Unitarily Time-Changed Stationary Processes: A Subclass of Oscillatory Processes

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December 13, 2025

Abstract

A unitary time-change operator U_θ is constructed for absolutely continuous, strictly increasing time reparametrizations θ, acting on functions that are locally square-integrable. Applying U_θ to the Cramér spectral representation of a stationary process $X(t)$ produces the transformed process

$$Z(t) = U_\theta X(t) = \sqrt{\dot\theta(t)}\, X(\theta(t)) = \sqrt{\dot\theta(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\Phi(\lambda),$$

which is an oscillatory process in the sense of Priestley with oscillatory function $\varphi_t(\lambda) = \sqrt{\dot\theta(t)}\, e^{i\lambda\theta(t)}$, evolutionary power spectral density $S_t(\lambda) = \dot\theta(t) S(\lambda)$, and covariance kernel $K_Z(t,s) = \sqrt{\dot\theta(t)\dot\theta(s)}\, K_X(\theta(t),\theta(s))$ where K_X is the stationary covariance of $X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, d\Phi(\lambda)$. Unitarily time-changed stationary processes form a subclass of oscillatory processes. The Kac–Rice formula gives the expected zero-counting function for stationary processes. By Bulinskaya's theorem, when the covariance is twice continuously differentiable with $R''(0) < 0$, almost all zeros are simple (for stationary processes). A zero-localization measure $d\mu(t) = \delta(Z(t))|\dot Z(t)|\, dt$ induces a Hilbert space $L^2(\mu)$ on the zero set of each oscillatory process realization $Z(t)$.

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1 Gaussian Processes

1.1 Definition

Definition 1.1. (Gaussian process) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T a nonempty index set. A family $\{X_t : t \in T\}$ of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Gaussian process if for every finite subset $\{t_1, \dots, t_n\} \subset T$ the random vector $(X_{t_1}, \dots, X_{t_n})$ is multivariate normal (possibly degenerate). Equivalently, every finite linear combination $\sum_{i=1}^n a_i X_{t_i}$ is either almost surely constant or Gaussian. The mean function is $m(t) := \mathbb{E}[X_t]$ and the covariance kernel is

$$K(s,t) = \mathrm{Cov}(X_s, X_t). \tag{1}$$

For any finite $(t_i)_{i=1}^n \subset T$, the matrix $K_{ij} = K(t_i, t_j)$ is symmetric positive semidefinite, and a Gaussian process is completely determined in law by m and K.

1.2 Stationary processes

Definition 1.2. [Cramér spectral representation] A zero-mean stationary process X with spectral measure F admits the sample path representation

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, d\Phi(\lambda) \tag{2}$$

which has covariance

$$R_X(t-s) = \int_{\mathbb{R}} e^{i\lambda(t-s)}\, dF(\lambda). \tag{3}$$

1.2.1 Sample path realizations

Definition 1.3. [Locally square-integrable functions] Define

$$L^2_{\mathrm{loc}}(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} : \int_K |f(t)|^2\, dt < \infty \text{ for every compact } K \subseteq \mathbb{R} \right\}. \tag{4}$$

Remark 1.4. Every bounded measurable set in $\mathbb{R}$ is compact or contained in a compact set; hence $L^2_{\mathrm{loc}}(\mathbb{R})$ contains functions that are square-integrable on every bounded interval, including functions with polynomial growth at infinity.

Theorem 1.5. ***[Sample paths in** $L^2_{\mathrm{loc}}(\mathbb{R})$**]*** *Let $\{X(t)\}_{t\in\mathbb{R}}$ be a second-order stationary process with*

$$\sigma^2 := \mathbb{E}[X(t)^2] < \infty. \tag{5}$$

Then almost every sample path lies in $L^2_{\mathrm{loc}}(\mathbb{R})$.

Proof. Fix a bounded interval $[a,b] \subset \mathbb{R}$ with $a < b$ and define

$$Y_{[a,b]} := \int_a^b X(t)^2\, dt. \tag{6}$$

By Tonelli's theorem,

$$\mathbb{E}[Y_{[a,b]}] = \int_a^b \mathbb{E}[X(t)^2]\, dt. \tag{7}$$

By stationarity, $\mathbb{E}[X(t)^2] = \sigma^2$, hence

$$\mathbb{E}[Y_{[a,b]}] = \sigma^2(b-a) < \infty. \tag{8}$$

Markov's inequality yields

$$\mathbb{P}(Y_{[a,b]} > M) \le \frac{\sigma^2(b-a)}{M}, \tag{9}$$

so $\mathbb{P}(Y_{[a,b]} < \infty) = 1$. If $K \subset \mathbb{R}$ is compact then $K \subseteq [-N, N]$ for some $N > 0$, so

$$\int_K X(t)^2\, dt \le \int_{-N}^N X(t)^2\, dt < \infty \quad \text{a.s.} \tag{10}$$

Thus $X(\cdot, \omega) \in L^2_{\mathrm{loc}}(\mathbb{R})$ for almost every ω. ∎

2 (Non-Stationary) Oscillatory Processes

Definition 2.1. [Oscillatory process] Let F be a finite nonnegative Borel measure on $\mathbb{R}$. Let

$$A_t \in L^2(F) \quad \forall t \in \mathbb{R} \tag{11}$$

be the gain function and

$$\varphi_t(\lambda) = A_t(\lambda)e^{i\lambda t} \tag{12}$$

the corresponding oscillatory function. An oscillatory process is a stochastic process represented as

$$Z(t) = \int_{\mathbb{R}} \varphi_t(\lambda)\, d\Phi(\lambda) = \int_{\mathbb{R}} A_t(\lambda)e^{i\lambda t}\, d\Phi(\lambda) \tag{13}$$

where Φ is a complex orthogonal random measure with spectral measure F satisfying

$$\mathbb{E}[\Phi(\lambda)\overline{\Phi(\mu)}] = \delta(\lambda-\mu)\, dF(\lambda) \tag{14}$$

and covariance

$$\begin{aligned} R_Z(t,s) = \mathbb{E}[Z(t)\overline{Z(s)}] &= \int_{\mathbb{R}} A_t(\lambda)\overline{A_s(\lambda)}e^{i\lambda(t-s)}\, dF(\lambda) \\ &= \int_{\mathbb{R}} \varphi_t(\lambda)\overline{\varphi_s(\lambda)}\, dF(\lambda). \end{aligned} \tag{15}$$

Definition 2.2. [Evolutionary power spectral density (EPSD)] If $dF(\lambda) = S(\lambda)\, d\lambda$, define

$$S_t(\lambda) := |A_t(\lambda)|^2 S(\lambda), \tag{16}$$

so $dF_t(\lambda) = S_t(\lambda)\, d\lambda = |A_t(\lambda)|^2\, dF(\lambda)$.

Theorem 2.3. *[Real-valuedness criterion for oscillatory processes] Let Z be an oscillatory process with $\varphi_t(\lambda) = A_t(\lambda)e^{i\lambda t}$ and spectral measure F. Then Z is real-valued if and only if*

$$A_t(-\lambda) = \overline{A_t(\lambda)} \quad \textit{for } F\textit{-a.e. } \lambda \in \mathbb{R}, \tag{17}$$

equivalently

$$\varphi_t(-\lambda) = \overline{\varphi_t(\lambda)} \quad \textit{for } F\textit{-a.e. } \lambda \in \mathbb{R}. \tag{18}$$

Proof. Taking complex conjugates in

$$Z(t) = \int A_t(\lambda)e^{i\lambda t}\, d\Phi(\lambda)$$

and applying the real-valued symmetry for the orthogonal random measure, with the change of variables $\mu = -\lambda$, yields $A_t(\lambda) = \overline{A_t(-\lambda)}$ F-a.e., which is (17). The equivalence with (18) follows from $\varphi_t(\lambda) = A_t(\lambda)e^{i\lambda t}$. Reversing the steps gives the converse. □

Theorem 2.4. *[Existence of oscillatory processes with explicit L^2-limit construction] Let F be absolutely continuous with density $S(\lambda)$ and let $A_t(\lambda) \in L^2(F)$ for all $t \in \mathbb{R}$, measurable jointly in (t,λ). Define*

$$\sigma_t^2 := \int_{\mathbb{R}} |A_t(\lambda)|^2\, dF(\lambda) < \infty. \tag{19}$$

Then there exists a complex orthogonal random measure Φ with spectral measure F such that for each fixed t the stochastic integral

$$Z(t) = \int_{\mathbb{R}} A_t(\lambda)e^{i\lambda t}\, d\Phi(\lambda) \tag{20}$$

is well-defined as an $L^2(\Omega)$-limit and has covariance (15).

Proof.

1. Let S be the set of simple functions $g(\lambda) = \sum_{j=1}^n c_j \mathbf{1}_{E_j}(\lambda)$ with disjoint Borel E_j and $F(E_j) < \infty$. Define $\int g\, d\Phi := \sum_{j=1}^n c_j \Phi(E_j)$. Orthogonality gives the isometry

$$\mathbb{E}\left[\left|\int g\, d\Phi\right|^2\right] = \int_{\mathbb{R}} |g(\lambda)|^2\, dF(\lambda).$$

2. For $h \in L^2(F)$, choose $g_n \in \mathsf{S}$ with $\|h - g_n\|_{L^2(F)} \to 0$. Then

$$\mathbb{E}\left[\left|\int g_n\, d\Phi - \int g_m\, d\Phi\right|^2\right] = \|g_n - g_m\|^2_{L^2(F)}, \qquad \lim_{n,m\to\infty} \|g_n - g_m\|^2_{L^2(F)} = 0. \tag{21}$$

3. Completeness of $L^2(\Omega)$ yields existence of the limit, and the isometry shows independence of the approximating sequence. □

3 Filter Representations and Invertibility for Oscillatory Processes

Definition 3.1. [Time-dependent filter and gain] The time-dependent filter $h(t,u)$ and gain function $A_t(\lambda)$ satisfy

$$A_t(\lambda) = \int_{-\infty}^{\infty} h(t,u) e^{-i\lambda(t-u)}\, du, \tag{22}$$

$$h(t,u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_t(\lambda) e^{i\lambda(t-u)}\, d\lambda, \tag{23}$$

with

$$\int_{-\infty}^{\infty} |h(t,u)|^2\, du < \infty \quad \forall t \in \mathbb{R}. \tag{24}$$

Theorem 3.2. ***[Forward and inverse filter representations for general oscillatory processes]*** *Let $Z(t)$ be an oscillatory process with $\varphi_t(\lambda) = A_t(\lambda)e^{i\lambda t}$. Then:*

1.
$$Z(t) = \int_{\mathbb{R}} h(t,\lambda)\, d\Phi(\lambda), \qquad h(t,\lambda) = A_t(\lambda)e^{i\lambda t}. \tag{25}$$

2. *If there is a white noise representation $dW(u)$ with*
$$\mathbb{E}[dW(u_1)\overline{dW(u_2)}] = \delta(u_1 - u_2)\, du_1, \tag{26}$$
then
$$Z(t) = \int_{-\infty}^{\infty} h(t,u)\, dW(u). \tag{27}$$

Proof. Item (1) is (13). Item (2) follows from the spectral relation

$$d\Phi(\lambda) = \frac{1}{2\pi} \int e^{-i\lambda u}\, dW(u)\, du$$

and the Fourier pair in Definition 3.1. □

Definition 3.3. [Amplitude nondegeneracy]

$$A_t(\lambda) \neq 0 \quad \text{for all } (t,\lambda) \text{ in the support of } F. \tag{28}$$

Definition 3.4. [Kernel orthonormality]

$$\int_{-\infty}^{\infty} A_t(\lambda_1) A_t(\lambda_2) e^{i(\lambda_2 - \lambda_1)t}\, dt = \delta(\lambda_1 - \lambda_2). \tag{29}$$

Theorem 3.5. ***[Fundamental invertibility for oscillatory processes]*** *The inversion formula*

$$d\Phi(\lambda) = \int_{-\infty}^{\infty} A_t(\lambda) e^{-i\lambda t} Z(t)\, dt \tag{30}$$

holds if and only if (28) *and* (29) *hold, and the gain function satisfies the additional consistency condition*

$$\int_{-\infty}^{\infty} A_t(\lambda) A_s(\mu) e^{i(\mu s - \lambda t)}\, dt = \delta(\lambda - \mu)\delta(s - t) \tag{31}$$

for all s, μ in the support of the respective measures.

Proof. The forward direction requires the strengthened orthonormality condition (31) to handle the mixed time variables arising from the substitution of $Z(t) = \int A_s(\mu)e^{i\mu s}\, d\Phi(\mu)$. Under this condition, the double integral reduces to $\delta(\lambda - \mu)\, d\Phi(\mu) = d\Phi(\lambda)$. The reverse direction follows by testing on $Z_{\lambda_0}(t) := A_t(\lambda_0)e^{i\lambda_0 t}$. □

Lemma 3.6. ***[Uniqueness of inversion]*** *If $\mathcal{I}_1 Z = d\Phi(\lambda) = \mathcal{I}_2 Z$ for all Z, then $\mathcal{I}_1 = \mathcal{I}_2$.*

Proof. Let $\mathcal{L} = \mathcal{I}_1 - \mathcal{I}_2$. Then $\mathcal{L}Z = 0$ for all Z, hence $\mathcal{L} = 0$. □

4 Unitarily Time-Changed Stationary Processes

4.1 Unitary time-change operator

Theorem 4.1. ***[Unitary time-change and local isometry]*** *Let $\theta : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, strictly increasing, and bijective with $\dot{\theta}(t) > 0$ a.e. For measurable f, define*

$$(U_\theta f)(t) = \sqrt{\dot{\theta}(t)}\, f(\theta(t)). \tag{32}$$

Define the inverse map

$$(U_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}}. \tag{33}$$

For every compact $K \subseteq \mathbb{R}$ and $f \in L^2_{\mathrm{loc}}(\mathbb{R})$,

$$\int_K |(U_\theta f)(t)|^2\, dt = \int_{\theta(K)} |f(s)|^2\, ds. \tag{34}$$

Moreover, for $f, g \in L^2_{\mathrm{loc}}(\mathbb{R})$,

$$U_\theta^{-1}(U_\theta f) = f, \qquad U_\theta(U_\theta^{-1} g) = g. \tag{35}$$

Proof. By (32),

$$\int_K |(U_\theta f)(t)|^2\, dt = \int_K \dot{\theta}(t) |f(\theta(t))|^2\, dt.$$

With $s = \theta(t)$ and $ds = \dot{\theta}(t)\, dt$, this equals $\int_{\theta(K)} |f(s)|^2\, ds$. Direct substitution proves the two identities in (35). □

4.2 The subclass and exact inversion

Definition 4.2. [Unitarily time-changed stationary process] Let $X = \{X(t)\}_{t\in\mathbb{R}}$ be a second-order stationary process with sample paths in $L^2_{\mathrm{loc}}(\mathbb{R})$. Let θ satisfy the hypotheses of Theorem 4.1. Define

$$Z(t) := (U_\theta X)(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)). \tag{36}$$

Then Z is called a unitarily time-changed stationary process.

Lemma 4.3. ***[Exact recovery of** X**]*** *If Z is defined by* (36)*, then*

$$X = U_\theta^{-1} Z. \tag{37}$$

Proof. This is the identity $U_\theta^{-1}(U_\theta X) = X$ from (35). □

4.3 Filter representations

Theorem 4.4. ***[Forward and inverse filter representations for unitarily time-changed stationary processes]*** *Let θ satisfy Theorem 4.1. Let $X(u) = \int_{\mathbb{R}} e^{i\lambda u}\, d\Phi(\lambda)$ and let $Z(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t))$. Then:*

1. $$h(t, u) = \sqrt{\dot{\theta}(t)}\, \delta(u - \theta(t)). \tag{38}$$

2. $$g(t, s) = \frac{\delta(s - \theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}}. \tag{39}$$

3. $$X(t) = \int_{\mathbb{R}} g(t, s)\, Z(s)\, ds = \frac{Z(\theta^{-1}(t))}{\sqrt{\dot{\theta}(\theta^{-1}(t))}}. \tag{40}$$

Proof. Item (1) is the sifting identity for δ applied to $X(\theta(t))$. Item (2) is the explicit inverse map U_θ^{-1} written as an integral kernel. Item (3) is substitution. □

4.4 Stationary to oscillatory

Theorem 4.5. ***[Unitary time change produces oscillatory process]*** *Let X be zero-mean stationary with*

$$X(t) = \int_{\mathbb{R}} e^{i\lambda t}\, d\Phi(\lambda).$$

Let θ satisfy Theorem 4.1. Define

$$Z(t) := (U_\theta X)(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)). \tag{41}$$

Then Z is an oscillatory process with oscillatory function

$$\varphi_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \tag{42}$$

and gain function

$$A_t(\lambda) = \sqrt{\dot{\theta}(t)}\, e^{i\lambda(\theta(t)-t)}. \tag{43}$$

Proof. Substitute $t \mapsto \theta(t)$ into the stationary representation and multiply by $\sqrt{\dot{\theta}(t)}$:

$$Z(t) = \sqrt{\dot{\theta}(t)} \int_{\mathbb{R}} e^{i\lambda\theta(t)}\, d\Phi(\lambda) = \int_{\mathbb{R}} \left(\sqrt{\dot{\theta}(t)}\, e^{i\lambda\theta(t)} \right) d\Phi(\lambda).$$

This is (13) with φ_t given by (42). □

Corollary 4.6. ***[EPSD for the unitary time change]*** *If $dF(\lambda) = S(\lambda)\, d\lambda$, then*

$$S_t(\lambda) = |A_t(\lambda)|^2 S(\lambda) = \dot{\theta}(t)\, S(\lambda). \tag{44}$$

Proof. From (43), $|A_t(\lambda)|^2 = \dot{\theta}(t)$. □

5 Zero Localization

5.1 Kac–Rice

Theorem 5.1. ***[Kac–Rice formula for stationary processes]*** *Let $Z(t)$ be a real-valued, zero-mean Gaussian process with covariance $K(t,s) = \mathbb{E}[Z(t)Z(s)]$. Assume Z is stationary with $K(t,s) = K(t-s)$, $K(0) > 0$, and that $K(h)$ is twice continuously differentiable in a neighborhood of $h = 0$ with $K''(0) < 0$. Define*

$$K(0) := \mathbb{E}[Z(t)^2], \qquad K''(0) := \left. \frac{d^2K(h)}{dh^2} \right|_{h=0}. \tag{45}$$

Then for $t \in [a,b]$,

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \frac{1}{\pi} \sqrt{\frac{-K''(0)}{K(0)}}\, dt. \tag{46}$$

Proof. For stationary processes, $\mathbb{E}[Z(t)\dot{Z}(t)] = 0$ and $\mathbb{E}[\dot{Z}(t)^2] = -K''(0)$. The joint density of $(Z(t), \dot{Z}(t))$ factors, and the standard Kac-Rice computation gives the result. □

5.2 Bulinskaya

Theorem 5.2. ***[Bulinskaya]*** *Let $X(t)$ be a real-valued, zero-mean stationary Gaussian process with covariance $R(h) = \mathbb{E}[X(t)X(t+h)]$. If R is twice continuously differentiable in a neighborhood of 0 and $R''(0) < 0$, then with probability 1 all zeros of X are simple.*

Proof. For fixed t_0, $(X(t_0), \dot{X}(t_0))$ is jointly Gaussian and stationarity gives $\mathbb{E}[X(t_0)\dot{X}(t_0)] = R'(0) = 0$. Thus $X(t_0)$ and $\dot{X}(t_0)$ are independent. Since $R''(0) < 0$, $\dot{X}(t_0)$ is non-degenerate, hence $\mathbb{P}(\dot{X}(t_0) = 0) = 0$. This yields $\mathbb{P}(X(t_0) = 0 \text{ and } \dot{X}(t_0) = 0) = 0$, and the conclusion follows. □

Remark 5.3. The Bulinskaya theorem applies only to stationary processes. For unitarily time-changed processes $Z(t) = \sqrt{\dot{\theta}(t)}X(\theta(t))$, the cross-covariance $\mathbb{E}[Z(t)\dot{Z}(t)] = \frac{\sigma^2}{2}\ddot{\theta}(t)$ is generally non-zero, so the theorem does not apply.

6 Example: The Hardy Z-Function

6.1 Definitions

Definition 6.1. [**Hardy Z-function**] Let $\zeta(s)$ be the Riemann zeta function and let $\theta(t)$ denote the Riemann–Siegel theta function. Define

$$Z(t) := e^{i\theta(t)}\,\zeta\big(\tfrac{1}{2} + it\big)\,. \tag{47}$$

Definition 6.2. [**Monotonized theta time change**] Let $a > 0$ be the unique critical point of θ on $(0, \infty)$. Define $\Theta : [0, \infty) \to [\Theta(0), \infty)$ by

$$\Theta(t) := \begin{cases} 2\theta(a) - \theta(t), & 0 \le t \le a, \\ \theta(t), & t \ge a. \end{cases} \tag{48}$$

6.2 Stationary candidate and exact inversion

Definition 6.3. [**Stationary candidate defined by U_Θ^{-1}**] Define

$$X(u) := (U_\Theta^{-1} Z)(u) = \frac{Z(\Theta^{-1}(u))}{\sqrt{\Theta'(\Theta^{-1}(u))}}, \qquad u \in [\Theta(0), \infty). \tag{49}$$

Lemma 6.4. ***[Exact reconstruction** $Z = U_\Theta X$**]*** *With X as in Definition 6.3,*

$$Z(t) = (U_\Theta X)(t) = \sqrt{\Theta'(t)}\,X(\Theta(t)), \qquad t \in [0, \infty). \tag{50}$$

Proof. This is the identity $U_\Theta(U_\Theta^{-1} Z) = Z$ from Theorem 4.1, applied on the domain where Θ is defined. □

6.3 L^2_{loc} identity on finite intervals

Lemma 6.5. ***[Finite-interval** L^2 **identity]*** *For every $T > 0$,*

$$\int_{\Theta(0)}^{\Theta(T)} |X(u)|^2\,du = \int_0^T |Z(t)|^2\,dt. \tag{51}$$

Proof. With $u = \Theta(t)$, $du = \Theta'(t)\,dt$, and $X(u) = Z(t)/\sqrt{\Theta'(t)}$,

$$\int_{\Theta(0)}^{\Theta(T)} |X(u)|^2\,du = \int_0^T \left|\frac{Z(t)}{\sqrt{\Theta'(t)}}\right|^2 \Theta'(t)\,dt = \int_0^T |Z(t)|^2\,dt.$$

□

Theorem 6.6. *[$X \in L^2_{\text{loc}}([\Theta(0), \infty))$]*

$$X \in L^2_{\text{loc}}([\Theta(0), \infty)). \tag{52}$$

Proof. Let $[c, d] \subset [\Theta(0), \infty)$ be compact. Then $[\Theta^{-1}(c), \Theta^{-1}(d)] \subset [0, \infty)$ is compact, and Z is bounded on that interval. Thus $\int_{\Theta^{-1}(c)}^{\Theta^{-1}(d)} |Z(t)|^2\,dt < \infty$. By the same change of variables as in Lemma 6.5,

$$\int_c^d |X(u)|^2\,du = \int_{\Theta^{-1}(c)}^{\Theta^{-1}(d)} |Z(t)|^2\,dt < \infty.$$

□

6.4 Limit-form mean-square statements

Theorem 6.7. ***[Hardy–Littlewood second moment (limit form)]***

$$\lim_{T\to\infty} \frac{\displaystyle\int_0^T \left|\zeta\big(\tfrac{1}{2} + it\big)\right|^2\,dt}{T\log T} = 1. \tag{53}$$

Equivalently,

$$\lim_{T\to\infty} \frac{\displaystyle\int_0^T |Z(t)|^2\,dt}{T\log T} = 1. \tag{54}$$

Theorem 6.8. ***[Ratio limit for Θ]***

$$\lim_{T\to\infty} \frac{\Theta(T)}{\frac{T}{2}\log T} = 1. \tag{55}$$

Theorem 6.9. ***[Mean-square limit for X]***

$$\lim_{T\to\infty} \frac{1}{\Theta(T)-\Theta(0)} \int_{\Theta(0)}^{\Theta(T)} |X(u)|^2\,du = 2. \tag{56}$$

Proof. By Lemma 6.5,

$$\frac{1}{\Theta(T)-\Theta(0)} \int_{\Theta(0)}^{\Theta(T)} |X(u)|^2\,du = \frac{\int_0^T |Z(t)|^2\,dt}{\Theta(T)-\Theta(0)}.$$

Write

$$\frac{\int_0^T |Z(t)|^2\,dt}{\Theta(T)-\Theta(0)} = \left(\frac{\int_0^T |Z(t)|^2\,dt}{T\log T}\right)\left(\frac{T\log T}{\Theta(T)-\Theta(0)}\right).$$

The first factor has limit 1 by (54). From (55),

$$\lim_{T\to\infty} \frac{T\log T}{\Theta(T)} = 2 \quad \Rightarrow \quad \lim_{T\to\infty} \frac{T\log T}{\Theta(T)-\Theta(0)} = 2.$$

Multiplying limits gives (56). □

7 Time-average covariance conjectures

Definition 7.1. [Empirical covariance kernel] For $U > \Theta(0)$ and $\tau \in \mathbb{R}$ define

$$K_U(\tau) := \frac{1}{U-\Theta(0)} \int_{\Theta(0)}^{U} X(u)\,X(u+\tau)\,du. \tag{57}$$

Conjecture 7.2. [Existence of a stationary covariance kernel] For each fixed $\tau \in \mathbb{R}$, the limit

$$K(\tau) := \lim_{U\to\infty} K_U(\tau) \tag{58}$$

exists in $\mathbb{R}$.

Conjecture 7.3. [Ergodic stationary realization] There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a stationary ergodic process $\{X_{\mathrm{st}}(u,\omega)\}_{u\in\mathbb{R}}$ such that there exists $\omega_0 \in \Omega$ with

$$X_{\mathrm{st}}(u,\omega_0) = X(u) \quad \forall u \ge \Theta(0),$$

and such that for every fixed $\tau \in \mathbb{R}$,

$$\mathbb{E}[X_{\mathrm{st}}(0,\omega)X_{\mathrm{st}}(\tau,\omega)] = \lim_{U\to\infty} \frac{1}{U-\Theta(0)} \int_{\Theta(0)}^{U} X(u)\,X(u+\tau)\,du.$$

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