Evaluation of an Integral Involving Hypergeometric Functions

Introduction

We consider the integral:

$$I = \int_{-1}^{1} {}_2F_1\left(-m, m+1; 1; \frac{1}{2} - \frac{x}{2}\right) {}_2F_1\left(-n, n+1; 1; \frac{1}{2} - \frac{x}{2}\right) e^{ixy}\, dx \tag{1}$$

where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function, and m, n are non-negative integers. This document provides a rigorous step-by-step derivation of the result.

Step 1: Expanding the Hypergeometric Functions

The hypergeometric function ${}_2F_1(a, b; c; z)$ has the finite series representation:

$${}_2F_1(-p, b; c; z) = \sum_{k=0}^{p} \frac{(-p)_k (b)_k}{(c)_k\, k!} z^k \tag{2}$$

when p is a non-negative integer. Here, $(a)_k = a\,(a+1)\,(a+2)\,\cdots\,(a+k-1)$ is the Pochhammer symbol.

For the integral, we expand both hypergeometric functions:

$${}_2F_1\left(-m, m+1; 1; \frac{1}{2} - \frac{x}{2}\right) = \sum_{k=0}^{m} \frac{(-m)_k\,(m+1)_k}{(1)_k\, k!} \left(\frac{1}{2} - \frac{x}{2}\right)^k \tag{3}$$

$${}_2F_1\left(-n, n+1; 1; \frac{1}{2} - \frac{x}{2}\right) = \sum_{l=0}^{n} \frac{(-n)_l\,(n+1)_l}{(1)_l\, l!} \left(\frac{1}{2} - \frac{x}{2}\right)^l \tag{4}$$

Substituting these into the integral:

$$I = \int_{-1}^{1} \left[\sum_{k=0}^{m} \frac{(-m)_k\,(m+1)_k}{k!} \left(\frac{1}{2} - \frac{x}{2}\right)^k\right] \left[\sum_{l=0}^{n} \frac{(-n)_l\,(n+1)_l}{l!} \left(\frac{1}{2} - \frac{x}{2}\right)^l\right] e^{ixy}\, dx \tag{5}$$

Expanding the double sum:

$$I = \sum_{k=0}^{m} \sum_{l=0}^{n} \frac{(-m)_k\,(m+1)_k}{k!}\,\frac{(-n)_l\,(n+1)_l}{l!} \int_{-1}^{1} \left(\frac{1}{2} - \frac{x}{2}\right)^{k+l} e^{ixy}\,d\,x \tag{6}$$

Step 2: Evaluating the Integral

Let $s = k + l$. The integral to evaluate is:

$$I_s = \int_{-1}^{1} \left(\frac{1}{2} - \frac{x}{2}\right)^{s} e^{ixy}\,d\,x \tag{7}$$

Rewriting $\frac{1}{2} - \frac{x}{2}$:

$$\left(\frac{1}{2} - \frac{x}{2}\right)^{s} = \frac{1}{2^s}\,(1-x)^s \tag{8}$$

Thus:

$$I_s = \frac{1}{2^s} \int_{-1}^{1} (1-x)^s\,e^{ixy}\,d\,x \tag{9}$$

Substitution: $u = 1 - x$

Set $u = 1 - x$, so $x = 1 - u$ and $d\,x = -d\,u$. The limits of integration change:

$$x = -1 \Longrightarrow u = 2, \quad x = 1 \Longrightarrow u = 0 \tag{10}$$

The integral becomes:

$$I_s = \frac{1}{2^s} \int_{2}^{0} u^s\,e^{iy(1-u)}\,(-d\,u) = \frac{1}{2^s} \int_{0}^{2} u^s\,e^{iy}\,e^{-iuy}\,d\,u \tag{11}$$

Factoring out e^{iy}:

$$I_s = \frac{e^{iy}}{2^s} \int_{0}^{2} u^s\,e^{-iuy}\,d\,u \tag{12}$$

Known Result for the Integral

The integral $\int_0^2 u^s\,e^{-iuy}\,d\,u$ is a standard result:

$$\int_{0}^{2} u^s\,e^{-iuy}\,d\,u = \frac{\Gamma\,(s+1)}{(-i\,y)^{s+1}} \left[1 - e^{-2iy} \sum_{j=0}^{s} \frac{(2\,i\,y)^j}{j!}\right] \tag{13}$$

Substituting this back:

$$I_s = \frac{e^{iy}\,\Gamma(s+1)}{2^s\,(-i\,y)^{s+1}}\left[1 - e^{-2iy}\sum_{j=0}^{s}\frac{(2\,i\,y)^j}{j!}\right] \tag{14}$$

Step 3: Combining Results

Returning to the full integral:

$$I = \sum_{k=0}^{m}\sum_{l=0}^{n}\frac{(-m)_k\,(m+1)_k}{k!}\,\frac{(-n)_l\,(n+1)_l}{l!}\cdot\frac{e^{iy}\,\Gamma(k+l+1)}{2^{k+l}\,(-i\,y)^{k+l+1}}\left[1 - e^{-2iy}\sum_{j=0}^{k+l}\frac{(2\,i\,y)^j}{j!}\right] \tag{15}$$

Let $s = k+l$. For fixed s, k ranges from $\max(0, s-n)$ to $\min(s, m)$. Rewriting the double sum:

$$I = e^{iy}\sum_{s=0}^{m+n}\frac{\Gamma(s+1)}{2^s\,(-i\,y)^{s+1}}\left[\sum_{k=\max(0,s-n)}^{\min(s,m)}\frac{(-m)_k\,(m+1)_k}{k!}\,\frac{(-n)_{s-k}\,(n+1)_{s-k}}{(s-k)!}\right]\left[1 - e^{-2iy}\sum_{j=0}^{s}\frac{(2\,i\,y)^j}{j!}\right] \tag{16}$$

The inner sum is recognized as a hypergeometric function:

$$\sum_{k=\max(0,s-n)}^{\min(s,m)}\frac{(-m)_k\,(m+1)_k}{k!}\,\frac{(-n)_{s-k}\,(n+1)_{s-k}}{(s-k)!} = {}_3F_2(-m,-n,-s;m+1,n+1;1) \tag{17}$$

Thus, the final result is:

$$I = e^{iy}\sum_{s=0}^{m+n}\frac{\Gamma(s+1)}{2^s\,(-i\,y)^{s+1}}\,{}_3F_2(-m,-n,-s;m+1,n+1;1)\left[1 - e^{-2iy}\sum_{j=0}^{s}\frac{(2\,i\,y)^j}{j!}\right] \tag{18}$$

Conclusion

The integral has been evaluated exactly in terms of hypergeometric functions and exponential terms. All results follow directly from standard mathematical formulas for hypergeometric functions and Fourier-like integrals.