# Measure-Preserving Bijective Time Changes of Stationary Gaussian Processes Generate Oscillatory Processes With Evolving Spectra

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**Abstract**

This article establishes that Gaussian processes obtained through measure-preserving bijective unitary time transformations of stationary processes constitute a subclass of oscillatory processes in the sense of Priestley. The transformation $Z(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t))$, where $X(t)$ is a realization of stationary Gaussian process and $\theta$ is a strictly increasing $C^1$ differentiable monotonic function, yields an oscillatory process with evolutionary power spectrum $d\,F_t(\omega) = \dot{\theta}(t)\, d\,\mu(\omega)$. An explicit unitary transformation between the input stationary process and the transformed oscillatory process is established, preserving the $L^2$-norm and providing a complete spectral characterization.

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# 1 Scaling Functions

**Definition 1.** *[Scaling Functions] Let $\mathcal{F}$ denote the set of functions $\theta\colon\mathbb{R}\to\mathbb{R}$ satisfying*

1. *$\theta$ is absolutely continuous with $\dot{\theta}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\theta(t) \geq 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of Lebesgue measure zero*

2. *$\theta$ is strictly increasing and bijective.*

**Remark 2.** The conditions in Definition 1 ensure that $\theta^{-1}$ exists and is absolutely continuous. By the inverse function theorem for absolutely continuous functions,

$$\frac{\mathrm{d}}{\mathrm{d}s}(\theta^{-1})(s) = \frac{1}{\dot{\theta}(\theta^{-1}(s))} \tag{1}$$

for almost all $s$ in the range of $\theta$. The condition that $\dot{\theta}(t) = 0$ only on sets of measure zero ensures that $\frac{1}{\dot{\theta}(\theta^{-1}(s))}$ is well-defined almost everywhere.

# 2 Oscillatory Processes

**Definition 3.** *[Oscillatory Process] A complex-valued, second-order process $\{X(t)\}_{t\in\mathbb{R}}$ is called* oscillatory *if there exist*

1. *a family of oscillatory basis functions $\{\phi_t(\omega)\}_{t\in\mathbb{R}}$ with*

$$\phi_t(\omega) = A_t(\omega)\, e^{i\omega t} \tag{2}$$

*and a given gain function*

$$A_t(\cdot) \in L^2(\mu) \tag{3}$$

2. *and a complex orthogonal random measure $\Phi(\omega)$ with*

$$E\,|d\,\Phi(\omega)|^2 = d\,\mu(\omega) = S(\omega) \tag{4}$$

*such that*

$$\begin{aligned} Z(t) &= \int_{-\infty}^{\infty} \phi_t(\omega)\ d\,\Phi(\omega) \\ &= \int_{-\infty}^{\infty} A_t(\omega)\, e^{i\omega t} d\,\Phi(\omega) \end{aligned} \tag{5}$$

*All stationary processes are oscillatory with $A_t(\omega) = 1$*

TODO: insert proof of this as well as representation of Z(t) as a time-dependent convolution of a stationary process with the time-dependent filter given by the Fourier transform of the oscillatory function

# 3 Stationary Reference Process

Let $\{X(t)\}_{t\in\mathbb{R}}$ be a stationary Gaussian process with continuous spectral representation

$$X(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, d\,\Phi(\omega) \tag{6}$$

where $\Phi(\omega)$ is an orthogonal-increment process with spectral density

$$E\,|d\,\Phi(\omega)|^2 = d\,\mu(\omega) = S(\omega) = <\text{fourier transform of } K_X > \tag{7}$$

and $\mu$ is a finite measure on $\mathbb{R}$.

# 4 Time-Changed Process

## 4.1 Definition and Unitary Operator

**Definition 4.** *[Unitary Time-Change Operator]For $\theta \in \mathcal{F}$, define the operator $M_\theta$: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by*

$$(M_\theta\, f)(t) = \sqrt{\dot{\theta}(t)}\;\; f(\theta(t)) \tag{8}$$

**Definition 5.** *[Unitarily Time-Changed Stationary Process] For $\theta \in \mathcal{F}$, apply the unitary time change operator $M_\theta$ from Definition-4 to a realization of a stationary process $X(t)$ from the ensemble $\{X(t)\}$ to define a realization of the unitarily time-changed process*

$$Z(t) = \sqrt{\dot{\theta}(t)}\;\; X(\theta(t)) \forall t \in \mathbb{R} \tag{9}$$

**Definition 6.** *[Inverse Unitary Time-Change Operator] The inverse operator $M_\theta^{-1}$: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ corresponding to the unitary time-change operator $(M_\theta\, f)(t)$ defined in Equation-8 is given by*

$$(M_\theta^{-1}\, g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{10}$$

**Lemma 7.** *[Well-Definedness of Inverse Operator] The operator $M_\theta^{-1}$ in Definition 6 is well-defined $\forall\theta\in\mathcal{F}$.*

**Proof.** Since $\dot{\theta}(t)=0$ only on sets of measure zero by Definition 1, and $\theta^{-1}$ maps sets of measure zero to sets of measure zero (as it preserves absolute continuity), the denominator $\sqrt{\dot{\theta}(\theta^{-1}(s))}$ is positive almost everywhere. The expression in equation (10) is therefore well-defined almost everywhere, which is sufficient for defining an element of $L^2(\mathbb{R})$. □

**Theorem 8.** *[Unitarity of Transformation Operator] The operator $M_\theta$ defined in equation (8) is unitary, i.e.,*

$$\int_{\mathbb{R}}|(M_\theta f)(t)|^2\,dt=\int_{\mathbb{R}}|f(s)|^2\,ds\forall f\in L^2(\mathbb{R}) \tag{11}$$

**Proof.** Let $f\in L^2(\mathbb{R})$. The $L^2$-norm of $M_\theta f$ is computed as follows:

$$\int_{\mathbb{R}}|(M_\theta f)(t)|^2\,dt=\int_{\mathbb{R}}\left|\sqrt{\dot{\theta}(t)}\;f(\theta(t))\right|^2\,dt \tag{12}$$

$$=\int_{\mathbb{R}}\dot{\theta}(t)\,|f(\theta(t))|^2\,dt \tag{13}$$

Apply the change of variables $s=\theta(t)$. Since $\theta$ is absolutely continuous and strictly increasing, its Jacobian is given by

$$ds=\dot{\theta}(t)\;dt \tag{14}$$

almost everywhere. As $t$ ranges over $\mathbb{R}$, $s=\theta(t)$ ranges over $\mathbb{R}$ due to the bijectivity of $\theta$. Therefore:

$$\int_{\mathbb{R}}\dot{\theta}(t)\,|f(\theta(t))|^2\,dt=\int_{\mathbb{R}}|f(s)|^2\,ds \tag{15}$$

This establishes equation (11). To complete the proof of unitarity, it remains to show that $M_\theta^{-1}$ is indeed the inverse of $M_\theta$. For any $f\in L^2(\mathbb{R})$:

$$(M_\theta^{-1}M_\theta f)(s)=(M_\theta^{-1})\left[\sqrt{\dot{\theta}(\cdot)}\;f(\theta(\,\cdot\,))\right](s) \tag{16}$$

$$=\frac{\sqrt{\dot{\theta}(\theta^{-1}(s))}\;f(\theta(\theta^{-1}(s)))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}} \tag{17}$$

$$=f(s) \tag{18}$$

where the last equality uses $\theta(\theta^{-1}(s)) = s$. Similarly, for any $g \in L^2(\mathbb{R})$:

$$
\begin{aligned}
(M_\theta M_\theta^{-1} g)(t) &= \sqrt{\dot{\theta}(t)}\ (M_\theta^{-1} g)(\theta(t)) && (19)\\
&= \sqrt{\dot{\theta}(t)}\ \frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\dot{\theta}(\theta^{-1}(\theta(t)))}} && (20)\\
&= \sqrt{\dot{\theta}(t)}\ \frac{g(t)}{\sqrt{\dot{\theta}(t)}} && (21)\\
&= g(t) && (22)
\end{aligned}
$$

Therefore

$$
M_\theta M_\theta^{-1} = M_\theta^{-1} M_\theta = I \tag{23}
$$

proving that $M_\theta$ is unitary. □

**Corollary 9.** *[Measure Preservation] The transformation $M_\theta$ preserves the $L^2$-measure in the sense that for any measurable set $A \subseteq \mathbb{R}$*

$$
\int_A |(M_\theta f)(t)|^2\, dt = \int_{\theta(A)} |f(s)|^2\, ds \tag{24}
$$

**Proof.** The proof follows the same change of variables argument as in Theorem 8, applied to the characteristic function of the set $A$. □

## 4.2 $L^2$-Norm Preservation

**Theorem 10.** *[Measure Preservation] The transformation defined in equation (9) preserves the $L^2$-norm in the sense that*

$$
\int_I \mathrm{var}(Z(t))\, dt = \int_{\theta(I)} \mathrm{var}(X(s))\, ds \tag{25}
$$

*for any measurable set $I \subseteq \mathbb{R}$.*

**Proof.** Using the change of variables $s = \theta(t)$ with $ds = \dot{\theta}(t)\, dt$:

$$
\begin{aligned}
\int_I \mathrm{var}(X(t))\, dt &= \int_I \mathrm{var}\Big(\sqrt{\dot{\theta}(t)}\ X(\theta(t))\Big)\, dt && (26)\\
&= \int_I \dot{\theta}(t)\, \mathrm{var}(X(\theta(t)))\, dt && (27)
\end{aligned}
$$

$$= \int_{\theta(I)} \mathrm{var}(X(s))\ d\,s \tag{28}$$

□

## 4.3 Oscillatory Representation

**Theorem 11.** *[Oscillatory Form] The process $\{Z(t)\}$ defined in equation (9) is oscillatory with oscillatory functions*

$$\phi_t(\omega) = A_t(\omega)\, e^{i\omega t} = \sqrt{\dot{\theta}(t)}\ e^{i\omega\theta(t)} \tag{29}$$

*and gain functions*

$$A_t(\omega) = \sqrt{\dot{\theta}(t)}\ e^{i\omega(\theta(t)-t)} \tag{30}$$

**Proof.** From the spectral representation (6) of the stationary process $X(t)$:

$$X(t) = \sqrt{\dot{\theta}(t)}\, X(\theta(t)) \tag{31}$$

$$= \sqrt{\dot{\theta}(t)} \int_{-\infty}^{\infty} e^{i\omega\theta(t)}\ d\,\Phi(\omega) \tag{32}$$

$$= \int_{-\infty}^{\infty} \sqrt{\dot{\theta}(t)}\ e^{i\omega\theta(t)}\ d\,\phi(\omega) \tag{33}$$

$$= \int_{-\infty}^{\infty} \phi_t(\omega)\ d\,\Phi(\omega) \tag{34}$$

where

$$\phi_t(\omega) = \sqrt{\dot{\theta}(t)}\ e^{i\omega\theta(t)} \tag{35}$$

To verify this is an oscillatory representation according to Definition 3, express $\phi_t(\omega)$ in the form of a function of the time-dependent gain $A_t(\lambda)$ as required

$$\begin{aligned} \phi_t(\omega) &= A_t(\omega)\, e^{i\omega t} \\ &= \sqrt{\dot{\theta}(t)}\ e^{i\omega(\theta(t)-t)}\ e^{i\omega t} \\ &= \sqrt{\dot{\theta}(t)} e^{i\omega(\theta(t)-t+t)} \\ &= \sqrt{\dot{\theta}(t)}\ e^{i\omega\theta(t)} \end{aligned} \tag{36}$$

where

$$A_t(\omega) = \sqrt{\dot{\theta}(t)}\ e^{i\omega(\theta(t)-t)} \tag{37}$$

.

Since $\dot{\theta}(t) \geq 0$ almost everywhere and $\dot{\theta}(t) = 0$ only on sets of measure zero, the function $A_t(\omega)$ is well-defined almost everywhere. Moreover, $A_t(\cdot) \in L^2(\mu)$ for each $t$ since:

$$\begin{aligned}\int_{-\infty}^{\infty} |A_t(\omega)|^2\ d\,\mu(\omega) =& \int_{-\infty}^{\infty} \dot{\theta}(t)\ d\,\mu(\omega) && (38)\\ =& \dot{\theta}(t) \int_{-\infty}^{\infty} d\,\mu(\omega) \\ =& \dot{\theta}(t)\ \mu(\mathbb{R}) < \infty && (39)\end{aligned}$$

where the finiteness follows from $\mu$ being a finite measure and $\dot{\theta}(t)$ being finite almost everywhere. □

## 4.4 Envelope and Evolutionary Spectrum

**Corollary 12.** *[Evolutionary Spectrum] The evolutionary power spectrum is*

$$\begin{aligned} d\,F_t(\omega)\ &= |A_t(\omega)|^2\ d\,\mu(\omega) \\ &= \dot{\theta}(t)\ d\,\mu(\omega) \end{aligned} \tag{40}$$

**Proof.** By Definition 3 and the envelope from Equation 3, the evolutionary power spectrum is:

$$\begin{aligned} d\,F_t(\omega) =& |A_t(\omega)|^2\ d\,\mu(\omega) && (41)\\ =& \left|\sqrt{\dot{\theta}(t)}\ e^{i\omega(\theta(t)-t)}\right|^2\ d\,\mu(\omega) && (42)\\ =& \dot{\theta}(t)\,|e^{i\omega(\theta(t)-t)}|^2\ d\,\mu(\omega) && (43)\\ =& \dot{\theta}(t)\ d\,\mu(\omega) && (44)\end{aligned}$$

since

$$|e^{i\alpha}| = 1 \forall \alpha \in \mathbb{R} \tag{45}$$

□

# 5 Operator Conjugation

**Theorem 13.** *[Operator Conjugation] Let $T_K$ be the integral covariance operator defined by*

$$(T_K f)(t) = \int_{-\infty}^{\infty} K(|t-s|)\, f(s)\; d\,s \tag{46}$$

*where $K(h)$ is the stationary kernel*

$$K(h) = \int_{-\infty}^{\infty} S(\lambda) e^{i\lambda h} \mathrm{d}\lambda \tag{47}$$

*, and let $T_{K_\theta}$ be the integral covariance operator defined by*

$$\begin{aligned}(T_{K_\theta} f)(t) &= \int_{-\infty}^{\infty} K_\theta(s,t) f(s)\; d\,s \\ &= \int_{-\infty}^{\infty} K(|\theta(t)-\theta(s)|)\,\sqrt{\dot{\theta}(t)\dot{\theta}(s)}\, f(s)\; d\,s\end{aligned} \tag{48}$$

*for the unitarily time-changed kernel*

$$K_\theta(s,t) = K(|\theta(t)-\theta(s)|)\sqrt{\dot{\theta}(t)\dot{\theta}(s)} \tag{49}$$

*. Then*

$$T_{K_\theta} = M_\theta\, T_K\, M_\theta^{-1} \tag{50}$$

**Proof.** For any $g \in L^2(\mathbb{R})$, compute $(M_\theta\, T_K\, M_\theta^{-1}\, g)(t)$:

$$(M_\theta^{-1}\, g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\theta'(\theta^{-1}(s))}}, \tag{51}$$

$$(T_K\, M_\theta^{-1}\, g)(t) = \int_{-\infty}^{\infty} K(|t-s|) \frac{g(\theta^{-1}(s))}{\sqrt{\dot{\theta}(\theta^{-1}(s))}}\, d\,s. \tag{52}$$

Apply the change of variables $u = \theta^{-1}(s)$, so $s = \theta(u)$ and $d\,s = \dot{\theta}(u)\, d\,u$:

$$(T_K\, M_\theta^{-1}\, g)(t) = \int_{-\infty}^{\infty} K(|t-\theta(u)|) \frac{g(u)}{\sqrt{\dot{\theta}(u)}}\, \dot{\theta}(u)\, d\,u \tag{53}$$

$$= \int_{-\infty}^{\infty} K(|t-\theta(u)|)\; g(u) \sqrt{\dot{\theta}(u)}\; d\,u. \tag{54}$$

Now apply $M_\theta$:

$$(M_\theta T_K M_\theta^{-1} g)(t) = \sqrt{\dot{\theta}(t)}\, (T_K M_\theta^{-1} g)(\theta(t)) \tag{55}$$

$$= \sqrt{\dot{\theta}(t)} \int_{-\infty}^{\infty} K(|\theta(t) - \theta(u)|)\, g(u) \sqrt{\dot{\theta}(u)}\, d u. \tag{56}$$

Apply the change of variables $s = \theta(u)$ in the reverse direction:

$$(M_\theta T_K M_\theta^{-1} g)(t) = \int_{-\infty}^{\infty} K(|\theta(t) - \theta(s)|)\, g(s)\, d s \tag{57}$$

$$= (T_{K_\theta} g)(t) \tag{58}$$

This establishes the conjugation relation (50). □

# 6 Expected Zero Count

**Theorem 14.** *[Expected Zero-Counting Function] Let $\theta \in \mathcal{F}$ and let*

$$K(\tau) = \operatorname{cov}(X(t), X(\tau)) \tag{59}$$

*be twice differentiable at $\tau = 0$. The expected number of zeros of the process $X_t$ in $[a, b]$ is*

$$\mathbb{E}[N_{[a,b]}] = \sqrt{-\ddot{K}(0)}\ (\theta(b) - \theta(a)) \tag{60}$$

**Proof.** The covariance function of the time-changed process is

$$K_\theta(s, t) = \operatorname{cov}(X_s, X_t) = \sqrt{\dot{\theta}(s)\, \dot{\theta}(t)}\ K(|\theta(t) - \theta(s)|) \tag{61}$$

For the zero-crossing analysis, consider the normalized process. By the Kac-Rice formula:

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{-\lim_{s \to t} \frac{\partial^2}{\partial s\, \partial t} K_\theta(s, t)}\ d t \tag{62}$$

Computing the mixed partial derivative:

$$\frac{\partial}{\partial t} K_\theta(s, t) = \frac{1}{2} \frac{\ddot{\theta}(t)}{\sqrt{\dot{\theta}(t)}} \sqrt{\theta'(s)}\, K(|\theta(t) - \theta(s)|) \tag{63}$$

$$+ \sqrt{\dot{\theta}(s)\, \dot{\theta}(t)}\, \dot{K}(|\theta(t) - \theta(s)|) \operatorname{sgn}(\theta(t) - \theta(s))\, \dot{\theta}(t). \tag{64}$$

Taking the limit as $s \to t$ and using the fact that $\dot{K}(0) = 0$ for stationary processes:

$$\lim_{s \to t} \frac{\partial^2}{\partial s \, \partial t} K_\theta(s,t) = \dot{\theta}(s)\, \dot{\theta}(t)\, \ddot{K}(0) \tag{65}$$

$$= \dot{\theta}(t)^2 \, \ddot{K}(0) \tag{66}$$

Substituting into the Kac-Rice formula:

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{-\dot{\theta}(t)^2 \, \ddot{K}(0)} \; d\,t \tag{67}$$

$$= \sqrt{-\ddot{K}(0)} \int_a^b \dot{\theta}(t) \; d\,t \tag{68}$$

$$= \sqrt{-\ddot{K}(0)} \; (\theta(b) - \theta(a)) \tag{69}$$

Here the second equality uses $\dot{\theta}(t) \geq 0$ almost everywhere. □

# 7 Conclusion

This analysis establishes that Gaussian processes generated by measure-preserving bijective time changes of stationary processes form a well-defined subclass of oscillatory processes. The key contributions include:

1. The rigorous construction of the unitary operator $M_\theta$ and its inverse, with proper treatment of the case where $\dot{\theta}(t) = 0$ on sets of measure zero.

2. The explicit oscillatory representation with envelope function $A_t(\omega) = \sqrt{\dot{\theta}(t)}\, e^{i\omega(\theta(t) - t)}$.

3. The evolutionary power spectrum formula $d\,F_t(\omega) = \dot{\theta}(t)\, d\,\mu(\omega)$.

4. The operator conjugation relationship $T_{K_\theta} = M_\theta \, T_K \, M_\theta^{-1}$.

5. A closed-form expression for the expected zero count in terms of the range of the time transformation.

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