L2 Norm Preservation Under Monotonic Substitutions

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Theorem 1. *[L2 Norm Preservation via Square Root Jacobian Factor] Let $g: I \to J$ be a strictly monotonic and differentiable function between intervals I, $J \subseteq \mathbb{R}$ (possibly unbounded), with $g'(y) \neq 0$ for all $y \in I$. For any $f \in L^2(J, dx)$, define the transformed function $\tilde{f}: I \to \mathbb{C}$ by*

$$\tilde{f}(y) = f(g(y))\sqrt{|g'(y)|}.$$

Then $\tilde{f} \in L^2(I, dy)$ and

$$\|\tilde{f}\|_{L^2(I,dy)} = \|f\|_{L^2(J,dx)}.$$

Proof. Without loss of generality, assume g is strictly increasing (the decreasing case follows by considering $-g$).

First, establish the change of variables formula. For any measurable set $E \subseteq J$:

$$\int_E |f(x)|^2 dx = \int_{g^{-1}(E)} |f(g(y))|^2 |g'(y)| \, dy.$$

This follows from the standard change of variables theorem, since g is strictly monotonic and differentiable with $g'(y) \neq 0$.

To handle potentially unbounded intervals, consider the norm computation:

$$\begin{aligned} \|\tilde{f}\|_{L^2(I,dy)}^2 &= \int_I |\tilde{f}(y)|^2 \, dy && (1)\\ &= \int_I |f(g(y))\sqrt{|g'(y)|}|^2 \, dy && (2)\\ &= \int_I |f(g(y))|^2 |g'(y)| \, dy. && (3) \end{aligned}$$

By the change of variables formula applied to $J = g(I)$:

$$\int_I |f(g(y))|^2 |g'(y)| \, dy = \int_J |f(x)|^2 \, dx = \|f\|_{L^2(J,dx)}^2.$$

For unbounded intervals, this equality holds by the monotone convergence theorem: approximate I by an increasing sequence of bounded intervals $I_n \uparrow I$, apply the result to each I_n, and take the limit.

Therefore:

$$\|\tilde{f}\|_{L^2(I,dy)} = \|f\|_{L^2(J,dx)}.$$

The integrability of $\tilde{f}$ follows immediately from the norm equality and the assumption that $f \in L^2(J, dx)$. $\square$

Lemma 2. *[Density of Transformed Functions] Under the conditions of Theorem 1, the set $\{f(g(\cdot)): f \in L^2(J, dx)\}$ is dense in $L^2(I, |g'(y)|\, dy)$, where $L^2(I, |g'(y)|\, dy)$ denotes the space of square-integrable functions with respect to the measure $|g'(y)|\, dy$.*

Proof. The map $f \mapsto f \circ g$ is an isometric isomorphism from $L^2(J, dx)$ to $L^2(I, |g'(y)|\, dy)$ by the change of variables formula. Since $L^2(J, dx)$ is complete, its image under an isometry is also complete, hence dense in itself. $\square$

Theorem 3. *[Necessity of Square Root Factor] Under the same conditions as Theorem 1, the factor $\sqrt{|g'(y)|}$ is necessary for L2 norm preservation. That is, if $h(y) = f(g(y)) \cdot \phi(y)$ for some measurable function $\phi: I \to \mathbb{R}^+$ satisfies $\|h\|_{L^2(I,dy)} = \|f\|_{L^2(J,dx)}$ for all $f \in L^2(J, dx)$, then $\phi(y) = \sqrt{|g'(y)|}$ almost everywhere.*

Proof. Suppose $\|f(g(\cdot)) \cdot \phi(\cdot)\|_{L^2(I,dy)} = \|f\|_{L^2(J,dx)}$ for all $f \in L^2(J, dx)$.

Then for any $f \in L^2(J, dx)$:

$$\int_I |f(g(y))|^2 |\phi(y)|^2\, dy = \|f\|^2_{L^2(J,dx)} = \int_I |f(g(y))|^2 |g'(y)|\, dy,$$

where the second equality follows from the change of variables formula.

Therefore:

$$\int_I |f(g(y))|^2 (|\phi(y)|^2 - |g'(y)|)\, dy = 0$$

for all $f \in L^2(J, dx)$.

By Lemma 1, functions of the form $f(g(y))$ are dense in $L^2(I, |g'(y)|\, dy)$. For any $u \in L^2(I, |g'(y)|\, dy)$, there exists a sequence $f_n \in L^2(J, dx)$ such that $f_n(g(y)) \to u(y)$ in $L^2(I, |g'(y)|\, dy)$.

Since $|\phi(y)|^2 - |g'(y)|$ is integrable with respect to $|g'(y)|\ dy$ (by the boundedness of the norm-preserving property), we have:

$$\int_I |u(y)|^2 \left(|\phi(y)|^2 - |g'(y)|\right)\ dy = 0$$

for all $u \in L^2(I, |g'(y)|\ dy)$.

In particular, taking $u(y) = \operatorname{sgn}(|\phi(y)|^2 - |g'(y)|) \cdot 1_{\{|\phi(y)|^2 \neq |g'(y)|\}}(y)$, we obtain:

$$\int_I || \phi(y) |^2 - |g'(y)|| \, | g'(y)|\ dy = 0.$$

Since $|g'(y)| > 0$ almost everywhere, this implies $|\phi(y)|^2 = |g'(y)|$ almost everywhere.

Taking $\phi(y) > 0$, we conclude $\phi(y) = \sqrt{|g'(y)|}$ almost everywhere. ☐

Theorem 4. *[Extension to General Measures] Let μ and ν be σ-finite measures on I and J respectively, and let $g: I \to J$ be a measurable bijection. If $\nu = \mu \circ g^{-1}$ (i.e., $\nu(E) = \mu(g^{-1}(E))$ for all measurable $E \subseteq J$), then for $f \in L^2(J, d\nu)$:*

$$\tilde{f}(y) = f(g(y)) \sqrt{\frac{d(\mu \circ g^{-1})}{d\mu}(y)}$$

satisfies $\|\tilde{f}\|_{L^2(I, d\mu)} = \|f\|_{L^2(J, d\nu)}$, where $\frac{d(\mu \circ g^{-1})}{d\mu}$ is the Radon-Nikodym derivative.

Proof. When μ and ν are both Lebesgue measure and g is differentiable, the Radon-Nikodym derivative is $|g'(y)|$, reducing to Theorem 1. The general case follows by the same change of variables argument using the definition of the push-forward measure. ☐