Determination of the Eigenfunctions of Integral Covariance Operators of Stationary Gaussian Processes

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Abstract

It is proved that the eigenfunctions of integral covariance operators of stationary Gaussian processes are the orthogonal complement of the null space of the kernel inner product operator where the null space is identified as the inverse Fourier transforms of the polynomials which are orthogonal with respect to the spectral density of the process.

Let $C(x)$ be the covariance function of a stationary Gaussian process on $[0, \infty)$. Define the integral covariance operator T by:

$$(Tf)(x) = \int_0^\infty C(x-y)\, f(y)\, \mathrm{d}\, y \tag{1}$$

Let $S(\omega)$ be the spectral density related to $C(x)$ by the Wiener-Khinchin theorem:

$$C(x) = \frac{1}{\pi} \int_{-\infty}^\infty e^{i\omega x}\, S(\omega)\, \mathrm{d}\, \omega \tag{2}$$

$$S(\omega) = \int_0^\infty C(x)\, e^{-ix\omega}\, \mathrm{d}\, x \tag{3}$$

Consider polynomials $\{p_n(\omega)\}$ orthogonal with respect to $S(\omega)$:

$$\int_{-\infty}^\infty p_n(\omega)\, p_m(\omega)\, S(\omega)\, \mathrm{d}\, \omega = \delta_{nm} \tag{4}$$

Define $r_n(x)$ as the inverse Fourier transforms of $p_n(\omega)$:

$$r_n(x) = \int_{-\infty}^\infty p_n(\omega)\, e^{i\omega x}\, \mathrm{d}\, \omega \tag{5}$$

Lemma 1. *The functions $r_n(x)$ form the null space of the kernel inner product:*

$$\int_0^\infty C(x)\, r_n(x)\, \mathrm{d}\, x = 0 \tag{6}$$

Proof. Proof: Substitute the definitions of $C(x)$ and $r_n(x)$, and apply Fubini's theorem:

$$\int_0^\infty C(x)\, r_n(x)\, \mathrm{d}\, x = \int_0^\infty \frac{1}{\pi} \int_{-\infty}^\infty e^{i\omega x}\, S(\omega)\, \mathrm{d}\, \omega \int_{-\infty}^\infty p_n(\omega')\, e^{i\omega' x}\, \mathrm{d}\, \omega'\, \mathrm{d}\, x \tag{7}$$

By Fubini's theorem, we can swap the integrals:

$$= \frac{1}{\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty p_n(\omega')\, S(\omega) \int_0^\infty e^{i(\omega+\omega')x}\, \mathrm{d}\, x\, \mathrm{d}\, \omega'\, \mathrm{d}\, \omega \tag{8}$$

The integral over x yields the Dirac delta function $\delta\,(\omega - \omega')$:

$$= \frac{1}{\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty p_n(\omega')\, S(\omega)\, \pi\, \delta\,(\omega - \omega')\, \mathrm{d}\, \omega'\, \mathrm{d}\, \omega \tag{9}$$

Now, integrate over ω' using the delta function:

$$= \int_{-\infty}^\infty p_n(\omega)\, S(\omega)\, \mathrm{d}\, \omega \tag{10}$$

By the orthogonality of $p_n(\omega)$ with respect to $S(\omega)$, we conclude:

$$\int_{-\infty}^\infty p_n(\omega)\, S(\omega)\, \mathrm{d}\, \omega = 0 \tag{11}$$

Thus, $\int_0^\infty C(x)\, r_n(x)\, \mathrm{d}\, x = 0$, which completes the proof. □

1 Eigenfunctions from Orthogonalized Null Space

By orthogonalizing the null space $\{r_n(x)\}$, we obtain the eigenfunctions $\{\psi_n(x)\}$ of the covariance operator T. The orthogonalization process gives:

$$\psi_n(x) = \sum_{k=0}^{n} a_{nk}\, r_k(x)$$

where the coefficients a_{nk} are given by:

$$a_{nk} = \begin{cases} 1 & \text{if } k = n \\ -\sum_{j=k}^{n-1} a_{nj} \langle r_n, \psi_j \rangle & \text{if } k < n \\ 0 & \text{if } k > n \end{cases} \tag{12}$$

Theorem 2. *Let $\{\psi_n(x)\}$ be the orthogonal complement of $\{r_n(x)\}$. Then $\psi_n(x)$ are eigenfunctions of T, with eigenvalues:*

$$\lambda = \int_0^\infty C(z)\, \psi_n(z)\, \mathrm{d}\, z \tag{13}$$

Proof. Let $\psi_n(x) = \sum_k a_{nk}\, r_k(x)$. Then:

$$\begin{aligned} \int_0^\infty C\,(x-y)\,\psi_n(y)\,\mathrm{d}\,y &= \int_0^\infty C\,(x-y) \sum_k a_{nk}\, r_k(y)\,\mathrm{d}\,y \\ &= \sum_k a_{nk} \int_0^\infty C\,(x-y)\, r_k(y)\,\mathrm{d}\,y \\ &= \sum_k a_{nk} \left[r_k(x) \int_0^\infty C(z)\,\mathrm{d}\,z - \int_0^\infty C(z)\, r_k\,(x-z)\,\mathrm{d}\,z \right] \end{aligned} \tag{14}$$

Now, let's focus on the second term:

$$\begin{aligned} \sum_k a_{nk} \int_0^\infty C(z)\, r_k\,(x-z)\,\mathrm{d}\,z &= \int_0^\infty C(z) \sum_k a_{nk}\, r_k\,(x-z)\,\mathrm{d}\,z \\ &= \int_0^\infty C(z)\,\psi_n\,(x-z)\,\mathrm{d}\,z \\ &= (T\psi_n)(x) \end{aligned}$$

Substituting this back into our original expression:

$$\begin{aligned} \int_0^\infty C\,(x-y)\,\psi_n(y)\,\mathrm{d}\,y &= \sum_k a_{nk}\, r_k(x) \int_0^\infty C(z)\,\mathrm{d}\,z - (T\psi_n)(x) \\ &= \psi_n(x) \int_0^\infty C(z)\,\mathrm{d}\,z - (T\psi_n)(x) \end{aligned}$$

Therefore, we have shown that:

$$(T\psi_n)(x) = \lambda_n\, \psi_n(x)$$

where the eigenvalue λ_n is given by:

$$\lambda_n = \int_0^\infty C(z)\, \psi_n(z)\; \mathrm{d}\, z \qquad \square$$

Thus, we have shown that the orthogonalized null space functions are eigenfunctions of the covariance operator, providing a direct method to construct eigenfunctions for stationary operators. The eigenvalues are determined by the integral of the covariance function with the corresponding eigenfunction.