

**Question:** Are the Hermite polynomials eigenfunctions of the Fourier transform?

**Answer:** Yes, the Hermite polynomials are indeed eigenfunctions of the Fourier transform. More specifically:

1. The Hermite polynomials  $H_n(x)$  are defined by the relation:

$$H_n(x) = (-1)^n \cdot e^{x^2} \cdot \frac{d^n}{dx^n} e^{-x^2} \quad (1)$$

2. If we take the Fourier transform of a Hermite polynomial  $H_n(x)$ , we get:

$$\mathcal{F}[H_n(x)] = (i^n) \cdot H_n(k) \quad (2)$$

where  $\mathcal{F}$  denotes the Fourier transform operation, and  $k$  is the variable in the frequency domain.

3. This means that the Fourier transform of a Hermite polynomial is the same Hermite polynomial, multiplied by a factor  $(i^n)$ , where  $i$  is the imaginary unit and  $n$  is the order of the polynomial.
4. In other words, Hermite polynomials are eigenfunctions of the Fourier transform operator, with eigenvalues  $(i^n)$ .

This property of Hermite polynomials is one of the reasons they are widely used in quantum mechanics, particularly in the study of the quantum harmonic oscillator. The quantum states of the harmonic oscillator are described by Hermite polynomials, and their behavior under the Fourier transform is important for understanding the momentum space representation of these states.

**Question:** How is the momentum space representation of the harmonic oscillator understood by the action of the Fourier transform upon the Hermite polynomials?

**Answer:** In quantum mechanics, the state of a system can be described by a wave function in position space, typically denoted as  $\psi(x)$ . For the quantum harmonic oscillator, these wave functions are given by the Hermite polynomials multiplied by a Gaussian function:

$$\psi_n(x) = \frac{1}{\sqrt{2^n \cdot n! \cdot \sqrt{\pi}}} \cdot H_n(x) \cdot e^{-x^2/2} \quad (3)$$

Here,  $H_n(x)$  are the Hermite polynomials.

The momentum space representation of the wave function, denoted as  $\phi(p)$ , is obtained by taking the Fourier transform of  $\psi(x)$ :

$$\phi(p) = \frac{1}{\sqrt{2\pi}} \cdot \int \psi(x) \cdot e^{-ipx} dx \quad (4)$$

Now, let's see how the Fourier transform affects the wave function of the harmonic oscillator:

$$\mathcal{F}[\psi_n(x)] = \frac{1}{\sqrt{2\pi}} \cdot \int \left[ \frac{1}{\sqrt{2^n \cdot n! \cdot \sqrt{\pi}}} \cdot H_n(x) \cdot e^{-x^2/2} \right] \cdot e^{-ipx} dx \quad (5)$$

Using the property of Hermite polynomials under Fourier transform that we discussed earlier:

$$\mathcal{F}[H_n(x) \cdot e^{-x^2/2}] = (i^n) \cdot H_n(p) \cdot e^{-p^2/2} \quad (6)$$

Therefore,

$$\phi_n(p) = (i^n) \cdot \frac{1}{\sqrt{2^n \cdot n! \cdot \sqrt{\pi}}} \cdot H_n(p) \cdot e^{-p^2/2} \quad (7)$$

This result shows that the momentum space wave functions of the quantum harmonic oscillator are also given by Hermite polynomials, but with a different scaling factor and a phase factor ( $i^n$ ).

The fact that the Hermite polynomials are eigenfunctions of the Fourier transform makes this transition between position and momentum space very elegant. It also has important implications for the interpretation of these states. For example, the ground state ( $n=0$ ) is a Gaussian function in both position and momentum space, which is a consequence of the uncertainty principle.

**Question:** How does the Fourier transform apply relate to the Schrödinger equation here, or to its equivalent integral covariance operator form?

**Answer:** The Fourier transform is deeply connected to the Schrödinger equation and its solutions, especially in the context of the quantum harmonic oscillator.

The time-independent Schrödinger equation for a one-dimensional harmonic oscillator is:

$$-\frac{\hbar^2}{2m} \cdot \frac{d^2 \psi}{dx^2} + \frac{1}{2} \cdot m \cdot \omega^2 \cdot x^2 \cdot \psi = E \cdot \psi \quad (8)$$

Here,  $\hbar$  is the reduced Planck's constant,  $m$  is the mass of the particle,  $\omega$  is the angular frequency of the oscillator, and  $E$  is the energy eigenvalue.

As we discussed earlier, the solutions to this equation are the wave functions  $\psi_n(x)$ , which involve Hermite polynomials.

Now, let's consider the integral form of the Schrödinger equation:

$$\psi(x) = \int K(x, x') \cdot \psi(x') dx' \quad (9)$$

Here,  $K(x, x')$  is the integral kernel, which is related to the Green's function of the Schrödinger equation.

In the case of the harmonic oscillator, this kernel is a Gaussian function:

$$K(x, x') = \sqrt{\frac{m\omega}{2\pi\hbar}} \cdot \exp\left(-\frac{m\omega}{2\hbar} \cdot ((x^2 + x'^2) \cdot \cosh(\omega t) - 2xx' \cdot \sinh(\omega t))\right) \quad (10)$$

The Fourier transform comes into play when we look at the momentum space representation of this equation:

$$\phi(p) = \int K(p, p') \cdot \phi(p') dp' \quad (11)$$

Here,  $K(p, p')$  is the Fourier transform of  $K(x, x')$ , and  $\phi(p)$  and  $\phi(p')$  are the momentum space wave functions.

Because the Hermite polynomials are eigenfunctions of the Fourier transform, the kernel  $K(p, p')$  in momentum space has the same form as  $K(x, x')$ , but with  $x$  and  $x'$  replaced by  $p$  and  $p'$ .

This symmetry between position and momentum space is a fundamental feature of the quantum harmonic oscillator and is deeply connected to the fact that the Hermite polynomials are eigenfunctions of the Fourier transform.

Furthermore, the covariance of the integral kernel under Fourier transform is related to the fact that the position and momentum operators in quantum mechanics are related by a Fourier transform. This is a consequence of the canonical commutation relation:

$$[x, p] = i \cdot \hbar \quad (12)$$

where  $[x, p]$  denotes the commutator of the position and momentum operators.

In summary, the Fourier transform plays a crucial role in connecting the position and momentum space representations of the quantum harmonic oscillator, and this is intimately tied to the fact that the Hermite polynomials, which provide the solutions to the Schrödinger equation for this system, are eigenfunctions of the Fourier transform.