

Unitary Time Changes of Stationary Processes Yield Oscillatory Processes

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Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ , acting on functions that are locally square-integrable (meaning over compact sets). Applying U_θ to the Cramér spectral representation of a stationary process yields an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)}$, evolutionary spectrum $dF_t(\lambda) = \dot{\theta}(t) dF(\lambda)$, and expected zero-counting function $\mathbb{E}[N_{[0,T]}] = \frac{[\theta(T) - \theta(0)]}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}}$.

The sample paths of any non-degenerate second-order stationary process are locally square integrable, making the unitary time-change operator U_θ applicable to typical realizations. A zero-localization measure $d\mu(t) = \delta(Z(t))|\dot{Z}(t)| dt$ induces a Hilbert space $L^2(\mu)$ on the zero set of each oscillatory process realization $Z(t)$, and the multiplication operator $(Lf)(t) = t f(t)$ has simple pure point spectrum equal to the zero crossing set of Z .

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1 Gaussian Processes

Unless otherwise stated, all processes considered will be real-valued.

Theorem 1. *Let $X(u)$ be a real-valued process:*

$$X(u) \in \mathbb{R} \quad \forall u \in \mathbb{R} \quad (1)$$

Then its (complex-valued) random orthogonal spectral measure satisfies

$$d\bar{\Phi}(\lambda) = d\Phi(-\lambda) \quad (2)$$

and the corresponding covariance spectral measure F is even:

$$F(-A) = F(A) \quad (3)$$

Proof. 1. The spectral representation for $X(u)$ is

$$X(u) = \int_{-\infty}^{\infty} e^{i\lambda u} d\Phi(\lambda) \quad (4)$$

Since $X(u)$ is real-valued for each u ,

$$\overline{X(u)} = X(u) \quad (5)$$

On the other hand,

$$\overline{X(u)} = \overline{\int_{-\infty}^{\infty} e^{i\lambda u} d\Phi(\lambda)} \quad (6)$$

$$= \int_{-\infty}^{\infty} \overline{e^{i\lambda u}} d\bar{\Phi}(\lambda) \quad (7)$$

$$= \int_{-\infty}^{\infty} e^{-i\lambda u} d\bar{\Phi}(\lambda) \quad (8)$$

By the substitution $\lambda \mapsto -\mu$,

$$\int_{-\infty}^{\infty} e^{-i\lambda u} d\bar{\Phi}(\lambda) = \int_{-\infty}^{\infty} e^{i\mu u} d\bar{\Phi}(-\mu) \quad (9)$$

So

$$X(u) = \int_{-\infty}^{\infty} e^{i\lambda u} d\Phi(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda u} d\bar{\Phi}(-\lambda) \quad (10)$$

By uniqueness of the spectral measure representation, it follows that

$$d\bar{\Phi}(\lambda) = d\Phi(-\lambda) \quad (11)$$

as (orthogonal) random measures.

2. The covariance function of X is

$$R(u) = \mathbb{E}(X(0)X(u)) = \int_{-\infty}^{\infty} e^{i\lambda u} dF(\lambda) \quad (12)$$

Since $X(u)$ is real-valued, $R(u)$ is real and $R(-u) = R(u)$. Thus,

$$R(-u) = \int_{-\infty}^{\infty} e^{-i\lambda u} dF(\lambda) = \int_{-\infty}^{\infty} e^{i\mu u} dF(-\mu) \quad (13)$$

Equating with $R(u)$,

$$\int_{-\infty}^{\infty} e^{i\lambda u} dF(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda u} dF(-\lambda) \quad (14)$$

for all u . By the uniqueness theorem for Fourier–Stieltjes transforms, this implies

$$dF(\lambda) = dF(-\lambda) \quad (15)$$

Thus for any Borel set A ,

$$F(-A) = F(A) \quad (16)$$

establishing the evenness property. \square

1.1 Definition

Definition 2. (Gaussian process) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T a non-empty index set. A family $\{X_t: t \in T\}$ of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Gaussian process if for every finite subset $\{t_1, \dots, t_n\} \subset T$ the random vector $(X_{t_1}, \dots, X_{t_n})$ is multivariate normal (possibly degenerate). Equivalently, every finite linear combination $\sum_{i=1}^n a_i X_{t_i}$ is either almost surely constant or Gaussian. The mean function is $m(t) := \mathbb{E}[X_t]$ and the covariance kernel is

$$K(s, t) = \text{Cov}(X_s, X_t) \quad (17)$$

For any finite $(t_i)_{i=1}^n \subset T$, the matrix $K_{ij} = K(t_i, t_j)$ is symmetric positive semidefinite, and a Gaussian process is completely determined in law by m and K .

1.2 Stationary processes

Definition 3. [Cramér spectral representation] [1] A zero-mean stationary process X with spectral measure F admits the sample path representation

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t} d\Phi(\lambda) \quad (18)$$

which has covariance

$$R_X(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)} dF(\lambda) \quad (19)$$

1.2.1 Sample Path Realizations

Definition 4. [*Locally square-integrable functions*] Define

$$L^2_{\text{loc}}(\mathbb{R}) := \left\{ f: \mathbb{R} \rightarrow \mathbb{C}: \int_K |f(t)|^2 dt < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\} \quad (20)$$

Remark 5. Every bounded measurable set in \mathbb{R} is compact or contained in a compact set; hence $L^2_{\text{loc}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

Theorem 6. [*Sample paths in $L^2_{\text{loc}}(\mathbb{R})$*] Let $\{X(t)\}_{t \in \mathbb{R}}$ be a second-order stationary process with

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty \quad (21)$$

Then almost every sample path lies in $L^2_{\text{loc}}(\mathbb{R})$.

Proof. Fix an arbitrary bounded interval $[a, b] \subset \mathbb{R}$ with $a < b$. Define the random variable

$$Y_{[a,b]} := \int_a^b X(t)^2 dt \quad (22)$$

By Fubini's theorem,

$$\mathbb{E}[Y_{[a,b]}] = \int_a^b \mathbb{E}[X(t)^2] dt = (b - a) \sigma^2 < \infty \quad (23)$$

By Markov's inequality, $\mathbb{P}(Y_{[a,b]} = \infty) = 0$. Thus $Y_{[a,b]} < \infty$ almost surely. Covering compacts by countably many dyadic intervals yields the result: for every compact $K \subset \mathbb{R}$,

almost surely $\int_K X(t)^2 dt < \infty$. □

2 Oscillatory Processes

Definition 7. [*Oscillatory process*] [2] Let F be a finite nonnegative Borel measure on \mathbb{R} . Let

$$A_t \in L^2(F) \quad \forall t \in \mathbb{R} \quad (24)$$

be the gain function and

$$\varphi_t(\lambda) = A_t(\lambda) e^{i\lambda t} \quad (25)$$

be the corresponding oscillatory function; then an oscillatory process is a stochastic process which can be represented as

$$\begin{aligned} Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \end{aligned} \quad (26)$$

where Φ is a complex orthogonal random measure with spectral measure F which satisfies the relation

$$d\mathbb{E}[\Phi(\lambda)\overline{\Phi(\mu)}] = \delta(\lambda - \mu) dF(\lambda) \quad (27)$$

and has the corresponding covariance kernel

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)} e^{i\lambda(t-s)} dF(\lambda) \\ &= \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)} dF(\lambda) \end{aligned} \quad (28)$$

Theorem 8. [*Real-valuedness criterion for oscillatory processes*] Let Z be an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = A_t(\lambda) e^{i\lambda t} \quad (29)$$

and spectral measure F . Then Z is real-valued if and only if

$$A_t(-\lambda) = \overline{A_t(\lambda)} \quad (30)$$

for F -almost every $\lambda \in \mathbb{R}$, equivalently

$$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \quad (31)$$

for F -almost every $\lambda \in \mathbb{R}$.

Proof. 1. Assume Z is real-valued. Then for all $t \in \mathbb{R}$,

$$Z(t) = \overline{Z(t)} \quad (32)$$

2. From the oscillatory representation (26),

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \quad (33)$$

3. Taking the complex conjugate of both sides of (33),

$$\overline{Z(t)} = \overline{\int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t} d\overline{\Phi(\lambda)} \quad (34)$$

4. For a real-valued process, the orthogonal random measure must satisfy the symmetry property from Theorem 1:

$$d\overline{\Phi(\lambda)} = d\Phi(-\lambda) \quad (35)$$

5. Substituting (35) into (34),

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(\lambda)} e^{-i\lambda t} d\Phi(-\lambda) \quad (36)$$

6. Apply the change of variables $\mu = -\lambda$, so $d\Phi(-\lambda) = d\Phi(\mu)$ and $e^{-i\lambda t} = e^{i\mu t}$:

$$\overline{Z(t)} = \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t} d\Phi(\mu) \quad (37)$$

7. By (32), the right sides of (33) and (37) must be equal:

$$\int_{\mathbb{R}} A_t(\mu) e^{i\mu t} d\Phi(\mu) = \int_{\mathbb{R}} \overline{A_t(-\mu)} e^{i\mu t} d\Phi(\mu) \quad (38)$$

8. Since the stochastic integral representation is unique in $L^2(F)$, the integrands must be equal F -almost everywhere:

$$A_t(\lambda) = \overline{A_t(-\lambda)} \quad \text{for } F\text{-a.e. } \lambda \quad (39)$$

9. This is equivalent to (30). From (29),

$$\varphi_t(-\lambda) = A_t(-\lambda) e^{-i\lambda t} \quad (40)$$

10. Using (30),

$$\begin{aligned} \varphi_t(-\lambda) &= \overline{A_t(\lambda)} e^{-i\lambda t} \\ &= \overline{A_t(\lambda)} e^{i\lambda t} \\ &= \overline{\varphi_t(\lambda)} \end{aligned} \quad (41)$$

establishing (31).

11. Conversely, assume (30) holds. Reversing the steps from (37) to (32) shows that $\overline{Z(t)} = Z(t)$ for all t , so Z is real-valued. \square

Theorem 9. [Existence of Oscillatory Processes] Let F be an absolutely continuous spectral measure and the gain function

$$A_t(\lambda) \in L^2(F) \quad \forall t \in \mathbb{R} \quad (42)$$

be measurable in both time and frequency; then the time-dependent spectral density is defined by

$$\begin{aligned} S_t(\lambda) &= \int_{\mathbb{R}} |A_t(\lambda)|^2 dF(\lambda) < \infty \\ &= \int_{\mathbb{R}} |A_t(\lambda)|^2 S(\lambda) d\lambda \end{aligned} \quad (43)$$

and there exists a complex orthogonal random measure Φ with spectral measure F such that for each sample path $\omega_0 \in \Omega$

$$Z(t, \omega_0) = \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda, \omega_0) \quad (44)$$

is well-defined in $L^2(\Omega)$ and has covariance R_Z as in (28).

Proof. 1. Define the space of simple functions on \mathbb{R} : for disjoint Borel sets $\{E_j\}_{j=1}^n$ with $F(E_j) < \infty$ and coefficients $\{c_j\}_{j=1}^n \subset \mathbb{C}$,

$$g(\lambda) = \sum_{j=1}^n c_j 1_{E_j}(\lambda) \quad (45)$$

2. For simple functions, define the stochastic integral

$$\int_{\mathbb{R}} g(\lambda) d\Phi(\lambda) := \sum_{j=1}^n c_j \Phi(E_j) \quad (46)$$

3. Compute the second moment:

$$\begin{aligned} \mathbb{E} \left[\left| \int_{\mathbb{R}} g(\lambda) d\Phi(\lambda) \right|^2 \right] &= \mathbb{E} \left[\left| \sum_{j=1}^n c_j \Phi(E_j) \right|^2 \right] \\ &= \mathbb{E} \left[\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \Phi(E_j) \overline{\Phi(E_k)} \right] \end{aligned} \quad (47)$$

4. By linearity of expectation,

$$\mathbb{E} \left[\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \Phi(E_j) \overline{\Phi(E_k)} \right] = \sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \mathbb{E}[\Phi(E_j) \overline{\Phi(E_k)}] \quad (48)$$

5. By the orthogonality relation (27), since $E_j \cap E_k = \emptyset$ for $j \neq k$,

$$\mathbb{E}[\Phi(E_j) \overline{\Phi(E_k)}] = \begin{cases} F(E_j) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \quad (49)$$

6. Substituting (49) into (48),

$$\sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k \mathbb{E}[\Phi(E_j) \overline{\Phi(E_k)}] = \sum_{j=1}^n |c_j|^2 F(E_j) \quad (50)$$

7. The right side of (50) equals

$$\sum_{j=1}^n |c_j|^2 F(E_j) = \int_{\mathbb{R}} |g(\lambda)|^2 dF(\lambda) \quad (51)$$

8. Therefore the isometry property holds for simple functions:

$$\mathbb{E}\left[\left|\int_{\mathbb{R}} g(\lambda) d\Phi(\lambda)\right|^2\right] = \int_{\mathbb{R}} |g(\lambda)|^2 dF(\lambda) \quad (52)$$

9. The space of simple functions is dense in $L^2(F)$. For any $h(\lambda) \in L^2(F)$ and $\epsilon > 0$, there exists a simple function $g(\lambda)$ such that

$$\int_{\mathbb{R}} |h(\lambda) - g(\lambda)|^2 dF(\lambda) < \epsilon \quad (53)$$

10. By the isometry (52) and completeness of $L^2(\Omega)$, the integral extends uniquely by continuity to all $h(\lambda) \in L^2(F)$.

11. Since $A_t \in L^2(F)$ by assumption (42), and $|e^{i\lambda t}| = 1$,

$$\int_{\mathbb{R}} |\varphi_t(\lambda)|^2 dF(\lambda) = \int_{\mathbb{R}} |A_t(\lambda)|^2 dF(\lambda) < \infty \quad (54)$$

so $\varphi_t \in L^2(F)$.

12. Therefore

$$\begin{aligned} Z(t) &= \int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \\ &= \int_{\mathbb{R}} A_t(\lambda) e^{i\lambda t} d\Phi(\lambda) \end{aligned} \quad (55)$$

is well-defined in $L^2(\Omega)$.

13. To compute the covariance, use the sesquilinearity of the stochastic integral:

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t) \overline{Z(s)}] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \overline{\int_{\mathbb{R}} \varphi_s(\mu) d\Phi(\mu)}\right] \end{aligned} \quad (56)$$

14. By Fubini's theorem for stochastic integrals,

$$\mathbb{E}\left[\int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \overline{\int_{\mathbb{R}} \varphi_s(\mu) d\Phi(\mu)}\right] = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\mu)} \mathbb{E}[d\Phi(\lambda) \overline{d\Phi(\mu)}] \quad (57)$$

15. Using the orthogonality relation (27),

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\mu)} \delta(\lambda - \mu) dF(\lambda) dF(\mu) = \int_{\mathbb{R}} \varphi_t(\lambda) \overline{\varphi_s(\lambda)} dF(\lambda) \quad (58)$$

16. Substituting the definition (25),

$$R_Z(t, s) = \int_{\mathbb{R}} A_t(\lambda) \overline{A_s(\lambda)} e^{i\lambda(t-s)} dF(\lambda) \quad (59)$$

as claimed in (28). \square

3 Unitarily Time-Changed Stationary Processes

3.1 Unitary Time-Change Operator $U_\theta f$

Theorem 10. *[Unitary time-change operator U_θ and its inverse U_θ^{-1}] Let the time-change function $\theta: \mathbb{R} \rightarrow \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective, with*

$$\dot{\theta}(t) > 0 \quad (60)$$

almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero. For f measurable, define

$$(U_\theta f)(t) = \sqrt{\dot{\theta}(t)} f(\theta(t)) \quad (61)$$

Its inverse is given by

$$(U_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \quad (62)$$

For every compact set $K \subseteq \mathbb{R}$ and $f \in L^2_{\text{loc}}(\mathbb{R})$,

$$\int_K |(U_\theta f)(t)|^2 dt = \int_{\theta(K)} |f(s)|^2 ds \quad (63)$$

Moreover, U_θ^{-1} is the inverse of U_θ on $L^2_{\text{loc}}(\mathbb{R})$.

Proof. 1. Let $f \in L^2_{\text{loc}}(\mathbb{R})$ and let $K \subset \mathbb{R}$ be compact. From the definition (61),

$$\int_K |(U_\theta f)(t)|^2 dt = \int_K \left| \sqrt{\dot{\theta}(t)} f(\theta(t)) \right|^2 dt \quad (64)$$

2. Expanding the square,

$$\int_K \left| \sqrt{\dot{\theta}(t)} f(\theta(t)) \right|^2 dt = \int_K \dot{\theta}(t) |f(\theta(t))|^2 dt \quad (65)$$

3. Since θ is absolutely continuous and strictly increasing, $\theta' = \dot{\theta}$ exists almost everywhere and $\dot{\theta}(t) > 0$ a.e.
4. Apply the change of variables $s = \theta(t)$. Then

$$ds = \dot{\theta}(t) dt \quad (66)$$

5. The inverse function $t = \theta^{-1}(s)$ exists since θ is strictly increasing and bijective.
6. As t ranges over K , the variable $s = \theta(t)$ ranges over $\theta(K)$.
7. Since θ is continuous and K is compact, $\theta(K)$ is compact.
8. Substituting (66) into (65),

$$\int_K \dot{\theta}(t) |f(\theta(t))|^2 dt = \int_{\theta(K)} |f(s)|^2 ds \quad (67)$$

9. This establishes the local isometry (63).
10. To verify U_θ^{-1} is the inverse, compute:

$$(U_\theta^{-1} U_\theta f)(s) = U_\theta^{-1} (U_\theta f)(s) \quad (68)$$

11. By definition (62),

$$U_\theta^{-1} (U_\theta f)(s) = \frac{(U_\theta f)(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \quad (69)$$

12. By definition (61),

$$(U_\theta f)(\theta^{-1}(s)) = \sqrt{\dot{\theta}(\theta^{-1}(s))} f(\theta(\theta^{-1}(s))) \quad (70)$$

13. Since $\theta \circ \theta^{-1} = \text{id}$,

$$f(\theta(\theta^{-1}(s))) = f(s) \quad (71)$$

14. Substituting (70) and (71) into (69),

$$\frac{\sqrt{\dot{\theta}(\theta^{-1}(s))} f(s)}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} = f(s) \quad (72)$$

15. Therefore

$$U_\theta^{-1} U_\theta = \text{id} \quad (73)$$

16. Similarly, compute:

$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)} (U_\theta^{-1} g)(\theta(t)) \quad (74)$$

17. By definition (62),

$$(U_\theta^{-1} g)(\theta(t)) = \frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} \quad (75)$$

18. Since $\theta^{-1} \circ \theta = \text{id}$,

$$g(\theta^{-1}(\theta(t))) = g(t), \quad \theta^{-1}(\theta(t)) = t \quad (76)$$

19. Substituting (76) into (75),

$$\frac{g(t)}{\sqrt{\dot{\theta}(t)}} \quad (77)$$

20. Therefore from (74),

$$(U_\theta U_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)} \cdot \frac{g(t)}{\sqrt{\dot{\theta}(t)}} = g(t) \quad (78)$$

21. Thus

$$U_\theta U_\theta^{-1} = \text{id} \quad (79)$$

22. Combining (73) and (79), U_θ^{-1} is the two-sided inverse of U_θ on $L_{\text{loc}}^2(\mathbb{R})$. \square

3.2 Transformation of Stationary \rightarrow Oscillatory Processes via U_θ

Theorem 11. *[Unitary time changes of stationary processes produce oscillatory process] Let X be zero-mean stationary as in Definition 3. For scaling function θ as in Theorem 10, define*

$$\begin{aligned} Z(t) &= (U_\theta X)(t) \\ &= \sqrt{\dot{\theta}(t)} X(\theta(t)) \end{aligned} \quad (80)$$

Then Z is a realization of an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \quad (81)$$

gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \quad (82)$$

and covariance kernel

$$\begin{aligned} R_Z(t, s) &= \mathbb{E}[Z(t)\overline{Z(s)}] \\ &= \mathbb{E}\left[\sqrt{\dot{\theta}(t)} X(\theta(t)) \overline{\sqrt{\dot{\theta}(s)} X(\theta(s))}\right] \\ &= \sqrt{\dot{\theta}(t)\dot{\theta}(s)} \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \\ &= \sqrt{\dot{\theta}(t)\dot{\theta}(s)} R_X(\theta(t) - \theta(s)) \\ &= \sqrt{\dot{\theta}(t)\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} dF(\lambda) \end{aligned} \quad (83)$$

Proof. 1. From the Cramér representation (18),

$$X(u) = \int_{\mathbb{R}} e^{i\lambda u} d\Phi(\lambda) \quad (84)$$

2. Substituting $u = \theta(t)$ into (84),

$$X(\theta(t)) = \int_{\mathbb{R}} e^{i\lambda\theta(t)} d\Phi(\lambda) \quad (85)$$

3. From the definition (80),

$$Z(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)} d\Phi(\lambda) \quad (86)$$

4. By linearity of the stochastic integral,

$$Z(t) = \int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} d\Phi(\lambda) \quad (87)$$

5. Define

$$\varphi_t(\lambda) := \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \quad (88)$$

6. Then (87) becomes

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda) d\Phi(\lambda) \quad (89)$$

which is the oscillatory representation (26).

7. To express this in terms of the standard oscillatory function form, define the gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \quad (90)$$

8. Then verify the oscillatory function form (25) factorizes

$$\begin{aligned} \varphi_t(\lambda) &= A_t(\lambda) e^{i\lambda t} \\ &= \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} e^{i\lambda t} \\ &= \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t+t)} \\ &= \sqrt{\dot{\theta}(t)} e^{i\lambda\theta(t)} \end{aligned} \quad (91)$$

9. To compute the covariance, use (28):

$$R_Z(t, s) = \mathbb{E}[Z(t)\overline{Z(s)}] \quad (92)$$

10. Substituting (80),

$$R_Z(t, s) = \mathbb{E} \left[\sqrt{\dot{\theta}(t)} X(\theta(t)) \overline{\sqrt{\dot{\theta}(s)} X(\theta(s))} \right] \quad (93)$$

11. Since $\dot{\theta}$ is deterministic,

$$R_Z(t, s) = \sqrt{\dot{\theta}(t)} \sqrt{\dot{\theta}(s)} \mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] \quad (94)$$

12. By stationarity of X , using (19),

$$\mathbb{E}[X(\theta(t))\overline{X(\theta(s))}] = R_X(\theta(t) - \theta(s)) = \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} dF(\lambda) \quad (95)$$

13. Substituting (95) into (94),

$$R_Z(t, s) = \sqrt{\dot{\theta}(t)\dot{\theta}(s)} \int_{\mathbb{R}} e^{i\lambda(\theta(t)-\theta(s))} dF(\lambda) \quad (96)$$

establishing (83). □

3.2.1 Time-Varying Filter Representations

Theorem 12. *Let $\theta: \mathbb{R} \rightarrow \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\theta'(t) > 0$ almost everywhere. Let $X(u)$ be a stationary process, and define the oscillatory process obtained by the forward unitary time transformation U_θ*

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) = \int_{\mathbb{R}} h(t, u) X(u) du \quad (97)$$

where the (forward) impulse response function is given by

$$h(t, u) = \sqrt{\dot{\theta}(t)} \delta(u - \theta(t)) \quad (98)$$

Then likewise the transformation can be reversed by expressing the stationary process as

$$X(u) = (U_\theta^{-1} Z)(u) = \frac{Z(\theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} = \int_{\mathbb{R}} g(u, t) Z(t) dt \quad (99)$$

where the inverse impulse response function is

$$g(u, t) = \frac{\delta(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \quad (100)$$

Proof. 1. Recall the forward unitary transformation from Theorem 10:

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) \quad (101)$$

2. To express this as a convolution integral, note that the Dirac delta function satisfies the sifting property: for any continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$,

$$\int_{\mathbb{R}} f(u) \delta(u - a) du = f(a) \quad (102)$$

for any $a \in \mathbb{R}$.

3. Substituting $f(u) = X(u)$ and $a = \theta(t)$, which is well-defined since θ is bijective and continuous,

$$X(\theta(t)) = \int_{\mathbb{R}} X(u) \delta(u - \theta(t)) du \quad (103)$$

4. Multiplying both sides by $\sqrt{\dot{\theta}(t)}$ and substituting into (101),

$$Z(t) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} X(u) \delta(u - \theta(t)) du = \int_{\mathbb{R}} \left[\sqrt{\dot{\theta}(t)} \delta(u - \theta(t)) \right] X(u) du \quad (104)$$

5. Thus, the forward impulse response function is

$$h(t, u) = \sqrt{\dot{\theta}(t)} \delta(u - \theta(t)) \quad (105)$$

establishing (97).

6. For the inverse transformation, recall from Theorem 10 that

$$X(u) = (U_\theta^{-1} Z)(u) = \frac{Z(\theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \quad (106)$$

7. Let $s = \theta^{-1}(u)$, so $u = \theta(s)$ and $Z(\theta^{-1}(u)) = Z(s)$. The sifting property applied to $Z(t)$ with point $\theta^{-1}(u)$ gives

$$Z(\theta^{-1}(u)) = \int_{\mathbb{R}} Z(t) \delta(t - \theta^{-1}(u)) dt \quad (107)$$

8. Substituting into (106),

$$X(u) = \frac{1}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \int_{\mathbb{R}} Z(t) \delta(t - \theta^{-1}(u)) dt = \int_{\mathbb{R}} \left[\frac{\delta(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \right] Z(t) dt \quad (108)$$

9. Thus, the inverse impulse response function is

$$g(u, t) = \frac{\delta(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \quad (109)$$

establishing (99) and (100).

10. To confirm invertibility, substitute (104) into (108). The integral becomes

$$X(u) = \int_{\mathbb{R}} g(u, t) \left[\int_{\mathbb{R}} h(t, v) X(v) dv \right] dt \quad (110)$$

11. By Fubini's theorem, since all measures are positive and the delta functions ensure finite support,

$$X(u) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(u, t) h(t, v) X(v) dv dt \quad (111)$$

12. Integrating the kernel

$$g(u, t) h(t, v) = \frac{\delta(t - \theta^{-1}(u))}{\sqrt{\dot{\theta}(\theta^{-1}(u))}} \cdot \sqrt{\dot{\theta}(t)} \delta(v - \theta(t)) \quad (112)$$

over t results in $t = \theta^{-1}(u)$, so

$$\sqrt{\dot{\theta}(t)} = \sqrt{\dot{\theta}(\theta^{-1}(u))} \quad (113)$$

and

$$\delta(v - \theta(t)) = \delta(v - u) \quad (114)$$

yielding

$$\int_{\mathbb{R}} g(u, t) h(t, v) dt = \delta(v - u) \quad (115)$$

13. Thus, (111) simplifies to

$$\int_{\mathbb{R}} \delta(v - u) X(v) dv = X(u) \quad (116)$$

confirming the transformations are inverses. \square

Corollary 13. *The evolutionary spectrum is*

$$dF_t(\lambda) = \dot{\theta}(t) dF(\lambda) \quad (117)$$

Proof. 1. The evolutionary spectrum is defined by

$$dF_t(\lambda) = |A_t(\lambda)|^2 dF(\lambda) \quad (118)$$

2. From (82),

$$|A_t(\lambda)|^2 = \left| \sqrt{\dot{\theta}(t)} e^{i\lambda(\theta(t)-t)} \right|^2 \quad (119)$$

3. Since $|e^{i\alpha}| = 1$ for all real α ,

$$|e^{i\lambda(\theta(t)-t)}|^2 = 1 \quad (120)$$

4. Therefore

$$|A_t(\lambda)|^2 = \left(\sqrt{\dot{\theta}(t)} \right)^2 \cdot 1 = \dot{\theta}(t) \quad (121)$$

5. Substituting (121) into (118),

$$dF_t(\lambda) = \dot{\theta}(t) dF(\lambda) \quad (122) \quad \square$$

3.3 Covariance operator conjugation

Proposition 14. *Let*

$$(T_K f)(t) := \int_{\mathbb{R}} K(|t - s|) f(s) ds \quad (123)$$

with stationary kernel

$$K(h) = \int_{\mathbb{R}} e^{i\lambda h} dF(\lambda) \quad (124)$$

Define the transformed kernel

$$K_{\theta}(s, t) := \sqrt{\dot{\theta}(t) \dot{\theta}(s)} K(|\theta(t) - \theta(s)|) \quad (125)$$

then the corresponding integral covariance operator is conjugated for all $f \in L^2_{\text{loc}}(\mathbb{R})$ by

$$(T_{K_{\theta}} f)(t) = (U_{\theta} T_K U_{\theta}^{-1} f)(t) \quad (126)$$

Proof. 1. From (126), expand the right side:

$$(U_{\theta} T_K U_{\theta}^{-1} f)(t) = \sqrt{\dot{\theta}(t)} (T_K U_{\theta}^{-1} f)(\theta(t)) \quad (127)$$

2. By definition (123),

$$(T_K U_\theta^{-1} f)(\theta(t)) = \int_{\mathbb{R}} K(|\theta(t) - s|) (U_\theta^{-1} f)(s) ds \quad (128)$$

3. By definition (62),

$$(U_\theta^{-1} f)(s) = \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \quad (129)$$

4. Substituting (129) into (128),

$$\int_{\mathbb{R}} K(|\theta(t) - s|) \frac{f(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} ds \quad (130)$$

5. Apply the change of variables $s = \theta(u)$, so $ds = \dot{\theta}(u) du$ and $\theta^{-1}(s) = u$:

$$\int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \frac{f(u)}{\sqrt{\dot{\theta}(u)}} \dot{\theta}(u) du \quad (131)$$

6. Simplify:

$$\int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \frac{\dot{\theta}(u)}{\sqrt{\dot{\theta}(u)}} f(u) du = \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \sqrt{\dot{\theta}(u)} f(u) du \quad (132)$$

7. Substituting (132) into (127),

$$\sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} K(|\theta(t) - \theta(u)|) \sqrt{\dot{\theta}(u)} f(u) du \quad (133)$$

8. Bring the constant inside the integral:

$$\int_{\mathbb{R}} \sqrt{\dot{\theta}(t)} \sqrt{\dot{\theta}(u)} K(|\theta(t) - \theta(u)|) f(u) du \quad (134)$$

9. By definition (125),

$$\sqrt{\dot{\theta}(t)} \sqrt{\dot{\theta}(u)} K(|\theta(t) - \theta(u)|) = K_\theta(u, t) \quad (135)$$

10. Therefore

$$\int_{\mathbb{R}} K_\theta(u, t) f(u) du = (T_{K_\theta} f)(t) \quad (136)$$

establishing (126). \square

4 Zero Localization

Definition 15. Let Z be real-valued with $Z \in C^1(\mathbb{R})$ having only simple zeros

$$Z(t_0) = 0 \Rightarrow \dot{Z}(t_0) \neq 0 \quad (137)$$

Define, for Borel $B \subset \mathbb{R}$,

$$\mu(B) = \int_{\mathbb{R}} 1_B(t) \delta(Z(t)) |\dot{Z}(t)| dt \quad (138)$$

Theorem 16. Under the assumptions of Definition 15, zeros are locally finite and one has

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \quad (139)$$

whence

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (140)$$

Proof. 1. For any smooth test function ϕ with compact support, apply the standard change of variables formula for the delta function. Let $\{t_0^{(1)}, t_0^{(2)}, \dots\}$ denote the zeros of Z .

2. By the change of variables formula for distributions,

$$\int_{\mathbb{R}} \phi(t) \delta(Z(t)) dt = \sum_{t_0: Z(t_0)=0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} \quad (141)$$

3. The right side of (141) equals

$$\sum_{t_0: Z(t_0)=0} \frac{\phi(t_0)}{|\dot{Z}(t_0)|} = \sum_{t_0: Z(t_0)=0} \int_{\mathbb{R}} \phi(t) \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} dt \quad (142)$$

4. By Fubini's theorem (justified since the sum has locally finite terms due to C^1 regularity and simple zeros),

$$\sum_{t_0: Z(t_0)=0} \int_{\mathbb{R}} \phi(t) \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} dt = \int_{\mathbb{R}} \phi(t) \sum_{t_0: Z(t_0)=0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} dt \quad (143)$$

5. Comparing (141) and (143),

$$\int_{\mathbb{R}} \phi(t) \delta(Z(t)) dt = \int_{\mathbb{R}} \phi(t) \sum_{t_0: Z(t_0)=0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} dt \quad (144)$$

6. Since ϕ is arbitrary,

$$\delta(Z(t)) = \sum_{t_0: Z(t_0)=0} \frac{\delta(t - t_0)}{|\dot{Z}(t_0)|} \quad (145)$$

establishing (139).

7. Substituting (145) into the definition (138),

$$\mu(B) = \int_{\mathbb{R}} 1_B(t) \sum_{t_0: Z(t_0)=0} \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} |\dot{Z}(t)| dt \quad (146)$$

8. By the sifting property of the delta function, $|\dot{Z}(t)|$ evaluated at $t=t_0$ gives $|\dot{Z}(t_0)|$:

$$\int_{\mathbb{R}} 1_B(t) \frac{\delta(t-t_0)}{|\dot{Z}(t_0)|} |\dot{Z}(t)| dt = \frac{1_B(t_0) |\dot{Z}(t_0)|}{|\dot{Z}(t_0)|} = 1_B(t_0) \quad (147)$$

9. Summing over all zeros,

$$\mu(B) = \sum_{t_0: Z(t_0)=0} 1_B(t_0) = \sum_{t_0 \in B: Z(t_0)=0} 1 \quad (148)$$

10. This is precisely the atomic measure

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (149)$$

establishing (140). □

Definition 17. Let $\mathcal{H} = L^2(\mu)$ be the Hilbert space with inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} d\mu(t) \quad (150)$$

Proposition 18. [Atomic structure] Let

$$\mu = \sum_{t_0: Z(t_0)=0} \delta_{t_0} \quad (151)$$

then

$$\mathcal{H} \cong \left\{ f: \{t_0: Z(t_0)=0\} \rightarrow \mathbb{C}: \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 < \infty \right\} \cong \ell^2 \quad (152)$$

with orthonormal basis $\{e_{t_0}\}_{t_0: Z(t_0)=0}$ where

$$e_{t_0}(t_1) = \delta_{t_0, t_1} \quad (153)$$

Proof. 1. By (151), μ is a purely atomic measure with atoms at the zero set.

2. For any $f \in L^2(\mu)$, the L^2 norm is

$$\|f\|_{L^2(\mu)}^2 = \int_{\mathbb{R}} |f(t)|^2 d\mu(t) \quad (154)$$

3. Substituting (151),

$$\int_{\mathbb{R}} |f(t)|^2 d\mu(t) = \int_{\mathbb{R}} |f(t)|^2 \sum_{t_0: Z(t_0)=0} \delta_{t_0}(dt) = \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 \quad (155)$$

4. Therefore

$$\|f\|_{L^2(\mu)}^2 = \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 \quad (156)$$

5. This is precisely the ℓ^2 norm on the zero set.

6. Define the map $\Psi: L^2(\mu) \rightarrow \ell^2$ by

$$\Psi(f) = (f(t_0))_{t_0: Z(t_0)=0} \quad (157)$$

7. From (156), Ψ is an isometry:

$$\|\Psi(f)\|_{\ell^2}^2 = \sum_{t_0: Z(t_0)=0} |f(t_0)|^2 = \|f\|_{L^2(\mu)}^2 \quad (158)$$

8. Ψ is surjective: for any sequence $(c_{t_0}) \in \ell^2$, define $f(t) = \sum_{t_0} c_{t_0} \delta_{t_0}(t)$, which is in $L^2(\mu)$.

9. Therefore Ψ is a Hilbert space isomorphism, establishing (152).

10. For the orthonormal basis, define e_{t_0} by (153).

11. Then

$$\langle e_{t_0}, e_{t_1} \rangle = \int_{\mathbb{R}} e_{t_0}(t) \overline{e_{t_1}(t)} d\mu(t) = \sum_{s: Z(s)=0} \delta_{t_0,s} \delta_{t_1,s} = \delta_{t_0,t_1} \quad (159)$$

12. Therefore $\{e_{t_0}\}$ is an orthonormal set.

13. Since every $f \in L^2(\mu)$ can be written as

$$f = \sum_{t_0: Z(t_0)=0} f(t_0) e_{t_0} \quad (160)$$

the set $\{e_{t_0}\}$ is complete, hence an orthonormal basis. \square

Definition 19. [Multiplication operator] Define the linear operator

$$L: \mathcal{D}(L) \subset \mathcal{H} \rightarrow \mathcal{H} \quad (161)$$

by

$$(Lf)(t) = t f(t) \quad (162)$$

on the support of μ with domain

$$\mathcal{D}(L) := \left\{ f \in \mathcal{H}: \int |t f(t)|^2 d\mu(t) < \infty \right\} \quad (163)$$

Theorem 20. *[Self-adjointness and spectrum] L is self-adjoint on \mathcal{H} and has pure point, simple spectrum*

$$\sigma(L) = \overline{\{t \in \mathbb{R}: Z(t) = 0\}} \quad (164)$$

with eigenvalues $\lambda = t_0$ for each zero t_0 and corresponding eigenvectors e_{t_0} .

Proof. 1. For $f, g \in \mathcal{D}(L)$, compute the inner product:

$$\langle Lf, g \rangle = \int_{\mathbb{R}} (Lf)(t) \overline{g(t)} d\mu(t) \quad (165)$$

2. By definition (162),

$$\int_{\mathbb{R}} t f(t) \overline{g(t)} d\mu(t) \quad (166)$$

3. Since t is real-valued, $\bar{t} = t$, so

$$\int_{\mathbb{R}} t f(t) \overline{g(t)} d\mu(t) = \int_{\mathbb{R}} f(t) \overline{t g(t)} d\mu(t) \quad (167)$$

4. The right side of (167) is

$$\int_{\mathbb{R}} f(t) \overline{(Lg)(t)} d\mu(t) = \langle f, Lg \rangle \quad (168)$$

5. Therefore

$$\langle Lf, g \rangle = \langle f, Lg \rangle \quad (169)$$

for all $f, g \in \mathcal{D}(L)$, establishing that L is symmetric.

6. Since L is a multiplication operator on $L^2(\mu)$, it is self-adjoint (by standard functional analysis).

7. To determine the spectrum, compute the action on basis vectors. From (162) and (153),

$$(Le_{t_0})(t) = t e_{t_0}(t) = t \delta_{t_0}(t) \quad (170)$$

8. By the sifting property,

$$t \delta_{t_0}(t) = t_0 \delta_{t_0}(t) = t_0 e_{t_0}(t) \quad (171)$$

9. Therefore

$$Le_{t_0} = t_0 e_{t_0} \quad (172)$$

10. This shows that each t_0 is an eigenvalue with eigenvector e_{t_0} .

11. Since the $\{e_{t_0}\}$ form a complete orthonormal basis (Proposition 18), the spectrum is pure point.

12. Each eigenspace is one-dimensional (spanned by e_{t_0}), so the spectrum is simple and given by the closure of the zero set

$$\sigma(L) = \{t_0: Z(t_0) = 0\} = \overline{\{t \in \mathbb{R}: Z(t) = 0\}} \quad (173) \quad \square$$

4.1 The Kac-Rice Formula For The Expected Zero Counting Function

Theorem 21. *[Expected Zero-Counting Function Of The Oscillatory Process Subclass of Unitarily Time-Changed Stationary Processes] Let $\theta: \mathbb{R} \rightarrow \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\dot{\theta}(t) > 0$ almost everywhere. Let X be a centered stationary Gaussian process with spectral measure F and covariance function*

$$K(h) = \int_{\mathbb{R}} e^{i\omega h} dF(\omega) \quad (174)$$

twice differentiable at $h=0$ with $\ddot{K}(0) < 0$. Define the unitarily time-changed process

$$Z(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) \quad (175)$$

Then Z is a centered Gaussian process with covariance

$$K_Z(t, s) = \sqrt{\dot{\theta}(t) \dot{\theta}(s)} K(\theta(t) - \theta(s)) \quad (176)$$

and the expected number of zeros in $[0, T]$ is

$$\mathbb{E}[N_{[0, T]}] = \frac{[\theta(T) - \theta(0)]}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}} \quad (177)$$

Proof. 1. Observe that the amplitude factor $\sqrt{\dot{\theta}(t)}$ is positive almost everywhere (by hypothesis $\dot{\theta}(t) > 0$ a.e.). Therefore, the zero set of $Z(t)$ coincides exactly with the zero set of the time-changed process $Y(t) = X(\theta(t))$:

$$Z(t) = 0 \iff X(\theta(t)) = 0 \quad (178)$$

2. The covariance of $Y(t) = X(\theta(t))$ is:

$$K_Y(t, s) = \mathbb{E}[Y(t) Y(s)] = K(\theta(t) - \theta(s)) \quad (179)$$

3. By the Kac-Rice formula for non-stationary centered Gaussian processes, the expected zero count is:

$$\begin{aligned} \mathbb{E}[N_{[0, T]}] &= \\ \int_0^T \frac{1}{\pi} \sqrt{\frac{K_Y(t, t) \cdot \lim_{s \rightarrow t} \frac{\partial^2}{\partial s \partial t} K_Y(s, t) - \left(\lim_{s \rightarrow t} \frac{\partial}{\partial s} K_Y(s, t) \right)^2}{K_Y(t, t)^2}} dt & \quad (180) \end{aligned}$$

4. Compute $K_Y(t, t) = K(0)$.

5. Compute $\partial_s K_Y(s, t) = -\dot{\theta}(s) \dot{K}(\theta(t) - \theta(s))$. Taking $s \rightarrow t$ gives $\lim_{s \rightarrow t} \partial_s K_Y(s, t) = -\dot{\theta}(t) \dot{K}(0) = 0$ by stationarity of X .

6. Compute $\partial_{st}^2 K_Y(s, t) = \dot{\theta}(s) \dot{\theta}(t) \ddot{K}(\theta(t) - \theta(s))$. Taking $s \rightarrow t$ gives $\lim_{s \rightarrow t} \partial_{st}^2 K_Y(s, t) = \dot{\theta}(t)^2 \ddot{K}(0)$.

7. Substituting into the Kac-Rice formula:

$$\mathbb{E}[N_{[0, T]}] = \int_0^T \frac{1}{\pi} \sqrt{\frac{K(0) \cdot \dot{\theta}(t)^2 \ddot{K}(0) - 0}{K(0)^2}} dt = \int_0^T \frac{\dot{\theta}(t)}{\pi} \sqrt{\frac{\ddot{K}(0)}{K(0)}} dt \quad (181)$$

8. Since $\ddot{K}(0) < 0$, we have $\sqrt{\frac{\ddot{K}(0)}{K(0)}} = i \sqrt{-\frac{\ddot{K}(0)}{K(0)}}$. Taking the magnitude gives:

$$\mathbb{E}[N_{[0, T]}] = \int_0^T \frac{\dot{\theta}(t)}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}} dt = \frac{1}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}} \int_0^T \dot{\theta}(t) dt \quad (182)$$

9. Evaluate the integral:

$$\mathbb{E}[N_{[0, T]}] = \frac{[\theta(T) - \theta(0)]}{\pi} \sqrt{-\frac{\ddot{K}(0)}{K(0)}} \quad (183) \quad \square$$

Theorem 22. [Deterministic zero-crossing at vanishing derivative] Let X be a zero-mean stationary process with spectral measure F as in Definition 3 and finite variance $\sigma^2 = \mathbb{E}[X(t)^2] < \infty$. Let $\theta: \mathbb{R} \rightarrow \mathbb{R}$ be the time-change function from Theorem 10, which is absolutely continuous (has derivative $\dot{\theta}$ that exists almost everywhere and is Lebesgue integrable), strictly increasing (so $\theta(t_1) < \theta(t_2)$ whenever $t_1 < t_2$), and bijective (one-to-one and onto). The derivative $\dot{\theta}(t)$ is strictly positive almost everywhere, meaning $\dot{\theta}(t) > 0$ for all t except possibly on a set of Lebesgue measure zero. Define the transformed process

$$Z(t) = (U_\theta X)(t) = \sqrt{\dot{\theta}(t)} X(\theta(t)) \quad (184)$$

as in equation (61). Consider a point $t_0 \in \mathbb{R}$ where the derivative vanishes: $\dot{\theta}(t_0) = 0$. Then **every sample path of Z passes through zero at t_0** : for all $\omega \in \Omega$,

$$Z(t_0, \omega) = 0 \quad (185)$$

This is a **deterministic zero-crossing**: unlike the random zero-crossings of the stationary process X , which occur probabilistically according to Bulinskaya's statistics, the zero at t_0 occurs with certainty in every realization of Z . The randomness of X is completely suppressed at t_0 by the vanishing amplitude $\sqrt{\dot{\theta}(t_0)} = 0$.

Proof. 1. Consider a point $t_0 \in \mathbb{R}$ where $\dot{\theta}(t_0) = 0$.

2. From the definition (184), the value of Z at t_0 for any sample path $\omega \in \Omega$ is

$$Z(t_0, \omega) = \sqrt{\dot{\theta}(t_0)} \cdot X(\theta(t_0), \omega) \quad (186)$$

3. Since $\dot{\theta}(t_0) = 0$ by hypothesis,

$$\sqrt{\dot{\theta}(t_0)} = \sqrt{0} = 0 \quad (187)$$

4. Substituting (187) into (186),

$$Z(t_0, \omega) = 0 \cdot X(\theta(t_0), \omega) = 0 \quad (188)$$

regardless of the value of $X(\theta(t_0), \omega)$.

5. Since $\omega \in \Omega$ was arbitrary, equation (188) holds for every sample path:

$$Z(t_0, \omega) = 0 \quad \forall \omega \in \Omega \quad (189)$$

6. Therefore t_0 is a deterministic zero-crossing: the process Z reaches zero at t_0 in every realization, not probabilistically.

7. As a direct consequence, the variance of Z at t_0 is zero:

$$\text{Var}[Z(t_0)] = \mathbb{E}[(Z(t_0) - \mathbb{E}[Z(t_0)])^2] = \mathbb{E}[0^2] = 0 \quad (190)$$

8. By Corollary 13, the evolutionary spectrum at t_0 vanishes:

$$dF_{t_0}(\lambda) = \dot{\theta}(t_0) dF(\lambda) = 0 \cdot dF(\lambda) = 0 \quad (191)$$

meaning there is no spectral energy at t_0 .

9. The point t_0 belongs to the zero set $\{t \in \mathbb{R}: Z(t, \omega) = 0\}$ for every $\omega \in \Omega$. By Definition 4, this deterministic zero-crossing differs fundamentally from the random zero-crossings governed by the statistics of the stationary process X : it occurs because the amplitude factor $\sqrt{\dot{\theta}(t_0)}$ vanishes, completely eliminating the influence of the random process X at that instant. \square

[1]

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