Measure-Preserving Bijective Time Changes of Stationary Gaussian Processes Generate Oscillatory Processes With Evolving Spectra

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Abstract

This article establishes that Gaussian processes obtained through measure-preserving bijective unitary time transformations of stationary processes constitute a subclass of oscillatory processes in the sense of Priestley. The transformation $Z(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t))$, where $X(t)$ is a realization of stationary Gaussian process and θ is a strictly increasing C^1 differentiable monotonic function, yields an oscillatory process with evolutionary power spectrum $d\,F_t(\omega) = \dot{\theta}(t)\, d\,\mu(\omega)$. An explicit unitary transformation between the input stationary process and the transformed oscillatory process is established, preserving the L^2-norm and providing a complete spectral characterization.

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1 Scaling Functions

Definition 1

[Scaling Functions] Let $\mathcal{F}$ denote the set of functions $\theta: \mathbb{R} \to \mathbb{R}$ satisfying

1. *θ is absolutely continuous with*

$$\dot{\theta}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\theta(t) \geq 0 \tag{1}$$

   almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero

2. *θ is strictly increasing and bijective.*

Remark 2. The conditions in Definition 1 ensure that θ^{-1} exists and is absolutely continuous. By the inverse function theorem for absolutely continuous functions,

$$\frac{\mathrm{d}}{\mathrm{d}s}(\theta^{-1})(s) = \frac{1}{\dot{\theta}(\theta^{-1}(s))} \tag{2}$$

for almost all s in the range of θ. The condition that $\dot{\theta}(t) = 0$ only on sets of measure zero ensures that $\frac{1}{\dot{\theta}(\theta^{-1}(s))}$ is well-defined almost everywhere.

2 Oscillatory Processes

Definition 3

[Oscillatory Process] A complex-valued, second-order process $\{X(t)\}_{t\in\mathbb{R}}$ *is called* oscillatory *if there exist*

1. *a family of oscillatory basis functions* $\{\phi_t(\omega)\}_{t\in\mathbb{R}}$ *with*

$$\phi_t(\omega) = A_t(\omega)\, e^{i\omega t} \tag{3}$$

   and a given gain function

$$A_t(\cdot) \in L^2(\mu) \tag{4}$$

2. *and a complex orthogonal random measure* $\Phi(\omega)$ *with*

$$E\,|d\,\Phi(\omega)|^2 = d\,\mu(\omega) = S(\omega) \tag{5}$$

such that

$$\begin{aligned} Z(t) &= \int_{-\infty}^{\infty} \phi_t(\omega)\ d\,\Phi(\omega) \\ &= \int_{-\infty}^{\infty} A_t(\omega)\, e^{i\omega t} d\,\Phi(\omega) \end{aligned} \tag{6}$$

All stationary processes are oscillatory with $A_t(\omega) = 1$

TODO: insert proof of this as well as representation of Z(t) as a time-dependent convolution of a stationary process with the time-dependent filter given by the Fourier transform of the oscillatory function

3 Stationary Reference Process

Let $\{X(t)\}_{t\in\mathbb{R}}$ be a stationary Gaussian process with continuous spectral representation

$$X(t) = \int_{-\infty}^{\infty} e^{i\omega t}\ d\,\Phi(\omega) \tag{7}$$

where $\Phi(\omega)$ is an orthogonal-increment process with spectral density

$$E\,|d\,\Phi(\omega)|^2 = d\,\mu(\omega) = S(\omega) = <\text{fourier transform of } K_X> \tag{8}$$

and μ is a finite measure on $\mathbb{R}$.

4 Time-Changed Process

4.1 Definition and Unitary Operator

Definition 4

[Unitary Time-Change Operator]For $\theta \in \mathcal{F}$, define the operator M_θ: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$(M_\theta f)(t) = \sqrt{\dot{\theta}(t)}\; f(\theta(t)) \tag{9}$$

Definition 5

[Unitarily Time-Changed Stationary Process] For $\theta \in \mathcal{F}$, apply the unitary time change operator M_θ from Definition-4 to a realization of a stationary process $X(t)$ from the ensemble $\{X(t)\}$ to define a realization of the unitarily time-changed process

$$Z(t) = \sqrt{\dot{\theta}(t)}\; X(\theta(t)) \forall t \in \mathbb{R} \tag{10}$$

Definition 6

[Inverse Unitary Time-Change Operator] The inverse operator M_θ^{-1}: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ corresponding to the unitary time-change operator $(M_\theta f)(t)$ defined in Equation-9 is given by

$$(M_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{11}$$

Lemma 7

[Well-Definedness of Inverse Operator] The operator M_θ^{-1} in Definition 6 is well-defined $\forall \theta \in \mathcal{F}$.

Proof. Since $\dot{\theta}(t) = 0$ only on sets of measure zero by Definition 1, and θ^{-1} maps sets of measure zero to sets of measure zero (as it preserves absolute continuity), the denominator $\sqrt{\dot{\theta}(\theta^{-1}(s))}$ is positive almost everywhere. The expression in equation (11) is therefore well-defined almost everywhere, which is sufficient for defining an element of $L^2(\mathbb{R})$. $\square$

Theorem 8

[Unitarity of Transformation Operator] The operator M_θ defined in equation (9) is unitary, i.e.,

$$\int_{\mathbb{R}} |(M_\theta f)(t)|^2 \, dt = \int_{\mathbb{R}} |f(s)|^2 \, ds \forall f \in L^2(\mathbb{R}) \tag{12}$$

Proof. Let $f \in L^2(\mathbb{R})$. The L^2-norm of $M_\theta f$ is computed as follows:

$$\int_{\mathbb{R}} |(M_\theta f)(t)|^2 \, dt = \int_{\mathbb{R}} \left| \sqrt{\dot{\theta}(t)} \; f(\theta(t)) \right|^2 dt \tag{13}$$

$$= \int_{\mathbb{R}} \dot{\theta}(t) \, |f(\theta(t))|^2 \, dt \tag{14}$$

Apply the change of variables $s = \theta(t)$. Since θ is absolutely continuous and strictly increasing, its Jacobian is given by

$$ds = \dot{\theta}(t) \; dt \tag{15}$$

almost everywhere. As t ranges over $\mathbb{R}$, $s = \theta(t)$ ranges over $\mathbb{R}$ due to the bijectivity of θ. Therefore:

$$\int_{\mathbb{R}} \dot{\theta}(t) \, |f(\theta(t))|^2 \, dt = \int_{\mathbb{R}} |f(s)|^2 \, ds \tag{16}$$

This establishes equation (12). To complete the proof of unitarity, it remains to show that M_θ^{-1} is indeed the inverse of M_θ. For any $f \in L^2(\mathbb{R})$:

$$(M_\theta^{-1} M_\theta f)(s) = (M_\theta^{-1}) \left[\sqrt{\dot{\theta}(\cdot)} \; f(\theta(\cdot)) \right](s) \tag{17}$$

$$= \frac{\sqrt{\dot{\theta}(\theta^{-1}(s))} \; f(\theta(\theta^{-1}(s)))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{18}$$

$$= f(s) \tag{19}$$

where the last equality uses $\theta(\theta^{-1}(s)) = s$. Similarly, for any $g \in L^2(\mathbb{R})$:

$$(M_\theta M_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)} \; (M_\theta^{-1} g)(\theta(t)) \tag{20}$$

$$= \sqrt{\dot{\theta}(t)} \; \frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} \tag{21}$$

$$= \sqrt{\dot{\theta}(t)} \; \frac{g(t)}{\sqrt{\dot{\theta}(t)}} \tag{22}$$

$$= g(t) \tag{23}$$

Therefore

$$M_\theta M_\theta^{-1} = M_\theta^{-1} M_\theta = I \tag{24}$$

proving that M_θ is unitary. ☐

Corollary 9

[Measure Preservation] The transformation M_θ preserves the L^2-measure in the sense that for any measurable set $A \subseteq \mathbb{R}$

$$\int_A |(M_\theta f)(t)|^2 \, dt = \int_{\theta(A)} |f(s)|^2 \, ds \tag{25}$$

Proof. The proof follows the same change of variables argument as in Theorem 8, applied to the characteristic function of the set A. ☐

4.2 L^2-Norm Preservation

Theorem 10

[Measure Preservation] The transformation defined in equation (10) preserves the L^2-norm in the sense that

$$\int_I \mathrm{var}(Z(t)) \, dt = \int_{\theta(I)} \mathrm{var}(X(s)) \, ds \tag{26}$$

for any measurable set $I \subseteq \mathbb{R}$.

Proof. Using the change of variables $s = \theta(t)$ with $ds = \dot{\theta}(t) \, dt$:

$$\int_I \mathrm{var}(X(t)) \, dt = \int_I \mathrm{var}\Big(\sqrt{\dot{\theta}(t)} \; X(\theta(t))\Big) \, dt \tag{27}$$

$$= \int_I \dot{\theta}(t) \, \mathrm{var}(X(\theta(t))) \, dt \tag{28}$$

$$= \int_{\theta(I)} \mathrm{var}(X(s)) \, ds \tag{29}$$

☐

4.3 Oscillatory Representation

Theorem 11

[Oscillatory Form] The process $\{Z(t)\}$ defined in equation (10) is oscillatory with oscillatory functions

$$\phi_t(\omega) = A_t(\omega)\, e^{i\omega t} = \sqrt{\dot\theta(t)}\ e^{i\omega\theta(t)} \tag{30}$$

and gain functions

$$A_t(\omega) = \sqrt{\dot\theta(t)}\ e^{i\omega(\theta(t)-t)} \tag{31}$$

Proof. From the spectral representation (7) of the stationary process $X(t)$:

$$\begin{aligned} X(t) &= \sqrt{\dot\theta(t)}\, X(\theta(t)) && (32)\\ &= \sqrt{\dot\theta(t)} \int_{-\infty}^{\infty} e^{i\omega\theta(t)}\ d\,\Phi(\omega) && (33)\\ &= \int_{-\infty}^{\infty} \sqrt{\dot\theta(t)}\ e^{i\omega\theta(t)}\ d\,\phi(\omega) && (34)\\ &= \int_{-\infty}^{\infty} \phi_t(\omega)\ d\,\Phi(\omega) && (35) \end{aligned}$$

where

$$\phi_t(\omega) = \sqrt{\dot\theta(t)}\ e^{i\omega\theta(t)} \tag{36}$$

To verify this is an oscillatory representation according to Definition 3, express $\phi_t(\omega)$ in the form of a function of the time-dependent gain $A_t(\lambda)$ as required

$$\begin{aligned} \phi_t(\omega) &= A_t(\omega)\, e^{i\omega t}\\ &= \sqrt{\dot\theta(t)}\ e^{i\omega(\theta(t)-t)}\ e^{i\omega t}\\ &= \sqrt{\dot\theta(t)} e^{i\omega(\theta(t)-t+t)}\\ &= \sqrt{\dot\theta(t)}\ e^{i\omega\theta(t)} \end{aligned} \tag{37}$$

where

$$A_t(\omega) = \sqrt{\dot\theta(t)}\ e^{i\omega(\theta(t)-t)} \tag{38}$$

.

Since $\dot{\theta}(t) \geq 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of measure zero, the function $A_t(\omega)$ is well-defined almost everywhere. Moreover, $A_t(\cdot) \in L^2(\mu)$ for each t since:

$$\begin{aligned}
\int_{-\infty}^{\infty} |A_t(\omega)|^2 \, d\,\mu(\omega) = & \int_{-\infty}^{\infty} \dot{\theta}(t) \; d\,\mu(\omega) && (39)\\
= & \dot{\theta}(t) \int_{-\infty}^{\infty} d\,\mu(\omega) \\
= & \dot{\theta}(t)\, \mu(\mathbb{R}) < \infty && (40)
\end{aligned}$$

where the finiteness follows from μ being a finite measure and $\dot{\theta}(t)$ being finite almost everywhere. □

4.4 Envelope and Evolutionary Spectrum

Corollary 12

[Evolutionary Spectrum] The evolutionary power spectrum is

$$\begin{aligned}
d\,F_t(\omega) \; & = |A_t(\omega)|^2 \; d\,\mu(\omega) \\
& = \dot{\theta}(t) \; d\,\mu(\omega)
\end{aligned} \tag{41}$$

Proof. By Definition 3 and the envelope from Equation 4, the evolutionary power spectrum is:

$$\begin{aligned}
d\,F_t(\omega) = & |A_t(\omega)|^2 \; d\,\mu(\omega) && (42)\\
= & \left| \sqrt{\dot{\theta}(t)} \; e^{i\omega(\theta(t)-t)} \right|^2 \; d\,\mu(\omega) && (43)\\
= & \dot{\theta}(t) \, |e^{i\omega(\theta(t)-t)}|^2 \; d\,\mu(\omega) && (44)\\
= & \dot{\theta}(t) \; d\,\mu(\omega) && (45)
\end{aligned}$$

since

$$|e^{i\alpha}| = 1 \forall \alpha \in \mathbb{R} \tag{46}$$

□

5 Operator Conjugation

Theorem 13

[Operator Conjugation] Let T_K be the integral covariance operator defined by

$$(T_K f)(t) = \int_{-\infty}^{\infty} K(|t-s|)\, f(s)\, ds \tag{47}$$

where $K(h)$ is the stationary kernel

$$K(h) = \int_{-\infty}^{\infty} S(\lambda) e^{i\lambda h} \mathrm{d}\lambda \tag{48}$$

, and let T_{K_θ} be the integral covariance operator defined by

$$\begin{aligned}(T_{K_\theta} f)(t) &= \int_{-\infty}^{\infty} K_\theta(s,t) f(s)\, ds \\ &= \int_{-\infty}^{\infty} K(|\theta(t)-\theta(s)|)\, \sqrt{\dot{\theta}(t)\dot{\theta}(s)}\, f(s)\, ds\end{aligned} \tag{49}$$

for the unitarily time-changed kernel

$$K_\theta(s,t) = K(|\theta(t)-\theta(s)|) \sqrt{\dot{\theta}(t)\dot{\theta}(s)} \tag{50}$$

. Then

$$T_{K_\theta} = M_\theta T_K M_\theta^{-1} \tag{51}$$

Proof. For any $g \in L^2(\mathbb{R})$, compute $(M_\theta T_K M_\theta^{-1} g)(t)$:

$$(M_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\theta'(\theta^{-1}(s))}}, \tag{52}$$

$$(T_K M_\theta^{-1} g)(t) = \int_{-\infty}^{\infty} K(|t-s|) \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}}\, ds. \tag{53}$$

Apply the change of variables $u = \theta^{-1}(s)$, so $s = \theta(u)$ and $ds = \dot{\theta}(u)\, du$:

$$(T_K M_\theta^{-1} g)(t) = \int_{-\infty}^{\infty} K(|t-\theta(u)|) \frac{g(u)}{\sqrt{\dot{\theta}(u)}}\, \dot{\theta}(u)\, du \tag{54}$$

$$= \int_{-\infty}^{\infty} K(|t-\theta(u)|)\, g(u) \sqrt{\dot{\theta}(u)}\, du. \tag{55}$$

Now apply M_θ:

$$(M_\theta T_K M_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)}\, (T_K M_\theta^{-1} g)(\theta(t)) \tag{56}$$

$$= \sqrt{\dot{\theta}(t)} \int_{-\infty}^{\infty} K(|\theta(t) - \theta(u)|)\, g(u) \sqrt{\dot{\theta}(u)}\, d u. \tag{57}$$

Apply the change of variables $s = \theta(u)$ in the reverse direction:

$$(M_\theta T_K M_\theta^{-1} g)(t) = \int_{-\infty}^{\infty} K(|\theta(t) - \theta(s)|)\, g(s)\, d s \tag{58}$$

$$= (T_{K_\theta} g)(t) \tag{59}$$

This establishes the conjugation relation (51). □

6 Expected Zero Count

Theorem 14

[Expected Zero-Counting Function] Let $\theta \in \mathcal{F}$ and let

$$K(\tau) = \operatorname{cov}(X(t), X(\tau)) \tag{60}$$

be twice differentiable at $\tau = 0$. The expected number of zeros of the process X_t in $[a, b]$ is

$$\mathbb{E}[N_{[a,b]}] = \sqrt{-\ddot{K}(0)}\ (\theta(b) - \theta(a)) \tag{61}$$

Proof. The covariance function of the time-changed process is

$$K_\theta(s, t) = \operatorname{cov}(X_s, X_t) = \sqrt{\dot{\theta}(s)\, \dot{\theta}(t)}\ K(|\theta(t) - \theta(s)|) \tag{62}$$

For the zero-crossing analysis, consider the normalized process. By the Kac-Rice formula:

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{-\lim_{s \to t} \frac{\partial^2}{\partial s\, \partial t} K_\theta(s, t)}\ d t \tag{63}$$

Computing the mixed partial derivative:

$$\frac{\partial}{\partial t} K_\theta(s, t) = \frac{1}{2} \frac{\ddot{\theta}(t)}{\sqrt{\dot{\theta}(t)}} \sqrt{\theta'(s)}\, K(|\theta(t) - \theta(s)|) \tag{64}$$

$$+ \sqrt{\dot{\theta}(s)\, \dot{\theta}(t)}\, \dot{K}(|\theta(t) - \theta(s)|) \operatorname{sgn}(\theta(t) - \theta(s))\, \dot{\theta}(t). \tag{65}$$

Taking the limit as $s \to t$ and using the fact that $\dot{K}(0) = 0$ for stationary processes:

$$\lim_{s \to t} \frac{\partial^2}{\partial s \, \partial t} K_\theta(s,t) = \dot{\theta}(s) \, \dot{\theta}(t) \, \ddot{K}(0) \tag{66}$$

$$= \dot{\theta}(t)^2 \, \ddot{K}(0) \tag{67}$$

Substituting into the Kac-Rice formula:

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{-\dot{\theta}(t)^2 \, \ddot{K}(0)} \; d t \tag{68}$$

$$= \sqrt{-\ddot{K}(0)} \int_a^b \dot{\theta}(t) \; d t \tag{69}$$

$$= \sqrt{-\ddot{K}(0)} \; (\theta(b) - \theta(a)) \tag{70}$$

Here the second equality uses $\dot{\theta}(t) \geq 0$ almost everywhere. □

7 Conclusion

This analysis establishes that Gaussian processes generated by measure-preserving bijective time changes of stationary processes form a well-defined subclass of oscillatory processes. The key contributions include:

1. The rigorous construction of the unitary operator M_θ and its inverse, with proper treatment of the case where $\dot{\theta}(t) = 0$ on sets of measure zero.

2. The explicit oscillatory representation with envelope function $A_t(\omega) = \sqrt{\dot{\theta}(t)} \, e^{i\omega(\theta(t) - t)}$.

3. The evolutionary power spectrum formula $d F_t(\omega) = \dot{\theta}(t) \, d \mu(\omega)$.

4. The operator conjugation relationship $T_{K_\theta} = M_\theta \, T_K \, M_\theta^{-1}$.

5. A closed-form expression for the expected zero count in terms of the range of the time transformation.

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