(x)$ as the inverse Fourier transforms of $p_n(\omega)$:

$$r_n(x) = \int_{-\infty}^{\infty} p_n(\omega)\, e^{i\omega x}\, \mathrm{d}\,\omega \tag{5}$$

Lemma 1. *The functions $r_n(x)$ form the null space of the kernel inner product:*

$$\int_0^{\infty} C(x)\, r_n(x)\, \mathrm{d}\,x = 0 \tag{6}$$

Proof. Proof: Substitute the definitions of $C(x)$ and $r_n(x)$, and apply Fubini's theorem:

$$\int_0^{\infty} C(x)\, r_n(x)\, \mathrm{d}\,x = \int_0^{\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\omega x}\, S(\omega)\, \mathrm{d}\,\omega \int_{-\infty}^{\infty} p_n(\omega')\, e^{i\omega' x}\, \mathrm{d}\,\omega'\, \mathrm{d}\,x \tag{7}$$

By Fubini's theorem, we can swap the integrals:

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_n(\omega')\, S(\omega) \int_0^{\infty} e^{i(\omega+\omega')x}\, \mathrm{d}\,x\, \mathrm{d}\,\omega'\, \mathrm{d}\,\omega \tag{8}$$

The integral over x yields the Dirac delta function $\delta\,(\omega - \omega')$:

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_n(\omega')\, S(\omega)\, \pi\, \delta\,(\omega - \omega')\, \mathrm{d}\,\omega'\, \mathrm{d}\,\omega \tag{9}$$

Now, integrate over ω' using the delta function:

$$= \int_{-\infty}^{\infty} p_n(\omega)\, S(\omega)\, \mathrm{d}\,\omega \tag{10}$$

By the orthogonality of $p_n(\omega)$ with respect to $S(\omega)$, we conclude:

$$\int_{-\infty}^{\infty} p_n(\omega)\, S(\omega)\, \mathrm{d}\,\omega = 0 \tag{11}$$

Thus, $\int_0^{\infty} C(x)\, r_n(x)\, \mathrm{d}\,x = 0$, which completes the proof. □

1 Eigenfunctions from Orthogonalized Null Space

By orthogonalizing the null space $\{r_n(x)\}$, we obtain the eigenfunctions $\{\psi_n(x)\}$ of the covariance operator T. The orthogonalization process gives:

$$r_n^{\perp}(y) = \psi_n(x) = \sum_{k=0}^{n} a_{nk}\, r_k(x) = r_n(y) - \sum_{m=0}^{n-1} \frac{\langle r_n(y), r_m^{\perp}(y)\rangle}{\langle r_m^{\perp}(y), r_m^{\perp}(y)\rangle}\, r_m^{\perp}(y)$$

where the coefficients a_{nk} are given by:

$$a_{nk} = \begin{cases} 1 & \text{if } k = n \\ -\sum_{j=k}^{n-1} a_{nj}\, \langle r_n, \psi_j\rangle & \text{if } k < n \\ 0 & \text{if } k > n \end{cases} \tag{12}$$

Theorem 2. *Let $\{\psi_n(x)\}$ be the orthogonal complement of $\{r_n(x)\}$. Then $\psi_n(x)$ are eigenfunctions of T, with eigenvalues:*

$$\lambda_n = \int_0^{\infty} C(z)\, \psi_n(z)\, \mathrm{d}\, z \tag{13}$$

Proof. This is not quite right, they have to be factorized as in Theorem 10. I think the infinite-dimensional version of this is the Hadamard product factorization? □

Definition 3. *Let $j_n(x)$ is the spherical Bessel function of the first kind,*

$$\begin{aligned} j_n(z) &= \sqrt{\frac{\pi}{2\, z}}\, J_{n+\frac{1}{2}}(x) \\ &= \frac{1}{\sqrt{z}} \left(\sin(z)\, R_{n,\frac{1}{2}}(z) - \cos(z)\, R_{n-1,\frac{3}{2}}(z)\right) \end{aligned} \tag{14}$$

where $R_{n,v}(z)$ are the (misnamed) Lommel polynomials [2]

$$R_{n,v}(z) = \frac{\Gamma(n+v)}{\Gamma(v)} \left(\frac{2}{z}\right)^n {}_2F_3\left(\left[-\frac{n}{2}, \frac{1}{2} - \frac{n}{2}\right]; [v, -n, 1-v-n]; -z^2\right) \tag{15}$$

where ${}_2F_3$ is a generalized hypergeometric function. The "Lommel polynomials" are actually rational functions of z, not polynomial; but rather "polynomial in $\frac{1}{z}$".

Theorem 4. *The eigenfunctions of the stationary integral covariance operator*

$$[T\psi_n](x) = \int_0^\infty J_0\,(x-y)\;\psi_n(x)\mathrm{d}x = \lambda_n\psi_n(x) \tag{16}$$

are given by

$$\psi_n(x)\psi_n(y) = (-1)^n\sqrt{\frac{8n+2}{\pi}}\;j_{2n}(x-y) \tag{17}$$

and the eigenvalues are given by

$$\begin{aligned}\lambda_n \;&= \int_0^\infty J_0(x)\;\psi_n(x)\,\mathrm{d}x\\ &= \sqrt{\frac{2n+\frac{1}{2}}{\pi}}\,\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\,(n+1)^2}\\ &= \sqrt{\frac{2n+\frac{1}{2}}{\pi}}\;(n+1)^2_{-\frac{1}{2}}\end{aligned} \tag{18}$$

where $(n+1)_{-\frac{1}{2}}$ *is the Pochhammer symbol(ascending/rising factorial).*

Theorem 5. *The series*

$$\begin{aligned}J_0(t)\;&= \sum_{k=0}^\infty\;\lambda_k\psi_k(t)\\ &= \sum_{k=0}^\infty\;\sqrt{\frac{2\,n+\frac{1}{2}}{\pi}}\;\sqrt{\frac{8\,n+2}{\pi}}(n+1)^2_{-\frac{1}{2}}(-1)^n\,j_{2n}(t)\\ &= \sum_{k=0}^\infty\;\frac{4\,n+1}{\pi}\;(n+1)^2_{-\frac{1}{2}}(-1)^n\,j_{2n}(t)\end{aligned} \tag{19}$$

converges uniformly for all complex t *except the origin where it has a regular singular point where* $\lim_{t\to 0}J_0(t) = 1$.

Proof. $\sqrt{\frac{2\,n+\frac{1}{2}}{\pi}}\,\sqrt{\frac{8\,n+2}{\pi}} = \frac{\sqrt{16n^2+8n+1}}{\sqrt{\pi^2}} = \frac{\sqrt{(4n+1)(4n+1)}}{\sqrt{\pi\,\pi}} = \frac{4n+1}{\pi}$. The rest is left as an excercise. ☐

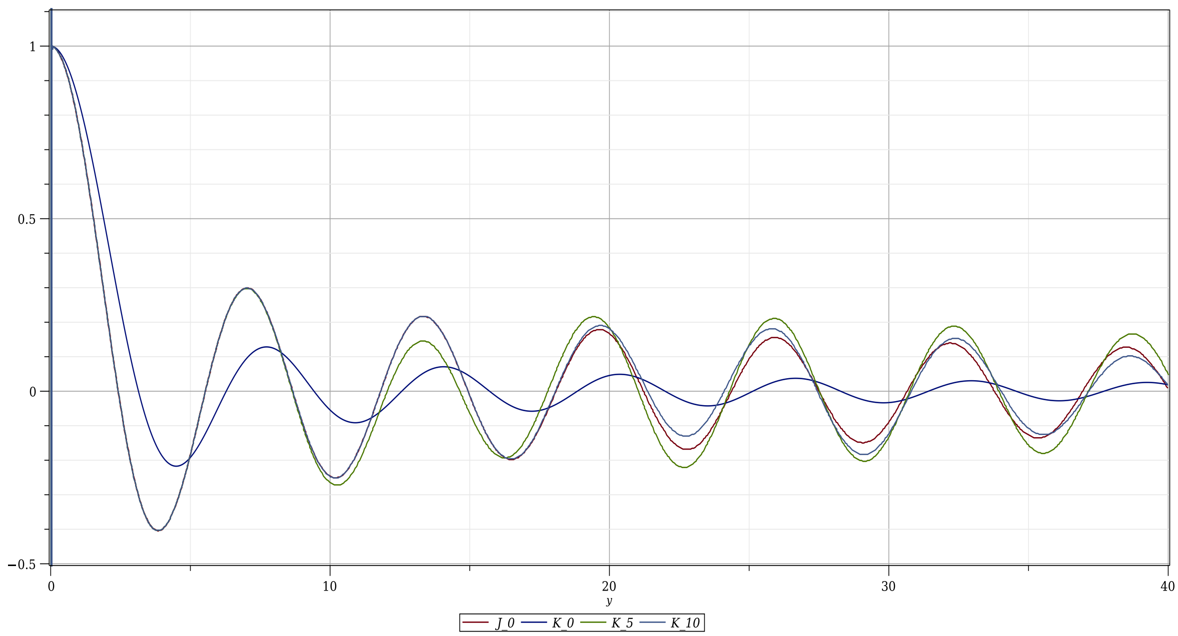

Figure 1. J_0 compared to the finite-rank approximations for rank $0, 5,$ and 10. The figure shows the excellent convergence properties of the proposed eigenfunction expansion $J_0(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \lambda_k \psi_k(x)$.

Theorem 6. *The operator defined by Equation (16) is compact relative to the canonical metric induced by the covariance kernel $J_0(|x-y|)$.*

Proof. The proof can be demonstrated by calculating the metric entropy integral and showing it is finite for all positive epsilon ☐

Definition 7. *The spectral density of a stationary process is the Fourier tranform of the covariance kernel due to Wiener-Khinchine theorem.*

Definition 8. *Let $S_n(x)$ be the orthogonal polynomials whose orthogonality measure is equal to the spectral density of the process. These polynomials shall be called the spectral polynomials corresponding to the process.*

Remark 9. If the spectral density does not equal the orthogonality measure of a known set of orthogonal polynomials then such a set can always be generated by applying the

Gram-Schmidt process to the monomials so that they are transformed into a set that is orthogonal with respect any given spectral density (of a stationary process).

Lemma 10. *For any rational function*

$$f(t-s) = \frac{P(t-s)}{Q(t-s)} \tag{20}$$

, where P and Q are polynomials, there exist rational functions $g(t)$ and $g(s)$ such that

$$f(t-s) = g(t)\, g(s) \tag{21}$$

Proof. Let

$$P(t-s) = c_p \prod_{i=1}^{n} (t-s-\alpha_i) \tag{22}$$

and

$$Q(t-s) = c_q \prod_{j=1}^{m} (t-s-\beta_j) \tag{23}$$

then define

$$g(t) = \sqrt{\frac{c_p}{c_q}}\, \frac{\prod_{i=1}^{n} (t-\alpha_i)}{\prod_{j=1}^{m} (t-\beta_j)} \tag{24}$$

such that

$$\begin{aligned} g(t)\, g(s) &= \frac{c_p}{c_q} \frac{\prod_{i=1}^{n} (t-\alpha_i)}{\prod_{j=1}^{m} (t-\beta_j)} \frac{\prod_{i=1}^{n} (s-\alpha_i)}{\prod_{j=1}^{m} (s-\beta_j)} \\ &= \frac{c_p}{c_q} \frac{\prod_{i=1}^{n} (t-\alpha_i)(s-\alpha_i)}{\prod_{j=1}^{m} (t-\beta_j)(s-\beta_j)} \\ &= \frac{c_p \prod_{i=1}^{n} ((t-s)-\alpha_i)}{c_q \prod_{j=1}^{m} ((t-s)-\beta_j)} \\ &= f(t-s) \end{aligned} \tag{25}$$

For complex roots, we pair each α_i or β_j with its complex conjugate in the factorization of $g(t)$. This ensures that the product $(t-\alpha_i)(t-\bar{\alpha_i})$ results in a quadratic polynomial with real coefficients, making $g(t)$ a real-valued function. □

Theorem 11. *Let $Y_n(y)$ be the normalized Fourier transforms of the spectral polynomials $Y_n(y) = \frac{\hat{S}_n(y)}{|\hat{S}_n|}$ where the sequence $\hat{S}_n(y)$ of inverse Fourier transforms of the spectral polynomials $S_n(x)$ is given by*

$$\hat{S}_n(y) = \int_{-1}^{1} S_n(x) e^{ixy} \mathrm{d}x \tag{26}$$

The eigenfunctions of the integral covariance operator (16) are given by the products

$$\psi_n(x)\psi_n(y) = Y_n^{\perp}(x-y) \tag{27}$$

of the elements of orthogonal complement of the normalized Fourier transforms $Y_n(y)$ of the spectral polynomials (via the Gram-Schmidt process)

$$\begin{aligned} \psi_n(x)\psi_n(y) &= Y_n^{\perp}(x-y) \\ &= Y_n(x-y) - \sum_{m=0}^{n-1} \frac{\langle Y_m(x-y), Y_m^{\perp}(x-y)\rangle}{\langle Y_m^{\perp}(x-y), Y_m^{\perp}(x-y)\rangle} Y_m^{\perp}(x-y) \end{aligned} \tag{28}$$

Example 12. Let the kernel function be given by $K(t,s) = J_0(t-s)$ then identify the orthogonal polynomial sequence associated with the spectral density

$$S(\omega) = \int_0^{\infty} J_0(x)\, e^{ix\omega}\, \mathrm{d}x = \begin{cases} \frac{1}{\sqrt{1-\omega^2}} & \omega \in (-1,1) \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

as being the orthogonality measure of the Type-I Chebyshev polynomials $T_n(x)$ so that the orthogonal polynomial sequence is identified as

$$S_n(x) = T_n(x) \tag{30}$$

so that

$$\int_{-1}^{1} S_n(\omega)\, S_m(\omega)\, S(\omega)\, \mathrm{d}\omega = \begin{cases} 0 & n \neq m \\ 2\pi & n = m = 0 \\ \pi & n = m \neq 0 \end{cases} \tag{31}$$

The finite Fourier transforms of the Chebyshev polynomials[1] are just the usual infinite Fourier transforms with the integration restricted to the range $-1 \dots 1$ since $T_n(x) = 0 \forall x \notin [-1,1]$). Equivalently, the spectral density function can be extended to take the value 0 outside the interval $[-1,1]$.

$$\begin{aligned}\hat{T}_n(y) &= \int_{-\infty}^{\infty} e^{-ixy} T_n(x) d y = \int_{-1}^{1} e^{-ixy} T_n(x) d x \\ &= \int_{-\infty}^{\infty} e^{-ixy} {}_2F_1\left(\begin{matrix} n, & -n \\ & \frac{1}{2} \end{matrix} \middle| \frac{1}{2} - \frac{x}{2} \right) d x \\ &= \frac{i}{y} \left(e^{-iy} F_n^{+}(y) - e^{i(\pi n + y)} F_n^{-}(y)\right)\end{aligned} \tag{32}$$

where

$$F_n^{\pm}(y) = {}_3F_1\left(\begin{matrix} 1, & n, & -n \\ & & \frac{1}{2} \end{matrix} \middle| \frac{\pm i y}{2} \right) \tag{33}$$

the spectral polynomials S_n are given by the Type-I Chebyshev polynomials

$$S_n(x) = T_n(x) \tag{34}$$

and their normalization is

$$\begin{aligned}Y_n(y) &= \frac{\hat{T}_n(y)}{|\hat{T}_n|} \\ &= \frac{i}{y} \left(\frac{e^{-iy} F_n^{+}(y) - e^{i(\pi n + y)} F_n^{-}(y)}{\sqrt{\frac{4(-1)^n \pi - (2n^2 - 1)}{4n^2 - 1}}} \right)\end{aligned} \tag{35}$$

where the L^2 norm of $\hat{T}_n(y)$ is given by

$$\begin{aligned}|\hat{T}_n| &= \sqrt{\int_{-\infty}^{\infty} \hat{T}_n(y)^2 \mathrm{d}y} \\ &= \sqrt{\frac{4(-1)^n \pi - (2n^2 - 1)}{4n^2 - 1}}\end{aligned} \tag{36}$$

Bibliography

[1] Atul Dixit, Lin Jiu, Victor H Moll, and Christophe Vignat. The finite fourier transform of classical polynomials. *Journal of the Australian Mathematical Society*, 98:145–160, 2015.

[2] R. Wong K.F. Lee. Asymptotic expansion of the modified lommel polynomials $h_{n,\nu}(x)$ and their zeros. *Proceedings of the American Mathematical Society*, 142(11):3953–3964, 2014.