The Random Wave Process

BY STEPHEN CROWLEY

January 18, 2025

The Bessel function of the first kind of order zero, derived from the Fourier transform of a uniform measure on the unit circle, plays a crucial role in describing random plane waves and has significant implications for Berry's random wave conjecture in quantum chaos theory.

1 Bessel Function Definition

The Bessel function of the first kind, denoted as $J_\alpha(x)$, is a solution to Bessel's differential equation:

$$x^2 \frac{d^2 y}{d x^2} + x \frac{d y}{d x} + (x^2 - \alpha^2) y = 0 \tag{1}$$

where α is the order of the Bessel function. For integer or positive α, $J_\alpha(x)$ is finite at the origin ($x = 0$), while for negative non-integer α, it diverges as x approaches zero. The function can be expressed as an infinite series:

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m + \alpha + 1)} \left(\frac{x}{2}\right)^{2m+\alpha} \tag{2}$$

where Γ is the gamma function.

2 Gaussian Process Kernel Derivation

The Gaussian process kernel $J_0(|t - s|)$ for the random wave model can be derived through two equivalent approaches. In one dimension:

$$J_0(|t - s|) = \int_{-1}^{1} e^{i\lambda(t-s)} \frac{1}{\pi\sqrt{1 - \lambda^2}} d\lambda \tag{3}$$

And in two dimensions:

$$J_0(|\lambda|) = \int_0^{2\pi} e^{i(\lambda_1 \cos\theta + \lambda_2 \sin\theta)} \frac{d\theta}{2\pi} \tag{4}$$

where $|\lambda| = \sqrt{\lambda_1^2 + \lambda_2^2}$.

3 Connection to Covariance Functions

The J_0 covariance function belongs to the family of radial basis functions, which are isotropic and depend only on the distance between points. This allows for uniform behavior in all directions and connects to broader classes of Gaussian processes through their spectral representations.

4 Stationarity and Isotropy in Random Plane Waves

Random plane waves exhibit stationarity:

$$\mathbb{E}\left[X(t)\,X(s)\right] = \mathbb{E}\left[X\,(t+h)\,X\,(s+h)\right] \tag{5}$$

and isotropy:

$$\mathbb{E}\left[X(t)\,X(s)\right] = J_0(|t-s|) \tag{6}$$

Key properties include:

1. Uniform energy distribution
2. Scale-invariance
3. Ergodicity

5 Spectral Analysis of J_0 Kernel

The spectral analysis reveals:

- Discrete spectrum over $[0, \infty)$
- Unknown eigenfunctions (no closed form, yet anyway)
- Slowly decaying eigenvalues
- Dimensionality dependence
- Orthonormal basis properties

The kernel's properties fundamentally shape the behavior of the random plane wave model and present ongoing challenges in spectral theory and stochastic processes.