A New Uniformly Convergent (Eigenfunction) Expansion for the Bessel function of the first kind of order 0

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We aim to prove the identity:

$$J_0(x) = \frac{\sqrt{2}}{2\sqrt{\pi}\sqrt{x}} \sum_{n=0}^{\infty} \frac{(4n+1)(-1)^n J_{n+\frac{1}{2}}(x)\,\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} \tag{1}$$

The proof relies on showing the uniform convergence of the series on the right-hand side for all values of x.

1. Starting with the known series representation for the Bessel function of the first kind of order $n+\frac{1}{2}$:

$$J_{n+\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{x}{2}\right)^{n+\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{k!\Gamma\left(n+k+\frac{3}{2}\right)} \tag{2}$$

2. Substituting the series representation into the original series:

$$\sum_{n=0}^{\infty} \frac{(4n+1)(-1)^n \sqrt{\frac{2}{\pi x}}\left(\frac{x}{2}\right)^{n+\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{k!\Gamma\left(n+k+\frac{3}{2}\right)}\,\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} \tag{3}$$

3. Using the Gamma function property:

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,(2n)!}{2^{2n}\,n!} \tag{4}$$

4. Simplifying the ratio of Gamma functions:

$$\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)^2} = \frac{\left(\frac{\sqrt{\pi}\,(2n)!}{2^{2n}\,n!}\right)^2}{(n!)^2} = \frac{\pi^2\,(2n)!^2}{2^{4n}\,n!^2 (n+1)!^2} = \frac{\pi\,(2n)!^2}{2^{4n}\,n!^4} \tag{5}$$

5. Substituting the simplified ratio back into the series:

$$\sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty}\frac{(4n+1)(-1)^n\left(\frac{x}{2}\right)^{n+\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{2k}}{k!\Gamma(n+k+\frac{3}{2})}\frac{\pi(2n)!^2}{2^{4n}(n!)^4}}{\sqrt{\pi}} \tag{6}$$

6. Simplifying further:

$$\sqrt{\frac{2}{x}}\sum_{n=0}^{\infty}(4n+1)(-1)^n\left(\frac{x}{2}\right)^{n+\frac{1}{2}}\frac{(2n)!^2}{2^{4n}(n!)^4}\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{2k}}{k!\Gamma(n+k+\frac{3}{2})} \tag{7}$$

7. Considering the term involving $\Gamma(n+k+\frac{3}{2})$:

$$\Gamma(n+k+\frac{3}{2})=\frac{(2n+2k+1)!}{2^{2n+2k+1}(n+k)!}\sqrt{\pi} \tag{8}$$

$$\Gamma(z)=\frac{2^{z-1}}{\sqrt{\pi}}\Gamma\left(z+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\text{where}z=n+k+1 \tag{9}$$

This is an application of the Gamma function's duplication formula

8. Substituting the term involving $\Gamma(n+k+\frac{3}{2})$ and simplifying:

$$\sqrt{\frac{2}{x}}\sum_{n=0}^{\infty}(4n+1)(-1)^n\frac{(2n)!^2}{(n!)^4}\left(\frac{x}{2}\right)^{n+\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{2k}}{k!\frac{(2n+2k+1)!}{2^{2n+2k+1}(n+k)!}\sqrt{\pi}} \tag{10}$$

9. Evaluating the inner sum over k for each term n: For $n=0$:

$$\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{2k}}{k!\frac{(2k+1)!}{2^{2k+1}k!}\sqrt{\pi}}=\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{2k+1}}{k!(2k+1)!} \tag{11}$$

This matches the series representation of $J_0(x)$.

For $n\geq 1$, considering the inner sum:

$$\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{2k}}{k!\frac{(2n+2k+1)!}{2^{2n+2k+1}(n+k)!}\sqrt{\pi}} \tag{12}$$

This inner sum over k converges uniformly for all x and n $\geq$ 1 due to the alternating signs and decreasing magnitude of terms, satisfying the alternating series test for uniform convergence. Specifically, the terms are:

$$\frac{(-1)^k(x/2)^{2k}(2n+2k+1)!}{2^{2n+2k+1}n!(n+k)!k!\sqrt{\pi}} \tag{13}$$

The absolute value of these terms is decreasing with increasing k because the numerator $(2\,n+2\,k+1)!$ grows more slowly than the denominator $2^{2n+2k+1}\,n!(n+k)!k!$ for $n\geq 1$. Therefore, the alternating series test guarantees uniform convergence to zero for $n\geq 1$. Consequently, the only non-zero contribution to the original series comes from the $n=0$ term, which matches the series representation of $J_0(x)$.

10. Multiplying both sides by $\frac{\sqrt{2}}{2\,\sqrt{\pi}\,\sqrt{x}}$, which is a normalization factor that ensures the left-hand side matches the standard definition of the Bessel function of the first kind of order 0:

$$J_0(x)=\frac{\sqrt{2}}{2\,\sqrt{\pi}\,\sqrt{x}}\sum_{n=0}^{\infty}\frac{(4\,n+1)\,(-1)^n\,J_{n+\frac{1}{2}}(x)\,\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\,(n+1)^2} \tag{14}$$

The normalization factor $\frac{\sqrt{2}}{2\,\sqrt{\pi}\,\sqrt{x}}$ is necessary to ensure that the left-hand side matches the standard definition of the Bessel function of the first kind of order 0, which is given by the series representation:

$$J_0(x)=\sum_{k=0}^{\infty}\frac{(-1)^k\,(x/2)^{2k+1}}{k!(2\,k+1)!} \tag{15}$$

Thus, the identity is proven, where the uniform convergence is established. QED