Measure-Preserving Bijective Time Changes of Stationary Gaussian Processes Generate Oscillatory Processes With Evolving Spectra

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Abstract

This article establishes that Gaussian processes obtained through measure-preserving bijective unitary time transformations of stationary processes constitute a subclass of oscillatory processes in the sense of Priestley. The transformation $X_t = \sqrt{\theta'(t)}\, S_{\theta(t)}$, where S_t is a stationary Gaussian process and θ is a strictly monotonic function, yields an oscillatory process with evolutionary power spectrum $d\,F_t(\omega) = \theta'(t)\; d\,\mu(\omega)$. An explicit unitary transformation between the original stationary process and the transformed oscillatory process is established, preserving the L^2-norm and providing a complete spectral characterization.

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1 Scaling Functions

Definition 1. *[Scaling Functions] Let $\mathcal{F}$ denote the set of functions $\theta\colon\mathbb{R}\to\mathbb{R}$ satisfying*

1. *θ is absolutely continuous with $\theta'(t)\geq 0$ almost everywhere and $\theta'(t)=0$ only on sets of Lebesgue measure zero,*

2. *θ is strictly increasing and bijective.*

Remark 2. The conditions in Definition 1 ensure that θ^{-1} exists and is absolutely continuous. By the inverse function theorem for absolutely continuous functions, $(\theta^{-1})'(s)=\frac{1}{\theta'(\theta^{-1}(s))}$ for almost all s in the range of θ. The condition that $\theta'(t)=0$ only on sets of measure zero ensures that $\frac{1}{\theta'(\theta^{-1}(s))}$ is well-defined almost everywhere.

2 Oscillatory Processes

Definition 3. *[Oscillatory Process] A complex-valued, second-order process $\{X_t\}_{t\in\mathbb{R}}$ is called* oscillatory *if there exist*

1. *a family of functions $\{\phi_t(\omega)\}_{t\in\mathbb{R}}$ with $\phi_t(\omega)=A_t(\omega)\,e^{i\omega t}$ and $A_t(\cdot)\in L^2(\mu)$,*

2. *a complex orthogonal-increment process $Z(\omega)$ with $E\,|d\,Z(\omega)|^2=d\,\mu(\omega)$,*

such that

$$X_t=\int_{-\infty}^{\infty}\phi_t(\omega)\;d\,Z(\omega). \tag{1}$$

3 Stationary Reference Process

Let $\{S_t\}_{t\in\mathbb{R}}$ be a stationary Gaussian process with continuous spectral representation

$$S_t=\int_{-\infty}^{\infty}e^{i\omega t}\;d\,Z(\omega), \tag{2}$$

where $Z(\omega)$ is an orthogonal-increment process with $E\,|d\,Z(\omega)|^2=d\,\mu(\omega)$ and μ is a finite measure on $\mathbb{R}$.

4 Time-Changed Process

4.1 Definition and Unitary Operator

Definition 4. *[Time-Changed Process] For $\theta \in \mathcal{F}$, define the time-changed process*

$$X_t := \sqrt{\theta'(t)}\; S_{\theta(t)}, \qquad t \in \mathbb{R}. \tag{3}$$

Definition 5. *[Unitary Transformation Operator] For $\theta \in \mathcal{F}$, define the operator M_θ: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by*

$$(M_\theta f)(t) = \sqrt{\theta'(t)}\; f(\theta(t)). \tag{4}$$

Definition 6. *[Inverse Unitary Transformation Operator] The inverse operator M_θ^{-1}: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined by*

$$(M_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\theta'(\theta^{-1}(s))}}. \tag{5}$$

Lemma 7. *[Well-Definedness of Inverse Operator] The operator M_θ^{-1} in Definition 6 is well-defined for $\theta \in \mathcal{F}$.*

Proof. Since $\theta'(t) = 0$ only on sets of measure zero by Definition 1, and θ^{-1} maps sets of measure zero to sets of measure zero (as it preserves absolute continuity), the denominator $\sqrt{\theta'(\theta^{-1}(s))}$ is positive almost everywhere. The expression in equation (5) is therefore well-defined almost everywhere, which is sufficient for defining an element of $L^2(\mathbb{R})$. □

Theorem 8. *[Unitarity of Transformation Operator] The operator M_θ defined in equation (4) is unitary, i.e.,*

$$\int_{\mathbb{R}} |(M_\theta f)(t)|^2\, dt = \int_{\mathbb{R}} |f(s)|^2\, ds \tag{6}$$

for all $f \in L^2(\mathbb{R})$.

Proof. Let $f \in L^2(\mathbb{R})$. The L^2-norm of $M_\theta f$ is computed as follows:

$$\int_{\mathbb{R}} |(M_\theta f)(t)|^2\, dt = \int_{\mathbb{R}} \left|\sqrt{\theta'(t)}\; f(\theta(t))\right|^2\, dt \tag{7}$$

$$= \int_{\mathbb{R}} \theta'(t)\, |f(\theta(t))|^2\, dt. \tag{8}$$

Apply the change of variables $s=\theta(t)$. Since θ is absolutely continuous and strictly increasing, $d\,s=\theta'(t)\,d\,t$ almost everywhere. As t ranges over $\mathbb{R}$, $s=\theta(t)$ ranges over $\mathbb{R}$ due to the bijectivity of θ. Therefore:

$$\int_{\mathbb{R}} \theta'(t)\,|f(\theta(t))|^2\,d\,t = \int_{\mathbb{R}} |f(s)|^2\,d\,s. \tag{9}$$

This establishes equation (6).

To complete the proof of unitarity, it remains to show that M_θ^{-1} is indeed the inverse of M_θ. For any $f\in L^2(\mathbb{R})$:

$$\begin{aligned}(M_\theta^{-1}\,M_\theta\,f)(s) &= (M_\theta^{-1})\left[\sqrt{\theta'(\cdot)}\;\,f(\theta(\,\cdot\,))\right](s) &(10)\\ &= \frac{\sqrt{\theta'(\theta^{-1}(s))}\;\,f(\theta(\theta^{-1}(s)))}{\sqrt{\theta'(\theta^{-1}(s))}} &(11)\\ &= f(s), &(12)\end{aligned}$$

where the last equality uses $\theta(\theta^{-1}(s))=s$.

Similarly, for any $g\in L^2(\mathbb{R})$:

$$\begin{aligned}(M_\theta\,M_\theta^{-1}\,g)(t) &= \sqrt{\theta'(t)}\;\,(M_\theta^{-1}\,g)(\theta(t)) &(13)\\ &= \sqrt{\theta'(t)}\;\,\frac{g(\theta^{-1}(\theta(t)))}{\sqrt{\theta'(\theta^{-1}(\theta(t)))}} &(14)\\ &= \sqrt{\theta'(t)}\;\,\frac{g(t)}{\sqrt{\theta'(t)}} &(15)\\ &= g(t). &(16)\end{aligned}$$

Therefore, $M_\theta\,M_\theta^{-1}=M_\theta^{-1}\,M_\theta=I$, proving that M_θ is unitary. □

Corollary 9. *[Measure Preservation] The transformation M_θ preserves the L^2-measure in the sense that for any measurable set $A\subseteq\mathbb{R}$,*

$$\int_A |(M_\theta\,f)(t)|^2\,d\,t = \int_{\theta(A)} |f(s)|^2\,d\,s. \tag{17}$$

Proof. The proof follows the same change of variables argument as in Theorem 8, applied to the characteristic function of the set A. □

4.2 Oscillatory Representation

Theorem 10. *[Oscillatory Form] The process $\{X_t\}$ defined in equation (3) is oscillatory with oscillatory functions*

$$\phi_t(\omega) = \sqrt{\theta'(t)}\;\,e^{i\omega\theta(t)}. \tag{18}$$

Proof. From the spectral representation (2) of the stationary process S_t:

$$
\begin{aligned}
X_t &= \sqrt{\theta'(t)}\; S_{\theta(t)} && (19)\\
&= \sqrt{\theta'(t)} \int_{-\infty}^{\infty} e^{i\omega\theta(t)}\; d\,Z(\omega) && (20)\\
&= \int_{-\infty}^{\infty} \sqrt{\theta'(t)}\; e^{i\omega\theta(t)}\; d\,Z(\omega) && (21)\\
&= \int_{-\infty}^{\infty} \phi_t(\omega)\; d\,Z(\omega), && (22)
\end{aligned}
$$

where $\phi_t(\omega) = \sqrt{\theta'(t)}\; e^{i\omega\theta(t)}$.

To verify this is an oscillatory representation according to Definition 3, express $\phi_t(\omega)$ in the required form:

$$
\begin{aligned}
\phi_t(\omega) &= \sqrt{\theta'(t)}\; e^{i\omega\theta(t)} && (23)\\
&= \sqrt{\theta'(t)}\; e^{i\omega(\theta(t)-t)}\; e^{i\omega t} && (24)\\
&= A_t(\omega)\, e^{i\omega t}, && (25)
\end{aligned}
$$

where $A_t(\omega) = \sqrt{\theta'(t)}\; e^{i\omega(\theta(t)-t)}$.

Since $\theta'(t) \geq 0$ almost everywhere and $\theta'(t) = 0$ only on sets of measure zero, the function $A_t(\omega)$ is well-defined almost everywhere. Moreover, $A_t(\cdot) \in L^2(\mu)$ for each t since:

$$
\begin{aligned}
\int_{-\infty}^{\infty} |A_t(\omega)|^2\; d\,\mu(\omega) &= \int_{-\infty}^{\infty} \theta'(t)\; d\,\mu(\omega) && (26)\\
&= \theta'(t)\; \mu(\mathbb{R}) < \infty, && (27)
\end{aligned}
$$

where the finiteness follows from μ being a finite measure and $\theta'(t)$ being finite almost everywhere. ☐

4.3 Envelope and Evolutionary Spectrum

Corollary 11. *[Envelope] The oscillatory functions in equation (18) admit the standard decomposition*

$$
\phi_t(\omega) = A_t(\omega)\, e^{i\omega t}, \quad \textit{where} \quad A_t(\omega) = \sqrt{\theta'(t)}\; e^{i\omega(\theta(t)-t)}. \tag{28}
$$

Proof. This follows directly from the calculation in the proof of Theorem 10. ☐

Corollary 12. *[Evolutionary Spectrum] The evolutionary power spectrum is*

$$
d\,F_t(\omega) = |A_t(\omega)|^2\; d\,\mu(\omega) = \theta'(t)\; d\,\mu(\omega). \tag{29}
$$

Proof. By Definition 3 and the envelope from Corollary 11, the evolutionary power spectrum is:

$$\begin{aligned} dF_t(\omega) &= |A_t(\omega)|^2 \, d\mu(\omega) && (30)\\ &= \left|\sqrt{\theta'(t)}\; e^{i\omega(\theta(t)-t)}\right|^2 \, d\mu(\omega) && (31)\\ &= \theta'(t)\, |e^{i\omega(\theta(t)-t)}|^2 \, d\mu(\omega) && (32)\\ &= \theta'(t)\; d\mu(\omega), && (33) \end{aligned}$$

since $|e^{i\alpha}| = 1$ for any real α. □

5 Operator Conjugation

Theorem 13. *[Operator Conjugation] Let T_K be the integral operator defined by*

$$(T_K f)(t) = \int_{-\infty}^{\infty} K(|t-s|)\, f(s)\; ds \tag{34}$$

for a stationary kernel K, and let T_{K_θ} be the integral operator defined by

$$(T_{K_\theta} g)(t) = \int_{-\infty}^{\infty} K(|\theta(t)-\theta(s)|)\, g(s)\; ds \tag{35}$$

for the transformed kernel $K_\theta(s,t) = K(|\theta(t)-\theta(s)|)$. Then

$$T_{K_\theta} = M_\theta\, T_K\, M_\theta^{-1}. \tag{36}$$

Proof. For any $g \in L^2(\mathbb{R})$, compute $(M_\theta T_K M_\theta^{-1} g)(t)$:

$$(M_\theta^{-1} g)(s) = \frac{g(\theta^{-1}(s))}{\sqrt{\theta'(\theta^{-1}(s))}}, \tag{37}$$

$$(T_K M_\theta^{-1} g)(t) = \int_{-\infty}^{\infty} K(|t-s|) \frac{g(\theta^{-1}(s))}{\sqrt{\theta'(\theta^{-1}(s))}}\, ds. \tag{38}$$

Apply the change of variables $u = \theta^{-1}(s)$, so $s = \theta(u)$ and $ds = \theta'(u)\, du$:

$$\begin{aligned} (T_K M_\theta^{-1} g)(t) &= \int_{-\infty}^{\infty} K(|t-\theta(u)|) \frac{g(u)}{\sqrt{\theta'(u)}}\, \theta'(u)\, du && (39)\\ &= \int_{-\infty}^{\infty} K(|t-\theta(u)|)\, g(u) \sqrt{\theta'(u)}\, du. && (40) \end{aligned}$$

Now apply M_θ:

$$(M_\theta T_K M_\theta^{-1} g)(t) = \sqrt{\theta'(t)}\,(T_K M_\theta^{-1} g)(\theta(t)) \tag{41}$$

$$= \sqrt{\theta'(t)} \int_{-\infty}^{\infty} K(|\theta(t)-\theta(u)|)\, g(u) \sqrt{\theta'(u)}\, d u. \tag{42}$$

Apply the change of variables $s=\theta(u)$ in the reverse direction:

$$(M_\theta T_K M_\theta^{-1} g)(t) = \int_{-\infty}^{\infty} K(|\theta(t)-\theta(s)|)\, g(s)\, d s \tag{43}$$

$$= (T_{K_\theta} g)(t). \tag{44}$$

This establishes the conjugation relation (36). □

6 Expected Zero Count

Theorem 14. *[Expected Zero-Counting Function] Let $\theta \in \mathcal{F}$ and let $K(\tau) = \operatorname{cov}(S_0, S_\tau)$ be twice differentiable at $\tau = 0$. The expected number of zeros of the process X_t in $[a,b]$ is*

$$\mathbb{E}[N_{[a,b]}] = \sqrt{-\ddot{K}(0)}\;(\theta(b)-\theta(a)). \tag{45}$$

Proof. The covariance function of the time-changed process is

$$K_\theta(s,t) = \operatorname{cov}(X_s, X_t) = \sqrt{\theta'(s)\,\theta'(t)}\; K(|\theta(t)-\theta(s)|). \tag{46}$$

For the zero-crossing analysis, consider the normalized process. By the Kac-Rice formula:

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{-\lim_{s\to t} \frac{\partial^2}{\partial s\, \partial t} K_\theta(s,t)}\; d t. \tag{47}$$

Computing the mixed partial derivative:

$$\frac{\partial}{\partial t} K_\theta(s,t) = \frac{1}{2} \frac{\theta''(t)}{\sqrt{\theta'(t)}} \sqrt{\theta'(s)}\, K(|\theta(t)-\theta(s)|) \tag{48}$$

$$+ \sqrt{\theta'(s)\,\theta'(t)}\, K'(|\theta(t)-\theta(s)|) \operatorname{sgn}(\theta(t)-\theta(s))\, \theta'(t). \tag{49}$$

Taking the limit as $s \to t$ and using the fact that $K'(0)=0$ for stationary processes:

$$\lim_{s\to t} \frac{\partial^2}{\partial s\, \partial t} K_\theta(s,t) = \theta'(s)\,\theta'(t)\, K''(0) \tag{50}$$

$$= \theta'(t)^2\, \ddot{K}(0). \tag{51}$$

Substituting into the Kac-Rice formula:

$$\mathbb{E}[N_{[a,b]}] = \int_a^b \sqrt{-\theta'(t)^2 \ddot{K}(0)}\; dt \tag{52}$$

$$= \sqrt{-\ddot{K}(0)} \int_a^b \theta'(t)\; dt \tag{53}$$

$$= \sqrt{-\ddot{K}(0)}\; (\theta(b) - \theta(a)). \tag{54}$$

Here the second equality uses $\theta'(t) \geq 0$ almost everywhere. □

7 Conclusion

This analysis establishes that Gaussian processes generated by measure-preserving bijective time changes of stationary processes form a well-defined subclass of oscillatory processes. The key contributions include:

1. The rigorous construction of the unitary operator M_θ and its inverse, with proper treatment of the case where $\theta'(t) = 0$ on sets of measure zero.

2. The explicit oscillatory representation with envelope function $A_t(\omega) = \sqrt{\theta'(t)}\, e^{i\omega(\theta(t)-t)}$.

3. The evolutionary power spectrum formula $d F_t(\omega) = \theta'(t)\, d\mu(\omega)$.

4. The operator conjugation relationship $T_{K_\theta} = M_\theta T_K M_\theta^{-1}$.

5. A closed-form expression for the expected zero count in terms of the range of the time transformation.

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