Testing for Harmonizability

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Abstract

Let $R(s,t)$ be a covariance function having the representation

$$R(s,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(isx-ity)}\ d^2\,G(x,y) \tag{1}$$

where $G(x,y)$ is continuous to the right in both variables and is of bounded variation in the plane; then $R(s,t)$ is harmonizable in that $G(x,y)$ is also a covariance. Examples are given in which this result is used to determine the harmonizability of new processes and covariances that are formed by operations on old processes and covariances. Specifically, if $X(t)$ is a real Gaussian harmonizable process, then $X^n(t)$ is harmonizable. If $X(t)$ is harmonizable, $G(x,y)$ has bounded support and $g(t)$ is a Fourier-Stieltjes transform, then $X(g(t))$ and $X\,(t+g(t))$ are harmonizable. If

$$X(t) = \int_{-\infty}^{\infty} f(t,u)\ d\,Z(u) \tag{2}$$

where $f(t,u) = f\,(t-u)$ is a Fourier-Stieltjes transform and

$$G(u,v) = \mathbb{E}\,\{Z(u)\,Z^*(v)\} \tag{3}$$

has finite total variation, then $X(t)$ is harmonizable. A sufficient condition for Priestley's oscillatory processes to be harmonizable is also obtained. The Bochner-Eberlein characterization of Fourier-Stieltjes transforms is particularly convenient for determining harmonizability in these cases.

1 Introduction

Let $\{X(t,\omega),\,-\infty<t<\infty,\,\omega\in\Omega\}$ be a second-order continuous-parameter stochastic process defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$. The process $X(t,\omega)$ is said to be harmonizable [loeve1955probability, p. 474] if it has the quadratic mean representation

$$X(t,\omega) = \int_{-\infty}^{\infty} e^{itx} d\,Z(x,\omega) \tag{4}$$

where $\{Z(x,\omega),-\infty<x<\infty\}$ is a process whose covariance is of bounded variation (BV) in the plane. Harmonizable processes are of engineering interest because decomposition relative to $\exp\,(i\,t\,x)$ admits the usual frequency interpretation of linear filtering. If $H(x)$ is the frequency response of a stable, linear time-invariant system, then the system output process $Y(t,\omega)$ is given by the quadratic mean integral

$$Y(t,\omega) = \int_{-\infty}^{\infty} e^{itx}\,H(x)\ d\,Z(x,\omega) \tag{5}$$

A detailed account may be found in [blanc-lapierre1968random, Ch. 8]. For recent results on harmonizable processes in engineering, see [cambanis1970harmonizable, donati1971spectra, ogura1971spectral].

The covariance functions for $X(t)$ and $Y(t)$ are

$$R(s,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(isx-ity)}\ d^2 G(x,y) \tag{6}$$

$$R_Y(s,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(isx-ity)}\, H(x)\, H^*(y)\ d^2 G(x,y) \tag{7}$$

where

$$d^2 G(x,y) = \mathbb{E}\left\{d\, Z(x)\, d\, Z^*(y)\right\} \tag{8}$$

Any covariance with representation (6), with $G(x,y)$ a covariance of bounded variation, is called harmonizable; harmonizable processes have harmonizable covariances. Conversely, processes with harmonizable covariances are themselves harmonizable [loeve1955probability, p. 476]. For brevity, we call those corresponding to G with finite total variation simply "harmonizable."

This paper addresses the determination of harmonizability for new processes or covariances constructed from old ones. The main results are as follows:

- If $X(t)$ is a real Gaussian harmonizable process, then $X^n(t)$ is harmonizable.

- If $X(t)$ is harmonizable with spectral decomposition of bounded support and $g(t)$ is a Fourier-Stieltjes transform, then $X\left(t + g(t)\right)$ and $X(g(t))$ are harmonizable.

- If R_1 and R_2 are harmonizable covariances, then for T of finite Lebesgue measure,

  $$R_3(s,t) = \int_T R_1(s,u)\, R_2(u,t)\ d\,u \tag{9}$$

  is harmonizable.

- If $X(t)$ is a moving average as in (2) with

  $$f(t,u) = f\,(t-u) \tag{10}$$

  a Fourier-Stieltjes transform and $G(u,v)$ of bounded variation, then $X(t)$ is harmonizable.

- If $X(t)$ is as above, $Z(u)$ has orthogonal increments with $d\,F(u) = \mathbb{E}\,|d\,Z(u)|^2$ and

  $$f(t,u) = e^{(iut)} \int_{-\infty}^{\infty} e^{(itx)}\ d\,H_u(x) \tag{11}$$

with $H_u(x)$ of bounded variation for every u, the resulting processes include Priestley's oscillatory processes [priestley1965evolutionary], which are harmonizable under suitable conditions.

The method is to use the following result.

Theorem 1. *If $R(s,t)$ is simultaneously a covariance and a Fourier-Stieltjes (FS) transform with respect to some $G(x,y)$ of bounded variation, then $R(s,t)$ is harmonizable in that G is necessarily a covariance.*

Proof. Sufficiency is immediate: one can find a process $Z(x,\omega)$ whose covariance is G and whose FS transform $X(t,\omega)$ as in (4) has covariance $R(s,t)$. Conversely, suppose R is both a covariance and an FS transform with respect to some G that is BV. Define

$$G_a(x,y) = G(x,y) - G(a,y) - G(x,a) + G(a,a) \tag{12}$$

For any n and sequence $\{x_j \geq a,\ j=1,\ldots,n\}$ and complex $\{c_j\}$

$$\sum_{j,k=1}^{n} c_j\, c_k^*\, G_a(x_j, x_k) \geq 0 \tag{13}$$

This follows by constructing

$$g_a(s) = \sum_{j=1}^{n} c_j \left[1 - \exp\left(-i\, s\, x_j\right)\right] \exp\left(-i\, s\, a\right) \tag{14}$$

and applying the inversion theorem [loeve1955probability, p. 475] then letting $a \to -\infty$ to see that

$$\lim_{a \to -\infty} G_a(x,y) = G(x,y) \tag{15}$$

Thus $G(x,y)$ is non-negative definite. □

Thus, any characterization of FS transforms, such as the Bochner-Eberlein theorem, also provides a characterization for harmonizable covariances [bochner1934fst, eberlein1955fst, rudin1962groups]. Cramér [cramer1939representation] and Dominguez [dominguez1940fst] provide alternative characterizations.

2 Mathematical Preliminaries

Assume $G(x,y)$ is normalized, e.g.,

$$G(x,y) = \frac{1}{4}\left[G(x+0,y+0) + G(x+0,y-0) + G(x-0,y+0) + G(x-0,y-0)\right] \tag{16}$$

and satisfies

$$\lim_{x \to -\infty} G(x, y) = \lim_{y \to -\infty} G(x, y) = 0 \tag{17}$$

We now state the key characterizations.

Theorem 2. *[Bochner]A necessary and sufficient condition that $f(t)$, $-\infty < t < \infty$, has the representation*

$$f(t) = \int_{-\infty}^{\infty} e^{ixt}\, d\,G(x) \tag{18}$$

for a complex measure G of bounded variation is that, for any n, any sequence $\{t_j\}_{j=1}^{n}$ and any complex $\{c_j\}_{j=1}^{n}$,

$$\left|\sum_{j=1}^{n} c_j\, f(t_j)\right| \leq M \sqrt{\sum_{j=1}^{n} \sum_{k=1}^{n} c_j\, c_k^* \exp\left(i\, x\, (t_j - t_k)\right)} \tag{19}$$

for some $M > 0$.

Theorem 3. *[Bochner-Eberlein]A necessary and sufficient condition for a function $R(s, t)$ to have representation (1) with*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |d^2\, G(x, y)| \leq M \tag{20}$$

is that for any n, any sequence of pairs $\{(s_j, t_j)\}_{j=1}^{n}$ and any complex $\{c_j\}_{j=1}^{n}$,

$$\left|\sum_{j=1}^{n} c_j\, R(s_j, t_j)\right| \leq M \sqrt{\sum_{j=1}^{n} \sum_{k=1}^{n} c_j\, c_k^* \exp\left(i\, (x\, s_j - y\, t_j - x\, s_k + y\, t_k)\right)} \tag{21}$$

for some $M > 0$.

3 Examples and Application

Example 4. Let $X(t)$ be a zero-mean real Gaussian harmonizable process with covariance $R(s, t)$. Then, for any $n \geq 1$, $X^n(t)$ is harmonizable.

Proof. An exercise with characteristic functions shows that

$$\mathbb{E}\left[X^n(s)\,X^n(t)\right] = \sum_{p,q,r\geq 0} c_z\,(p,q,r,n)\,R^p(s,s)\,R^q(s,t)\,R^r(t,s)\,R^p(t,t) \tag{22}$$

where the sum is over all $p,q,r\geq 0$ with $n=2\,p+q+r$ and $c_z\,(p,q,r,n)$ are combinatorial coefficients.

Since both FS transforms [bochner1934fst, p.151] and covariances [loeve1955probability, p.468] are closed under products, $R^q(s,t)\,R^r(t,s)=R^{q+r}(s,t)$ is an FS transform and a covariance. The product $R^p(s,s)\,R^p(t,t)$ is also an FS transform, as for $f(s)=R(s,s)$

$$f^p(s)\,f^p(t) \tag{23}$$

is NND and an FS transform by Theorem 2. The sum in (22) is harmonizable since FS transforms and covariances are closed under positive sums. □

Example 5. Suppose $X(t)$ is harmonizable with spectral decomposition supported by a bounded set A and $g(t)$ is the FS transform of some $G(x)$ of finite variation. Then $X\,(t+g(t))$ and $X(g(t))$ are harmonizable.

Proof. Set

$$Y(t)=X\,(t+g(t)) \tag{24}$$

so

$$\begin{aligned} R_Y(s,t) &= R_X(s+g(s),t+g(t)) \\ &= \iint_A e^{(ix(s+g(s))-iy(t+g(t)))}\,d^2\,G(x,y) \end{aligned} \tag{25}$$

Let M_A denote the variation over $A\times A$ of $G(x,y)$. For any complex $\{c_j\}$ and parameter pairs,

$$Q=\left|\sum_{j=1}^n c_j\,R_Y(s_j,t_j)\right| \leq M_A\left|\sum_{j=1}^n c_j\,e^{(ix's_j+ix'g(s_j)-iy't_j-iy'g(t_j))}\right| \tag{26}$$

where x',y' in closure of A. The mappings

$$t\mapsto e^{ix'g(t)} \tag{27}$$

and

$$t\mapsto e^{-iy'g(t)} \tag{28}$$

are FS transforms, so by repeated application of Bochner's condition this is bounded, and $R_Y(s,t)$ is an FS transform. □

Example 6. Suppose R_1 and R_2 are harmonizable covariances, and for T of finite Lebesgue measure define $R_3(s,t)$ as in (9). Then R_3 is harmonizable.

Proof. By the Bochner-Eberlein condition, for any $\{c_j\}$,

$$Q = \left|\sum_{j=1}^{n} c_j \int_T R_1(s_j,u)\, R_2(u,t_j)\; d\,u\right| \leq M_1 \sup_{x,y} \left|\sum_{j=1}^{n} c_j \int_T R_2(u,t_j) \exp\left(i\, s_j x - i\, u\, y\right) d\,u\right| \tag{29}$$

With M_2 the variation bound for R_2,

$$Q \leq M_1\, M_2\, m(T) \sup_{x,y} \left|\sum_{j=1}^{n} c_j\, e^{(i s_j x - i t_j y)}\right| \tag{30}$$

where $m(T)$ is Lebesgue measure. Thus R_3 is an FS transform. □

Example 7. Suppose $X(t)$ is a moving average as in (2) with

$$f(t,u) = f\,(t-u) \tag{31}$$

a Fourier-Stieltjes transform and $G(u,v)$ of bounded variation. Then $X(t)$ is harmonizable.

Proof. From (2) and the bounded variation of $H(x)$ and $G(u,v)$,

$$R(s,t) = \iint e^{i(x(s-u) - y(t-v))}\; d^2\, G(u,v)\; d\,H(x)\; d\,H^*(y) \tag{32}$$

This is an FS transform with variation bound $M_G\, M_H^2$, where M_G bounds $G(u,v)$ and M_H bounds $H(x)$. □

Example 8. Suppose $X(t)$ is as above, $Z(u)$ has orthogonal increments with

$$d\,F(u) = \mathbb{E}[|d\,Z(u)|^2] \tag{33}$$

and $f(t,u)$ as in (11). Then $X(t)$ is a Priestley oscillatory process, and is harmonizable provided

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |d\,H_u(x)|\,|\,d\,H_u^*(y)|\;\, d\,F(u) < \infty \tag{34}$$

Proof. The covariance is

$$R(s,t) = \int_{-\infty}^{\infty} e^{iu(s-t)}\, f(u,s)\, f^*(u,t)\, d\,F(u) \tag{35}$$

By repeated application of Bochner's and Fubini's theorems, provided (34) holds, $R(s,t)$ is an FS transform and thus harmonizable. □

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