Osterwalder-Schrader Theory

1 Introduction

The Osterwalder-Schrader (OS) theory provides a rigorous framework for constructing quantum field theories by transforming problems from Minkowski space to Euclidean space. This transformation is facilitated by the Wick rotation, and the theory is defined by a set of axioms (OS axioms) that ensure the reconstruction of a corresponding quantum field theory in Minkowski space.

2 Wick Rotation

The Wick rotation transforms Minkowski space-time with metric signature $(+,-,-,-)$ into Euclidean space with metric $(+,+,+,+)$ by replacing the real-time coordinate t with an imaginary time coordinate τ:

$$t \to -i\,\tau \tag{1}$$

Under this transformation, the Minkowski metric:

$$d\,s^2 = d\,t^2 - d\,x^2 - d\,y^2 - d\,z^2 \tag{2}$$

becomes the Euclidean metric:

$$d\,s_E^2 = d\,\tau^2 + d\,x^2 + d\,y^2 + d\,z^2 \tag{3}$$

This simplification converts the relativistic Klein-Gordon equation in Minkowski space:

$$(\partial_t^2 - \nabla^2 + m^2)\,\phi(t, \mathbf{x}) = 0 \tag{4}$$

to the Euclidean Klein-Gordon equation:

$$(\partial_\tau^2 + \nabla^2 - m^2)\,\phi(\tau, \mathbf{x}) = 0 \tag{5}$$

3 OS Axioms

The OS axioms are conditions on the Euclidean Green's functions $G_n(x_1, \ldots, x_n)$, where x_i are points in Euclidean space.

3.1 Euclidean Invariance

The Green's functions must be invariant under the Euclidean group (translations and rotations):

$$G_n(x_1, \ldots, x_n) = G_n(x_1 + a, \ldots, x_n + a) \tag{6}$$

$$G_n(R\,x_1, \ldots, R\,x_n) = G_n(x_1, \ldots, x_n) \tag{7}$$

where a is a translation vector and R is a rotation matrix.

3.2 Reflection Positivity

For any test function $f(x_1, \ldots, x_n)$ with support in $\tau \geq 0$:

$$\int d^d x_1 \ldots d^d x_n\, \overline{f(x_1, \ldots, x_n)} G_n(\theta\, x_1, \ldots, \theta\, x_n)\, f(x_1, \ldots, x_n) \geq 0 \tag{8}$$

where θ is the reflection $\theta\,(\tau, \mathbf{x}) = (-\tau, \mathbf{x})$.

3.3 Regularity Conditions

The Green's functions must be distributions with respect to their arguments and have certain analyticity properties.

3.4 Growth and Smoothness Conditions

The Green's functions should not grow too rapidly at infinity and should be sufficiently smooth. Specifically, they must satisfy:

$$|G_n(x_1, \ldots, x_n)| \leq C\, e^{-\mu \sum_{i<j} |x_i - x_j|} \tag{9}$$

for some constants C and μ.

4 Reconstruction Theorem

Given a set of Euclidean Green's functions G_n satisfying the OS axioms, one can construct the corresponding Wightman functions W_n in Minkowski space.

4.1 Inverse Wick Rotation

Perform the inverse Wick rotation:

$$\tau \rightarrow i\,t \tag{10}$$

to obtain the Wightman functions $W_n(t_1, \mathbf{x}_1, \ldots, t_n, \mathbf{x}_n)$ from the Euclidean Green's functions $G_n(\tau_1, \mathbf{x}_1, \ldots, \tau_n, \mathbf{x}_n)$.

4.2 Construction of the Hilbert Space

Define the Hilbert space using the Wightman functions. The inner product is given by:

$$\langle \Phi | \Psi \rangle = \int d^d x_1 \ldots d^d x_n \, \overline{\Phi(x_1, \ldots, x_n)} W_n(x_1, \ldots, x_n) \, \Psi(x_1, \ldots, x_n) \tag{11}$$

4.3 Quantum Field Operators

Construct the quantum field operators $\phi(x)$ such that:

$$\langle 0 | \phi(x_1) \, \phi(x_2) \ldots \phi(x_n) | 0 \rangle = W_n(x_1, \ldots, x_n) \tag{12}$$

These operators act on the Hilbert space constructed from the Wightman functions and satisfy the Wightman axioms, ensuring that the resulting quantum field theory is well-defined.

5 Summary

The Osterwalder-Schrader theory bridges Euclidean and Minkowski quantum field theories using the Wick rotation and a set of axioms that ensure the physical consistency of the reconstructed theory. The key mathematical tools involve Euclidean Green's functions, reflection positivity, and the reconstruction theorem, enabling the formulation of rigorous quantum field theories.