Unitary Time Changes of Stationary Processes Yield Oscillatory Processes and a Functional Framework Toward a Hilbert–Pólya Construction

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are square-integrable over σ-compact sets. Applying U_θ to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\theta'(t)}\, e^{i\lambda\theta(t)}$ and evolutionary spectrum $dF_t(\lambda) = \theta'(t)\, dF(\lambda)$. It is proved that sample paths of any non-degenerate second-order stationary process almost surely lie in $L^2_{\sigma\text{-comp}}(\mathbb{R})$, making the operator applicable to typical realizations. A zero-localization measure $\mu(dt) = \delta(Z(t))\,|Z'(t)|\, dt$ induces a Hilbert space $L^2(\mu)$ on the zero set of an oscillatory process Z, and the multiplication operator $(Lf)(t) = t\, f(t)$ has pure point, simple spectrum equal to the zero set of Z. This produces a concrete operator scaffold consistent with a Hilbert–Pólya-type viewpoint.

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1 Function Spaces and Unitary Time Change

1.1 σ-compact sets and local L^2

Definition 1. *[σ-compact sets] A subset $U \subseteq \mathbb{R}$ is σ-compact if $U = \bigcup_{n=1}^{\infty} K_n$ with each K_n compact.*

Definition 2. *[Square-integrability on σ-compact sets] Define*

$$L^2_{\sigma\text{-}comp}(\mathbb{R}) := \left\{ f\colon \mathbb{R} \to \mathbb{C}\colon \int_U |f(t)|^2 \, dt < \infty \text{ for every } \sigma\text{-compact } U \subseteq \mathbb{R} \right\} \tag{1}$$

Remark 3. Every bounded measurable set in $\mathbb{R}$ is σ-compact; hence $L^2_{\sigma\text{-comp}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

1.2 Unitary time-change operator

Definition 4. *[Unitary time-change] Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with $\theta'(t) > 0$ almost everywhere and $\theta'(t) = 0$ only on sets of Lebesgue measure zero. The function θ maps σ-compact sets to σ-compact sets. Define, for f measurable,*

$$(U_\theta f)(t) := \sqrt{\theta'(t)} \; f(\theta(t)) \tag{2}$$

Proposition 5. *[Inverse map] The inverse map is given by*

$$(U_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\theta'(\theta^{-1}(s))}} \tag{3}$$

which is well-defined almost everywhere on every σ-compact set.

Proof. Since $\theta'(t) = 0$ only on sets of measure zero, and θ^{-1} maps sets of measure zero to sets of measure zero (as absolutely continuous bijective functions preserve measure-zero sets), the denominator $\sqrt{\theta'(\theta^{-1}(s))}$ is positive almost everywhere. The expression is therefore well-defined almost everywhere on every σ-compact set, which suffices for defining an element of $L^2_{\sigma\text{-comp}}(\mathbb{R})$. □

Theorem 6. *[Local unitarity on σ-compact sets] For every σ-compact set $U \subseteq \mathbb{R}$ and $f \in L^2_{\sigma\text{-}comp}(\mathbb{R})$,*

$$\int_U |(U_\theta f)(t)|^2 \, dt = \int_{\theta(U)} |f(s)|^2 \, ds \tag{4}$$

Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\sigma\text{-}comp}(\mathbb{R})$.

Proof. Let $f \in L^2_{\sigma\text{-comp}}(\mathbb{R})$ and let U be any σ-compact set. The local L^2-norm of $U_\theta f$ over U is:

$$\int_U |(U_\theta f)(t)|^2 \, dt = \int_U \left|\sqrt{\theta'(t)}\, f(\theta(t))\right|^2 dt \tag{5}$$

$$= \int_U \theta'(t) |f(\theta(t))|^2 \, dt \tag{6}$$

Since θ is absolutely continuous and strictly increasing, applying the change of variables $s = \theta(t)$ gives $ds = \theta'(t)\, dt$ almost everywhere. Since θ maps σ-compact sets to σ-compact sets, as t ranges over U, $s = \theta(t)$ ranges over $\theta(U)$, which is σ-compact. Therefore:

$$\int_U \theta'(t)\, |f(\theta(t))|^2 \, dt = \int_{\theta(U)} |f(s)|^2 \, ds \tag{7}$$

To verify that U_θ^{-1} is indeed the inverse, we compute explicitly. For any $f \in L^2_{\sigma\text{-comp}}(\mathbb{R})$:

$$(U_\theta^{-1} U_\theta f)(s) = (U_\theta^{-1}) \left[\sqrt{\theta'(\cdot)}\, f(\theta(\cdot))\right](s) \tag{8}$$

$$= \frac{[\sqrt{\theta'(\theta^{-1}(s))}\, f(\theta(\theta^{-1}(s)))])}{\sqrt{\theta'(\theta^{-1}(s))}} \tag{9}$$

$$= \frac{\sqrt{\theta'(\theta^{-1}(s))}\, f(s)}{\sqrt{\theta'(\theta^{-1}(s))}} \tag{10}$$

$$= f(s) \tag{11}$$

where we used $\theta(\theta^{-1}(s)) = s$.

Similarly, for any $g \in L^2_{\sigma\text{-comp}}(\mathbb{R})$:

$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\theta'(t)}\, (U_\theta^{-1} g)(\theta(t)) \tag{12}$$

$$= \sqrt{\theta'(t)}\, \frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\theta'(\theta^{-1}(\theta(t)))}} \tag{13}$$

$$= \sqrt{\theta'(t)}\, \frac{g(t)}{\sqrt{\theta'(t)}} \tag{14}$$

$$= g(t) \tag{15}$$

where we used

$$\theta^{-1}(\theta(t)) = t \tag{16}$$

Therefore $U_\theta U_\theta^{-1} = U_\theta^{-1} U_\theta = I$ on $L^2_{\sigma\text{-comp}}(\mathbb{R})$. $\square$

Theorem 7. *[Unitarity on $L^2(\mathbb{R})$] $U_\theta: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is unitary:*

$$\int_{\mathbb{R}} |(U_\theta f)(t)|^2 \, dt = \int_{\mathbb{R}} |f(s)|^2 \, ds \tag{17}$$

and U_θ^{-1} is its inverse.

Proof. For $f \in L^2(\mathbb{R})$, we have:

$$\int_{\mathbb{R}} |(U_\theta f)(t)|^2 \, dt = \int_{\mathbb{R}} \theta'(t) |f(\theta(t))|^2 \, dt \tag{18}$$

By the change of variables $s = \theta(t)$ with $ds = \theta'(t) \, dt$, and since $\theta: \mathbb{R} \to \mathbb{R}$ is bijective:

$$\int_{\mathbb{R}} \theta'(t) \, |f(\theta(t))|^2 \, dt = \int_{\mathbb{R}} |f(s)|^2 \, ds \tag{19}$$

The inverse relationship follows from the same computation as in Theorem 6, applied globally. $\square$

2 Oscillatory Processes (Priestley)

Definition 8. *[Oscillatory process] Let F be a finite nonnegative Borel measure on $\mathbb{R}$. For each $t \in \mathbb{R}$, let $A_t \in L^2(F)$ and set $\varphi_t(\lambda) := A_t(\lambda) \, e^{i\lambda t}$. An* oscillatory process *is a stochastic process*

$$Z(t) := \int_{\mathbb{R}} \varphi_t(\lambda) \, \Phi(d\lambda) = \int_{\mathbb{R}} A_t(\lambda) \, e^{i\lambda t} \, \Phi(d\lambda) \tag{20}$$

where Φ is a complex orthogonal random measure with spectral measure F, that is,

$$\mathbb{E}[\Phi(d\lambda) \, \overline{\Phi(d\mu)}] = \delta(\lambda - \mu) \, dF(\lambda) \tag{21}$$

Its covariance kernel is

$$R_Z(t,s) = \mathbb{E}[Z(t) \overline{Z(s)}] = \int_{\mathbb{R}} A_t(\lambda) \, \overline{A_s(\lambda)} \, e^{i\lambda(t-s)} \, dF(\lambda) \tag{22}$$

Remark 9. [Real-valuedness] Z is real-valued if and only if $A_t(-\lambda) = \overline{A_t(\lambda)}$ for F-a.e. λ, equivalently $\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)}$ for F-a.e. λ.

Theorem 10. *[Existence] If F is finite and $(A_t)_{t\in\mathbb{R}}$ is measurable in t with $\int_{\mathbb{R}} |A_t(\lambda)|^2 \, dF(\lambda) < \infty$ for each t, then there exists a complex orthogonal random measure Φ with spectral measure F such that*

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda) \, e^{i\lambda t} \, \Phi(d\lambda) \tag{23}$$

is well-defined in $L^2(\Omega)$ and has covariance R_Z as above.

Proof. We construct the stochastic integral using the standard extension procedure. First, define the integral for simple functions of the form $g(\lambda) = \sum_{j=1}^n c_j \, 1_{E_j}(\lambda)$ where $\{E_j\}$ are disjoint Borel sets with $F(E_j) < \infty$ and $c_j \in \mathbb{C}$:

$$\int_{\mathbb{R}} g(\lambda) \, \Phi(d\lambda) := \sum_{j=1}^n c_j \, \Phi(E_j) \tag{24}$$

For such simple functions, the isometry property holds:

$$\begin{aligned}
\mathbb{E}\left[\left|\int_{\mathbb{R}} g(\lambda) \, \Phi(d\lambda)\right|^2\right] &= \mathbb{E}\left[\left|\sum_{j=1}^n c_j \, \Phi(E_j)\right|^2\right] && (25)\\
&= \sum_{j=1}^n \sum_{k=1}^n c_j \bar{c_k} \mathbb{E}[\Phi(E_j)\overline{\Phi(E_k)}] && (26)\\
&= \sum_{j=1}^n |c_j|^2 \, F(E_j) && (27)\\
&= \int_{\mathbb{R}} |g(\lambda)|^2 \, dF(\lambda) && (28)
\end{aligned}$$

Since simple functions are dense in $L^2(F)$, we extend by continuity to all $g \in L^2(F)$. For each t, since $\varphi_t(\lambda) = A_t(\lambda) \, e^{i\lambda t}$ and $A_t \in L^2(F)$, we have $\varphi_t \in L^2(F)$. Therefore $Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda) \, \Phi(d\lambda)$ is well-defined in $L^2(\Omega)$.

The covariance is computed as:

$$\begin{aligned}
R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] && (29)\\
&= \mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda) \, \Phi(d\lambda) \int_{\mathbb{R}} \overline{\varphi_s(\mu)} \, \overline{\Phi(d\mu)}\right] && (30)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\mu)} \mathbb{E}[\Phi(d\lambda)\overline{\Phi(d\mu)}] && (31)\\
&= \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)} \, dF(\lambda) && (32)
\end{aligned}$$

$$= \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)} e^{i\lambda(t-s)}\, dF(\lambda) \tag{33}$$

□

3 Stationary Processes and Time Change

3.1 Stationary processes

Definition 11. *[Cramér representation] A zero-mean stationary process X with spectral measure F admits*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, \Phi(d\lambda), \qquad R_X(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)}\, dF(\lambda).$$

3.2 Stationary → oscillatory via U_θ

Theorem 12. *[Time change yields oscillatory process] Let X be zero-mean stationary as in Definition 11. For θ as in Definition 4, define*

$$Z(t) := (U_\theta X)(t) = \sqrt{\theta'(t)}\; X(\theta(t)) \tag{34}$$

Then Z is oscillatory with oscillatory function

$$\varphi_t(\lambda) = \sqrt{\theta'(t)}\; e^{i\lambda\theta(t)} \tag{35}$$

, gain function

$$A_t(\lambda) = \sqrt{\theta'(t)}\; e^{i\lambda(\theta(t)-t)} \tag{36}$$

, and covariance

$$R_Z(t,s) = \int_{\mathbb{R}} \sqrt{\theta'(t)\,\theta'(s)}\; e^{i\lambda(\theta(t)-\theta(s))}\, dF(\lambda).$$

Proof. Applying the unitary time change operator to the spectral representation of $X(t)$:

$$
\begin{aligned}
Z(t) &= (U_\theta X)(t) && (37)\\
&= \sqrt{\theta'(t)}\; X(\theta(t)) && (38)\\
&= \sqrt{\theta'(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, \Phi(d\lambda) && (39)\\
&= \int_{\mathbb{R}} \sqrt{\theta'(t)}\, e^{i\lambda\theta(t)}\, \Phi(d\lambda) && (40)\\
&= \int_{\mathbb{R}} \varphi_t(\lambda)\; \Phi(d\lambda) && (41)
\end{aligned}
$$

where

$$\varphi_t(\lambda) = \sqrt{\theta'(t)}\, e^{i\lambda\theta(t)} \tag{42}$$

To verify this constitutes an oscillatory representation according to Definition 8, we must write $\varphi_t(\lambda)$ in the form $A_t(\lambda)\, e^{i\lambda t}$:

$$\begin{aligned} \varphi_t(\lambda) =& \sqrt{\theta'(t)}\, e^{i\lambda\theta(t)} & (43)\\ =& \sqrt{\theta'(t)}\, e^{i\lambda(\theta(t)-t)}\, e^{i\lambda t} & (44)\\ =& A_t(\lambda)\, e^{i\lambda t} & (45) \end{aligned}$$

where

$$A_t(\lambda) = \sqrt{\theta'(t)}\, e^{i\lambda(\theta(t)-t)} \tag{46}$$

Since $\theta'(t) \geq 0$ almost everywhere and $\theta'(t) = 0$ only on sets of measure zero, $A_t(\lambda)$ is well-defined almost everywhere. Moreover, $A_t \in L^2(F)$ for each t since:

$$\begin{aligned} \int_{\mathbb{R}} |A_t(\lambda)|^2\, dF(\lambda) =& \int_{\mathbb{R}} \left|\sqrt{\theta'(t)}\, e^{i\lambda(\theta(t)-t)}\right|^2\, dF(\lambda) & (47)\\ =& \int_{\mathbb{R}} \theta'(t) |e^{i\lambda(\theta(t)-t)}|^2\, dF(\lambda) & (48)\\ =& \theta'(t) \int_{\mathbb{R}} dF(\lambda) & (49)\\ =& \theta'(t)\, F(\mathbb{R}) < \infty & (50) \end{aligned}$$

where we used $|e^{i\alpha}| = 1$ for all real α.

The covariance is computed as:

$$\begin{aligned} R_Z(t,s) =& \mathbb{E}[Z(t)\overline{Z(s)}] & (51)\\ =& \mathbb{E}\left[\sqrt{\theta'(t)}\, X(\theta(t)) \sqrt{\theta'(s)}\, \overline{X(\theta(s))}\right] & (52)\\ =& \sqrt{\theta'(t)\,\theta'(s)}\; \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] & (53)\\ =& \sqrt{\theta'(t)\,\theta'(s)}\; R_X(\theta(t)-\theta(s)) & (54)\\ =& \sqrt{\theta'(t)\,\theta'(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\, dF(\lambda) & (55) \end{aligned}$$

□

Corollary 13. *[Evolutionary spectrum] The evolutionary spectrum is*

$$dF_t(\lambda) = |A_t(\lambda)|^2\, dF(\lambda) = \theta'(t)\, dF(\lambda) \tag{56}$$

Proof. By definition of the evolutionary spectrum and using the gain function from Theorem 12:

$$
\begin{aligned}
d\,F_t(\lambda) &= |A_t(\lambda)|^2 \; d\,F(\lambda) && (57)\\
&= \left|\sqrt{\theta'(t)}\, e^{i\lambda(\theta(t)-t)}\right|^2 \; d\,F(\lambda) && (58)\\
&= \theta'(t)|e^{i\lambda(\theta(t)-t)}|^2 \; d\,F(\lambda) && (59)\\
&= \theta'(t) \; d\,F(\lambda) && (60)
\end{aligned}
$$

since $|e^{i\alpha}| = 1$ for all real α. □

3.3 Covariance operator conjugation

Proposition 14. *[Operator conjugation] Let* $(T_K\, f)(t) := \int_{\mathbb{R}} K(|t-s|) \; f(s) \; d\,s$ *with stationary kernel* $K(h) = \int_{\mathbb{R}} e^{i\lambda h} \; d\,F(\lambda)$. *Define the transformed kernel*

$$K_\theta(s,t) := \sqrt{\theta'(t)\,\theta'(s)} \;\; K(|\theta(t)-\theta(s)|) \tag{61}$$

and operator

$$(T_{K_\theta} f)(t) := \int_{\mathbb{R}} K_\theta(s,t) \;\; f(s) \; d\,s \tag{62}$$

Then

$$T_{K_\theta} = U_\theta \; T_K \; U_\theta^{-1} \tag{63}$$

on $L^2_{\sigma\text{-}comp}(\mathbb{R})$.

Proof. For any $g \in L^2_{\sigma\text{-comp}}(\mathbb{R})$, compute $(U_\theta\, T_K\, U_\theta^{-1}\, g)(t)$ step by step.

First,

$$(U_\theta^{-1}\, g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\theta'(\theta^{-1}(s))}} \tag{64}$$

Second,

$$(T_K\, U_\theta^{-1}\, g)(t) = \int_{\mathbb{R}} K(|t-s|) \frac{g(\theta^{-1}(s))}{\sqrt{\theta'(\theta^{-1}(s))}} \; d\,s \tag{65}$$

Apply change of variables $u = \theta^{-1}(s)$, so $s = \theta(u)$ and $d\,s = \theta'(u) \; d\,u$:

$$
\begin{aligned}
(T_K\, U_\theta^{-1}\, g)(t) &= \int_{\mathbb{R}} K(|t-\theta(u)|) \frac{g(u)}{\sqrt{\theta'(u)}}\, \theta'(u) \; d\,u && (66)\\
&= \int_{\mathbb{R}} K(|t-\theta(u)|) \; g(u) \sqrt{\theta'(u)} \; d\,u && (67)
\end{aligned}
$$

Third, $(U_\theta T_K U_\theta^{-1} g)(t) = \sqrt{\theta'(t)}\,(T_K U_\theta^{-1} g)(\theta(t))$:

$$= \sqrt{\theta'(t)} \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|)\, g(u) \sqrt{\theta'(u)}\; d u \tag{68}$$

$$= \int_{\mathbb{R}} \sqrt{\theta'(t)\,\theta'(u)}\, K(|\theta(t) - \theta(u)|)\, g(u)\; d u \tag{69}$$

Finally, changing variables back with $s = \theta(u)$:

$$= \int_{\mathbb{R}} \sqrt{\theta'(t)\,\theta'(s)}\, K(|\theta(t) - \theta(s)|)\, g(s)\; d s \tag{70}$$

$$= \int_{\mathbb{R}} K_\theta(s, t)\, g(s)\; d s \tag{71}$$

$$= (T_{K_\theta} g)(t) \tag{72}$$

This establishes the conjugation relation $T_{K_\theta} = U_\theta T_K U_\theta^{-1}$. □

4 Sample Paths Live in $L^2_{\sigma\text{-comp}}$

Theorem 15. *[Sample paths in $L^2_{\sigma\text{-comp}}(\mathbb{R})$] Let $\{X(t)\}_{t\in\mathbb{R}}$ be a second-order stationary process with $\sigma^2 := \mathbb{E}[X(t)^2] < \infty$. Then, almost surely, every sample path $t \mapsto X(\omega, t)$ belongs to $L^2_{\sigma\text{-comp}}(\mathbb{R})$.*

Proof. Fix any bounded interval $[a, b]$ and consider the random variable

$$Y_{[a,b]} := \int_a^b X(t)^2\; d t \tag{73}$$

By stationarity and Fubini's theorem:

$$\mathbb{E}[Y_{[a,b]}] = \mathbb{E}\left[\int_a^b X(t)^2\; d t\right] = \int_a^b \mathbb{E}[X(t)^2]\; d t = \int_a^b \sigma^2\; d t = \sigma^2\,(b - a) < \infty \tag{74}$$

By Markov's inequality, for any $M > 0$:

$$P\,(Y_{[a,b]} > M) \leq \frac{\mathbb{E}[Y_{[a,b]}]}{M} = \frac{\sigma^2\,(b - a)}{M} \tag{75}$$

Taking $M \to \infty$, we conclude $P\,(Y_{[a,b]} < \infty) = 1$, i.e., almost surely the sample path is square-integrable on $[a, b]$.

Since $\mathbb{R}$ is the countable union of bounded intervals:

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n] \tag{76}$$

by countable subadditivity of probability:

$$P\left(\bigcap_{n=1}^{\infty}\left\{\int_{-n}^{n} X(t)^2\,dt<\infty\right\}\right)=1 \tag{77}$$

Now let U be any σ-compact set. Then $U=\bigcup_{m=1}^{\infty} K_m$ where each K_m is compact. Each compact set K_m is bounded, so $K_m\subseteq[-N_m,N_m]$ for some N_m. Therefore:

$$\int_U X(t)^2\,dt=\int_{\bigcup_{m=1}^{\infty}K_m} X(t)^2\,dt\leq\sum_{m=1}^{\infty}\int_{K_m} X(t)^2\,dt\leq\sum_{m=1}^{\infty}\int_{-N_m}^{N_m} X(t)^2\,dt \tag{78}$$

Since each integral $\int_{-N_m}^{N_m} X(t)^2\,dt<\infty$ almost surely, and the sum of countably many finite terms is finite, we have $\int_U X(t)^2\,dt<\infty$ almost surely.

This holds for every σ-compact set U, so almost surely every sample path lies in $L^2_{\sigma\text{-comp}}(\mathbb{R})$. □

5 Zero Localization and Hilbert–Pólya Scaffold

5.1 Zero localization measure

Definition 16. *[Zero localization measure] Let Z be real-valued with $Z\in C^1(\mathbb{R})$ and only simple zeros $Z(t_0)=0\Rightarrow Z'(t_0)\neq 0$. Define, for Borel $B\subset\mathbb{R}$,*

$$\mu(B):=\int_{\mathbb{R}} 1_B(t)\;\delta(Z(t))\,|Z'(t)|\,dt \tag{79}$$

Theorem 17. *[Atomicity on the zero set] For every $\phi\in C_c^{\infty}(\mathbb{R})$,*

$$\int_{\mathbb{R}}\phi(t)\;\delta(Z(t))\,|Z'(t)|\,dt=\sum_{t_0:Z(t_0)=0}\phi(t_0) \tag{80}$$

hence

$$\mu=\sum_{t_0:Z(t_0)=0}\delta_{t_0} \tag{81}$$

Proof. Since all zeros of Z are simple and $Z\in C^1(\mathbb{R})$, by the inverse function theorem each zero t_0 is isolated. Near each zero t_0, Z is locally monotonic, so we can apply the one-dimensional change of variables formula for the Dirac delta.

Specifically, near t_0 where $Z(t_0)=0$ and $Z'(t_0)\neq 0$, we have locally $Z(t)=(t-t_0)\,Z'(t_0)+O((t-t_0)^2)$. The distributional identity for the Dirac delta under smooth changes of variables gives:

$$\delta(Z(t))=\sum_{t_0:Z(t_0)=0}\frac{\delta\,(t-t_0)}{|Z'(t_0)|} \tag{82}$$

Therefore:

$$\begin{aligned}
\int_{\mathbb{R}}\phi(t)\;\delta(Z(t))\,|Z'(t)|\;d\,t &= \int_{\mathbb{R}}\phi(t)\,|Z'(t)|\sum_{t_0:Z(t_0)=0}\frac{\delta\,(t-t_0)}{|Z'(t_0)|}\;d\,t && (83)\\
&= \sum_{t_0:Z(t_0)=0}\int_{\mathbb{R}}\phi(t)\frac{|Z'(t)|\,\delta\,(t-t_0)}{|Z'(t_0)|}\;d\,t && (84)\\
&= \sum_{t_0:Z(t_0)=0}\frac{|Z'(t_0)|\;\phi(t_0)}{|Z'(t_0)|} && (85)\\
&= \sum_{t_0:Z(t_0)=0}\phi(t_0) && (86)
\end{aligned}$$

This shows that μ is the discrete measure $\mu=\sum_{t_0:Z(t_0)=0}\delta_{t_0}$ assigning unit mass to each zero. □

5.2 Hilbert space on zeros and multiplication operator

Definition 18. *[Hilbert space on the zero set] Let $\mathcal{H}:=L^2(\mu)$ with inner product $\langle f,g\rangle=\int f(t)\overline{g(t)}\;\mu\,(d\,t)$.*

Proposition 19. *[Atomic structure] With $\mu=\sum_{t_0:Z(t_0)=0}\delta_{t_0}$,*

$$\mathcal{H}\cong\left\{f\colon\{t_0\colon Z(t_0)=0\}\to\mathbb{C}\colon\sum_{t_0:Z(t_0)=0}|f(t_0)|^2<\infty\right\}\cong\ell^2 \tag{87}$$

with orthonormal basis $\{e_{t_0}\}_{t_0:Z(t_0)=0}$, where $e_{t_0}(t_1)=\delta_{t_0t_1}$.

Proof. By the atomic form of μ, for any $f\in L^2(\mu)$:

$$\begin{aligned}
\|f\|_{\mathcal{H}}^2 &= \int|f(t)|^2\;\mu\,(d\,t) && (88)\\
&= \int|f(t)|^2\sum_{t_0:Z(t_0)=0}\delta_{t_0}\,(d\,t) && (89)\\
&= \sum_{t_0:Z(t_0)=0}|f(t_0)|^2 && (90)
\end{aligned}$$

This shows the isomorphism with ℓ^2. The functions e_{t_0} defined by $e_{t_0}(t_1) = \delta_{t_0 t_1}$ satisfy:

$$\langle e_{t_0}, e_{t_1} \rangle = \int e_{t_0}(t) \overline{e_{t_1}(t)} \, \mu(d t) = \sum_{t: Z(t) = 0} \delta_{t_0 t} \, \delta_{t_1 t} = \delta_{t_0 t_1} \tag{91}$$

so they form an orthonormal set. Any $f \in \mathcal{H}$ can be written as

$$f = \sum_{t_0: Z(t_0) = 0} f(t_0) \, e_{t_0} \tag{92}$$

proving they form a basis. □

Definition 20. *[Multiplication operator] Define $L: \mathcal{D}(L) \subset \mathcal{H} \rightarrow \mathcal{H}$ by $(L f)(t) = t \ f(t)$ on $\operatorname{supp}(\mu)$ with domain*

$$\mathcal{D}(L) := \left\{ f \in \mathcal{H}: \int |t \ f(t)|^2 \ \mu(d t) < \infty \right\} \tag{93}$$

Theorem 21. *[Self-adjointness and spectrum] L is self-adjoint on $\mathcal{H}$ and has pure point, simple spectrum*

$$\sigma(L) = \{ t \in \mathbb{R}: Z(t) = 0 \} \tag{94}$$

with eigenvalues $\lambda = t_0$ and eigenvectors e_{t_0}.

Proof. First, we verify self-adjointness. For $f, g \in \mathcal{D}(L)$:

$$\begin{aligned} \langle L f, g \rangle &= \int (L f)(t) \overline{g(t)} \, \mu(d t) && (95) \\ &= \int t \, f(t) \overline{g(t)} \, \mu(d t) && (96) \\ &= \int f(t) \overline{t \, g(t)} \, \mu(d t) && (97) \\ &= \int f(t) \overline{(L g)(t)} \, \mu(d t) && (98) \\ &= \langle f, L g \rangle && (99) \end{aligned}$$

Thus L is symmetric.

In the atomic representation, L acts as

$$(L f)(t_0) = t_0 \, f(t_0) \tag{100}$$

for each t_0 where $Z(t_0)=0$. This is unitarily equivalent to the diagonal operator on ℓ^2 with diagonal entries $\{t_0: Z(t_0)=0\}$. Such diagonal operators are self-adjoint.

For the spectrum calculation: We have

$$L\, e_{t_0} = t_0\, e_{t_0} \tag{101}$$

so each t_0 where $Z(t_0)=0$ is an eigenvalue of L with eigenvector e_{t_0}. Since $\{e_{t_0}\}$ forms an orthonormal basis, L has pure point spectrum.

To show there are no other spectral points, suppose $\lambda \notin \{t_0: Z(t_0)=0\}$. Then for any $f \in \mathcal{D}(L)$, $(L-\lambda I)\, f$ has components

$$((L-\lambda I)\, f)(t_0) = (t_0-\lambda)\, f(t_0) \tag{102}$$

Since $t_0-\lambda \neq 0$ for all zeros t_0, we can solve

$$(L-\lambda I)\, f = g \tag{103}$$

uniquely for any $g \in \mathcal{H}$ by setting

$$f(t_0) = \frac{g(t_0)}{t_0-\lambda} \tag{104}$$

This shows $L-\lambda I$ is invertible, so $\lambda \notin \sigma(L)$. Therefore

$$\sigma(L) = \{t_0: Z(t_0)=0\} \tag{105}$$

and the eigenvalues are simple. □

Remark 22. [Operator scaffold] The construction

$$\text{stationary } X \xrightarrow{U_\theta} \text{oscillatory } Z \xrightarrow{\mu=\delta(Z)|Z'|\, dt} L^2(\mu) \xrightarrow{L:t\cdot} (L, \sigma(L)) \tag{106}$$

produces a concrete self-adjoint operator whose spectrum equals the zero set of Z, determined by the choice of time-change θ and spectral measure F. This provides an explicit realization consistent with Hilbert–Pólya approaches to encoding arithmetic information in operator spectra.

6 Appendix: Regularity and Simple Zeros

Definition 23. *[Regularity and simplicity] Assume $Z \in C^1(\mathbb{R})$ and every zero is simple: $Z(t_0)=0 \Rightarrow Z'(t_0) \neq 0$.*

Lemma 24. *[Local finiteness and delta decomposition] Under Definition 23, zeros are locally finite and*

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta(t-t_0)}{|Z'(t_0)|} \tag{107}$$

whence $\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0}$.

Proof. Since $Z \in C^1(\mathbb{R})$ and $Z'(t_0) \neq 0$ at each zero t_0, the inverse function theorem implies that Z is locally invertible near each zero. Specifically, there exists a neighborhood U_{t_0} of t_0 such that $Z|_{U_{t_0}}$ is strictly monotonic and invertible.

This implies zeros are isolated: if $Z(t_0) = 0$ and $Z'(t_0) \neq 0$, then there exists $\epsilon > 0$ such that $Z(t) \neq 0$ for $0 < |t - t_0| < \epsilon$. Therefore zeros are locally finite (finitely many in any bounded interval).

For the distributional identity, consider the one-dimensional change of variables formula for the Dirac delta. If $g: I \to \mathbb{R}$ is C^1 on interval I with $g'(x) \neq 0$ for all $x \in I$, then

$$\delta(g(x)) = \sum_{x_0: g(x_0)=0} \frac{\delta(x-x_0)}{|g'(x_0)|} \tag{108}$$

Applying this locally around each zero t_0 of Z, and since zeros are isolated, we can patch together the local results to obtain the global identity:

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta(t-t_0)}{|Z'(t_0)|} \tag{109}$$

Consequently:

$$\mu(dt) = \delta(Z(t))|Z'(t)|\ dt = \sum_{t_0: Z(t_0)=0} \frac{|Z'(t)|}{|Z'(t_0)|} \delta(t-t_0)\ dt = \sum_{t_0: Z(t_0)=0} \delta_{t_0}(dt) \tag{110}$$

where the last equality uses the fact that $|Z'(t_0)|/|Z'(t_0)| = 1$ when evaluating at $t = t_0$. □