Theorem 1. (Real Spectral Representation for Stationary Processes) *Let $\{\xi(t), t \in \mathbb{R}\}$ be a real-valued, zero-mean, second-order stationary process with covariance function $r(t) = \mathbb{E}[\xi(t)\,\xi(0)]$ and spectral distribution function $F(\omega)$. Then there exist real-valued random measures $\{U(\omega), \omega \geq 0\}$ and $\{V(\omega), \omega \geq 0\}$ with orthogonal increments such that:*

1. ***Process Representation:***

$$\xi(t) = \int_0^\infty [\cos(\omega t)\; dU(\omega) + \sin(\omega t)\; dV(\omega)] \tag{1}$$

2. ***Covariance Representation:***

$$r(t) = \int_0^\infty \cos(\omega t)\; dF(\omega) \tag{2}$$

3. ***Orthogonality Properties:***

$$\mathbb{E}[U(\omega)] = \mathbb{E}[V(\omega)] = 0 \tag{3}$$
$$\mathbb{E}[dU(\omega_1)\; dU(\omega_2)] = \mathbb{E}[dV(\omega_1)\; dV(\omega_2)] = \delta(\omega_1 - \omega_2)\, dF(\omega_1) \tag{4}$$
$$\mathbb{E}[dU(\omega_1)\; dV(\omega_2)] = 0 \quad \textit{for all } \omega_1, \omega_2 \geq 0 \tag{5}$$

Proof.

1. **Construction from Complex Representation:** From the complex spectral representation theorem, there holds

$$\xi(t) = \int_{-\infty}^\infty e^{i\omega t}\, d\zeta(\omega) \tag{6}$$

where $\zeta(\omega)$ is a complex-valued random measure with orthogonal increments and $\mathbb{E}[|d\zeta(\omega)|^2] = \frac{1}{2}\, dF(\omega)$ for the two-sided representation.

2. **Reality Condition:** As $\xi(t)$ is real-valued,

$$\xi(t) = \overline{\xi(t)} = \int_{-\infty}^\infty e^{-i\omega t}\, d\overline{\zeta(\omega)} \tag{7}$$

3. **Symmetry Property:** This reality condition requires the spectral random mea-

sure to satisfy

$$d\,\zeta\,(-\omega) = d\overline{\zeta(\omega)} \tag{8}$$

for all ω.

4. **Factorization into Real Random Measures:** For $\omega > 0$, define

$$\begin{aligned} d\,U(\omega) &= 2\ \Re\,[d\,\zeta(\omega)] \qquad (9)\\ d\,V(\omega) &= 2\ \Im\,[d\,\zeta(\omega)] \qquad (10)\end{aligned}$$

where $\Re$ and $\Im$ denote the real and imaginary parts.

5. **Derivation of Real Spectral Representation:**

$$\begin{aligned}\xi(t) &= \int_0^\infty e^{i\omega t}\,d\,\zeta(\omega) + \int_0^\infty e^{-i\omega t}\,d\,\zeta\,(-\omega)\\ &= \int_0^\infty e^{i\omega t}\,d\,\zeta(\omega) + \int_0^\infty e^{-i\omega t}\,d\overline{\zeta(\omega)}\\ &= \int_0^\infty [e^{i\omega t} + e^{-i\omega t}]\,\Re\,[d\,\zeta(\omega)] + \int_0^\infty i\,[e^{i\omega t} - e^{-i\omega t}]\,\Im\,[d\,\zeta(\omega)]\\ &= \int_0^\infty 2\cos\,(\omega\,t)\,\Re\,[d\,\zeta(\omega)] + 2\sin\,(\omega\,t)\,\Im\,[d\,\zeta(\omega)]\\ &= \int_0^\infty \cos\,(\omega\,t)\,d\,U(\omega) + \sin\,(\omega\,t)\,d\,V(\omega)\end{aligned} \tag{11}$$

6. **Orthogonality Verification:** For the two-sided complex representation,

$$\mathbb{E}[|d\,\zeta(\omega)|^2] = \frac{1}{2}\,d\,F(\omega) \tag{12}$$

Since $|d\,\zeta(\omega)|^2 = [\Re\,[d\,\zeta(\omega)]]^2 + [\Im\,[d\,\zeta(\omega)]]^2$ and the real and imaginary parts are orthogonal with equal variances,

$$\mathbb{E}[[\Re\,[d\,\zeta(\omega)]]^2] = \mathbb{E}[[\Im\,[d\,\zeta(\omega)]]^2] = \frac{1}{4}\,d\,F(\omega) \tag{13}$$

Therefore,

$$\mathbb{E}\,[d\,U(\omega)^2] = \mathbb{E}\,[d\,V(\omega)^2] = 4\cdot\frac{1}{4}\,d\,F(\omega) = d\,F(\omega) \tag{14}$$

7. **Covariance Function:** The covariance is given by

$$\begin{aligned} r(t) &= \mathbb{E}\left[\xi(t)\,\xi(0)\right] \\ &= \mathbb{E}\left[\left(\int_0^\infty \cos(\omega t)\,dU(\omega) + \sin(\omega t)\,dV(\omega)\right)\int_0^\infty dU(\omega')\right] \\ &= \int_0^\infty \cos(\omega t)\,\mathbb{E}\left[dU(\omega)^2\right] \end{aligned} \tag{15}$$

where cross-terms vanish by orthogonality and the sine term vanishes since $\mathbb{E}[dV(\omega)] = 0$. Using $\mathbb{E}[dU(\omega)^2] = dF(\omega)$:

$$r(t) = \int_0^\infty \cos(\omega t)\,dF(\omega) \tag{16}$$ □

Corollary 2. (Physical Interpretation) *In the real spectral representation:*

1. $\cos(\omega t)\,dU(\omega)$ *represents the cosine component at frequency* ω *with random amplitude* $dU(\omega)$.

2. $\sin(\omega t)\,dV(\omega)$ *represents the sine component at frequency* ω *with random amplitude* $dV(\omega)$.

3. $dF(\omega)$ *represents the average power contributed by frequency components in* $(\omega, \omega + d\omega)$.

4. *The random measures* $U(\omega)$ *and* $V(\omega)$ *are uncorrelated and have equal variance increments.*

Theorem 3. (U and V Random Measures) *For a real-valued stationary process $\xi(t)$ with mean-square continuous sample paths and spectral representation*

$$\xi(t) = \int_0^\infty [\cos(\omega t)\ dU(\omega) + \sin(\omega t)\ dV(\omega)] \tag{17}$$

the random measures $U(\omega)$ and $V(\omega)$ are given explicitly by:

1. ***U-process formula:***

$$U(\omega) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{1-\cos(\omega t)}{t}\, \xi(t)\ dt \tag{18}$$

2. ***V-process formula:***

$$V(\omega) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{\sin(\omega t)}{t}\, \xi(t)\ dt \tag{19}$$

3. ***Alternative forms using sine and cosine integrals:***

$$U(\omega) = \lim_{T\to\infty} \frac{2}{\pi} \int_{0}^{T} \frac{1-\cos(\omega t)}{t}\, \xi(t)\ dt \tag{20}$$

$$V(\omega) = \lim_{T\to\infty} \frac{2}{\pi} \int_{0}^{T} \frac{\sin(\omega t)}{t}\, \xi(t)\ dt \tag{21}$$

4. ***Incremental form:***

$$U(\omega_2) - U(\omega_1) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{\cos(\omega_1 t) - \cos(\omega_2 t)}{t}\, \xi(t)\ dt \tag{22}$$

$$V(\omega_2) - V(\omega_1) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{\sin(\omega_2 t) - \sin(\omega_1 t)}{t}\, \xi(t)\ dt \tag{23}$$

Proof.

1. Starting from the complex inversion formula:

$$\zeta(\lambda) - \zeta(0) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{1-e^{-it\lambda}}{-it}\, \xi(t)\ dt \tag{24}$$

2. For real processes, the following relations hold from our definitions:

$$d\zeta(\omega) = \frac{1}{2}[dU(\omega) - i\ dV(\omega)] \quad \text{for } \omega > 0 \tag{25}$$

$$d\zeta(-\omega) = \frac{1}{2}[dU(\omega) + i\ dV(\omega)] \quad \text{for } \omega > 0 \tag{26}$$

3. Therefore,

$$U(\omega) - U(0) = 2\,([\zeta(\omega) - \zeta(0)] + [\zeta\,(-\omega) - \zeta(0)]) \tag{27}$$
$$V(\omega) - V(0) = 2\,i\,([\zeta(\omega) - \zeta(0)] - [\zeta\,(-\omega) - \zeta(0)]) \tag{28}$$

4. Substituting the inversion formula and using $1 - e^{-it\lambda} = 1 - \cos(\lambda\,t) + i\sin(\lambda\,t)$:

$$U(\omega) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{1 - \cos(\omega\,t)}{t}\,\xi(t)\;d\,t \tag{29}$$

$$V(\omega) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{\sin(\omega\,t)}{t}\,\xi(t)\;d\,t \tag{30}$$

   where $U(0) = V(0) = 0$ is used.

5. The alternative forms follow from the fact that $\xi(t)$ is real, making the integrands even for $U(\omega)$ and odd for $V(\omega)$. □

Remark 4. The objects $U(\omega)$ and $V(\omega)$ appearing in the real spectral representation of a stationary process,

$$\xi(t) = \int_0^\infty \cos(\omega\,t)\;d\,U(\omega) + \int_0^\infty \sin(\omega\,t)\;d\,V(\omega) \tag{31}$$

are *random measures* (or random set functions) on the frequency axis $[0,\infty)$. Their main property is that their increments over disjoint frequency intervals are orthogonal, i.e., uncorrelated (and independent if Gaussian). The notation $U(\omega)$ denotes the cumulative random measure up to frequency ω:

$$U(\omega) = U([0,\omega]) \qquad V(\omega) = V([0,\omega]) \tag{32}$$

For a stationary process with mean-square continuous sample paths, each sample path uniquely determines the corresponding random measures through the inversion formulas given above.