The Radial Solution of the Two-Dimensional Schrödinger Equation with Circular Symmetry

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August 18, 2025

Theorem 1. *[Separation of Variables for 2D Schrödinger Equation] Consider a particle of mass m in a two-dimensional radially symmetric potential $V(r)$. The time-independent Schrödinger equation in polar coordinates (r, θ) admits separable solutions of the form $\psi(r, \theta) = R(r)\,\Theta(\theta)$.*

Proof. The time-independent Schrödinger equation in polar coordinates is:

$$-\frac{\hbar^2}{2\,m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\psi(r, \theta) + V(r)\,\psi(r, \theta) = E\,\psi(r, \theta) \tag{1}$$

Substituting $\psi(r, \theta) = R(r)\,\Theta(\theta)$ and dividing by $R(r)\,\Theta(\theta)$:

$$-\frac{\hbar^2}{2\,m}\left(\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)}\right) + V(r) = E \tag{2}$$

Multiplying by r^2 and rearranging:

$$r^2\left[-\frac{\hbar^2}{2\,m}\left(\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)}\right) + V(r) - E\right] = -\frac{\hbar^2}{2\,m}\frac{\Theta''(\theta)}{\Theta(\theta)} \tag{3}$$

Since the left side depends only on r and the right side depends only on θ, both sides must equal a constant. Let this separation constant be $\frac{\hbar^2\,m_l^2}{2\,m}$ where m_l is an integer. □

Theorem 2. *[Angular Part Solution] The angular part of the separated wave function satisfies $\Theta(\theta) = e^{i m_l \theta}$ where $m_l \in \mathbb{Z}$.*

Proof. From the separation procedure, the angular equation is:

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -m_l^2 \tag{4}$$

This gives the differential equation:

$$\Theta''(\theta) + m_l^2\,\Theta(\theta) = 0 \tag{5}$$

The general solution is:

$$\Theta(\theta) = A\,e^{i m_l \theta} + B\,e^{-i m_l \theta} \tag{6}$$

For single-valued wave functions, periodicity requires $\Theta(\theta + 2\,\pi) = \Theta(\theta)$, which implies:

$$e^{i m_l \cdot 2\pi} = 1 \tag{7}$$

This condition is satisfied if and only if $m_l \in \mathbb{Z}$. Without loss of generality, one can choose the normalized form $\Theta(\theta) = \frac{1}{\sqrt{2\,\pi}}\, e^{i m_l \theta}$. □

Theorem 3. *[Radial Equation for Free Particle] For a free particle in two dimensions ($V(r) = 0$), the radial part of the wave function satisfies Bessel's differential equation of integer order $|m_l|$.*

Proof. From the separation of variables with $V(r) = 0$, the radial equation becomes:

$$-\frac{\hbar^2}{2\,m}\left(R''(r) + \frac{1}{r}\,R'(r)\right) + \frac{\hbar^2\,m_l^2}{2\,m\,r^2}\,R(r) = E\,R(r) \tag{8}$$

Rearranging and defining $k^2 = \frac{2\,m\,E}{\hbar^2}$:

$$R''(r) + \frac{1}{r}\,R'(r) + \left(k^2 - \frac{m_l^2}{r^2}\right) R(r) = 0 \tag{9}$$

Making the substitution $x = k\,r$, let $u(x) = R(r) = R\,(x\,/\,k)$. Then:

$$\frac{d\,R}{d\,r} = k\,\frac{d\,u}{d\,x}\,\frac{d^2\,R}{d\,r^2} = k^2\,\frac{d^2\,u}{d\,x^2} \tag{10}$$

Substituting into the radial equation:

$$k^2\,\frac{d^2\,u}{d\,x^2} + \frac{k}{r}\,k\,\frac{d\,u}{d\,x} + \left(k^2 - \frac{m_l^2}{r^2}\right) u = 0 \tag{11}$$

Since $r = x\,/\,k$, this becomes:

$$x^2\,\frac{d^2\,u}{d\,x^2} + x\,\frac{d\,u}{d\,x} + (x^2 - m_l^2)\,u = 0 \tag{12}$$

This is precisely Bessel's differential equation of order $|m_l|$. □

Theorem 4. *[General Solution in Terms of Bessel Functions] The general solution to the radial equation for a free particle in two dimensions is:*

$$R(r) = A\,J_{|m_l|}\,(k\,r) + B\,Y_{|m_l|}\,(k\,r) \tag{13}$$

where $J_{|m_l|}$ and $Y_{|m_l|}$ are Bessel functions of the first and second kind, respectively, of order $|m_l|$.

Proof. From Theorem 3, the radial equation is Bessel's differential equation of order $|m_l|$. The standard theory of Bessel functions establishes that the general solution to:

$$x^2\,y'' + x\,y' + (x^2 - \nu^2)\,y = 0 \tag{14}$$

is given by:

$$y(x) = c_1\,J_\nu(x) + c_2\,Y_\nu(x) \tag{15}$$

where J_ν and Y_ν are linearly independent solutions for non-integer ν, and for integer ν, Y_ν is defined as the appropriate limit. Since $|m_l|$ is a non-negative integer, and with $x = k\,r$, the general solution is:

$$R(r) = A\,J_{|m_l|}\,(k\,r) + B\,Y_{|m_l|}\,(k\,r) \tag{16}$$ □

Corollary 5. *[Regular Solution at Origin] For wave functions that must be finite at the origin $r = 0$, the coefficient $B = 0$, yielding:*

$$R(r) = A\,J_{|m_l|}\,(k\,r) \tag{17}$$

Proof. The Bessel function of the second kind $Y_{|m_l|}\,(k\,r)$ has a logarithmic singularity at $r = 0$ for $m_l = 0$ and diverges as $r^{-|m_l|}$ for $m_l \neq 0$. Since physical wave functions must be square-integrable near the origin, one requires $B = 0$. □

Theorem 6. *[Complete Solution] The complete separable solution for a free particle in two dimensions with circular symmetry is:*

$$\psi(r, \theta) = A\,J_{|m_l|}\,(k\,r)\,e^{i m_l \theta} \tag{18}$$

where $m_l \in \mathbb{Z}$, $k = \sqrt{\frac{2\,m\,E}{\hbar^2}}$, and A is a normalization constant.

Proof. This follows directly from combining Theorems 2, 4, and Corollary 1. The angular part contributes $e^{i m_l \theta}$ with integer m_l, and the radial part contributes $A\,J_{|m_l|}\,(k\,r)$ for regularity at the origin. □