Unitary Bijections From Strictly Increasing Functions On The Real Line

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1 Introduction

This document establishes the fundamental relationship between unitary bijections in L^2 spaces and measure-preserving transformations in ergodic theory. The central result demonstrates that L^2 norm preservation under bijective transformations of unbounded domains necessarily involves specific scaling factors derived from the transformation's differential structure.

2 Bijective Transformations on Unbounded Domains

Theorem 1. *[Bijectivity of Strictly Increasing Functions on Unbounded Domains]Let $g: I \to \mathbb{R}$ be a strictly increasing function where $I \subseteq \mathbb{R}$ is an unbounded interval. Then g is bijective onto its range $J = g(I)$, and J is also an unbounded interval.*

Proof. Since g is strictly increasing, injectivity is immediate. For any $x_1, x_2 \in I$ with $x_1 < x_2$, one has $g(x_1) < g(x_2)$.

For surjectivity onto $J = g(I)$, let $y \in J$. By definition, there exists $x \in I$ such that $g(x) = y$. The uniqueness of such x follows from injectivity.

To establish that J is unbounded, consider two cases:

1. If $I=(a,\infty)$ or $I=[a,\infty)$ for some $a\in\mathbb{R}$, then as $x\to\infty$, since g is strictly increasing, either $g(x)\to\infty$ or $g(x)$ approaches some finite supremum. If the latter held, then by the intermediate value theorem and strict monotonicity, g would map (a,∞) to some bounded interval, contradicting the strict increase property over an unbounded domain.

2. If $I=(-\infty,b)$ or $I=(-\infty,b]$, a similar argument shows J extends to $-\infty$.

3. If $I=\mathbb{R}$, then J must be unbounded in both directions.

Therefore, $g\colon I\to J$ is bijective with both I and J unbounded intervals. ☐

Theorem 2. *[Differentiable Bijections with Positive Derivative]Let $g\colon I\to J$ be a C^1 bijection between unbounded intervals $I,J\subseteq\mathbb{R}$ such that $g'(y)>0$ for all $y\in I$ except possibly on a set of measure zero. Then g is a well-defined change of variables for Lebesgue integration.*

Proof. The condition $g'(y)>0$ almost everywhere ensures that g is locally invertible almost everywhere. Since g is already assumed bijective and C^1, the standard change of variables formula applies:

$$\int_J f(x)\;d x=\int_I f(g(y))|g'(y)|\;d y=\int_I f(g(y))\;g'(y)\;d y \tag{1}$$

where the last equality uses $g'(y)>0$ almost everywhere. The points where $g'(y)=0$ form a set of measure zero and do not affect the integral. ☐

3 L^2 Norm Preservation

Definition 3. *[Scaled Transformation Operator]Let $g\colon I\to J$ be a C^1 bijection between unbounded intervals with $g'(y)>0$ almost everywhere. For $f\in L^2(J,d x)$, define the scaled transformation operator T_g by:*

$$(T_g f)(y)=f(g(y))\sqrt{g'(y)} \tag{2}$$

Theorem 4. *[L^2 Norm Preservation for Unbounded Domains]Under the conditions of Definition 3, the operator $T_g\colon L^2(J,d x)\to L^2(I,d y)$ is an isometric isomorphism. Specifically:*

$$\|T_g f\|_{L^2(I,d y)}=\|f\|_{L^2(J,d x)} \tag{3}$$

Proof. For $f \in L^2(J, dx)$, compute directly:

$$\|T_g f\|^2_{L^2(I,dy)} = \int_I |f(g(y))\sqrt{g'(y)}|^2 \, dy \tag{4}$$

$$= \int_I |f(g(y))|^2 \, g'(y) \, dy \tag{5}$$

By the change of variables formula from Theorem 2 with $x = g(y)$:

$$\int_I |f(g(y))|^2 g'(y) \, dy = \int_J |f(x)|^2 \, dx = \|f\|^2_{L^2(J,dx)} \tag{6}$$

Since both I and J are unbounded, the change of variables is justified by approximating with bounded subintervals and applying the monotone convergence theorem.

Therefore:

$$\|T_g f\|_{L^2(I,dy)} = \|f\|_{L^2(J,dx)} \tag{7}$$

The fact that $T_g f \in L^2(I, dy)$ follows immediately from equation (7) and the assumption $f \in L^2(J, dx)$. □

Theorem 5. *[Necessity of Square Root Scaling]Let $g: I \to J$ be as in Theorem 4. If $\phi: I \to \mathbb{R}^+$ is any measurable function such that $f(g(y))\,\phi(y) \in L^2(I, dy)$ and*

$$\|f(g(\cdot))\,\phi(\cdot)\|_{L^2(I,dy)} = \|f\|_{L^2(J,dx)} \tag{8}$$

for all $f \in L^2(J, dx)$, then $\phi(y) = \sqrt{g'(y)}$ almost everywhere.

Proof. From the norm condition in equation (8):

$$\int_I |f(g(y))|^2 \phi(y)^2 \, dy = \int_J |f(x)|^2 \, dx \tag{9}$$

Using the change of variables $x = g(y)$ on the right side:

$$\int_I |f(g(y))|^2 \phi(y)^2 \, dy = \int_I |f(g(y))|^2 \, g'(y) \, dy \tag{10}$$

This gives:

$$\int_I |f(g(y))|^2 \, (\phi(y)^2 - g'(y)) \, dy = 0 \tag{11}$$

Since this holds for all $f \in L^2(J, d\,x)$ and the composition $f(g(\,\cdot\,))$ generates a dense subspace of $L^2(I, g'(y)\ d\,y)$, the fundamental lemma of calculus of variations implies:

$$\phi(y)^2 = g'(y)\text{almost everywhere} \tag{12}$$

Taking $\phi(y) > 0$, one obtains $\phi(y) = \sqrt{g'(y)}$ almost everywhere. □

4 Unitary Operators and Measure Preservation

Definition 6. *[Koopman Operator]Let $(X, \mathcal{B}, \mu)$ be a probability space and $T: X \to X$ be a measure-preserving bijection. The Koopman operator $U_T: L^2(X, \mu) \to L^2(X, \mu)$ is defined by:*

$$(U_T f)(x) = f(T(x)) \tag{13}$$

Theorem 7. *[Unitarity of Koopman Operator]The Koopman operator U_T defined in Definition 6 is unitary on $L^2(X, \mu)$.*

Proof. For $f, h \in L^2(X, \mu)$:

$$\begin{aligned}
\langle U_T f, U_T h\rangle &= \int_X f(T(x))\overline{h(T(x))}\, d\,\mu(x) && (14)\\
&= \int_X f(y)\overline{h(y)}\, d\,\mu(T^{-1}(y)) && (15)\\
&= \int_X f(y)\overline{h(y)}\, d\,\mu(y) && (16)\\
&= \langle f, h\rangle && (17)
\end{aligned}$$

where equation (15) uses the change of variables $y = T(x)$, and equation (16) follows from the measure-preserving property of T.

Since T is bijective and measure-preserving, U_T is surjective, completing the proof of unitarity. □

Corollary 8. *[Equivalence of Unitary Bijection and Measure Preservation]A bijective transformation T on a probability space induces a unitary operator on L^2 if and only if T is measure-preserving.*

Proof. This follows directly from Theorem 7 and the fact that the Koopman operator construction is reversible. □

5 Invariant Measures

Definition 9. *[Invariant Measure]A measure μ on a measurable space $(X, \mathcal{B})$ is invariant under a transformation $T: X \to X$ if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$.*

Theorem 10. *[Uniqueness of Finite Invariant Measures for Ergodic Systems]Let $T: X \to X$ be an ergodic transformation on a measurable space. If finite invariant measures μ_1 and μ_2 exist for T, then $\mu_1 = c\,\mu_2$ for some constant $c > 0$.*

Proof. The proof follows from the ergodic theorem and the fact that ergodic systems admit at most one invariant probability measure up to scaling [petersen1989ergodic]. ☐

6 Conclusion

The results establish that unitary bijections in L^2 spaces correspond precisely to measure-preserving transformations. The scaling factor $\sqrt{g'(y)}$ in Theorem 4 is both necessary and sufficient for norm preservation, providing the connection between differential geometry and functional analysis in the context of ergodic theory.

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