A Constructive Solution for the Exact Eigenfunctions of Stationary Gaussian Processes

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Abstract

A constructive method yielding exact eigenfunctions through uniform expansions is presented for stationary Gaussian processes. While Mercer's theorem guarantees existence and Aronszajn's theory provides the RKHS framework, this method explicitly constructs these objects through finite expressions. The triangular structure that emerges reveals deep connections to the Cameron-Martin space.

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1 Introduction

For a stationary Gaussian process with covariance operator Q, there exists a unique positive operator $Q^{1/2}$ such that $(Q^{1/2})^2 = Q$. The Cameron-Martin space H is defined as:

$$H = \{Q^{1/2} h: h \in L^2(\mathbb{R})\}$$

This space exists for every Gaussian process, though explicit construction of $Q^{1/2}$ is unnecessary. The method works directly with uniform expansions of the kernel $K(t-s)$ and the Fourier transform of $\sqrt{S(\omega)}$, where $S(\omega)$ is the spectral density.

2 Novel Expansion Method

A new technique is presented for expanding positive definite functions uniformly. For a stationary Gaussian process with kernel $K(t-s)$, construct an orthonormal basis $\{\phi_n\}$ whose partial sums converge uniformly:

$$K(t-s) = \sum_{n=0}^{\infty} \langle K, \phi_n \rangle \, \phi_n(t-s)$$

The kernel also admits a Mercer expansion:

$$K\,(t-s)=\sum_{n=0}^{\infty}\,\lambda_n\,f_n(t)\,f_n(s)$$

where the Mercer basis elements $\{f_n\}$ are given by:

$$f_n(t)=\int g\,(t-s)\,\phi_n(s)\,d\,s$$

with $g(t)=\mathcal{F}[\sqrt{S(\omega)}]$.

Rather than computing this convolution directly, $g(t)$ is expanded in the Cameron-Martin basis $\{\psi_n\}$ whose partial sums converge uniformly:

$$g(t)=\sum_{n=0}^{\infty}\,\langle g,\psi_n\rangle\,\psi_n(t)$$

This allows exact computation of the Mercer basis elements through term-by-term convolution, yielding the exact eigenfunctions rather than an approximation.

3 The Uniform Expansion Method

The construction begins with orthogonal polynomials with respect to $S(\omega)$. For $n\geq 0$:

$$p_n(\omega)=\omega^n-\sum_{k=0}^{n-1}\frac{\int_{-\infty}^{\infty}\omega^n\,p_k(\omega)\,S(\omega)\,d\,\omega}{\int_{-\infty}^{\infty}p_k^2(\omega)\,S(\omega)\,d\,\omega}\,p_k(\omega)$$

Taking their Fourier transforms:

$$\hat{p}_n(t)=\int_{-\infty}^{\infty}p_n(\omega)\,e^{it\omega}\,d\,\omega$$

The orthogonal complement construction yields the basis, and the kernel expands as:

$$u_n=K-\sum_{k=0}^{n-1}\frac{\int_{-\infty}^{\infty}K(t)\,\hat{p}_k(t)\,d\,t}{\int_{-\infty}^{\infty}\hat{p}_k^2(t)\,d\,t}\,\hat{p}_k$$

$$\phi_n=\frac{u_n}{\|u_n\|}$$

$$K(t)=\sum_{n=0}^{\infty}\,\langle K,\phi_n\rangle\,\phi_n(t)$$

Similarly for the Cameron-Martin basis, using $\sqrt{S(\omega)}$:

$$q_n(\omega) = \omega^n - \sum_{k=0}^{n-1} \frac{\int_{-\infty}^{\infty} \omega^n \, q_k(\omega) \sqrt{S(\omega)} \, d\,\omega}{\int_{-\infty}^{\infty} q_k^2(\omega) \sqrt{S(\omega)} \, d\,\omega} \, q_k(\omega)$$

Their Fourier transforms:

$$\hat{q}_n(t) = \int_{-\infty}^{\infty} q_n(\omega) \, e^{it\omega} \, d\,\omega$$

And the Cameron-Martin basis with its expansion:

$$v_n = g - \sum_{k=0}^{n-1} \frac{\int_{-\infty}^{\infty} g(t) \, \hat{q}_k(t) \, d\,t}{\int_{-\infty}^{\infty} \hat{q}_k^2(t) \, d\,t} \, \hat{q}_k$$

$$\psi_n = \frac{v_n}{\|v_n\|}$$

$$g(t) = \sum_{n=0}^{\infty} \langle g, \psi_n \rangle \, \psi_n(t)$$

4 Triangular Structure

Theorem 1. *For a stationary Gaussian process with kernel $K(t)$ and spectral factor $g(t) = \mathcal{F}[\sqrt{S(\omega)}]$, let:*

$$K(t) = \sum_{n=0}^{\infty} \langle K, \phi_n \rangle \, \phi_n(t)$$

$$g(t) = \sum_{n=0}^{\infty} \langle g, \psi_n \rangle \, \psi_n(t)$$

be their respective uniform expansions. Then the infinite convolution sum reduces to a finite sum due to triangularity.

Proof. Fix k. The Mercer basis elements are given by the convolution of ϕ_k with $g(t) = \mathcal{F}[\sqrt{S(\omega)}]$:

$$\int g\,(t - s) \, \phi_k(s) \, d\,s$$

Using the uniform expansion of g, this integral equals:

$$\int \left(\sum_{n=0}^{\infty} \langle g, \psi_n \rangle \, \psi_n \, (t-s) \right) \phi_k(s) \, d\, s = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \langle g, \psi_n \rangle \, \psi_n \, (t-s) \, \phi_k(s) \, d\, s$$

By Fubini's theorem:

$$\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \langle g, \psi_n \rangle \, \psi_n \, (t-s) \, \phi_k(s) \, d\, s = \sum_{n=0}^{\infty} \langle g, \psi_n \rangle \int_{-\infty}^{\infty} \psi_n \, (t-s) \, \phi_k(s) \, d\, s$$

We now prove that $\int \psi_n(t) \, \phi_k(t) \, d\, t = 0$ for n > k:

1) Recall $\phi_k(t) = \mathcal{F}[p_k(\omega)]$ and $\psi_n(t) = \mathcal{F}[q_n(\omega)]$ where p_k and q_n are orthogonal polynomials w.r.t. $S(\omega)$ and $\sqrt{S(\omega)}$ respectively.

2) By Parseval's theorem: $\int \psi_n(t) \, \phi_k(t) \, d\, t = \int q_n(\omega) \overline{p_k(\omega)} \sqrt{S(\omega)} \, d\, \omega$

3) Expand $q_n(\omega)$ in terms of $\{p_j(\omega)\}_{j=0}^{n}$: $q_n(\omega) = \sum_{j=0}^{n} a_{nj} \, p_j(\omega)$

4) Substitute this into the integral:

$$\int q_n(\omega) \overline{p_k(\omega)} \sqrt{S(\omega)} \, d\, \omega = \sum_{j=0}^{n} a_{nj} \int p_j(\omega) \overline{p_k(\omega)} \sqrt{S(\omega)} \, d\, \omega \tag{1}$$

5) For j > k, $\int p_j(\omega) \overline{p_k(\omega)} S(\omega) \, d\, \omega = 0$ by orthogonality of $\{p_j\}$.

6) Therefore, $\int p_j(\omega) \overline{p_k(\omega)} \sqrt{S(\omega)} \, d\, \omega = 0$ for j > k as well.

7) Since n > k, all terms in the sum for j > k vanish, leaving only terms up to k.

8) But q_n is orthogonal to all polynomials of degree < n w.r.t. $\sqrt{S(\omega)}$, so these remaining terms must sum to zero.

Therefore, $\int \psi_n(t) \, \phi_k(t) \, d\, t = 0$ for n > k, establishing the triangular structure. □