Unitary Time Changes of Stationary Processes Yield Oscillatory Processes

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are locally square-integrable(meaning over compact sets). Applying U_θ to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}$, evolutionary spectrum $dF_t(\lambda) = \dot{\theta}(t)\, dF(\lambda)$ and expected zero-counting function $\mathbb{E}[N_{[0,T]}] = \sqrt{-\ddot{K}(0)}\, [\theta(T) - \theta(0)]$. The sample paths of any non-degenerate second-order stationary process are locally square integrable, making the unitary time-change operator U_θ applicable to typical realizations. A zero-localization measure $d\mu(t) = \delta(Z(t))|\dot{Z}(t)|\, dt$ induces a Hilbert space $L^2(\mu)$ on the zero set of each oscillatory process realization $Z(t)$, and the multiplication operator $(L\,f)(t) = t\,f(t)$ has simple pure point spectrum equal to the zero crossing set of Z.

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1 Gaussian Processes

1.1 Definition

Definition 1. (*Gaussian process*) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T a nonempty index set. A family $\{X_t : t \in T\}$ of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Gaussian process if for every finite subset $\{t_1, \ldots, t_n\} \subset T$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ is multivariate normal (possibly degenerate). Equivalently, every finite linear combination $\sum_{i=1}^n a_i X_{t_i}$ is either almost surely constant or Gaussian. The mean function is $m(t) := \mathbb{E}[X_t]$ and the covariance kernel is*

$$K(s,t) = \mathrm{Cov}(X_s, X_t) \tag{1}$$

For any finite $(t_i)_{i=1}^n \subset T$, the matrix $K_{ij} = K(t_i, t_j)$ is symmetric positive semidefinite, and a Gaussian process is completely determined in law by m and K.

Definition 2. *The canonical metric associated with a Gaussian process is*

$$d(s,t) = \sqrt{\mathbb{E}[(X_s - X_t)^2]} = \sqrt{K(s,s) + K(t,t) - 2\,K(s,t)} \tag{2}$$

1.2 Sample Path Realizations

Definition 3. *[Locally square-integrable functions]* *Define*

$$L_{\mathrm{loc}}^2(\mathbb{R}) := \left\{ f \colon \mathbb{R} \to \mathbb{C} \colon \int_K |f(t)|^2 \, dt < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \tag{3}$$

Remark 4. Every bounded measurable set in $\mathbb{R}$ is compact or contained in a compact set; hence $L_{\mathrm{loc}}^2(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

Theorem 5. *[Sample paths in $L_{\mathrm{loc}}^2(\mathbb{R})$]* *Let $\{X(t)\}_{t \in \mathbb{R}}$ be a second-order stationary process with*

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \tag{4}$$

Then almost every sample path lies in $L_{\mathrm{loc}}^2(\mathbb{R})$. However, for non-degenerate processes with $\sigma^2 > 0$, sample paths are not globally square-integrable.

Proof. Fix an arbitrary bounded interval $[a,b] \subset \mathbb{R}$ with $a < b$. Define the random variable

$$Y_{[a,b]} := \int_a^b X(t)^2 \, dt \tag{5}$$

1. By Tonelli's theorem, since $X(t)^2 \geq 0$,

$$\mathbb{E}[Y_{[a,b]}] = \mathbb{E}\left[\int_a^b X(t)^2 \, dt \right] = \int_a^b \mathbb{E}[X(t)^2] \, dt \tag{6}$$

2. By stationarity of X, $\mathbb{E}[X(t)^2] = \sigma^2$ for all $t \in \mathbb{R}$. Therefore

$$\mathbb{E}[Y_{[a,b]}] = \int_a^b \sigma^2 \, dt = \sigma^2\,(b-a) \tag{7}$$

3. Since $b - a < \infty$ and $\sigma^2 < \infty$ by assumption (4),

$$\mathbb{E}[Y_{[a,b]}] < \infty \tag{8}$$

4. By Markov's inequality, for any $M > 0$,

$$\mathbb{P}\,(Y_{[a,b]} > M) \leq \frac{\mathbb{E}[Y_{[a,b]}]}{M} = \frac{\sigma^2\,(b-a)}{M} \tag{9}$$

5. Taking $M \to \infty$ in (9),

$$\mathbb{P}\,(Y_{[a,b]} < \infty) = 1 \tag{10}$$

6. Now let $K \subset \mathbb{R}$ be an arbitrary compact set. Since K is compact in $\mathbb{R}$, it is closed and bounded. Therefore there exists $N > 0$ such that

$$K \subseteq [-N, N] \tag{11}$$

7. By (10) applied to $[a,b] = [-N, N]$,

$$\mathbb{P}\left(\int_{-N}^N X(t)^2 \, dt < \infty \right) = 1 \tag{12}$$

8. Since $K \subseteq [-N, N]$ by (11),

$$\int_K X(t)^2 \, dt \leq \int_{-N}^N X(t)^2 \, dt \tag{13}$$

9. Combining (12) and (13),

$$\mathbb{P}\left(\int_K X(t)^2 \, dt < \infty \right) = 1 \tag{14}$$

10. Since K was arbitrary, (14) holds for every compact set $K \subset \mathbb{R}$. Therefore, almost every sample path $t \mapsto X(t, \omega)$ satisfies

$$\int_K |X(t, \omega)|^2 \, dt < \infty \quad \forall \text{compact } K \subset \mathbb{R} \tag{15}$$

which means almost every sample path lies in $L^2_{\text{loc}}(\mathbb{R})$.

11. For the global divergence statement, assume $\sigma^2 > 0$. For each $n \in \mathbb{N}$, by (7),

$$\mathbb{E}\left[\int_{-n}^n X(t)^2 \, dt \right] = 2 \, n \, \sigma^2 \tag{16}$$

12. As $n \to \infty$, the right side of (16) diverges:

$$\lim_{n \to \infty} \mathbb{E}\left[\int_{-n}^n X(t)^2 \, dt \right] = \lim_{n \to \infty} 2 \, n \, \sigma^2 = \infty \tag{17}$$

13. By monotone convergence theorem, since $\int_{-n}^n X(t)^2 \, dt$ increases monotonically with n,

$$\mathbb{E}\left[\lim_{n \to \infty} \int_{-n}^n X(t)^2 \, dt \right] = \lim_{n \to \infty} \mathbb{E}\left[\int_{-n}^n X(t)^2 \, dt \right] = \infty \tag{18}$$

14. Therefore

$$\mathbb{P}\left(\int_{-\infty}^\infty X(t)^2 \, dt = \infty \right) = 1 \tag{19}$$

15. Thus sample paths are not in $L^2(\mathbb{R})$, only in $L^2_{\text{loc}}(\mathbb{R})$. $\qquad\square$

1.3 Stationary processes

Definition 6. *[Cramér spectral representation][1] A zero-mean stationary process X with spectral measure F admits the sample path representation*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t} \, d\Phi(\lambda) \tag{20}$$

which has covariance

$$R_X(t - s) = \int_{\mathbb{R}} e^{i\lambda(t-s)} \, dF(\lambda) \tag{21}$$

1.4 Oscillatory Processes

Definition 7. *[Oscillatory process][2] Let F be a finite nonnegative Borel measure on $\mathbb{R}$. Let*

$$A_t \in L^2(F) \quad \forall t \in \mathbb{R} \tag{22}$$

be the gain function and

$$\varphi_t(\lambda) = A_t(\lambda) \, e^{i\lambda t} \tag{23}$$

be the corresponding oscillatory function then an oscillatory process is a stochastic process which can be represented as

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda) \, d\Phi(\lambda) = \int_{\mathbb{R}} A_t(\lambda) \, e^{i\lambda t} \, d\Phi(\lambda) \tag{24}$$

where Φ is a complex orthogonal random measure with spectral measure F which satisfies the relation

$$d\,\mathbb{E}[\Phi(\lambda)\overline{\Phi(\mu)}] = \delta(\lambda - \mu) \, dF(\lambda) \tag{25}$$

and has the corresponding covariance kernel

$$R_Z(t, s) = \mathbb{E}[Z(t)\overline{Z(s)}] = \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)} e^{i\lambda(t-s)} \, dF(\lambda) = \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)} \, dF(\lambda) \tag{26}$$

> **Theorem 8. [Real-valuedness criterion for oscillatory processes]** *Let Z be an oscillatory process with oscillatory function*
>
> $$\varphi_t(\lambda) = A_t(\lambda)\, e^{i\lambda t} \tag{27}$$
>
> *and spectral measure F. Then Z is real-valued if and only if*
>
> $$A_t(-\lambda) = \overline{A_t(\lambda)} \tag{28}$$
>
> *for F-almost every $\lambda \in \mathbb{R}$, equivalently*
>
> $$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \tag{29}$$
>
> *for F-almost every $\lambda \in \mathbb{R}$.*

Proof.   1. Assume Z is real-valued. Then for all $t \in \mathbb{R}$,

$$Z(t) = \overline{Z(t)} \tag{30}$$

2. From the oscillatory representation (24),

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda) \tag{31}$$

3. Taking the complex conjugate of both sides of (31),

$$\overline{Z(t)} = \overline{\int_{\mathbb{R}} A_t(\lambda)\, e^{i\lambda t}\, d\,\Phi(\lambda)} = \int_{\mathbb{R}} \overline{A_t(\lambda)}e^{-i\lambda t}\, d\overline{\Phi(\lambda)} \tag{32}$$

4. For a real-valued process, the orthogonal random measure must satisfy the symmetry property

$$d\overline{\Phi(\lambda)} = -d\,\Phi(-\lambda) \tag{33}$$

5. Substituting (33) into (32),

$$\overline{Z(t)} = -\int_{\mathbb{R}} \overline{A_t(\lambda)}e^{-i\lambda t}\, d\,\Phi(-\lambda) \tag{34}$$

6. Apply the change of variables $\mu = -\lambda$, so $d\,\Phi(-\lambda) = -d\,\Phi(\mu)$ and $e^{-i\lambda t} = e^{i\mu t}$:

$$\overline{Z(t)} = -\int_{\mathbb{R}} \overline{A_t(-\mu)}e^{i\mu t}\,(-d\,\Phi(\mu)) = \int_{\mathbb{R}} \overline{A_t(-\mu)}e^{i\mu t}\, d\,\Phi(\mu) \tag{35}$$

7. By (30), the right sides of (31) and (35) must be equal:

$$\int_{\mathbb{R}} A_t(\mu)\, e^{i\mu t}\, d\,\Phi(\mu) = \int_{\mathbb{R}} \overline{A_t(-\mu)}e^{i\mu t}\, d\,\Phi(\mu) \tag{36}$$

8. Since the stochastic integral representation is unique in $L^2(F)$, the integrands must be equal F-almost everywhere:

$$A_t(\lambda) = \overline{A_t(-\lambda)} \quad \text{for } F\text{-a.e. } \lambda \tag{37}$$

9. This is equivalent to (28). From (27),

$$\varphi_t(-\lambda) = A_t(-\lambda)\, e^{-i\lambda t} \tag{38}$$

10. Using (28),

$$\varphi_t(-\lambda) = \overline{A_t(\lambda)}e^{-i\lambda t} = \overline{A_t(\lambda)\, e^{i\lambda t}} = \overline{\varphi_t(\lambda)} \tag{39}$$

establishing (29).

11. Conversely, assume (28) holds. Reversing the steps from (35) to (30) shows that $\overline{Z(t)} = Z(t)$ for all t, so Z is real-valued. $\square$

> **Theorem 9. [Existence of Oscillatory Processes]** *Let F be an absolutely continuous spectral measure and the gain function*
>
> $$A_t(\lambda) \in L^2(F) \quad \forall t \in \mathbb{R} \tag{40}$$
>
> *be measurable in both time and frequency then the time-dependent spectral density is defined by*
>
> $$S_t(\lambda) = \int_{\mathbb{R}} |A_t(\lambda)|^2 \, dF(\lambda) < \infty = \int_{\mathbb{R}} |A_t(\lambda)|^2 \, S(\lambda) \, d\lambda \tag{41}$$
>
> *and there exists a complex orthogonal random measure Φ with spectral measure F such that for each sample path $\omega_0 \in \Omega$*
>
> $$Z(t, \omega_0) = \int_{\mathbb{R}} A_t(\lambda) \, e^{i\lambda t} \, d\Phi(\lambda, \omega_0) \tag{42}$$
>
> *is well-defined in $L^2(\Omega)$ and has covariance R_Z as in (26).*

Proof. 1. Define the space of simple functions on $\mathbb{R}$: for disjoint Borel sets $\{E_j\}_{j=1}^n$ with $F(E_j) < \infty$ and coefficients $\{c_j\}_{j=1}^n \subset \mathbb{C}$,

$$g(\lambda) = \sum_{j=1}^n c_j \, 1_{E_j}(\lambda) \tag{43}$$

2. For simple functions, define the stochastic integral

$$\int_{\mathbb{R}} g(\lambda) \, d\Phi(\lambda) := \sum_{j=1}^n c_j \, \Phi(E_j) \tag{44}$$

3. Compute the second moment:

$$\mathbb{E}\left[\left| \int_{\mathbb{R}} g(\lambda) \, d\Phi(\lambda) \right|^2 \right] = \mathbb{E}\left[\left| \sum_{j=1}^n c_j \, \Phi(E_j) \right|^2 \right] = \mathbb{E}\left[\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \Phi(E_j) \overline{\Phi(E_k)} \right] \tag{45}$$

4. By linearity of expectation,

$$\mathbb{E}\left[\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \Phi(E_j) \overline{\Phi(E_k)} \right] = \sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \mathbb{E}[\Phi(E_j) \overline{\Phi(E_k)}] \tag{46}$$

5. By the orthogonality relation (25), since $E_j \cap E_k = \emptyset$ for $j \neq k$,

$$\mathbb{E}[\Phi(E_j) \overline{\Phi(E_k)}] = \begin{cases} F(E_j) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \tag{47}$$

6. Substituting (47) into (46),

$$\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \mathbb{E}[\Phi(E_j) \overline{\Phi(E_k)}] = \sum_{j=1}^n |c_j|^2 \, F(E_j) \tag{48}$$

7. The right side of (48) equals

$$\sum_{j=1}^n |c_j|^2 \, F(E_j) = \int_{\mathbb{R}} |g(\lambda)|^2 \, dF(\lambda) \tag{49}$$

8. Therefore the isometry property holds for simple functions:

$$\mathbb{E}\left[\left| \int_{\mathbb{R}} g(\lambda) \, d\Phi(\lambda) \right|^2 \right] = \int_{\mathbb{R}} |g(\lambda)|^2 \, dF(\lambda) \tag{50}$$

9. The space of simple functions is dense in $L^2(F)$. For any $h \in L^2(F)$ and $\epsilon > 0$, there exists a simple function g such that

$$\int_{\mathbb{R}} |h(\lambda) - g(\lambda)|^2 \, dF(\lambda) < \epsilon \tag{51}$$

10. By the isometry (50) and completeness of $L^2(\Omega)$, the integral extends uniquely by continuity to all $h \in L^2(F)$.

11. Since $A_t \in L^2(F)$ by assumption (40), and $|e^{i\lambda t}| = 1$,

$$\int_{\mathbb{R}} |\varphi_t(\lambda)|^2 \, d\, F(\lambda) = \int_{\mathbb{R}} |A_t(\lambda)|^2 \, d\, F(\lambda) < \infty \tag{52}$$

so $\varphi_t \in L^2(F)$.

12. Therefore

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda) \, d\, \Phi(\lambda) = \int_{\mathbb{R}} A_t(\lambda) \, e^{i\lambda t} \, d\, \Phi(\lambda) \tag{53}$$

is well-defined in $L^2(\Omega)$.

13. To compute the covariance, use the sesquilinearity of the stochastic integral:

$$R_Z(t, s) = \mathbb{E}[Z(t)\overline{Z(s)}] = \mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda) \, d\, \Phi(\lambda) \, \overline{\int_{\mathbb{R}} \varphi_s(\mu) \, d\, \Phi(\mu)} \right] \tag{54}$$

14. By Fubini's theorem for stochastic integrals,

$$\mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda) \, d\, \Phi(\lambda) \, \overline{\int_{\mathbb{R}} \varphi_s(\mu) \, d\, \Phi(\mu)} \right] = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\mu)} \mathbb{E}\left[d\, \Phi(\lambda) \overline{d\, \Phi(\mu)} \right] \tag{55}$$

15. Using the orthogonality relation (25),

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\mu)} \delta(\lambda - \mu) \, d\, F(\lambda) \, d\, F(\mu) = \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\lambda)} d\, F(\lambda) \tag{56}$$

16. Substituting the definition (23),

$$R_Z(t, s) = \int_{\mathbb{R}} A_t(\lambda) \overline{A_s(\lambda)} e^{i\lambda(t-s)} \, d\, F(\lambda) \tag{57}$$

as claimed in (26). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2 Unitarily Time-Changed Stationary Processes

2.1 Unitary Time-Change Operator $U_\theta f$

> **Theorem 10.** *[Unitary time-change and local unitarity]* *Let the time-scaling function $\theta \colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with*
>
> $$\dot{\theta}(t) > 0 \tag{58}$$
>
> *almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero. For f measurable, define*
>
> $$(U_\theta f)(t) = \sqrt{\dot{\theta}(t)} \, f(\theta(t)) \tag{59}$$
>
> *Its inverse is given by*
>
> $$(U_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{60}$$
>
> *For every compact set $K \subseteq \mathbb{R}$ and $f \in L^2_{\mathrm{loc}}(\mathbb{R})$,*
>
> $$\int_K |(U_\theta f)(t)|^2 dt = \int_{\theta(K)} |f(s)|^2 \, d\, s \tag{61}$$
>
> *Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\mathrm{loc}}(\mathbb{R})$.*

Proof.   1. Let $f \in L^2_{\mathrm{loc}}(\mathbb{R})$ and let $K \subset \mathbb{R}$ be compact. From the definition (59),

$$\int_K |(U_\theta f)(t)|^2 \, dt = \int_K \left| \sqrt{\dot{\theta}(t)} \, f(\theta(t)) \right|^2 dt \tag{62}$$

2. Expanding the square,

$$\int_K \left| \sqrt{\dot{\theta}(t)} \, f(\theta(t)) \right|^2 dt = \int_K \dot{\theta}(t) |f(\theta(t))|^2 \, dt \tag{63}$$

3. Since θ is absolutely continuous and strictly increasing, $\theta' = \dot{\theta}$ exists almost everywhere and $\dot{\theta}(t) > 0$ a.e.

4. Apply the change of variables $s = \theta(t)$. Then

$$ds = \dot{\theta}(t) \, dt \tag{64}$$

5. The inverse function $t = \theta^{-1}(s)$ exists since θ is strictly increasing and bijective.

6. As t ranges over K, the variable $s = \theta(t)$ ranges over $\theta(K)$.

7. Since θ is continuous and K is compact, $\theta(K)$ is compact.

8. Substituting (64) into (63),

$$\int_K \dot{\theta}(t) \, |f(\theta(t))|^2 dt = \int_{\theta(K)} |f(s)|^2 \, ds \tag{65}$$

9. This establishes the local isometry (61).

10. To verify U_θ^{-1} is the inverse, compute:

$$(U_\theta^{-1} U_\theta f)(s) = U_\theta^{-1} (U_\theta f)(s) \tag{66}$$

11. By definition (60),

$$U_\theta^{-1} (U_\theta f)(s) = \frac{(U_\theta f)(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{67}$$

12. By definition (59),

$$(U_\theta f)(\theta^{-1}(s)) = \sqrt{\dot{\theta}(\theta^{-1}(s))} \, f(\theta(\theta^{-1}(s))) \tag{68}$$

13. Since $\theta \circ \theta^{-1} = \mathrm{id}$,

$$f(\theta(\theta^{-1}(s))) = f(s) \tag{69}$$

14. Substituting (68) and (69) into (67),

$$\frac{\sqrt{\dot{\theta}(\theta^{-1}(s))} \, f(s)}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} = f(s) \tag{70}$$

15. Therefore

$$U_\theta^{-1} U_\theta = \mathrm{id} \tag{71}$$

16. Similarly, compute:

$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)} \, (U_\theta^{-1} g)(\theta(t)) \tag{72}$$

17. By definition (60),

$$(U_\theta^{-1} g)(\theta(t)) = \frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} \tag{73}$$

18. Since $\theta^{-1} \circ \theta = \mathrm{id}$,

$$g(\theta^{-1}(\theta(t))) = g(t), \quad \theta^{-1}(\theta(t)) = t \tag{74}$$

19. Substituting (74) into (73),

$$\frac{g(t)}{\sqrt{\dot{\theta}(t)}} \tag{75}$$

20. Therefore from (72),

$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)} \cdot \frac{g(t)}{\sqrt{\dot{\theta}(t)}} = g(t) \tag{76}$$

21. Thus

$$U_\theta \, U_\theta^{-1} = \mathrm{id} \tag{77}$$

22. Combining (71) and (77), U_θ^{-1} is the two-sided inverse of U_θ on $L^2_{\mathrm{loc}}(\mathbb{R})$. $\square$

2.2 Inverse Filter for Unitary Time Transformations

Theorem 11. *[Inverse Filter for Unitary Time Transformations] Let $\theta \colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\theta'(t) > 0$ almost everywhere. Let $Y(u)$ be a stationary process with unit variance, and define*

$$Z(t) = \sqrt{\dot{\theta}(t)} \, Y(\theta(t)) \tag{78}$$

as the oscillatory process obtained by the unitary time transformation. Then:

   1. The forward filter kernel is

$$h(t, u) = \sqrt{\dot{\theta}(t)} \, \delta\left(u - \theta(t)\right) \tag{79}$$

   2. The inverse filter kernel is

$$g(t, s) = \frac{\delta\left(s - \theta^{-1}(t)\right)}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \tag{80}$$

   3. The composition $(g \circ h)$ recovers the identity:

$$Y(t) = \int_{\mathbb{R}} g(t, s) \, Z(s) \, d\,s = \frac{Z(\theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \tag{81}$$

Proof.   1. From (78), the forward transformation is

$$Z(t) = \int_{\mathbb{R}} h(t, u) \, Y(u) \, d\,u \tag{82}$$

2. Substituting (79),

$$\int_{\mathbb{R}} h(t, u) \, Y(u) \, d\,u = \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} \, \delta\left(u - \theta(t)\right) Y(u) \, d\,u \tag{83}$$

3. By the sifting property of the Dirac delta,

$$\int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} \, \delta\left(u - \theta(t)\right) Y(u) \, d\,u = \sqrt{\dot{\theta}(t)} \, Y(\theta(t)) \tag{84}$$

4. This confirms (78).

5. For the inverse, compute:

$$\int_{\mathbb{R}} g(t, s) \, Z(s) \, d\,s = \int_{\mathbb{R}} \frac{\delta\left(s - \theta^{-1}(t)\right)}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \, Z(s) \, d\,s \tag{85}$$

6. By the sifting property,

$$\int_{\mathbb{R}} \frac{\delta\left(s - \theta^{-1}(t)\right)}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \, Z(s) \, d\,s = \frac{Z(\theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} \tag{86}$$

7. Substituting (78) with t replaced by $\theta^{-1}(t)$,

$$Z(\theta^{-1}(t)) = \sqrt{\dot{\theta}(\theta^{-1}(t))} \, Y(\theta(\theta^{-1}(t))) \tag{87}$$

8. Since $\theta \circ \theta^{-1} = \mathrm{id}$,
$$Y(\theta(\theta^{-1}(t))) = Y(t) \tag{88}$$

9. Substituting (87) and (88) into (86),
$$\frac{Z(\theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} = \frac{\sqrt{\dot{\theta}(\theta^{-1}(t))}\, Y(t)}{\sqrt{\dot{\theta}(\theta^{-1}(t))}} = Y(t) \tag{89}$$

10. This establishes (81), confirming that $g \circ h = \mathrm{id}$. $\qquad\square$

2.3 Transformation of Stationary $\rightarrow$ Oscillatory Processes via U_θ

Theorem 12. [Unitary time change yields oscillatory process] *Let X be zero-mean stationary as in Definition 6. For scaling function θ as in Theorem 10, define*
$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) \tag{90}$$

Then Z is a realization of an oscillatory process with oscillatory function
$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \tag{91}$$
gain function
$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)} \tag{92}$$
and covariance
$$\begin{aligned}
R_Z(t,s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\
&= \mathbb{E}\left[\sqrt{\dot{\theta}(t)}\, X(\theta(t))\overline{\sqrt{\dot{\theta}(s)}\, X(\theta(s))}\right] \\
&= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\, \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \qquad R_Z(t,s) = \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\, dF(\lambda) \\
&= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)}\, R_X(\theta(t)-\theta(s)) \\
&= \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))}\, dF(\lambda)
\end{aligned} \tag{93}$$

Proof. 1. From the Cramér representation (20),
$$X(u) = \int_{\mathbb{R}} e^{i\lambda u}\, d\Phi(\lambda) \tag{94}$$

2. Substituting $u = \theta(t)$ into (94),
$$X(\theta(t)) = \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\Phi(\lambda) \tag{95}$$

3. From the definition (90),
$$Z(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\Phi(\lambda) \tag{96}$$

4. By linearity of the stochastic integral,
$$Z(t) = \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)}\, d\Phi(\lambda) \tag{97}$$

5. Define
$$\varphi_t(\lambda) := \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \tag{98}$$

6. Then (97) becomes

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda) \, d\,\Phi(\lambda) \tag{99}$$

which is the oscillatory representation (24).

7. To express this in terms of the standard oscillatory function form, write

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} \, e^{i\lambda\theta(t)} = \sqrt{\dot{\theta}(t)} \, e^{i\lambda(\theta(t)-t)} \, e^{i\lambda t} \tag{100}$$

8. Define the gain function

$$A_t(\lambda) := \sqrt{\dot{\theta}(t)} \, e^{i\lambda(\theta(t)-t)} \tag{101}$$

9. Then

$$\varphi_t(\lambda) = A_t(\lambda) \, e^{i\lambda t} \tag{102}$$

confirming the oscillatory function form (23).

10. To compute the covariance, use (26):

$$R_Z(t,s) = \mathbb{E}[Z(t)\overline{Z(s)}] \tag{103}$$

11. Substituting (90),

$$R_Z(t,s) = \mathbb{E}\left[\sqrt{\dot{\theta}(t)} \, X(\theta(t)) \overline{\sqrt{\dot{\theta}(s)} \, X(\theta(s))} \right] \tag{104}$$

12. Since $\dot{\theta}$ is deterministic,

$$R_Z(t,s) = \sqrt{\dot{\theta}(t)} \, \sqrt{\dot{\theta}(s)} \, \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \tag{105}$$

13. By stationarity of X, using (21),

$$\mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] = R_X\left(\theta(t) - \theta(s)\right) = \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} \, d\,F(\lambda) \tag{106}$$

14. Substituting (106) into (105),

$$R_Z(t,s) = \sqrt{\dot{\theta}(t)\,\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} \, d\,F(\lambda) \tag{107}$$

establishing (93). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Corollary 13. *[Evolutionary spectrum of unitarily time-changed stationary process]* *The evolutionary spectrum is*

$$d\,F_t(\lambda) = \dot{\theta}(t) \, d\,F(\lambda) \tag{108}$$

Proof.   1. The evolutionary spectrum is defined by

$$d\,F_t(\lambda) = |A_t(\lambda)|^2 \, d\,F(\lambda) \tag{109}$$

2. From (92),

$$|A_t(\lambda)|^2 = \left| \sqrt{\dot{\theta}(t)} \, e^{i\lambda(\theta(t)-t)} \right|^2 \tag{110}$$

3. Since $|e^{i\alpha}| = 1$ for all real α,

$$|e^{i\lambda(\theta(t)-t)}|^2 = 1 \tag{111}$$

4. Therefore

$$|A_t(\lambda)|^2 = \left(\sqrt{\dot{\theta}(t)} \right)^2 \cdot 1 = \dot{\theta}(t) \tag{112}$$

5. Substituting (112) into (109),

$$d\,F_t(\lambda) = \dot{\theta}(t) \, d\,F(\lambda) \tag{113}\ \square$$

2.4 Covariance operator conjugation

Proposition 14. [Operator conjugation] *Let*

$$(T_K f)(t) := \int_{\mathbb{R}} K(|t - s|) \, f(s) \, d\,s \tag{114}$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h} \, d\,F(\lambda) \tag{115}$$

Define the transformed kernel

$$K_\theta(s, t) := \sqrt{\dot{\theta}(t)\, \dot{\theta}(s)} \; K(|\theta(t) - \theta(s)|) \tag{116}$$

then the corresponding integral covariance operator is conjugated for all $f \in L^2_{\mathrm{loc}}(\mathbb{R})$ by

$$(T_{K_\theta} f)(t) = (U_\theta \, T_K \, U_\theta^{-1} \, f)(t) \tag{117}$$

Proof.   1. From (117), expand the right side:

$$(U_\theta \, T_K \, U_\theta^{-1} \, f)(t) = \sqrt{\dot{\theta}(t)} \, (T_K \, U_\theta^{-1} \, f)(\theta(t)) \tag{118}$$

2. By definition (114),

$$(T_K \, U_\theta^{-1} \, f)(\theta(t)) = \int_{\mathbb{R}} K(|\theta(t) - s|) \, (U_\theta^{-1} \, f)(s) \, d\,s \tag{119}$$

3. By definition (60),

$$(U_\theta^{-1} \, f)(s) = \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{120}$$

4. Substituting (120) into (119),

$$\int_{\mathbb{R}} K(|\theta(t) - s|) \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \, d\,s \tag{121}$$

5. Apply the change of variables $s = \theta(u)$, so $d\,s = \dot{\theta}(u) \, d\,u$ and $\theta^{-1}(s) = u$:

$$\int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \frac{f(u)}{\sqrt{\dot{\theta}(u)}} \dot{\theta}(u) \, d\,u \tag{122}$$

6. Simplify:

$$\int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \frac{\dot{\theta}(u)}{\sqrt{\dot{\theta}(u)}} \, f(u) \, d\,u = \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \sqrt{\dot{\theta}(u)} \, f(u) \, d\,u \tag{123}$$

7. Substituting (123) into (118),

$$\sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \sqrt{\dot{\theta}(u)} \, f(u) \, d\,u \tag{124}$$

8. Bring the constant inside the integral:

$$\int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} \, \sqrt{\dot{\theta}(u)} \, K(|\theta(t) - \theta(u)|) \, f(u) \, d\,u \tag{125}$$

9. By definition (116),

$$\sqrt{\dot{\theta}(t)} \, \sqrt{\dot{\theta}(u)} \, K(|\theta(t) - \theta(u)|) = K_\theta(u, t) \tag{126}$$

10. Therefore

$$\int_{\mathbb{R}} K_\theta(u, t) \, f(u) \, d\,u = (T_{K_\theta} f)(t) \tag{127}$$

establishing (117). $\square$

3 Zero Localization

Definition 15. *[Zero localization measure] Let Z be real-valued with $Z \in C^1(\mathbb{R})$ having only simple zeros*

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \tag{128}$$

Define, for Borel $B \subset \mathbb{R}$,

$$\mu(B) = \int_{\mathbb{R}} 1_B(t)\, \delta(Z(t))|\dot{Z}(t)|\, d\,t \tag{129}$$

Theorem 16. *[Atomicity and local finiteness of zeros and delta decomposition] Under the assumptions of Definition 15, zeros are locally finite and one has*

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0) = 0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \tag{130}$$

whence

$$\mu = \sum_{t_0 : Z(t_0) = 0} \delta_{t_0} \tag{131}$$

Proof.   1. For any smooth test function ϕ with compact support, apply the standard change of variables formula for the delta function. Let $\{t_0^{(1)}, t_0^{(2)}, \dots\}$ denote the zeros of Z.

2. By the change of variables formula for distributions,

$$\int_{\mathbb{R}} \phi(t)\, \delta(Z(t))\, d\,t = \sum_{t_0 : Z(t_0) = 0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} \tag{132}$$

3. The right side of (132) equals

$$\sum_{t_0 : Z(t_0) = 0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} = \sum_{t_0 : Z(t_0) = 0} \int_{\mathbb{R}} \phi(t) \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|}\, d\,t \tag{133}$$

4. By Fubini's theorem (justified since the sum has locally finite terms due to C^1 regularity and simple zeros),

$$\sum_{t_0 : Z(t_0) = 0} \int_{\mathbb{R}} \phi(t) \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|}\, d\,t = \int_{\mathbb{R}} \phi(t) \sum_{t_0 : Z(t_0) = 0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|}\, d\,t \tag{134}$$

5. Comparing (132) and (134),

$$\int_{\mathbb{R}} \phi(t)\, \delta(Z(t))\, d\,t = \int_{\mathbb{R}} \phi(t) \sum_{t_0 : Z(t_0) = 0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|}\, d\,t \tag{135}$$

6. Since ϕ is arbitrary,

$$\delta(Z(t)) = \sum_{t_0 : Z(t_0) = 0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \tag{136}$$

establishing (130).

7. Substituting (136) into the definition (129),

$$\mu(B) = \int_{\mathbb{R}} 1_B(t) \sum_{t_0 : Z(t_0) = 0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|}|\dot{Z}(t)|\, d\,t \tag{137}$$

8. By the sifting property of the delta function, $|\dot{Z}(t)|$ evaluated at $t = t_0$ gives $|\dot{Z}(t_0)|$:

$$\int_{\mathbb{R}} 1_B(t) \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|}|\dot{Z}(t)|\, d\,t = \frac{1_B(t_0)|\dot{Z}(t_0)|}{|\dot{Z}(t_0)|} = 1_B(t_0) \tag{138}$$

9. Summing over all zeros,

$$\mu(B) = \sum_{t_0: Z(t_0)=0} 1_B(t_0) = \sum_{t_0 \in B: Z(t_0)=0} 1 \tag{139}$$

10. This is precisely the atomic measure

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \tag{140}$$

establishing (131). □

Definition 17. *[Hilbert space on the zero set] Let $\mathcal{H} = L^2(\mu)$ with inner product*

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)} d\mu(t) \tag{141}$$

Proposition 18. *[Atomic structure] Let*

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \tag{142}$$

then

$$\mathcal{H} \cong \left\{ f: \{t_0: Z(t_0)=0\} \to \mathbb{C}: \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \tag{143}$$

with orthonormal basis $\{e_{t_0}\}_{t_0: Z(t_0)=0}$ where

$$e_{t_0}(t_1) = \delta_{t_0, t_1} \tag{144}$$

Proof. 1. By (142), μ is a purely atomic measure with atoms at the zero set.

2. For any $f \in L^2(\mu)$, the L^2 norm is

$$\|f\|_{L^2(\mu)}^2 = \int_{\mathbb{R}} |f(t)|^2 d\mu(t) \tag{145}$$

3. Substituting (142),

$$\int_{\mathbb{R}} |f(t)|^2 d\mu(t) = \int_{\mathbb{R}} |f(t)|^2 \sum_{t_0: Z(t_0)=0} \delta_{t_0}(dt) = \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 \tag{146}$$

4. Therefore

$$\|f\|_{L^2(\mu)}^2 = \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 \tag{147}$$

5. This is precisely the ℓ^2 norm on the zero set.

6. Define the map $\Psi: L^2(\mu) \to \ell^2$ by

$$\Psi(f) = (f(t_0))_{t_0: Z(t_0)=0} \tag{148}$$

7. From (147), Ψ is an isometry:

$$\|\Psi(f)\|_{\ell^2}^2 = \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 = \|f\|_{L^2(\mu)}^2 \tag{149}$$

8. Ψ is surjective: for any sequence $(c_{t_0}) \in \ell^2$, define $f(t) = \sum_{t_0} c_{t_0} \delta_{t_0}(t)$, which is in $L^2(\mu)$.

9. Therefore Ψ is a Hilbert space isomorphism, establishing (143).

10. For the orthonormal basis, define e_{t_0} by (144).

11. Then

$$\langle e_{t_0}, e_{t_1} \rangle = \int_{\mathbb{R}} e_{t_0}(t)\overline{e_{t_1}(t)} d\mu(t) = \sum_{s: Z(s)=0} \delta_{t_0, s} \delta_{t_1, s} = \delta_{t_0, t_1} \tag{150}$$

12. Therefore $\{e_{t_0}\}$ is an orthonormal set.

13. Since every $f \in L^2(\mu)$ can be written as

$$f = \sum_{t_0 : Z(t_0) = 0} f(t_0)\, e_{t_0} \tag{151}$$

the set $\{e_{t_0}\}$ is complete, hence an orthonormal basis. $\square$

Definition 19. [Multiplication operator] *Define the linear operator*

$$L \colon \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H} \tag{152}$$

by

$$(L\,f)(t) = t\,f(t) \tag{153}$$

on the support of μ with domain

$$\mathcal{D}(L) := \left\{ f \in \mathcal{H} \colon \int |t\,f(t)|^2\, d\mu(t) < \infty \right\} \tag{154}$$

Theorem 20. [Self-adjointness and spectrum] *L is self-adjoint on $\mathcal{H}$ and has pure point, simple spectrum*

$$\sigma(L) = \overline{\{t \in \mathbb{R} \colon Z(t) = 0\}} \tag{155}$$

with eigenvalues $\lambda = t_0$ for each zero t_0 and corresponding eigenvectors e_{t_0}.

Proof.   1. For $f, g \in \mathcal{D}(L)$, compute the inner product:

$$\langle L\,f, g \rangle = \int_{\mathbb{R}} (L\,f)(t)\,\overline{g(t)}\,d\mu(t) \tag{156}$$

2. By definition (153),

$$\int_{\mathbb{R}} t\,f(t)\,\overline{g(t)}\,d\mu(t) \tag{157}$$

3. Since t is real-valued, $\bar{t} = t$, so

$$\int_{\mathbb{R}} t\,f(t)\,\overline{g(t)}\,d\mu(t) = \int_{\mathbb{R}} f(t)\,\overline{t\,g(t)}\,d\mu(t) \tag{158}$$

4. The right side of (158) is

$$\int_{\mathbb{R}} f(t)\,\overline{(L\,g)(t)}\,d\mu(t) = \langle f, L\,g \rangle \tag{159}$$

5. Therefore

$$\langle L\,f, g \rangle = \langle f, L\,g \rangle \tag{160}$$

for all $f, g \in \mathcal{D}(L)$, establishing that L is symmetric.

6. Since L is a multiplication operator on $L^2(\mu)$, it is self-adjoint (by standard functional analysis).

7. To determine the spectrum, compute the action on basis vectors. From (153) and (144),

$$(L\,e_{t_0})(t) = t\,e_{t_0}(t) = t\,\delta_{t_0}(t) \tag{161}$$

8. By the sifting property,

$$t\,\delta_{t_0}(t) = t_0\,\delta_{t_0}(t) = t_0\,e_{t_0}(t) \tag{162}$$

9. Therefore

$$L\,e_{t_0} = t_0\,e_{t_0} \tag{163}$$

10. This shows that each t_0 is an eigenvalue with eigenvector e_{t_0}.

11. Since the $\{e_{t_0}\}$ form a complete orthonormal basis (Proposition 18), the spectrum is pure point.

12. Each eigenspace is one-dimensional (spanned by e_{t_0}), so the spectrum is simple.

13. Therefore

$$\sigma(L) = \{t_0 \colon Z(t_0) = 0\} \tag{164}$$

14. If the zero set accumulates, the spectrum is the closure:

$$\sigma(L) = \overline{\{t \in \mathbb{R} \colon Z(t) = 0\}} \tag{165} \ \square$$

3.1 The Kac-Rice Formula For The Expected Zero Counting Function

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Theorem 21. [Kac-Rice Formula for Zero Crossings] *Let $Z(t)$ be a centered Gaussian process on $[a,b]$ with covariance $K(s,t) = \mathbb{E}[Z(s)\,Z(t)]$ then the expected number of zeros in $[a,b]$ is*

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{\frac{2}{\pi}} \, \frac{\sqrt{K(t,t)\,K_{\dot{Z}}(t,t) - K_{Z,\dot{Z}}(t,t)^2}}{K(t,t)} \, dt \tag{166}$$

where

$$K(t,t) = \mathbb{E}[Z(t)^2] \tag{167}$$

$$K_{\dot{Z}}(t,t) = -\partial_s^2 \, \partial_t \, K(s,t)|_{s=t} \tag{168}$$

and

$$K_{Z,\dot{Z}}(t,t) = \partial_s \, K(s,t)|_{s=t} \tag{169}$$

Proof.   1. The expected number of zeros is given by the Kac-Rice formula for the level crossing density at level zero.

2. For a Gaussian process $Z(t)$, the joint distribution of $(Z(t), \dot{Z}(t))$ is bivariate normal with covariance matrix

$$\Sigma(t) = \begin{pmatrix} K(t,t) & K_{Z,\dot{Z}}(t,t) \\ K_{Z,\dot{Z}}(t,t) & K_{\dot{Z}}(t,t) \end{pmatrix} \tag{170}$$

3. The determinant of $\Sigma(t)$ is

$$\det \Sigma(t) = K(t,t)\,K_{\dot{Z}}(t,t) - K_{Z,\dot{Z}}(t,t)^2 \tag{171}$$

4. The Kac-Rice formula states that the expected number of zeros in $[a,b]$ is

$$\mathbb{E}[N_{[a,b]}] = \int_a^b p_{Z(t),\dot{Z}(t)}(0,v)|v| \, dv \, dt \tag{172}$$

   where $p_{Z(t),\dot{Z}(t)}$ is the joint density of $(Z(t), \dot{Z}(t))$.

5. For a centered bivariate normal distribution with covariance $\Sigma(t)$,

$$p_{Z(t),\dot{Z}(t)}(z,v) = \frac{1}{2\pi \sqrt{\det \Sigma(t)}} \exp\left(-\frac{1}{2} \begin{pmatrix} z \\ v \end{pmatrix}^T \Sigma(t)^{-1} \begin{pmatrix} z \\ v \end{pmatrix} \right) \tag{173}$$

6. At $z = 0$,

$$p_{Z(t),\dot{Z}(t)}(0,v) = \frac{1}{2\pi \sqrt{\det \Sigma(t)}} \exp\left(-\frac{v^2 \, K(t,t)}{2 \det \Sigma(t)} \right) \tag{174}$$

7. Integrating (174) against $|v|$,

$$\int_{-\infty}^{\infty} p_{Z(t),\dot{Z}(t)}(0,v)|v| \, dv = \frac{1}{2\pi \sqrt{\det \Sigma(t)}} \int_{-\infty}^{\infty} |v| \exp\left(-\frac{v^2 \, K(t,t)}{2 \det \Sigma(t)} \right) dv \tag{175}$$

8. Using symmetry, $\int_{-\infty}^{\infty} |v| \, e^{-av^2} \, dv = 2 \int_0^{\infty} v \, e^{-av^2} \, dv = \frac{1}{a} \sqrt{\frac{\pi}{a}}$:

$$\int_{-\infty}^{\infty} |v| \exp\left(-\frac{v^2 \, K(t,t)}{2 \det \Sigma(t)} \right) dv = \sqrt{\frac{2\pi \det \Sigma(t)}{K(t,t)}} \tag{176}$$

9. Substituting (176) into (175),

$$\frac{1}{2\,\pi\,\sqrt{\det\Sigma(t)}}\cdot\sqrt{\frac{2\,\pi\det\Sigma(t)}{K(t,t)}}=\sqrt{\frac{2}{\pi}}\,\frac{\sqrt{\det\Sigma(t)}}{K(t,t)} \tag{177}$$

10. Using (171),

$$\sqrt{\frac{2}{\pi}}\,\frac{\sqrt{K(t,t)\,K_{\dot{Z}}(t,t)-K_{Z,\dot{Z}}(t,t)^2}}{K(t,t)} \tag{178}$$

11. Integrating (178) over $[a,b]$ yields (166). $\square$

3.1.1   The Expected Zero Counting Function for Unitarily Time-Changed Stationary Processes

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Theorem 22. (Expected Zero-Counting Function Of The Oscillatory Process Subclass of Unitarily Time-Changed Stationary Processes) *Let $\theta\colon\mathbb{R}\to\mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\dot{\theta}(t)>0$ almost everywhere. Let X be a centered stationary Gaussian process with spectral measure F and covariance function*

$$K(h)=\int_{\mathbb{R}}e^{i\omega h}\,dF(\omega) \tag{179}$$

twice differentiable at $h=0$. Define the unitarily time-changed process

$$Z(t)=\sqrt{\dot{\theta}(t)}\;X(\theta(t)) \tag{180}$$

Then Z is a centered Gaussian process with covariance

$$K_Z(t,s)=\sqrt{\dot{\theta}(t)\dot{\theta}(s)}\;K\big(\theta(t)-\theta(s)\big) \tag{181}$$

and the expected number of zeros in $[0,T]$ is

$$\mathbb{E}[N_{[0,T]}]=\sqrt{\frac{-\ddot{K}(0)}{\pi\,K(0)}}\;[\theta(T)-\theta(0)] \tag{182}$$

Proof.   1. By the Kac-Rice formula:

$$\mathbb{E}[N_{[0,T]}]=\int_0^T\sqrt{-\lim_{s\to t}\frac{\partial^2}{\partial s\,\partial t}K_Z(s,t)}\;dt \tag{183}$$

2. Differentiate (181) with respect to s:

$$\frac{\partial}{\partial s}K_Z(s,t)=\frac{\ddot{\theta}(s)}{2\sqrt{\dot{\theta}(s)\dot{\theta}(t)}}\;K\big(\theta(t)-\theta(s)\big)-\dot{\theta}(s)\sqrt{\dot{\theta}(s)\dot{\theta}(t)}\;\dot{K}\big(\theta(t)-\theta(s)\big) \tag{184}$$

3. Differentiate (184) with respect to t and take $s\to t$. Since $\dot{K}(0)=0$ by stationarity:

$$\lim_{s\to t}\frac{\partial^2}{\partial s\,\partial t}K_Z(s,t)=-\dot{\theta}(t)^2\,\ddot{K}(0) \tag{185}$$

4. Substitute (185) into (183):

$$\mathbb{E}[N_{[0,T]}]=\int_0^T\sqrt{-(-\dot{\theta}(t)^2\,\ddot{K}(0))}\;dt=\int_0^T\sqrt{\dot{\theta}(t)^2\,(-\ddot{K}(0))}\;dt \tag{186}$$

5. Since $\dot{\theta}(t)>0$ and $\ddot{K}(0)<0$:

$$\mathbb{E}[N_{[0,T]}]=\int_0^T\dot{\theta}(t)\sqrt{-\ddot{K}(0)}\;dt=\sqrt{-\ddot{K}(0)}\int_0^T\dot{\theta}(t)\;dt \tag{187}$$

6. Evaluate the integral:

$$\mathbb{E}[N_{[0,T]}] = \sqrt{-\ddot{K}(0)} \ [\theta(T) - \theta(0)] \tag{188} \ \square$$

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