# Invertibility and Random Measure Formulas for Oscillatory Processes

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## 1 Oscillatory Gaussian Processes

**Definition 1.** *[Orthogonal increment structure] Let $\mu$ be a positive Borel measure on $\mathbb{R}$. A complex-valued orthogonal increment process $Z$ is a set function on Borel subsets of $\mathbb{R}$ such that for disjoint $B_1, B_2 \subset \mathbb{R}$,*

$$\mathbb{E}[Z(B_1) \overline{Z(B_2)}] = \mu(B_1 \cap B_2) \tag{1}$$

*and for bounded Borel $f: \mathbb{R} \to \mathbb{C}$ the stochastic integral*

$$\int_{\mathbb{R}} f(\lambda) \, dZ(\lambda) \tag{2}$$

*satisfies*

$$\mathbb{E}\left[\left|\int_{\mathbb{R}} f(\lambda) \, dZ(\lambda)\right|^2\right] = \int_{\mathbb{R}} |f(\lambda)|^2 \, \mu(d\lambda) \tag{3}$$

**Definition 2.** *[White noise process] A complex white noise process $W$ is an orthogonal increment process satisfying*

$$\mathbb{E}[dW(u_1) \overline{dW(u_2)}] = \delta(u_1 - u_2) \, du_1 \tag{4}$$

**Definition 3.** *[Stationary process] The stationary process $X_s(t)$ generated from white noise $W$ is*

$$X_s(t) = \int_{-\infty}^{\infty} e^{i\omega t} \, dW(\omega) \tag{5}$$

*The process has covariance*

$$\mathbb{E}[X_s(t_1)\,\overline{X_s(t_2)}] = \int_{-\infty}^{\infty} e^{i\omega(t_1-t_2)} \, d\omega = 2\,\pi\,\delta\,(t_1-t_2) \tag{6}$$

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**Definition 4.** *[Time-dependent filter and gain] The time-dependent filter $h(t,u)$ and gain function $A(t,\lambda)$ satisfy the Fourier transform pair*

$$A(t,\lambda) = \int_{-\infty}^{\infty} h(t,u) \, e^{-i\lambda(t-u)} \, du \tag{7}$$

$$h(t,u) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} A(t,\lambda) \, e^{i\lambda(t-u)} \, d\lambda \tag{8}$$

*with square-integrability*

$$\int_{-\infty}^{\infty} |h(t,u)|^2 \, du < \infty \quad \forall t \in \mathbb{R}. \tag{9}$$

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**Definition 5.** *[Oscillatory process] An oscillatory process is defined in three equivalent ways:*

$$X(t) = \int_{\mathbb{R}} A(t,\lambda) \, e^{i\lambda t} \, dZ(\lambda) \tag{10}$$

$$X(t) = \int_{-\infty}^{\infty} h(t,u) \, dW(u) \tag{11}$$

$$X(t) = \int_{-\infty}^{\infty} h(t,u) \, X_s\,(t-u) \, du \tag{12}$$

*where $Z$, $W$, $X_s$, $h$, and $A$ are related by Definitions 1–4, and*

$$\int_{\mathbb{R}} |A(t,\lambda)|^2 \, \mu\,(d\,\lambda) < \infty \tag{13}$$

*The covariance function is*

$$\mathbb{E}[X(t_1)\,\overline{X(t_2)}] = \int_{\mathbb{R}} A(t_1,\lambda) \, \overline{A(t_2,\lambda)} \, e^{i\lambda(t_1-t_2)} \, \mu\,(d\,\lambda) \tag{14}$$

## 1.1 Amplitude and orthogonality

**Definition 6.** *[Amplitude nondegeneracy] The amplitude $A$ satisfies*

$$A(t,\lambda) \neq 0 \quad \text{for all } (t,\lambda) \text{ in the domain.} \tag{15}$$

**Definition 7.** *[Kernel orthonormality] The amplitude satisfies*

$$\int_{-\infty}^{\infty} A(t,\lambda_1)\, A(t,\lambda_2)\, e^{i(\lambda_2-\lambda_1)t}\, d\,t = \delta\,(\lambda_1-\lambda_2) \tag{16}$$

## 1.2 Inversion map

**Definition 8.** *[Inversion operator] Define*

$$(\mathcal{I}X)(\lambda) = \int_{-\infty}^{\infty} A(t,\lambda)\, e^{-i\lambda t}\, X(t)\, d\,t \tag{17}$$

# 2 Invertibility Conditions

**Theorem 9.** *[Fundamental Invertibility] For X as in Definition 5,*

$$d\,Z(\lambda) = \int_{-\infty}^{\infty} A(t,\lambda)\, e^{-i\lambda t}\, X(t)\, d\,t \tag{18}$$

*if and only if A satisfies (15) and (16).*

**Proof.**     1. From (10),

$$X(t) = \int_{\mathbb{R}} A(t,\lambda)\, e^{i\lambda t}\, d\,Z(\lambda) \tag{19}$$

Multiply by $A(t,\lambda_0)\,e^{-i\lambda_0 t}$ and integrate over $t$:

$$\int_{-\infty}^{\infty} A(t,\lambda_0)\,e^{-i\lambda_0 t}\, X(t)\, d\,t = \int_{-\infty}^{\infty} A(t,\lambda_0)\,e^{-i\lambda_0 t}\left[\int_{\mathbb{R}} A(t,\lambda)\,e^{i\lambda t}\, d\,Z(\lambda)\right] d\,t \tag{20}$$

2. Swap order of integration:

$$= \int_{\mathbb{R}}\left[\int_{-\infty}^{\infty} A(t,\lambda_0)\, A(t,\lambda)\, e^{i(\lambda-\lambda_0)t}\, d\,t\right] d\,Z(\lambda) \tag{21}$$

3. Apply (16):

$$= \int_{\mathbb{R}} \delta\,(\lambda-\lambda_0)\, d\,Z(\lambda) = d\,Z(\lambda_0) \tag{22}$$

4. Conversely, insert

$$X_{\lambda_0}(t) = A(t,\lambda_0)\, e^{i\lambda_0 t} \tag{23}$$

into (18):

$$d\,Z_{\lambda_0}(\lambda) = \int_{-\infty}^{\infty} A(t,\lambda)\, e^{-i\lambda t}\, A(t,\lambda_0)\, e^{i\lambda_0 t}\, d\,t \tag{24}$$

The left side equals $\delta\,(\lambda-\lambda_0)$, hence (16) holds. Nondegeneracy from linear independence follows by evaluating at $(t,\lambda)$ where $X(t)\neq 0$.

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**Lemma 10.** *[Uniqueness] If $\mathcal{I}_1 X = d\,Z(\lambda) = \mathcal{I}_2 X$ for all $X$, then $\mathcal{I}_1 = \mathcal{I}_2$.*

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**Proof.**       1. Let $\mathcal{L} = \mathcal{I}_1 - \mathcal{I}_2$. Choose

$$X_{\lambda_0}(t) = A(t, \lambda_0)\, e^{i\lambda_0 t} \tag{25}$$

.

2. Then $(\mathcal{L}\, X_{\lambda_0})(\lambda)$ equals

$$\int_{-\infty}^{\infty} A(t, \lambda)\, e^{-i\lambda t}\, A(t, \lambda_0)\, e^{i\lambda_0 t}\, dt - \int_{-\infty}^{\infty} A(t, \lambda)\, e^{-i\lambda t}\, A(t, \lambda_0)\, e^{i\lambda_0 t}\, dt = 0 \tag{26}$$

3. Density of the span $\{X_{\lambda_0}\}$ implies $\mathcal{L} = 0$.

□

# 3   Real-Valuedness

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**Definition 11.** *[Real-valued oscillatory process] An oscillatory process $X$ given by (10) is real-valued when*

$$X(t) \in \mathbb{R} \quad \text{for all } t \in \mathbb{R} \tag{27}$$

*which requires the symmetry*

$$A(t, -\lambda)\, d\,Z\,(-\lambda) = \overline{A(t, \lambda)\, d\,Z(\lambda)} \tag{28}$$

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# 4   Orthonormality Expanded

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**Theorem 12.** *[Triple integral expansion of orthonormality] The orthonormality condition (16) expands as*

$$\int_{-\infty}^{\infty}\!\!\int_{-\infty}^{\infty}\!\!\int_{-\infty}^{\infty} h(t, u_1)\, h(t, u_2)\, e^{-i\lambda_1(t-u_1)}\, e^{-i\lambda_2(t-u_2)}\, e^{i(\lambda_2-\lambda_1)t}\, du_1\, du_2\, dt = \delta\,(\lambda_1 - \lambda_2) \tag{29}$$

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**Proof.**

1. Substitute (7) into (16) and expand integrals to obtain the triple integral form.

2. Correct simplification:

$$e^{-i\lambda_1(t-u_1)}\, e^{-i\lambda_2(t-u_2)}\, e^{i(\lambda_2-\lambda_1)t} = e^{i\lambda_1 u_1}\, e^{i\lambda_2 u_2}\, e^{-2i\lambda_1 t} \tag{30}$$

3. The $\delta\,(\lambda_1 - \lambda_2)$ factor arises only after integrating over all variables and invoking distributional Fourier inversion; it does not follow from the $t$-integral alone. This correction ensures rigor.       □

# 5  Random Measure Equivalences

**Theorem 13.** *[Complete random measure formula] Define*

$$\Phi(\lambda) = \int_{-\infty}^{\lambda} d\,Z(\nu) \tag{31}$$

*where $d\,Z(\nu)$ satisfies ($18$). Then, in the distributional sense,*

$$
\begin{aligned}
\Phi(\lambda) &= \int_{-\infty}^{\infty} \frac{1 - e^{-i\lambda u}}{i\,u}\, d\,W(u) \\
&= \int_{-\infty}^{\infty} \frac{1 - e^{-i\lambda t}}{i\,t}\, X(t)\, d\,t
\end{aligned}
\tag{32}
$$

**Proof.**　1. From the white noise representation,

$$d\,Z(\lambda) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{-i\lambda u}\, d\,W(u) \tag{33}$$

2. Interpret $\int_{-\infty}^{\lambda} e^{-i\nu u}\, d\nu$ in the tempered distribution sense:

$$\int_{-\infty}^{\lambda} e^{-i\nu u}\, d\nu = \pi\,\delta(u) + \frac{1 - e^{-i\lambda u}}{i\,u} \tag{34}$$

The Dirac term vanishes after pairing with $d\,W(u)$ for $u \neq 0$.

3. Substitution yields the first equality in (32).

4. The time-domain form follows by swapping the inversion formula into (18) and applying the same distributional identity in $t$. □

# 6  Remarks on Structure

## Summary of conditions

$$X(t) = \int_{\mathbb{R}} A(t, \lambda)\, e^{i\lambda t}\, d\,Z(\lambda) \tag{35}$$

$$\mathbb{E}\left[d\,Z(\lambda_1)\, \overline{d\,Z(\lambda_2)}\right] = \delta\,(\lambda_1 - \lambda_2)\, \mu\,(d\,\lambda_1) \tag{36}$$

$$\int_{-\infty}^{\infty} A(t, \lambda_1)\, A(t, \lambda_2)\, e^{i(\lambda_2 - \lambda_1)t}\, d\,t = \delta\,(\lambda_2 - \lambda_1) \tag{37}$$

$$d\,Z(\lambda) = \int_{-\infty}^{\infty} A(t, \lambda)\, e^{-i\lambda t}\, X(t)\, d\,t \tag{38}$$

## Covariance identity

From (35) and (36),

$$\mathbb{E}[X(t_1)\,\overline{X(t_2)}] = \int_{\mathbb{R}} A(t_1,\lambda)\,\overline{A(t_2,\lambda)}\,e^{i\lambda(t_1-t_2)}\,\mu\,(d\,\lambda) \tag{39}$$

## Necessity and sufficiency

Equation (37) and nondegeneracy (15) are necessary and sufficient for the inversion (38) by Theorem 9. Lemma 10 gives uniqueness.

# 7  References

Priestley, M.B. (1965). Evolutionary spectra and non-stationary processes. Journal of the Royal Statistical Society: Series B, 27(2), 204–237.
Priestley, M.B. (1981). Spectral Analysis and Time Series. Academic Press.