Relationship Between U( ), V( ) and White Noise Components

The orthogonal processes U( ) and V( ) are **direct linear transformations** of the underlying white noise components, scaled by the square root of the power spectral density. This relationship embodies the fundamental connection between time-domain randomness and frequency-domain spectral structure.

Mathematical Foundation

In the spectral representation theorem, a real-valued stationary Gaussian process has the form:

$X(t) = \int_0^\infty [\cos(\lambda t)\, dU(\lambda) + \sin(\lambda t)\, dV(\lambda)]$

The orthogonal increment processes U( ) and V( ) are constructed from **complex-valued white noise measures** as described in the literature[1].

Direct Construction Relationship

In the discrete implementation, the relationship is explicit:

White Noise Components:

- $W_k^{re} \sim \mathcal{N}(0,1)$ (real part)
- $W_k^{im} \sim \mathcal{N}(0,1)$ (imaginary part)
- Independent across frequencies and between real/imaginary parts

Spectral Scaling: $Z_k = \sqrt{S(\lambda_k)\Delta\lambda} \cdot (W_k^{re} + iW_k^{im})$

Orthogonal Process Increments:

- $dU(\lambda_k) = \mathrm{Re}(Z_k) = \sqrt{S(\lambda_k)\Delta\lambda} \cdot W_k^{re}$
- $dV(\lambda_k) = \mathrm{Im}(Z_k) = \sqrt{S(\lambda_k)\Delta\lambda} \cdot W_k^{im}$

Key Properties

Isometry Preservation: The white noise isometry property[2] ensures that orthogonal white noise components map to orthogonal increments in U( ) and V( ). This means:

$E[dU(\lambda_i)dU(\lambda_j)] = E[dV(\lambda_i)dV(\lambda_j)] = 0 \text{ for } i \neq j$ $E[dU(\lambda_i)dV(\lambda_j)] = 0 \text{ for all } i, j$

Independence Structure: Since the underlying white noise components are independent Gaussians, and linear transformations preserve Gaussian distributions,

[1] https://arxiv.org/pdf/2111.01084.pdf
[2] https://www.math.utah.edu/~davar/math7880/S15/Chapter6.pdf

the orthogonal increments maintain independence across frequencies[3,4].

Spectral Coloring: The power spectral density S( ) acts as a **frequency-dependent amplification factor** that transforms white (flat spectrum) noise into colored noise with the desired spectral characteristics.

Physical Interpretation

U( ) Process: Captures the **cosine components** of the spectral decomposition. Each increment $dU(\lambda_k)$ represents the contribution of frequency λ_k to the "even" or "symmetric" part of the process.

V( ) Process: Captures the **sine components** of the spectral decomposition. Each increment $dV(\lambda_k)$ represents the contribution of frequency λ_k to the "odd" or "antisymmetric" part of the process.

Randomness Inheritance: The statistical properties (Gaussianity, independence, zero mean) are **inherited directly** from the white noise, while the frequency-dependent variance structure comes from the spectral density.

Computational Implementation

In the code implementation:

```
// White noise generation (innovation)
element.re().set(random.nextGaussian());  // W_k^re
element.im().set(random.nextGaussian());  // W_k^im

// Spectral scaling (coloring)
complexSignal.get(k).set(element).mul(mag, bits);  // Z_k = sqrt(S( _k)) * W_k

// Orthogonal process extraction
uProcess[k] = complexSignal.get(k).re().doubleValue();  // dU( _k)
vProcess[k] = complexSignal.get(k).im().doubleValue();  // dV( _k)
```

This reveals that U( ) and V( ) are **not independent random processes**, but rather **deterministic linear functionals** of the same underlying white noise field, differentiated only by their real versus imaginary parts and their trigonometric roles in the spectral representation.

The white noise provides the fundamental **innovation** or **unpredictability**, while the spectral density determines how this innovation is **distributed across frequencies** to create the desired correlation structure in the time domain.

[3] https://dsp.stackexchange.com/questions/35802/gaussian-white-noise-relation-between-distribution-and-correlation

[4] https://www.math.utah.edu/~davar/math7880/S15/Chapter6.pdf