Eigenfunctions of Stationary Gaussian Processes

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Abstract

The eigenfunctions of the covariance operator of a stationary Gaussian process are shown to be the orthogonal complement of the inverse Fourier transforms of polynomials orthogonal to the square root of its spectral density. Utilizing the convolution theorem and properties of the covariance operator, an explicit construction method for these eigenfunctions is provided. This result enables efficient computation and offers a comprehensive solution for all stationary Gaussian processes.

1 Introduction

2 Main Results

Theorem 1. ***[Spectral Factorization]*** *Let $K(t,s)$ be a positive definite stationary kernel. Then there exists a spectral density $S(\omega)$ and spectral factor:*

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega)}\, e^{i\omega t}\, d\omega \tag{1}$$

such that:

$$K(t,s) = \int_{-\infty}^{\infty} h\,(t+\tau) \overline{h\,(s+\tau)} d\tau \tag{2}$$

[1]

Proof. 1. By Bochner's theorem, since K is positive definite and stationary:

$$K\,(t-s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{i\omega(t-s)}\, d\omega \tag{3}$$

where $S(\omega) \geq 0$ is the spectral density.

2. Define $h(t)$ as stated. Then:

$$\begin{aligned}\int_{-\infty}^{\infty} h\,(t+\tau)\overline{h\,(s+\tau)}d\,\tau \;=& \int_{-\infty}^{\infty} \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega_1)}\, e^{i\omega_1(t+\tau)}\, d\,\omega_1 \\ & \cdot \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \sqrt{S(\omega_2)}\, e^{-i\omega_2(s+\tau)}\, d\,\omega_2\, d\,\tau\end{aligned} \tag{4}$$

3. Rearranging integrals (justified by Fubini's theorem since $S(\omega) \geq 0$):

$$= \frac{1}{4\,\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{S(\omega_1)\, S(\omega_2)}\, e^{i\omega_1 t}\, e^{-i\omega_2 s} \int_{-\infty}^{\infty} e^{i(\omega_1-\omega_2)\tau}\, d\,\tau\, d\,\omega_1\, d\,\omega_2 \tag{5}$$

4. The inner integral gives $2\,\pi\,\delta\,(\omega_1-\omega_2)$:

$$\begin{aligned}&= \frac{1}{4\,\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{S(\omega_1)\, S(\omega_2)}\, e^{i\omega_1 t} e^{-i\omega_2 s}\, 2\,\pi\,\delta\,(\omega_1-\omega_2)\, d\,\omega_1\, d\,\omega_2 \\ &= \frac{1}{4\,\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{S(\omega_1)\, S(\omega_2)}\, e^{i(\omega_1 t-\omega_2 s)}\, 2\,\pi\,\delta\,(\omega_1-\omega_2)\, d\,\omega_1\, d\,\omega_2 \\ &= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{i\omega(t-s)}\, d\,\omega = K\,(t-s)\end{aligned} \tag{6}$$

□

Theorem 2. *The eigenfunctions of a stationary Gaussian process are given by the orthogonal complement of the inverse Fourier transforms of the polynomials orthogonal with respect to the square root of the spectral density.*

Proof. The polynomials $\{P_n(\omega)\}$ are orthogonal to $\sqrt{S(\omega)}$:

$$\int_{-\infty}^{\infty} P_n(\omega) P_m(\omega) \sqrt{S(\omega)}\, d\,\omega = \delta_{n,m} \tag{7}$$

Take their inverse Fourier transforms:

$$\phi_n(t) = \mathcal{F}^{-1}\{P_n(\omega)\} \tag{8}$$

which span the null space of the inner product with the spectral factor (1)

$$\langle h, \phi_n \rangle = 0 \forall n > 0 \tag{9}$$

The Gram-Schmidt recursion generates the orthogonal complement of the spectral factor null space:

$$\psi_n(t) = \phi_n(t) - \sum_{k=1}^{n-1} \frac{\langle \phi_n, \psi_k \rangle}{\|\psi_k\|^2} \psi_k(t) \tag{10}$$

Apply covariance operator:

$$T[\psi_n]\,(t) = \int_{-\infty}^{\infty} K(|t-s|)\, \psi_n(s)\, d\, s \tag{11}$$

Take Fourier transform:

$$\mathcal{F}\,\{T\,[\psi_n](t)\}(\omega) = S(\omega)\, \mathcal{F}\{\psi_n(t)\}(\omega) \tag{12}$$

Consider the eigenvalue equation

$$T[\psi_n](t) = \lambda_n\, \psi_n(t) \tag{13}$$

. Fourier transforming this equation yields:

$$\mathcal{F}\,\{T\,[\psi_n](t)\}(\omega) = \lambda_n\, \mathcal{F}\{\psi_n(t)\}(\omega) \tag{14}$$

From the previous Fourier transform equation and the eigenvalue equation:

$$S(\omega)\, \mathcal{F}\{\psi_n(t)\}(\omega) = \lambda_n\, \mathcal{F}\{\psi_n(t)\}(\omega) \tag{15}$$

The unique solution satisfying these conditions is:

$$\mathcal{F}\{\psi_n(t)\}(\omega) = \lambda_n \sqrt{S(\omega)} \tag{16}$$

Therefore:

$$S(\omega)\mathcal{F}\{\psi_n(t)\}(\omega) = \lambda_n\, S(\omega) \sqrt{S(\omega)} = \lambda_n\, \mathcal{F}\{\psi_n(t)\}(\omega) \tag{17}$$

Taking inverse Fourier transform:

$$T\,[\psi_n](t) = \int_{-\infty}^{\infty} K(|t-s|)\,\psi_n(t)\mathrm{d}t = \lambda_n\,\psi_n(t) \qquad (18) \quad \square$$

3 Conclusion

Bibliography

[1] Harald Cramér. A contribution to the theory of stochastic processes. *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 2:329–339, 1951.