Green's function for the Laplace operator on the unit disk

To compute the exact Green's function for the Laplace operator on the unit disk, we consider the boundary value problem for Laplace's equation where the boundary of the disk is held at zero potential, and there is a point source at a location within the disk.

Problem Setup: The Green's function $G(z, w)$ on a unit disk centered at the origin in the complex plane satisfies:

$$\Delta G(z, w) = \delta(z - w) \tag{1}$$

where δ is the Dirac delta function, z and w are points in the disk, and Δ is the Laplace operator.

Solution in Polar Coordinates: Since the domain is a unit disk, it's convenient to use polar coordinates. Let $z = r\,e^{i\theta}$ and $w = \rho\,e^{i\phi}$. The solution exploits the symmetry and the properties of the Laplace equation in polar coordinates.

Green's Function Formula: The Green's function in a unit disk with Dirichlet boundary conditions (potential zero at the boundary) is given by:

$$G(z, w) = -\frac{1}{2\pi} \log\left|\frac{z - w}{1 - \bar{w}z}\right| \tag{2}$$

This formula uses the logarithmic potential and a correction term that ensures the Green's function vanishes at the boundary of the disk. The term $\frac{z - w}{1 - \bar{w}z}$ accounts for the mirror charge placed outside the disk to enforce the boundary condition.

Breakdown:

- **Logarithmic Potential**: The $-\frac{1}{2\pi} \log|z - w|$ term represents the fundamental solution to Laplace's equation in two dimensions, corresponding to the potential due to a point source.

- **Mirror Charge Adjustment**: The $\frac{1}{1 - \bar{w}z}$ factor adjusts for the presence of the disk's boundary by introducing a mirror charge effect, ensuring the potential is zero on the boundary $|z| = 1$.

This Green's function is key in potential theory and electrostatics for modeling fields within circular regions and finds applications in various areas of physics and engineering where boundary value problems in circular geometries are prevalent.

To connect the Green's function for the Laplace operator on the unit disk to the Legendre polynomials, we need to consider the series representation of the Green's function in terms of these polynomials. This relationship becomes evident when examining the azimuthal symmetry and the separation of variables technique used in polar coordinates.

Expanded Form of Green's Function Using Legendre Polynomials:

In polar coordinates, $z = r\,e^{i\theta}$ and $w = \rho\,e^{i\phi}$, the complex variables z and w are represented in terms of their magnitudes r and ρ and their angles θ and ϕ. The distance term in the Green's function can be expanded using the addition theorem for Legendre polynomials:

$$\log\left|\frac{z-w}{1-\bar{w}z}\right| = \log\left|\frac{r\,e^{i\theta}-\rho\,e^{i\phi}}{1-\rho\,r\,e^{-i(\theta-\phi)}}\right| \tag{3}$$

which can be written as

$$\log|r\,e^{i\theta}-\rho\,e^{i\phi}| - \log|1-\rho\,r\,e^{-i(\theta-\phi)}| \tag{4}$$

Using the Legendre polynomial expansion for $\log|1-x|$ and the addition theorem, the terms can be expanded to:

$$\log\sqrt{r^2+\rho^2-2\,r\,\rho\cos(\theta-p\,h\,i)} = \log\sqrt{(r-\rho)^2} + \sum_{n=1}^{\infty}\frac{r^n\,\rho^n}{n}P_n(\cos(\theta-\phi)) \tag{5}$$

where P_n are Legendre polynomials.

Incorporating Orthogonality and Completeness:

The Legendre polynomials $P_n(\cos(\theta-\phi))$ are particularly useful because of their orthogonality and completeness, which allow the representation of the kernel of the Green's function as an infinite series:

$$G(z,w) = -\frac{1}{2\,\pi}\left(\log\sqrt{(r-\rho)^2} + \sum_{n=1}^{\infty}\frac{r^n\,\rho^n}{n}P_n(\cos(\theta-\phi))\right) \tag{6}$$

This series essentially decomposes the influence of the point source at w into a series of angular harmonics around the point z, with each term in the series corresponding to a Legendre polynomial. Each P_n term in the series represents a mode of the potential influenced by the boundary conditions and the geometry of the problem