

Delta Functions, Heaviside Steps, and Level Crossing Counts for Differentiable Paths

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1 Foundations of Distributions on Real Line

Definition 1 (Schwartz Test Function Space) *The Schwartz space $\mathcal{S}(\mathbb{R})$ is the space of all infinitely differentiable functions $\phi : \mathbb{R} \rightarrow \mathbb{R}$ such that for every pair of nonnegative integers m, n ,*

$$\sup_{x \in \mathbb{R}} |x^m \phi^{(n)}(x)| < \infty \quad (1)$$

Functions in $\mathcal{S}(\mathbb{R})$ are called rapidly decreasing smooth test functions.

Definition 2 (Tempered Distribution) *A tempered distribution is a continuous linear functional*

$$T : \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{R} \quad (2)$$

Definition 3 (Dirac Delta Distribution) *The Dirac delta distribution $\delta_a \in \mathcal{S}'(\mathbb{R})$ centered at $a \in \mathbb{R}$ is defined by*

$$\langle \delta_a, \phi \rangle = \phi(a) \quad (3)$$

for all $\phi \in \mathcal{S}(\mathbb{R})$. When $a = 0$, one writes $\delta = \delta_0$.

Definition 4 (Heaviside Step Function) *The Heaviside step function $H : \mathbb{R} \rightarrow \{0, 1\}$ is defined by*

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \quad (4)$$

Definition 5 (Distributional Derivative) For a tempered distribution $T \in \mathcal{S}'(\mathbb{R})$, its distributional derivative $T' \in \mathcal{S}'(\mathbb{R})$ is defined by

$$\langle T', \phi \rangle = -\langle T, \phi' \rangle$$

for all $\phi \in \mathcal{S}(\mathbb{R})$.

2 Basic Identities

Theorem 1 (Heaviside Derivative) The Heaviside step function H satisfies

$$H' = \delta \tag{5}$$

as distributions on $\mathcal{S}'(\mathbb{R})$.

Proof For all $\phi \in \mathcal{S}(\mathbb{R})$,

$$\langle H', \phi \rangle = -\langle H, \phi' \rangle \tag{6}$$

$$= -\int_{-\infty}^{\infty} H(x) \phi'(x) dx \tag{7}$$

$$= -\int_0^{\infty} \phi'(x) dx \tag{8}$$

$$= -[\phi(x)]_0^{\infty} \tag{9}$$

$$= -\left(\lim_{x \rightarrow \infty} \phi(x) - \phi(0)\right) \tag{10}$$

$$= \phi(0) \tag{11}$$

where the limit vanishes since $\phi \in \mathcal{S}(\mathbb{R})$ decays rapidly at infinity. Thus

$$\langle H', \phi \rangle = \phi(0) = \langle \delta, \phi \rangle \tag{12}$$

□

Theorem 2 (Integral of Delta) For any $a \in \mathbb{R}$ and $T \in \mathbb{R}$,

$$\int_{-\infty}^T \delta(t - a) dt = H(T - a) \tag{13}$$

Proof Define

$$F(T) = \int_{-\infty}^T \delta(t - a) dt \tag{14}$$

Taking the distributional derivative with respect to T :

$$F'(T) = \frac{d}{dT} \int_{-\infty}^T \delta(t - a) dt = \delta(T - a) \tag{15}$$

Since $F(-\infty) = 0$ and

$$F'(T) = \delta(T - a) = H'(T - a) \quad (16)$$

from the previous theorem, one has

$$F(T) = H(T - a) + C \quad (17)$$

for some constant C . The boundary condition

$$F(-\infty) = 0 = H(-\infty) + C \quad (18)$$

implies $C = 0$, thus

$$F(T) = H(T - a) \quad (19)$$

□

3 Delta of a Smooth Function

Theorem 3 (Delta under Change of Variables) *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable with isolated, simple zeros $\{x_i\}$ such that $g(x_i) = 0$ and $g'(x_i) \neq 0$. Then the identity*

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|} \quad (20)$$

holds in $\mathcal{S}'(\mathbb{R})$.

Proof For $\phi \in \mathcal{S}(\mathbb{R})$,

$$\langle \delta(g(x)), \phi \rangle = \int_{-\infty}^{\infty} \phi(x) \delta(g(x)) dx \quad (21)$$

Near each zero x_i , where g is locally monotone by the implicit function theorem, the change of variables $u = g(x)$ gives

$$\begin{aligned} \int_{I_i} \phi(x) \delta(g(x)) dx &= \int_{g(I_i)} \frac{\phi(g^{-1}(u))}{|g'(g^{-1}(u))|} \delta(u) du \\ &= \frac{\phi(x_i)}{|g'(x_i)|} \end{aligned} \quad (22)$$

by the sifting property of δ . Summing over all zeros yields

$$\langle \delta(g(x)), \phi \rangle = \sum_i \frac{\phi(x_i)}{|g'(x_i)|} = \left\langle \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}, \phi \right\rangle \quad (23)$$

Since this holds for all $\phi \in \mathcal{S}(\mathbb{R})$, the distributional equality follows. □

4 Counting Function for Level Crossings

Let $x : \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable, and fix $u \in \mathbb{R}$. Assume the zeros of $g(t) := x(t) - u$ are isolated and simple; that is, for every zero t_i ,

$$g'(t_i) = x'(t_i) \neq 0 \quad (24)$$

Definition 6 [*Level Crossing Counting Function*] Define the counting function

$$N(T) := \text{the number of zeros } t_i \text{ of } x(t) - u \text{ with } t_i \leq T \quad (25)$$

Theorem 4 (Counting Function as Integral Over Delta) For every $T \in \mathbb{R}$,

$$N(T) = \int_{-\infty}^T |x'(t)| \delta(x(t) - u) dt \quad (26)$$

Proof Using the delta change of variables theorem with

$$g(t) = x(t) - u \quad (27)$$

one finds that

$$|x'(t)| \delta(x(t) - u) = |x'(t)| \sum_i \frac{\delta(t - t_i)}{|x'(t_i)|} \quad (28)$$

$$= \sum_i |x'(t)| \frac{\delta(t - t_i)}{|x'(t_i)|} \quad (29)$$

Since $x'(t_i) \neq 0$ by assumption, and $\delta(t - t_i)$ picks out the value at $t = t_i$,

$$\begin{aligned} |x'(t)| \delta(x(t) - u) &= \sum_i \frac{|x'(t_i)|}{|x'(t_i)|} \delta(t - t_i) \\ &= \sum_i \delta(t - t_i) \end{aligned} \quad (30)$$

Therefore,

$$\begin{aligned} \int_{-\infty}^T |x'(t)| \delta(x(t) - u) dt &= \sum_i \int_{-\infty}^T \delta(t - t_i) dt \\ &= \sum_{t_i \leq T} 1 \\ &= N(T) \end{aligned} \quad (31)$$

□

Theorem 5 (Counting Function as Sum of Heaviside Steps) The counting function (6) is given by

$$N(T) = \sum_i H(T - t_i) \forall T \in \mathbb{R} \quad (32)$$

where the sum runs over all zero crossing times t_i .

Proof By definition of the Heaviside function,

$$H(T - t_i) = 1 \quad (33)$$

if and only if $T \geq t_i$, and

$$H(T - t_i) = 0 \quad (34)$$

otherwise. Therefore,

$$\begin{aligned} \sum_i H(T - t_i) &= \sum_{t_i \leq T} 1 \\ &= N(T) \end{aligned} \quad (35)$$

□

Theorem 6 (Equivalence of Representations) *The delta integral representation and the Heaviside step sum representation are equivalent:*

$$\int_{-\infty}^T |x'(t)| \delta(x(t) - u) dt = \sum_i H(T - t_i) \quad (36)$$

Proof This follows immediately from the two previous theorems, since both expressions equal $N(T)$. □