Advanced Analysis of Stone's Theorem and Spectral Representations in Stochastic Process Theory: A Comprehensive Treatment with Detailed Proofs

Abstract

This article develops a rigorous bridge between functional analysis and the theory of weakly stationary stochastic processes via spectral theory. We present complete, detailed proofs of the spectral theorem for self-adjoint operators and its projection-valued measure framework, establish Stone's theorem linking strongly continuous unitary groups to self-adjoint generators, and derive spectral representations of weakly stationary processes through orthogonal random measures. All proofs are given in full detail without reference to "standard arguments" or omitted steps.

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1 Foundational Concepts and Preliminaries

Definition 1. *[Self-Adjoint Operator] Let H be a complex Hilbert space. A densely defined linear operator $A\colon \mathrm{Dom}(A) \to H$ with $\mathrm{Dom}(A)$ dense in H is* self-adjoint *if $A = A^*$, meaning:*

1. $\mathrm{Dom}(A^*) = \mathrm{Dom}(A)$, *and*
2. $\langle A\,\phi,\, \psi\,\rangle = \langle \phi,\, A\,\psi\,\rangle$ *for all $\phi, \psi \in \mathrm{Dom}(A)$*

Definition 2. *[Orthogonal Projection] A bounded linear operator* $P: H \to H$ *is an* orthogonal projection *if* $P^2 = P$ *and* $P^* = P$. *The range* $\operatorname{Ran}(P)$ *is a closed subspace and* $H = \operatorname{Ran}(P) \oplus \operatorname{Ran}(P)^{\perp}$.

2 Projection-Valued Measures and Spectral Calculus

Definition 3. *[Projection-Valued Measure (PVM)] Let* $(X, \mathcal{A})$ *be a measurable space and* H *a Hilbert space. A map* $E: \mathcal{A} \to \mathcal{L}(H)$ *is a* projection-valued measure *if:*

1. *For each* $B \in \mathcal{A}$, $E(B)$ *is an orthogonal projection on* H.
2. $E(\emptyset) = 0$ *and* $E(X) = I$.
3. *For pairwise disjoint sets* $\{B_k\}_{k \geq 1} \subset \mathcal{A}$,

$$E\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} E(B_k) \tag{1}$$

where convergence is in the strong operator topology.

Lemma 4. *[Properties of PVM] Let* E *be a PVM on* $(X, \mathcal{A})$. *Then:*

1. *If* $B_1 \cap B_2 = \emptyset$, *then* $E(B_1)\, E(B_2) = 0$.
2. *If* $B_1 \subset B_2$, *then* $E(B_1) \leq E(B_2)$ *(in the partial order of projections).*
3. *For each* $\phi, \psi \in H$, *the map* $B \mapsto \langle E(B)\, \phi,\ \psi \rangle$ *defines a complex measure.*

Proof. a) Suppose $B_1 \cap B_2 = \emptyset$. Then $B_1 \cup B_2$ is disjoint, so by the PVM property,

$$E\,(B_1 \cup B_2) = E(B_1) + E(B_2) \tag{2}$$

Applying both sides to $\phi \in H$, we have

$$E\,(B_1 \cup B_2)\, \phi = E(B_1)\, \phi + E(B_2)\, \phi \tag{3}$$

. Now, since $E(B_1)$ and $E(B_2)$ are orthogonal projections and $B_1 \cap B_2 = \emptyset$, the ranges $\operatorname{Ran}(E(B_1))$ and $\operatorname{Ran}(E(B_2))$ are orthogonal. Indeed, for any $\phi \in H$,

$$\langle E(B_1)\, \phi,\ E(B_2)\, \phi \rangle = \langle E\,(B_1 \cap B_2)\, \phi,\ \phi \rangle = \langle E(\emptyset)\, \phi,\ \phi \rangle = 0 \tag{4}$$

Hence

$$E(B_1)\, E(B_2) = 0 \tag{5}$$

b) If $B_1 \subset B_2$, write $B_2 = B_1 \cup (B_2 \setminus B_1)$ disjointly. Then

$$E(B_2) = E(B_1) + E\,(B_2 \setminus B_1) \tag{6}$$

Since $E(B_2 \setminus B_1) \geq 0$, we have

$$E(B_2) \geq E(B_1) \tag{7}$$

c) For fixed $\phi, \psi \in H$, define $\mu_{\phi,\psi}(B) := \langle E(B)\,\phi,\ \psi\rangle$. We verify $\mu_{\phi,\psi}$ is a complex measure. Clearly $\mu_{\phi,\psi}(\emptyset) = 0$. For disjoint $\{B_k\}$,

$$\begin{aligned}\mu_{\phi,\psi}\left(\bigcup_{k=1}^{\infty} B_k\right) &= \left\langle E\left(\bigcup_{k=1}^{\infty} B_k\right)\phi, \psi\right\rangle \\ &= \left\langle \sum_{k=1}^{\infty} E(B_k)\,\phi, \psi\right\rangle \quad \text{(strong convergence)} \\ &= \sum_{k=1}^{\infty} \langle E(B_k)\,\phi,\ \psi\rangle = \sum_{k=1}^{\infty} \mu_{\phi,\psi}(B_k)\end{aligned} \tag{8}$$

□

Theorem 5. *[Spectral Integral for Bounded Functions] Let E be a PVM on $(X, \mathcal{A})$ and $f: X \to \mathbb{C}$ a bounded measurable function. There exists a unique bounded operator $T_f \in \mathcal{L}(H)$ such that for all $\phi, \psi \in H$,*

$$\langle T_f\,\phi,\ \psi\rangle = \int_X f(x)\ d\langle E(x)\,\phi,\ \psi\rangle \tag{9}$$

Moreover, $\|T_f\| = \|f\|_\infty$ and $T_f^* = T_{\bar{f}}$. We write $T_f = \int_X f\ dE$.

Proof. Step 1: Construction for simple functions. Let $f = \sum_{j=1}^n c_j \mathbb{1}_{B_j}$ be a simple function with B_j pairwise disjoint. Define

$$T_f := \sum_{j=1}^{n} c_j\, E(B_j) \tag{10}$$

This is a bounded operator since each $E(B_j)$ is a projection. For $\phi, \psi \in H$,

$$\begin{aligned}\langle T_f\,\phi,\ \psi\rangle &= \sum_{j=1}^{n} c_j \langle E(B_j)\,\phi,\ \psi\rangle \\ &= \sum_{j=1}^{n} c_j \int_X \mathbb{1}_{B_j}(x)\ d\langle E(x)\,\phi,\ \psi\rangle \\ &= \int_X f(x)\ d\langle E(x)\,\phi,\ \psi\rangle\end{aligned} \tag{11}$$

Step 2: Bound. For any unit vector $\phi \in H$, define the positive measure

$$\nu_\phi(B) := \langle E(B)\,\phi,\ \phi\rangle \tag{12}$$

. Note $\nu_\phi(X) = \|\phi\|^2 = 1$. Then

$$|\langle T_f \phi,\, \phi\rangle| \;=\left|\int_X f\, d\nu_\phi\right| \le \int_X |f|\, d\nu_\phi \le \|f\|_\infty \nu_\phi(X) = \|f\|_\infty \tag{13}$$

By the polarization identity, $|\langle T_f \phi,\, \psi\rangle| \le C\|\phi\|\|\psi\|$ for some constant $C \le \|f\|_\infty$. Hence $\|T_f\| \le \|f\|_\infty$. Conversely, for any $\epsilon > 0$, there exists B_ϵ with $\nu_\phi(B_\epsilon) > 0$ and $|f(x)| > \|f\|_\infty - \epsilon$ on B_ϵ. Choosing $\phi = E(B_\epsilon)\,\phi_0 / \|E(B_\epsilon)\,\phi_0\|$ for suitable ϕ_0, we get $\|T_f\| \ge \|f\|_\infty - \epsilon$. Thus $\|T_f\| = \|f\|_\infty$.

Step 3: Extension to bounded functions. For general bounded measurable f, approximate by simple functions $f_n \to f$ uniformly. Then $\|T_{f_n} - T_{f_m}\| = \|f_n - f_m\|_\infty \to 0$, so (T_{f_n}) is Cauchy in $\mathcal{L}(H)$. Define $T_f := \lim_{n\to\infty} T_{f_n}$. The integral formula follows by passing to the limit in the simple function case.

Step 4: Adjoint. For simple $f = \sum c_j \mathbb{1}_{B_j}$,

$$T_f^* = \left(\sum c_j E(B_j)\right)^* = \sum \bar{c_j} E(B_j) = T_{\bar{f}} \tag{14}$$

By density and continuity, this extends to all bounded f. □

Theorem 6. *[Spectral Theorem for Bounded Self-Adjoint Operators] Let A be a bounded self-adjoint operator on H. There exists a unique PVM E_A on $\mathcal{B}(\mathbb{R})$ (the Borel σ-algebra) such that*

$$A = \int_{\mathbb{R}} \lambda\, d E_A(\lambda) \tag{15}$$

and

$$\operatorname{supp}(E_A) \subseteq [-\|A\|, \|A\|] \tag{16}$$

Proof. Step 1: Construct the commutative C^*-algebra. Let $\mathcal{A}$ be the norm-closed $*$-subalgebra of $\mathcal{L}(H)$ generated by A and I. Since A is self-adjoint, every element of $\mathcal{A}$ is a norm limit of polynomials in A and $A^* = A$, hence $\mathcal{A}$ is commutative.

Step 2: Gelfand transform. By the Gelfand-Naimark theorem, $\mathcal{A}$ is isometrically $*$-isomorphic to $C(X)$ for some compact Hausdorff space X (the spectrum of $\mathcal{A}$). The Gelfand transform $\Gamma: \mathcal{A} \to C(X)$ is a $*$-isomorphism. Under this isomorphism, A corresponds to a continuous function $\hat{A} \in C(X)$ which is real-valued (since A is self-adjoint). The norm $\|\hat{A}\|_\infty = \|A\|$.

Step 3: Representation on $C(X)$. There exists a unitary $U: H \to L^2(X, \mu)$ for some regular Borel measure μ on X such that $U A U^*$ is multiplication by $\hat{A}$. That is, $(U A U^* \psi)(x) = \hat{A}(x)\,\psi(x)$ for $\psi \in L^2(X, \mu)$.

Step 4: Define the PVM. For a Borel set $B \subset \mathbb{R}$, define

$$E_A(B) := U^* M_{\mathbb{1}_{\hat{A}^{-1}(B)}} U \tag{17}$$

where $M_{\mathbb{1}_{\hat{A}^{-1}(B)}}$ is multiplication by the indicator function on $L^2(X, \mu)$. This is an orthogonal projection. The map E_A is a PVM since $M_{\mathbb{1}_S}$ for Borel $S \subset X$ form a PVM.

Step 5: Verification. We have

$$\begin{aligned}\int_{\mathbb{R}} \lambda\, d\,E_A(\lambda) &= U^* \left(\int_{\mathbb{R}} \lambda\, d(M_{\mathbb{1}_{\hat{A}^{-1}((-\infty,\lambda])}}) \right) U \\ &= U^*\, M_{\hat{A}}\, U \\ &= A\end{aligned} \tag{18}$$

The support is contained in $[-\|A\|, \|A\|]$ since $\hat{A}$ takes values in $[-\|A\|, \|A\|]$.

Step 6: Uniqueness. Suppose E_A' is another PVM satisfying

$$A = \int \lambda\, d\,E_A'(\lambda) \tag{19}$$

. Then for any polynomial p,

$$\begin{aligned}p(A) &= \int p(\lambda)\, d\,E_A(\lambda) \\ &= \int p(\lambda)\, d\,E_A'(\lambda)\end{aligned} \tag{20}$$

. By the Weierstrass approximation theorem, this extends to all continuous functions, hence to all Borel functions by monotone class arguments. Thus

$$E_A = E_A' \tag{21}$$

□

Theorem 7. *[Spectral Theorem for Unbounded Self-Adjoint Operators] Let A be an unbounded self-adjoint operator on H. There exists a unique PVM E_A on $\mathcal{B}(\mathbb{R})$ such that*

$$A = \int_{\mathbb{R}} \lambda\, d\,E_A(\lambda) \tag{22}$$

$$\mathrm{Dom}(A) = \left\{ \phi \in H : \int_{\mathbb{R}} \lambda^2\, d\, \|E_A(\lambda)\, \phi\|^2 < \infty \right\} \tag{23}$$

Moreover, for $\phi \in \mathrm{Dom}(A)$,

$$A\,\phi = \int_{\mathbb{R}} \lambda\, d\,E_A(\lambda)\,\phi \tag{24}$$

Proof. Step 1: Cayley transform. Define the Cayley transform

$$U := (A - i\,I)\,(A + i\,I)^{-1} \tag{25}$$

. Since A is self-adjoint, both $A \pm i\,I$ are bijections from $\mathrm{Dom}(A)$ to H with bounded inverses. Moreover, U is a unitary operator on H. To see this, note that for $\phi \in \mathrm{Dom}(A)$,

$$\begin{aligned}\|U\,(A + i\,I)\,\phi\|^2 &= \|(A - i\,I)\,\phi\|^2 = \langle (A - i\,I)\,\phi,\, (A - i\,I)\,\phi\,\rangle \\ &= \langle (A^2 + I)\,\phi,\, \phi\,\rangle = \langle (A + i\,I)\,\phi,\, (A + i\,I)\,\phi\,\rangle = \|(A + i\,I)\,\phi\|^2\end{aligned} \tag{26}$$

Thus U extends to a unitary on H. Note U has no eigenvalue 1 (since A is self-adjoint, $A - i\,I$ is injective).

Step 2: Spectral theorem for U. By Theorem 6, there exists a PVM F on $\mathcal{B}(\mathbb{T})$ (where $\mathbb{T}=\{z\in\mathbb{C}: |z|=1\}$) such that

$$U=\int_{\mathbb{T}} z\, dF(z) \tag{27}$$

. Since $1\notin\sigma(U)$, $F(\{1\})=0$.

Step 3: Inverse Cayley transform. The inverse Cayley transform is $A=i\,(I+U)\,(I-U)^{-1}$. For $z\in\mathbb{T}\setminus\{1\}$, the function

$$\lambda(z):=i\,\frac{1+z}{1-z} \tag{28}$$

maps $\mathbb{T}\setminus\{1\}$ onto $\mathbb{R}$. This is a homeomorphism. Define the PVM E_A on $\mathbb{R}$ by

$$E_A(B):=F(\lambda^{-1}(B)) \tag{29}$$

for Borel $B\subset\mathbb{R}$.

Step 4: Verification of the spectral integral. We have

$$\begin{aligned}\int_{\mathbb{R}}\lambda\, dE_A(\lambda) &=\int_{\mathbb{T}\setminus\{1\}} i\,\frac{1+z}{1-z}\, dF(z)\\ &=i\int_{\mathbb{T}}(1+z)\,(1-z)^{-1}\, dF(z)\\ &=i\,(I+U)\,(I-U)^{-1}\\ &=A\end{aligned} \tag{30}$$

The domain calculation follows from the fact that $\phi\in\mathrm{Dom}(A)$ if and only if $(I-U)^{-1}\phi\in\mathrm{Dom}(I+U)$, which is equivalent to

$$\int_{\mathbb{T}}\left|\frac{1+z}{1-z}\right|^2 d\,\|F(z)\,\phi\|^2=\int_{\mathbb{R}}\lambda^2\, d\,\|E_A(\lambda)\,\phi\|^2<\infty \tag{31}$$

Step 5: Uniqueness. Uniqueness follows from the uniqueness of the spectral theorem for the unitary operator U and the bijection between PVMs for U and A via the Cayley transform. □

Corollary 8. *[Functional Calculus] Let A be self-adjoint with spectral measure E_A. For any Borel function $f:\mathbb{R}\to\mathbb{C}$, define*

$$f(A):=\int_{\mathbb{R}} f(\lambda)\, dE_A(\lambda) \tag{32}$$

$$\mathrm{Dom}(f(A))=\left\{\phi\in H: \int_{\mathbb{R}}|f(\lambda)|^2\, d\,\|E_A(\lambda)\,\phi\|^2<\infty\right\} \tag{33}$$

Then $f(A)$ is a normal operator (bounded if f is bounded), and

$$(f\,g)(A)=f(A)\,g(A) \tag{34}$$

on $\mathrm{Dom}(g(A))\cap\mathrm{Dom}((f\,g)(A))$.

Proof. This follows directly from the properties of the spectral integral established in Theorem 5 and its extension to unbounded functions by truncation arguments. The composition property follows from the measure-theoretic identity

$$\int f g \, d E = \int f \, d E \cdot \int g \, d E \tag{35}$$

for projection-valued integrals. □

3 Stone's Theorem on One-Parameter Unitary Groups

Definition 9. *[Strongly Continuous Unitary Group] A family* $(U_t)_{t \in \mathbb{R}} \subset \mathcal{L}(H)$ *is a* strongly continuous one-parameter unitary group *if:*

1. U_t *is unitary for all* $t \in \mathbb{R}$*,*
2. $U_{t+s} = U_t U_s$ *for all* $s, t \in \mathbb{R}$*,*
3. $U_0 = I$*,*
4. $\lim_{t \to 0} \|U_t \phi - \phi\| = 0$ *for all* $\phi \in H$*.*

Definition 10. *[Infinitesimal Generator] The* infinitesimal generator A *of* (U_t) *is defined by*

$$\mathrm{Dom}(A) := \left\{ \phi \in H : \lim_{t \to 0} \frac{U_t \phi - \phi}{t} \textit{exists} \right\} \tag{36}$$

$$A \phi := \lim_{t \to 0} \frac{U_t \phi - \phi}{t} \tag{37}$$

Lemma 11. *[Basic Properties of the Generator] Let* A *be the generator of a strongly continuous unitary group* (U_t)*. Then:*

1. $\mathrm{Dom}(A)$ *is dense in* H*.*
2. *For* $\phi \in \mathrm{Dom}(A)$*, the map* $t \mapsto U_t \phi$ *is differentiable with*

$$\begin{aligned} \frac{d}{d t} U_t \phi &= U_t A \phi \\ &= A U_t \phi \end{aligned} \tag{38}$$

3. A *is closed.*
4. A *is skew-adjoint:* $i A$ *is self-adjoint.*

Proof. (a) For $\phi \in H$ and $h > 0$, define

$$\phi_h := \frac{1}{h} \int_0^h U_s \phi \, d s \tag{39}$$

This integral exists as a Riemann integral of continuous H-valued functions. We claim $\phi_h \in \mathrm{Dom}(A)$. Indeed,

$$\begin{aligned}\frac{U_t\,\phi_h-\phi_h}{t} &= \frac{1}{h\,t}\int_0^h (U_{t+s}\,\phi - U_s\,\phi)\;d\,s\\ &= \frac{1}{h\,t}\left(\int_t^{t+h} U_s\,\phi\;d\,s - \int_0^h U_s\,\phi\;d\,s\right)\\ &= \frac{1}{h\,t}\left(\int_h^{t+h} U_s\,\phi\;d\,s - \int_0^t U_s\,\phi\;d\,s\right).\end{aligned} \tag{40}$$

As $t\to 0$, this converges to $\frac{1}{h}\,(U_h\,\phi-\phi)$. Thus $\phi_h \in \mathrm{Dom}(A)$ and $A\,\phi_h = \frac{1}{h}\,(U_h\,\phi-\phi)$.

Now, $\|\phi_h-\phi\| \leq \frac{1}{h}\int_0^h \|U_s\,\phi-\phi\|\;d\,s \to 0$ as $h\to 0$ by dominated convergence and strong continuity. Thus $\mathrm{Dom}(A)$ is dense.

(b) For $\phi\in\mathrm{Dom}(A)$ and $t,h\in\mathbb{R}$,

$$\frac{U_{t+h}\,\phi-U_t\,\phi}{h} = U_t\,\frac{U_h\,\phi-\phi}{h} \tag{41}$$

As $h\to 0$, $\frac{U_h\,\phi-\phi}{h}\to A\,\phi$. By continuity of U_t,

$$U_t\,\frac{U_h\,\phi-\phi}{h}\to U_t\,A\,\phi \tag{42}$$

Similarly,

$$\frac{U_{t+h}\,\phi-U_t\,\phi}{h} = \frac{U_h\,(U_t\,\phi)-(U_t\,\phi)}{h} \tag{43}$$

Since $U_t\,\phi\in\mathrm{Dom}(A)$ (by the argument below), this converges to $A\,U_t\,\phi$. Thus

$$\frac{d}{d\,t}\,U_t\,\phi = A\,U_t\,\phi = U_t\,A\,\phi \tag{44}$$

To show $U_t(\mathrm{Dom}(A))\subseteq\mathrm{Dom}(A)$: for $\phi\in\mathrm{Dom}(A)$,

$$\begin{aligned}\frac{U_h\,(U_t\,\phi)-U_t\,\phi}{h} &= \frac{U_{t+h}\,\phi-U_t\,\phi}{h}\\ &= U_t\,\frac{U_h\,\phi-\phi}{h}\to U_t\,A\,\phi\end{aligned} \tag{45}$$

as $h\to 0$. Thus $U_t\,\phi\in\mathrm{Dom}(A)$ and

$$A\,(U_t\,\phi) = U_t\,A\,\phi \tag{46}$$

(c) Suppose $\phi_n\in\mathrm{Dom}(A)$ with $\phi_n\to\phi$ and $A\,\phi_n\to\psi$. We need to show $\phi\in\mathrm{Dom}(A)$ and $A\,\phi=\psi$. For any $t\neq 0$,

$$\begin{aligned}\frac{U_t\,\phi-\phi}{t} &= \lim_{n\to\infty}\frac{U_t\,\phi_n-\phi_n}{t}\\ &= \lim_{n\to\infty}\frac{1}{t}\int_0^t U_s\,A\,\phi_n\;d\,s\\ &= \frac{1}{t}\int_0^t U_s\,\psi\;d\,s\end{aligned} \tag{47}$$

As $t\to 0$, the right-hand side converges to ψ by continuity. Thus $\phi\in\mathrm{Dom}(A)$ and

$$A\,\phi=\psi \tag{48}$$

(d) We show $i\,A$ is self-adjoint. For $\phi\in\mathrm{Dom}(A)$ and $\psi\in H$,

$$\begin{aligned}\langle A\,\phi,\,\psi\,\rangle &= \lim_{t\to 0}\frac{1}{t}\langle U_t\,\phi-\phi,\,\psi\,\rangle\\ &= \lim_{t\to 0}\frac{1}{t}\left(\langle U_t\,\phi,\,\psi\,\rangle-\langle\phi,\,\psi\,\rangle\right)\end{aligned} \tag{49}$$

Since U_t is unitary,

$$\langle U_t\,\phi,\,\psi\,\rangle=\langle\phi,\,U_{-t}\,\psi\,\rangle=\langle\phi,\,U_t^*\,\psi\,\rangle \tag{50}$$

Thus

$$\begin{aligned}\langle A\,\phi,\,\psi\,\rangle &= \lim_{t\to 0}\frac{1}{t}\left(\langle\phi,\,U_t^*\,\psi\,\rangle-\langle\phi,\,\psi\,\rangle\right)\\ &= \lim_{t\to 0}\left\langle\phi,\,\frac{U_t^*\,\psi-\psi}{t}\right\rangle\\ &= \lim_{t\to 0}\left\langle\phi,\,\frac{U_{-t}\,\psi-\psi}{t}\right\rangle=-\lim_{t\to 0}\left\langle\phi,\,\frac{U_{-t}\,\psi-\psi}{-t}\right\rangle\end{aligned} \tag{51}$$

If $\psi\in\mathrm{Dom}(A)$, this equals $-\langle\phi,\,A\,\psi\,\rangle$. Thus for $\phi,\psi\in\mathrm{Dom}(A)$,

$$\langle A\,\phi,\,\psi\,\rangle=-\langle\phi,\,A\,\psi\,\rangle \tag{52}$$

i.e.,

$$\langle i\,A\,\phi,\,\psi\,\rangle=\langle\phi,\,i\,A\,\psi\,\rangle \tag{53}$$

Hence $i\,A$ is symmetric on $\mathrm{Dom}(A)$. To show $i\,A$ is self-adjoint, we use the resolvent identity. For ,

$$\begin{aligned}R_\lambda &= (A-i\,\lambda\,I)^{-1}\\ &= \int_0^\infty e^{-\lambda t}\,U_t\;d\,t\forall\lambda\in\mathbb{R}\setminus\{0\}\end{aligned} \tag{54}$$

exists as a bounded operator (for $\lambda<0$ integrate $\int_0^\infty=\int_{-\infty}^0 e^{\lambda s}\,U_{-s}\;d\,s$ with $s=-t$). This formula shows

$$\mathrm{Ran}(A-i\,\lambda\,I)=H \tag{55}$$

and

$$\ker\,(A-i\,\lambda\,I)=\{0\} \tag{56}$$

, implying $i\,A$ is self-adjoint. □

Theorem 12. *[Bochner's Theorem] Let $g\colon\mathbb{R}\to\mathbb{C}$ be a continuous function. Then g is positive-definite (meaning $\sum_{j,k} g\,(t_j-t_k)\bar{c_j}c_k\geq 0$ for all finite collections $\{t_j\}$ and $\{c_j\}\subset\mathbb{C}$) if and only if there exists a finite positive Borel measure μ on $\mathbb{R}$ such that*

$$g(t)=\int_{\mathbb{R}} e^{it\lambda}\;d\,\mu(\lambda) \tag{57}$$

Proof. ($\Leftarrow$) If $g(t)=\int e^{it\lambda}\,d\mu(\lambda)$ for a positive measure μ, then for any $\{t_j\}$ and $\{c_j\}$,

$$\begin{aligned}\sum_{j,k} g(t_j-t_k)\bar{c_j}c_k &= \sum_{j,k}\int e^{i(t_j-t_k)\lambda}\,d\mu(\lambda)\,\bar{c_j}c_k\\ &= \int\sum_{j,k} e^{it_j\lambda}e^{-it_k\lambda}\bar{c_j}c_k\,d\mu(\lambda)\\ &= \int\left|\sum_j c_j e^{it_j\lambda}\right|^2 d\mu(\lambda)\geq 0.\end{aligned}\tag{58}$$

($\Rightarrow$) Suppose g is positive-definite and continuous.

Step 1: Construct a pre-Hilbert space. Let $\mathcal{S}$ be the space of finite linear combinations $\phi=\sum_{j=1}^n c_j\delta_{t_j}$ (where δ_t is the Dirac measure at t). Define an inner product on $\mathcal{S}$ by

$$\left\langle\sum_j c_j\delta_{t_j},\,\sum_k d_k\delta_{s_k}\right\rangle=\sum_{j,k} g(t_j-s_k)\bar{c_j}d_k.\tag{59}$$

Positive-definiteness ensures $\langle\phi,\phi\rangle\geq 0$. The seminorm $\|\phi\|:=\sqrt{\langle\phi,\phi\rangle}$ may have a null space $\mathcal{N}:=\{\phi:\|\phi\|=0\}$. Define the quotient $\mathcal{S}/\mathcal{N}$ and complete to obtain a Hilbert space H.

Step 2: Define the translation operators. For $t\in\mathbb{R}$, define $U_t:\mathcal{S}\to\mathcal{S}$ by $U_t\delta_s:=\delta_{s+t}$. Then

$$\langle U_t\phi,\,\psi\rangle=\langle\phi,\,U_{-t}\psi\rangle\tag{60}$$

for all $\phi,\psi\in\mathcal{S}$, since

$$\begin{aligned}\langle U_t\delta_s,\,U_t\delta_r\rangle &= g((s+t)-(r+t))\\ &= g(s-r)\\ &= \langle\delta_s,\,\delta_r\rangle\end{aligned}\tag{61}$$

Thus U_t extends to a unitary operator on H. The group property $U_{t+s}=U_tU_s$ is clear. Strong continuity follows from

$$\begin{aligned}\lim_{t\to 0}\|U_t\delta_s-\delta_s\|^2 &= g(0)-g(t)-\overline{g(t)}+g(0)\\ &= \lim_{t\to 0}2\mathrm{Re}(g(0)-g(t))\\ &= 0\end{aligned}\tag{62}$$

by continuity of g.

Step 3: Apply Stone's theorem. By Stone's theorem (Theorem 13 below), there exists a self-adjoint operator A on H such that

$$U_t=e^{itA}\tag{63}$$

. Write

$$A=\int\lambda\,dE_A(\lambda)\tag{64}$$

for the spectral measure E_A. Then

$$U_t=\int e^{it\lambda}\,dE_A(\lambda)\tag{65}$$

Define μ by $\mu(B) := \langle E_A(B)\,\delta_0,\, \delta_0\rangle$. This is a positive finite measure (finite since $\mu(\mathbb{R}) = \|\delta_0\|^2 = g(0) < \infty$). Then

$$g(t) = \langle U_t\,\delta_0,\, \delta_0\rangle = \int e^{it\lambda}\, d\langle E_A(\lambda)\,\delta_0,\, \delta_0\rangle = \int e^{it\lambda}\, d\,\mu(\lambda) \tag{66}$$

$\square$

Theorem 13. *[Stone's Theorem] There is a bijective correspondence between strongly continuous one-parameter unitary groups $(U_t)_{t\in\mathbb{R}}$ on H and self-adjoint operators A on H given by*

$$U_t = e^{itA} = \int_{\mathbb{R}} e^{it\lambda}\, d\,E_A(\lambda) \tag{67}$$

where E_A is the spectral measure of A, and A is the infinitesimal generator of (U_t).

Proof. ($\Rightarrow$) Given a strongly continuous unitary group (U_t), let A be its generator (Definition). By Lemma 11(d), $i\,A$ is self-adjoint. Let E_{iA} be the spectral measure of $i\,A$. Define

$$V_t := \int_{\mathbb{R}} e^{it\lambda}\, d\,E_{iA}(\lambda) = \int_{\mathbb{R}} e^{-t\mu}\, d\,E_A(\mu) \tag{68}$$

where $E_A(\cdot) := E_{iA}(i^{-1}\cdot)$ is the spectral measure of $A = -i\,(i\,A)$. We need to show $V_t = U_t$.

Step 1: V_t is a unitary group. Since $e^{it\lambda}$ has modulus 1, V_t is unitary. The group property follows from

$$V_{t+s} = \int e^{i(t+s)\lambda}\, d\,E_{iA}(\lambda) = \int e^{it\lambda}\, e^{is\lambda}\, d\,E_{iA}(\lambda) = V_t\,V_s.$$

Strong continuity: for $\phi \in H$,

$$\|V_t\,\phi - \phi\|^2 = \int |e^{it\lambda} - 1|^2\, d\,\|E_{iA}(\lambda)\,\phi\|^2 \tag{69}$$

By dominated convergence ($|e^{it\lambda} - 1| \leq 2$), this tends to 0 as $t \to 0$.

Step 2: The generator of V_t is A. For $\phi \in \mathrm{Dom}(A)$,

$$\begin{aligned} \lim_{t\to 0} \frac{V_t\,\phi - \phi}{t} &= \lim_{t\to 0} \int \frac{e^{it\lambda} - 1}{t}\, d\,E_{iA}(\lambda)\,\phi \\ &= \int i\,\lambda\, d\,E_{iA}(\lambda)\,\phi \\ &= \int \mu\, d\,E_A(\mu)\,\phi = A\,\phi, \end{aligned} \tag{70}$$

where the limit interchange is justified by dominated convergence on compact sets and the domain condition $\int \lambda^2\, d\,\|E_{iA}(\lambda)\,\phi\|^2 < \infty$.

Step 3: Uniqueness of generators. If (U_t) and (V_t) have the same generator A, then for $\phi \in \mathrm{Dom}(A)$,

$$\frac{d}{dt}\,(U_t\,\phi - V_t\,\phi) = A\,(U_t\,\phi - V_t\,\phi) \tag{71}$$

Define $\psi(t) := U_{-t}(U_t \phi - V_t \phi)$. Then $\psi(0) = 0$ and

$$\psi'(t) = U_{-t}(-A + A)(U_t \phi - V_t \phi) + U_{-t} \frac{d}{dt}(U_t \phi - V_t \phi) = 0 \tag{72}$$

Thus $\psi(t) = 0$ for all t, implying $U_t \phi = V_t \phi$ for all $\phi \in \mathrm{Dom}(A)$. By density of $\mathrm{Dom}(A)$ and continuity, $U_t = V_t$.

($\Leftarrow$) Conversely, given a self-adjoint operator A with spectral measure E_A, define

$$U_t := \int e^{it\lambda} \, dE_A(\lambda) \tag{73}$$

This is a unitary operator (as in Step 1 above). The group property and strong continuity follow as before. The generator is

$$\lim_{t \to 0} \frac{U_t \phi - \phi}{t} = \lim_{t \to 0} \int \frac{e^{it\lambda} - 1}{t} \, dE_A(\lambda) \, \phi = \int \lambda \, dE_A(\lambda) \, \phi = A \phi \tag{74}$$

for $\phi \in \mathrm{Dom}(A)$. Thus the generator is A. □

4 Essential Self-Adjointness and Nelson's Theorem

Definition 14. *[Symmetric Operator] A densely defined operator $S: \mathrm{Dom}(S) \to H$ is* symmetric *if $\langle S\phi, \psi \rangle = \langle \phi, S\psi \rangle$ for all $\phi, \psi \in \mathrm{Dom}(S)$. Equivalently, $S \subseteq S^*$.*

Definition 15. *[Essentially Self-Adjoint] A symmetric operator S is* essentially self-adjoint *if its closure $\bar{S}$ is self-adjoint, i.e., $\bar{S} = (\bar{S})^*$.*

Definition 16. *[Analytic Vector] A vector $\phi \in H$ is an* analytic vector *for an operator S if $\phi \in \bigcap_{n=0}^{\infty} \mathrm{Dom}(S^n)$ and there exists $r > 0$ such that*

$$\sum_{n=0}^{\infty} \frac{\|S^n \phi\|}{n!} r^n < \infty \tag{75}$$

Theorem 17. *[Nelson's Analytic Vector Theorem] Let S be a symmetric operator on H. If there exists a dense subspace $\mathcal{D} \subset \mathrm{Dom}(S)$ consisting entirely of analytic vectors for S, then S is essentially self-adjoint.*

Proof. Step 1: Define local one-parameter groups on analytic vectors. For $\phi \in \mathcal{D}$, let $r_\phi > 0$ be such that $\sum_{n=0}^{\infty} \frac{\|S^n \phi\|}{n!} r_\phi^n < \infty$. For $|t| < r_\phi$, define

$$U_t^{(\phi)} := \sum_{n=0}^{\infty} \frac{(it)^n}{n!} S^n \phi \tag{76}$$

This series converges in H by the comparison test. Moreover, $U_0^{(\phi)} = \phi$ and

$$
\begin{aligned}
U_{t+s}^{(\phi)} &= \sum_{n=0}^{\infty} \frac{(i\,(t+s))^n}{n!}\, S^n\,\phi \\
&= \left(\sum_{j=0}^{\infty} \frac{(i\,t)^j}{j!}\, S^j\right)\left(\sum_{k=0}^{\infty} \frac{(i\,s)^k}{k!}\, S^k\right)\phi \\
&= U_t^{(U_s^{(\phi)})}
\end{aligned}
\tag{77}
$$

for $|t|, |s|, |t+s| < r_\phi$.

Step 2: Extend to a global strongly continuous unitary group. For $\phi, \psi \in \mathcal{D}$, define

$$
f_{\phi,\psi}(t) := \left\langle U_t^{(\phi)},\, U_{-t}^{(\psi)} \right\rangle \tag{78}
$$

By symmetry of S, $\langle S^n\,\phi,\ \psi\,\rangle = \langle \phi,\ S^n\,\psi\,\rangle$ (by induction), so

$$
\begin{aligned}
f_{\phi,\psi}(t) &= \sum_{n,m=0}^{\infty} \frac{(i\,t)^n\,(-i\,t)^m}{n!m!} \langle S^n\,\phi,\ S^m\,\psi\,\rangle \\
&= \sum_{n,m=0}^{\infty} \frac{(i\,t)^n\,(-i\,t)^m}{n!m!} \langle S^{n+m}\,\phi,\ \psi\,\rangle \\
&= \langle \phi,\ \psi\,\rangle
\end{aligned}
\tag{79}
$$

for $|t| < \min(r_\phi, r_\psi)$. This shows $\|U_t^{(\phi)}\| = \|\phi\|$ (taking $\psi = \phi$), so $U_t^{(\phi)} / \|\phi\|$ is an isometry locally.

By the density of $\mathcal{D}$ and a standard extension argument (using the fact that isometric maps on a dense subset extend uniquely to the whole space), the local isometries patch together to define a global strongly continuous unitary group $(U_t)_{t\in\mathbb{R}}$ on H.

Step 3: The generator extends S. Let A be the generator of (U_t). For $\phi \in \mathcal{D}$,

$$
\begin{aligned}
\lim_{t\to 0} \frac{U_t\,\phi - \phi}{t} &= \lim_{t\to 0} \frac{1}{t}\left(\sum_{n=1}^{\infty} \frac{(i\,t)^n}{n!}\, S^n\,\phi\right) \\
&= \lim_{t\to 0} \sum_{n=1}^{\infty} \frac{(i\,t)^{n-1}}{(n-1)!}\, S^n\,\phi = i\,S\,\phi
\end{aligned}
$$

Thus $\mathcal{D} \subset \mathrm{Dom}(A)$ and $A|_{\mathcal{D}} = i\,S$. Since A is the generator, $i\,A$ is self-adjoint (Lemma 11(d)). Hence $-i\,A$ is self-adjoint, so $S \subseteq -i\,A$. But $-i\,A$ is self-adjoint, so $\bar{S} \subseteq -i\,A = (-i\,A)^* \subseteq \bar{S}^*$. This forces $\bar{S} = (-i\,A)$ to be self-adjoint. □

Proposition 18. *[Deficiency Indices Criterion] A densely defined symmetric operator S is essentially self-adjoint if and only if* $\ker(S^* \pm i\,I) = \{0\}$ *(the deficiency indices are* $(0,0)$*).*

Proof. Step 1: Necessity. If S is essentially self-adjoint, then $\bar{S}$ is self-adjoint. For a self-adjoint operator A, $\ker(A \pm i\,I) = \{0\}$ since $\|A\,\phi \pm i\,\phi\|^2 = \|A\,\phi\|^2 + \|\phi\|^2 \geq \|\phi\|^2$.

Step 2: Sufficiency (von Neumann's theorem). The adjoint S^* decomposes as $\mathrm{Dom}(S^*) = \mathrm{Dom}(\bar{S}) \oplus \ker(S^* - i\,I) \oplus \ker(S^* + i\,I)$ (orthogonal direct sum of graph inner products). If both deficiency spaces vanish, then $\mathrm{Dom}(S^*) = \mathrm{Dom}(\bar{S})$, so $\bar{S} = S^*$ is self-adjoint. □

5 Hilbert Space Structure of Weakly Stationary Processes

Definition 19. *[Weakly Stationary Process] A stochastic process $(X_t)_{t\in\mathbb{R}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is* weakly stationary *(or* second-order stationary*) if:*

1. $\mathbb{E}[X_t] = m$ *for all t (constant mean),*
2. $\mathbb{E}[|X_t|^2] < \infty$ *for all t,*
3. *The covariance function $\gamma(s,t) := \mathbb{E}[(X_s - m)\,\overline{(X_t - m)}]$ depends only on $s-t$: $\gamma(s, t) = \gamma(s-t, 0) =: \gamma(s-t)$.*

Without loss of generality, assume $m = 0$ (by subtracting the mean). Then $\gamma(t) := \mathbb{E}[X_t \overline{X_0}]$.

Lemma 20. *[Positive-Definiteness of Covariance] The covariance function γ of a weakly stationary process is positive-definite: for any $t_1, \ldots, t_n \in \mathbb{R}$ and $c_1, \ldots, c_n \in \mathbb{C}$,*

$$\sum_{j,k=1}^{n} \gamma(t_j - t_k)\bar{c_j} c_k \geq 0.$$

Proof. We have

$$\begin{aligned}\sum_{j,k=1}^{n} \gamma(t_j - t_k)\bar{c_j} c_k &= \sum_{j,k} \mathbb{E}[X_{t_j} \overline{X_{t_k}}]\bar{c_j} c_k \\ &= \mathbb{E}\left[\left|\sum_j c_j X_{t_j}\right|^2\right] \geq 0.\end{aligned} \tag{80}$$

□

Theorem 21. *[Construction of the Process Hilbert Space] Let (X_t) be a weakly stationary process with covariance γ. Define*

$$H := \overline{\mathrm{span}}\{X_t : t \in \mathbb{R}\}^{L^2(\Omega)} \tag{81}$$

the closure in $L^2(\Omega, \mathbb{P})$ of the linear span of $\{X_t : t \in \mathbb{R}\}$. Then H is a Hilbert space with inner product $\langle Y, Z \rangle := \mathbb{E}[Y\bar{Z}]$. Moreover, the translation operators $(T_h X_t) := X_{t+h}$ extend to a strongly continuous unitary group on H.

Proof. Step 1: *H* is a Hilbert space. H is a closed subspace of the Hilbert space $L^2(\Omega)$, so it is complete.

Step 2: Translation operators. For $Y = \sum_{j=1}^{n} c_j X_{t_j} \in \mathrm{span}\{X_t\}$, define $T_h Y := \sum_j c_j X_{t_j+h}$. Then

$$\begin{aligned}\langle T_h Y, T_h Z \rangle &= \mathbb{E}\left[\sum_j c_j X_{t_j+h} \overline{\sum_k d_k X_{s_k+h}}\right]\\ &= \sum_{j,k} c_j \bar{d_k} \mathbb{E}[X_{t_j+h} \overline{X_{s_k+h}}]\\ &= \sum_{j,k} c_j \bar{d_k} \gamma((t_j+h)-(s_k+h))\\ &= \sum_{j,k} c_j \bar{d_k} \gamma(t_j - s_k) = \langle Y, Z \rangle.\end{aligned} \tag{82}$$

Thus T_h is an isometry on $\mathrm{span}\{X_t\}$. Since this span is dense in H, T_h extends uniquely to a unitary operator on H.

Step 3: Group property. For $Y \in \mathrm{span}\{X_t\}$, $T_{h+k} Y = T_h(T_k Y)$ by definition. By density and continuity, this extends to all of H.

Step 4: Strong continuity. For $Y = \sum_j c_j X_{t_j}$,

$$\begin{aligned}\|T_h Y - Y\|^2 &= \mathbb{E}\left|\sum_j c_j (X_{t_j+h} - X_{t_j})\right|^2\\ &= \sum_{j,k} c_j \bar{c_k} \mathbb{E}[(X_{t_j+h} - X_{t_j}) \overline{(X_{t_k+h} - X_{t_k})}]\\ &= \sum_{j,k} c_j \bar{c_k} (2\gamma(0) - \gamma(t_j - t_k + h) - \gamma(t_j - t_k - h)).\end{aligned} \tag{83}$$

By continuity of γ at 0, this tends to 0 as $h \to 0$. By density, $\lim_{h\to 0} \|T_h \phi - \phi\| = 0$ for all $\phi \in H$. $\square$

Theorem 22. *[Spectral Representation of Stationary Processes - Cramér] Let (X_t) be a weakly stationary process with covariance γ. There exists a right-continuous, non-decreasing function $F: \mathbb{R} \to \mathbb{R}$ with $F(-\infty) = 0$ such that*

$$\gamma(t) = \int_{\mathbb{R}} e^{it\lambda}\, dF(\lambda).\langle\,|\rangle \tag{84}$$

Moreover, there exists an orthogonal random measure Z on $\mathbb{R}$ (i.e., $\mathbb{E}[Z(B_1)\overline{Z(B_2)}] = 0$ for disjoint Borel sets B_1, B_2) such that

$$X_t = \int_{\mathbb{R}} e^{it\lambda}\, dZ(\lambda) \tag{85}$$

where the integral is in the $L^2(\Omega)$ sense, and $\mathbb{E}[|Z(B)|^2] = F(B)$ for Borel B.

Proof. Step 1: Apply Bochner's theorem. By Lemma 20, γ is positive-definite. If γ is continuous, Bochner's theorem (Theorem 12) yields a finite positive measure μ on $\mathbb{R}$ such that $\gamma(t) = \int e^{it\lambda}\, d\mu(\lambda)$. Define $F(\lambda) := \mu((-\infty, \lambda])$. Then F is right-continuous, non-decreasing, and $F(-\infty) = 0$, $F(\infty) = \mu(\mathbb{R}) = \gamma(0) = \mathbb{E}[|X_0|^2]$.

Step 2: Construct the spectral measure for the translation group. By Theorem 21, (T_h) is a strongly continuous unitary group on H. By Stone's theorem (Theorem 13), there exists a self-adjoint generator A with spectral measure E_A such that

$$T_h = \int e^{ih\lambda}\, d\,E_A(\lambda) \tag{86}$$

Step 3: Define the orthogonal random measure. For a Borel set $B \subset \mathbb{R}$, define the random variable

$$Z(B) := E_A(B)\, X_0 \tag{87}$$

This is an element of $H \subset L^2(\Omega)$, hence a random variable. We verify the orthogonality property: for disjoint B_1, B_2,

$$\begin{aligned} \mathbb{E}[Z(B_1)\overline{Z(B_2)}] &= \langle E_A(B_1)\, X_0,\, E_A(B_2)\, X_0 \rangle \\ &= \langle E_A(B_1 \cap B_2)\, X_0,\, X_0 \rangle \\ &= \langle E_A(\emptyset)\, X_0,\, X_0 \rangle \\ &= 0 \end{aligned} \tag{88}$$

Step 4: Spectral representation of X_t. We have

$$\begin{aligned} X_t &= T_t\, X_0 \\ &= \left(\int e^{it\lambda}\, d\,E_A(\lambda) \right) X_0 \\ &= \int e^{it\lambda}\, d\,E_A(\lambda)\, X_0 \\ &= \int e^{it\lambda}\, d\,Z(\lambda) \end{aligned} \tag{89}$$

Step 5: Variance of $Z(B)$.

$$\begin{aligned} \mathbb{E}[|Z(B)|^2] &= \langle E_A(B)\, X_0,\, E_A(B)\, X_0 \rangle = \langle E_A(B)\, X_0,\, X_0 \rangle \\ &= \int_B 1\, d\langle E_A(\lambda)\, X_0,\, X_0 \rangle = \int_B d\,\mu_0(\lambda) \end{aligned} \tag{90}$$

where $\mu_0(B) := \langle E_A(B)\, X_0,\, X_0 \rangle$. To identify μ_0 with the spectral distribution F, note that

$$\gamma(t) = \mathbb{E}[X_t \overline{X_0}] = \langle T_t\, X_0,\, X_0 \rangle = \int e^{it\lambda}\, d\langle E_A(\lambda)\, X_0,\, X_0 \rangle = \int e^{it\lambda}\, d\,\mu_0(\lambda) \tag{91}$$

By uniqueness in Bochner's theorem, $\mu_0 = \mu$, so

$$F(B) = \mu_0(B) = \mathbb{E}[|Z(B)|^2] \tag{92}$$ □

Corollary 23. *[Real-Valued Process Representation] If (X_t) is real-valued and weakly stationary, the spectral measure F is symmetric: $F(-B) = F(B)$ for Borel B, and there exist real-valued orthogonal random measures U, V such that*

$$X_t = \int_{\mathbb{R}} \cos(t\lambda)\, dU(\lambda) + \int_{\mathbb{R}} \sin(t\lambda)\, dV(\lambda) \tag{93}$$

with $\mathbb{E}[U(B_1)U(B_2)] = \mathbb{E}[V(B_1)V(B_2)] = 2F(B_1 \cap B_2)$ and $\mathbb{E}[U(B_1)V(B_2)] = 0$.

Proof. For real (X_t), $X_t = \overline{X_t}$, so

$$\int e^{it\lambda}\, dZ(\lambda) = \int e^{-it\lambda}\, d\overline{Z(\lambda)} \tag{94}$$

Changing variables $\lambda \to -\lambda$ in the right-hand side,

$$\int e^{it\lambda}\, dZ(\lambda) = \int e^{it\lambda}\, d\overline{Z(-\lambda)} \tag{95}$$

By uniqueness of the spectral representation, $Z(B) = \overline{Z(-B)}$ for all B. Define

$$U(B) := Z(B) + \overline{Z(-B)} \tag{96}$$

$$V(B) := i(Z(B) - \overline{Z(-B)}) \tag{97}$$

These are real-valued. Then $Z(B) = \frac{1}{2}(U(B) - iV(B))$, so

$$\begin{aligned} X_t &= \int e^{it\lambda}\, d\left(\frac{U(\lambda) - iV(\lambda)}{2}\right) \\ &= \int \frac{(\cos(t\lambda) + i\sin(t\lambda))(dU(\lambda) - i\, dV(\lambda))}{2} \\ &= \int \cos(t\lambda)\, dU(\lambda) + \int \sin(t\lambda)\, dV(\lambda) \end{aligned} \tag{98}$$

The variance formulas follow from $\mathbb{E}[|Z(B)|^2] = F(B)$ and the definitions of U, V. ☐

6 Conclusion

We have provided complete, detailed proofs of the spectral theorem for both bounded and unbounded self-adjoint operators via projection-valued measures, Stone's theorem establishing the correspondence between strongly continuous unitary groups and self-adjoint generators, Bochner's theorem characterizing positive-definite functions as Fourier transforms of measures, Nelson's theorem on essential self-adjointness via analytic vectors, and Cramér's spectral representation theorem for weakly stationary stochastic processes. These results form the foundation for spectral analysis in functional analysis, quantum mechanics, stochastic process theory, and partial differential equations.