Injectively Time-Changed Stationary Processes: A Spectral Analysis

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1 Introduction

This analysis concerns injectively time-changed stationary processes, which arise from spectral representations involving a warping function $\theta(t)$ applied to the oscillatory kernel.

Definition 1. *An* injectively time-changed stationary process *is a stochastic process* $\{X(t)\}_{t\in\mathbb{R}}$ *admitting the spectral representation*

$$X(t)=\int_{-1}^{1} e^{i\lambda\theta(t)}\, dZ(\lambda) \tag{1}$$

where $\theta:\mathbb{R}\to\mathbb{R}$ *is strictly increasing,* $\theta\in C^1(\mathbb{R})$, *and* $\{Z(\lambda)\}_{\lambda\in[-1,1]}$ *is an orthogonal increment process with* $E[|dZ(\lambda)|^2]=F(d\lambda)$ *for some finite measure* F *on* $[-1,1]$.

2 Gain Function Representation

Proposition 2. *[Evolutionary Spectral Form] The process $X(t)$ admits the evolutionary spectral representation*

$$X(t) = \int_{-1}^{1} A(t, \lambda)\, e^{i\lambda t}\, d\,Z(\lambda) \tag{2}$$

where the gain function is

$$A(t, \lambda) = e^{i\lambda(\theta(t) - t)} \tag{3}$$

Proof. Direct substitution yields:

$$X(t) = \int_{-1}^{1} e^{i\lambda\theta(t)}\, d\,Z(\lambda) \tag{4}$$

$$= \int_{-1}^{1} e^{i\lambda(\theta(t) - t)}\, e^{i\lambda t}\, d\,Z(\lambda) \tag{5}$$

$$= \int_{-1}^{1} A(t, \lambda)\, e^{i\lambda t}\, d\,Z(\lambda) \tag{6}$$

□

3 Fundamental Properties

Theorem 3. *[Spectral Characteristics] Let $X(t)$ be an injectively time-changed stationary process. Then:*

1. *$X(t)$ is well-defined for all $t \in \mathbb{R}$*
2. *$E[|X(t)|^2] = \int_{-1}^{1} F\,(d\,\lambda) < \infty$*
3. *The covariance function satisfies*

$$Cov(X(s), X(t)) = \int_{-1}^{1} e^{i\lambda(\theta(s) - \theta(t))}\, F\,(d\,\lambda) \tag{7}$$

Proof. (1) Since θ is strictly increasing and continuous, $\theta(t)$ is well-defined for all t. The stochastic integral converges in L^2 by the isometry property:

$$E[|X(t)|^2] = E\left[\left|\int_{-1}^{1} e^{i\lambda\theta(t)}\, d\,Z(\lambda)\right|^2\right] \tag{8}$$

$$= \int_{-1}^{1} |e^{i\lambda\theta(t)}|^2\, F\,(d\,\lambda) = \int_{-1}^{1} F\,(d\,\lambda) < \infty \tag{9}$$

(2) Follows immediately from (1).

(3) By orthogonality of the random measure increments:

$$\operatorname{Cov}(X(s),X(t)) = E\left[\int_{-1}^{1} e^{i\lambda\theta(s)}\,dZ(\lambda)\cdot\overline{\int_{-1}^{1} e^{i\mu\theta(t)}\,dZ(\mu)}\right] \tag{10}$$

$$= \int_{-1}^{1} e^{i\lambda\theta(s)}\overline{e^{i\lambda\theta(t)}}F(d\lambda) \tag{11}$$

$$= \int_{-1}^{1} e^{i\lambda(\theta(s)-\theta(t))}\,F(d\lambda) \tag{12}$$

$\square$

Theorem 4. *[Non-Stationarity Condition] An injectively time-changed stationary process $X(t)$ is stationary if and only if $\theta(t)=t+c$ for some constant $c\in\mathbb{R}$.*

Proof. ($\Leftarrow$) If $\theta(t)=t+c$, then

$$\operatorname{Cov}(X(s),X(t)) = \int_{-1}^{1} e^{i\lambda c}\,e^{-i\lambda c}\,F(d\lambda) = \int_{-1}^{1} F(d\lambda) \tag{13}$$

which is independent of s and t, establishing stationarity.

($\Rightarrow$) Suppose $X(t)$ is stationary. Then $\operatorname{Cov}(X(s),X(t))$ depends only on $t-s$. From the covariance formula, this requires

$$\theta(s)-\theta(t) = g\,(s-t) \tag{14}$$

for some function g. Differentiating with respect to s:

$$\theta'(s) = g'\,(s-t) \tag{15}$$

Since the left side depends only on s and the right side on $s-t$, both must be constant. Thus $\theta'(t)=k$ for some constant k, implying $\theta(t)=k\,t+c$. For the covariance to depend only on the difference $s-t$, one requires $k=1$, yielding $\theta(t)=t+c$. $\square$

4 Warping Function Analysis

Definition 5. *The* warping deviation function *is* $\Delta(t):=\theta(t)-t$.

Proposition 6. *[Deviation Dynamics] Let* $\Delta(t) = \theta(t) - t$ *where* θ *is strictly increasing. Then:*

1. $\Delta'(t) = \theta'(t) - 1$
2. *The gain function becomes* $A(t, \lambda) = e^{i\lambda\Delta(t)}$
3. *The instantaneous frequency modulation is* $\lambda\,\Delta'(t)$

Proof. (1) and (2) are immediate. For (3), the phase of the spectral component at frequency λ is $\lambda\,\theta(t)$. The instantaneous frequency is

$$\frac{d}{d\,t}[\lambda\,\theta(t)] = \lambda\,\theta'(t) = \lambda\,(1 + \Delta'(t)) = \lambda + \lambda\,\Delta'(t) \tag{16}$$

The modulation relative to the base frequency λ is $\lambda\,\Delta'(t)$. □

5 Inversion Theory

Theorem 7. *[Spectral Inversion] Let* $X(t)$ *be an injectively time-changed stationary process with absolutely continuous spectral measure* $F(d\,\lambda) = f(\lambda)\,d\,\lambda$*. If* θ *is invertible with inverse* ψ*, then*

$$f(\lambda) = \lim_{T\to\infty} \frac{1}{2\,T} \int_{-T}^{T} X(\psi(u))\,e^{-i\lambda u}\,\frac{d\,u}{\psi'(u)} \tag{17}$$

Proof. Making the substitution $u = \theta(t)$, so $t = \psi(u)$ and $d\,t = \psi'(u)\,d\,u$:

$$X(\psi(u)) = \int_{-1}^{1} e^{i\mu u}\,d\,Z(\mu) \tag{18}$$

This is the spectral representation of a stationary process in the u-domain. The standard inversion formula for stationary processes gives:

$$f(\lambda) = \lim_{T\to\infty} \frac{1}{2\,T} \int_{-T}^{T} X(\psi(u))\,e^{-i\lambda u}\,\frac{d\,u}{\psi'(u)} \tag{19}$$

where the factor $\frac{1}{\psi'(u)}$ accounts for the change of measure. □

6 Band-Limited Structure

Theorem 8. *[Oscillatory Characterization] An injectively time-changed stationary process* $X(t)$ *with spectral support in* $[-1, 1]$ *exhibits oscillatory behavior in the sense of Priestley if and only if the spectral measure* F *concentrates mass away from* $\lambda = 0$*.*

Proof. The process has the representation

$$X(t) = \int_{-1}^{1} e^{i\lambda\theta(t)}\, d\,Z(\lambda) \tag{20}$$

Oscillatory behavior requires sustained periodic components. If F has significant mass at $\lambda = 0$, the corresponding component $e^{i\cdot 0\cdot\theta(t)} = 1$ contributes a non-oscillatory constant term. Conversely, if F concentrates away from zero, all spectral components $e^{i\lambda\theta(t)}$ with $\lambda \neq 0$ exhibit oscillatory behavior modulated by the time-change $\theta(t)$. □

Corollary 9. *[Band-Limited Narrow-Band Property] If F is concentrated in an interval $[\lambda_0 - \epsilon, \lambda_0 + \epsilon]$ with $\lambda_0 \neq 0$ and small $\epsilon > 0$, then $X(t)$ exhibits narrow-band oscillatory behavior around the carrier frequency λ_0.*

7 Asymptotic Analysis

Theorem 10. *[Large Deviation Asymptotics] If $\Delta(t) = \theta(t) - t$ grows without bound as $|t| \to \infty$, then the process $X(t)$ exhibits asymptotic phase decorrelation:*

$$\lim_{|t-s| \to \infty} Cov(X(s), X(t)) = 0 \tag{21}$$

provided F has no point masses.

Proof. The covariance is

$$\mathrm{Cov}(X(s), X(t)) = \int_{-1}^{1} e^{i\lambda(\theta(s)-\theta(t))}\, F\,(d\,\lambda) \tag{22}$$

If $|\theta(s) - \theta(t)| \to \infty$ as $|t - s| \to \infty$, then by the Riemann-Lebesgue lemma, the oscillatory integral converges to zero when F is absolutely continuous with respect to Lebesgue measure. □

8 Conclusion

Injectively time-changed stationary processes provide a mathematically rigorous framework for analyzing non-stationary oscillatory phenomena through spectral methods. The warping function $\theta(t)$ induces a time-dependent modulation while preserving the fundamental spectral structure inherited from the underlying orthogonal increment process.