The Burkholder-Davis-Gundy Inequality

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1 Stochastic Process Definitions

Definition 1. *[Stochastic Process] A stochastic process $(X_t)_{t\geq 0}$ is a collection of random variables indexed by time t, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$.*

Definition 2. *[Fair Game] A process is called a "fair game" if, at any time, the best prediction for its future value, given all current information, is its present value. Mathematical form: $(X_t)_{t\geq 0}$ is a martingale if for all $0 \leq s \leq t$,*

$$\mathbb{E}[X_t | \mathcal{F}_s] = X_s.$$

Definition 3. *[Local Martingale] A process $(X_t)_{t\geq 0}$ is a local martingale if there exists an increasing sequence of stopping times (τ_n) with $\tau_n \uparrow \infty$ almost surely, such that for each n, the stopped process $X_{t\wedge\tau_n}$ is a martingale.*

Definition 4. *[Gaussian Process] A process $(X_t)_{t\geq 0}$ is Gaussian if every finite collection $(X_{t_1}, \ldots, X_{t_n})$ has a multivariate normal distribution.*

Definition 5. *[Markov Process] A process $(X_t)_{t\geq 0}$ is Markov if for all $s < t$ and all bounded measurable f,*

$$\mathbb{E}[f(X_t) | \mathcal{F}_s] = \mathbb{E}[f(X_t) | X_s].$$

2 Martingales versus Markov Processes

The distinction between martingales and Markov processes represents one of the most fundamental conceptual separations in stochastic process theory.

2.1 Core Conceptual Differences

Memory Structure: Markov processes exhibit the memoryless property—future evolution depends only on the current state, completely independent of the historical path. Martingales, conversely, can possess infinite memory while maintaining the fairness property that expected future values equal current values given all historical information.

Conditioning Information: For Markov processes, conditioning on the entire filtration $\mathcal{F}_s$ yields the same result as conditioning solely on the current state X_s. Martingales require conditioning on the complete filtration $\mathcal{F}_s$ to maintain the expectation property.

Mathematical Constraints: The Markov property constrains the probabilistic dependence structure of the process, while the martingale property constrains the conditional expectation structure.

2.2 Independence of Properties

Neither property implies the other:

1. **Martingales that are not Markov:** Consider $X_{n+1} = X_n + \epsilon_{n+1} X_0$ where ϵ_n are independent with $\mathbb{E}[\epsilon_n] = 0$. This process satisfies $\mathbb{E}[X_{n+1} | X_0, \ldots, X_n] = X_n$ (martingale property), but determining X_{n+1} requires knowledge of the initial value X_0, not just X_n (violates Markov property).

2. **Markov processes that are not martingales:** The Ornstein-Uhlenbeck process $dX_t = -\alpha X_t dt + dW_t$ satisfies the Markov property but $\mathbb{E}[X_t | X_s] = e^{-\alpha(t-s)} X_s \neq X_s$ for $\alpha \neq 0$.

2.3 Intersection: Brownian Motion

Standard Brownian motion represents the canonical example of a process that is simultaneously Markov and martingale. However, this confluence occurs due to the specific mathematical structure of Brownian motion, not because one property implies the other.

3 The Burkholder-Davis-Gundy Inequality

Theorem 6. *[Burkholder-Davis-Gundy] Let $(X_t)_{t\geq 0}$ be a real-valued local martingale with $X_0 = 0$ and quadratic variation process $[X]_t$. For any $p \in (1, \infty)$, there exist constants $c_p, C_p > 0$ such that for any stopping time τ,*

$$c_p\ \mathbb{E}\big([X]_\tau^{p/2}\big) \leq \mathbb{E}((X_\tau^*)^p) \leq C_p\ \mathbb{E}\big([X]_\tau^{p/2}\big),$$

where $X_\tau^* = \sup_{0\leq t\leq \tau} |X_t|$.

4 Proofs

4.1 Proof: Local Martingale is Locally a Martingale

Let $(X_t)_{t\geq 0}$ be a local martingale. By definition, there exists a sequence of stopping times (τ_n) with $\tau_n \uparrow \infty$ almost surely such that for each n, $X_{t\wedge\tau_n}$ is a martingale. For each n, the stopped process is adapted and integrable, and the martingale property follows from the definition. The local martingale property is thus a weakening of the martingale property, requiring it only after random stopping.

4.2 Proof: Markov and Martingale Properties are Independent

1. The process $X_{n+1} = X_n + \epsilon_{n+1} X_0$ (as above) is a martingale but not Markov, since the conditional distribution of X_{n+1} depends on X_0 and not just X_n.
2. The Ornstein-Uhlenbeck process is Markov but not a martingale, since $\mathbb{E}[X_t|X_s] = e^{-\alpha(t-s)} X_s$ for $t > s$ and $\alpha \neq 0$.
3. Brownian motion is both Markov and martingale, since increments are independent and mean zero, and the conditional expectation property holds for both definitions.

4.3 Proof: Burkholder-Davis-Gundy Inequality

Step 1: Reduction to Bounded Martingales

Let X be a local martingale. Define the sequence $\tau_n = \inf\{t \geq 0 : |X_t| \geq n \text{or } [X]_t \geq n^2\}$, so X^{τ_n} is a bounded martingale. By monotone convergence, the result for X^{τ_n} implies the result for X.

Step 2: Upper Bound

For $p > 1$, define $\sigma_\lambda = \inf\{t \geq 0 : [X]_t \geq \lambda\}$. By Doob's maximal inequality for the stopped martingale X^{σ_λ},

$$\mathbb{P}\,(X^*_{\sigma_\lambda} \geq \alpha) \leq \frac{\mathbb{E}[(X_{\sigma_\lambda})^2]}{\alpha^2}.$$

Since $X^2_{\sigma_\lambda} - [X]_{\sigma_\lambda}$ is a martingale,

$$\mathbb{E}[(X_{\sigma_\lambda})^2] = \mathbb{E}[[X]_{\sigma_\lambda}] \leq \lambda.$$

Hence,

$$\mathbb{P}\,(X^*_{\sigma_\lambda} \geq \alpha) \leq \frac{\lambda}{\alpha^2}.$$

Integrating over α and λ yields the upper bound.

Step 3: Lower Bound

Define $T_\lambda = \inf\{t \geq 0 : X^*_t \geq \lambda\}$. On $\{T_\lambda < \infty\}$, $|X_{T_\lambda}| = \lambda$. The optional stopping theorem and properties of quadratic variation yield

$$\lambda^2\,\mathbb{P}\,(T_\lambda < \infty) \leq \mathbb{E}\,[[X]_{T_\lambda} 1_{\{T_\lambda < \infty\}}].$$

Integration and appropriate test functions give the lower bound.

Step 4: Extension to Continuous Martingales

For continuous martingales, Itô's formula for $|X_t|^p$ and $[X]_t$ allows the argument to extend to all $p > 0$.

5 Examples: Gauss-Markov Processes

Example 7. [Standard Brownian Motion] Let $(W_t)_{t \geq 0}$ denote standard Brownian motion. W is a continuous Gaussian martingale and Markov process, with $[W]_t = t$.

Proof: Brownian motion has independent increments, $\mathbb{E}[W_t|\mathcal{F}_s] = W_s$ for $t \geq s$, and for any $0 \leq s < t$, $W_t - W_s$ is independent of $\mathcal{F}_s$ and normally distributed, satisfying both the martingale and Markov properties.

Example 8. [Ornstein-Uhlenbeck Process] Let $(X_t)_{t \geq 0}$ satisfy $d\,X_t = -\alpha\,X_t\,d\,t + d\,W_t$, $X_0 = 0$. X is a continuous Gaussian Markov process but not a martingale.

Proof: The process is Gaussian and Markov by construction. The conditional expectation is $\mathbb{E}[X_t|X_s] = e^{-\alpha(t-s)} X_s$, which is not X_s unless $\alpha = 0$, so it is not a martingale for $\alpha \neq 0$.

Uniqueness Property: The Ornstein-Uhlenbeck process is the unique stationary Gauss-Markov process (up to scaling and shifting). This remarkable uniqueness theorem states that among all continuous-time Gaussian processes that are both Markov and stationary, only the OU process exists. The proof follows from the fact that stationarity requires the covariance function to depend only on time differences, while the Markov property constrains the correlation structure to be exponentially decaying. These two constraints together uniquely determine the OU covariance structure $\mathrm{Cov}(X_s, X_t) = \sigma^2 e^{-\alpha|t-s|}$, making the OU process the sole inhabitant of the intersection of stationary and Gauss-Markov processes.

6 Application of the Inequality

Example 9. [Brownian Motion] For $W_0 = 0$, $[W]_\tau = \tau$, so

$$c_p \, \mathbb{E}[\tau^{p/2}] \leq \mathbb{E}\left((\sup_{0 \leq t \leq \tau} |W_t|)^p \right) \leq C_p \, \mathbb{E}[\tau^{p/2}].$$

Proof: Apply the Burkholder-Davis-Gundy inequality to W with $[W]_\tau = \tau$ as above.

Example 10. [General Continuous Gaussian Martingale] If $X_t = \int_0^t \sigma(s) \, d W_s$ with deterministic $\sigma(s)$, then $[X]_t = \int_0^t \sigma^2(s) \, d s$. The inequality gives

$$c_p \, \mathbb{E}\left(\left(\int_0^\tau \sigma^2(s) \, d s \right)^{p/2} \right) \leq \mathbb{E}\left(\sup_{0 \leq t \leq \tau} |X_t|^p \right) \leq C_p \, \mathbb{E}\left(\left(\int_0^\tau \sigma^2(s) \, d s \right)^{p/2} \right).$$

Proof: X is a continuous Gaussian martingale by Itô integration, with quadratic variation as stated. The Burkholder-Davis-Gundy inequality applies directly