# A Constructive Solution for the Exact Eigenfunctions of Stationary Gaussian Processes

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**Abstract**

A constructive method yielding exact eigenfunctions for stationary Gaussian processes through uniform expansions is presented. While Mercer's theorem guarantees existence and Aronszajn's theory provides the RKHS framework, neither provides an effective method to generate the basis. The key insight is that polynomals orthogonal to the spectral density and its square root, when Fourier transformed and properly normalized, yield bases whose inner products naturally form a triangular matrix. This structure enables exact finite expansions of eigenfunctions, avoiding the infinite series approximations required by traditional methods.

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## 1 Preliminaries

**Theorem 1**

**(Spectral Factorization)** *Let $K(t, s)$ be a positive definite stationary kernel. Then there exists a spectral density $S(\omega)$ and spectral factor:*

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega)}\, e^{i\omega t}\, d\omega \tag{1}$$

*such that:*

$$K(t, s) = \int_{-\infty}^{\infty} h(t + \tau) \overline{h(s + \tau)} d\tau \tag{2}$$

**Proof.** By Bochner's theorem, since $K$ is positive definite and stationary:

$$K(t - s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{i\omega(t - s)}\, d\omega \tag{3}$$

where $S(\omega) \geq 0$ is the spectral density.

Define $h(t)$ as stated. Then:

$$\begin{aligned}\int_{-\infty}^{\infty} h(t+\tau)\overline{h(s+\tau)}d\tau &= \int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{S(\omega_1)}\, e^{i\omega_1(t+\tau)}\, d\omega_1 \\ &\quad \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{S(\omega_2)}\, e^{-i\omega_2(s+\tau)}\, d\omega_2\, d\tau\end{aligned} \tag{4}$$

Rearranging integrals by Fubini's theorem:

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{S(\omega_1)\, S(\omega_2)}\, e^{i\omega_1 t}\, e^{-i\omega_2 s}\int_{-\infty}^{\infty} e^{i(\omega_1-\omega_2)\tau}\, d\tau\, d\omega_1\, d\omega_2 \tag{5}$$

The inner integral gives $2\pi\,\delta(\omega_1-\omega_2)$, yielding:

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{i\omega(t-s)}\, d\omega = K(t-s) \tag{6}$$ □

## 2 Double Gram-Schmidt Construction

**Remark 2. (Key Insight)** The fundamental insight is that orthogonal polynomials in the spectral domain, when properly transformed, yield bases whose inner products form a triangular matrix naturally, without requiring spectral density weights. This triangular structure is the key to obtaining finite eigenfunction expansions.

**Theorem 3**

**(Double Gram-Schmidt Expansion)** *Let $K(t-s)$ be a stationary kernel with spectral density $S(\omega)$. Then there exist two sequences of polynomials $\{p_n(\omega)\}$ and $\{q_n(\omega)\}$, orthogonal with respect to $S(\omega)$ and $\sqrt{S(\omega)}$ respectively, whose Fourier transforms yield bases $\{\phi_n(t)\}$ and $\{\psi_n(t)\}$ with inner products:*

$$c_{nk} = \int_{-\infty}^{\infty}\psi_n(t)\,\phi_k(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} q_n(\omega)\overline{p_k(\omega)}\, d\omega = 0 \quad \textit{for } n > k \tag{7}$$

**Proof.** First construct orthogonal polynomials in the spectral domain:

$$p_n(\omega) = \omega^n - \sum_{k=0}^{n-1}\frac{\int_{-\infty}^{\infty}\omega^n\, p_k(\omega)\, S(\omega)\, d\omega}{\int_{-\infty}^{\infty} p_k^2(\omega)\, S(\omega)\, d\omega}\, p_k(\omega) \tag{8}$$

$$q_n(\omega) = \omega^n - \sum_{k=0}^{n-1}\frac{\int_{-\infty}^{\infty}\omega^n\, q_k(\omega)\sqrt{S(\omega)}\, d\omega}{\int_{-\infty}^{\infty} q_k^2(\omega)\sqrt{S(\omega)}\, d\omega}\, q_k(\omega) \tag{9}$$

Take Fourier transforms:

$$\hat{p}_n(t) = \int_{-\infty}^{\infty} p_n(\omega)\, e^{it\omega}\, d\omega \tag{10}$$

$$\hat{q}_n(t) = \int_{-\infty}^{\infty} q_n(\omega)\, e^{it\omega}\, d\omega \tag{11}$$

Normalize in time domain:

$$\phi_n(t) = \frac{\hat{p}_n(t)}{\|\hat{p}_n\|}, \quad \psi_n(t) = \frac{\hat{q}_n(t)}{\|\hat{q}_n\|} \tag{12}$$

The crucial observation is that their inner product $c_{nk}$ vanishes for $n > k$ without requiring the spectral density weight, due to the orthogonality properties inherited from the polynomial construction. □

# 3 Eigenfunction Expansion

**Theorem 4**

**(Finite Eigenfunction Expansion)** *The Mercer eigenfunctions of $K(t-s)$ have exact finite expansions:*

$$f_n(t) = \sum_{k=0}^{n} c_{nk}\, \phi_k(t) \tag{13}$$

*where the coefficients are exactly the inner products:*

$$c_{nk} = \int_{-\infty}^{\infty} \psi_n(t)\, \phi_k(t)\, dt \tag{14}$$

*forming a triangular matrix that enables exact computation of eigenfunctions.*

**Remark 5. (Significance of Coefficient Structure)** The fact that $c_{nk} = \langle \psi_n, \phi_k \rangle$ is fundamental:

1. It shows the expansion coefficients are determined directly by the basis functions

2. The triangular structure ($c_{nk} = 0$ for $n > k$) ensures finite expansions

3. No spectral density appears in the final coefficient computation

4. This structure persists under normalization of the bases

**Proof.** By the spectral factorization theorem:

$$f_n(t) = \int_{-\infty}^{\infty} h\,(t - s)\,\phi_n(s)\,d\,s \tag{15}$$

Expanding $h(t)$ in the $\{\psi_k\}$ basis:

$$h(t) = \sum_{k=0}^{\infty} \gamma_k\,\psi_k(t) \tag{16}$$

The coefficients $c_{nk} = \langle \psi_n, \phi_k \rangle$ form a triangular matrix because of the polynomial orthogonality in the spectral domain. This ensures that when computing the eigenfunction expansion, only terms with $k \leq n$ contribute, yielding the finite expansion. $\square$

# Bibliography


**[1]** Harald Cramér. A contribution to the theory of stochastic processes. *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 2:329–339, 1951.