Resolutions of Identities Are Spectral Eigenfunction Representations

In the case of a spectral decomposition of an operator with a discrete spectrum, the discrete measure can be represented as an integral over Dirac delta functions shifted by the eigenvalues of the operator. This integral representation uses the spectral measure associated with the operator.

For an operator A with eigenvalues $\{\lambda_n\}$ and corresponding eigenfunctions $\{\psi_n\}$, the spectral measure E can be written as:

$$E(\lambda) = \sum_n \delta(\lambda - \lambda_n) |\psi_n\rangle\langle\psi_n| \tag{1}$$

Here, $\delta(\lambda - \lambda_n)$ is the Dirac delta function centered at λ_n. This expression represents the spectral projection operator for the eigenvalue λ_n, and the integral of any function f over this spectral measure gives the operator:

$$f(A) = \int f(\lambda)\ d\,E(\lambda) = \sum_n f(\lambda_n) |\psi_n\rangle\langle\psi_n| \tag{2}$$

This integral sums over the eigenvalues, applying the function f to each eigenvalue and weighting by the corresponding spectral projection. This representation is particularly useful in quantum mechanics and functional analysis for defining functions of operators and studying their properties.