The Eigenfunctions of Stationary and Oscillatory Processes on the Real Line

by Stephen Crowley

August 5, 2025

Definition 1. (Stationary Process) *A stochastic process* $\{X(t), t \in \mathbb{R}\}$ *is called stationary if its covariance function satisfies*

$$R(s,t) = R(t-s)$$

for all $s, t \in \mathbb{R}$.

Definition 2. (Oscillatory Process (Priestley)) *A stochastic process* $\{X(t), t \in \mathbb{R}\}$ *is called oscillatory if it possesses an evolutionary spectral representation*

$$X(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, d\, Z(\omega)$$

where $A(t,\omega)$ *is the evolutionary amplitude function and* $Z(\omega)$ *is an orthogonal increment process.*

Theorem 3. (Eigenfunction Property for Stationary Processes) *Let* $\{X(t), t \in \mathbb{R}\}$ *be a stationary process with covariance function* $R(\tau)$ *and covariance operator*

$$(K\, f)(t) = \int_{-\infty}^{\infty} R(t-s)\, f(s)\, d\, s$$

Then the complex exponentials $e^{i\omega t}$ *are eigenfunctions of* K *with eigenvalues equal to the power spectral density* $S(\omega)$.

Proof. Consider the action of K on $e^{i\omega t}$:

$$(K\, e^{i\omega t})(t) = \int_{-\infty}^{\infty} R(t-s)\, e^{i\omega s}\, d\, s$$

Substituting $\tau = t - s$:

$$\begin{aligned}&= e^{i\omega t} \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau\\&= e^{i\omega t} \cdot S(\omega)\end{aligned}$$

where

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau$$

is the power spectral density by the Wiener-Khintchine theorem. □

Theorem 4. (Eigenfunction Property for Oscillatory Processes) *Let $\{X(t), t \in \mathbb{R}\}$ be an oscillatory process with evolutionary spectral representation*

$$X(t) = \int_{-\infty}^{\infty} A(t, \omega)\, e^{i\omega t}\, dZ(\omega)$$

and covariance function

$$C(s, t) = \int_{-\infty}^{\infty} A(s, \omega)\, A^*(t, \omega)\, dF(\omega)$$

where $F(\omega)$ is the spectral measure. Then the oscillatory functions

$$\phi(t, \omega) = A(t, \omega)\, e^{i\omega t}$$

are eigenfunctions of the covariance operator

$$(Kf)(t) = \int_{-\infty}^{\infty} C(t, s)\, f(s)\, ds$$

with eigenvalues $dF(\omega)$.

Proof. Consider the action of K on the oscillatory function $\phi(s, \omega) = A(s, \omega)\, e^{i\omega s}$:

$$(K\phi)(t) = \int_{-\infty}^{\infty} C(t, s)\, A(s, \omega)\, e^{i\omega s}\, ds$$

Substitute $C(t, s) = \int A(t, \lambda)\, A^*(s, \lambda)\, dF(\lambda)$:

$$\begin{aligned}(K\phi)(t) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} A(t, \lambda)\, A^*(s, \lambda)\, dF(\lambda) \right] A(s, \omega)\, e^{i\omega s}\, ds\\&= \int_{-\infty}^{\infty} A(t, \lambda) \left[\int_{-\infty}^{\infty} A^*(s, \lambda)\, A(s, \omega)\, e^{i\omega s}\, ds \right] dF(\lambda)\end{aligned}$$

By Fubini's theorem, the order of integration may be exchanged:

$$=\int_{-\infty}^{\infty} A(t,\lambda)\left[\int_{-\infty}^{\infty} A^*(s,\lambda)\, A(s,\omega)\, e^{i\omega s}\, d\, s\right] d\, F(\lambda)$$

The inner integral represents the orthogonality condition in the evolutionary spectral representation:

$$\int_{-\infty}^{\infty} A^*(s,\lambda)\, A(s,\omega)\, e^{i\omega s}\, d\, s = \delta\,(\lambda-\omega)$$

Therefore

$$(K\,\phi)(t) = \int_{-\infty}^{\infty} A(t,\lambda)\,\delta\,(\lambda-\omega)\, d\, F(\lambda) = A(t,\omega)\, d\, F(\omega) = \phi(t,\omega)\cdot d\, F(\omega) \qquad \square$$

Lemma 5. (Orthogonality Property) *For the evolutionary spectral representation, the orthogonality condition*

$$\int_{-\infty}^{\infty} A^*(s,\lambda)\, A(s,\omega)\, e^{i\omega s}\, d\, s = \delta\,(\lambda-\omega)$$

follows from the requirement that $d\, Z(\omega)$ be an orthogonal increment process.

Proof. The orthogonality of $d\, Z(\omega)$ requires

$$\mathbb{E}[d\, Z(\lambda)\, d\, Z^*(\omega)] = \delta\,(\lambda-\omega)\, d\, F(\lambda)$$

This condition, with the evolutionary spectral representation, directly implies the stated orthogonality property for the amplitude functions. $\square$

Theorem 6. (Real-Valued Oscillatory Processes) *Let $Z(t)$ be a sample path realization of an oscillatory process (with evolutionary spectral representation)*

$$X(t) = \int_{-\infty}^{\infty} A_\lambda(t)\, e^{i\lambda t}\, d\,\Phi(\lambda) \tag{1}$$

where $A_t(\omega)$ is the gain function and $\Phi(\omega)$ is an orthogonal increment process. Then $X(t)$ is real-valued if and only if the following conditions hold:

$$A(t,\omega) = A^*(t,-\omega) \qquad \textit{(Gain Conjugate Symmetry)} \tag{2}$$

$$d\, Z\,(-\omega) = d\, Z^*(\omega) \qquad \textit{(Increment Conjugate Symmetry)} \tag{3}$$

Proof. **Necessity:** Assume $X(t)$ is real-valued, so

$$X(t) = X^*(t) \forall t \in \mathbb{R} \tag{4}$$

Taking the complex conjugate of the evolutionary spectral representation:

$$X^*(t) = \left[\int_{-\infty}^{\infty} A(t, \omega)\, e^{i\omega t}\, d\,Z(\omega) \right]^* = \int_{-\infty}^{\infty} A^*(t, \omega)\, e^{-i\omega t}\, d\,Z^*(\omega) \tag{5}$$

Making the substitution $\omega \mapsto -\omega$ in this integral:

$$X^*(t) = \int_{-\infty}^{\infty} A^*(t, -\omega)\, e^{i\omega t}\, d\,Z^*\,(-\omega) \tag{6}$$

Since $X(t) = X^*(t)$, we have:

$$\int_{-\infty}^{\infty} A(t, \omega)\, e^{i\omega t}\, d\,Z(\omega) = \int_{-\infty}^{\infty} A^*(t, -\omega)\, e^{i\omega t}\, d\,Z^*\,(-\omega) \tag{7}$$

By the uniqueness of the evolutionary spectral representation, this equality holds for all t if and only if:

$$A(t, \omega) = A^*(t, -\omega) \tag{8}$$

$$d\,Z(\omega) = d\,Z^*\,(-\omega) \tag{9}$$

Sufficiency: Assume the two conjugate symmetry conditions hold. Then:

$$X^*(t) = \int_{-\infty}^{\infty} A^*(t, \omega)\, e^{-i\omega t}\, d\,Z^*(\omega) \tag{10}$$

$$= \int_{-\infty}^{\infty} A(t, -\omega)\, e^{-i\omega t}\, d\,Z\,(-\omega) \tag{11}$$

Substituting $\omega \mapsto -\omega$:

$$X^*(t) = \int_{-\infty}^{\infty} A(t, \omega)\, e^{i\omega t}\, d\,Z(\omega) = X(t)$$

Therefore, $X(t)$ is real-valued. □

Theorem 7. (Eigenfunction Conjugate Pairs) *Under the conditions for real-valued oscillatory processes, the eigenfunctions* $\phi(t,\omega) = A(t,\omega)\, e^{i\omega t}$ *satisfy the conjugate symmetry relation*

$$\phi^*(t,\omega) = \phi(t,-\omega) \tag{12}$$

Proof. Given that $A(t,\omega) = A^*(t,-\omega)$, we compute:

$$\begin{aligned} \phi^*(t,\omega) &= [A(t,\omega)\, e^{i\omega t}]^* \\ &= A^*(t,\omega)\, e^{-i\omega t} \\ &= A(t,-\omega)\, e^{-i\omega t} \quad \text{(by amplitude symmetry)} \\ &= \phi(t,-\omega) \end{aligned} \tag{13}$$

□

Theorem 8. (Equivalence of Evolutionary Spectral and Filter Representations) *Let* $X(t)$ *be a stochastic process. The evolutionary spectral representation*

$$X(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, dZ(\omega) \tag{14}$$

where $A(t,\omega)$ *is the gain function and* $dZ(\omega)$ *is an orthogonal increment process, is equivalent to the time-domain filter representation*

$$X(t) = \int_{-\infty}^{\infty} h_t(t-s)\, dW(s) \tag{15}$$

where $h_t(t-s)$ *is a time-dependent filter kernel and* $dW(s)$ *is an orthogonal increment process.*

Proof. The filter kernel $h_t(t-s)$ is related to the gain function and oscillatory function by the Fourier transform relationships:

$$\begin{aligned} h_t(t-s) &= \int_{-\infty}^{\infty} \phi(t,\omega)\, e^{-i\omega(t-s)}\, d\omega && (16) \\ &= \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, e^{-i\omega(t-s)}\, d\omega && (17) \\ &= \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega s}\, d\omega && (18) \end{aligned}$$

where $\phi(t,\omega) = A(t,\omega)\, e^{i\omega t}$ is the oscillatory function.

The inverse relationships are:

$$A(t,\omega) = \int_{-\infty}^{\infty} h_t\,(t-s)\,e^{-i\omega s}\,d\,s \tag{19}$$

$$\phi(t,\omega) = \int_{-\infty}^{\infty} h_t(u)\,e^{-i\omega(t-u)}\,d\,u \tag{20}$$

To establish the equivalence of the two representations, substitute the orthogonal increment relationship $d\,Z(\omega) = \int_{-\infty}^{\infty} e^{-i\omega s}\,d\,W(s)$ into the evolutionary spectral representation:

$$\begin{aligned}
X(t) &= \int_{-\infty}^{\infty} A(t,\omega)\,e^{i\omega t}\,d\,Z(\omega) && (21)\\
&= \int_{-\infty}^{\infty} A(t,\omega)\,e^{i\omega t}\left[\int_{-\infty}^{\infty} e^{-i\omega s}\,d\,W(s)\right]d\,\omega && (22)\\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} A(t,\omega)\,e^{i\omega t}\,e^{-i\omega s}\,d\,\omega\right]d\,W(s) && (23)\\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} A(t,\omega)\,e^{i\omega(t-s)}\,d\,\omega\right]d\,W(s) && (24)\\
&= \int_{-\infty}^{\infty} h_t\,(t-s)\,d\,W(s) && (25)
\end{aligned}$$

where the last equality follows from the definition of $h_t\,(t-s)$ with $u = t-s$. □

Theorem 9. (Fourier Transform Relationships) *The gain function $A(t,\omega)$, oscillatory function $\phi(t,\omega)$, and filter kernel $h_t(u)$ satisfy the following Fourier transform relationships:*

$$\begin{aligned}
A(t,\omega) &= \int_{-\infty}^{\infty} h_t\,(t-s)\,e^{-i\omega s}\,d\,s && (26)\\
\phi(t,\omega) &= A(t,\omega)\,e^{i\omega t} = \int_{-\infty}^{\infty} h_t(u)\,e^{-i\omega(t-u)}\,d\,u && (27)\\
h_t\,(t-s) &= \int_{-\infty}^{\infty} A(t,\omega)\,e^{i\omega s}\,d\,\omega = \int_{-\infty}^{\infty} \phi(t,\omega)\,e^{-i\omega(t-s)}\,d\,\omega && (28)
\end{aligned}$$

Proof. The proof establishes each transform relationship directly.

For the first relationship, apply the inverse Fourier transform to $h_t\,(t-s)$:

$$\begin{aligned}
A(t,\omega) &= \mathcal{F}_s^{-1}\,[h_t\,(t-s)] && (29)\\
&= \int_{-\infty}^{\infty} h_t\,(t-s)\,e^{-i\omega s}\,d\,s && (30)
\end{aligned}$$

For the oscillatory function relationship, substitute the definition $\phi(t,\omega)=A(t,\omega)\,e^{i\omega t}$:

$$\begin{aligned}\phi(t,\omega) &= A(t,\omega)\,e^{i\omega t} && (31)\\ &= \left[\int_{-\infty}^{\infty} h_t\,(t-s)\,e^{-i\omega s}\,d\,s\right] e^{i\omega t} && (32)\\ &= \int_{-\infty}^{\infty} h_t\,(t-s)\,e^{-i\omega s}\,e^{i\omega t}\,d\,s && (33)\\ &= \int_{-\infty}^{\infty} h_t\,(t-s)\,e^{-i\omega(s-t)}\,d\,s && (34)\\ &= \int_{-\infty}^{\infty} h_t(u)\,e^{-i\omega(t-u)}\,d\,u && (35)\end{aligned}$$

where $u=t-s$ in the last step.

For the inverse relationships, apply the Fourier transform to recover $h_t\,(t-s)$:

$$\begin{aligned}h_t\,(t-s) &= \mathcal{F}_\omega^{-1}\,[A(t,\omega)\,e^{i\omega s}] && (36)\\ &= \int_{-\infty}^{\infty} A(t,\omega)\,e^{i\omega s}\,d\,\omega && (37)\end{aligned}$$

Similarly:

$$\begin{aligned}h_t\,(t-s) &= \mathcal{F}_\omega^{-1}\,[\phi(t,\omega)\,e^{-i\omega t}] && (38)\\ &= \int_{-\infty}^{\infty} \phi(t,\omega)\,e^{-i\omega t}\,e^{i\omega(t-s)}\,d\,\omega && (39)\\ &= \int_{-\infty}^{\infty} \phi(t,\omega)\,e^{-i\omega(t-s)}\,d\,\omega && (40)\end{aligned}$$

$\square$